Second Order Constant Coefficient ODE

We would next like to write down solutions for second-order constant coefficient linear ODE. These have the form:

$$ay'' + by' + cy = f(x).$$

Here, the coefficients a, b and c are constant, and we assume that $a \neq 0$ so that the equation will indeed be second order. We will first focus on homogeneous equations:

$$ay'' + by' + cy = 0.$$

Let us seek some inspiration by studying the similar problem for first-order equations.

The general solution of the first order homogeneous constant-coefficient linear equation

$$ay' + by = 0, \quad a \neq 0.$$

is

$$y = Ce^{-bt/a},$$

which can be verified by the method of integrating factors. If $b = 0$, then the solution is just a constant function $y = C$. Notice that if $y = Ae^{rt}$ satisfies the ODE $ay' + by = 0$, then the constant r satisfies the algebraic equation $ar + b = 0$. This will serve as our starting point for trying to understand second order equations.

EXERCISE 1: Prove that if $y = Ae^{rx}$ satisfies the differential equation $ay'' + by' + cy = 0$, then r is a solution of the algebraic equation $ar^2 + br + c = 0$.

The algebraic equation $ar^2 + br + c = 0$ is called the **characteristic equation** for the ODE $ay'' + by' + cy = 0$. The previous exercise indicates that there is a connection between the solutions of the ODE and the solutions of the corresponding characteristic equation. The following exercise completes the description of that connection.

EXERCISE 2: Prove that if r is a root of $ar^2 + br + c = 0$, then for any constant coefficient A, the function $y = Ae^{rt}$ satisfies the differential equation $ay'' + by' + cy = 0$. *(Note that r might equal zero.)*

Also, because the ODE $ay'' + by' + cy = 0$ is linear in (y, y', y''), we know that if y_1 and y_2 are both functions that satisfy the differential equation, then so does the sum $y = y_1 + y_2$. This and the results of the previous exercises demonstrate that the following is true: If r_1 and r_2 are roots of the characteristic equation $ar^2 + br + c = 0$, then functions of the form $y = Ae^{r_1 t} + Be^{r_2 t}$ satisfy the ODE $ay'' + by' + c = 0$.

In fact:

> If the characteristic equation for
> $$ay'' + by' + cy = 0$$
> has two distinct roots r_1 and r_2, then the formula
> $$y = Ae^{r_1 t} + Be^{r_2 t}$$
> provides us with the **general solution** on $\mathbb{R}$ of this differential equation.

By distinct, we mean that $r_1 \neq r_2$. (The fact that it is indeed the *general solution* is explored in the problem set at the end of this chapter.) We still need to investigate what to do if the characteristic equation has a repeated root (that is to say, if it is equivalent to the equation $a(r - r_1)^2 = 0$). But first let us explore a few examples involving non-repeated roots.

EXAMPLE 1: Find the solution of the initial value problem $y''+5y'+6 = 0$, $y(0) = 0$, $y'(0) = 2$.

First we identify the characteristic equation for this ODE: $r^2+5r+6 = 0$. Solving this algebraic equation gives us the solutions $r_1 = -2$ and $r_2 = -3$. Therefore, the general solution of the ODE is

$$y = Ae^{-2x} + Be^{-3x}.$$

If we substitute in the given initial conditions, we obtain the system of equations:

$$0 = A + B, \quad 2 = -2A - 3B$$

Solving this system of equations lead to the values $A = 2, B = -2$. Consequently, the solution of this initial value problem is

$$y = 2e^{-2t} - 2e^{-3t}.$$

□

EXERCISE 3: Solve the following initial value problems:

- $y'' - y' + 6 = 0, \quad y(0) = 2, \quad y'(0) = 0$
- $2y'' - 5y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 2$

The process identified above even works when the solutions of the characteristic equation are complex numbers, though in that case it is often more convenient to write the solutions in a different form.

Recall that if a complex number is written in the form $\alpha + i\beta$, where α and β are real, then $e^{\alpha+i\beta} = e^{\alpha}(\cos(\beta) + i\sin(\beta))$. Also, if the characteristic equation has real coefficients but complex roots, the the roots must be complex conjugates of one another. Therefore the general solution has the form:

$$
\begin{aligned}
y &= Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} \\
&= Ae^{\alpha x}(\cos(\beta x)_i \sin(\beta x)) + Be^{\alpha x}(\cos(-\beta x) + i\sin(-\beta x) \\
&= Ae^{\alpha x}(\cos(\beta x)_i \sin(\beta x)) + Be^{\alpha x}(\cos(-\beta x) - i\sin(-\beta x) \\
&= (A+B)e^{\alpha x}\cos(\beta x) + (A-B)ie^{\alpha x}\sin(\beta x)
\end{aligned}
$$

If we introduce new coefficients C and D satisfying $C = A + B$ and $D = (A - B)i$, then we obtain the form

$$y = Ce^{\alpha x}\cos(\beta x) + De^{\alpha x}\sin(\beta x).$$

This gives us:

> If the characteristic equation $ar^2+br+c=0$ has complex roots of the form $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, then the general solution on $\mathbb{R}$ of the ODE $ay'' + by' + cy = 0$ can be written in the form
>
> $$y = Ce^{\alpha x}\cos(\beta x) + De^{\alpha x}\sin(\beta x).$$

EXERCISE 4: Solve the following initial value problems.

