# Concepts of Ordinary Differential Equations

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Edition 1

# Contents

# Preface

This text is intended for a first course in ordinary differential equations, typically taken at the sophomore or junior level by science and engineering majors. Many calculus students are introduced to differential equations and separation of variables, so Chapter 1 begins assuming that material has been seen before. (If it has not, or if students need a review, they should work through Chapter 1 and Appendix A pretty much simultaneously.)

Paging through the book will reveal that some of the Exercises are embedded within the reading, and others are at the end of each chapter. Don't skip any Exercise that is embedded in the reading! They form an integral part of the text, and the insights gained from working these Exercises is often necessary to completely appreciate what follows them. Instructors may choose to cover some of the Exercises in lecture, but students should make sure they can recreate the solutions on their own. The last several Additional Exercises at the ends of the chapters tend to be more challenging, and it is up to an instructor to decide which problems are appropriate for his or her students.

I intend for students to *actually read* this text. With the internet providing nearly limitless access to information, it no longer seems necessary to me for a textbook to be a complete reference source. Therefore, I have tried to emphasize clarity rather than completeness. There are only a few footnotes and no marginal notes to break up the flow of reading.

Chapters 1-4 and 7-9 are pretty much fundamental for this kind of course. Chapter 5 on Taylor methods can be done at any time after Chapter 4, and nothing else in the text assumes it. Chapters 10-11-12 can also be covered immediately after Chapter 4 if the instructor wants to get to Laplace transforms early. Great practical utility of Laplace transforms is found with discontinuous driving functions, which are discussed in Chapter 11. I therefore believe that, if you're going to do Chapter 10, you really should do Chapter 11, also. Chapter 12 assumes Chapter 10. Chapters 13-14 can be covered any time after Chapter 4, and doing so before covering Chapter 7 would provide an alternative route to deriving the general solutions to second order constant coefficient equations (instead of using reduction of order).

Chapter 6, on existence and uniqueness, it very much subject to an instructor's taste. Some might want to go through a detailed proof of the results. At the other extreme, one might just cite the theorem and then look at an example of what can happen when the hypotheses are not met. My personal preference is somewhere in between: I like students to learn how to calculate Picard iterates and to get a sense of how they might converge to a solution by carefully working through an easy example; then I wave my hands a bit in describing how to generalize from the example to a real proof. There's a lot of flexibility here.

The appendices contain important information, but some of it is likely already known by students (in particular, the material on matrices and separation of variables might be review). It is probably a good idea to spend a day on Appendix B (complex numbers) before starting either Chapter 7, 10 or 13, (whichever is done first). If you cover reduction of order in Appendix C, you have choices: you can do it before Chapter 7, so that it is an available tool which can be used to find a general solution when the characteristic equation has a repeated root; alternatively, you can introduce the necessary idea in the process of finding general solutions in Chapter 7 and then, at a later time, generalize that trick into a larger technique.

My sincere thanks go to my students who have taken this course as I piloted these materials. Many students contributed suggestions and improvements. In particular, I would like to thank a few who have given me substantial feedback: Brianna Kuypers, Gail Scott, Chris Nason, Ryan Smoots, and Kelly Sindelar. I am also indebted to my colleagues Adrienne Palmer and Sarah Massengill.

# Part 1

# First Order Equations

CHAPTER 1

# The Nature of Differential Equations

> **Prototype Question:** A large tank contains 100 gallons of pure water. Brine solution that contains 50 grams of salt per gallon of water is added to the tank at a rate of 3 gallons per minute. The liquid in the tank is thoroughly mixed, and it drains from the tank at 3 gallons per minute as well. How long will it take until there is one kilogram of salt in the tank? And how much salt will there be in the tank in the long term?

The problem above describes a quantity (the mass of salt in a tank) which is changing over time. The description tells us how that rate of change depends on other factors. Salt is added to the tank as part of a brine solution. Simultaneously, salt leaves the tank as part of the liquid that drains. We can use the language of calculus (in particular, derivative notation) to describe this rate of change precisely.

Suppose we let $g(t)$ represent the number of grams of salt in the tank after $t$ minutes have elapsed. If we can find an explicit formula for $g(t)$ then we should be able to answer the questions posed above. As a first step to coming up with such a formula for $g(t)$, we're going to write down a formula that describes its derivative.

The net rate of change of $g$ at any instant is

$$\frac{dg}{dt} = (\text{rate in}) - (\text{rate out}),$$

where "rate in" describes how fast salt is entering the tank and "rate out" describes how fast salt is leaving it. The rate at which salt is entering the tank is determined by multiplying the rate at which brine solution enters the tank by the concentration of salt in the brine:

$$\text{rate in} = \frac{50 \text{ grams}}{1 \text{ gallon}} \times \frac{3 \text{ gallons}}{1 \text{ minute}} = 150 \frac{\text{grams}}{\text{minute}}.$$

Similarly, we can find the rate out by multiplying the rate at which liquid is leaving the tank by the concentration of salt in that liquid. But the concentration of salt already in the tank is changing – it depends on how much salt is in the tank at that instant. The volume of liquid in the tank remains constant at 100 gallons, and the mass of salt is represented by the function $g$. Therefore we can write

$$\text{rate out} = \frac{g \text{ grams}}{100 \text{ gallons}} \times \frac{3 \text{ gallons}}{1 \text{ minute}} = \frac{3g}{100} \frac{\text{grams}}{\text{minute}}.$$

Therefore the instantaneous rate of change over time for the function $g$ is described by the equation

$$\frac{dg}{dt} = 150 - \frac{3g}{100} \qquad \frac{\text{grams}}{\text{minute}}.$$

This equation is a precise mathematical description of how the mass of salt in the tank is changing over time. There is one other fact given in the problem statement that is necessary to find a solution to this question: because the tank begins with only pure water, the initial mass of salt in the tank is zero grams, so that

$$g(0) = 0.$$

These two facts – the equation describing the rate of change and the initial value of the function – will allow us to find a formula for $g(t)$.

The technique we will employ to find a formula for $g(t)$ is called separation of variables. It is usually taught in a second course on calculus, and a reader who wishes to review this technique will find it in Appendix A.

Beginning with the equation $\frac{dg}{dt} = 150 - \frac{3g}{100}$, we first rewrite the right side as a single quotient,

$$\frac{dg}{dt} = \frac{15000 - 3g}{100},$$

and then we formally separate the variables $g$ and $t$ as follows:

$$\frac{dg}{15000 - 3g} = \frac{dt}{100}.$$

The previous step doesn't make sense all by itself – it is a notational shorthand – until we anti-differentiate both sides:

$$\int \frac{dg}{15000 - 3g} = \int \frac{dt}{100}.$$

Completing the integration gives us

$$-\frac{1}{3} \ln |15000 - 3g| = \frac{t}{100} + C.$$

Using the fact that $g(0) = 0$ (that is to say, $g = 0$ when $t = 0$) to solve for the unknown constant of integration gives us

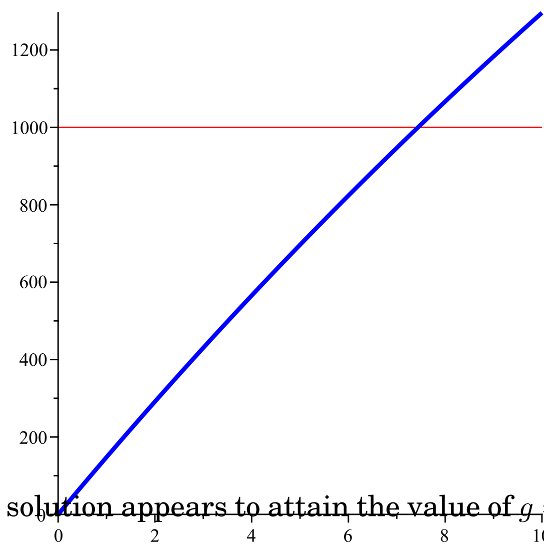
$$-\frac{1}{3}\ln|15000| = C,$$

and solving for $g$ in terms of $t$ yields

$$g = 5000 - 5000e^{-3t/100}.$$

EXERCISE 1: Fill in the missing details in the calculation above to solve for $g$.

Here's a graph of the solution we calculated above:



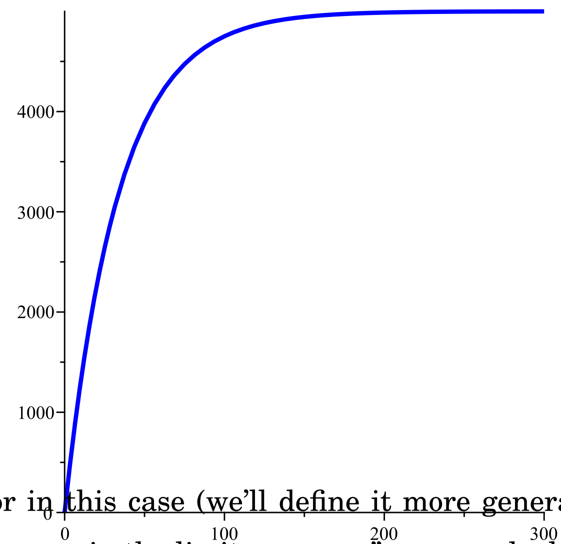

We can see that the solution appears to attain the value of $g = 1000$ grams (1 kilogram) sometime between $t = 7$ and $t = 8$ seconds. We can calculate the exact value using this formula:

$$1000 = 5000 - 5000e^{-3t/100}$$

and solving for $t$ algebraically gives us:

$$t = \frac{-100}{3}\ln\left(\frac{4}{5}\right) \approx 7.4 \text{ minutes.}$$

The other issue we want to address is the long-term behavior of the function $g$: what happens to the value of $g(t)$ as $t$ continues to increase? A graph of the solution over a longer time interval illustrates this behavior:




Long-term behavior in this case (we'll define it more generally later in the chapter) really means "what happens in the limit as $t \mapsto \infty$", so we calculate:

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} 5000 - 5000e^{-3t/100} = 5000 \text{ grams.}$$

That is to say, as time passes the mass of salt in the tank will get closer and closer to 5 kilograms. □

We were able to find a precise solution for the problem above because we were able to write down the function $g$ that satisfies the equation $\frac{dg}{dt} = 150 - \frac{3g}{100}$ and the condition $g(0) = 0$. Equations like this one which describe the rate of change of a function are called differential equations.

A **differential equation** is an equation involving an unknown function and its derivatives. Here are a few examples:

- $\frac{du}{dt} = 3u + 2t$, where $u(t)$ is the unknown function
- $\frac{dy}{dx} = y^2$, where $y(x)$ is the unknown function
- $\frac{\partial f}{\partial x} = 2 + \frac{\partial f}{\partial y}$, where $f(x, y)$ is the unknown function
- $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial t}$, where $f(x, y, t)$ is the unknown function.

The first two examples above are **ordinary differential equations** because the unknown functions are functions of just one variable, hence the derivatives are ordinary derivatives as studied in single variable calculus; this is in contrast to the last two examples where the unknown is a function of two or more variables so that the derivatives are partial derivatives, as studied in multivariable calculus. These last two are examples of **partial differential equations**. Generally speaking, the study of partial differential equations requires more mathematical background and is usually reserved for a second course on differential equations. The abbreviation ODE is used to mean either an ordinary differential equation or equations (it can be either singular or plural, depending on context). The abbreviation PDE is used similarly for a partial differential equation or equations.

EXERCISE 2: Classify each of the following as either an ODE or a PDE.

(1) $\left(\frac{dy}{dx}\right)^2 = y^2 + \frac{d^2y}{dx^2}$
(2) $u_x = u_y$

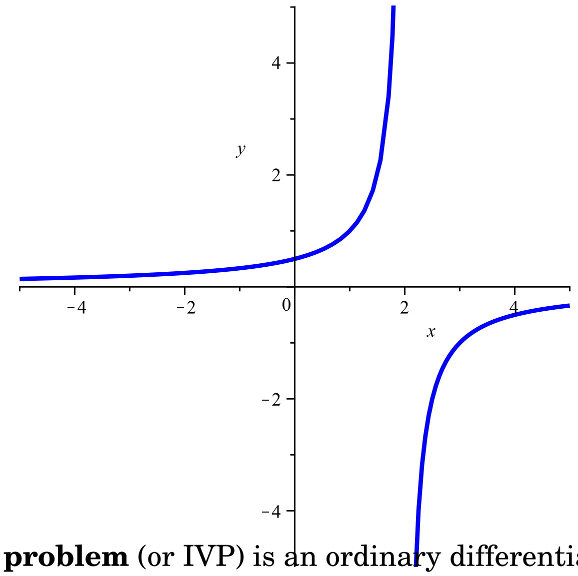
For the remainder of this text, we will only concern ourselves with ordinary differential equations.

A **solution** of an ODE is a function, say $y(x)$, such that $y$ and its derivatives satisfy the differential equation for all $x \in I$, where $I$ is an interval in $\mathbb{R}$. In particular, $y(x)$ must be defined at every point $x \in I$ for us to say that it is a solution on $I$. We call $x$ the **independent variable**, and the symbol representing the function, $y$, is called the **dependent variable**.

EXAMPLE 1: Consider the function $y = \frac{1}{2-x}$. This function is a solution of the differential equation $y' = y^2$ on the interval $I = (-\infty, 2)$ because

$$y' = \frac{d}{dx}\left[(2-x)^{-1}\right] = -(2-x)^{-2}(-1) = \left(\frac{1}{2-x}\right)^2 = y^2.$$

It is also a solution on the interval $I = (2, \infty)$, but notice that because of the discontinuity of $y(x)$ at $x = 2$, this function is *not* a solution in the interval $I = \mathbb{R}$. The following graph shows the function $y = \frac{1}{2-x}$, which has two components separated by the discontinuity at $x = 2$. Either component can be considered a solution of the differential equation, but not both together, because we only consider a function to be a solution if it is defined throughout an entire connected interval.




□

An **initial-value problem** (or IVP) is an ordinary differential equation together with an **initial condition** of the form $y(x_0) = y_0$, where $x_0$ and $y_0$ are given. The value of $y_0$ is called the **initial value**. Here's an example:

$$\frac{dy}{dx} = 2y + x^2, \quad y(0) = 1.$$

A **solution** of an initial value problem is a solution of the differential equation defined on an interval $I$ in $\mathbb{R}$ that contains $x_0$ and such that the initial condition $y(x_0) = y_0$ holds true. The largest interval $I$ containing $x_0$ for which the function $y$ is defined is called the **domain of definition** (or the **interval of definition**) for the solution of the IVP.

Once we know the domain of definition for a solution, we can discuss the solution's **long-term behavior**: if a solution $y(t)$ has as its domain of definition an interval $(a, b)$, then the long-term behavior of $y$ is $\lim_{t\to b} y(t)$. It is often (but not always) the case that $b = \infty$.

EXERCISE 3: Consider the initial value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = 1.$$

Prove that the functions $y_1(t) = \frac{1}{1-t}$ and $y_2(t) = \frac{1}{2-t}$ both satisfy the differential equation, but that only one of these also satisfies the initial condition. Which one is it, and what is its domain of definition?

The prototype problem that began this chapter allowed us to illustrate how one can sometimes find a solution to an initial value problem using separation of variables. Here is another example with a different application.

EXAMPLE 2: The population of a colony of bacteria grows in such a way that its instantaneous rate of change is proportional to the size of the population at that time. This is because bacteria have a predictable life cycle, and at any given moment in a large population of bacteria, a certain fraction are ready to reproduce. If there are 4 million bacteria to start with, and after 20 hours there are 4.2 million, find a function that predicts the size of the bacteria population after $t$ hours.

**Solution:** The assumption that the instantaneous growth rate is proportional to the size of the population can be stated in terms of a derivative:

$$\frac{dP}{dt} = kP,$$

where $P(t)$ is the size of the population (let's use units of millions of bacteria) at time $t$ (in hours). The symbol $k$ here represents a constant of proportionality. We will need to find the appropriate value of $k$ as part of our solution. Separating variables gives us

$$\int \frac{dP}{P} = \int k\ dt,$$

and anti-differentiating produces the equation

$$\ln|P| = kt + C.$$

(Two anti-derivatives of the same function must differ by a constant, hence the presence of the $C$ in this equation.) We can exponentiate both sides to obtain

$$|P| = e^{kt+C},$$

and we can remove the absolute value notation if we introduce a plus/minus symbol on the other side:

$$P = \pm e^{kt+C}.$$

Now because $P = P(t)$ is a function, it must have just one output for each input, and therefore we cannot leave the $\pm$ symbol in place.[1] We will need to make a choice whether $P = e^{kt+C}$ or $P = -e^{kt+C}$. Because $P$ represents population and is thus a positive quantity,

[1]We could imagine needing to make a different choice of $\pm$ at each point in the domain for $t$, however because solutions must be continuous functions, the choice will need to be the same for all $t$ in the domain.

and the expression $e^{kt+C}$ is necessarily positive, we can conclude that the former expression is appropriate. Thus

$$P = e^{kt+C} = e^C e^{kt}.$$

We have rewritten the equation in this form to point out that we don't really need to solve for $C$, just for $e^C$: insert the initial condition that $P = 4$ when $t = 0$ to obtain

$$4 = e^C e^0 = e^C,$$

and now we have

$$P = 4e^{kt}.$$

Now we are in a position to determine the appropriate value of $k$ using the initial condition $P(20) = 4.2$:

$$4.2 = 4e^{k(20)}$$

implies that

$$k = \frac{\ln(1.05)}{20}.$$

Therefore

$$P = 4e^{\ln(1.05)t/20},$$

and this can be written in a variety of ways, but the simplest form is probably

$$P = 4(1.05)^{t/20}.$$

That is to say, after $t$ hours there will be $P(t) = 4(1.05)^{t/20}$ million bacteria. □

In the preceding example, the unknown function was $P(t)$, and it was determined by two facts:

- it satisfied the differential equation $\frac{dP}{dt} = kP$ and
- it satisfied the initial condition $P(0) = 4$.

These two conditions constituted the initial value problem. Before we used the initial condition $P(0) = 4$, we had come up with a formula that could be written as $P = Ae^{kt}$. Any value we choose for $A$ would give us a solution of the differential equation, and any initial value could be satisfied by selecting an appropriate value for $A$ (for example, to satisfy $y(0) = y_0$, use $A = y_0$ in this formula). Because we can satisfy *any* initial condition by choosing an appropriate value for the parameter $A$, the formula $y = Ae^{kt}$ is called a **general solution** for the differential equation.

EXAMPLE 3: Find a general solution of $y' = xy^2$, and then solve the initial value problem with $y(0) = 0$ and then with $y(0) = 4$.

**Solution:** We start with separation of variables:

$$\int \frac{dy}{y^2} = \int x dx.$$

Integrating gives us

$$-\frac{1}{y} = \frac{x^2}{2} + C.$$

Isolating $y$ gives us

$$y = \frac{-1}{\frac{x^2}{2} + C}.$$

This is a cumbersome way to write the solution, so let's replace the symbol $C$ with $D = -2C$ so that we can write

$$y = \frac{2}{D - x^2}.$$

Observe that this formula can be used to satisfy any initial condition for $y(x_0)$ *except* $y(x_0) = 0$. The constant function $y(x) = 0$, however, gives us a solution of the differential equation with zero as an initial value. Therefore, the general solution can be expressed as

$$y = \begin{cases} 0 & \text{if } y(x_0) = 0 \\ 2(D - x^2)^{-1} & \text{otherwise} \end{cases}.$$

Consequently, the solution of the initial value problem $y' = xy^2,\ y(0) = 0$ is the constant function $y(x) = 0$. The solution of the initial value problem $y' = xy^2, y(0) = 4$ can be found by using the general formula and solving for $D$:

$$4 = 2(D - (0)^2)^{-1} \implies D = \frac{1}{2},$$

so $y(x) = \frac{2}{\frac{1}{2} - x^2}$. □

EXERCISE 4: Solve the initial value problem $\frac{dy}{dx} = x(y^2 + 1),\ y(0) = 1$. What is the domain of definition for the solution?

EXERCISE 5: Use separation of variables to find a function that satisfies the differential equation $\frac{dy}{dx} = xe^x y$ and the initial condition $y(0) = 1$.

EXERCISE 6: Find a function $x(t)$ that satisfies the differential equation $\frac{dx}{dt} = x^2 - 1$ and the initial condition $x(0) = 0$.

The last few examples and exercises above all contain **first-order** differential equations, because the first derivative is the highest order of derivative that appears in the equation. In general, we call a differential equation $n^{th}$ **order** if the $n^{th}$ derivative is the highest order derivative in the equation. With this terminology, the equation

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = 7$$

is $3^{rd}$ order.

EXERCISE 7: Classify the order of each of the following differential equations:

(1) $\frac{dy}{dt} = \left(\frac{d^2y}{dt^2}\right) + y^3$
(2) $\frac{du}{dv} + \left(\frac{du}{dv}\right)^3 = u - v^4$
(3) $\frac{d^4y}{dx^4} = (x+y)^2$

Look back at Example 1.2 again, where we examined the differential equation $\frac{dP}{dt} = kP$. The solution we obtained using the initial condition was a function $P(t)$, but it included an unknown constant, $k$. We really don't want to think of $k$ as a variable because it was determined by the physical facts of the situation. In particular, it was determined by the growth rate of this particular kind of bacteria. A different species of bacteria might have a different reproductive rate and therefore its population growth would be modeled using a different value for $k$.

To make this kind of distinction clear, we use the term **parameter** to describe an unknown constant in a differential equation. This terminology distinguishes it from the independent variables – the differential equation contains derivatives with respect to an independent variable, but not with respect to a parameter.

With ordinary differential equations, we can use various notations to indicate the derivatives, and we usually draw conclusions from context about what symbol represents the independent variable. For example, the differential equation

$$y' = 3yx^4$$

indicates the presence of 2 variables, $x$ and $y$, and since we have a derivative of $y$ present, $x$ must be the independent variable. Therefore the unknown here is the function $y(x)$ (and

$y$ is the dependent variable). On the other hand, the equation

$$y' = 3y$$

shows us only one variable, $y$, which is clearly a dependent variable because the term $y'$ appears in the equation. Since no independent variable is named, we are usually free to choose whatever we like. We might decide to write the function $y$ in terms of a variable $x$, in which case separation of variables would give us solutions of the form $y(x) = Ae^{3x}$. But we could just as easily decide to call the independent variable something else, say $t$, in which case the solutions would have the form $y(t) = Ae^{3t}$. If we knew from context that this differential equation describes a quantity changing over time, that would be a strong reason to choose $t$ as the independent variable.

Another way to express a derivative is with 'dot notation', as in the following ODE:

$$\dot{y} = 3 + t$$

The dot indicates a first derivative with respect to *time*. This is always the convention with dot notation: the independent variable must represent time. Otherwise, we should use prime notation like $y'$ or Leibniz notation like $\frac{dy}{dx}$.

Dot notation can be extended to higher derivatives. The equation

$$\ddot{y} + 3\dot{y} + 2y = 0$$

involves both first and second derivatives of $y$ with respect to time.

The equation $y' = 2kx$ has both $x$ and $k$ on the right side, either of which could be the independent variable. However, they cannot both be independent variables, otherwise $y$ would be a function of two variables, and the notation $y'$ indicates an ordinary derivative, so $y$ can only be a function of one variable. Therefore at least one of $x$ and $k$ must be a parameter. It would be reasonable to assume that $x$ is the variable because we so often use it as such. However, if the equation were written as $y' = kl$, nothing would be so clear: $k$ could be the independent variable, or $l$ could be the independent variable, or both $k$ and $l$ could be constants while the independent variable is something else entirely. It is therefore a good idea, whenever an equation involves a parameter to state clearly which is which. Alternatively, one can make the independent variable visible in the derivative: the notation $\frac{dy}{dl} = kl$ would make it clear that $l$ is the independent variable, and therefore $k$ must be a parameter. The equation $y'(l) = kl$ would provide the same information.

Note also that the notation $\dot{y} = tm$ shows that $t$ is the variable because dot notation always means derivative with respect to time, so $m$ must be a parameter. (That is, unless the author had made a truly bizarre choice of having $m$ represent time and $t$ represent something else – don't ever do that!)

EXAMPLE 4: Solve $\frac{dy}{dx} = n,\ y(1) = 4$.

**Solution:** The independent variable is $x$, and therefore the right side of the ODE is just a constant. We can thus find the general solution by anti-differentiation:

$$y = nx + C.$$

The initial condition implies $C = 4 - n$, so

$$y = nx + 4 - n.$$

□

EXERCISE 8: Find a solution of the ODE $\dot{y} = ky$ subject to the initial condition $y(0) = y_0$. Here, $k$ and $y_0$ are both unknown constants.

Parameters can arise in two different ways: they can be part of the differential equation, in which case each value of the parameter actually corresponds to a *different* ODE; or parameters can show up as part of the problem solving process, such as the constant of integration does when we use separation of variables, in which case each value of the parameter gives a different solution of the *same* ODE.

For example, the general solution of $\dot{y} = yt$ is $y = Ae^{t^2/2}$. Each value for the parameter $A$ singles out a particular solution of the ODE. The set of functions $\{y = Ae^{t^2/2}; A \in \mathbb{R}\}$ is called a **one-parameter family of solutions** to the ODE because each and every choice of value for the parameter $A$ gives a solution of the same ODE:

$$\begin{aligned} & y = Ae^{t^2/2} \\ \Longrightarrow\ & \dot{y} = Ae^{t^2/2}\frac{d}{dt}\left[\frac{t^2}{2}\right] \\ & \quad = Ae^{t^2/2}t \\ & \quad = yt \\ \Longrightarrow\ & \dot{y} = yt \end{aligned}$$

First-order equations usually have a one-parameter family of solutions, but higher order equations typically need more than that.

EXERCISE 9: Verify that every member of the one-parameter family $\{y = Ae^{-x} + 2x - 2; A \in \mathbb{R}\}$ is a solution of the ODE $y' = 2x - y$.

EXERCISE 10: Prove that the members of the **two-parameter family** of functions $\{y = Ae^t + Be^{2t}; A, B \in \mathbb{R}\}$ all solve the second-order ODE $\ddot{y} - 3\dot{y} + 2y = 0$. *(You do not need to prove that every solution of this ODE is a member of this two-parameter family – that will be taken up in a later chapter – you are just being asked to verify that every member of the family is a solution of the ODE.)*

EXERCISE 11: Show that all members of the family $\{y = A\cos(t) + B\sin(t); A, B \in \mathbb{R}\}$ are solutions of the ODE $\ddot{y} + y = 0$.

There is one subtle point of language here about which we have ought to be clear. We have consistently referred to *the* solution of an IVP, rather than saying *a* solution. This suggests that an IVP has one, and only one, solution. Indeed, such a condition is an important prerequisite for using these methods to solve problems. For instance, if an IVP for population growth had two different solutions, we might find one of them but not realize that nature would actually behave according to the predictions of the other. Fortunately, this is usually not the case. For most ODE of interest, solutions to initial value problems are unique, as stated in the following theorem (which we will studied in Chapter 6):

> Suppose that $f(x, y)$ and $f_y(x, y)$ are defined and continuous on an open set containing $(x_0, y_0)$. Then there is an open interval $I$ containing $x_0$ such that the initial value problem
> $$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$
> has a unique solution $y(x)$ defined on $I$.

There are other versions of theorems on existence and uniqueness, but this version is enough to get us started.

# Additional Exercises

Solve each of the following initial-value problems, and determine when the value of the solution will be equal to the given value of $a$.

**12** $y' = 4 + y^2,\ y(0) = 1,\ a = 2$

**13** $y' = xy^2,\ y(0) = 2,\ a = 5$

**14** $y' = \frac{y-1}{x^2-1},\ y(0) = 0,\ a = 1$

**15** $4\dot{x} = x,\ x(1) = 2,\ a = 4$

**16** $\dot{u} + 2u = 1,\ u(1) = 0,\ a = -1$

**17** $\dot{y} = e^{2t+y},\ y(0) = 0,\ a = e$

**18** $\frac{dy}{dx} = \frac{xy}{e^x},\ y(1) = -2,\ a = -1$

**19** $\frac{du}{dv} = u^2 - 3u + 2,\ u(0) = 1,\ a = 3$

Find the interval of definition for the solution of each of the following initial value problems.

**20** $y' = y^2,\ y(0) = 1$

**21** $y' = y^2,\ y(1) = 2$

**22** $y' = y^2,\ y(0) = 0$

**23** $y' = \sqrt[3]{y},\ y(0) = 1$

Solve each of the following initial-value problems, and determine the long-term behavior $\lim_{t\to\infty} y(t)$.

**24** $\dot{y} = 2y,\ y(0) = 3$

**25** $\dot{y} = -2y,\ y(0) = 3$

**26** $\dot{y} = 2 - 3y,\ y(0) = 0$

**27** $\dot{y} = 2 - 3y,\ y(0) = 1$

**28** $\dot{y} = 2 + 3y,\ y(0) = 0$

**29** $\dot{y} = 2 + 3y,\ y(0) = -1$

**30** $\dot{y} = 2xy,\ y(0) = 1$

**31** $\dot{y} = \frac{t}{y},\ y(0) = 1$

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**32** Envision a population of, say, bacteria in a lab experiment. If the organisms have a predictable, periodic life cycle of reproduction and death, then we can model the rate at which the size of this population grows with the simple differential equation

$$\dot{P} = kP,$$

where $P(t)$ the the number of bacteria after $t$ units of time. The parameter $k$ is called the **relative growth rate** of the population. It is the ratio of the instantaneous rate of growth of a population to the size of the population: $k = \frac{\dot{P}}{P}$. In a simple model of population growth, it is reasonable to expect that this ratio will be constant, for if you double the number of bacteria in the population, you will expect to double the number of bacteria which are also reproducing at that instant.

Suppose that $P(0) = P_0$ is a positive number. Find the solution of this differential equation and initial condition. *(Your answer will depend on the independent variable $t$ as wells as the parameters $k$ and $P_0$.)*

**33** A differential equation of the form $\dot{y} = ky$, as in the previous exercise, can be used to model other phenomena besides population growth. For example, the balance of a savings account that earns compound interest might be this kind differential equation. In such a context, the relative growth rate $k$ is often called a **continuous growth rate**.

Suppose that a savings account begins with a positive balance $B_0$ and earns an annual interest rate $r$ compounded $n$ times per year. (For example, if the annual interest rate is 6% compounded monthly, then $r = 0.06$ and $n = 12$.) Then the balance after $t$ years will be $B_0\left(1 + \frac{r}{n}\right)^{nt}$. Prove that as $n \to \infty$, the balance converges to the solution of the initial value problem $\dot{B} = rB,\ B(0) = B_0$.

**34** A retirement account begins with a balance of \$500,000 and earns 3% annual interest. Meanwhile, withdraws of \$40,000 are made each year. Assume that the interest is compounded continuously and the withdraws are made continuously throughout the year. Then the balance will satisfy the differential equation $\dot{B} = 0.03B - 40000$. Explain this model this using a 'rate-in-minus-rate-out' approach. Then find the solution using the initial value $B(0) = 500,000$, and determine how long the account will last before the balance reaches zero.

**35** A radioactive element (for example, plutonium) decays into lighter elements over time at a rate that is proportional to the mass of the radioactive element present. Express this as a differential equation for the mass $m$ of the element present after $t$ units of time, using $k$ as your constant of proportionality. Then prove that the amount of time it takes for the mass to decay in half depends only on $k$ – not on the initial mass! (This period of time is called the **half-life** of the element.)

**36** A tank contains 100 liters of fresh water. Water containing $s$ grams of salt per liter enters the tank at a rate of 5 liters per minute, and the well-mixed solution leaves the tank at the same rate. Suppose that after 10 minutes, the concentration of salt in the tank is 3 grams per liter. Find $s$.

**37** If an object sits in surroundings that are a constant temperature, then Newton's Law of Cooling tells us that the rate of change of the object's temperature is proportional to the difference in temperature between the object and its surroundings:

$$\frac{dT}{dt} = k(T - A).$$

Here, $T(t)$ is the object's temperature, $A$ is the ambient temperature of the object's surroundings, and $k$ is a constant of proportionality. (This constant depends on the material of the object and its surroundings, as well as which units of time are used.)

**(a)** Find a general solution of the differential equation above. (You answer will contain three parameters: $A$, $k$, and $C$, where $C$ arises from the process of antidifferentiation.)

**(b)** A hot turkey comes out of the oven and has an initial temperature of $170$ degrees Fahrenheit. The turkey sits in a room whose temperature is $65$ degree Fahrenheit. After 10 minutes, the turkey's temperature is $168$ degrees. How much longer will it take until the turkey's temperature is $140$ degrees Fahrenheit?

38 The exponential model of population, $\dot{P} = kP$, growth asserts that a population will grow at a rate that is proportional to its size. However, populations (whether they be people, rabbits or bacteria) usually cannot grow indefinitely because they need resources from the environment to thrive. When the population gets too large, the resources of the environment will not be enough to support rapid growth. One mathematical model of population growth that takes this into account is the so-called **logistic growth model**:

$$\dot{P} = \frac{k}{M} P(M - P).$$

The constant $M$ in this differential equation represents a **carrying capacity** – as the size of the population $P$ approaches the carrying capacity $M$, the rate of growth will slow down because the factor $(M - P)$ will be small. Notice that when the population $P$ is small, the right side of the differential equation is approximately equal to $kP$, the same as the exponential growth model.

**(a)** Find a general solution of the logistic growth model. *(Hint: When you isolate $P$ algebraically, you will need to do some simplification that employs either rules of exponents or rules of logarithms.)*

**(b)** Imagine a population of bacteria that would, in the absence of resource limitations, double in size every two days. Find the value of the parameter $k$ that models this population growth in the exponential growth model $\dot{P} = kP$.

**(c)** Using the same value of $k$ you found in part (b), solve the logistic growth model assuming that the initial population is $P = 1$ million, and the carrying capacity of the environment is $20$ million.

**(d)** How long does it take for the population in part (c) to reach 99% of the carrying capacity?

39 Certain chemical reactions can be modeled by the differential equation

$$\dot{x} = k(a - x)(b - x),$$

where $a,\ b$ and $k$ are positive constants and $x(t)$ is the mass of a compound produced by the reaction during the time interval $[0,t]$. Find a formula for $x(t)$ given that $k = 0.01$, $a = 2$, $b = 4$ and $x(0) = 1$. What is the long-term behavior of $x(t)$?

**40** As an object falls, it encounters two forces: gravity and air resistance. If we assume that the force of the air resistance is proportional to the object's speed, then the velocity of the falling object would be modeled by

$$\dot{v} = g - kv.$$

Here, a positive velocity indicates *downward* motion, $g$ is the acceleration due to gravity, and $k > 0$ is a constant of proportionality. Find a solution of this differential equation subject to the parameters $g = 9.8$ and $k = 0.04$ and the initial condition $v(0) = 0$. *(The units of distance here are meters, time is measured in seconds, the acceleration due to gravity has units of $\frac{meters}{second^2}$, so the constant $k$ must then have units of $\frac{1}{second}$.)*

**41** Modify the differential equation in the previous problem to represent the assumption that the force of air resistance is proportional to the square of the speed of the falling object. Then find a solution using the same parameters and initial conditions.

**42** A find a function $y(x)$ that is continuous on all of $\mathbb{R}$ and that satisfies

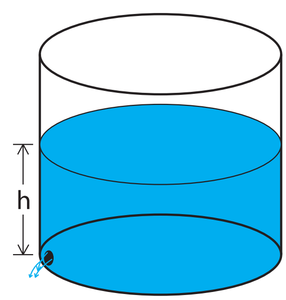
$$\begin{cases} y' = yg(x) \\ y(0) = 1 \end{cases}$$

where

$$g(x) = \begin{cases} 1 & \text{for } x \leq 1 \\ x & \text{for} x > 1 \end{cases}.$$

*(Hint: Start by solving the differential equation on the interval $x \leq 1$. Then use that function's value when $x = 1$ as an initial condition to find a solution on the interval $x \geq 1$. Use piecewise notation to 'glue' these solutions together.)*

**43** Consider a liquid draining from a hole in the side (or bottom) of a cylindrical tank. Torricelli's Law states that the velocity of water exiting through the hole is proportional to the square root of the depth of the water in the tank above the hole. Let's write $v(t)$ for the velocity (in $\frac{m}{s}$) of the water at time $t$ seconds, and let's let $h(t)$ represent the depth (in $m$) of water above the hole. Then we have




$$v = c\sqrt{h}.$$

If we now let the volume of water in the tank above the hole be written as $V(t)$, we have $V = Ah$, where $A$ is the cross-sectional area of the tank, and

$$\frac{dV}{dt} = A\frac{dh}{dt}.$$

But $\frac{dV}{dt} = kv$, where $k$ is the area of the hole, because the change in volume is just due to water flowing out of the hole. This gives us

$$A\frac{dh}{dt} = \frac{dV}{dt} = kv = kc\sqrt{h}.$$

Introducing a new constant $C$, we can write

$$\frac{dh}{dt} = C\sqrt{h}.$$

Use this formula to determine how long it will take a full cylindrical tank to completely drain through a hole in the bottom if the tank is $0.25m$ tall and the water level decreases by $0.05m$ in the first minute.

