# Reduction of Order

With this chapter we begin our study of second order ODE. Sometimes it is easy to find one solution of a differential equation, and reduction of order con sometimes provide us with a way of using that one solution to find a formula for the general solution.

EXAMPLE 1: Consider the second order differential equation $\ddot{y} - y = 0$. We observe that the function $y_1(t) = e^t$ is a solution on the interval $\mathbb{R}$, and that this solution is non-zero for all $t \in \mathbb{R}$. If $y(t)$ is *any* solution of the ODE, let $u(t)$ be defined by $u(t) = \frac{y(t)}{y_1(t)}$, or $y = uy_1$. We substitute this into the ODE to see that

$$\begin{aligned} 0 &= \ddot{y} - y \\ &= \frac{d^2}{dt^2}\left[ue^t\right] - (ue^t) \\ &= (\ddot{u}e^t + 2\dot{u}e^t + ue^t) - (ue^t) \\ &= \ddot{u}e^t + 2\dot{u}e^t. \end{aligned}$$

Dividing by $e^t$, which is never zero, gives us the following differential equation for $u$:

$$\ddot{u} + 2\dot{u} = 0.$$

Make the substitution $v = \dot{u}$ to obtain

$$\dot{v} + 2v = 0.$$

This equation can be solved using the method of integrating factors (the integrating factor is $e^{2t}$):

$$\begin{aligned} \frac{d}{dt}\left[e^{2t}v\right] &= 0 \\ e^{2t}v &= C \\ v &= Ce^{-2t} \end{aligned}$$

Integrating this shows that $u = Ce^{-2t} + D$, and inserting this into the equation $y = uy_1$ we see that

$$y(t) = (Ce^{-2t} + D)e^t = Ce^{-t} + De^t$$

is the general solution of the ODE. $\square$

The general process is:

- Find a solution $y_1$ of the ODE;
- set $y = uy_1$, and apply this substitution for $y$ in the ODE;
- simplify to find a differential equation for $u$;
- find a general solution for $u$;

- the product $y = uy_1$ gives the general solution for the ODE on the set where $y_1 \neq 0$.

The last point is am important one: because we define $u$ by $u = \frac{y}{y_1}$, this process is only guaranteed to give a formula for a general solution on the set where $y_1 \neq 0$. One might get lucky and obtain a general solution on a larger domain, but there is no guarantee that will happen in general.

EXERCISE 1: Verify that $y_1(x) = e^{-x}$ is a solution of the differential equation $y'' + 3y' + 2y = 0$. Then use reduction of order to find a general solution of this ODE defined on $\mathbb{R}$.

EXERCISE 2: Verify that $y_1(x) = e^{2x}$ is a solution of $y'' - 2y' = 0$. The use reduction of order to find a general solution on $\mathbb{R}$.

EXERCISE 3: Verify that $y_1(t) = t$ is a solution of the ODE $t^2\ddot{y} + 2t\dot{y} - 2y = 0$. Then use reduction of order to find the general solution of this ODE defined on the interval $(0, \infty)$.

EXERCISE 4: Verify that $y_1(t) = \sin(2t)$ is a solution of the ode $\ddot{y} + 4y = 0$. Then use reduction of order to find a general solution on $\mathbb{R}$. *(Note: Because $y_1(t) = 0$ for $t = \frac{k\pi}{2}$, the method only guarantees a solution on an interval of the form $\left(\frac{k\pi}{2}, \frac{(k+1)\pi}{2}\right)$; therefore you will need to verify directly that the formula you obtain is a solution on $\mathbb{R}$.)*

Now we will explore the theory of this method – that is to say, we will discuss why it works.

Begin with an ODE of the form

$$a(x)y'' + b(x)y' + c(x)y = 0,$$

and a function $y_1(x)$ which is a solution of this equation. Let $I$ be an open interval where $y_1 \neq 0$. Then if $y(x)$ is *any* solution of this ODE on $I$, we can define $u = \frac{y}{y_1}$ on $I$; thus $y = uy_1$. The product rule gives us $y' = u'y_1 + uy_1'$ and $y'' = u''y_1 + 2u'y_1' + uy_1''$. Inserting these into the ODE yields

$$\begin{aligned} 0 =& a(x)y'' + b(x)y' + c(x)y \\ &= a(x)\left(u''y_1 + 2u'y_1' + uy_1''\right) + b(x)\left(u'y_1 + uy_1'\right) + c(x)(uy_1) \\ &= a(x)y_1u'' + (2a(x)y_1' + b(x)y_1)u' + (a(x)y_1'' + b(x)y_1' + c(x)y_1)u, \end{aligned}$$

and the last term in the last line is zero on $I$ since $y_1$ solve the ODE there. we are thus left with

$$a(x)y_1u'' + (2a(x)y_1' + b(x)y_1)u' = 0.$$

If we make the subsitution $v = u'$, we get

$$a(x)y_1(x)v' + (2a(x)y_1'(x) + b(x)y_1(x)v = 0.$$

This is a first order equation that can typically be solved to find a general formula for $v$, and integrating that solution gives us a general formula for $u$; inserting that formula for $u$ into the equation $y = uy_1$ gives us a general formula for $y$ on $I$.

It is because of the fact we always obtain an equation of the form $\tilde{a}(x)u'' + \tilde{b}(x)u' = 0$, which can be reduced to a first order equation via the substitution $v = u'$, that this method gets it name.

# Problems

PROBLEM 1: Verify that the function $y_1(t) = e^t$ is a solution of the third order ODE $y''' - y = 0$. Then let $y$ be any other solution of the ODE, and use reduction of order to show that $y = ue^x$, where $u$ is a solution of the ODE $u''' + 3u'' + 3u' = 0$. *(Do not try to solve this ODE.)*

PROBLEM 2: Find a power function $y_1(x) = x^n$ that solves the differential equation $x^2y'' - 3xy' + 4y = 0$. Then use reduction of order to find a general solution on the interval $(0, \infty)$.

PROBLEM 3: Find a power function $y_1(x) = x^n$ that solves the differential equation $x^2y'' - 7xy' - 6y = 0$. Then use reduction of order to find a general solution on the interval $(0, \infty)$.

PROBLEM 4: Consider the ODE $y'' - 2\alpha y' + a^2y = 0$, where $\alpha$ is a constant. **(a)** Find a value of $r$ such that $y_1(x) = e^{rx}$ is a solution of this ODE. *(Hint: Do this by substituting $e^{rx}$ for $y$ and solving for $r$.)* **(b)** Use the solution you found in part (a) and reduction of order to find a general solution of the ODE.

PROBLEM 5: Consider the ODE $y'' - (\alpha + \beta)y' + \alpha\beta y = 0$, where $\alpha,\ \beta$ are constants and $\alpha \neq \beta$. **(a)** Prove that the only values of $r$ such that $y_1(x) = e^{rx}$ solves the ODE are $r = \alpha$ and $r = \beta$. **(b)** Use the solution $y_1(x) = e^{\alpha x}$ and reduction of order to find the general solution of the ODE. Simplify your solution.