- $y'' + 2y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$
- $3\ddot{y} + 5\dot{y} + 2y = 0, \quad y(0) = 2, \quad y'(0) = 0.$

Finally, we need to determine how to find a general solution to $ay'' + by' + cy = 0$ when the characteristic equation yields only one root, r_1. In this case, we know that the expression $e^{r_1 x}$ gives one solution of the ODE which is never zero. We will use reduction or order to find the general solution. Let y be any solution of the ODE, and write $y = ue^{r_1 x}$.

The product rule gives us $y'(x) = u'e^{r_1 x} + r_1 ue^{r_1 x}$ and $y''(x) = u''e^{r_1 x} + 2r_1 u'e^{r_1 x} + r_1^2 ue^{r_1 x}$. Now we can substitute $ue^{r_1 x}$ for $y(x)$ in the differential equation:

$$
\begin{aligned}
0 &= ay'' + by' + cy \\
&= a(u''e^{r_1 x} + 2r_1 u'e^{r_1 x} + r_1^2 ue^{r_1 x}) \\
&\qquad + b(u'e^{r_1 x} + r_1 ue^{r_1 x}) + c(ue^{r_1 x}) \\
&= au''e^{r_1 x} + (2ar_1 + b)u'e^{r_1 x} + (ar_1^2 + br_1 + c)ue^{r_1 x} \\
&= au''e^{r_1 x}.
\end{aligned}
$$

In the last line we used the facts that $ar_1^2 + br_1 + c = 0$, which is true since r_1 is a root of the characteristic equation, and $2ar_1 + b = 0$, which follows because r_1 is a *double root* of the characteristic equation:

$$ar^2 + br + c = a(r - r_1)^2,$$

and expanding the right side yields

$$ar^2 + br + c = ar^2 - 2ar_1r + ar_1^2;$$

equating coefficients gives us

$$b = 2ar_1 \quad \text{and} \quad c = ar_1^2.$$

Now we have the differential equation $au''e^{r_1x} = 0$, or just $u'' = 0$, and therefore $u(x) = Ax + B$ for some constants A and B. Consequently, $y = (Ax + B)e^{r_1x}$, and this is the general solution when the characteristic equation has a double root.

> If the characteristic equation $ar^2+br+c = 0$ has a double root r_1, then the general solution on $\mathbb{R}$ of the ODE $ay''+by'+cy = 0$ can be written in the form
>
> $$y = Axe^{r_1x} + Be^{r_1x}.$$

EXERCISE 5: Solve the following initial value problems.

- $y'' - 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 4$
- $3\ddot{y} + 18\dot{y} + 27y = 0, \quad y(0) = 2, \quad y'(0) = 3.$

EXERCISE 6: Solve the following initial value problems.

- $y'' + 9y = 0,\ y(0) = 2,\ y'(0) = -2$
- $\frac{d^2y}{dv^2} + y = 0,\ y(0) = 0,\ y'(0) = 3$
- $\ddot{w} - 3\dot{w} - 4w = 0,\ w(1) = 0,\ w'(1) = 2$
- $4y'' - 4y' + y = 0,\ y(0) = 0,\ y'(0) = 0$
- $\ddot{v} - 4\dot{v} + 4v = 0,\ v(0) = 1,\ \dot{v}(0) = 2$
- $y'' + 4y' + 5y = 0,\ y(0) = 0,\ y'(0) = 3$

Problems

PROBLEM 1: Let $y(t)$ be the solution of the initial value problem $\ddot{y} + 2\dot{y} + \gamma y = 0$, where γ is a real constant. Find $\lim_{t\to\infty} y(t)$. Does the answer depend on the value of γ?

PROBLEM 2: *In this problem, you will verify that our formula for the case when the characteristic equation has two distinct coefficients is in fact the general solution – that is to say , that any solution of the ODE can be written in this form.*

Suppose that $ay'' + by' + cy = 0$ has a characteristic equation $ar^2 + br + c$ with two distinct roots, r_1 and r_2. **(a)** Verify directly that $y_1 = e^{r_1 x}$ is a solution of the ODE. **(b)** Let y be an arbitrary solution of the ODE, and write $y(x) = u(x)e^{r_1 x}$. Use reduction-of-order to prove that $u'' + \left(2r_1 + \frac{b}{a}\right) u' = 0$. **(c)** Use the substitution $v = u'$ and the method of integrating factors to deduce that the general solution for u is $u(x) = Ce^{-(2r_1 + b/a)x} + D$. **(d)** Conclude that $y = Ce^{-(r_1 + b/a)x} + De^{-r_1 x}$. **(e)** Because r_1 and r_2 are both solutions of the characteristic equation, it must be true that $ar^2 + br + c = a(r - r_1)(r - r_2)$. Equate coefficients here to prove that $r_2 = -(r_1 + b/a)$. **(f)** Conclude that $y(x) = Ce^{r_2 x} + De^{r_1 x}$.

PROBLEM 3: Find a general solution for the differential equation $y''' + 3y'' + 3y' + y = 0$.

PROBLEM 4: Solve the initial value problem $y^{(4)} - 5y^{(2)} + 4y = 0,\ y(0) = 4, y'(0) = 4, y''(0) = 10, y'''(0) = 16$.