FOCUS ON MODELING

# Air Resistance

A typical example of a physical process we can model with first-order differential equations is that of a falling body. For example, one might consider a skydiver in free fall after jumping out of an airplane, or particle of dust or pollen falling through the air. In order to keep things simple, we will assume that the motion occurs in only one dimension, the vertical one.

However, it turns out that these two falling objects – the skydiver and the dust particle – require different mathematical models in order to accurately describe their motions, and the nature of the difference will probably surprise you.

Let's begin with the skydiver. Let $v(t)$ denote the velocity of the skydiver at time $t$. If the skydiver has mass $m$, then Newton's second law tells us that the acceleration of the skydiver, $\dot{v}$, satisfies $m\dot{v} = F$, where $F$ is the sum of the forces acting on the object. One of those forces is gravity, which has a magnitude of $|F_{gravity}| = mg$. (Here, $g$ is the acceleration of an object close to the Earth's surface due to gravity.)

The other force we wish to take into account in this model is air resistance, or **inertial drag**. Drag is actually a very complicated phenomenon, but we can try to build a reasonable model by thinking about how the air interacts with the falling skydiver. As he falls through the air, he impacts molecules of air, and the total force of these impacts will depend on their relative speed (which is the same as his own speed relative to the ground) *and* the frequency of these impacts, which is also proportional to the speed. Therefore the total force of these impacts with air molecules is proportional to the square of the velocity: $|F_{inertial-drag}| = cv^2$. (Note that we assumed here that the skydiver remains in the same physical orientation during most of his fall (perhaps in the spread eagle, belly-towards-the-ground position; if that is not the case, then his orientation will also play a role in determining the frequency of impact with molecules of air.)

This force acts in the upward direction, because it is acting in the *opposite* direction of the skydiver's fall. The force due to gravity acts downward. If we choose coordinates so that a falling object has positive velocity, then Newton's second law gives us

$$m\dot{v} = F_{gravity} + F_{inertial-drag} = mg - cv^2.$$

Dividing through by $m$ and introducing $k = \frac{c}{m}$ gives us

$$\dot{v} = g - kv^2.$$

This should match the mathematical model developed in Problem 1.5. However, this model is incomplete! There is also a friction-like force, called **viscous drag**, which impedes the motion of an object moving through a fluid (like air or water) by acting on the object laterally as it moves though the fluid. You can experience this force by trying to drag a long piece of paper through a swimming pool edge-on; even though there is a very small cross-sectional area where the paper's edge impacts water molecules, the sides of the paper experience viscous drag as the water moves laterally across them.

Like friction, this viscous force is proportional to the speed of the object: $|F_{viscous-drag}| = b|v|$, where $b$ is a positive constant. This force always acts in the opposite direction of the object's motion, so we can write it is $F_{viscous-drag} = -bv$ (in our chosen coordinates, $v$ will be positive). The coefficient $b$ depends on the viscosity of the fluid through which the object moves. If we were to use this type of drag in our model instead of inertial drag, we would obtain an ODE of the form

$$\dot{v} = g - bv.$$

One might try to combine both of these drag effects into a single differential equation, but that isn't always necessary. It turns out that when objects move very fast, or when the viscosity of the fluid through which they move is comparatively small, then the inertial drag is the dominant effect and viscous drag can often be ignored. On the other hand, when the velocity is very low, or when the viscosity of the fluid is comparatively high, then viscous drag is dominant and inertial drag may be ignored.

For a skydiver, the large velocities at hand can be accurately modeled by the inertial-drag equation above, wherein air resistance is proportional to the square of the velocity. For the relatively low terminal velocities of dust particles, viscous drag remain the dominant force and better predictions are made by the viscous-drag equation in which air-resistance varies in proportion to the velocity.

To learn more about these different models, read [**1**].

A detailed treatment of these ideas belong to a course in fluid mechanics and derives from a system of partial differential equations known as the 'Navier-Stokes equations'. This is far beyond the scope of this text. In fact, we still don't have a complete understanding of the solutions of Navier-Stokes equations: even though these equations were introduced nearly two centuries ago, many open questions remain.

CHAPTER 2

# Graphical Methods

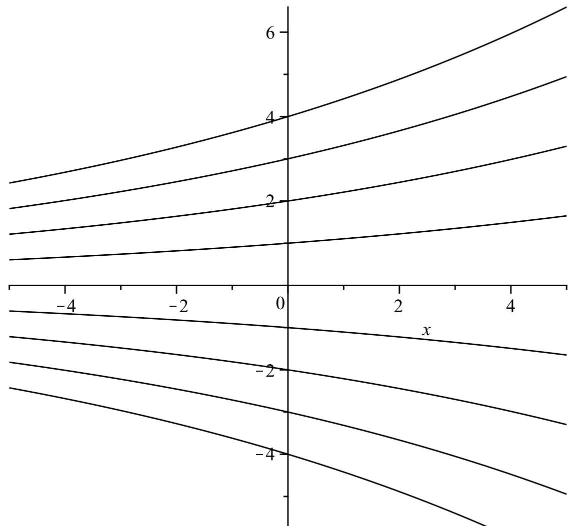
**Prototype Question:** Certain chemical reactions can be modeled by the differential equation

$$\dot{x} = k(a - x)(b - x),$$

where $a,\ b$ and $k$ are positive constants and $x(t)$ is the total mass of a compound produced by the reaction during the time interval $[0, t]$. Assume that that $k = 0.01$, $a = 2$, $b = 4$ and $x(0) = 1$. What is the long-term behavior of $x(t)$?

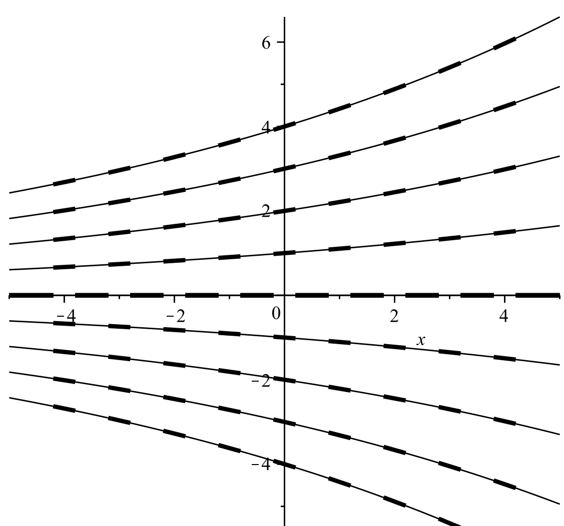
The initial value problem in the question above can be solved **analytically**, which means that it is possible to find an explicit formula for the solution (see Problem 1.3). However, the algebra involved is somewhat complicated, and once you have a solution written down, there is still some work in determining the limit. The point of this chapter is to introduce techniques that make it possible to answer some questions about the qualitative behavior of solutions without the necessity of finding an explicit formula for the solution. This is especially useful when it is difficult or impossible to find such a formula.

Let's begin our discussion by considering a simpler example, such as the ODE $y' = \frac{1}{10}y$. We can separate variables and show that the general solution of this equation is $y = Ae^{x/10}$, for $A \in \mathbb{R}$. Here's a graph of several of the solution curves to this equation, for various values of $A$.

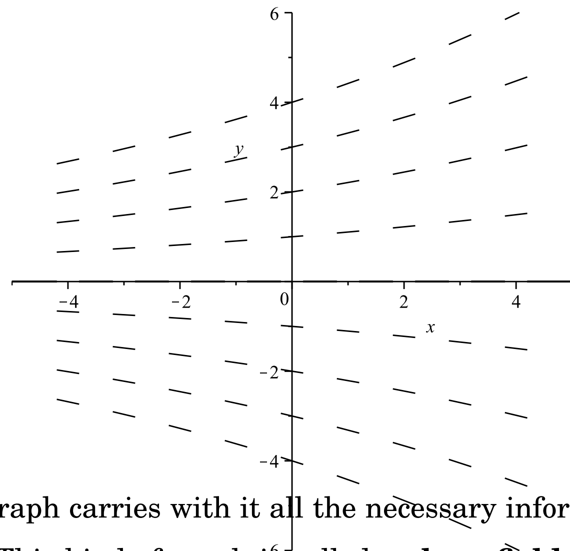



Of course, there are infinitely many such curves – one for each value of the parameter $A$ – including $A = 0$, as the constant function $y = 0$ is also a solution of the differential equation. Whenever a constant function solves an ODE, we call it an **equilibrium solution**.

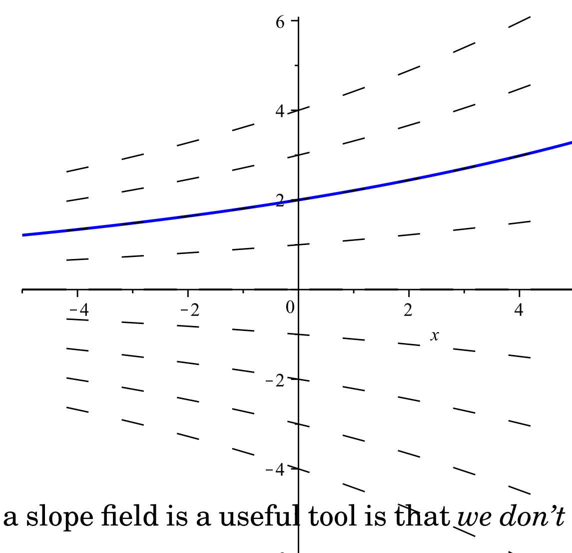
Now let's sketch small segments of tangent lines to each of these curves on the same coordinate plane:




And then let's remove the graphs of the solution curves, leaving just the little line segments:



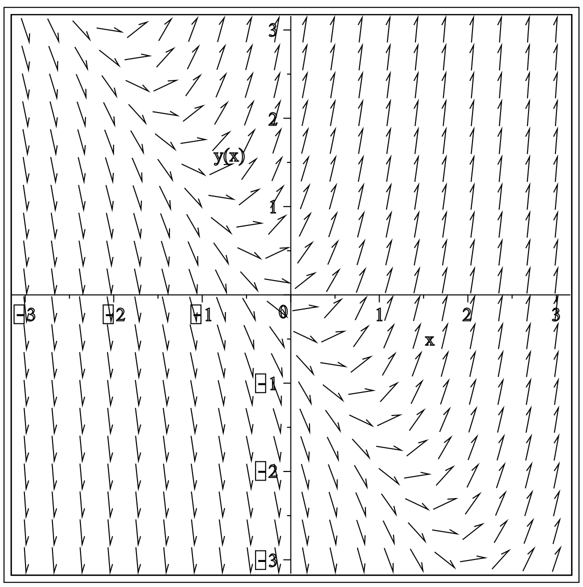

Notice how this graph carries with it all the necessary information for us to visualize the solution curves. This kind of graph is called a **slope field** (or **direction field**), and using it we could sketch a solution curve by following the little line segments. For example, if we want to see what the solution looks like which satisfies the condition $y(0) = 2$, we can start at the point $(0, 2)$ and draw a curve that remains tangent to each little line segment it touches:




A big reason why a slope field is a useful tool is that *we don't need to know the solutions of the ODE in order to draw it!* All we need is a first order ODE written in the form $\frac{dy}{dx} = f(x, y)$. Then we can evaluate the right side at a bunch of points $(x, y)$ and use those values as the slopes when we draw the little segments of tangent lines.

We can also see some qualitative information about the solutions from the slope field, even without a formula for the solution. For example, we can see in the slope field above that the solutions will either be positive and increasing or negative and decreasing. Furthermore, for any solution $y(x)$ of $\frac{dy}{dx} = \frac{y}{10}$, we can see that $\lim_{x\to-\infty} y(x) = 0$, because the solutions will approach the $x$-axis asymptotically as $x \to -\infty$. The positive solutions will satisfy $\lim_{x\to\infty} y(x) = \infty$, and the negative solutions will satisfy $\lim_{x\to\infty} y(x) = -\infty$.

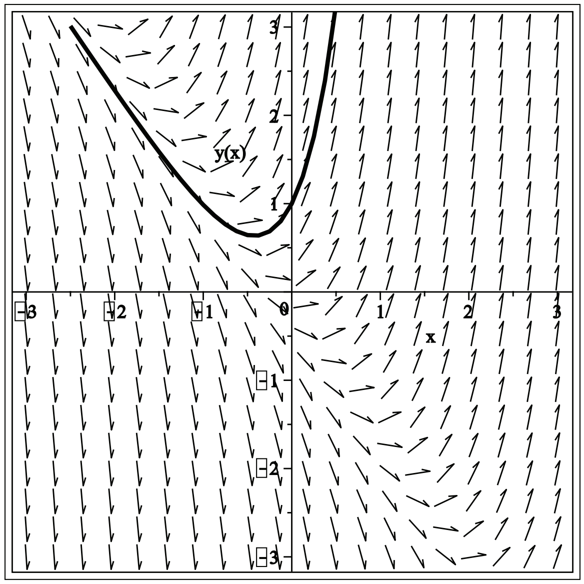
EXAMPLE 1: Consider the ODE $y' = 3y + x$. The following plot is a computer-generated slope field for this differential equation.




(This graph was generated using the *dfieldplot* command on Maple. This particular software program draws little vectors instead of line segments, but that won't bother us.)

If a solution $y(x)$ of this ODE satisfies the condition $y(0) = 1$, then its graph passes through the point $(0, 1)$, and therefore the slope of that curve at that point will be 3 because the differential equation $y' = 3y + x$ tells us that $y' = 3(1) + (0) = 3$. Therefore the direction field shows a tiny vector of slope 3 at the point $(0, 1)$. The software does the same thing at a bunch of other points, and the result is a slope field.

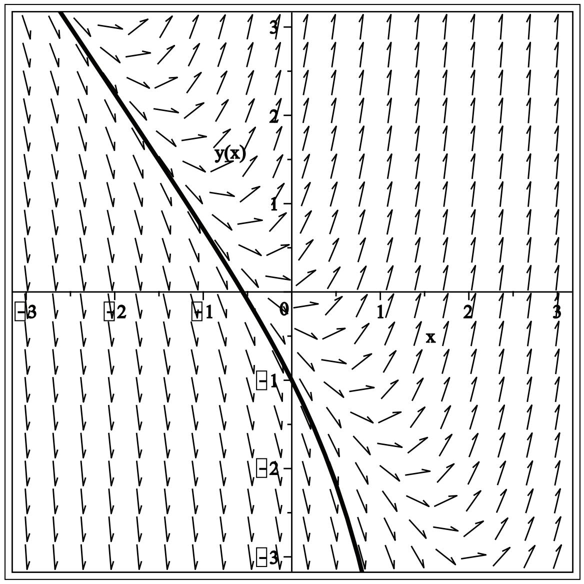
In Chapter 4, we will discuss an analytic technique for solving the initial-value problem $y' = 3x + y$, $y(0) = 1$ to get the function $y = -\frac{1}{9} - \frac{x}{3} + \frac{10}{9}e^{3x}$, and the following plot shows the graph of this function superimposed on the slope field.



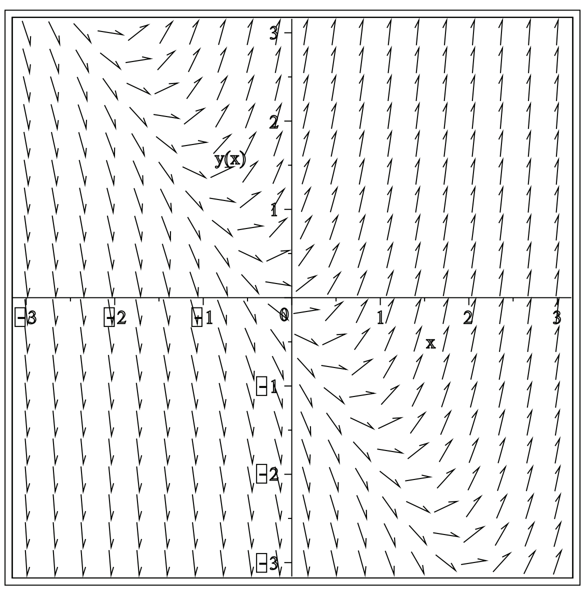

As you can see, the solution curves in such as way as to follow the slopes of the little vectors tangentially. Knowing this, we can actually sketch the solution curve *without having an explicit formula for the solution*. All we have to do is sketch a curve by following the direction of the little vectors or line segments.

For example, we could see from the slope field above that if a solution passes through the point $(0, 1)$, then $y$ will be increasing near $x = 0$. So we draw a little upward curve from there until we get near another direction vector, which tells us in which direction to draw the next segment. We can also trace the direction field going to the left from $(0, 1)$ Doing this, we can draw a complete curve a little bit at a time, and we'll get (roughly) the same picture as above.

Next, imagine a solution of the initial-value problem $y' = 3y + x,\ y(0) = -1$ is graphed on top of the same slope field. Then the solution $y$ will be decreasing as $x$ increases. The plot below shows a curve that passes through $(0, -1)$ and whose tangent lines at each point are parallel to the slope field at each point. The curve represents a solution of the initial value problem.




EXERCISE 1: Use the computer-generated slope field below for $y' = 3y + x$ to sketch a solution curve that passes through the point $(0, 0)$. (The result will be a sketch of the solution to the initial-value problem $y' = 3y + x,\ y(0) = 0$.)



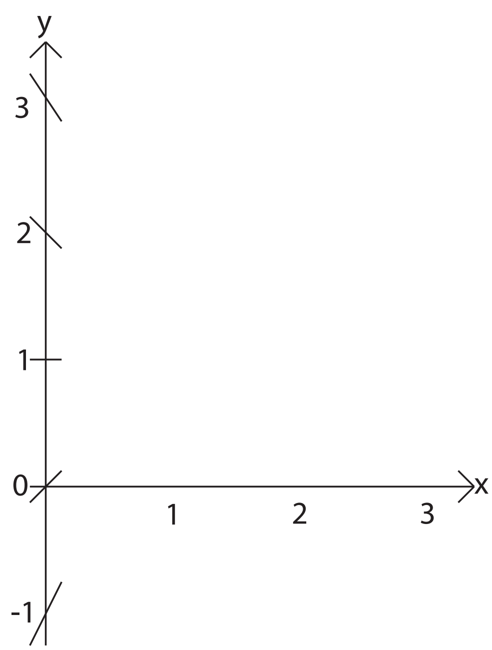

Slope fields also provide us with some general intuition regarding the behavior of solutions. In particular, they tell us that two solutions curves for an ODE $\frac{dy}{dx} = f(x, y)$ cannot cross one another transversely (i.e. at an angle). That's because, if they did, there would be two different slopes at the points where they cross (imagine the two little tangent vectors crossing), but there can only be one slope because the function $f(x, y)$ gives us a single output for each input.

The easiest slope fields to plot, if we need to do so by hand, are the ones where the right side of the differential equation depends on only the dependent variable (so that $\frac{dy}{dx} = f(y)$), because all the direction vectors along a horizontal line have the same slope, as illustrated in the following example.

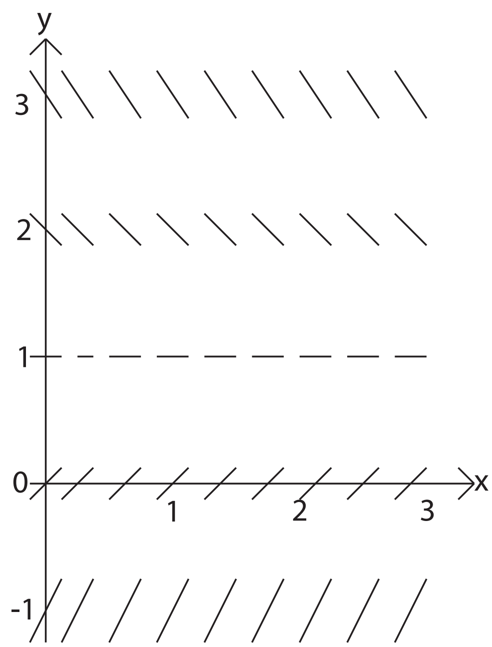
EXAMPLE 2: Let us manually sketch a slope field for the ODE $\frac{dy}{dx} = 1 - y$. We start by evaluating the right side of the differential equation at several points along the $y$-axis:

| $(x, y)$ | $y' = 1 - y$ |
|---|---|
| $(0, 3)$ | $-2$ |
| $(0, 2)$ | $-1$ |
| $(0, 1)$ | $0$ |
| $(0, 0)$ | $1$ |
| $(0, -1)$ | $2$ |

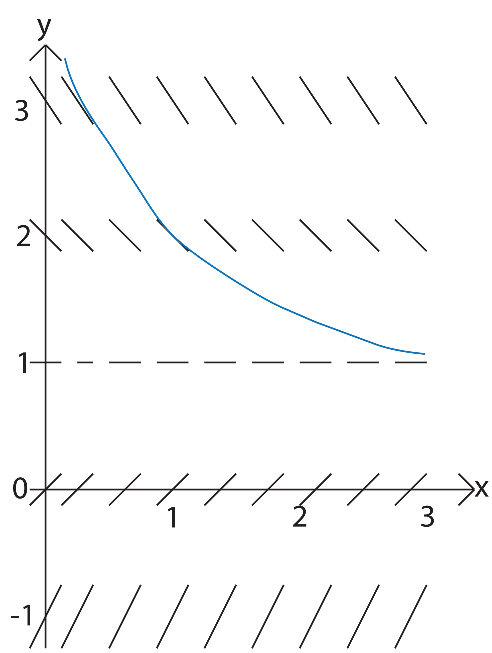
Let's now plot small line segments (there's no need for arrowheads) with these slopes at the indicated points on an $xy$-plane:




We can fill in more of this slope field without computing the values of $y'$ at any more points because the structure of the differential equation shows that the slope does not depend on $x$, only on $y$, and thus the slopes will be the same when we shift our attention left or right. Thus we can just copy horizontal translations of the line segments we have already drawn:




Finally, with a slope field in hand, we can try to analyze the behavior of solutions. For example, if we wanted to know the long-term behavior of a solution to the IVP $y' = 1 - y$, $y(1) = 2$, we can sketch a curve that passes through the point $(1, 2)$ and that remains tangent to the slope field at each point:




Based on this, we would guess that the solution of this IVP has the long-term behavior $\lim_{x\to\infty} y(x) = 1$. □

Differential equations like this last one have a special name: we say that an ODE of the form $y' = f(y)$ is **autonomous** (that is to say, the function $f$ on the right side of the equation only depends on the dependent variable, $y$, not the independent variable $x$). In addition to it being easier to plot a slope field for autonomous ODE, they are always separable (thus autonomous ODE are prime candidates for separation of variables, provided it is feasible to calculate all the anti-derivatives involved).

Incidentally, equations of the form $y' = g(x)$ (where the right side depends only on the independent variable, $x$) also have slope fields which are easy to plot, since the direction vectors along any vertical line have the same slope; however, these equations are not of as much interest to us in this course since they were studied extensively in calculus – a solution of $y' = g(x)$ is just an anti-derivative of the function $g$.

The last example illustrated how slope fields can be generated by hand, but it is usually much more efficient to use a computer program to generate them. The reader should try to generate one or two slope fields by hand for the sake of experience, but after that it will be a more efficient use of time to employ a computer.

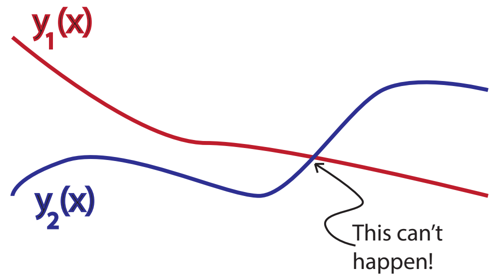
**Exercise 2:** Generate a slope field for $y' = y(y-2)$. Then sketch several solution curves on the plot, one for each of the following initial conditions: $y(0) = -1$, $y(0) = 0$, $y(0) = 1$, and $y(0) = 2$. In each case, use the behavior you see on the slope field to predict the value of $\lim_{x\to\infty} y(x)$.

**Exercise 3:** Use a slope field to predict the behavior of $\lim_{x\to\infty} y(x)$, where $y$ is a solution of $y' = y + 4$. Explain how this limit depends on the initial value $y(0)$.

**Exercise 4:** Use a slope field to predict the behavior of $\lim_{x\to\infty} y(t)$ where $y$ is a solution of $\dot{y} = y - t$, $y(0) = 0$.

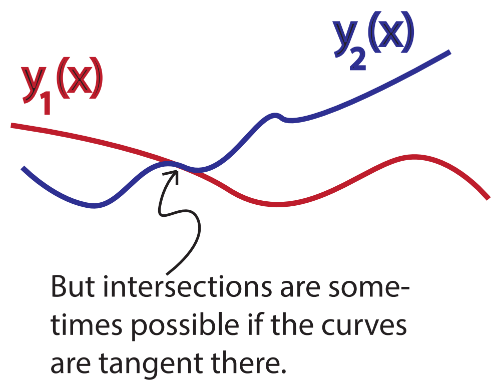
**Exercise 5:** Use a slope field to predict the behavior of solutions to $y' = y^2$. Then confirm your prediction by finding a formula for the solution of the initial value problem $y' = y^2,\ y(0) = y_0$.

Slope fields can give us enormous insight into the behavior of solutions without having to explicitly solve the differential equation first. In fact, they can give us one very general insight that applies to all solutions of ordinary differential equations: *two solutions of the same ODE $y = f(x, y)$ cannot have graphs that cross one another at a non-zero angle.* To see this, suppose that there were two solutions that did cross, as shown in the figure below:




At the point of intersection, these two graphs have different slopes: but that cannot be, because the differential equation determines the slope based on the coordinates of the point – the slope would need to be the same for both solutions at that point (if the point is $(x_0, y_0)$, then the slope would be $f(x_0, y_0)$).

On the other hand, it is still possible for solutions to cross, as long as they are tangent to one another at the point of intersection:




It is sometimes possible to also rule out such intersections, but to do so, we need to know a more about the function $f(x, y)$. For more about this topic, see Chapter 6.

## Phase Lines

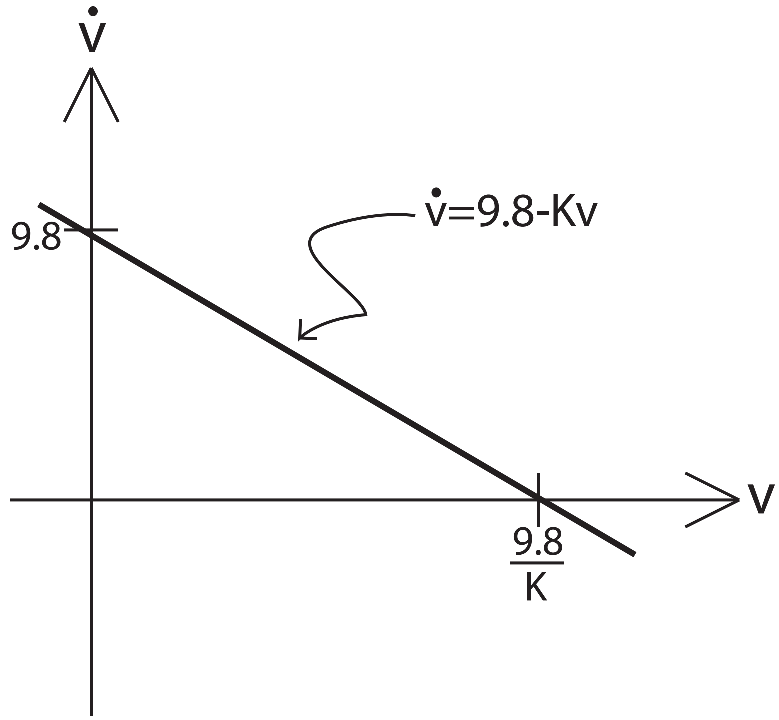
Next, we turn our attention to another graphical approach for understanding solutions of differential equations that is specifically applicable to autonomous equations.

EXAMPLE 3: Suppose the velocity $v(t)$ (in meters per second) of a falling object that encounters air resistance is modeled by the differential equation

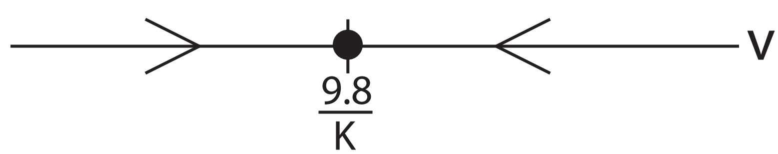
$$\dot{v} = 9.8 - Kv$$

where $K > 0$ is a constant. The assumption here is that air resistance is a force that is proportional to the speed of the object, and so the constant of proportionality K depends on both the speed and the mass of the falling object; the quantity $9.8$ accounts for the acceleration due to gravity, and by selecting a positive acceleration here, we have implicitly selected the convention that positive velocities correspond to *downward* motion.

Let us graph $\dot{v}$ as a function of $v$ as expressed by the equation $\dot{v} = 9.8 - Kv$:




The line intersects the $v$-axis at $v = \frac{9.8}{K}$. According to this graph, whenever the velocity is less than $\frac{9.8}{K}$, $\dot{v}$ will be positive, and therefore the object will continue to increase its velocity. Similarly, if the velocity $v$ were to start out greater than $\frac{9.8}{K}$, then $\dot{v}$ will be negative, and therefore the velocity will decrease. In both cases, the velocity will tend toward the value $v = \frac{9.8}{K}\frac{m}{s}$. And if the initial velocity is exactly $\frac{9.8}{K}\ \frac{m}{s}$, then $\dot{v} = 0$, so the velocity will remain constant. We use arrows on a number line to illustrate the behavior of the solution as follows:



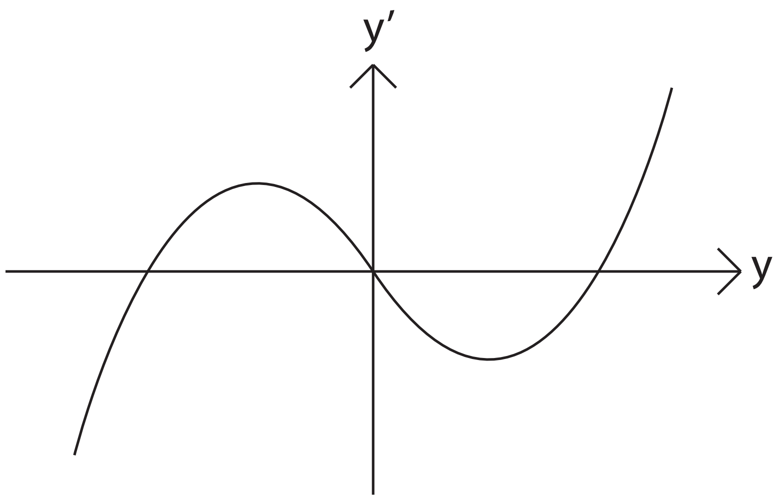

This figure is called the **phase line** for the differential equation: it indicates the equilibrium (i.e. constant) solutions of the differential equation with dots (in this case, $v = \frac{9.8}{K}$ is the only equilibrium solution), and arrows indicate the behavior of solutions with other initial conditions (right-pointing arrows indicate that a solution is increasing while left-pointing arrows indicate that a solution is decreasing). In this example, if a solution starts out with a value that is greater than $\frac{9.8}{K}$, then the left-pointing arrows indicates that the velocity will decrease and approach $\frac{9.8}{K}$. (The reader can verify this by calculating the explicit solutions of the ODE.)

In this context of a falling object, the constant solution is called the **terminal velocity** of the falling object – it is the limiting value of the velocity as $t$ increases:

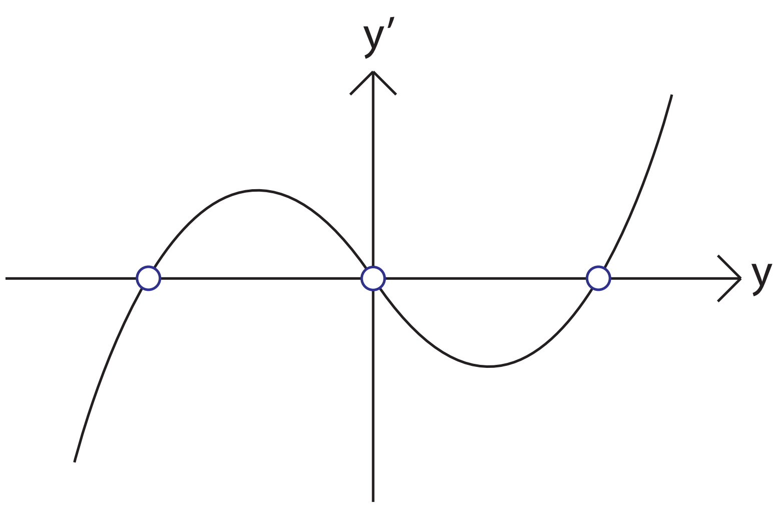
$$\lim_{t\to\infty} v(t) = \frac{9.8}{K}\ \frac{m}{s}.$$

□

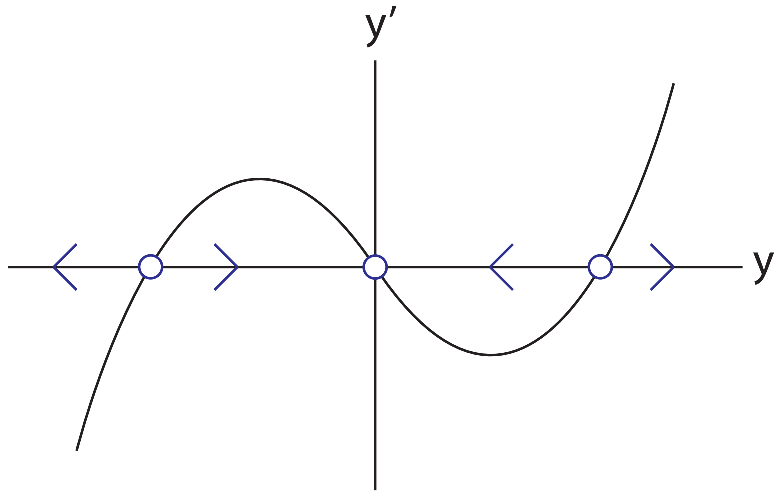
To draw a phase line for an autonomous ODE $y' = f(y)$, we usually start by graphing the relationship described by this equation on a y-y'-coordinate plane. For example, if the ODE is $y' = y^3 - y$, the graph would look like this:



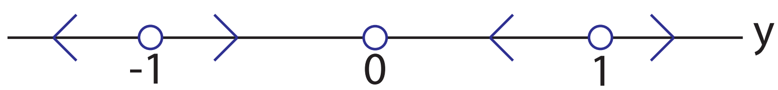

Each $y$-intercept of this graph corresponds to a value of $y$ for which $y' = 0$, and this means that a constant function with this $y$-value will be a solution to the IVP. These are the equilibrium solutions, and we highlight these values on the $y$-axis by drawing circles there:



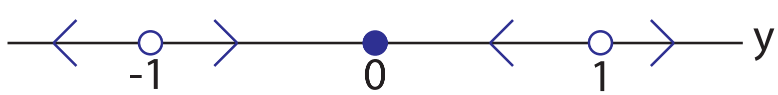
These circles divide the $y$-axis into several intervals, and in each interval we draw arrows to indicate whether solutions that start with $y$-values in those intervals will be increasing or decreasing. If the graph of $y'$ against $y$ is above the $y$-axis for an interval, then $y'$ is positive, thus $y$ is increasing, and we denote this with right-pointing arrows. On the other hand, if the graph is below the $y$-axis, then $y'$ is negative, so $y$ is decreasing, and we indicate that behavior using left-pointing arrows. Also, to avoid any confusion, we should remove any arrowheads on the coordinate axes at this time.




Now we can remove the $y'$-axis and the graph of the relationship between $y$ and $y'$, and we indicate the $y$-values along the phase line where we have drawn circles for equilibrium solutions:




Finally, it is convention to add detail to the circles that indicate equilibrium solutions. An equilibrium solution is called **stable** if solutions to the ODE with nearby initial values tend towards that equilibrium value, as is the case for the equilibrium solution $y = 0$ of $y' = y^3 - y$. We indicate this by shading in the circle. An equilibrium solution is called **unstable** if solutions that start with nearby initial values tend away from the equilibrium value, as is the case here for $y = -1$ and $y = 1$; these are indicated by leaving the circles hollow:




This is a complete phase line for the ODE $y' = y^3 - y$. It tells us that:

(1) The constant functions $y = -1$, $y = 0$ and $y = 1$ are all equilibrium solutions.

(2) If $y(0) < -1$, then $y$ will be a decreasing function of $t$, and it will decrease toward $-\infty$

(3) If $-1 < y(0) < 0$ then $y$ will be an increasing function of $t$, and $\lim_{t\to\infty} y(t) = 0$

(4) If $0 < y(0) < 1$ then $y$ will be a decreasing function of $t$, and $\lim_{t\to\infty} y(t) = 0$

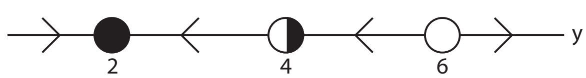
(5) If $y(t) > 1$, then $y$ will be an increasing function of $t$, and it will increase toward $\infty$.

That's a lot of qualitative information, and we didn't need to calculate an explicit general solution of the ODE in order to obtain it!

**EXERCISE 6:** Create a slope field and sketch some solution curves for $y' = y^3 - y$ to confirm the conclusions above.

It is also possible for an equilibrium solution to be **half-stable**, meaning that nearby solutions tend toward the equilibrium value on one side but away on the other. This is usually indicated on the phase line by drawing a half-shaded circle. The icon ◑ would be used for an equilibrium value for which solutions that begin with slightly greater initial values tend toward the equilibrium value but solutions that begin with slightly lesser initial values tend away from it. The reverse situation would be indicated by ◐.

**EXERCISE 7:** Interpret the following phase line:

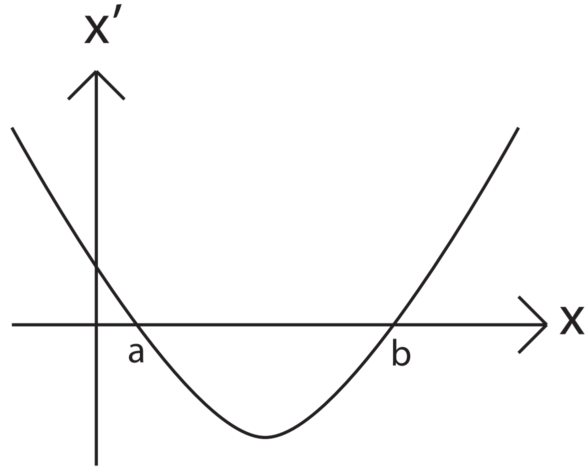



**EXERCISE 8:** A tank contains a changing mixture of pure water and brine (salt water solution). The differential equation that models the quantity of salt in the tank after $t$ minutes have passed is

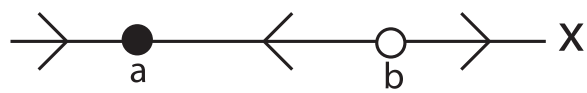
$$\dot{S} = 8 - \frac{4S}{25}\ \frac{\text{grams}}{\text{min}}$$

where $S(t)$ is measured in grams. Use a phase line analysis to determine the long-term behavior, $\lim_{t\to\infty} S(t)$. *(Note: Because $S$ represents mass, which cannot be negative, it makes sense in this context to restrict our attention to a positive $S$-axis for the phase line.)*

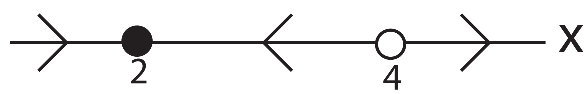
Let's finish this chapter by answering the prototype question that began it. Consider the differential equation $\dot{x} = k(a-x)(b-x)$ where $a$, $b$ and $k$ are positive constants. Assume that $a$ is less than $b$, just so that we can draw a graph (if $b$ is less than $a$, then the graph will be the same but with the labels switched). Then the graph of the relationship between $\dot{x}$ and $x$ which we obtain is:




This leads to the phase line




In the prototype question, we had $a = 2$ and $b = 4$, giving us




We also have $x(0) = 1$, which means the $x$-value begins to the left of the stable equilibrium shown on the phase line. Therefore, a solution of the differential equation with the initial condition $x(0) = 1$ will increase and have the property that $\lim_{t\to\infty} x(t) = 2$.

A reader who takes the time to compare this analysis with the work necessary to find an explicit solution of the IVP and then compute the limit will certainly come to appreciate the usefulness of this graphical approach.

# Additional Exercises

Sketch a slope field for the following differential equations on the domain $-5 \le x \le 5$ and $-5 \le y \le 5$.

**9** $y' = 2 - y$

**10** $y' = y + 3$

**11** $y' = x - y$

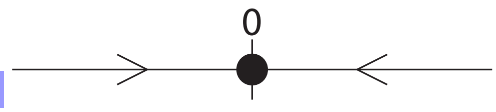
**12** $y' = 2x + 4y$

**13** $y' = y(y - 3)$

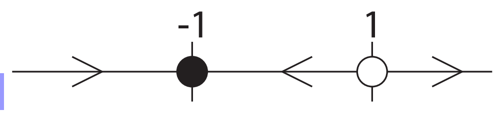
**14** $y' = y^2(y + 2)$

Use the phase lines below to determine the long term behavior of $y(t)$ for the initial conditions **(a)** $y_0 = -1$, **(b)** $y_0 = 1$ and **(c)** $y_0 = 3$.

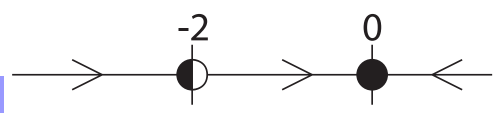
**17**




**18**

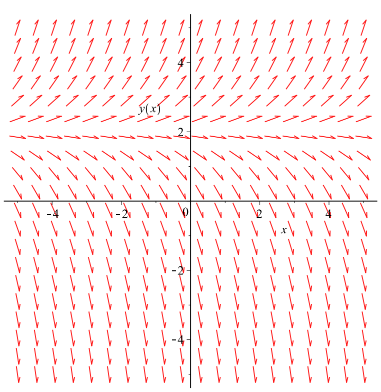



**19**

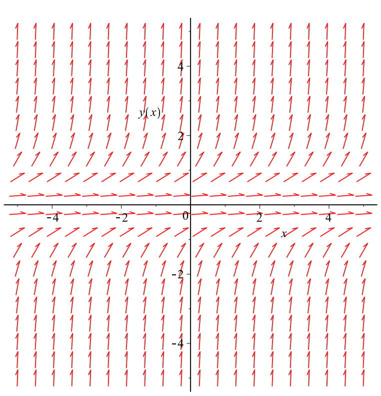



Write down a differential equation that is (approximately) consistent with the following slope fields.

**15**




**16**




Sketch a phase line for each of the following differential equations.

**20** $\dot{y} = y(2 - y)$

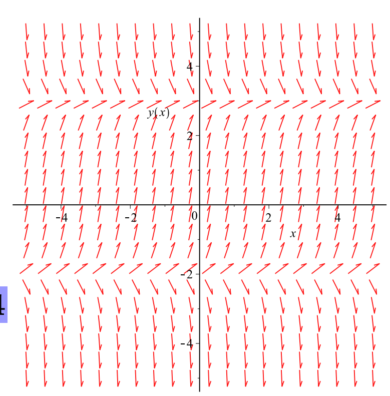
**21** $\dot{y} = y^2 + 1$

**22** $\dot{y} = y^2(y + 2)$

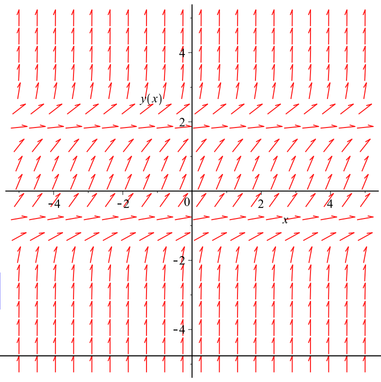
**23** $\dot{y} = 4y^4 - y^2$

Sketch a phase line that is consistent with the following slope fields.

**24**



**25**



**26** Consider a falling body that experiences inertial drag and whose velocity (in meters per second) is modeled by the ODE $\dot{v} = 9.8 - Kv^2$. Draw a phase line for $v \geq 0$. Use the phase line analysis to determine the equilibrium solution of the ODE. If the object's terminal velocity is measured as $v = 9.4 \times 10^1 \frac{m}{s}$, what is the value of the parameter $K$? Round your answer to two significant figures, and include units.

**27** Use a phase-line analysis to determine the long-term behavior of solutions to the differential equation

$$\dot{y} = \sin(y).$$

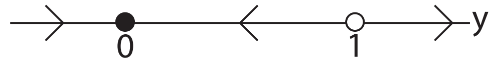
How does the behavior depend on the initial value $y(0)$? Give a complete answer for any possible initial value.

**28** Perform a phase-line analysis for the logistic growth model,

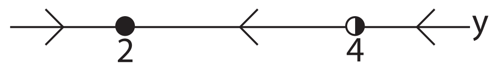
$$\dot{P} = \frac{k}{M} P(M - P).$$

The parameters $k$ and $M$ are positive constants.

**29** Find a differential equation $\dot{y} = f(y)$ which is consistent with the phase line shown below (there is more than one correct answer):




**30** Find a differential equation $\dot{y} = f(y)$ which is consistent with the phase line shown below (there is more than one correct answer):




**31** This problem illustrates a weakness of relying on graphical approaches alone – namely, that we may not be able to determine the domains of *unbounded* solutions without using an analytical method. Suppose that $u(t)$ is the solution of $\dot{u} = |u|$, $u(0) = 1$, and suppose that $v(t)$ is the solution of $\dot{v} = v^2$, $v(0) = 1$. Illustrate that the differential equations for $u$ and $v$ give identical phase lines and similar slope fields. In particular, notice that the solutions of both initial value problems will be increasing functions of $t$ that approach $\infty$. Then solve the initial value problems analytically to prove that the intervals of definition for these two solutions are not the same. Thus while it makes sense to discuss $\lim_{t\to\infty} u(t)$, it does not make sense to discuss the same limit of $v(t)$. What limit for $v(t)$ should be considered instead as the long-term behavior? *(Hint: Since the solution $u$ is a positive* *function, you can drop the absolute value sign when you use separation of variables to solve for $u(t)$. Verify directly that the formula you end up with for $u$ is indeed a solution of the initial value problem.)*

**32** The command `dfieldplot` can be used to generate a slope field using the computer algebra software Maple. First, call up the necessary subroutines by executing the command `with(DEtools)`. Here's the command to generate the slope field in Example 2.1:

```
dfieldplot(y'(x)=3y(x)+x, y(x),
                    x=-3..3, y=-3..3)
```

Modify this command to generate a slope field for $y' = \sin(y)$ on the domain $0 \leq x \leq 6$, $-4 \leq y \leq 4$. Use the resulting graph to describe the long-term behavior of a solution satisfying the initial condition $y(0) = 1$. Compare with your answer to Exercise 27.

CHAPTER 3

# Numerical Methods

**Prototype Question:** Consider an object whose shape changes as it falls against air resistance (for example, a raindrop). The changing shape means the drag coefficient will change as well. Assume we can model this behavior with the differential equation

$$\dot{v} = g - k(v)v^2,$$

where $k(v)$ denotes the drag coefficient as a function of the object's instantaneous velocity. For a falling object with a drag coefficient $k(v) = e^v$, find the velocity 3 seconds after it begins to fall from rest.

The prototype question asks us find the value of $v(3)$ for the solution of the initial-value problem $\dot{v} = 9.8 - e^v v^2,\ v(0) = 0$. Although this looks like a number of problems we have solved already, it is different in that *we cannot find an explicit solution to this differential equation*. The reader can attempt to use separation of variables to find a solution but will get stuck at the step of calculating $\int \frac{1}{9.8 - e^v v^2}\, dv$.

We will not discuss any method in this textbook for finding an explicit general solution of $\dot{v} = 9.8 - e^v v^2$. (In fact, the author is unaware of any technique that could accomplish this.) We could do a graphical analysis of this ODE to try to discern the long-term behavior, but those methods are not terribly useful for estimating the value at a specific input.

Instead, we will explore a method for finding approximate values of solutions to ODE even when it is not possible to find formulas for solutions analytically. That is to say, we will find an approximate value of $v(3)$, even though we will not be able to find an explicit formula for $v(t)$.

The discussion below will illustrate the basic idea of our approach, and the example afterward will demonstrate the fully developed idea with a more efficient organization of the calculations.

Consider the initial value problem

$$\begin{cases} y' = y^2 - x \\ y(0) = 1 \end{cases}.$$

Suppose we want to know the value of $y(1)$, but we are unable to calculate an exact solution for the ODE. We can find an approximate solution as follows.

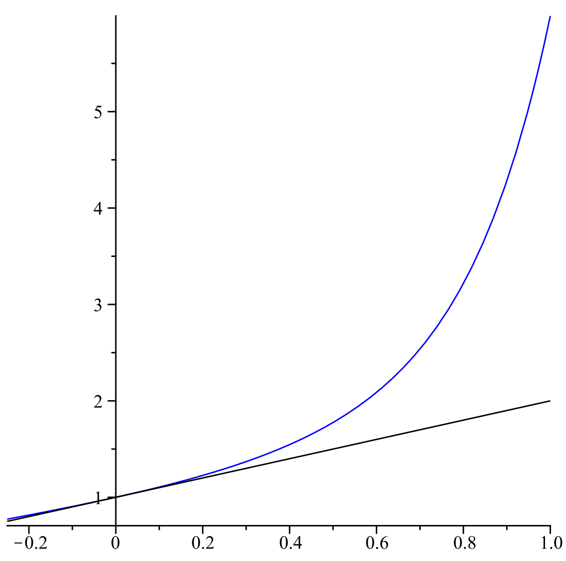
If $y(x)$ is the solution, then the initial condition implies that $y(0) = 1$, and if we insert $x = 0$ and $y = 1$ into the differential equation, we see that $y'(0) = (y(0))^2 - (0) = (1)^2 - 0 = 1$. Therefore the tangent-line approximation to $y(x)$ at the point $(0, 1)$ is

$$y(x) \approx x + 1.$$

(This is the line through $(0, 1)$ with slope $y'(0) = 1$.)

EXERCISE 1: Suppose that $y(t)$ satisfies the initial value problem $y' = y^3 + 3x,\ y(1) = 2$. Without solving the differential equation, find $y'(1)$, and find the equation of the tangent line to the graph of $y(x)$ at the point $(1, 2)$.

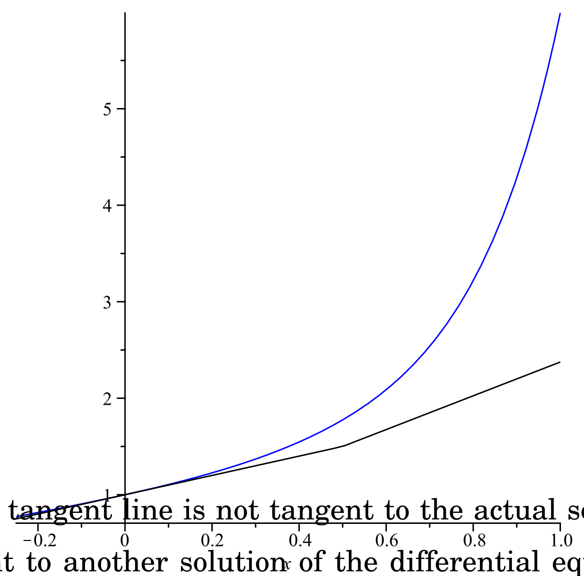
We can use the tangent line approximation $y(x) \approx x + 1$ to calculate that $y(1) \approx 2$. This picture illustrates the solution curve and the tangent line we just calculated:




EXERCISE 2: Suppose that $y(t)$ satisfies the initial value problem $y' = y^3 + 3x,\ y(1) = 2$. Use the tangent line approximation to estimate the value of $y(1.5)$.

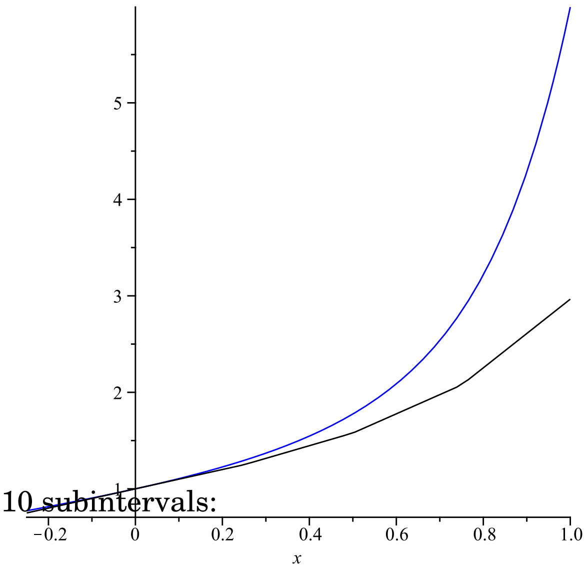
The idea was that, because we know the slope of the solution curve at the initial point $(0, 1)$, we can use that to project what happens as $x$ increases. However, it is not hard to see that the linear approximation is only good for small values of $x$ – the actual solution curve grows quickly as $x$ increases, and the difference between the curve and the tangent line will only worsen away from the initial point. The tangent line approximation will only give a good approximation if $\Delta x$ is small.

We can work around that limitation by using the tangent line approximation for a small interval, say, for $0 \leq x \leq 0.5$, and then creating another tangent line approximation at the new point where we find ourselves. According to the first tangent line approximation, when $x = 0.5$, we get $y \approx 1.5$; inserting this into the differential equation gives us $y' \approx (1.5)^2 - (0.5) = 1.75$. This becomes the slope for the second tangent line approximation, the segment from $x = 0.5$ to $x = 1$:

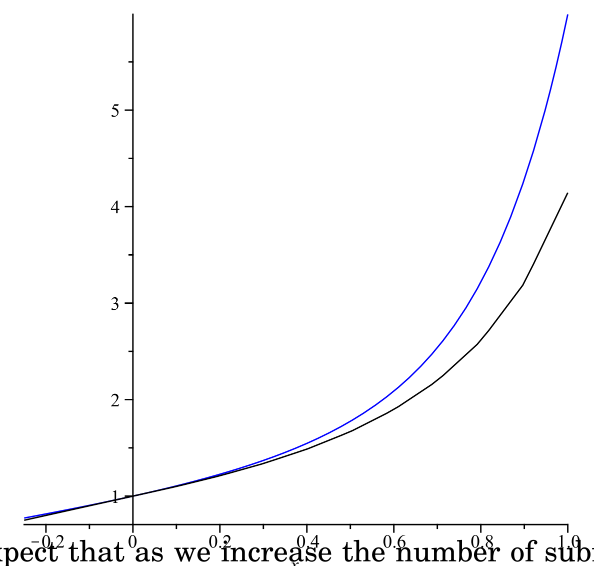



Note that our second tangent line is not tangent to the actual solution satisfying $y(0) = 1$, rather it is tangent to another solution of the differential equation, say $y_2(x)$, which satisfies $y_2(0.5) = 1.5$. But if $(0.5, 1.5)$ is sufficiently close to the first solution curve we drew, then it should give us a good approximation for $y(1)$ anyway. Based on this, we calculate that $y(1) \approx 1.5 + 0.5((1.5)^2 - 0.5)) = 2.375$.

Now our approach for finding an approximate solution has taken shape: by using a finer subdivision of intervals, we can obtain a better approximation for the value of $y(1)$. The following graph shows how dividing the interval $[0, 1]$ into 4 subintervals gives an even better approximation:




And here we use 10 subintervals:




Intuitively, we expect that as we increase the number of subintervals, the piecewise-defined function will bear an ever increasing resemblance to the actual solution curve. Indeed, it is possible to make a rigorous statement out of this (using limits) and to prove it for functions that satisfy the hypotheses of the Existence and Uniqueness Theorem.

We don't need the graphs in order to apply this method. All we need to do is to keep track of what happens to the $y$-value each time we increment the $x$-value. We start by fixing a value of $\Delta x$, which is called the **step size**. Let $(x_0, y_0)$ represent the initial condition. Let $m_i$ represent the slope of a tangent line approximation to a solution curve at $(x_i, y_i)$, which we find by plugging $(x_i, y_i)$ into the differential equation. Then let $x_{i+1} = x_i + \Delta x$, and $y_{i+1} = y_i + m_i \Delta x$. We can keep track of all this information in a table. Here's such a table for the ODE $y' = y^2 - x$ with the initial condition $(0, 1)$ and step size $\Delta x = 0.1$:

| $x_i$ | $y_i$ | $m_i = y_i^2 - x_i$ | $y_{i+1} = y_i + m_i \Delta x$ |
|---|---|---|---|
| 0 | 1 | 1 | 1.1 |
| 0.1 | 1.1 | 1.11 | 1.211 |
| 0.2 | 1.211 | 1.2665 | 1.3377 |
| 0.3 | 1.3377 | 1.4893 | 1.4866 |
| 0.4 | 1.4866 | 1.8099 | 1.6676 |
| 0.5 | 1.6676 | 2.2808 | 1.8957 |
| 0.6 | 1.8957 | 2.9935 | 2.1950 |
| 0.7 | 2.1950 | 4.1180 | 2.6068 |
| 0.8 | 2.6068 | 5.9955 | 3.2064 |
| 0.9 | 3.2064 | 9.3808 | 4.144 |

Notice that the entry for $y_{i+1}$ is also the entry for $y_i$ in the next row, because it represents the $y$-value that goes with the next $x$-value. The very last entry in the table is the $y$-value that goes with $x = 1$. Based on this information, we estimate that $y(1) \approx 4.144$. $\square$

The approach described here is known as **Euler's method**. It is not a highly efficient algorithm, but it is the basic foundation upon which methods in the numerical analysis of differential equations are built.

Our final estimate in the last example was an underestimate. We could have improved it by reapplying the algorithm with a smaller step size $\Delta x$. For example, using $\Delta x = 0.01$ leads to the approximation $y(1) \approx 7.8$. Using $\Delta x = 0.001$ gives us $y(1) \approx 9.16$. Using $\Delta x = 0.0001$ gives us $y(1) \approx 9.35$, which is the same value (to two decimal places) which we get when we use $\Delta x = 0.000001$. The spreadsheet used to obtain this last result had 1 million lines of calculations. Clearly there is a trade-off between accuracy and how computationally-intensive the method will be to implement as one considers various choices of $\Delta x$. There are variations on this method which will converge faster, meaning they give similar accuracy with a larger increment $\Delta x$, and they can therefore be calculated more quickly. The computational cost of a numerical method is an important area of study in applied mathematics. Many other algorithms have been discovered which are more efficient than Euler's method, even if they are just refinements of the same idea.

Several refinements are discussed in the problem set for this chapter. Learning Euler's method is preparation to begin our study of the more advanced techniques.

Before proceeding to another example, we should take a moment to notice that there is another, more symbolic way of understanding Euler's method. If $y$ satisfies the IVP $y' = f(x, y)$ and $y(x_0) = y_0$, then integrating both sides of the differential equation from $x_0$ to $x_1 = x_0 + \Delta x$ gives us

$$y(x_1) - y(x_0) = \int_{x_0}^{x_1} f(x, y(x))\, dx,$$

or

$$y(x_1) = y_0 + \int_{x_0}^{x_1} f(x, y(x))\, dx.$$

If $\Delta x$ is small and $f$ is continuous, then the integrand is approximately constant on the domain of integration, which has length $\Delta x$, and therefore

$$y(x_1) \approx y_0 + f(x_0, y_0)\Delta x.$$

Similarly,

$$y_{i+1} \approx y_i + f(x_i, y_i)\Delta x,$$

and this is exactly the recursion we used to find approximate values of $y$.

EXAMPLE 1: Use Euler's method with a step size of $\Delta t = 0.25$ to estimate $y(1)$, where $y$ is the solution of the IVP $\dot{y} = \sin(y),\ y(0) = 2$. Maintain 6 decimal places of accuracy at each step of the calculation, and report the final answer rounded to 2 decimal places.

**Solution:** We construct a table of values for $t_i = 0 + 0.25(i-1)$ and the corresponding values of $y_i$, $m_i$ and $y_{i+1}$:

| $t_i$ | $y_i$ | $m_i = \sin(y_i)$ | $y_{i+1} = y_i + m_i\Delta t$ |
|---|---|---|---|
| 0 | 2 | 0.909297 | 2.227324 |
| 0.25 | 2.227324 | 0.792116 | 2.425353 |
| 0.5 | 2.425353 | 0.656553 | 2.589492 |
| 0.75 | 2.589492 | 0.524477 | 2.720611 |

Therefore $y(1) \approx 2.72$. □

EXERCISE 3: Use Euler's method with a step size of $\Delta x = 0.5$ to estimate $y(1.5)$, where $y$ satisfies $y' = y^2,\ y(0) = 2$. Do all the calculations by hand. Draw a slope field to try to predict whether your approximate answer is an overestimate or an underestimate of the true solution value.

EXERCISE 4: Use Euler's method with a step size of $\Delta x = 0.25$ to estimate $y(1)$, where $y$ satisfies $y' = x^2 + y^2$ and $y(0) = 0$. Use a slope field to try to determine whether your solution is an overestimate or an underestimate of the true solution value.

## RUNGE-KUTTA

A very popular numerical method for finding approximate values of solutions to initial value problems is the Runge-Kutta method described below. The formula for computing the values of $y_i$ appears to be much more complicated that the formula for Euler's method. However, in exchange for this complexity, we obtain an algorithm that is much more efficient in that it gives better approximations with fewer arithmetic computations.

Let's begin by looking at the formula itself.

**The Runge-Kutta Method**

For an initial value problem $y' = f(x, y),\ y(x_0) = y_0$ and a step-size $\Delta x > 0$, define $x_n = x_0 + n\Delta x$ and

$$y_{n+1} = y_n + \Delta x \left( \frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6} \right)$$

where

$$\begin{aligned}
k_{n1} &= f(x_n, y_n) \\
k_{n2} &= f\left(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}\Delta x k_{n1}\right) \\
k_{n3} &= f\left(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}\Delta x k_{n2}\right) \\
k_{n4} &= f\left(x_n + \Delta x, y_n + \Delta x k_{n3}\right)
\end{aligned}$$

This formula can be thought of as an attempt to improve on estimating the value of $\int_{x_n}^{x_n+\Delta x} f(x, y(x))\ dx$. Euler's Method approximates this integral using the left endpoint approximation, since $f(x_n, y_n)$ is just the value of the integrand at the left endpoint. As you can find in the problem set for this chapter, there is a refinement of this idea (called the **improved Euler formula**) that attempts to use a Trapezoid Rule approximation to estimate the integral; but since we don't know the value of $f(x, y(x))$ at the right endpoint, we approximate it first using Euler's Method, then plug that approximation back in to estimate the integral. Recall then the the trapezoid rule basically averages the values at the left and right endpoints of the interval (and multiplies this average by the length of the interval) to approximate the integral. So the improved Euler formula gives us

$$y_{n+1} = y_n + \Delta x \left( \frac{f(x_n, y_n) + f(x_n + \Delta x, y_n + \Delta x f(x_n, y_n))}{2} \right).$$

The Runge-Kutta method is in turn a refinement of this idea. It uses a weighted average of the value of $f(x, y)$ approximated at several points throughout the interval. Notice that if $f(x, y)$ didn't depend on $y$, then the weighted average in the Runge-Kutta formula would simplify to

$$\frac{\Delta x}{6} \left( f(x_n) + 4f\left(x_n + \frac{1}{2}\Delta x\right) + f(x_n + \Delta x) \right),$$

which is precisely Simpson's Rule for approximating the integral.

EXAMPLE 2: Use the Runge-Kutta method to find an approximate value of $y(0.4)$ for the solution of $y' = 2y - x,\ y(0) = 0$ using two subintervals. Carry 5 decimal places throughout the calculations. Round the final answer to three decimal places.

**Solution:** Dividing the interval $[0, 0.4]$ into two subintervals gives us a step size of $\Delta x = 0.2$. We begin with $x_0 = 0$ and $y_0 = 0$. Using $f(x, y) = 2y - x$ gives us

$$\begin{aligned}
k_{01} &= f(0, 0) = 0 \\
k_{02} &= f(0 + \frac{1}{2}(0.2), 0 + \frac{1}{2}(0.2)(0)) = -0.1 \\
k_{03} &= f(0 + \frac{1}{2}(0.2), 0 + \frac{1}{2}(0.2)(-0.1)) = -0.12 \\
k_{04} &= f(0 + 0.2, 0 + (0.2)(-0.12)) = -0.248
\end{aligned}$$

Thus

$$y_1 = 0 + (0.2) \left( \frac{(0) + 2(-0.1) + 2(-0.12) + (-0.248)}{6} \right) = -0.02293$$

Repeating this process gives us

$$\begin{aligned}
k_{11} &= f(0.2, -0.02293) = -0.24586 \\
k_{12} &= f(0.2 + \frac{1}{2}(0.2), -0.02293 + \frac{1}{2}(0.2)(-0.24586)) = -0.34760 \\
k_{13} &= f(0.2 + \frac{1}{2}(0.2), -0.02293 + \frac{1}{2}(0.2)(-0.34760)) = -0.41538 \\
k_{14} &= f(0.2 + 0.2, -0.02293 + (0.2)(-0.41538)) = -0.61201
\end{aligned}$$

and

$$y_2 = -0.02293 + (0.2) \left( \frac{(-0.24586) + 2(-0.34760) + 2(-0.41538) + (-0.61201)}{6} \right) = -0.10239.$$

That is, $y(0.4) \approx -0.10239$. □

EXERCISE 5: Redo Example 3.2 using just a single subinterval.

The approximate solution found in Example 3.2 is within 3.8% of the correct value of $y(0.4)$. To obtain similar accuracy, Newton's method would require at least 44 subintervals! As a rough estimate of the computational cost of these algorithms, observe that 44 steps in Newton's method would require 44 evaluations of $f(x, y)$, whereas the Runge-Kutta calculation required only 8 (plus 2 more calculations to obtain the weighted average). The difference in computational cost can grow quickly as the length of the interval increases, especially when $f(x, y)$ is nonlinear. Thus the Runge-Kutta method can obtain similar results more efficiently than Newton's method, or, it can be used to obtain more accurate results for the same computational investment.

This kind of computational efficiency is especially important in applications that must compute solutions in "real time", such as in graphics-intensive video games and automated piloting systems. Much work is invested in industry to develop and implement efficient algorithms, as that is often less expensive than attempting to construct computers that would need to be orders of magnitude faster to implement the less efficient algorithms.

# Additional Exercises

Use Euler's method to find an approximate value of $y(0.2)$ using a step size of **(a)** $\Delta x = 0.1$, and **(b)** $\Delta x = 0.05$ (or $\Delta t$ when appropriate). **(c)** Then solve the initial value problem using separation of variables and find the exact value of $y(0.2)$. Compare your results.

**6** $y' = xy, \ y(0) = 1$

**7** $y' = \frac{x}{y}, \ y(0) = 2$

**8** $\dot{y} = ty + t, \ y(0) = 0$

**9** $4\dot{y} + e^{t+y} = 0, \ y(0) = 0$

Use Euler's method to find an approximate value of $y(0.4)$ using a step size of **(a)** $\Delta x = 0.4$, **(b)** $\Delta x = 0.2$ and **(c)** $\Delta x = 0.1$.

**10** $y' = 2 - \sqrt{y}, \ y(0) = 0$

**11** $y' = x + y, \ y(0) = 0$

**12** $2y' = x + y, \ y(0) = 1$

**13** $y' = xy + y^3, \ y(0) = 1$

Use the Runge-Kutta method to find an approximate value of $y(0.2)$ using a step size of $\Delta x = 0.1$.

**14** $y' = x + y, \ y(0) = 0$

**15** $y' = y^2, \ y(0) = 1$

---

**16** Set up a calculation on a spreadsheet, or write a short computer program in a high-level language (such as C++, Java or Python) to approximate $y(2)$ where $y' = \sin(y) + x$ and $y(0) = 1$, using Euler's method with a step size of $0.001$.

**17** Consider the following initial value problem: $y' = y, \ y(0) = y_0$. **(a)** Use Euler's method to find an approximate value for $y(x)$ by dividing the interval $[0, x]$ into $N$ subintervals of equal width. (That is to say, you will use $\Delta x = \frac{x}{N}$). *(Hint: First prove that* $y_i = y_0(1 + \Delta x)^i$*.)* **(b)** Take a limit of the result in (a) as $N \to \infty$ to get the exact value of $y(x)$.

**18** It was noted in the text that Euler's method can be thought of as calculating

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} f(x, y(x))\, dx$$

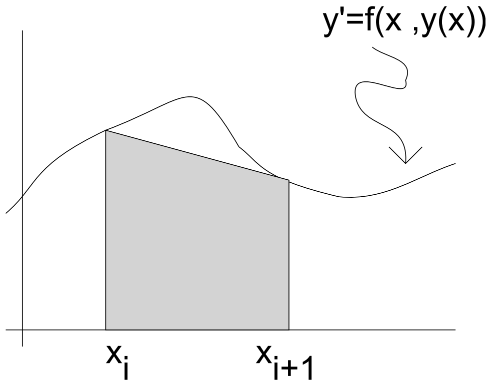
by approximating the integrand $f(x, y(x))$ with its value at the left endpoint:

$$y_{i+1} \approx y_i + f(x_i, y_i)\Delta x.$$

We can usually get a better approximation for the integral, however, if we approximate the integrand by the average of its values at the left and right endpoints:

$$y_{i+1} \approx y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2}\Delta x.$$

This is equivalent to using the Trapezoid Rule to approximate the integral, as illustrated at below.



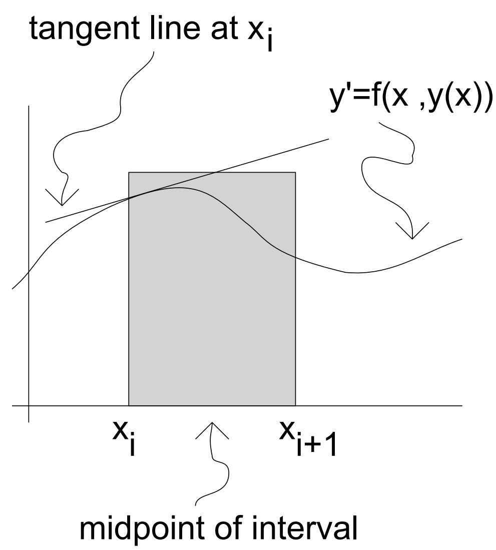

The difficulty with using this directly is that we would need to already have the value of $y_{i+1}$ to evaluate the quotient in the last term. However, we can approximate the y-value at the right endpoint by using the value that Euler's method would give us, namely $y_i + f(x_i, y_i)\Delta x$:

$$y_{i+1} \approx y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_i + f(x_i, y_i)\Delta x)}{2}\Delta x.$$

This formula is known as the **improved Euler formula**, as it usually produces better accuracy then the regular Euler's method when using the same step size.

Use the improved Euler formula to approximate $y(1)$ for the function satisfying $y' = y^2$, $y(0) = 1$ using a step size of $0.25$. Also calculate the approximate value obtained by the regular Euler's method, and find the exact value by solving the IVP with separation of variables. Compare the results.

19 Find a formula for approximating $y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} f(x, y(x))\ dx$ by approximating the integral with the Midpoint Rule from calculus, using the tangent line approximation at the left endpoint to obtain an approximate value of $y$ at the interval's midpoint (illustrated in the figure below). (This is called the **modified Euler formula**.) Use it to estimate $y(0.5)$ for the function $y' = \sqrt{y} + x$ and $y(0) = 1$ with a step size $\Delta x = 0.25$.




20 Answer the prototype question from the beginning of the chapter. Experiment with step sizes until you are satisfied with the results. Feel free to use Euler's method, Runge-Kutta or either of the modifications of that method described in the previous problems. Implement your calculations by using a spreadsheet or by writing a simple computer program in a high-level language.

21 This question illustrates the idea of **sensitive dependence on initial conditions**. Consider the initial-value problem $y' = \sin(y),\ y(0) = \pi$. **(a)** Use a numerical method (or a numerical 'solver' on a computer) to estimate the value of $y(1)$ using the initial value $y(0) = 3.1416$. **(b)** Redo part (a) with the initial value $y(0) = 3.14159$. **(c)** Explain the discrepancy between the results in (a) and (b). Use a slope field or a phase line analysis to illustrate. **(d)** What is the actual value of $y(1)$ when the initial condition is exactly $y(0) = \pi$?

CHAPTER 4

# First Order Linear Equations

**Prototype Question:** A large tank begins with 100 liters of pure water. A brine solution containing 30 grams of salt per liter is pumped into the tank at a rate of 3 liters per minute. The solution in the tank is thoroughly mixed, and it drains at a rate of 1 liter per minute. How much salt will be in the tank after 22 minutes?

At first glance, this problem seems quite similar to a mixing problem we were able to solve earlier using separation of variables. Indeed, the appropriate differential equation can be obtained using the same rate-in-minus-rate-out approach that we have already studied. But the key difference here is that the volume of liquid in the tank is not constant. Because it is draining slower than liquid is being pumped into the tank, the volume after $t$ minutes will be $100 + 2t\ L$. Letting $y(t)$ denote the number of grams of salt in the tank after $t$ minutes have passed, we can write the concentration of salt in the tank at time $t$ as $\frac{y}{100+2t}$ grams per liter. This gives us

$$\begin{aligned}\frac{dy}{dt} &= \text{(rate in) - (rate out)}\\ &= \left(\frac{30}{1}\frac{\text{g}}{\text{L}}\right)\left(\frac{3}{1}\frac{\text{L}}{\text{min}}\right) - \left(\frac{y}{100+2t}\frac{\text{g}}{\text{L}}\right)\left(\frac{1}{1}\frac{\text{L}}{\text{min}}\right)\\ &= \left(90 - \frac{y}{100+2t}\right)\frac{\text{g}}{\text{min}}.\end{aligned}$$

Thus answering the prototype question above requires us to find $y(22)$ where $y(t)$ is the solution of the IVP

$$\frac{dy}{dt} = 90 - \frac{y}{100+2t}, \quad y(0) = 0.$$

This ODE is not separable! Try as you might, you will not be able to algebraically rewrite the differential equation in the form $\frac{dy}{dt} = F(y)G(t)$, it just can't be done. However, this differential equation does have a particularly simple structure: it can be written in the form $\frac{dy}{dt} + p(t)y = g(t)$, and these are exactly the kinds of differential equations we will learn how to solve in this chapter.

We say an ordinary differential equation of order 1 is **linear** if it can be written in the form:

$$a(t)y'(t) + b(t)y(t) = f(t) \tag{1}$$

If $a(t) \neq 0$ on the domain of interest, then dividing by this quantity allows us to rewrite the equation in the **standard form**:

$$\frac{dy}{dt} + p(t)y = q(t).$$

In this chapter, we will explore a technique for analytically solving these differential equations and other equations that can be converted to this form by a change of variable.

EXAMPLE 1: Solve the initial value problem $\frac{dy}{dx} = 3 - 4y,\ y(0) = 1$.

First, let us rewrite the differential equation in the form:

$$\frac{dy}{dx} + 4y = 3.$$

Next, mutiply both sides of the equation by $e^{4x}$ to obtain

$$e^{4x}\frac{dy}{dx} + 4e^{4x}y = 3e^{4x}.$$

The point of this last step is that the left side of the equation is now the derivative of $e^{4x}y$:

$$\frac{d}{dx}\left[e^{4x}y\right] = 3e^{4x}.$$

If we anti-differentiate both sides with respect to $x$, we obtain

$$e^{4x}y = \int 3e^{4x}\, dx = \frac{3}{4}e^{4x} + C.$$

Isolating $y$ gives us

$$y = \frac{3}{4} + Ce^{-4x}.$$

The initial condition $y(0) = 1$ implies that $C = \frac{1}{4}$. Therefore the solution of the IVP is

$$y = \frac{3}{4} + \frac{1}{4}e^{-4x}.$$

□

Multiplying by the expression $e^{4x}$ is what allowed us to recognize the left side of the equation as a derivative (which would have come from using the product rule), and that was what allowed us to simplify when we integrated both sides of the equation. For that reason, an expression fulfilling this purpose is referred to as an **integrating factor**. The basic idea behind our approach to solving first-order linear equations is to multiply both sides of the equation by an integrating factor that will allow us to "reverse the product rule" on the left side.

Any first-order linear differential equation written in standard form,

$$\frac{dy}{dx} + p(x)y = q(x), \tag{2}$$

is a candidate for this **method of integrating factors**. Once an equation is written in this form, we multiply both sides by the integrating factor $e^{\int p(x)dx}$:

$$e^{\int p(x)dx}\frac{dy}{dx} + p(x)e^{\int p(x)dx}y = q(x)e^{\int p(x)dx}.$$

Now we can reverse the product rule to recognize the left side as a derivative:

$$\frac{d}{dx}\left[e^{\int p(x)dx}y\right] = q(x)e^{\int p(x)dx}.$$

Anti-differentiate both sides to get

$$e^{\int p(x)dx}y = \int q(x)e^{\int p(x)dx}\ dx,$$

and then isolate y:

$$y = e^{-\int p(x)dx}\int q(x)e^{\int p(x)dx}\ dx.$$

The reader should not try to memorize this formula. Instead, think of this as a general process that can be applied to solve the differential equation.

**Solving First-Order Linear ODE**

(1) Write the first-order linear equation in standard form
(2) Multiply by an appropriate integrating factor of the form $e^{\int p(x)dx}$
(3) Reverse the product rule to rewrite the left side as a derivative
(4) Anti-differentiate both sides
(5) Isolate $y$

Note that, in practice, any anti-derivative of $p(x)$ will suffice when you construct an integrating factor, so we may ignore the constant of integration when we find $e^{\int p(x)dx}$.

EXAMPLE 2: Solve the IVP $\dot{y} = \frac{y}{t} + 2$, $y(1) = 2$ on the domain $t > 0$.

We begin by rewriting the ODE in the form

$$\dot{y} - \frac{1}{t}y = 2,$$

and then we multiply both sides by the integrating factor $e^{\int -\frac{1}{t}\,dt} = e^{-\ln|t|} = \frac{1}{|t|} = \frac{1}{t}$ (since $t > 0$ by hypothesis) to get

$$\frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2}y = \frac{2}{t}.$$

Now the left side is a derivative of $\frac{1}{t}y$:

$$\frac{d}{dt}\left[\frac{1}{t}y\right] = \frac{2}{t}.$$

Integrating yields

$$\frac{1}{t}y = 2\ln t + C$$

(where we have again used the fact that $t > 0$), hence

$$y = 2t\ln t + Ct.$$

The initial condition $y(1) = 2$ implies $C = 2$, so we have

$$y = 2t\ln t + 2t.$$

□

EXERCISE 1: Solve the initial-value problem $y' = y + e^x,\ y(0) = 3.$

EXERCISE 2: Solve the initial-value problem $\dot{y} = ty + t,\ y(0) = 1.$

EXERCISE 3: Solve the initial-value problem $x^2y' + y = 1,\ y(1) = 2.$ *(Hint: Start by writing the first-order linear differential equation in standard form.)*

Let's return now to the differential equation which was motivated by our prototype question.

EXAMPLE 3: Find $y(22)$, where $\frac{dy}{dt} = 90 - \frac{y}{100+2t}$ and $y(0) = 0$.

Write the equation in the form

$$\frac{dy}{dt} + \frac{1}{100+2t}y = 90.$$

Use the integrating factor $e^{\int \frac{1}{100+2t}\,dt} = e^{\frac{1}{2}\ln|100+2t|} = \sqrt{100+2t}$ to obtain

$$\sqrt{100+2t}\frac{dy}{dt} + \frac{1}{\sqrt{100+2t}}y = 90\sqrt{100+2t}.$$

(We assumed that $100+2t$ was positive so that we could avoid writing absolute value signs, but that is acceptable since we only need a solution on the interval $0 \le t \le 22$.) Therefore

$$\frac{d}{dt}\left[\sqrt{100+2t}y\right] = 90\sqrt{100+2t},$$

and integrating gives us

$$\sqrt{100+2t}y = 30(100+2t)^{\frac{3}{2}} + C.$$

Consequently,

$$y = 30(100+2t) + \frac{C}{\sqrt{100+2t}}.$$

The initial condition $y(0) = 0$ implies that $C = -30000$, so we have

$$y = 30(100+2t) - \frac{30000}{\sqrt{100+2t}},$$

and from this we can calculate

$$y(22) = 30(100+2(22)) - \frac{30000}{\sqrt{100+2(22)}} = 1820.$$

□

In the context of the prototype question for this chapter, this calculation reveals that there will $1820$ $g$ of salt in the tank after 22 minutes.

The next example will illustrate how we can sometimes solve a non-linear differential equation by converting it into a related linear equation.

EXAMPLE 4: Find a solution of the initial value problem $\dot{y} = \frac{y}{t} + y^2,\ y(1) = \frac{1}{2}$.

This differential equation is not separable, and it is not linear because of the presence of the term $y^2$. However, we can find a related linear differential equation in the following way. Letting $u = \frac{1}{y}$, we can write the differential equation in terms of this new variable:

$$\begin{aligned}
\dot{u} &= -\frac{1}{y^2}\dot{y} \quad \text{(by the chain rule)}\\
&= -\frac{1}{y^2}\left(\frac{y}{t} + y^2\right) \quad \text{(by the differential equation } y \text{ must satisfy)}\\
&= -\frac{1}{ty} - 1\\
&= -\frac{u}{t} - 1 \quad \text{(since } u = y^{-1}\text{)}.
\end{aligned}$$

Now we have a differential equation that $u$ must satisfy: $\dot{u} = -\frac{u}{t} - 1$. If we can solve this differential equation to find $u$, then we can take the reciprocal of that solution to find a formula for $y$. Rewrite this as

$$\dot{u} + \frac{1}{t}u = -1.$$

Multiply both sides by the integrating factor $e^{\int \frac{1}{t}dt} = e^{\ln|t|} = |t| = t$. (Since the initial condition corresponds to $t = 1$, it will be enough if we find a solution whose interval of definition is only defined on a set of positive numbers containing 1, so we can simplify our calculations by assuming that $t > 0$.) This will give us

$$t\dot{u} + u = -t.$$

Reversing the product rule on the left side gives us

$$\frac{d}{dt}[tu] = -t.$$

Integrate both sides with respect to $t$:

$$tu = \int -t\,dt = -\frac{t^2}{2} + C.$$

Isolate $u$:

$$u = -\frac{t}{2} + \frac{C}{t}$$

Because $u(1) = \frac{1}{y(1)} = \frac{1}{1/2} = 2$, we obtain $C = \frac{5}{2}$. This gives us the formula $u = -\frac{t}{2} + \frac{5}{2t}$, and taking the reciprocal (because $y = u^{-1}$) yields

$$y = \frac{1}{-\frac{t}{2} + \frac{5}{2t}},$$

or

$$y = \frac{2t}{5 - t^2}.$$

□

Observe that the interval of definition for this solution is $(-\sqrt{5}, \sqrt{5})$, even though we imagined that we'd be satisfied with $t > 0$ to simplify our calculations. Sometimes you get more than you ask for.

The process above can be modified for any differential equation of the form

$$\frac{dy}{dx} = p(x)y + q(x)y^N,$$

where $N$ is a positive integer. These are called **Bernoulli equations.** For any such equation, the substitution $u = y^{1-N}$ leads to the differential equation

$$\frac{du}{dx} = (1 - N)p(x)u + (1 - N)q(x),$$

which is a candidate for the method of integrating factors. Again, the reader should not think of this as a formula to memorize but as a general procedure.

### Solving Bernoulli Equations

For an ODE of the form $\frac{dy}{dx} = p(x)y + q(x)y^N$,

(1) Let $u = y^{1-N}$; use the chain rule and the differential equation for $y$ to find a differential equation for $u$
(2) Solve for $u$ (be sure to modify the initial condition for $y$ appropriately)
(3) Use the solution for $u$ to obtain a formula for $y$

**EXERCISE 4:** Solve the Bernoulli equation $\frac{dy}{dx} = y + y^5$ subject to the initial condition $y(1) = 3$.

**EXERCISE 5:** Solve the Bernoulli equation $\dot{y} = 2y + ty^2$ subject to the initial condition $y(0) = 1$.

# Additional Exercises

Use the method of integrating factors to find the general solution of each of the following differential equations.

**6** $y' = 3y + 2x$

**7** $y' = 2xy + x^3$

**8** $\dot{y} = t - ty$

**9** $4\dot{y} + y = e^{2t}$

**10** $\frac{dx}{dt} = -\frac{x}{t} + \sin(t)$ on $(0, \infty)$

Use the method of integrating factors to find the solution of each of the following initial value problems.

**11** $\dot{y} = -y + t^2,\ y(0) = 0$

**12** $\dot{u} = 2tu,\ y(0) = 1$

**13** $\dot{x} + \frac{x}{t} = \sqrt{1 + t^2},\ x(1) = 1$

Find the solution of each of the following initial value problems, and use it to determine the long-term behavior as $t \to \infty$.

**14** $\dot{y} = t + y,\ y(0) = 1$

**15** $\dot{y} = t + y,\ y(0) = -1$

**16** $t^2\dot{u} + u = t^{-1},\ y(1) = 1$

Solve each of the following initial value problems for Bernoulli equations by first making a substitution to convert it into a linear differential equation.

**17** $y' = y + 3y^2,\ y(0) = 1$

**18** $\dot{x} = 4x + x^3,\ x(1) = \frac{1}{2}$

**19** $u' = xu + xu^2,\ u(1) = 1$

**20** $\dot{y} = y + 2ty^2,\ y(2) = 2$

**21** Find the general solution of the differential equation $a\dot{y} + by = c$, for any constant coefficients $a,\ b,\ c$, with $a \neq 0$. *(Hint: You should consider the cases $b = 0$ and $b \neq 0$ separately.)*

**22** A large tank begins with 50 gallons of water into which is dissolved 10 grams of salt. Salt water solution with a concentration of 5 grams of salt per gallon is added to the tank at a rate of 4 gallons per minute. Meanwhile, the solution in the tank is thoroughly mixed and drains at a rate of 2 gallons per minute. How long will it take until there are 1000 grams of salt in the tank? How much liquid will be in the tank at that instant?

**23** The population of Freedonia reproduces at a rate of 3% per year and dies at a rate of 1% per year. Also, 100,000 immigrate into Freedonia and 40,000 emigrate out each year. Model this population over time using a differential equation. Then use an initial population size of $P(0) = 4$ million to find an explicit formula for the population $P(t)$ using **(a)** separation of variables, and **(b)** the method of integrating factors. Verify that both methods produce the same final result.

**24** Your retirement account begins with half a million dollars. It earns 0.25% interest per month (compounded continuously). You withdrawal \$4,000 per month (withdrawn continuously). Model the balance of the retirement account using a differential equation. Then determine how long the savings will last before you empty the account entirely.

**25** The Springfield P.P. company dumps toxic waste in a lake in the nearby town of Shelbyville. The rate at which radioactive material is dumped increases linearly over time and is given by $r(t) = 100 + 2t$ grams per year, where $t$ is the number of years after 1987. The radioactive material decays continuously at a rate of 0.7877% per year. How much radioactive material will there be in the lake by the end of 2020?

**26** A large object begins to sink in a deep lake of water. The vertical velocity $v$, measured in meters per second, satisfies (roughly) a differential equation of the form

$$\dot{v} = 9.8 - Kv,$$

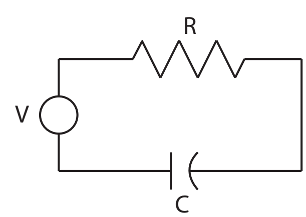
where $K > 0$ is a constant. If the object is falling at $0.2\frac{m}{s}$ after 10 seconds, determine how fast it will be falling after 60 seconds. *(This ODE can be solved using separation of variables, but the method of integrating factors should be easier.)*

**27** A large object begins to sink in a lake that is 100 meters deep. The vertical velocity $v$, measured in meters per second, satisfies

$$\dot{v} = 9.8 - Kv,$$

where $K > 0$ is a constant. If the object falls $0.2$ meters in the first 5 seconds, estimate when the object will hit the bottom of the lake. *(You will encounter an algebraic equation that cannot be solved analytically. Solve it approximately, using a graph or table of values on a calculator or computer, but keep as many decimal places of accuracy as you can until the end of the problem-solving process.)*

**28** The figure below depicts a schematic diagram of a simple electrical circuit containing a resistor, a capacitor and a voltage source, wired in series.




The charge $q$ on the capacitor changes over time, and it can be modeled by the differential equation

$$R\frac{dq}{dt} + \frac{1}{C}q = V,$$

where $R$ is the resistance in Ohms, $C$ is the capacitance in Farads and $V$ is the voltage in Volts; time $t$ is measured in seconds, and the charge $q$ is measured in Coulombs. Assume that $R$, $C$ and $V$ are all positive constants, and find a forumla for $q(t)$ using the initial condition $q(0) = 0$. What is the long-term behavior of the solution?

**29** Solve the logistic differential equation $\dot{P} = \frac{k}{M}P(M-P)$ by treating it as a Bernoulli equation and making a substitution.

**30** The idea of substitution has application beyond Bernoulli equations. For example, any differential equation of the form $y' = f(ax + by + c)$ can be transformed into a separable differential equation by means of the substitution $u = ax + by + c$. Use this idea to solve the initial-value problem:

$$y' = (x + 2y)^2, \quad y(0) = 1.$$

**31** Solve the initial-value problem $y' = \sin^2(x - y),\ y(0) = 1$.

**32** Consider the second-order initial value problem $y'' + 3y' = x$. Make the substitution $u = y'$ to create a first-order differential equation for $u(x)$, find its general solution, and then integrate that solution to find a general solution for $y(x)$. Then find a particular solution that satisfies the initial conditions $y(0) = 2$, $y'(0) = 4$.

**33** When we start with a first-order linear equation in the general form $a(t)y'(t) + b(t)y = f(t)$, we must divide through by $a(t)$ to put it into standard form, and this can cause problems if $a(t) = 0$. Values of $t$ where $a(t) = 0$ are called **singular points** of the ODE, and dividing by $a(t)$ can cause the function $p(t)$ in the standard-form equation $\frac{dy}{dt} + p(t)y = q(t)$ to have discontinuities at these singular points, which can be problematic. For example, it may not be possible to solve an arbitrary IVP with an initial condition at the singular point. To illustrate this, prove that if $y$ satisfies the ODE $ty' + y = 2t$ on a domain of definition that includes the singular point $t = 0$, then the only value that $y(0)$ can take is $0$. *(Therefore, $ty' + y = 2t$ cannot be solved for arbitrary initial conditions on $y(0)$.)*

CHAPTER 5

# Taylor Solutions

> **Prototype Question:** Consider an object whose shape changes as it falls against air resistance (for example, a raindrop). The changing shape means the drag coefficient will change as well. We can model this behavior with the differential equation
>
> $$\dot{v} = g - k(v)v^2,$$
>
> where $k(v)$ denotes the drag coefficient as a function of the object's instantaneous velocity. For a falling object with a drag coefficient $k(v) = e^v$, find an approximate formula for the velocity $t$ seconds after it begins to fall from rest.

This prototype question is very similar to the one we had to begin Chapter 3 on numerical methods, except that it doesn't specify a particular instant in time. Instead, we are to come up with a formula the velocity at a arbitrary time $t$. We will not be able to solve the problem analytically, so we won't be able to find a formula for the exact velocity. Instead, our goal is to find a formula that gives the approximate velocity, at least for a small period of time. This may sound rather similar to a topic in calculus – Taylor approximations and Taylor series. In fact, that's exactly the set of tools we are going to use.

A function $f$ is called **analytic** at $x_0$ if it can be written as a **power series** (or a **Taylor series**),

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

for $x$ in an open interval containing $x_0$. (Recall that the convention with power series is to treat $(x - x_0)^0$ as the constant $1$, so the first term of the power series is just $a_0$.) For example, the function $f(x) = \frac{1}{1-x}$ is analytic at $0$ because

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for all } x \in (-1, 1).$$

(The reader should recall from calculus that this is the geometric series formula.) Similarly, the exponential function $exp(x) = e^x$ is analytic at $0$ because

$$exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{for all } x \in \mathbb{R}.$$

Representing a function as a power series gives us another method for finding solutions to differential equations.

EXAMPLE 1: Solve the initial-value problem $y' - y = x,\ y(0) = 4$ using power series.

Suppose there is a solution that is analytic near $0$ (the x-value of the initial condition). Let us write the solution as $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$. Insert these into the differential equation to get

$$\sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = x.$$

Without the sigma notation, we can write this as

$$(0 + 1a_1 + 2a_2 x + 3a_3 x^2 + \cdots) - (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = x.$$

Rearranging to combine like terms yields

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + (4a_4 - a_3)x^3 + \cdots = x.$$

Equating coefficients gives us the following system of equations:

$$\begin{aligned} a_1 - a_0 &= 0 \\ 2a_2 - a_1 &= 1 \\ 3a_3 - a_2 &= 0. \\ 4a_4 - a_3 &= 0 \\ &\vdots \end{aligned}$$

The initial condition tells us that $y(0) = 4$, and if we insert this into the power series representation for $y$ we get

$$4 = \sum_{n=0}^{\infty} a_n (0)^n = a_0.$$

So $a_0 = 4$, and the first equation above tells us that $a_1 - a_0 = 0$, so $a_1 = 4$ also. The second equation tells us that $2a_2 - a_1 = 1$, so $a_2 = \frac{a_1+1}{2} = \frac{5}{2}$. The third equation tells us that $a_3 = \frac{1}{3}a_2 = \frac{5}{3\cdot 2}$, the next equation tells us that $a_4 = \frac{1}{4}a_3 = \frac{5}{4\cdot 3\cdot 2}$, and so on. That is to say, for $n \geq 2$,

$$a_n = \frac{5}{n(n-1)\cdots(2)} = \frac{5}{n!}.$$

So we have

$$y = 4 + 4x + \frac{5}{2}x^2 + \frac{5}{6}x^3 + \cdots = 4 + 4x + \sum_{n=2}^{\infty} \frac{5x^n}{n!}.$$

$\square$

**EXERCISE 1:** Use the Ratio test to verify that the power series $y = 4 + 4x + \sum_{n=2}^{\infty} \frac{5x^n}{n!}$ converges for $x \in (-1, 1)$. Then verify that this function is a solution by inserting it into the differential equation.

**EXERCISE 2:** Solve the initial-value problem in the previous example using the method of integrating factors. Then find a power series representation for your solution. Verify that it is the same as the solution found above.

The process above relies on the assumption that there is an analytic solution of the given initial value problem. If not, then this process will not find a solution, or it may produce nonsense. However, it is often a reasonable assumption, since so many of the elementary functions we meet in mathematics are in fact analytic.

**EXAMPLE 2:** Consider the initial value problem $y' + xy = 0,\ y(0) = 1$. Suppose there is a solution that is analytic near 0 (the x-value of the initial condition). Let us write the solution as $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$. Inserting these representations into the differential equation gives us

$$\sum_{n=0}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = 0,$$

or

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = -\sum_{n=0}^{\infty} a_n x^{n+1}.$$

If we write this without the sigma notation, we get

$$0 + a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = -a_0 x - a_1 x^2 - a_2 x^3 - a_3 x^4 - \cdots .$$

Equating coefficients of powers of $x$ gives us the following system of equations:

$$\begin{aligned} a_1 &= 0 \\ 2a_2 &= -a_0 \\ 3a_3 &= -a_1. \\ 4a_4 &= -a_2 \\ &\vdots \end{aligned}$$

The first equation tells us $a_1 = 0$, and the third equation tells us that $a_3 = 0$ also. Furthermore, we can see from the pattern of the equations that $a_n = 0$ whenever $n$ is odd. One way to express this is by writing $a_{2n+1} = 0$ for $n = 0, 1, 2, , , , .$

Next, we turn to the even-index coefficients. The second equation tells us that $a_2 = -\frac{1}{2}a_0$. The fourth equation gives us $a_4 = -\frac{1}{4}a_2$, which combined with the previous formula results in $a_4 = \frac{1}{8}a_0$. Again, we see a pattern in the equations for the coefficients $a_{2n}$:

$$\begin{aligned} a_{2n} &= -\frac{1}{2n}a_{2(n-1)} \\ &= \left(-\frac{1}{2n}\right)\left(-\frac{1}{2(n-1)}\right)a_{2(n-2)} \\ &= \cdots \\ &= \left(-\frac{1}{2n}\right)\left(-\frac{1}{2(n-1)}\right)\cdots\left(-\frac{1}{2}\right)a_0 \\ &= \frac{(-1)^n}{2^n n!}a_0. \end{aligned}$$

That is to say, all the even-index coefficients can be written in terms of $a_0$. Notice that we have not yet used the initial condition $y(0) = 1$. If we insert this into the power series representation for $y$, we get

$$1 = \sum_{n=0}^{\infty} a_n(0)^n = a_0.$$

If we insert this value into the previous formula, we obtain

$$a_{2n} = \frac{(-1)^n}{2^n n!}.$$

Hence

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}x^{2n}.$$

(Notice that our power series omits odd-index powers of $x$; that is because we already identified that all the odd-index coefficients are zero.) This formula for $y$ gives us a solution of the initial value problem. □

EXERCISE 3: Use the Ratio Test to prove that the series above converges for all $x \in \mathbb{R}$. Then verify that the given function really is a solution by inserting it into the differential equation.

EXERCISE 4: Use power series to solve the initial-value problem $y' - xy = 0$, $y(0) = 1$. Compare your result with the solution you obtain by separation of variables or the method of integrating factors. Are they the same?

It can be difficult to come up with a nice formula for the coefficients $a_n$, and without such a formula, it is usually not feasible to write out a full series representation for a function. However, often we don't need the whole infinite series but will be satisfied with the first few terms as an approximation. One way of expressing a function this way is to use "little-oh" notation: we write $f(x) = o(g(x))$ as $x \to a$ whenever

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

This is read out loud as follows: "$f(x)$ is little-oh of $g(x)$ as x approaches a".

For example, if $f(x) = x^3$, then $f(x) = o(x^2)$ as $a \to 0$ because

$$\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{x^3}{x^2} = 0.$$

Similarly, $f(x) = o(x)$ as $x \to 0$; however, $f(x) \neq o(x^3)$ as $x \to 0$.

We then extend this notation by writing $f(x) = h(x) + o(g(x))$ as $x \to a$ if

$$\lim_{x \to a} \frac{f(x) - h(x)}{g(x)} = 0.$$

(This is the same as saying that $f(x) - h(x) = o(g(x))$ as $x \to a$.)

For example, $\sin(x) = x + o(x^2)$ as $x \to 0$ because L'Hospital's Rule allows us to calculate

$$\begin{aligned}\lim_{x \to 0} \frac{\sin(x) - x}{x^2} &= \lim_{x \to 0} \frac{\cos(x) - 1}{2x} \\ &= \lim_{x \to 0} \frac{-\sin(x)}{2} \\ &= 0.\end{aligned}$$

The purpose of little-oh notation is that it allows us to say things like "$\sin(x) \approx x$ for small $x$" more precisely by saying just how good the approximation is.

Here's the connection with power series: Let $N$ be a positive integer, and suppose $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$; then

$$f(x) = \sum_{n=0}^{N} a_n (x - x_0)^n + o((x - x_0)^N) \text{ as } x \to x_0.$$

That is to say, we can replace the infinite sum by a finite sum if we append the little-oh notation. In this setting, the little-oh notation represents the error when you drop the suppressed terms of the series, and it expresses the degree of the error. If the error is $o((x-x_0)^N)$, that means the missing terms are of degree greater than $N$ (thus they are very small if $x$ is close to $x_0$). The explicit part of the series, $\sum_{n=0}^{N} a_n(x-x_0)^n$, is called the $N^{th}$ **degree Taylor polynomial of** $f$ or the $N^{th}$ **degree Taylor approximation of** $f$.

EXERCISE 5: Use the power series representation for $e^x$ at 0 to show that $e^{(x^2)} = 1+x^2+\frac{1}{2}x^4+o(x^5)$ as $x \to 0$.

EXERCISE 6: Prove that, if $k > l$, then $x^k = o(x^l)$ as $x \to 0$.

The next example shows this notation in action.

EXAMPLE 3: Let's find a $2^{nd}$ degree Taylor approximation for the solution of $yy' = x,\ y(0) = 2$. Write $y(x) = a_0+a_1x+a_2x^2+a_3x^3+o(x^3)$ as $x \to 0$. Then $y'(x) = a_1+2a_2x+3a_3x^2+o(x^2)$ as $x \to 0$. Inserting these into the differential equation gives us

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + o(x^3))(a_1 + 2a_2x + 3a_3x^2 + o(x^2)) = x.$$

Let's distribute, but every time we run into a power of $x$ with exponent 3 or greater, we will just 'consume' it in the notation $o(x^2)$:

$$a_0a_1 + 2a_0a_2x + 3a_0a_3x^2 + a_1^2x + 2a_1a_2x^2 + a_2a_1x^2 + o(x^2) = x.$$

Combining like terms results in

$$(a_0a_1) + (2a_0a_2 + a_1^2)x + (3a_0a_3 + 3a_1a_2)x^2 + o(x^2) = x + o(x^2).$$

Equating coefficients gives us

$$\begin{aligned} a_0a_1 &= 0 \\ 2a_0a_2 + a_1^2 &= 1 \\ 3a_0a_3 + 3a_1a_2 &= 0 \end{aligned} \quad .$$

The suppressed equations correspond to powers of $x$ that are greater than 2, but since we're seeking a $2^{nd}$ degree Taylor approximation, we don't need to worry about those. If we needed a $3^{rd}$ degree Taylor approximation, we would need to go a step further with our equations, and we would need to only suppress terms of degree $x^4$ and higher.

The initial condition $y(0) = 2$ tells us that $a_0 = 2$. Inserting this into the first equation above tells us $a_1 = 0$. Then the second equation simplifies to $4a_2 = 1$, so $a_2 = \frac{1}{4}$. That's all we need! The last equation would tell us what $a_3$ is, but we don't need it, since we're only seeking a $2^{nd}$ degree Taylor approximation.

Using these coefficients, we have

$$y = 2 + \frac{1}{4}x^2 + o(x^2) \text{ as } x \to 0.$$

□

Note that we didn't write "as $x \to 0$" in every single line of the calculation above. That is acceptable, provided that we make it explicit earlier in the argument and in our final solution.

EXERCISE 7: Find a $3^{rd}$ degree Taylor approximation for the solution of $yy' = x,\ y(0) = 3$.

EXERCISE 8: Find a $2^{nd}$ degree Taylor approximation for the solution of $(y')^2 = y,\ y(0) = 1$. *(Hint: There are actually two solutions, because you'll have some flexibility in choosing one of the coefficients.)*

# Additional Exercises

Use power series to solve the following initial value problems.

**9** $y' = 3y,\ y(0) = 2$

**10** $y' = xy,\ y(0) = -1$

**11** $y'' = y,\ y(0) = 4,\ y'(0) = 0$

**12** $y'' = xy,\ y(0) = 1, y'(0) = 0$

**13** $(x+1)y''+y'-xy = 0,\ y(0) = 0,\ y'(0) = 2$

**14** $(x^2 + 1)y'' = xy,\ y(0) = 1, y'(0) = 2$

Find Taylor approximations of degree $n$ for the following initial value problems.

**15** $y' = y^2 - x,\ y(0) = 1,\ n = 3$

**16** $y' = y^2 + x^2,\ y(0) = -1, n = 3$

**17** $y'' = y^2,\ y(0) = 1, y'(0) = 0, n = 4$

**18** $y'' + sin(x)y = 0,\ y(0) = 0, y'(0) = 1, n = 4$ *(Hint: Write* $\sin(x)$ *as a power series.)*

---

**19** Try to find a $2^{nd}$ order Taylor approximation for the solution of $(y')^2 = xy,\ y(0) = 1$. You will encounter a contradiction as you try to calculate the values of the coefficients in the power series. What does this contradiction tell you?

**20** Find an alternative approach to the initial value problem in Exercise 8 that does not use Taylor series or Taylor approximations. Explain how that approach also gives you two different solutions of the initial value problem.

**21** Suppose that $f(x) = o(x^k)$ as $x \to 0$ and $h(x) = o(x^l)$ as $x \to 0$, where $k, l > 0$. **(a)** Prove that $(fh)(x) = o(x^{k+l})$ as $x \to 0$. **(b)** Prove that $f(x)+h(x) = o(x^{\min(k,l)})$ as $x \to 0$.

**22** Solve the prototype question for this chapter using a $2^{nd}$ order Taylor approximation. *(Hint: You'll also want to express the exponential function in the form* $e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$ *as* $x \to 0$.*)*

**23** The computer software MAPLE can be used to find Taylor solutions for differential equations. For example, to compute an approximate solution to $ay'(x) + by(x) = g(x)$ with the initial condition $y(0) = c$, type

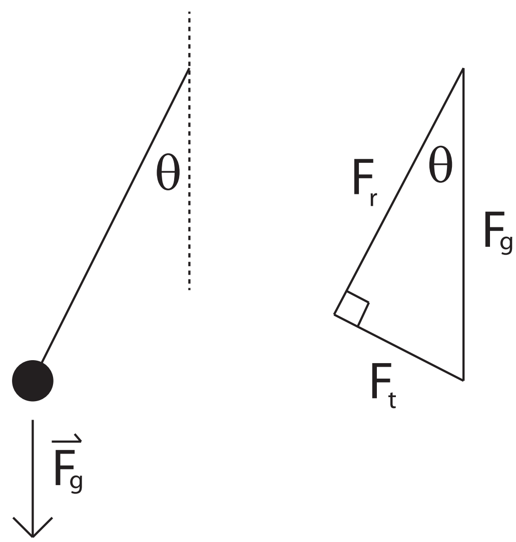
```
dsolve({ay'(x)+by(x)=g(x),y(0)=c},
                          y(x),series)
```

and press Enter. Use this command to find a Taylor approximation for the solution of $y' + y^2 = 1,\ y(0) = 0$. *(Note: MAPLE reports the answer using "big-oh" notation instead of "little-oh" notation. Big-oh notation indicates the smallest degree of the suppressed terms. So, for example we could write* $e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$ *as* $x \to 0$*, or we could instead write* $e^x = 1 + x + \frac{1}{2}x^2 + O(x^3)$ *as* $x \to 0$.*)*

**24** Use MAPLE to find a Taylor approximation for the solution of the second order equation $y'' + \sin(y) = 0$ with the initial conditions $y(0) = 0.1,\ y'(0) = 0$. *(This differential equation is related to the motion of an ideal pendulum. See the Focus on Modeling section that follows this chapter to learn how.)*

FOCUS ON MODELING
# Pendulums

Attach a mass to the end of a stiff rod that is allowed to swing from a fixed point, and you have a pendulum. Historically, pendulums have been used as accurate timekeeping pieces and accelerometers. Let's analyze the behavior of a pendulum by finding a differential equation governing the rate of change of the angle $\theta$ between the rod and the vertical. In our model, we will use a massless rod of length $L$, with a mass $m$ attached to the end.




The only external force acting on our pendulum is gravity, denoted by $\vec{F_g}$, which points downward with magnitude $mg$. Let us decompose this vector into a sum of two vectors: one that is parallel to the rod, $\vec{F_r}$ ($r$ for 'radial'); then the other vector, which we denote by $\vec{F_t}$, must be tangential to the path of the swinging mass. For a given value of $\theta$, this decomposition is unique. Trigonometric considerations tell us that

$$|\vec{F_r}| = |\vec{F_g}|\cos(\theta) \quad \text{and} \quad |\vec{F_t}| = |\vec{F_g}|\sin(\theta)|.$$

The tangential force $\vec{F_t}$ causes an acceleration of the mass along the circular path centered at the pendulum's fixed point. We know from precalculus that the linear velocity of the mass is equal to the radius of the circle multiplied by the angular velocity, i.e., $L\dot{\theta}$. Differentiating this expression gives us the acceleration, $L\ddot{\theta}$. Newton's second law then tells us that the tangential force equal the mass times the acceleration:

$$mL\ddot{\theta} = -mg\sin(\theta).$$

Notice the negative sign on the right side of the equation: it is there because the direction of the acceleration will be opposite the direction of the displacement from $\theta = 0$, since gravity will work to bring the mass back toward that position. Dividing through by $m$ and rearranging terms gives us the differential equation

$$\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0.$$

There were three important assumptions made in deriving this model:

(1) the rod remains taught and straight;

(2) the pendulum moves in only two dimensions; and

(3) the motion is not subject to resistance or friction.

Even with these simplifications, the resulting differential equation is not simple to solve. It can be analyzed using numerical or Taylor methods. But in order to obtain analytic solutions, we usually make one more assumption: *the angle $\theta$ remains small.* The point of this assumption is that, when $\theta$ is small, $\sin(\theta) \approx \theta$ (provided $\theta$ is measured in radians) which can be seen by neglecting the non-linear terms in the power series representation of sine: $\sin(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{(2n+1)}}{(2n+1)!} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$ . Replacing $\sin(\theta)$ with $\theta$ gives us the simpler, approximate differential equation:

$$\ddot{\theta} + \frac{g}{L}\theta = 0.$$

This second-order linear differential equation can be solved analytically using the methods we will discuss in Chapter 7.

CHAPTER 6

# Existence and Uniqueness

**Prototype Question:** If we can't find an explicit formula for a solution to an initial value problem, how do we know there is a solution at all?

We investigated the behavior of solutions to various ODE using graphical methods in Chapter 2, and we found approximate values of solutions using numerical method in Chapter 3. The whole time, we assumed that there *were* solutions to the given equations, even when we admitted that we wouldn't be able to produce formulas for said solutions. But that assumption really requires proof if we are going to trust any of the conclusions we draw from graphical and numerical methods. If the assumption is flawed, then those conclusions will be meaningless.

The next theorem is the main result of this chapter.

**Existence and Uniqueness Theorem (Autonomous)**
Suppose that $f(y)$ and $f'(y)$ are defined and continuous on an open interval containing $y_0$. Then there is an open interval $I$ containing $x_0$ such that the initial value problem

$$\frac{dy}{dx} = f(y), \quad y(x_0) = y_0$$

has a unique solution $y(x)$ defined on $I$.

When we say that a solution to an initial-value problem is **unique** on an interval $I$, we mean that if $y$ and $z$ are both functions that satisfy the initial-value problem on $I$, then $y(t) = z(t)$ for all $t \in I$. As the reader will find in the problems at the end of this chapter, uniqueness isn't always guaranteed. But it is guaranteed (at least *near* $x_0$) when $f$ and $f'$ are both continuous.

As we discuss the proof of this theorem, we will use a simple example to illustrate how the proof works. We will consider initial-value problem

$$\begin{cases} \frac{dy}{dx} = 3y \\ y(0) = 2 \end{cases} \qquad (\dagger)$$

The problem in (†) is easy to solve using techniques already discussed, but this will work well to illustrate the relevant ideas.

## Picard Iterates and Uniform Norms

A central idea in both the uniqueness and existence arguments is a procedure called Picard iteration. Suppose that $y$ is a function defined on an interval $I$. A **Picard iterate** of $y$ is another function, $\tilde{y}$, defined according to the following formula:

$$\tilde{y}(x) = y_0 + \int_{x_0}^{x} f(y(s))\, ds$$

Here, $x_0$, $y_0$ and $f$ are given – they correspond to the data for a given initial-value problem.

Exercise 1: Suppose that $x_0 = 0$, $y_0 = 2$, $f(y) = y^2$ and $y(x) = x$. Prove that $\tilde{y}(x) = 2 + \frac{x^3}{3}$.

Exercise 2: Let $x_0 = 0$, $y_0 = 0$ and $f(y) = e^y$. Find the Picard iterate of the function $y(x) = 2x$.

Exercise 3: Prove that $y(x)$ solves the initial value problem

$$\frac{dy}{dx} = f(y), \quad y(0) = y_0$$

if and only if $y = \tilde{y}$. *(Hint: Recall the Fundamental Theorem of Calculus.)*

The last exercise reveals the relationship between Picard iteration and initial-value problems. It allows us to recast the differential equation as an **integral equation**: finding a solution $y(x)$ of the equation

$$y(x) = y_0 + \int_{x_0}^{x} f(y(s))\, ds$$

is equivalent to finding a solution of the initial-value problem $\frac{dy}{dx} = f(y),\ y(x_0) = y_0$.

We need to define one more item of notation before we can proceed. For a bounded function $f$ defined on an interval $I$, define

$$\| f \|_I = \min \{a; |f(x)| \le a \text{ for all } x \in I\}.$$

This quantity is called the **uniform norm** of $f$ on $I$, or just the **norm** of $f$ for short. Notice that if $|f|$ attains a maximum value on $I$, then $\| f \|_I = \max_{x\in I} |f(x)|$.

EXERCISE 4: Let $f(x) = x^2 - x$ on the interval $I = [0, 1]$. Show that $\| f \|_I = \frac{1}{4}$.

EXERCISE 5: Let $f(x) = e^{(-x^2)}$ and let $I = [-1, 1]$. Find $\| f \|_I$ and $\| f' \|_I$.

EXERCISE 6: Prove that if $\| f - g \|_I = 0$, then $f(x) = g(x)$ for all $x \in I$.

The last exercise shows us how we can use the uniform norm to identify when two functions are equal on a domain $I$: they are equal if the uniform norm of their difference is zero.

UNIQUENESS

Let's look at our example problem (†). Suppose that $y(x)$ and $z(x)$ are two bounded functions that both satisfy (†) on some interval $I$ centered around $x_0 = 0$, say $I = (-k, k)$. Then $y = \tilde{y}$, and $z = \tilde{z}$. Consequently,

$$\begin{aligned} |\tilde{y}(x) - \tilde{z}(x)| &= \left| y_0 + \int_{x_0}^{x} f(y(s))\, ds - y_0 - \int_{x_0}^{x} f(z(s))\, ds \right| \\ &= \left| \int_{x_0}^{x} f(y(s)) - f(z(s))\, ds \right|. \end{aligned}$$

Using the fact that $f(y) = 3y$, we can rewrite this as

$$|\tilde{y}(x) - \tilde{z}(x)| = \left| \int_{x_0}^{x} 3y(s) - 3z(s)\, ds \right|.$$

We can also use the general fact about definite integrals that $\left| \int_a^b g(t)\, dt \right| \leq \int_a^b |g(t)| dt$ (provided $a \leq b$) to obtain for $x \geq x_0$

$$|\tilde{y}(x) - \tilde{z}(x)| \leq \int_{x_0}^{x} 3|y(s) - z(s)|\, ds \leq 3 \int_{x_0}^{x} \| y - z \|_I \;\; ds = 3 \| y - z \|_I \, |x - x_0|.$$

Similarly, if $x \leq x_0$, we get

$$|\tilde{y}(x) - \tilde{z}(x)| \leq \int_{x}^{x_0} 3|y(s) - z(s)|\, ds \leq 3 \int_{x}^{x_0} \| y - z \|_I \;\; ds = 3 \| y - z \|_I \, |x - x_0|.$$

Either way, we see that $|\tilde{y}(x) - \tilde{z}(x)| \leq 3 \| y - z \|_I \, |x - x_0|$. Now let's focus our attention on intervals of the form $I = (-k, k)$, where $0 < k < 3$. We then have

$$|\tilde{y}(x) - \tilde{z}(x)| \leq 3k \| y - z \|_I \quad \text{for all } x \in I,$$

and therefore

$$\| \tilde{y} - \tilde{z} \|_I \leq 3k \| y - z \|_I \ . \tag{3}$$

But since $y = \tilde{y}$ and $z = \tilde{z}$, that means

$$\| y - z \|_I \leq 3k \| y - z \|_I,$$

and because $3k < 1$, if $\| y - z \|_I \neq 0$, this is a contradiction! To see why, just divide both sides by $\| y - z \|_I$ to get $1 \leq 3k$, which implies $1 < 1$, which is utter nonsense. What does this contradiction tell us? It says that $\| y - z \|_I = 0$ on $I = (-k, k)$ for any positive $k < \frac{1}{3}$. That is to say, $y(x) = z(x)$ on $I$, and therefore there is only one solution of (†) in $I$. This is exactly what we mean by 'uniqueness' on $I$. (Notice that we have said nothing about whether $y$ and $z$ are equal outside of $I$.)

Incidentally, because $k$ can be *any* positive number less than $\frac{1}{3}$, we can really conclude that $y = z$ on the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$, because these functions could only fail to be equal on this interval by failing to be equal on some smaller interval.

The calculations above can be generalized to prove the following important result:

**Uniqueness Theorem with Lipschitz Condition**

Suppose that $f$ and $f'$ are defined and continuous on $\mathbb{R}$. Also suppose that $|f'| \leq K$ on $\mathbb{R}$. Then if $y_1$ and $y_2$ are both functions that satisfy the initial-value problem

$$\frac{dy}{dx} = f(y), \quad y(x_0) = y_0$$

on the interval $I = (x_0 - \frac{1}{K}, x_0 + \frac{1}{K})$, it must be true that $y_1(x) = y_2(x)$ for all $x \in I$.

The statement that $|f'| \leq K$ implies $|f(y_2) - f(y_1)| \leq K|y_2 - y_1|$ (by the Mean Value Theorem), and this inequality is known as a **Lipschitz condition** on $f$, which explains the name of the boxed result above.

This result supposes that $f$ and $f'$ are continuous on all of $\mathbb{R}$. In fact, that is not necessary. It is enough to assume that $f$ and $f'$ are continuous on *some* open interval containing $y_0$. However, under that weakened hypothesis, we can no longer guarantee the uniqueness on the entire interval $\left(x_0 - \frac{1}{K}, x_0 + \frac{1}{K}\right)$. Instead, we can just say that there is *some* open interval $I$ containing $x_0$ on which solutions must be unique. To say how large such an interval is requires delicate analysis which is outside the scope of this text.

EXERCISE 7: What is the largest interval on which the Uniqueness Theorem with Lipschitz Condition guarantees uniqueness of the solution to the initial value problem $y' = \sin(2y),\ y(0) = 1$?

The next example illustrates how uniqueness might fail if the condition that $f'$ exists and is continuous on an open interval containing $y_0$ is not met.

EXAMPLE 1: Consider the initial value problem $y' = 3y^{\frac{2}{3}}$, $y(0) = 0$. The constant function $y(x) = 0$ for all $x$ is a solution of this problem, but it is not the only one. Consider, for example, the function defined by

$$y_1(x) = \begin{cases} 0 & \text{if } x \le 1 \\ (x-1)^3 & \text{if } x > 1 \end{cases}.$$

Observe that $y_1'(x) = 0$ if $x < 1$ and $y_1'(x) = 3(x-1)^2$ if $x > 1$. Because $y_1$ is stitched together from two elementary functions using a piecewise definition, we need to find $y_1'(1)$ using the limit definition of derivative:

$$\lim_{h \to 0^-} \frac{y_1(1+h) - y_1(1)}{h} = \lim_{h \to 0^-} \frac{0-0}{h} = 0,$$

and

$$\lim_{h \to 0^+} \frac{y_1(1+h) - y_1(1)}{h} = \lim_{h \to 0^+} \frac{(1+h-1)^3 - 0}{h} = \lim_{h \to 0^+} \frac{h^3}{h} = \lim_{h \to 0^+} h^2 = 0.$$

Since the one-sided limits are equal, we have $y_1'(1) = \lim_{h \to 0} \frac{y_1(1+h) - y_1(1)}{h} = 0$. Combining these derivative facts gives us

$$y_1'(x) = \begin{cases} 0 & \text{if } x \le 1 \\ 3(x-1)^2 & \text{if } x > 1 \end{cases}.$$

Observe also that

$$3y_1^{\frac{2}{3}}(x) = \begin{cases} 3(0)^{\frac{2}{3}} & \text{if } x \le 1 \\ 3((x-1)^3)^{\frac{2}{3}} & \text{if } x > 1 \end{cases} = \begin{cases} 0 & \text{if } x \le 1 \\ 3(x-1)^2 & \text{if } x > 1 \end{cases},$$

so $y_1'(x) = 3y_1^{\frac{2}{3}}(x)$ for all $x$. Since $y_1(0) = 0$, we see that $y_1$ also satisfies this initial value problem.

In fact. there are infinitely many solutions of this initial value problem: for any parameter $a \ge 0$, the function

$$y_a = \begin{cases} 0 & \text{if } x \le a \\ (x-a)^3 & \text{if } x > a \end{cases}$$

will satisfy the differential equation and the initial condition. (If $a < 0$, only the differential equation is satisfied.)

Why does the uniqueness argument not apply to this problem? If we think of the differential equation in the form $y' = f(y)$, then the right side is $f(y) = 3y^{\frac{2}{3}}$. This function $f(y)$ is defined for all $y \in \mathbb{R}$. However, $f'(y)$ does not exist at the initial value $y_0 = 0$, and therefore there is no open interval containing $y_0 = 0$ on which we can say that $f$ and $f'$ are both defined and continuous throughout. $\square$

EXERCISE 8: Use the limit definition of derivative to verify that, for $f(y) = 3y^{\frac{2}{3}}$, the derivative $f'(0)$ does not exist.

Uniqueness results are of practical interest because many problems in industry are too complicated to admit analytic techniques for their solution, and numerical methods must be relied upon to find approximate solutions. In such circumstances, it is important to know that the solution one has approximated is the *only* solution to the problem at hand.

EXISTENCE

How do we know that there is a solution to a given initial-value problem at all? Let's again look at our model problem (†). Consider the initial-value $y_0$ as a *constant function*: in this case, $y_0(x) = 2$ for all $x$. Define a sequence of functions $y_j$ according to the recursion formula $y_j = \tilde{y}_{j-1}$ for all integers $j \geq 1$.

For example,

$$\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f(y_0(s))\, ds \\ &= 2 + \int_0^x 3(2)\, ds \\ &= 2 + 6x \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= y_0 + \int_{x_0}^{x} f(y_1(s))\, ds \\ &= 2 + \int_0^x 3(2 + 6s)\, ds \\ &= 2 + 6x + 9x^2. \end{aligned}$$

EXERCISE 9: Verify that $y_3(x) = 2 + 6x + 9x^2 + 9x^3$ and $y_4(x) = 2 + 6x + 9x^2 + 9x^3 + \frac{27}{4}x^4$.

This sequence of functions is converging to a limit! Observe that

$$y_n(x) = 2\sum_{j=0}^{n} \frac{(3x)^j}{j!},$$

so that

$$\lim_{n\to\infty} y_n(x) = 2\sum_{j=0}^{\infty} \frac{(3x)^j}{j!} = 2e^{3x}.$$

Notice that this limit function, $y(x) = 2e^{3x}$, is a solution of (†)! We have constructed a solution of the initial-value problem by generating a sequence of Picard iterates. Each function in the sequence turns out to be a kind of approximate solution. The limit of the sequence is an exact solution.

We can verify that this will work for a more general initial value problem by looking at the differences between consecutive terms. If we revisit our uniqueness argument leading to the inequality (3) and replace $y$ with $y_{j-1}$ and $z$ with $y_{j-2}$, we get

$$\| y_j - y_{j-1} \|_I \leq 3k \| y_{j-1} - y_{j-2} \|_I,$$

and iterating this inequality $j-1$ times leads to

$$\| y_j - y_{j-1} \|_I \leq (3k)^{j-1} \| y_1 - y_0 \|_I .$$

When we assume that $3k < 1$, this tells us that the sequence of $y_j$'s is contracting – the difference between consecutive terms decreases geometrically, and that's enough to guarantee that the sequence converges for every $x$ in $I$. Here's why:

$$\begin{aligned}
\lim_{n\to\infty} y_n(x) &= \lim_{n\to\infty} y_n(x) - y_{n-1}(x) + y_{n-1}(x) - y_{n-2}(x) + \cdots - y_1(x) + y_1(x) - y_0(x) + y_0(x) \\
&= \lim_{n\to\infty} y_0 + \sum_{j=1}^{n} (y_j(x) - y_{j-1}(x)) \\
&= y_0 + \sum_{j=1}^{\infty} (y_j(x) - y_{j-1}(x)),
\end{aligned}$$

and the infinite series at the end converges absolutely because (using the Comparison Test twice)

$$\begin{aligned}
\sum_{j=1}^{\infty} |y_j(x) - y_{j-1}(x)| &\leq \sum_{j=1}^{\infty} \| y_j - y_{j-1} \|_I \\
&\leq \sum_{j=1}^{\infty} (3k)^{j-1} \| y_1 - y_0 \|_I,
\end{aligned}$$

where the sum in the very last line is a convergent geometric series.

That is to say, there is a function defined by the formula $y(x) = \lim_{n\to\infty} y_n(x)$ for all $x \in I$. The definition of the sequence of functions gives us

$$y_n(x) = y_0 + \int_{x_0}^{x} f(y_{n-1}(s))\, ds,$$

and taking limits on both sides as $n \to \infty$ gives us

$$y(x) = y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f(y_{n-1}(s))\, ds.$$

If it is permissible[1] to exchange the order of the limit and the integral, we get

$$\begin{aligned} y(x) &= y_0 + \int_{x_0}^{x} \lim_{n\to\infty} f(y_{n-1}(s))\, ds \\ &= y_0 + \int_{x_0}^{x} f\left(\lim_{n\to\infty} y_{n-1}(s)\right)\, ds \\ &= y_0 + \int_{x_0}^{x} f(y(s))\, ds. \end{aligned}$$

That is to say, $y = \tilde{y}$, so $y$ is a solution of $y' = f(y),\ y(x_0) = y_0$ on $I$. We have argued the following existence result:

> **Existence Theorem with Lipschitz Condition**
> Suppose that $f$ and $f'$ are continuous on $\mathbb{R}$, and that $|f'| \le K$ on $\mathbb{R}$. Then there is a function $y(x)$ defined on $I = \left(x_0 - \frac{1}{K}, x_0 + \frac{1}{K}\right)$ such that $y(x_0) = y_0$ and $\frac{dy}{dx} = f(y)$.

As with our earlier uniqueness result, it is possible to loosen the hypotheses. As long as $f$ and $f'$ are continuous on some open interval containing $y_0$, then there is some open interval $I$ containing $x_0$ on which a solution of $y' = f(y),\ y(x_0) = y_0$ is guaranteed to exist. Making these adjustments to the proofs in this chapter gives us the general Existence and Uniqueness Theorem stated at the beginning of this chapter.

Exercise 24 explores an example of an initial value problem for which these conditions are not met and for which one can prove that solutions do not exist at all.

## NON-AUTONOMOUS EQUATIONS

[1]It is not permissible! However, a result from advanced calculus called the Arzela-Ascoli Theorem tells us that it is possible to exchange the order of the limit and the integral if we switch to an appropriate subsequence of the $y_j$'s; the subsequence has the same limit $y(x)$, so we end up with the same result. See [**3**].

All of the arguments above can be modified to deal with initial-value problems of the form

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}.$$

In that setting, the appropriate version of the Picard iterate is

$$\tilde{y} = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds.$$

So, for example, given the initial-value problem

$$\begin{cases} y' = yx \\ y(0) = 4 \end{cases},$$

the first Picard iterate would be

$$y_1(x) = 4 + \int_0^x 4s\, ds = 4 + 2x^2.$$

EXERCISE 10: Find the Picard iterates $y_2$ and $y_3$ for $y' = yx,\ y(0) = 4$.

The determining factors for existence and uniqueness are how the function $f(x,y)$ depends on $y$. The reader will see this by comparing the Existence and Uniqueness Theorem with the following. Note that an **open rectangle** in $\mathbb{R}^2$ is a Cartesian product of open intervals in $\mathbb{R}$:

$$R = (a,b) \times (c,d) \text{ means } R = \left\{(x,y) \in \mathbb{R}^2; a < x < b \text{ and } c < y < d\right\}.$$

**Existence and Uniqueness for Non-autonomous ODE**

Suppose that $f(x,y)$ and $f_y(x,y)$ are defined and continuous on an open rectangle $R$ containing $(x_0, y_0)$. Then there is an open interval $I$ containing $x_0$ such that the initial-value problem

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

has a unique solution $y(x)$ defined on $I$.

### VECTOR-VALUED FUNCTIONS

The arguments presented in this chapter can further be modified to prove existence and uniqueness for differential equations involving vector-valued functions. Vectors turn out to be a very useful language for working with systems of differential equations, as we'll see in Chapter 13.

Let us denote vectors in $\mathbb{R}^n$ by capital letters, such as $X$ and $Y$. It will be most convenient later if we think of these as column vectors and denote the components of these vectors by the corresponding lower-case letters with subscripts. For example, $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$.

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function, then the derivative of $f$ is a matrix $Df$ whose components represent all the partial derivatives of all the components of $f$:

$$\text{for } f(Y) = \begin{bmatrix} f_1(Y) \\ f_2(Y) \\ \vdots \\ f_n(Y) \end{bmatrix}, \text{ the derivative is } Df = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix}.$$

For example, consider the function $f$ defined on $\mathbb{R}^2$ by

$$f\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 3y_1 + 2y_2 \\ 4y_1 - y_2^2 \end{bmatrix}.$$

The derivative is

$$Df\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & -2y_2 \end{bmatrix}.$$

Such functions are said to be continuous if all their component functions are continuous.

**EXERCISE 11:** Find the derivative of the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{bmatrix} y_1 + y_3^2 \\ y_2 - y_3 \\ y_1 y_2 y_3 \end{bmatrix}.$$

We will use vector-valued functions to represent systems of ordinary differential equations in Chapter 13, and here is the statement of these ideas which we will need in that context.

**Existence and Uniqueness for Systems**

Suppose that $f$ and $Df$ are defined and continuous on an open set $R \subset \mathbb{R}^n$ containing $Y_0$. Then there is an open interval $I$ containing $x_0$ such that the initial-value problem

$$\frac{dY}{dx} = f(Y), \quad Y(x_0) = Y_0$$

has a unique solution on the interval $I$.

Here's a sample application of this theorem.

EXAMPLE 2: Consider the following initial-value problem for a vector-valued function $Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$:

$$\begin{bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{bmatrix} = \begin{bmatrix} -4y_2(x) \\ y_1(x) \end{bmatrix}, \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The function $f$ in this case is

$$f\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} -4y_2 \\ y_1 \end{bmatrix},$$

and the derivative of this function is

$$Df = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}.$$

The matrix function $Df$ is constant, so it is clearly continuous, as is $f$. Therefore, according to this Existence and Uniqueness Theorem for systems, there must be an interval in $\mathbb{R}$ containing $0$ on which there is a unique solution of this differential equation. (Indeed, the solution turns out to be $Y(x) = \begin{bmatrix} \cos(2x) \\ \frac{1}{2}\sin(2x) \end{bmatrix}$. We will explore methods for finding such explicit solutions in Chapter 13.)

# Additional Exercises

Find the Picard iterates $y_1 = \tilde{y}_0$ and $y_2 = \tilde{y}_1$ for each of the following initial value problems.

**12** $y' = 2y + x,\ y(0) = -2,\ y_0 = -2$

**13** $y' = y^2 - x,\ y(0) = 1,\ y_0 = 1$

Calculate $\| \ f \ \|_I$ for the function $f$ on the given interval $I$.

**14** $f(x) = x^3 - x,\ I = [0, 1]$

**15** $f(x) = x^4 - x,\ I = [0, 2]$

For each of the initial value problems below, find the interval on which solutions are guaranteed to be unique according to the Uniqueness Theorem with Lipschitz Condition.

**16** $y' = \cos(3y), y(0) = 2$

**17** $y' = \tan^{-1}(y),\ y(1) = 0$

**18** $y' = \frac{1}{1+y^2},\ y(0) = 0$

**19** $y' = y\tan^{-1}(y) - \frac{1}{2}\ln\left(1 + y^2\right), y(0) = 1$

---

**20** Find the solution of the initial value problem $y' = y + x,\ y(0) = 1$ using the method of integrating factors. Then verify directly that the solution satisfies $\tilde{y} = y$ by calculating $\tilde{y}$.

**21** The constant function $y = 0$ is a solution of the initial value problem

$$y' = \sqrt{y}, \quad y(0) = 0$$

for $x \geq 0$. However, it is not the only solution. Use separation of variables to find another solution of this initial value problem, $y_0 = \frac{1}{4}x^2$. This will show that solutions of this initial value problem are not unique. Why does this example not contradict the Existence and Uniqueness Theorem?

**22** Verify that the functions

$$y_a(x) = \begin{cases} 0 & \text{for } x \leq a \\ \frac{1}{4}(x-a)^2 & \text{for } x > a \end{cases}.$$

for $a > 0$ each satisfy the initial value problem

$$y' = \sqrt{y}, \quad y(0) = 0$$

for all $x \in \mathbb{R}$. *(Hint: When you calculate the derivative of $y$ to verify that it satisfies the differential equation, you can use derivative shortcuts to find $y'(x)$ when $x < a$ and when $x > a$, but you need to use the limit definition of derivative at $x = a$, similar to the calculation in Example 1.)*

**23** Using the same differential equation and initial condition as in Exercise 22 above, what solution does the sequence of Picard iterates converge to, starting with $y_0 = 0$?

**24** Suppose $p(x)$ and $q(x)$ are continuous functions on the interval $I$ containing $x_0$. Use the method of integrating factors to prove that any initial value problem of the form

$$\frac{dy}{dx} + p(x)y = q(x),\ y(x_0) = y_0$$

has a solution on $I$. Then explain how the calculations in that method also guarantee that this solution is unique.

25 Prove that the initial value problem

$$xy' + y = 1, \quad y(0) = 4$$

does not have a solution on any open interval containing $x_0 = 0$. Why does this example not contradict the Existence and Uniqueness Theorem? For what initial conditions $(x_0, y_0)$ do solutions exist?

# Part 2

# Second Order Equations

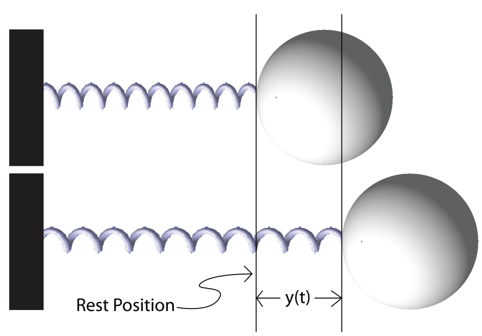
CHAPTER 7

# Constant-Coefficient Equations

**Prototype Question:** Consider a mass attached to one end of a spring whose other end is mounted in place, and imagine that this whole system is submerged in a viscous medium that resists a moving mass. The mass is $0.1$ kg, the spring constant is $2\frac{\text{N}}{\text{m}}$, and the resistance imparted by the viscous fluid is proportional to the velocity of the mass, with a constant of proportionality of $0.4\frac{\text{N}\cdot\text{s}}{\text{m}}$. If we stretch the spring, so that the end holding the mass is displaced from its rest position, and then let go, we can model the displacement $y(t)$ of the free end with the equation

$$0.1\ddot{y} + 0.4\dot{y} + 2y = 0.$$

If the initial displacement is $y(0) = 0.05$ m, and the initial velocity is $\dot{y}(0) = 0\ \frac{\text{m}}{\text{s}}$, determine how long it will take before the spring's free end first returns to its natural rest position. Also, how fast will the mass be moving at that instant?




We would next like to write down solutions for second-order **constant coefficient** linear ODE. These have the form:

$$ay'' + by' + cy = f(x).$$

Here, the coefficients $a$, $b$ and $c$ are constant, and we assume that $a \neq 0$ so that the equation will indeed be second order. We will first focus on **homogeneous equations** which are those that have $f(x) = 0$ for all $x$:

$$ay'' + by' + cy = 0.$$

(Note that this is the form of the differential equation in the prototype question; we will return to that in the problem set.) Let us seek some inspiration for solving this type of ODE by first reviewing the similar problem for first-order equations.

The general solution of the first order homogeneous constant-coefficient linear equation

$$ay' + by = 0, \quad a \neq 0.$$

is

$$y = Ce^{-bt/a},$$

which can be verified by the method of integrating factors. If $b = 0$, then the solution is just a constant function $y = C$. Notice that if $y = Ae^{rt}$ satisfies the ODE $ay' + by = 0$, then the constant $r$ satisfies the algebraic equation $ar + b = 0$. This will serve as our starting point for trying to understand second order equations.

EXERCISE 1: Prove that if $y = Ae^{rx}$ (with $A \neq 0$) satisfies the differential equation $ay'' + by' + cy = 0$, then $r$ is a solution of the algebraic equation $ar^2 + br + c = 0$.

The algebraic equation $ar^2 + br + c = 0$ is called the **characteristic equation** for the ODE $ay'' + by' + cy = 0$. The previous exercise indicates that there is a connection between the solutions of the ODE and the solutions of the corresponding characteristic equation. The following exercise completes the description of that connection.

EXERCISE 2: Prove that if $r$ is a root of $ar^2 + br + c = 0$, then for any constant coefficient $A$, the function $y = Ae^{rt}$ satisfies the differential equation $ay'' + by' + cy = 0$. *(Note that $r$ might equal zero.)*

EXERCISE 3: Prove that if $y_1$ and $y_2$ are both solutions of the differential equation $ay'' + by' + cy = 0$, then so is $(y_1 + y_2)$.

The results of the previous three exercises demonstrate that the following is true: If $r_1$ and $r_2$ are roots of the characteristic equation $ar^2 + br + c = 0$, then functions of the form $y = Ae^{r_1 t} + Be^{r_2 t}$ satisfy the ODE $ay'' + by' + c = 0$.

We can actually say even more than this: if $r_1$ and $r_2$ are distinct (meaning that $r_1 \neq r_2$), then *all* solutions of the differential equation $ay'' + by' + c = 0$ can be written in the form $y = Ae^{r_1 t} + Be^{r_2 t}$ for some appropriate choice of coefficients $A$ and $B$! One way to prove this claim is to observe that, by choosing $A$ and $B$ appropriately, we can satisfy any initial conditions for $y(t_0)$ and $y'(t_0)$, and then we appeal to a version of the existence and uniqueness theorem to show that there is only one function that satisfies the solution and these initial values, so any solution must therefore agree with one obtained this way. Another proof which doesn't require as much knowledge of ODE theory is explored in the problem set at the end of this chapter.

This is an appropriate moment to introduce some terminology. If $y_1, ..., y_k$ are functions on a domain $I$, then a **linear combination** of these functions is any function of the form $c_1 y_1 + \cdots c_k y_k$, where $c_1, \cdots, c_k$ are constants. If the only linear combination of $y_1, \cdots, y_k$ that gives us the constant function $0$ is the linear combination where $c_1 = \cdots = c_k = 0$, then we say the set $\{y_1, \cdots, y_k\}$ is **linearly independent** on $I$. And if every solution of a given differential equation on $I$ can be written as a linear combination of a linearly independent set of solutions $\{y_1, \cdots, y_k\}$, then we call that set a **fundamental set of solutions** for the differential equation on $I$.

According to this terminology, if $r_1$ and $r_2$ are distinct roots of the characteristic equations for $ay'' + by' + cy = 0$, then $\{e^{r_1 t}, e^{r_2 t}\}$ is a fundamental set of solutions on $\mathbb{R}$.

Whenever we have a fundamental set of solutions, the general solution of the differential equation can be written as linear combination of its members.

### Second Order Equations with Distinct Roots

If the characteristic equation for

$$ay'' + by' + cy = 0$$

has two distinct roots $r_1$ and $r_2$, then the formula

$$y = Ae^{r_1 t} + Be^{r_2 t}$$

provides us with the general solution on $\mathbb{R}$ of this differential equation.

We still need to investigate what to do if the characteristic equation has a repeated root (that is to say, if it is equivalent to the equation $a(r-r_1)^2 = 0$). But first let us explore a few examples involving non-repeated roots.

EXAMPLE 1: Find the solution of the initial value problem $y'' + 5y' + 6 = 0$, $y(0) = 0$, $y'(0) = 2$.

First we identify the characteristic equation for this ODE: $r^2 + 5r + 6 = 0$. Solving this algebraic equation gives us the roots $r_1 = -2$ and $r_2 = -3$. Therefore, the general solution of the ODE is

$$y = Ae^{-2x} + Be^{-3x}.$$

If we substitute in the given initial conditions, we obtain the system of equations:

$$\begin{cases} 0 = A + B \\ 2 = -2A - 3B \end{cases}$$

Solving this system of equations lead to the values $A = 2, B = -2$. Consequently, the solution of this initial value problem is

$$y = 2e^{-2t} - 2e^{-3t}.$$

□

EXERCISE 4: Solve the following initial value problems:

- $y'' - y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = 0$
- $2y'' - 5y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 2$

The process identified above even works when the solutions of the characteristic equation are complex numbers, though in that case it is often more convenient to write the solutions in a different form.

Recall that if a complex number is written in the form $\alpha + i\beta$, where $\alpha$ and $\beta$ are real, then $e^{\alpha+i\beta} = e^{\alpha}(\cos(\beta) + i\sin(\beta))$ (this is called Euler's Formula, and it can be found in Appendix 2). Also, if the characteristic equation has real coefficients but complex roots, then the roots must be complex conjugates of one another. Therefore the general solution has the form:

$$\begin{aligned}
y &= Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} \\
&= Ae^{\alpha x}(\cos(\beta x) + i\sin(\beta x)) + Be^{\alpha x}(\cos(-\beta x) + i\sin(-\beta x) \\
&= Ae^{\alpha x}(\cos(\beta x) + i\sin(\beta x)) + Be^{\alpha x}(\cos(-\beta x) - i\sin(-\beta x) \\
&= (A+B)e^{\alpha x}\cos(\beta x) + (A-B)ie^{\alpha x}\sin(\beta x)
\end{aligned}$$

If we introduce new coefficients $C$ and $D$ satisfying $C = A + B$ and $D = (A - B)i$, then we obtain the form

$$y = Ce^{\alpha x}\cos(\beta x) + De^{\alpha x}\sin(\beta x).$$

That is to say, if $\{e^{(\alpha+\beta i)t}, e^{(\alpha-\beta i)t}\}$ is a fundamental set of solutions for $ay''+by'+cy = 0$, then so is $\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$ (and vice versa).

This allows us to write the general solutions without introducing complex numbers into the solutions:

**Second Order Equations with Complex Roots**

If the characteristic equation $ar^2 + br + c = 0$ has complex roots of the form $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, then the general solution on $\mathbb{R}$ of the ODE $ay'' + by' + cy = 0$ can be written in the form

$$y = Ce^{\alpha x}\cos(\beta x) + De^{\alpha x}\sin(\beta x).$$

EXAMPLE 2: Solve the initial value problem $y'' + 4y = 0,\ y(0) = 1,\ y'(0) = 4$.

The characteristic equation is $r^2 + 4 = 0$, which has complex roots $r_1 = 0 + 2i$ and $r_2 = 0 - 2i$. Thus the general solution is

$$y(x) = Ce^{0x}\cos(2x) + De^{0x}\sin(2x) = C\cos(2x) + D\sin(2x).$$

Inserting the initial condition $y(0) = 1$ gives us the equation $1 = C$. The derivative of $y(x)$ is $y'(x) = -2C\sin(2x) + 2D\cos(2x)$, and inserting the initial condition $y'(0) = 2$ yields $4 = 2D$, so that $D = 2$. Therefore the solution of the initial value problem is

$$y(x) = \cos(2x) + 2\sin(2x).$$

□

The previous example illustrated the following useful observation: the roots of $r^2 + A^2 = 0$ are $r = \pm Ai$, and therefore the solutions of $y'' + A^2y = 0$ are of the form $y =$

$c_1 \cos(Ax) + c_2 \sin(Ax)$. That is to say, if the roots of the characteristic equation are *purely imaginary*, then the general solution does not require any exponential factors.

EXERCISE 5: Solve the following initial value problems.

- $y'' + 2y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$
- $y'' + 25y = 0, \quad y(0) = 2, \quad y'(0) = 5$
- $8\ddot{y} + 4\dot{y} + y = 0, \quad y(0) = 2, \quad \dot{y}(0) = 0$

Finally, we need to determine how to find a general solution to $ay'' + by' + cy = 0$ when the characteristic equation yields only one root, $r_1$. That is to say, sometimes the characteristic equation might be factored as $a(r - r_1)^2$, in which case we call $r_1$ a **double root** of the equation. (It is also sometimes called a "root of multiplicity two".)

In this case, we know that the expression $e^{r_1 x}$ gives one solution of the ODE which is never zero. We will use a technique called 'reduction of order' to find the general solution from this one known solution. Readers who are interested in learning more about this technique will find it in the appendix.

Suppose that $y$ is any solution of the ODE, and let $u = \frac{y}{e^{r_1 x}}$, so that $y = ue^{r_1 x}$. The product rule gives us $y'(x) = u'e^{r_1 x} + r_1 ue^{r_1 x}$ and $y''(x) = u''e^{r_1 x} + 2r_1 u'e^{r_1 x} + r_1^2 ue^{r_1 x}$. Now we can substitute $ue^{r_1 x}$ for $y(x)$ in the differential equation:

$$\begin{aligned}
0 &= ay'' + by' + cy \\
&= a(u''e^{r_1 x} + 2r_1 u'e^{r_1 x} + r_1^2 ue^{r_1 x}) \\
&\quad + b(u'e^{r_1 x} + r_1 ue^{r_1 x}) + c(ue^{r_1 x}) \\
&= au''e^{r_1 x} + (2ar_1 + b)u'e^{r_1 x} + (ar_1^2 + br_1 + c)ue^{r_1 x} \\
&= au''e^{r_1 x}.
\end{aligned}$$

In the last line we used the facts that $ar_1^2 + br_1 + c = 0$, which is true since $r_1$ is a root of the characteristic equation, and we used $2ar_1 + b = 0$, which follows because $r_1$ is a *double root* of the characteristic equation:

$$ar^2 + br + c = a(r - r_1)^2,$$

and expanding the right side yields

$$ar^2 + br + c = ar^2 - 2ar_1 r + ar_1^2;$$

so that equating coefficients gives us

$$b = -2ar_1 \quad \text{and} \quad c = ar_1^2.$$

Now we have the differential equation $au''e^{r_1 x} = 0$, or just $u'' = 0$, and therefore $u(x) = Ax + B$ for some constants $A$ and $B$. Consequently, $y = (Ax+B)e^{r_1 x}$, and this is the general solution when the characteristic equation has a double root.

**Second Order Equations with Repeated Roots**

If the characteristic equation $ar^2 + br + c = 0$ has a double root $r_1$, then the general solution on $\mathbb{R}$ of the ODE $ay'' + by' + cy = 0$ can be written in the form

$$y = Axe^{r_1 x} + Be^{r_1 x}.$$

This result can also be stated as follows: if $r$ is a double root of the characteristic equation for $ay'' + by' + cy = 0$, then $\{e^{rx}, xe^{rx}\}$ is a fundamental set of solutions for this differential equation.

EXAMPLE 3: Find the general solution of the ODE $\ddot{y} + 4\dot{y} + 4y = 0$.

The characteristic equation is $r^2 + 4r + 4 = 0$, or $(r+2)^2 = 0$, so $r = -2$ is a double root. Therefore the general solution of this ODE is

$$y(t) = Ate^{-2t} + Be^{-2t}.$$

$\square$

EXERCISE 6: Solve the following initial value problems.

- $y'' - 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 4$
- $3\ddot{y} + 18\dot{y} + 27y = 0, \quad y(0) = 2, \quad \dot{y}(0) = 3.$

EXERCISE 7: Solve the following initial value problems.

- $y'' + 9y = 0,\ y(0) = 2,\ y'(0) = -2$
- $\frac{d^2y}{dv^2} + y = 0,\ y(0) = 0,\ y'(0) = 3$
- $\ddot{w} - 3\dot{w} - 4w = 0,\ w(1) = 0,\ w'(1) = 2$
- $4y'' - 4y' + y = 0,\ y(0) = 0,\ y'(0) = 0$
- $\ddot{v} - 4\dot{v} + 4v = 0,\ v(0) = 1,\ \dot{v}(0) = 2$
- $y'' + 4y' + 5y = 0,\ y(0) = 0,\ y'(0) = 3$

### Higher Order Equations

Higher-order constant coefficient linear ordinary differential equations can be treated similarly.

EXAMPLE 4: Consider the initial-value problem $\dddot{y} - \dot{y} = 0$, $y(0) = 1$, $\dot{y}(0) = 4$, $\ddot{y}(0) = 0$. The characteristic equation is $r^3 - r = 0$, which has roots $r = 0, 1, -1$. Therefore the general solution of the ODE is

$$y(t) = A + Be^t + Ce^{-t}.$$

(The first term on the right side is the same as $Ae^{0t}$.) Its derivatives are $\dot{y}(0) = Be^t - Ce^{-t}$ and $\ddot{y}(t) = Be^t + Ce^{-t}$. Inserting the initial conditions gives us the equations

$$\begin{cases} A + B + C = 1 \\ B - C = 4 \\ B + C = 0 \end{cases}.$$

The solution of this system of equations is $A = 1$, $B = 2$, $C = -2$, so the solution of the initial value problem is

$$y(t) = 1 + 2e^t - 2e^{-t}.$$

□

For a root $r$ of a polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$, we say that $r$ has **multiplicity** $m$ if $(x - r)^m$ is a factor. For example, the polynomial $x^3 - x^2$ can be factored as $x^2(x - 1)$, from which we see that it has roots $x = 0$ and $x = 1$; the root $x = 0$ has multiplicity 2, and the root $x - 1$ has multiplicity 1.

EXERCISE 8: Find the roots and the multiplicities of the following polynomials.

(1) $x^4 - x^2$
(2) $x^3 + 3x^2 + 3x + 1$
(3) $x^5 - 3x^4 + 3x^3 - x^2$

In order to describe complicated or abstract products, it is useful to use $\Pi$ notation, which is similar to the $\Sigma$ notation used for sums:

$$\Pi_{k=1}^{K} a_k = a_1 a_2 a_3 \cdots a_K.$$

### Repeated Roots of Multiplicity Three or More

If the characteristic equation for a constant coefficient linear homogeneous ordinary differential equation is

$$a\Pi_{k=1}^{K}(r-r_k)^{m_k},$$

(that is to say, if the roots are $r_1, r_2, ..., r_K$ with corresponding multiplicities $m_1, m_2, ...m_K$), then the general solution on $\mathbb{R}$ of the ODE is

$$y = \sum_{k=1}^{K}\sum_{l=1}^{m_k} x^{l-1}e^{(r_k x)}.$$

EXAMPLE 5: If the characteristic equation for a constant coefficient homogeneous ODE is $r^2(r-3)^2(r+1)^3$, then the general solution is

$$y = A_0 + A_1 t + B_0 e^{3t} + B_1 t e^{3t} + B_2 t^2 e^{3t} + C_0 e^{-t} + C_1 t e^{-t}.$$

□

EXERCISE 9: Find a general solution for the differential equation $y''' + 3y'' + 3y' + y = 0$.

EXERCISE 10: Solve the initial value problem $y^{(4)} - 5y^{(2)} + 4y = 0,\ y(0) = 4, y'(0) = 4, y''(0) = 10, y^{(3)}(0) = 16$.

## HYPERBOLIC TRIGONOMETRIC FUNCTIONS

When the characteristic equation for a second-order ODE has roots $r = \pm a$, the general solution has the form $y = Ae^{ax} + Be^{-ax}$. It is often useful to write these solutions in a slightly different way using the following notation.

### Hyperbolic Trigonometric Functions

The **hyperbolic sine** function is

$$\sinh(x) = \frac{e^x - e^{-x}}{2},$$

and the **hyperbolic cosine** function is

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

EXERCISE 11: Prove that $\frac{d}{dx}[\sinh(x)] = \cosh(x)$ and $\frac{d}{dx}[\cosh(x)] = \sinh(x)$. Also verify that $\sinh(0) = 0$ and $\cosh(0) = 1$.

Here's how these can be used profitably to express solutions of ODE. If the characteristic equation has roots $r = \pm a$, then the general solution is

$$\begin{aligned}
y &= Ae^{ax} + Be^{-ax} \\
&= \frac{A+B}{2}e^{ax} + \frac{A-B}{2}e^{ax} + \frac{A+B}{2}e^{-ax} - \frac{A-B}{2}e^{-ax} \\
&= (A+B)\left(\frac{e^{ax}+e^{-ax}}{2}\right) + (A-B)\left(\frac{e^{ax}-e^{-ax}}{2}\right) \\
&= C\cosh(ax) + D\sinh(ax) \quad \text{(with } C = A+B \text{ and } D = A-B\text{).}
\end{aligned}$$

That is, we can write the general solution as $y = C\cosh(ax) + D\sinh(ax)$. Furthermore, according to the content of the next exercise, this form of writing the solution makes it particularly easy to write down the solution of initial value problems when the initial conditions are given at $x = 0$.

EXERCISE 12: For the function $y = C\cosh(ax) + D\sinh(ax)$, verify that $y(0) = C$ and $y'(0) = aD$.

These facts can save us the trouble of having to solve a system of linear equations to find the right coefficients from the initial conditions.

EXAMPLE 6: Solve the initial value problem $\ddot{y} - 5y = 0,\ y(0) = 3,\ \dot{y}(0) = 2$.

The characteristic equation is $r^2 - 5 = 0$, which has roots $r = \pm\sqrt{5}$. The general solution of this equation can be written in the form $y = C\cosh\left(\sqrt{5}t\right) + D\sinh\left(\sqrt{5}t\right)$. The initial conditions tell us that $C = 3$ and $D = \frac{2}{\sqrt{5}}$. Therefore

$$y = 3\cosh\left(\sqrt{5}t\right) + \frac{2}{\sqrt{5}}\sinh\left(\sqrt{5}t\right).$$

□

EXERCISE 13: Use hyperbolic trigonometric functions to solve the following initial value problems:

- $y'' - 2y = 0,\ y(0) = 2,\ y'(0) = 2$
- $y'' - 4y = 0,\ y(0) = 0,\ y'(0) = 8$
- $y'' - 3y = 0,\ y(0) = 1,\ y'(0) = 2$

## Boundary Value Problems

In each the examples done so far, we found a general solution and then used a given value of the solution and its derivative at some point to specify a unique solution. Because there are two unknown parameters in the general solutions, we needed two such pieces of information to specify their values. However, there are other ways to specify the values of the parameters by giving other information. For example, we could specify the value of the solution at two different points (instead of the solution and its derivative at a single point), as the following example illustrates. Such descriptions are called **boundary value problems**.

EXAMPLE 7: Solve the following boundary-value problem $y'' - y = 0,\ y(0) = 1,\ y(1) = e.$

The characteristic equation is $r^2 - 1 = 0$, and this has roots $r = \pm 1$. So the general solutions is

$$y = Ae^x + Be^{-x}.$$

The condition $y(0) = 1$ implies $1 = A + B$, and the condition $y(1) = e$ implies $e = Ae - Be$. The solution of this pair of equations for $A$ and $B$ is $A = 1,\ B = 0$. Thus the solution of this boundary value problem is $y = e^x$.

EXERCISE 14: Solve the following boundary value problems:

- $\ddot{y} + 4\dot{y} + 4y = 0,\ y(0) = 3,\ y(1) = 5$
- $4\ddot{y} + y = 0,\ y(0) = 1,\ y(\pi) = 0$

In contrast to initial value problems, boundary value problems don't always have solutions, and when they do, they may not be unique. This is explored in exercises 7.55-7.56.

## Additional Exercises

Find the root(s) of the characteristic equation for each of the following differential equations. Simplify your answers, and identify any repeated roots as such. (You may need to rewrite the differential equation in standard form before you begin.)

**15** $y'' - 6y' + 8y = 0$

**16** $2y'' + 6y' - 8u = 0$

**17** $3y'' + 8y' + 5y = 0$

**18** $4\theta'' + 2\theta = 0$

**19** $\ddot{x} - \dot{x} + x = 0$

**20** $\ddot{y} = 4y$

**21** $4\ddot{v} = \dot{v}$

**22** $\frac{d^2y}{dx^2} + 25y = 10\frac{dy}{dx}$

Find a general solution for each of the following differential equations.

**23** $y'' + 3y' + 2y = 0$

**24** $y'' - 6y' + 9y = 0$

**25** $w'' - 2w' + 5y = 0$

**26** $4\ddot{u} + 4\dot{u} = 3$

**27** $\ddot{\theta} + 16\theta = 0$

**28** $2x'' + 10x = 6x'$

**29** $2\ddot{v} + 4\dot{v} + 2v = 0$

**30** $\frac{d^2x}{dt^2} = 9x$

Find the solution of each of the following initial value problems.

**31** $y'' + 12y' + 36y = 0,\ y(0) = 0,\ y'(0) = 2$

**32** $\ddot{x} - 3\dot{x} - 4x = 0,\ x(0) = 1,\ x'(0) = 2$

**33** $w'' = w,\ w(0) = -2,\ w'(0) = 1$

**34** $\ddot{y} - 4\dot{y} = -4y,\ y(0) = 1,\ y'(0) = -1$

**35** $\frac{d^2v}{dz^2} + 8v = 4\frac{dv}{dz},\ v(0) = 1,\ \frac{dv}{dz}(0) = 0$

**36** $\omega'' + 4\omega' + 8\omega = 0,\ \omega(0) = 2,\ \omega'(0) = 0$

**37** $20\ddot{z} + 5z = 0,\ z(0) = 1,\ \dot{z}(0) = -1$

**38** $y'' = -3y',\ y(0) = 1,\ y'(0) = 1$

Find a differential equation whose general solution is the two-parameter family of functions given.

**39** $y = c_1e^{2x} + c_2e^{-5x}$

**40** $u = c_1e^{-x} + c_2xe^{-x}$

**41** $\theta = A\cos(2t) + B\sin(2t)$

**42** $y = c_1e^{3t}\cos(t) + c_2e^{3t}\sin(t)$

**43** $w = A\sinh(2t) + B\cosh(2t)$

**44** $x = At + B$

Solve the following boundary value problems.

**45** $y'' - y = 0,\ y(0) = 1,\ y(1) = 2$

**46** $y'' + y' - 2y = 0,\ y(0) = 0,\ y(1) = 1$

**47** $y'' + 4y = 0,\ y(0) = 1,\ y(\pi/2) = 2$

**48** $\ddot{y} - 4\dot{y} + 4y = 0,\ y(0) = 1,\ y(1) = 3e$

---

**49** Suppose $y(t)$ is the solution of the initial value problem $\ddot{y} + 4\dot{y} + 4y = 0$, $y(0) = 2$,

$\dot{y}(0) = 1$. Find the absolute maximum value of $y$ on the interval $[0, \infty)$.

**50** Solve the Prototype Question from the beginning of this chapter.

**51** Find a value of $\alpha$ so that the solution of the initial value problem $y'' + y' - 2y = 0$, $y(0) = \alpha$, $y'(0) = 2$ satisfies $\lim_{t\to\infty} y(t) = 0$.

**52** Let $y(t)$ be the solution of the initial value problem $\ddot{y} + 2\dot{y} + \gamma y = 0$, $y(0) = 1$, $\dot{y}(0) = 0$, where $\gamma$ is a real constant. Find $\lim_{t\to\infty} y(t)$. Does the answer depend on the value of $\gamma$? *(Hint: You will need to separate the solution of this initial value problem into several cases, depending on the value of $\gamma$, and then find the limit for each one.)*

**53** *In this problem, you will verify that our formula for the case when the characteristic equation has two distinct coefficients is in fact the general solution – that is to say , that any solution of the ODE can be written in this form.*

Suppose that $ay'' + by' + cy = 0$ has a characteristic equation $ar^2 + br + c$ with two distinct roots, $r_1$ and $r_2$ (which implies $a \neq 0$). **(a)** Verify directly that $y_1 = e^{r_1 x}$ is a solution of the ODE. **(b)** Let $y$ be an arbitrary solution of the ODE, and write $y(x) = u(x)e^{r_1 x}$. Use reduction-of-order to prove that $u'' + \left(2r_1 + \frac{b}{a}\right) u' = 0$. *(Review Appendix C if needed.)* **(c)** Use the substitution $v = u'$ and the method of integrating factors to deduce that the general solution for $u$ is $u(x) = Ce^{-(2r_1+b/a)x} + D$. **(d)** Conclude that $y = Ce^{-(r_1+b/a)x} + De^{r_1 x}$. **(e)** Because $r_1$ and $r_2$ are both solutions of the characteristic equation, it must be true that $ar^2 + br + c = a(r - r_1)(r - r_2)$. Equate coefficients here to prove that $r_2 = -(r_1 + b/a)$. **(f)** Conclude that $y(x) = Ce^{r_2 x} + De^{r_1 x}$.

**54** The motion of an ideal pendulum is governed by the differential equation $\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0$, where $\theta$ is the angle that the pendulum arm makes with the vertical, $L$ is the length of the (massless) pendulum arm and $g$ is the acceleration due to gravity acting on a mass at the end of the rod. If the angle $\theta$ is measured in radians and is sufficiently small, then $\sin(\theta) \approx \theta$, so the motion of the pendulum can be approximately modeled by the differential equation $\ddot{\theta} + \frac{g}{L}\theta = 0$. Use this equation to find the (approximate) period of a pendulum with arm length $L$ *meters* near the Earth's surface with a small initial displacement $\theta_0 > 0$ an initial velocity of $\dot{\theta} = 0$. (Notice that the period does not depend on the initial displacement! It will also not depend on a sufficiently small initial velocity.)

**55** Not every boundary-value problem has a solution. Verify that there is no solution to

$$\begin{cases} \ddot{y} + y = 0 \\ y(0) = 4 \\ y(\pi) = 0 \end{cases} .$$

**56** Some boundary-value problems have solutions, but the solutions are not unique. Verify that there are infinitely many solutions to

$$\begin{cases} \ddot{y} + y = 0 \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}.$$

**57** Find all solutions of the boundary value problem

$$\begin{cases} \ddot{y} + 4y = 0 \\ \dot{y}(0) = 0 \\ \dot{y}(\pi) = 0 \end{cases}.$$

**58** Find all real values of $\lambda$ such that the boundary value problem

$$\begin{cases} \ddot{y} + \lambda y = 0 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$$

has infinitely many solutions. What are the solutions? *(This is an example of an* ***eigenvalue problem****.)*

**59** Use power series for $e^x$, $\sin(x)$ and $\cos(x)$ to prove that $\cos(ix) = \cosh(x)$ and $\sin(ix) = i\sinh(x)$, where $i$ is a complex number satisfying $i^2 = -1$.

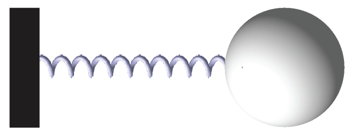
**60** Prove that $\{e^{rx}, e^{-rx}\}$ is a fundamental set of solutions if and only if $\{\cosh(rx), \sinh(rx)\}$ is a fundamental set of solutions for the same differential equation.

FOCUS ON MODELING

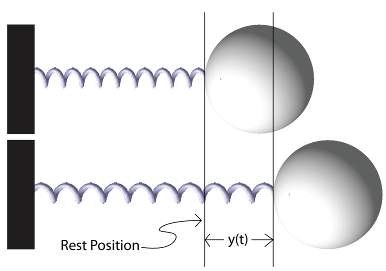
# Spring-Mass Systems

Second-order ODE arise when we model the behavior of a mass attached to a freely-moving end of an ideal spring, possibly subject to a damping effect (imagine the spring and mass are submerged in molasses). Understanding this model is a first step toward being able to analyze more complicated systems of physical oscillators.

Let us begin with a figure illustrating our physical system:



The free end of the spring is allowed to move, and we need to impose coordinates on the figure to measure this motion. There are many ways we could choose to do this. The natural point to choose as an origin is the **rest position** of the free end of the spring – that is to say, the point where the free end sits when the spring is not in a state of internal tension. From this point, we can measure the displacement of the free end of the spring, and we shall adopt the convention that a stretched spring corresponds to a positive displacement, while a compressed spring corresponds to a negative displacement.




To model the physical behavior of this system, our starting point is Newton's second law, $F = ma$ (force equals mass times acceleration). If we let $y(t)$ denote the displacement of the free end of the spring from its rest position as a function of time, then the acceleration is given by $\ddot{y}$. There will also be at least two forces acting on the mass. One is the spring's restoring force, which Hooke's Law tells us we can model by assuming it is proportional to the displacement from rest position: $F_s = -ky$. (Here, the spring constant $k$ is positive, and the direction of the spring's restoring force is in the direction opposite the displacement.) We will model the damping force by assuming it is proportional to the velocity of the mass (like viscous drag) and in the opposite direction: $F_d = -C\dot{y}$. Let us denote any other external driving force by $F_e$, and suppose this driving force is described by a (possibly constant) function of time, $F_e = f(t)$.

With these conventions we have:

$$ma = F_s + F_d + F_e$$

or

$$m\ddot{y} = -ky - C\dot{y} + f(t),$$

which we rearrange as

$$m\ddot{y} + C\dot{y} + ky = f(t).$$

We now see that this is a second order constant coefficient linear ODE, so we can study the behavior of this system using the mathematical techniques now available to us.

A standard choice of units for force would be Newtons, and a standard choice for measuring displacement $y$ would be meters. Thus the spring constant could have units of $\frac{N}{m}$, indicating that the magnitude of the spring's restoring force is $k$ Newtons for each meter the spring is displaced from rest position. If these units are used, then the last term on the left side of our ODE will have units of Newtons, which is consistent with the kind of units we would see on the right side of the equation for an external driving force $F_e$. To maintain consistency with the other terms on the left side of the equation, we should select mass $m$ to be measured in kilograms, and time should be measured in seconds; that way the units of $m\ddot{y}$ will be $\frac{kg \cdot m}{s^2}$, which are the same as Newtons. Similarly, the units of the damping coefficient will have to be $\frac{N \cdot s}{m}$.

**EXAMPLE:** Consider a mass of $3\ kg$ attached to the end of a spring with spring constant $9\frac{N}{m}$. If there is no damping or outside driving force, and the mass is initially stretched $0.05\ m$ from its rest position then released, determine how long it will take before the spring first returns to its rest position. What will the velocity be at that instant?

With the parameters $m = 3$, $C = 0$ and $k = 9$, and the driving force $f(t) = 0$, we are faced with the differential equation

$$3\ddot{y} + 9y = 0$$

and the initial conditions $y(0) = 0.05$ and $\dot{y}(0) = 0$. The solution of this IVP is

$$y(t) = 0.05\cos(3t).$$

The free end of the spring will be at the rest position when $y(t) = 0$, which will occurs when $3t = \frac{\pi}{2} + n\pi$, or $t = \frac{(2n+1)\pi}{6}$. The smallest positive solution will be $t = \frac{\pi}{6} \approx 0.524\ s$. At that instant, the velocity will be $\dot{y}\left(\frac{\pi}{6}\right) = -0.15\sin\left(\frac{\pi}{2}\right) = -0.15\frac{m}{s}$. $\square$

CHAPTER 8

# Non-homogeneous Equations

**Prototype Question:** A simple electrical circuit component contains a 2 ohm resistor, a 3 henry inductor and a 4 farad capacitor connected in series. If there is an oscillating voltage source connected that supplies $12\sin(4t)$ volts at time $t$, then the charge on the capacitor $q(t)$ can be modeled by the differential equation

$$3\ddot{q} + 2\dot{q} + \frac{1}{4}q = 12\sin(4t).$$

Here, $q$ is measured in amperes, and time is measured in seconds. The current initially satisfies $q(0) = 0$ and $\dot{q}(0) = 0$. Graph the current $q(t)$ on the time interval $0 \leq t \leq \pi$ seconds.

Now that we can solve ODE of the form

$$a\ddot{y} + b\dot{y} + cy = 0,$$

we would like to be able to solve the **non-homogeneous** equations:

$$a\ddot{y} + b\dot{y} + cy = f(t).$$

It is possible to write down general representation formulas for any continuous **driving function** $f(t)$, but we will mostly be interested in the special cases when $f(t)$ is a polynomial, exponential or trigonometric function.

We will develop the idea of our technique in the following example. Later examples will illustrate the streamlined version of this process.

EXAMPLE 1: Consider the differential equation

$$y'' + 2y' + y = x^2.$$

We would like to find a general solution of this differential equation. We will start by trying to find *one* solution.

What kinds of functions might satisfy the equation? The driving function is a power function, and in order that a function, its derivative and second derivative might simplify on the left side of the ODE to just $x^2$, it would be a reasonable guess that some polynomial function might work as $y(x)$. We will therefore try to find a function of the form $y_p = Ax^2 + Bx + C$ that satisfies the ODE. (We call the function $y_p$ because it is a *particular* solution of the nonhomogeneous diferential equation, not a general solution.) Notice that we don't want to try a polynomial of degree 3 or higher because there would be no way for the higher degree terms to cancel out and leave just $x^2$. Substitute this into the ODE to obtain

$$\begin{aligned}
x^2 &= y_p'' + 2y_p' + y_p \\
&= (2A) + 2(2Ax + B) + (Ax^2 + Bx + C) \\
&= Ax^2 + (4A + B)x + (2A + 2B + C)
\end{aligned}$$

Equating the polynomial coefficients on both sides of the equation gives

$$A = 1, \quad 4A + B = 0, \quad 2A + 2B + C = 0.$$

Consequently $A = 1$, $B = -4$ and $C = 6$. This gives us the function

$$y_p(x) = x^2 - 4x + 6$$

as one solution of the ODE.

Next, suppose that $y(x)$ is any solution of the equation, and define $y_h = y - y_p$. Inserting this into the differential equation, we see that

$$\begin{aligned}
x^2 &= y'' + 2y' + y \\
&= (y_h + y_p)'' + 2(y_h + y_p)' + (y_h + y_p) \\
&= y_h'' + 2y_h' + y_h + y_p'' + 2y_p' + y_p \\
&= y_h'' + 2y_h' + y_h + x^2
\end{aligned}$$

Subtracting $x^2$ from both sides, we see that

$$0 = y_h'' + 2y_h' + y_h,$$

and we know how to find the general solution of this equation:

$$y_h(x) = Ae^{-x} + Bxe^{-x}.$$

(There was a bit of foresight here in calling the difference $y - y_p$ by the name $y_h$, as the above calculation shows that $y_h$ is a solution of the corresponding *homogeneous* differential equation that has the same coefficients as our nonhomogeneous equation does.) Consequently,

$$y(x) = y_p(x) + y_h(x) = x^2 - 4x + 6 + Ae^{-x} + Bxe^{-x}.$$

All solutions of the ODE can be written in this form, so this is the general solution of the differential equation. □

In the previous example, we took advantage of the following important idea:

**Second Order Non-homogeneous Equations**

If $y_p$ satisfies the non-homogeneous ordinary differential equation

$$ay'' + by' + cy = f(x)$$

and if $y_h$ is the general solution of the corresponding homogeneous equation

$$ay'' + by' + cy = 0,$$

then $y = y_p + y_h$ is the general solution of the non-homogeneous ODE.

Based on this fact, we can try to find general solutions of non-homogeneous equations by finding just one solution (which we call a **particular solution)** and then adding to it the general solution of the related homogeneous equation.

Our method for finding a particular solution was to guess a form of a particular solution (such as the polynomial $Ax^2 + Bx + C$ we tried in the first example), and then by substituting it into the ODE we find the appropriate values for the unknown coefficients. This approach is called the **method of undetermined coefficients**.

It is usually a good idea to solve the related homogeneous equation first, because the form of that general solution might affect our guess for a particular solution of the non-homogeneous ODE, as we'll see in Example 4.

EXAMPLE 2: Solve the initial-value problem $y'' + y' - 6y = 3x + 4$, $y(0) = 1$, $y'(0) = 0$.

The related homogeneous equation $y'' + y' - 6y = 0$ has characteristic equation $r^2 + r - 6 = 0$, and the roots of this are $r = -3, 2$. Thus the homogeneous equation has the general solution $y_h = Ae^{-3x} + Be^{2x}$. We guess that there might be a particular solution of the non-homogeneous equation of the form $y_p = Cx + D$. Inserting this into the non-homogeneous equation yields

$$3x + 4 = (0) + (C) - 6(Cx + D) = -6Cx + (C - 6D).$$

Equating coefficients tells us $C = -\frac{1}{2}$ and then $D = -\frac{3}{4}$. This gives us $y_p = -\frac{1}{2}x - \frac{3}{4}$, and adding this to the general solution of the related homogeneous equation yields the general solution of the non-homogeneous equation:

$$y = -\frac{1}{2}x - \frac{3}{4} + Ae^{-3x} + Be^{2x}.$$

The initial conditions allow us to solve for $A$ and $B$:

$$y(0) = 1 \implies -\frac{3}{4} + A + B = 1$$
$$y'(0) = 0 \implies -\frac{1}{2} - 3A + 2B = 0$$

The solution of this system of algebraic equations is $A = \frac{3}{5}$ and $B = \frac{23}{20}$. This gives us the solution of the IVP:

$$y = -\frac{1}{2}x - \frac{3}{4} + \frac{3}{5}e^{-3x} + \frac{23}{20}e^{2x}.$$

□

**WARNING:** Don't try to find coefficients for the homogeneous equation that satisfy the initial conditions – wait until you add in the particular solution for the non-homogeneous equation. Doing otherwise will usually produce the wrong answer because it will not take into account the initial values of the particular solution.

EXERCISE 1: Solve the initial value problem $y'' - 5y' + 6y = x$, $y(0) = 0$, $y'(0) = 0$.

The previous exercise asks for a solution satisfying the initial conditions $y(0) = 0$ and $y'(0) = 0$. We often refer to such initial values as **rest initial conditions**, particularly when the differential equation is describing physical behavior, such as that of a spring-and-mass system.

EXAMPLE 3: Find the general solution of $y'' + 2y' + y = e^{2x}$.

The characteristic equation is $r^2 + 2r + 1 = 0$, which has a repeated root $r = -1$. Thus the general solution of the related homogeneous equation is $y_h = Ae^{-x} + Bxe^{-x}$. Next we guess that a particular solution of the non-homogeneous equation will have the form $y_p = Ce^{2x}$:

$$e^{2x} = (4Ce^{2x}) + 2(2Ce^{2x}) + (Ce^{2x}) = 9Ce^{2x},$$

so that $C = \frac{1}{9}$. Therefore the general solution of the non-homogeneous equation is

$$y = \frac{1}{9}e^{2x} + Ae^{-x} + Bxe^{-x}.$$

$\square$

EXERCISE 2: Solve the initial value problem $\ddot{y} - y = e^{2t},\ y(0) = 1, \dot{y}(0) = 1$. *(Hint: Guess that this ODE has a particular solution of the form $y_p = Ce^{2t}$. Convince yourself that this is a reasonable thing to guess.)*

EXERCISE 3: Try to find a particular solution to $y'' + 6y' = x$ of the form $y_p = Cx + D$. End up proving that no such solution exists.

The last exercise shows us how we might need to be more clever when guessing the form of our particular solution. If *any term* in the driving function is a solution of the related homogeneous equation, we will need to modify the form of our guess. For the differential equation in the last exercise, the correct form of the guess is actually a degree-two polynomial.

EXAMPLE 4: Consider the ODE $y'' + 6y' = x$. Let us seek a solution of the form $y_p = Cx^2 + Dx$. Inserting this into the differential equation produces

$$x = (2C) + 6(2Cx + D) = 12Cx + (2C + 6D).$$

Equating coefficients gives us $C = \frac{1}{12}$ and then $D = -\frac{1}{36}$. Now we see that the function $y_p = \frac{1}{12}x^2 - \frac{1}{36}x$ is a solution. $\square$

The general principle we follow is this: if the driving term of the non-homogeneous equation is a polynomial (or a monomial) of degree $N$, then our guess for the form of a particular solution is

$$y_p = x^S q(x),$$

where $q(x)$ is a polynomial of degree $N$, and where $S \geq 0$ is the smallest non-negative integer such that no term in the polynomial $x^S q(x)$ is a solution of the related homogeneous equation.

This is why it is a good practice to find the general solution of the related homogeneous equation first, so that we can compare our guess for a particular solution of the non-homogeneous equation with solutions of the related homogeneous equation.

EXERCISE 4: Find the general solution of $y'' + 2y' = x^2$.

EXAMPLE 5: Find the general solution of $y'' + 2y' + y = e^{-x}$.

The general solution of the related homogeneous equation is $y_h = Ae^{-x} + Bxe^{-x}$. Therefore no multiple of $e^{-x}$ can be a solution of the non-homogeneous equations. Neither can any multiple of $xe^{-x}$. However, we can find a solution by looking for a multiple of $x^2e^{-x}$. Let $y_p = Cx^2e^{-x}$. Then $y_p' = C(2x - x^2)e^{-x}$ and $y_p'' = C(2 - 4x + x^2)e^{-x}$. Insert these into the ODE:

$$\begin{aligned} e^{-x} &= \left(C(2-4x+x^2)e^{-x}\right) + 2\left(C(2x-x^2)e^{-x}\right) + \left(Cx^2e^{-x}\right) \\ &= 2Ce^{-x} \end{aligned}$$

Thus $C = \frac{1}{2}$. So $y_p = \frac{1}{2}x^2e^{-x}$ is a particular solution, and therefore the general solution is

$$y = \frac{1}{2}x^2e^{-x} + Ae^{-x} + Bxe^{-x}.$$

□

As in Example 3, when we recognized that the natural guess would be a solution of the homogeneous equation, we modified it by multiplying by the smallest (integer) power of $x$ such that the product would not be a homogeneous solution. This same approach can applied when the driving terms is a sine or cosine function. In general, if the driving term is $\sin(mx)$ or $\cos(mx)$, our guess will be a function of the form $y_p = A\sin(mx) + B\cos(mx)$, unless we need to multiply by a power of $x$ to ensure that no term in our guess is a homogeneous solution.

EXAMPLE 6: Find a general solution of $y'' - y = \sin(2x)$.

The characteristic equation is $r^2 - 1 = 0$, which has solutions $r = \pm 1$; thus the solution of the homogeneous equation is $y_h = Ae^x + Be^{-x}$. Next we guess that a solution of the non-homogeneous equation might have the form $y_p = C\sin(2x) + D\cos(2x)$. Inserting this into the ODE yields

$$\begin{aligned}\sin(2x) &= (-4C\sin(2x) - 4D\cos(2x)) - (C\sin(2x) + D\cos(2x))\\ &= -5C\sin(2x) - 5D\cos(2x).\end{aligned}$$

Equating coefficients gives us $C = -\frac{1}{5}$ and $D = 0$, so $y_p = -\frac{1}{5}\sin(2x)$. The general solution is thus

$$y = -\frac{1}{5}\sin(2x) + Ae^x + Be^{-x}.$$

□

**EXAMPLE 7:** Find a general solution of $y'' + y = \sin(2x)$.

The characteristic equation is $r^2 + 1 = 0$, which has solutions $r = \pm i$. We thus write the general solution of the homogeneous equation as $y_h = A\sin(x) + B\cos(x)$. Suppose a particular solution is $y_p = C\sin(2x) + D\cos(2x)$. Then

$$\begin{aligned}\sin(2x) &= (-4C\sin(2x) - 4D\cos(2x)) + (C\sin(2x) + D\cos(2x))\\ &= -3C\sin(2x) - 3D\cos(2x).\end{aligned}$$

So $C = -\frac{1}{3}$ and $D = 0$. Thus $y_p = -\frac{1}{3}\sin(2x)$ and

$$y = -\frac{1}{3}\sin(2x) + A\sin(x) + B\cos(x).$$

□

**EXAMPLE 8:** Find a general solution of $y'' + y = \sin(x)$.

As in the previous example, $y_h = A\sin(x) + B\cos(x)$. But because the driving function $\sin(x)$ is a solution of the homogeneous equation, we use the guess $y_p = Cx\sin(x) + Dx\cos(x)$:

$$\begin{aligned}\sin(x) &= (2C\cos(x) - Cx\sin(x) - 2D\sin(x) - D\sin(x))\\ &\quad + (Cx\sin(x) + Dx\cos(x))\\ &= 2C\cos(x) - 2D\sin(x).\end{aligned}$$

Therefore $C = 0$ and $D = -\frac{1}{2}$, $y_p = -\frac{1}{2}x\cos(x)$ and

$$y = -\frac{1}{2}x\cos(x) + A\sin(x) + B\cos(x).$$

□

The following table summarizes some of the most common forms of guesses for particular solutions when we employ this technique.

### Standard Guesses for the Method of Undetermined Coefficients

For $ay'' + by' + cy = f(x)$, we guess that a particular solution has the form $y_p(x)$ as follows:

| $f(x)$ | $y_p(x)$ |
|---|---|
| $A_n x^n + \cdots + A_1 x + A_0$ | $B_n x^n + \cdots + B_1 x + B_0$ |
| $e^{Ax}$ | $Be^{Ax}$ |
| $x^n e^{Ax}$ | $(B_n x^n + \cdots + B_1 x + B_0)\, e^{Ax}$ |
| $A_1 \cos(\alpha x + \beta) + A_2 \sin(\alpha x + \beta)$ | $B_1 \cos(\alpha x + \beta) + B_2 \sin(\alpha x + \beta)$ |
| $A_1 x^n \cos(\alpha x + \beta)$ $+A_2 x^n \sin(\alpha x + \beta)$ | $(B_n x^n + \cdots + B_1 x + B_0)\,(\cos(\alpha x + \beta))$ $+\,(C_n x^n + \cdots + C_1 x + C_0)\,(\sin(\alpha x + \beta))$ |
| $A_1 e^{Ax} \cos(\alpha x + \beta) + A_2 e^{Ax} \sin(\alpha x + \beta)$ | $B_1 e^{Ax}\,(\cos(\alpha x + \beta)) + B_2 e^{Ax}\,(\sin(\alpha x + \beta))$ |
| $A_1 x^n e^{Ax} \cos(\alpha x + \beta)$ $+A_2 x^n e^{Ax} \sin(\alpha x + \beta)$ | $e^{Ax}\,(B_n x^n + \cdots + B_1 x + B_0)\,(\cos(\alpha x + \beta))$ $+e^{Ax}\,(C_n x^n + \cdots + C_1 x + C_0)\,(\sin(\alpha x + \beta))$ |

Whenever necessary, multiply the recommended guess for $y_p(x)$ by $x^s$, where $s$ is the smallest positive integer such that the guess does not contain terms that satisfy the related homogeneous equation $ay'' + by' + cy = 0$.

EXERCISE 5: Use the method of undetermined coefficients to find one solution for each of the following differential equations.

(a) $y'' - 3y' + 4y = x^2 + 1$
(b) $\ddot{y} + 2\dot{y} + y = \sin(2t)$
(c) $y'' + 9y = \cos(x)$
(d) $y'' + 9y = \cos(3x)$
(e) $v'' + v' = \sin(x)$
(f) $v'' + v' = x^2$
(g) $\ddot{w} - \dot{w} - 3w = e^t$
(h) $y'' - 5y' + 6y = e^{2x}$
(i) $\ddot{y} - 4\dot{y} + 4y = 2e^{2t}$
(j) $\ddot{y} - 4\dot{y} + 4y = 2xe^{2t}$
(k) $\ddot{y} - 2\dot{y} + y = t^2 + e^t$
(l) $\ddot{x} + 9x = \sin(t) + \sin(2t) + \sin(3t)$

**Exercise 6:** Solve the following initial value problems.

(a) $y'' - y = \sin(x),\ y(0) = 1,\ y'(0) = 0$

(b) $y'' - y = e^x,\ y(0) = 1,\ y'(0) = 0$

(c) $\ddot{u} + 3\dot{u} + 2u = 2e^t,\ u(0) = 0,\ \dot{u}(0) = 0$

(d) $\ddot{u} - 3\dot{u} + 2u = 2e^t,\ u(0) = 0,\ \dot{u}(0) = 0$

(e) $v'' + 4v = \cos(x),\ v(0) = 0,\ v'(0) = 0$

(f) $v'' + 4v = \cos(2x),\ v(0) = 0,\ v'(0) = 0$

(g) $\ddot{x} + x = e^t + \cos(t),\ x(0) = 0,\ \dot{x}(0) = 0$

## Additional Exercises

Use the method of undetermined coefficients to find a particular solution of the differential equation.

**7** $y'' - 5y' + 6y = x^2 + 2x + 3$

**8** $2y'' + 6y' - 8y = 2\sin(x)$

**9** $3y'' + 8y' + 5y = e^{-x}$

**10** $\ddot{\theta} + 4\theta = \cos(2t)$

**11** $\ddot{x} - \dot{x} + x = 2e^{2t}$

**12** $\ddot{y} = 9y + e^{3t}$

**13** $\ddot{v} = \dot{v} + 2t$

**14** $\frac{d^2y}{dx^2} + 25y = 10\frac{dy}{dx} + e^{4t} - e^{5t}$

Find a general solution for each of the following differential equations.

**15** $y'' - 3y' + 2y = \sin(2x)$

**16** $y'' - 6y' + 9y = e^x + e^{3x}$

**17** $w'' - 2w' + 5y = t^2 - 4$

**18** $4\ddot{u} + 4\dot{u} = 3 + t + e^{-t}$

**19** $\ddot{\theta} + 4\theta = \sin(t) + \cos(2t)$

**20** $2x'' + 10x = 6x' - 2t - 1$

**21** $2\ddot{v} + 4\dot{v} + 2v = 4e^{-t}$

**22** $\frac{d^2x}{dt^2} = 4x + \cos(2t)$

Find the solution of each of the following initial value problems.

**23** $y'' + 12y' + 36y = t,\ y(0) = 0,\ y'(0) = 0$

**24** $\ddot{x} - 3\dot{x} - 4x = e^{-t},\ x(0) = 1,\ x'(0) = 0$

**25** $w'' = w + 2\sin(x),\ w(0) = 0,\ w'(0) = 1$

**26** $\ddot{y} - 6\dot{y} = e^{3t} - 9y,\ y(0) = 1,\ y'(0) = 0$

**27** $\frac{d^2v}{dz^2} + 8v = 4\frac{dv}{dz} + e^{2t},\ v(0) = 0,\ v'(0) = 0$

**28** $\omega'' + 2\omega' + 8\omega = 2\cos(x),\ \omega(0) = 1,\ \omega'(0) = 0$

**29** $4\ddot{z} + z = e^t + 4,\ z(0) = 1,\ \dot{z}(0) = -1$

**30** $y'' = x + e^{3x} + e^{-3x} - 3y',\ y(0) = 0,\ y'(0) = 0$

---

**31** Solve the initial value problem described in the prototype question at the beginning of this chapter. Then graph the function on the time interval $0 \le t \le \pi$.

**32** Consider the differential equation

$$2\ddot{q} + \dot{q} + \frac{1}{8}q = f(t),$$

where $f$ is the function

$$f(t) = \begin{cases} 0 & \text{if } t < 1 \\ 12 & \text{if } t > 1 \end{cases}.$$

This models a circuit with a 2 ohm resistor, a 1 henry inductor and an 8 farad capacitor connected to a voltage source which is only 'switched on' starting at time $t = 1$. (The circuit remains closed the whole time, but the voltage source is not constant.) Find a formula for a continuous function $q$ defined on $(-\infty, \infty)$ that satisfies this equation subject to the initial conditions $q(0) = 1$ and $\dot{q}(0) = 0$. (Note that we don't care what the derivatives of $q$ do when $t = 1$, since $f$ isn't defined at that instant.) Your answer will be a piecewise defined function. You'll need to solve this problem in two stages. First, solve the initial value problem with the equation $2\ddot{q} + \dot{q} + \frac{1}{8}q = 0$ to get a solution on the interval $t \leq 1$. Then, use the values of $q(1)$ and $\dot{q}(1)$ determined by this function as initial conditions on the interval $t \geq 1$, where the differential equation is $2\ddot{q} + \dot{q} + \frac{1}{8}q = 12$. Summarize the results in a single, piecewise formula.

33 Solve the following non-homogeneous boundary-value problem by first finding a general solution of the non-homogeneous differential equation:

$$\begin{cases} \ddot{x} + x = t^2 \\ x(0) = 1 \\ x(\pi) = 0 \end{cases} .$$

34 The function $y(t) = t + e^{-2t} + 2te^{-2t}$ is a solution of an initial-value problem for a second-order, non-homogeneous, constant-coefficient linear differential equation, with all non-zero coefficients. Find it.

35 Suppose that $y_p$ is a solution of $a(x)y'' + b(x)y' + c(x)y = f(x)$ on an interval $(x_1, x_2) \subset \mathbb{R}$. Show that any other solution $y$ of this same differential equation on this interval can be written as $y = y_p + y_h$, where $y_h$ is some solution of the corresponding homogeneous differential equation $a(x)y'' + b(x)y' + c(x)y = 0$, provided that solutions exist for all initial conditions. *(This extends the theory developed in this chapter to second-order ODE with non-constant coefficient functions $a(x)$, $b(x)$ and $c(x)$.)*

CHAPTER 9

# Vibrations

**Prototype Question:** How do the mass, restoring force and damping coefficient determine the long-term behavior of a spring-mass system? How do the ratings of the resistor, inductor and capacitor in a RLC electrical circuit determine its long-term behavior?

In this chapter we study in depth a classic application of second-order constant coefficient systems: simple harmonic oscillators. This class of mathematical objects includes both the spring-mass systems and the RLC electrical circuits which have already been introduced.

We will usually not show all the steps involved in solving each initial value problem in this chapter. The reader is strongly encouraged to keep a pencil and paper handy in order to fill in all the missing steps. For particularly ugly calculations, the reader may do well to use a computer algebra system.

The model we use for a spring-mass system is the ODE

$$m\ddot{y} + \gamma\dot{y} + ky = f(t),$$

where $y$ is the position (or displacement) of the mass at the end of the spring from its natural position. Compare this with the model for the current $i$ in a circuit with a resistor, inductor and capacitor in series, together with a voltage source $v(t)$:

$$R\ddot{i} + L\dot{i} + \frac{1}{C}i = v(t).$$

The obvious similarity between these mathematical models becomes even more pronounced when we observe that, for both systems, all of the coefficients must be positive numbers.

Our goal in this chapter is to explore the various possible behaviors of solutions to these equations. Because most readers will likely find the physical model of the spring-mass system offers more intuition than the electrical circuits, we will emphasize that point of view in our discussion, but it is worth remembering that our analysis can be applied to *any* constant-coefficient second order ODE with positive coefficients.

## Damped Vibrations

Before reading further in this chapter, it is probably a good idea to review the section "Focus on Modeling: Spring-Mass Systems" that follows Chapter 7.

Exercise 1: A mass $m\ kg$ is attached to the free end of a spring with spring constant $k\frac{N}{m}$, and the system is subject to a damping coefficient $\gamma\frac{Ns}{m}$. The spring is stretched $0.5$ meters from its natural length and released. Using the model $m\ddot{y}+\gamma\dot{y}+ky=0$, determine how long it will take for the spring to first return to its natural length for each of the following sets of conditions. (There are no outside forces, such as gravity, acting on the mass.)

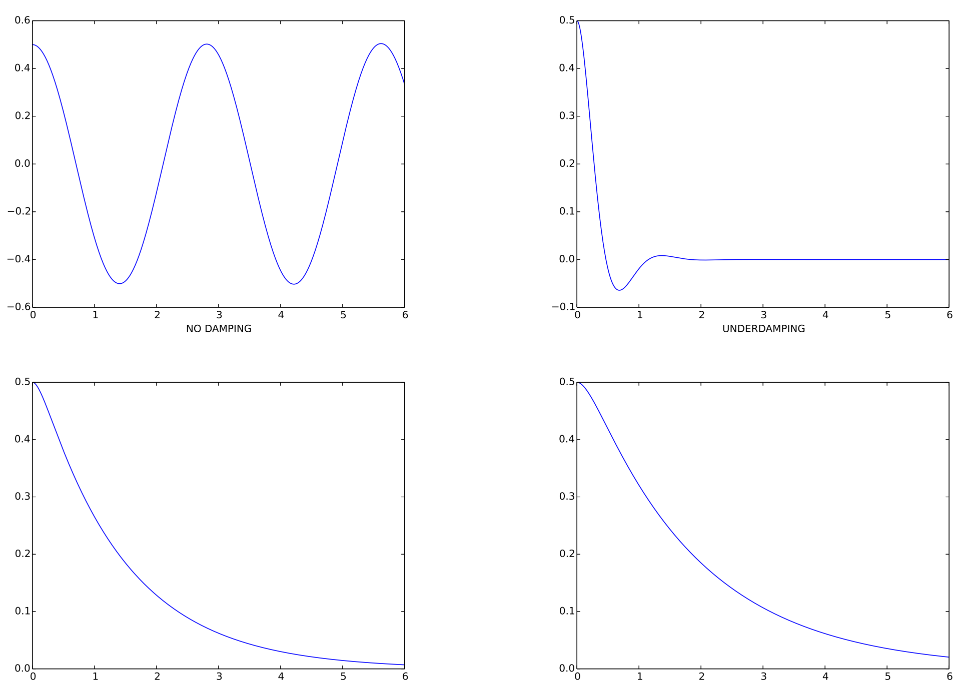
**(a)** $m=3,\ \gamma=0,\ k=15$

**(b)** $m=3,\ \gamma=6,\ k=30$

**(c)** $m=3,\ \gamma=9,\ k=6$

**(d)** $m=3,\ \gamma=6,\ k=3$

The following graphs illustrate the various solutions $y(t)$ from the previous example and the previous exercise.



The captions for these graphs also include terminology which we will explain now. The term in the first graph, **no damping**, means the damping coefficient (that is to say, the coefficient in front of $\dot{y}$, which represents resistance to the spring's motion) is equal to zero. Notice that the mass oscillates infinitely many times, always returning to the same maximum displacement as on the previous cycle.

The term describing the second graph, **underdamping**, indicates that even though the magnitude of the oscillations decreases, the damping coefficient is too small (relative to the other parameters) to stop the solution from oscillating forever – regardless of the initial conditions. That is to say, no matter how small the initial displacement, or whether there is any initial velocity imparted, the solution will always oscillate through the rest position infinitely many times. This is because the general solution has the form $y = Ae^{-t}\cos(3t) + Be^{-t}\sin(3t)$, and any initial condition (other than the trivial one $y(0) = \dot{y}(0) = 0$) will result in infinitely many such oscillations.

This stands in stark contrast to the third situation, which is described as **overdamping**. In this setting, the damping coefficient is so large that *no initial displacement or velocity* can cause more than one oscillation! We can see this by analyzing the form of the general solution: $y(t) = Ae^{-t} + Be^{-2t}$. If we factor out $e^{-2t}$, we can write this as $y(t) = e^{-2t}(Ae^{t} + B)$, which will only be zero when $t = \ln\left(\frac{-B}{A}\right)$. Depending on the choice of initial conditions, this value of $t$ may or may not exist (depending on whether $\frac{-B}{A}$ is positive or not), and even if it does exist, $\ln\left(\frac{-B}{A}\right)$ may not be positive, in which case that $t$ value would not be relevant to our model of the situation (since we typically assume the motion "starts" at $t = 0$).

The last graph looks very similar to the naked eye, but we have given it a different name: **critical damping**. That is because this is the borderline case – if the damping coefficient is reduced by any positive amount whatsoever, no matter how small, the situation will switch to underdamping, while if the damping coefficient is increased by any small amount whatsoever, the system will experience overdamping. Also, the form of the general solution is slightly different, as the double root of the characteristic equation produces $y(t) = Ae^{-t} + Bte^{-t}$. For this kind of function, the equation $y(t) = 0$ definitely has a solution – when $t = \frac{-A}{B}$ – though this may still be a negative value and thus irrelevant to the model.

These differences in behavior depend on the general solution of the equation $m\ddot{y} + \gamma\dot{y} + ky = 0$, which in turn is determined by the characteristic equation. We can therefore use the characteristic equation to classify the type of damping in any setup. To obtain infinitely many oscillations, our general solution must contain sinusoidal functions, and those occur when the characteristic equation has complex roots with non-zero imaginary parts; there will be no sinusoidal behavior if the roots are real. The borderline situation of critical damping is precisely the case of a double root. If we solve the characteristic equation using the quadratic formula,

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m},$$

then we can detect the types of solutions by looking directly at the discriminant (the expression inside the square root), as there will be real roots when the discriminant is positive (or zero) and complex roots when the discriminant is negative. A zero discriminant is the borderline case.

**Classifying Simple Harmonic Oscillators**

An oscillating system modeled by the ODE

$$a\ddot{y} + b\dot{y} + cy = f(t)$$

with $a, c > 0$ and $b \geq 0$ exhibits:

**No damping** if $b = 0$

**Underdamping** if $b^2 - 4ac < 0$

**Critical Damping** if $b^2 - 4ac = 0$

**Overdamping** if $b^2 - 4ac > 0$

In the case of a spring-mass system modeled by the equation $m\ddot{y} + \gamma\dot{y} + ky = f(t)$, the quantity $b^2 - 4ac$ described in the classification above becomes $\gamma^2 - 4mk$.

EXERCISE 2: Classify the type of damping for each of the following spring-mass situations.

(1) $m = 2,\ \gamma = 12,\ k = 16$

(2) $m = 2,\ \gamma = 12,\ k = 18$

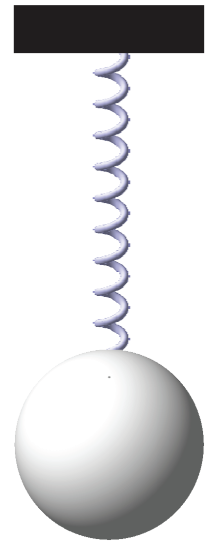
(3) $m = 3,\ \gamma = 12,\ k = 18$

(4) $m = 2,\ \gamma = 8,\ k = 16$

EXERCISE 3: Suppose a mass of $4\ kg$ is attached to the free end of a spring whose spring constant is $k = 10\frac{N}{m}$. Find the exact value of the damping coefficient $\gamma$ that will result in critical damping.

## Forced Vibrations

Next we turn our attention to what happens when we introduce an external driving force to the oscillating system.



EXAMPLE 1: A mass of $2\ kg$ is hung from a spring whose constant is $34\frac{N}{m}$. The other end of the spring is anchored to the ceiling. The system is subject to viscous damping with coefficient $\gamma = 4\frac{N\cdot s}{m}$. Gravity acts on the mass and, if not for the spring holding it, would accelerate the mass at $9.8\frac{m}{s^2}$. The mass is pushed up so that the spring is compressed 0.03 meters from its natural length. Then the mass is released. Determine the long-term behavior of the position of the mass (i.e., if the position over time is $y(t)$, find $\lim_{t\to\infty} y(t)$).

The external driving force due to gravity acts downward on the mass, so its effect is to lengthen the spring. The magnitude of this force is $F_e = mg = (2\ kg)\left(9.8\frac{m}{s^2}\right) = 19.6\ N$. Our initial value problem is thus:

$$2\ddot{y} + 4\dot{y} + 34y = 19.6, \quad y(0) = -0.03, \quad \dot{y}(0) = 0.$$

(Note that since we are treating the downward force of gravity as being in the positive direction, the initial displacement of the compressed spring must therefore be negative.)

The solution of this IVP is

$$y(t) = \frac{49}{85} - \frac{1031}{1700}e^{-t}\cos(4t) - \frac{1031}{6800}e^{-t}\sin(4t).$$

From this, we see that

$$\lim_{t\to\infty} y(t) = \frac{49}{85}.$$

That is to say, in the long term the mass settles toward a position that is $\frac{49}{85} \approx 0.58\ m$ below the spring's natural rest position. □

The position that the mass (i.e. the free end of the spring) tends toward in the previous example is called the **equilibrium position** because that is the position where the force due to gravity and the internal restoring force of the spring are in equilibrium with one another – the downward force due to gravity is equal in magnitude to the upward force of the spring. The reader should verify that the constant function $y(t) = \frac{49}{85}$ is an equilibrium solution of the differential equation in that example.

EXERCISE 4: The equilibrium position of a hanging spring-mass system is $0.12$ meters below the spring's natural rest position when a mass of $4\ kg$ is attached to the spring. Determine the spring constant.

EXAMPLE 2: The spring-mass system in Example 1 begins at rest in the equilibrium position, where spring and gravitational forces are balanced. An earthquake then begins to shake the building up and down, imparting a force on the system that is transferred to the mass. If this force is modeled by the function $f(t) = 0.2\sin(0.4t)\ N$, find formula for $y(t)$, and graph the solution over the time interval $0 \le t \le 300$.

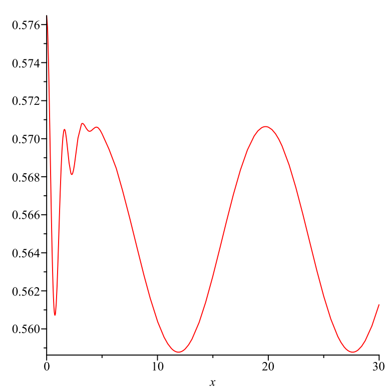
From what we obtained in Example 1, we see that we need to solve the initial value problem

$$2\ddot{y} + 4\dot{y} + 34y = 19.6 + 0.2\sin(0.4t), \quad y(0) = \frac{49}{85}, \quad \dot{y}(0) = 0$$

The solution of this is

$$y(t) = \frac{49}{85} - \frac{50}{177641}\cos(0.04t) + \frac{2105}{355282}\sin(0.04t) + \frac{181891}{15099485}e^{-t}\cos(4t) + \frac{73053}{30198970}e^{-t}\sin(4t).$$

This is a graph of the solution:




□

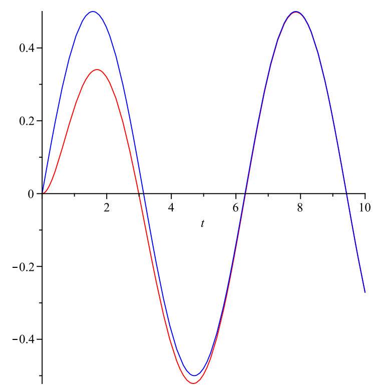
Notice that in each of the examples we've explored so far in this chapter, we have encountered solutions of the form

$$y(t) = y_S(t) + y_T(t),$$

where $y_S$ is periodic and $y_T(t) \to 0$ as $t \to \infty$. Thus the long-term behavior of $y$ matches whatever the long-term behavior is of $y_S$, while $y_T$ becomes negligible. (We can see this in the previous example, where the graph shows a function whose behavior appears to approach a simple sinusoidal oscillation after the first few seconds pass.) For this reason, the term $y_T$ is called a **transient solution** of the IVP, while $y_S$ is called the **steady-state solution**, or the **steady-state response**. For example, the solution of the IVP

$$\ddot{y} + 2\dot{y} + y = \cos(t), \quad y(0) = 0, \quad y'(0) = 0$$

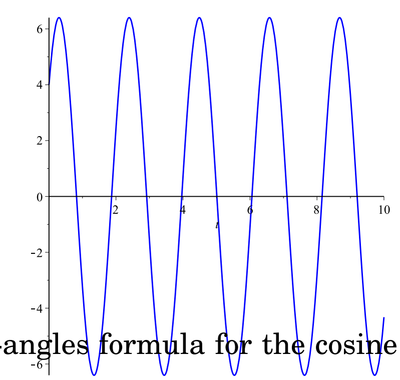
is the function $y(t) = \frac{1}{2}\sin(t) - \frac{1}{2}te^{-t}$. The transient solution is $y_T = -\frac{1}{2}te^{-t}$, and the steady-state solution is $y_S = \frac{1}{2}\sin(t)$. The following graph of $y$ and $y_S$ indicates the increasing similarity between these function as $t$ increases.




EXERCISE 5: Suppose that a mass of $6\ kg$ is attached to the free end of a spring whose spring constant is $12\frac{N}{m}$, and the mass when moving experiences viscous damping with a coefficient of $6\frac{Ns}{m}$. The spring is stretched $0.05$ meters from its rest position and released. The system also experiences forced vibrations of $0.5\cos(2t)\ N$. Find the steady-state and transient solutions for this system.

## Phase-Amplitude Form

Pick any real values of $a$, $b$ (not both 0) and any positive value of $\omega$, and then graph $y = a\cos(\omega t) + b\sin(\omega t)$. What you see will look just like a sinusoidal function, possibly with a horizontal shift. That is to say, it seems like we ought to be able to write the function in the form $y = A\cos(\omega t - \delta)$ for some coefficients $A$ and $\delta$. The graph at right shows one example, but the reader should verify this by trying a few of his or her own values for $a$ and $b$.




In fact, we can do exactly that. Recall the difference-of-angles formula for the cosine function:

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).$$

If we apply this identity with $\alpha = \omega t$ and $\beta = \delta$, we obtain

$$\cos(\omega t - \delta) = \cos(\omega t)\cos(\delta) + \sin(\omega t)\sin(\delta).$$

Trying to obtain values of $A$ and $\delta$ such that $A\cos(\omega t - \delta) = a\cos(\omega t) + b\sin(\omega t)$ amounts to finding values such that

$$A\left(\cos(\omega t)\cos(\delta) + \sin(\omega t)\sin(\delta)\right) = a\cos(\omega t) + b\sin(\omega t).$$

Equating coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ gives us

$$A\cos(\delta) = a \quad \text{and} \quad A\sin(\delta) = b. \tag{4}$$

Squaring each side of each of these equations and adding them gives us

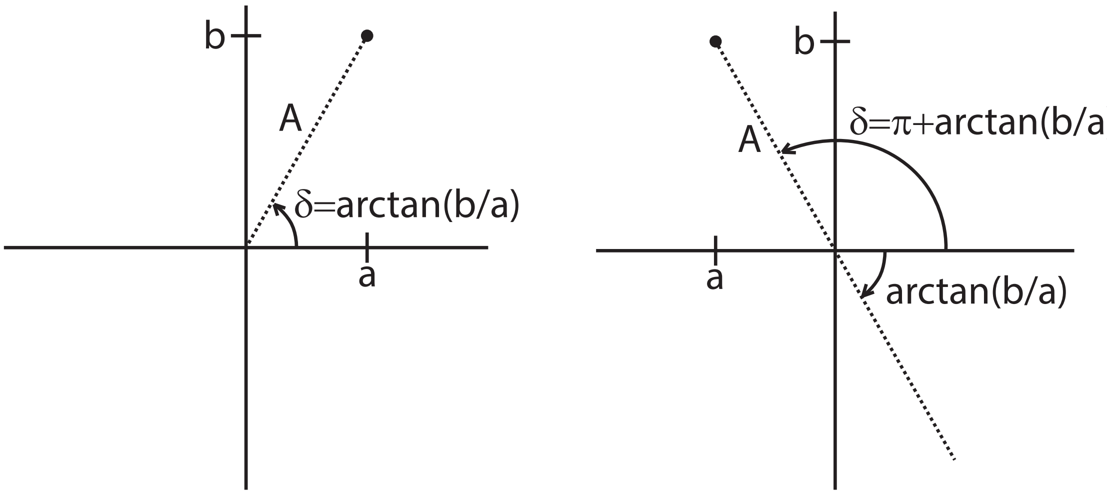
$$A^2\cos^2(\delta) + A^2\sin^2(\delta) = a^2 + b^2,$$

and therefore $A^2 = a^2 + b^2$. Let's take $A = \sqrt{a^2 + b^2}$. Now if $a = 0$, then $A = |b|$, so we can satisfy the system of equations (4) above by taking either $\delta = \frac{\pi}{2}$ (if $b > 0$) or $\delta = \frac{3\pi}{2}$ (if $b < 0$). On the other hand, if $a$ is not zero, then we can divide the second equation in (4) by the first equation to obtain

$$\tan(\delta) = \frac{b}{a},$$

and we now see that we can satisfy the system (4) by taking either $\delta = \arctan\left(\frac{b}{a}\right)$ (if $a > 0$) or $\delta = \pi + \arctan\left(\frac{b}{a}\right)$ (if $a < 0$). If we wish to ensure that $\delta$ is positive, then we can use $\delta = 2\pi + \arctan\left(\frac{b}{a}\right)$ when $a > 0$ and $b < 0$.

The algebraic manipulations we've been performing here have really just been an effort to find a polar-coordinate representation of the point with Cartesian coordinates $(a, b)$. The apparent complexity occurs only because the range of the inverse tangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so that it always wants to produce an angle pointing into the first or fourth quadrant, and the addition of $\pi$ in certain instances is necessary to correct for this, as illustrated in the following figures.




EXERCISE 6: Find values of $A$ and $\delta$ such that $A\cos(3t-\delta) = 4\cos(3t) + 5\sin(3t)$.

We can also use these observations to write solutions $y = ae^{\alpha t}\cos(\omega t) + be^{\alpha t}\sin(\omega t)$ in the **phase-amplitude form** $y = Ae^{\alpha t}\cos(\omega t - \delta)$. The dimensionless parameter $\delta$ is called the **phase** of the oscillations; if $\alpha = 0$ (meaning the solution does not decay, which occurs when there is no damping), then the coefficient $A$ is the **amplitude** of the oscillations.

EXAMPLE 3: Suppose a mass of $0.01kg$ is attached to the free end of a spring whose spring constant is $2\frac{N}{m}$, and there is viscous damping described by the coefficient $\gamma = 0.2\frac{N\cdot s}{m}$. The mass is stretched $0.01m$ from the natural position and released. Write a formula for the displacement of the mass from the natural position in the phase-amplitude form.

The initial value problem we need to solve here is

$$0.01\ddot{y} + 0.2\dot{y} + 2y = 0,\ y(0) = 0.01,\ \dot{y}(0) = 0.$$

Using the standard approach, we obtain the solution

$$y(t) = 0.01e^{-10t}\cos(10t) + 0.01e^{-10t}\sin(10t).$$

Factor out the exponential $e^{-10t}$ to obtain

$$y(t) = e^{-10t}\left(0.01\cos(10t) + 0.01\sin(10t)\right).$$

Let $A = \sqrt{(0.01)^2 + (0.01)^2} = \sqrt{0.0002}$ and $\delta = \arctan\left(\frac{0.01}{0.01}\right) = \arctan(1) = \frac{\pi}{4}$. Now we can write

$$y(t) = \sqrt{0.0002}e^{-10t}\cos\left(10t - \frac{\pi}{4}\right).$$

EXERCISE 7: Rework Example 9.3 assuming that the spring is initially compressed (instead of being stretched).

EXERCISE 8: Rework Example 9.3 assuming that, in addition to being stretched $0.01m$, the mass is also given an initial velocity of $1\frac{m}{s}$ in the direction of further stretching the spring.

EXAMPLE 4: A simple electrical circuit contains a 0.2 henry inductor and a 0.05 farad capacitor connected in series. The charge on the capacitor at time $t = 0$ is 2.5 coulombs, and the initial current in the circuit is $I(0) = \dot{Q}(0) = 0.2$ amperes. Determine the first time $t$ when the charge $Q(t)$ will reach 1.5 coulombs.

**Solution:** Since the reciprocal of the capacitance is $\frac{1}{C} = \frac{1}{0.005} = 200$, the charge $Q$ satisfies the initial value problem

$$0.2\ddot{Q} + 20Q = 0, \quad Q(0) = 2.5,\ \dot{Q}(0) = 0.$$

The solution is given by

$$Q(t) = 2.5\cos(10t) + 0.02\sin(10t).$$

Writing this in phase-amplitude form, we obtain

$$Q(t) = \sqrt{6.2504}\cos\left(10t - \tan^{-1}(0.008)\right).$$

This form makes it easier to isolate $t$ when we solve the equation $Q(t) = 1.5$:

$$\sqrt{6.2504}\cos\left(10t - \tan^{-1}(0.008)\right) = 1.5,$$

so

$$t = \frac{1}{10}\left(\cos^{-1}\left(\frac{1.5}{\sqrt{6.2504}}\right) + k\pi + \tan^{-1}(0.008)\right)$$

Plugging in a few consecutive values for $k$, we can see that the smallest positive solution occurs when $k = 0$, and in that case

$$\frac{1}{10}\left(\cos^{-1}\left(\frac{1.5}{\sqrt{6.2504}}\right) + \tan^{-1}(0.008)\right) \approx 0.092\ s.$$

□

## RESONANCE

It may seem obvious to some that modifying the amplitude $F_0$ of a driving function $F_0\cos(\omega t)$ can have a direct effect on the amplitude of the steady-state response. What is likely less obvious is that modifying the *frequency* $\omega$ of the driving function can also affect the amplitude of the response, often in very dramatic ways. We will explore this phenomenon through two examples before stating the general results. We will begin with the simplified example of an undamped oscillator.

EXAMPLE 5: Suppose a spring-mass system is modeled by the initial value problem

$$\ddot{y} + y = \cos(\omega t),\ y(0) = 0, \dot{y}(0) = 0.$$

Explore the consequences of various values of the driving frequency $\omega > 0$ on the solutions of this initial value problem.

The characteristic equation is $r^2 + 1 = 0$, which has roots $r = \pm i$. Therefore the solution of the related homogeneous equation is

$$y_h(t) = A\cos(t) + B\sin(t).$$

We will need to be careful when we guess the form of a particular solution to non-homogeneous equation, because the form of our guess depends upon the value of $\omega$.

First, if $\omega \neq 1$, then we guess $y_p(t) = C\sin(\omega t) + D\cos(\omega t)$, and the method of undetermined coefficients leads us to the solution

$$y_p(t) = \frac{1}{1-\omega^2}\cos(\omega t).$$

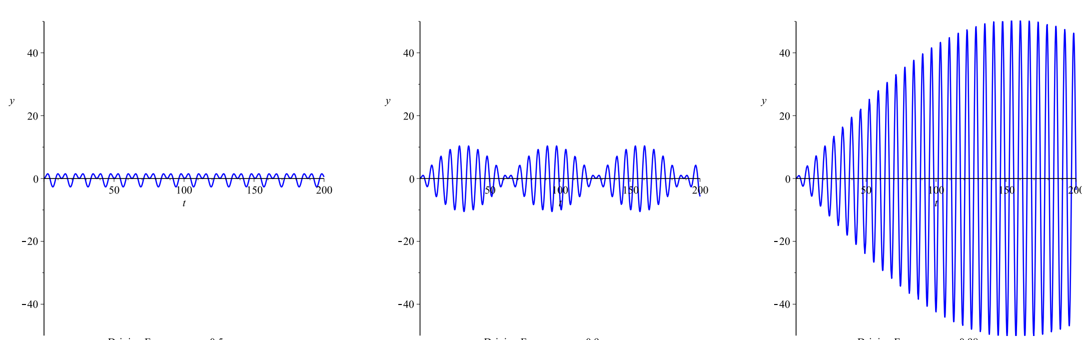
Therefore the general solution of the non-homogeneous problem is

$$y(t) = \frac{1}{1-\omega^2}\cos(\omega t) + A\sin(t) + B\cos(t),$$

and the initial conditions $y(0) = \dot{y}(0) = 0$ imply $A = 0$ and $B = -\frac{1}{1-\omega^2}$. Consequently we have

$$y(t) = \frac{1}{1-\omega^2}\cos(\omega t) + -\frac{1}{1-\omega^2}\cos(t).$$

Notice that the amplitude coefficients here all become larger as $\omega$ gets closer to $1$. The following three graphs show the solutions corresponding to the driving frequencies $\omega = 0.5,\ 0.9$ and $0.98$:

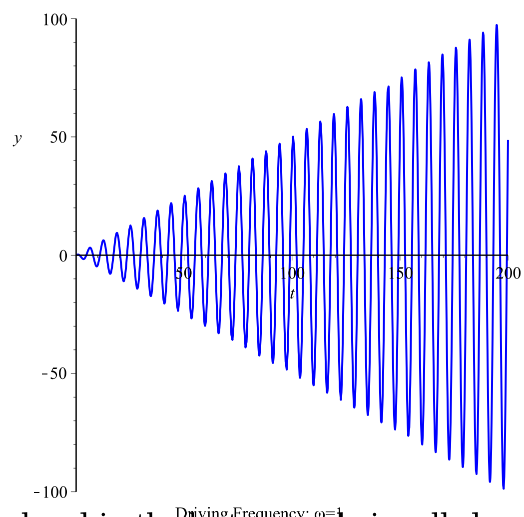


Notice how the magnitude of the vibrations have changed even though the amplitude of the driving function has not changed, only its frequency has been adjusted.

To complete this analysis, observe that if we start out with $\omega = 1$, then the initial guess for a particular solution of the non-homogeneous equation will have the form $y_p(t) = Ct\cos(t) + Dt\sin(t)$, and the reader should verify that this together with the initial conditions eventually leads us to the complete solution

$$y(t) = \frac{1}{2}t\sin(t).$$

In this case, as time progresses, the magnitude of the oscillations increases without bound:




□

The phenomenon explored in the last example is called **resonance**. The idea is that the spring-mass system (or other oscillator) has a **natural frequency** (or **resonant frequency**) at which it wants to oscillate, namely, the frequency[1] of the solutions to the corresponding homogeneous differential equation. If the driving force oscillates at close to this frequency, the resulting oscillations in the system will be larger in amplitude than they would be if the frequencies were not close (assuming the amplitude of the driving force is not changed). If the driving force is applied to an undamped oscillator at exactly the resonant frequency, then the oscillations will grow in magnitude instead of tending to a steady-state (i.e. periodic) behavior. In Example 9.4, we observed resonance when the

[1]Here, we are calling $\omega$ the 'frequency', but a more precise name would be 'angular frequency'. Calling it that would distinguish it from the so-called 'temporal frequency' $f$ that is a measurement of oscillations per second. The relationship between angular frequency and temporal frequency is $\omega = 2\pi f$.

frequency was $\omega = 1$, which corresponded to a driving function of $\cos(t)$. When there is no damping in a spring-mass system, the resonant frequency is $\omega_0$ defined by $\omega_0^2 = \frac{k}{m}$.

EXERCISE 9: Find the resonant frequency of a spring-mass system with mass $2\ kg$, a spring constant of $\frac{1}{2}\frac{N}{m}$ and no damping.

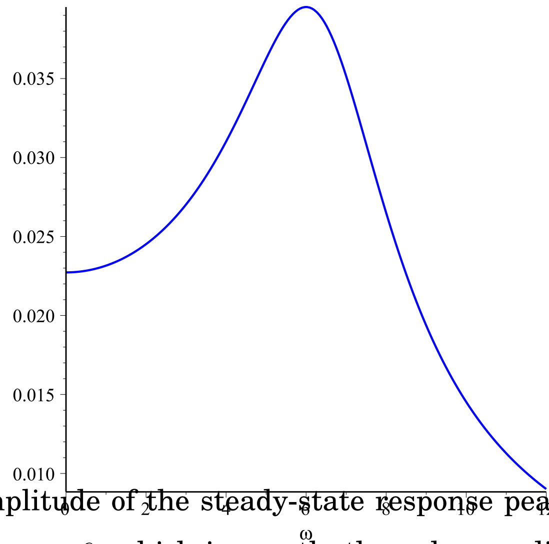
The example above used to illustrate this behavior assumed a damping coefficient of $\gamma = 0$ to make this phenomenon stand out dramatically. If $\gamma > 0$, then the homogeneous solutions will include decaying exponential factors so that they will, over time, tend to 0. However, if $\gamma$ is very small, then resonance behavior can can still be observed. When damping is present, we won't observe vibrations growing unbounded as $t \to 0$ – that can only happen in the undamped setting. Instead, the resonant frequency for a damped system is the frequency at which the steady-state response has the greatest amplitude. If the damping coefficient is small, then the amplitude of vibrations forced at the resonant frequency can be quite large.

**Resonant Frequency**

For a driven oscillating system $m\ddot{y} + \gamma\dot{y} + ky = F_0\cos(\omega t)$, the **resonant frequency** $\omega_{max}$ is given by

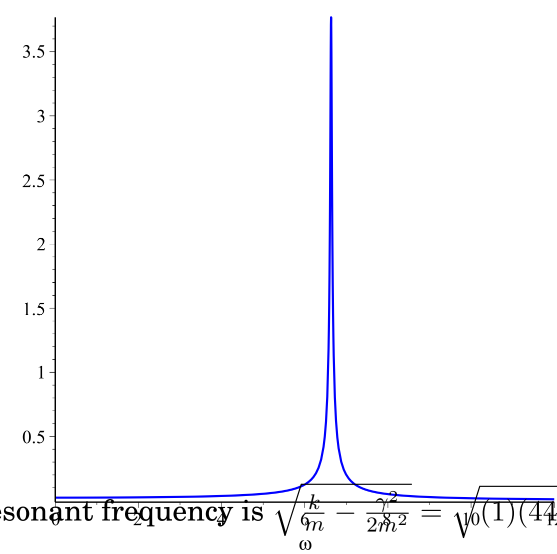
$$\omega_{max}^2 = \frac{k}{m} - \frac{\gamma^2}{2m^2}.$$

The derivation of this formula is outlined in the problem set at the end of this chapter. The following graph illustrates the resonance phenomenon for a driven, damped oscillator modeled by the differential equation $\ddot{y} + 4\dot{y} + 44y = \cos(\omega t)$; the graph illustrates the amplitude of the steady-state response to this equation as a function of $\omega$. (A formula for the amplitude of the steady-state response is also covered in the problem set at this chapter's end.)




Amplitude of Steady-State Response for m=1,γ=4,k=44

Notice that the amplitude of the steady-state response peaks when the frequency of the driving function is $\omega = 6$, which is exactly the value predicted for the resonant frequency. The possibility of a large amplitude in the steady state response becomes even more dramatic if we use a small coefficient of viscous resistance (say, $\gamma = 0.04$):




Amplitude of Steady-State Response for m=1,γ=0.04,k=44

In this case, the resonant frequency is $\sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}} = \sqrt{(1)(44) - \frac{(0.04)^2}{2(1)^2}} \approx 6.633$.

EXERCISE 10: Find the resonant frequency of a spring-mass system with mass $2\ kg$, a spring constant of $200\frac{N}{m}$ and viscous damping whose coefficient is $24\frac{Ns}{m}$.

Resonance phenomena must be taken seriously in the design of building structures which could shake themselves apart if they were to resonate at the same frequency as, say, an earthquake. Engineers can also take advantage of resonance to build devices that amplify the driving oscillations, such as seismographs and electronic signal amplifiers.

# Additional Exercises

Classify the type of damping for each the following combination of mass $m\ kg$, viscous damping coefficient $\gamma \frac{Ns}{m}$ and spring constant $k\frac{N}{m}$.

**11** $m = 4,\ \gamma = 4,\ k = 2$

**12** $m = 4,\ \gamma = 4,\ k = 1$

**13** $m = 2,\ \gamma = 4,\ k = 2$

**14** $m = 3,\ \gamma = 5,\ k = 2$

**15** $m = 2,\ \gamma = 0,\ k = 10$

**16** $m = 4,\ \gamma = 1,\ k = 0.1$

Determine the resonant frequency for a spring-mass system with the following combination of mass $m\ kg$, viscous damping coefficient $\gamma \frac{Ns}{m}$ and spring constant $k\frac{N}{m}$.

**17** $m = 0.5,\ \gamma = 4,\ k = 10$

**18** $m = 4,\ \gamma = 3,\ k = 20$

**19** $m = 3,\ \gamma = 0,\ k = 120$

**20** $m = 0.2,\ \gamma = 20,\ k = 90$

Find the steady state solution $y_S$ for the given initial value problem.

**21** $2\ddot{y} + 8y = \cos(t),\ y(0) = 0, \dot{y}(0) = 0$

**22** $3\ddot{y} + y = \sin(2t),\ y(0) = 1, \dot{y} = 0$

Express the solution of the given initial value problem in phase-amplitude form.

**23** $\ddot{y} + y = 0,\ y(0) = 8,\ \dot{y}(0) = 6$

**24** $\ddot{y} + 4y = 0,\ y(0) = 1,\ doty(0) = -4$

**25** $\ddot{y} + 2\dot{y} + 5y = 0,\ y(0) = 0,\ \dot{y}(0) = 1$

**26** $\ddot{y} + 4\dot{y} + 5y = 0,\ y(0) = 1,\ \dot{y} = 0$

**27** Prove that, if there is no external driving force and any damping at all (i.e. $b > 0$) for a spring-mass system, then $\lim_{t\to\infty} y(t) = 0$.

**28** Prove that, if there is no damping and no external driving force, a spring mass system will oscillate with period $\frac{2\pi}{\omega_0}$, where $\omega_0^2 = \frac{k}{m}$.

**29** Consider a critically damped spring-mass system subject to the following parameters: $m = 2,\ b = 8,\ k = 8$. If the initial displacement is $y(0) = 1$ and the initial velocity is $\dot{y} = v_0$, find a condition on $v_0$ that determines whether or not the spring will ever pass through its natural length during the time interval $t > 0$.

**30** Repeat the previous problem for the overdamped spring-mass system: $m = 2,\ b = 10,\ k = 8$.

**31** *This problem outlines a derivation of the resonant frequency in the simplified case of no damping.* Consider an undamped spring-mass system with forced vibrations described by the differential equation $m\ddot{y} + ky = F_0\cos(\omega t)$ (where $F_0 > 0$ is the amplitude of the driving vibrations). **(a)** Verify that the solutions of this differential equation are unbounded when $\omega^2 = \frac{k}{m}$. **(b)** Find a formula for the amplitude of the steady-state solution as a function of $F_0$ and $\omega$, when $\omega^2 \neq \frac{k}{m}$. **(c)** Verify that the amplitude of the solutions in part (b) approach $\infty$ as $\omega \to \frac{k}{m}$.

**32** *This problem outlines the derivation of the resonant frequency for a damped oscillator.* Consider a damped spring-mass system with forced vibrations described by the differential equation $m\ddot{y} + \gamma\dot{y} + ky = F_0\cos(\omega t)$ (where $F_0 > 0$ is the amplitude of the driving vibrations). **(a)** Verify, using the method of undetermined coefficients, that the steady-state solution is $y_S = A\cos(\omega t) + B\sin(\omega t)$, where $A = \frac{F_0(k-m\omega^2)}{(k-m\omega^2)^2+\omega^2\gamma^2}$ and $B = \frac{F_0\omega\gamma}{(k-m\omega^2)^2+\omega^2\gamma^2}$. **(b)** Verify that the amplitude of the steady-state solution is $F_0/\sqrt{(k-m\omega^2)^2 + \gamma^2\omega^2}$. **(c)** Regard the amplitude from part (b) as a function of $\omega$. Use differential calculus to verify that this function has a maximum value when $\omega^2 = \frac{k}{m} - \frac{\gamma^2}{2m^2}$.

**33** You wish to build a damped oscillator that whose resonant frequency will be $\omega_{max} = 5.00$. You also want the amplitude of the steady state response at the resonant frequency to be twice as large as the amplitude of the driving vibrations. Your oscillator will sit in a medium that exerts viscous damping given by the coefficient $\gamma = 0.300\frac{Ns}{m}$. Determine the appropriate mass $m$ and spring constant $k$ to use in the oscillator's construction. Report your answers to three significant figures. *(Note: There will be two solutions. Refer to Problem 9.6 for the necessary formulas.)*

# Part 3

# Laplace Transforms

CHAPTER 10

# Laplace Transforms

**Prototype Question:** A simple electrical circuit component contains a 2 ohm resistor, a 3 henry inductor and a 4 farad capacitor connected in series. If there is a voltage source connected that supplies $f(t)$ volts at time $t$, where

$$f(t) = \begin{cases} 2 & \text{if } \pi \leq t \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

then the charge $q(t)$ on the capacitor can be modeled by the differential equation

$$3\ddot{q} + 2\dot{q} + \frac{1}{4}q = f(t).$$

Here, $q$ is measured in coulombs, and time is measured in seconds. The current initially satisfies $q(0) = 0$ and $\dot{q}(0) = 0$. Graph the current $q(t)$ on the time interval $0 \leq t \leq 4\pi$ seconds.

In this chapter we will introduce the idea of a transform method. The basic idea is this: we begin with an initial value problem for a differential equation, and we transform this equation into an algebraic equation; once we solve for the unknown in the algebraic equation, we then transform back to find a corresponding solution of the IVP.

We will see how this transform can be used to solve second order constant coefficient ODE. We already know how to solve some of these equations using the method of undetermined coefficients, so one might wonder at first why we need a new method. The point is that our new approach will make it much easier to solve problems with discontinuous driving functions (such as we see in the prototype question above). In fact, this is the preferred method in many electrical engineering problems where discontinuous driving functions are extremely common.

The tool we will use for this is the **Laplace Transform** of a function, defined by

$$L[f] = \int_0^\infty f(t)e^{-st}\,dt.$$

Here, $f$ is a function defined on $[0,\infty)$ and $L[f]$ is a function of $s$ defined for whatever values of $s$ lead to a convergent integral.

EXAMPLE 1: The Laplace Transform of $e^t$ is

$$\begin{aligned} L[e^t] &= \int_0^\infty e^t e^{-st}\,dt \\ &= \lim_{T\to\infty} \int_0^T e^{(1-s)t}\,dt \\ &= \lim_{T\to\infty} \left.\frac{e^{(1-s)t}}{1-s}\right|_0^T \\ &= \lim_{T\to\infty} \frac{e^{(1-s)T}}{1-s} - \frac{1}{1-s} \\ &= \frac{1}{s-1} \quad \text{for } s > 1. \end{aligned}$$

□

EXERCISE 1: Calculate the Laplace Transform of the functions $t^2$, $\sin(t)$ and $e^{at}$ (where $a$ is a constant).

EXERCISE 2: Prove that the Laplace Transform is linear: for any functions $f$ and $g$ and for any constant coefficients $a$ and $b$, $L[af + bg] = aL[f] + bL[g]$. *(Equality only needs to hold on the set of $s$-values for which $L[f]$ and $L[g]$ are both defined.)*

It is typical to denote a transform of a function with a capital letter. For example, when it is useful to display the variable, we will often denote the Laplace Transform of a function $f(t)$ by $F(s)$; otherwise we will write it as $L[f]$. We usually do not care what the exact domain is for $F(s)$ – it will be enough to know that there is some interval for $s$ on which the integral defining the transform converges. The next theorem provides such a guarantee.

### Existence of the Laplace Transform

Suppose there exists $M \geq 0$ and any real number $N$ such that $|f(t)| \leq Me^{Nt}$ for all $t \geq 0$. Then the integral defining the Laplace Transform converges for all $s > N$.

A function that satisfies the hypothesis of this theorem is said to be of **exponential order**, because it does not grow any faster than exponential functions can grow.

PROOF. Observe that for $s > N$ we have

$$\begin{aligned}\int_0^\infty |f(t)e^{-st}|\,dt &= \int_0^\infty |f(t)|e^{-st}\,dt \\ &\leq \int_0^\infty Me^{Nt}e^{-st}\,dt \\ &= \int_0^\infty Me^{(N-s)t}\,dt \\ &= \frac{M}{s-N} \\ &< \infty.\end{aligned}$$

This proves that the integral defining $L[f]$ converges absolutely for all $s > N$. □

Next, we introduce the key fact which allows us to use Laplace Transforms for solving initial value problems. There is a close relationship between the Laplace transform of a function and that of its derivative: *If* $L[f]$ *exists on some s-interval* $(a, \infty)$*, where* $f$ *is a differentiable function, and if* $\lim_{t\to\infty} f(t)e^{-st} = 0$ *for* $s > a$*, then* $L[f']$ *also exists for* $s > a$*, and* $L[f'] = sL[f] - f(0)$*.*

Notice that any function of exponential order satisfies both hypotheses of this theorem. We call this a reduction formula for the Laplace Transform because it allows us to "reduce" $L[f']$ to an expression involving $L[f]$. The following box highlights this result.

### Reduction Formula: Laplace Transform of a Derivative

$$L[f'] = sL[f] - f(0)$$

PROOF. We use integration by parts, integrating $f'(t)$ and differentiating $e^{-st}$:

$$\begin{aligned} L[f'] &= \int_0^\infty f'(t)e^{-st}\,dt \\ &= \lim_{T\to\infty}\int_0^T f'(t)e^{-st}\,dt \\ &= \lim_{T\to\infty}\left[e^{-st}f(t) - \int -se^{-st}f(t)\,dt\right]_0^T \\ &= \lim_{T\to\infty} e^{-sT}f(T) - e^{0t}f(0) + s\int_0^T f(t)e^{-st}\,dt \\ &= -f(0) + s\int_0^\infty f(t)e^{-st}\,dt \\ &= -f(0) + sL[f]. \end{aligned}$$

□

In practice, when faced with an unknown function we will always assume that it is of exponential order and therefore satisfies hypotheses of these two theorems. Of course, in theory such an assumption could lead to erroneous results, but in practical applications this rarely happens. And because the process we illustrate in the next few examples furnishes us with a concrete function, we can always check it to make sure it satisfies the differential equation at hand.

To make use of the Laplace Transform to solve an initial value problem, we need to make use of one more fact which we will not prove: *If $f$ and $g$ are continuous functions on $[0,\infty)$ and $L[f] = L[g]$, then $f = g$ on $[0,\infty)$.* Thus the Laplace Transform is invertible. We denote the **Inverse Laplace Transform** by $L^{-1}$. Because $L$ is linear, so is $L^{-1}$:

$$L^{-1}[aF(s) + bG(s)] = aL^{-1}[F(s)] + bL^{-1}[G(s)]$$

EXAMPLE 2: Since $L[e^{2t}] = \frac{1}{s-2}$, it follows that $L^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$. □

EXERCISE 3: Find $L^{-1}\left[\frac{1}{s^3}\right]$. *(Hint: Refer to Exercise 1.)*

We now have enough machinery to use the Laplace Transform for solving an initial value problem.

EXAMPLE 3: Solve $y' + 2y = 0,\ y(0) = 3$ using Laplace Transforms.

**Solution:** Suppose $y$ is a solution of $y' + 2y = 0,\ y(0) = 3$ on the domain $[0, \infty)$. We take the Laplace Transform of both sides of the ODE:

$$L[y' + 2y] = L[0].$$

Then we use the facts that $L$ is linear and $L[0] = 0$:

$$L[y'] + 2L[y] = 0.$$

Next we apply the formula for the Laplace Transform of a derivative:

$$sL[y] - y(0) + 2L[y] = 0.$$

Insert the initial condition $y(0) = 3$ and collect like terms:

$$(s + 2)L[y] - 3 = 0.$$

Isolate $L[y]$:

$$L[y] = \frac{3}{s+2}.$$

Finally, isolate $y$ by taking the inverse Laplace Transform of both sides:

$$\begin{aligned} y &= L^{-1}\left[\frac{3}{s+2}\right] \\ &= 3L^{-1}\left[\frac{1}{s-(-2)}\right] \\ &= 3e^{-2t}. \end{aligned}$$

This is the solution of the initial value problem above. □

Clearly it will be useful to have a list of functions and their corresponding Laplace Transforms. Here is a short list of such correspondences.

**Brief Table of Laplace Transforms**

| $f(t)$ | $F(s)$ |
|---|---|
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\sin(kt)$ | $\frac{k}{s^2+k^2}$ |
| $\cos(kt)$ | $\frac{s}{s^2+k^2}$ |
| $\sinh(kt)$ | $\frac{k}{s^2-k^2}$ |
| $\cosh(kt)$ | $\frac{s}{s^2-k^2}$ |
| $e^{at}\sin(bt)$ | $\frac{b}{(s-a)^2+b^2}$ |
| $e^{at}\cos(bt)$ | $\frac{s-a}{(s-a)^2+b^2}$ |
| $t^n e^{at}$ | $\frac{n!}{(s-a)^{n+1}}$ |

EXERCISE 4: Use Laplace Transforms to solve the initial value problem $y' + 4y = 6,\ y(0) = 2$.

Higher-order ODE can be solved in the same way. When we transform $y''$, we just use the reduction formula twice:

$$L[y''] = sL[y'] - y'(0) = s(sL[y] - y(0)) - y'(0) = s^2L[y] - sy(0) - y'(0).$$

The reader may choose to memorize this formula as well, or just to use the first-order formula repeatedly when required.

EXAMPLE 4: Solve the IVP $y'' + 9y = 2,\ y(0) = 1,\ y'(0) = 0$.

**Solution:** Transform both sides of the equation, rewrite all the Laplace Transforms in terms of $L[y]$, and then isolate $L[y]$:

$$\begin{aligned}
L[y'' + 9y] &= L[2] \\
L[y''] + 9L[y] &= \frac{2}{s} \\
sL[y'] - y'(0) + 9L[y] &= \frac{2}{s} \\
s(sL[y] - y(0)) - y'(0) + 9L[y] &= \frac{2}{s} \\
(s^2 + 9)L[y] - s - 0 &= \frac{2}{s} \\
(s^2 + 9)L[y] &= s + \frac{2}{s} \\
L[y] &= \frac{s}{s^2 + 9} + \frac{2}{s(s^2 + 9)}
\end{aligned}$$

Use a partial fractions decomposition to rewrite the right side of the equation:

$$\begin{aligned}
L[y] &= \frac{s}{s^2 + 9} + \frac{(2/9)}{s} + \frac{(-2/9)s}{s^2 + 9} \\
&= \frac{(7/9)s}{s^2 + 9} + \frac{(2/9)}{s}
\end{aligned}$$

Then isolate $y$ using the inverse transform:

$$\begin{aligned}
y &= L^{-1}\left[\frac{(7/9)s}{s^2 + 9} + \frac{(2/9)}{s}\right] \\
&= \frac{7}{9}L^{-1}\left[\frac{s}{s^2 + 9}\right] + \frac{2}{9}L^{-1}\left[\frac{1}{s}\right] \\
&= \frac{7}{9}\cos(3t) + \frac{2}{9}
\end{aligned}$$

□

EXAMPLE 5: Solve the IVP $y'' + 4y' + 13y = 0,\ y(0) = 1,\ y'(0) = 1.$

**Solution:** Transform both sides of the equation, rewrite all the Laplace Transforms in terms of $L[y]$, and then isolate $L[y]$:

$$\begin{aligned}
L[y'' + 4y' + 13y] &= L[0] \\
sL[y'] - y'(0) + 4(sL[y] - y(0)) + 13L[y] &= 0 \\
s(sL[y] - y(0)) - y'(0) + 4(sL[y] - y(0)) + 13L[y] &= 0 \\
(s^2 + 4s + 13)L[y] - sy(0) - y'(0) - 4y(0) &= 0 \\
(s^2 + 4s + 13)L[y] - s - 5 &= 0 \\
L[y] &= \frac{s+5}{s^2 + 4s + 13}
\end{aligned}$$

The denominator does not factor over the real numbers, so we don't want to try to use a partial fraction decomposition. Instead, will will use the algebraic technique of **completing the square** to rewrite the expression. Completing the square on a quadratic expression such as $x^2 + bx + d$ means rewriting it in the form $(x + h)^2 + d$. In this case, that would be

$$s^2 + 4s + 13 = s^2 + 4s + 4 + 9 = (s + 2)^2 + 9.$$

So now we have

$$L[y] = \frac{s+5}{(s+2)^2 + 9},$$

which doesn't exactly match any of the forms in our table; however, we can split up the numerator to obtain two fractions whose forms do match entries in our table:

$$L[y] = \frac{s+2}{(s+2)^2 + 9} + \frac{3}{(s+2)^2 + 9}.$$

Consequently,

$$y = e^{(-2t)} \cos(3t) + e^{(-2t)} \sin(3t).$$

□

EXERCISE 5: Use Laplace Transforms to solve the following initial value problems:

**(a)** $y'' + 25y = t,\ y(0) = 0,\ y'(0) = 3.$

**(b)** $y'' + 4y' = 6,\ y(0) = 0,\ y'(0) = 1.$

**(c)** $y'' - 6y' + 8y = 6,\ y(0) = 2, y'(0) = 0.$

**(d)** $y'' + 5y' + 4y = \sin(t),\ y(0) = 0, y'(0) = 0.$

**(e)** $y'' + 4y' + 8y = 0,\ y(0) = 1,\ y'(0) = 0.$

**(f)** $y'' - 4y = t,\ y(0) = 0,\ y'(0) = 2.$

We end this chapter with one more useful fact about Laplace Transforms which will allow use to easily compute many of them:

### Derivative of a Laplace Transform

$$L[tf(t)] = -F'(s), \text{ where } F(s) = L[f(t)]$$

This equality assumes that all of the necessary integrals are convergent. The following calculation contains the essence of the proof.

$$\begin{aligned} F'(s) &= \frac{d}{ds}\left[\int_0^\infty f(t)e^{-st}\,dt\right] \\ &= \int_0^\infty \frac{\partial}{\partial s}\left[f(t)e^{-st}\right]\,dt \\ &= \int_0^\infty f(t)(-te^{-st})\,dt \\ &= -\int_0^\infty (tf(t))e^{-st}\,dt \\ &= -L[tf(t)]. \end{aligned}$$

To call this a proof, we would need to justify the act of "differentiating under the integral sign", for *it is not always true* that $\frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b f_y(x,y)\,dy$. However, if $f$ is of exponential order, then it is possible to justify this step by using a powerful theorem from the subject of Real Analysis called the Lebesgue Dominated Convergence Theorem (see reference [**2**] in the bibliography). However, the details of this are outside the scope of this course.

Let's use this result to calculate some Laplace Transforms of functions.

EXAMPLE 6: The Laplace Transform of $e^{at}$ is $F(s) = \frac{1}{s-a}$; therefore

$$\begin{aligned} L[te^{at}] &= -\frac{d}{ds}\left[\frac{1}{s-a}\right] \\ &= \frac{1}{(s-a)^2}. \end{aligned}$$

□

EXERCISE 6: Find the Laplace Transforms of $t\sin(bt)$ and $t\cos(bt)$.

EXERCISE 7: Find the Laplace Transform of $t^k e^t$, where $k$ is a positive integer.

# Additional Exercises

Use the definition of the Laplace Transform to calculate the following.

**8** $L[e^{3t}]$

**9** $L[4t]$

**10** $L[6t^2]$

**11** $L[\cosh(t)]$

**12** $L[te^t]$

**13** $L[t^2e^t]$

Use the brief table of Laplace Transforms to find the following.

**14** $L[e^{-10t}]$

**15** $L[t^2 - t^3]$

**16** $L[sin(4t)]$

**17** $L[3\sinh(2t)]$

**18** $L[te^{-t}]$

**19** $L[4e^{-3t}\cos(2t)]$

Find the Inverse Laplace Transform $L^{-1}[F(s)]$ for the given function $F(s)$.

**20** $F(s) = \frac{1}{s^4}$

**21** $F(s) = \frac{1}{4s}$

**22** $F(s) = \frac{s}{s^2+4}$

**23** $F(s) = \frac{s+2}{s^2+9}$

**24** $F(s) = \frac{1}{s^2+2s+1}$

**25** $F(s) = \frac{1}{s^2+4s+20}$

Solve the initial value problem using that Laplace Transform.

**26** $\dot{y} + y = 0,\ y(0) = 3$

**27** $2\dot{y} - y = 1,\ y(0) = 0$

**28** $3\ddot{y} + \dot{y} = 2,\ y(0) = 1,\ doty(0) = 1$

**29** $\ddot{y} + y = \sin\left(\sqrt{2}t\right),\ y(0) = 4,\ \dot{y}(0) = 0$

---

**30** Prove the formula $L\left[t^n\right] = \frac{n!}{s^{n+1}}$ three different ways: **(a)** directly from the definition of Laplace Transform, **(b)** by using the reduction formula for the Laplace Transform of a derivative, and **(c)** by taking advantage of the formula for the derivative of a Laplace Transform.

**31** Prove the following Laplace Transform formulas: **(a)** $L[e^{at}] = \frac{1}{s-a}$, **(b)** $L[\sinh(kt)] = \frac{k}{s^2-k^2}$, **(c)** $L[\cosh(kt)] = \frac{s}{s^2-k^2}$. *(Hint: Use part (a) to help with parts (b) and (c).)*

**32** **(a)** Prove $L[\sin(kt)] = \frac{k}{s^2+k^2}$ using the definition of the Laplace Transform. **(b)** Prove $L[\cos(kt)] = \frac{s}{s^2+k^2}$ by taking advantage of the result in part (a) and the reduction formula for the Laplace Transform of a derivative. *(Hint: For part (a), you'll need to use a "double integration by parts".)*

**33** Suppose $f$ and $g$ are continuous functions on $[0,\infty)$ with $L[f] = F(s)$, and $L[g] = F(s-a)$ (that is to say, the Laplace Transform of $g$ is a translation of the Laplace Transform of $f$). Prove that $g(t) = e^{at}f(t)$.

**34** Prove the following Laplace Transform formulas by taking advantage of the result of Problem 2 above: , **(a)** $L[e^{at}\sin(bt)] = \frac{b}{(s-a)^2+b^2}$, **(b)** $L[e^{at}\cos(bt)] = \frac{(s-a)}{(s-a)^2+b^2}$, **(c)** $L[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}$.

**35** Another useful transform in the study of differential equations is the Fourier Transform which can be defined for a function $f(t)$ by the formula

$$F[f] = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\xi t}\ dt.$$

(Here, the transform is a function of $\xi$.) Verify the following reduction formula for differentiable functions $f$ that satisfy $\lim_{t\to\pm\infty} f(t) = 0$:

$$F[f'] = 2\pi i\xi F[f].$$

**36** Prove that $f(t) = e^{(t^2)}$ *is not* of exponential order.

**37** If $L[f(t)] = F(s)$, it can be proved (but you are not being asked to prove) that $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\sigma)\ d\sigma$, provided that $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists and is finite. Verify that this condition holds for $\frac{\sin(t)}{t}$, and then use this formula to find $L\left[\frac{\sin(t)}{t}\right]$.

**38** The Gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\ dt.$$

Verify that $\Gamma(1) = 1$. Use integration by parts to verify that $\Gamma(x+1) = x\Gamma(x)$. What other mathematical operation do these two properties remind you of?[1]

**39** Use the definition of the Laplace Transform and integrate by substitution to prove

$$L\left[t^a\right] = \frac{\Gamma(a+1)}{s^{a+1}}$$

for all $a > 0$.

[1]These properties should remind you of the factorial. In fact, it turns out that if $n$ is a non-negative integer then $n! = \Gamma(n+1)$.

CHAPTER 11

# Discontinuous Driving Functions

**Prototype Question:** Model the effect on a spring-mass system when the mass is hit with a hammer.

In this chapter we explore the type of initial value problems for which Laplace Transforms are our best-suited tool: non-homogeneous equations with discontinuous driving functions.

## Unit Step Functions and Characteristic Functions

The **unit step function** is defined by

$$\mathcal{U}(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}.$$

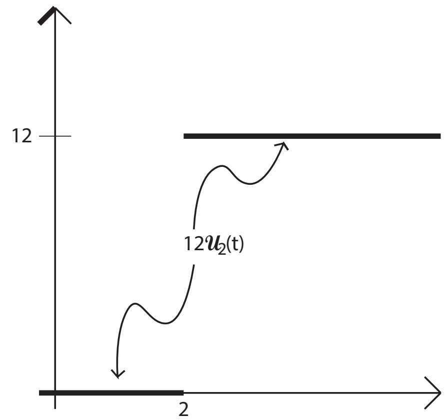
Notice that we do not bother defining $\mathcal{U}(0)$. That is because there is no natural way to define it that will be of practical value. Furthermore, we will mainly use these step functions inside integrands, and the value of a function at one point will not affect the definite integral.

We also define $\mathcal{U}_a$ as a translation of the unit step function $a$ units to the right (if $a < 0$, the translation would actually be to the left):

$$\mathcal{U}_a(t) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases}.$$

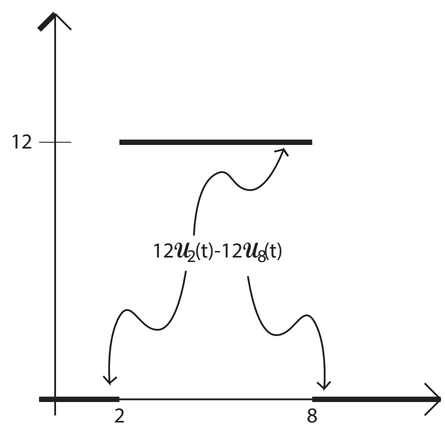
EXERCISE 1: Prove that $L[\mathcal{U}_a] = \begin{cases} \frac{1}{s} & \text{for } a \leq 0 \\ \frac{e^{-as}}{s} & \text{for } a > 0 \end{cases}.$

Unit step functions can be used to describe driving functions which are "switched on" at a certain moment in time. For example, a differential equation of the form $a\ddot{y} + b\dot{y} + cy = E(t)$ can be used to model the current in a simple electrical circuit, where $E(t)$ is the driving term corresponding to an external voltage source. If a 12-volt source is "turned on" at time $t = 2$ seconds, then we could model this with a driving term $E(t) = 12\mathcal{U}_2(t)$, which is illustrated in the following figure.




**EXERCISE 2:** Sketch the graphs of **(a)** $f(t) = 2\mathcal{U}_1(t)$, **(b)** $g(t) = 1 + 2\mathcal{U}_1(t)$ and **(c)** $h(t) = 3 - 2\mathcal{U}_1(t)$.

Expanding on the example of a voltage source described above, we could also imagine that the voltage source is "turned off" at, say, $t = 8$ seconds, as shown here:




The function $E(t)$ shown in the last figure can be represented as a difference of step functions by writing $E(t) = 12\mathcal{U}_2(t) - 12\mathcal{U}_8(t)$. We can think of the first term, $12\mathcal{U}_2(t)$, as "stepping up" by 12 units at $t = 2$, and the second term, $-12\mathcal{U}_8(t)$, as "stepping back down" at $t = 8$.

More generally, we can represent any function of the form

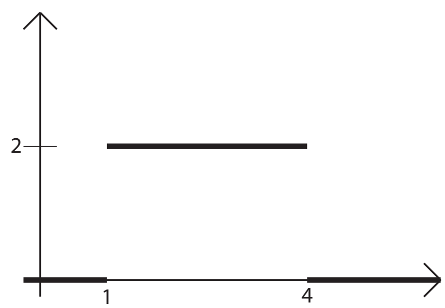
$$f(t) = \begin{cases} c & \text{if } a < t < b \\ 0 & \text{otherwise} \end{cases}$$

as a difference of step functions: $f(t) = c\,\mathcal{U}_a - c\,\mathcal{U}_b$. It is often useful to think of this as a basic building block for other functions, so we give it a name and its own notation: the function $\mathcal{U}_{a,b}(t)$ defined by

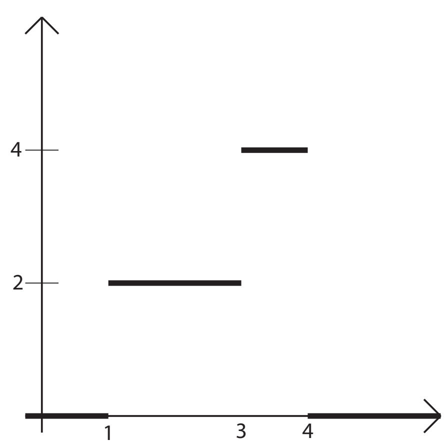
$$\mathcal{U}_{a,b}(t) = \mathcal{U}_a(t) - \mathcal{U}_b(t) = \begin{cases} 1 & \text{if } a < t < b \\ 0 & \text{otherwise} \end{cases}$$

is called the **characteristic function** (or the **indicator function**) **of the interval** $(a, b)$. Although we will use this notation at times to help us come up with a formula for a function, we will always choose to write our final answers in terms of step functions instead of characteristic functions.

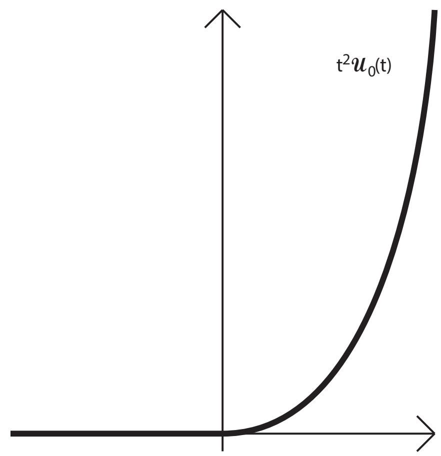
EXERCISE 3: Find a formula in terms of step functions for the function shown in the figure below. *(Hint: Begin by thinking of this as a sum of two characteristic functions. Then write the characteristic functions in terms of step functions and simplify.)*




EXERCISE 4: Find a formula in terms of step functions for the function shown in the figure below.



When we multiply a function $f(t)$ by a unit step function $\mathcal{U}_a(t)$, the resulting product gives us the same output as $f$ when $t > a$, and the output is $0$ when $t < a$. For example, here's a sketch of the graph of $g(t) = t^2\mathcal{U}_0(t)$:




**EXERCISE 5:** Sketch the graphs of **(a)** $f(t) = t\mathcal{U}_3(t)$ and **(b)** $f(t) = (t-3)\mathcal{U}_3(t)$.

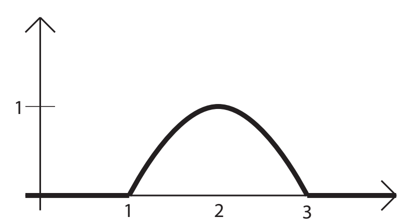
**EXAMPLE 1:** Sketch a graph of the function

$$f(t) = \begin{cases} 1-(t-2)^2 & \text{if } 1 < t < 3 \\ 0 & \text{otherwise} \end{cases},$$

and write a formula for $f(t)$ in terms of step functions.

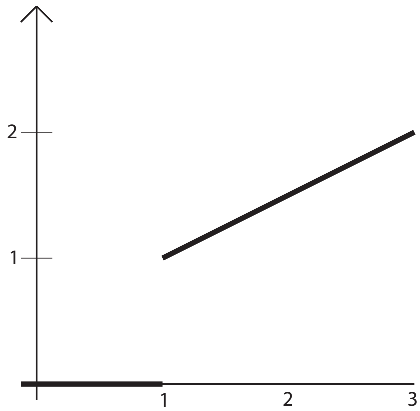
The graph of $f$ is shown in the figure at right. This function can be thought of as a product with the characteristic function on the interval $(1,3)$:

$$\begin{aligned} f(t) &= (1-(t-2)^2)\mathcal{U}_{1,3}(t) \\ &= (1-(t-2)^2)\left(\mathcal{U}_1(t) - \mathcal{U}_3(t)\right). \end{aligned}$$

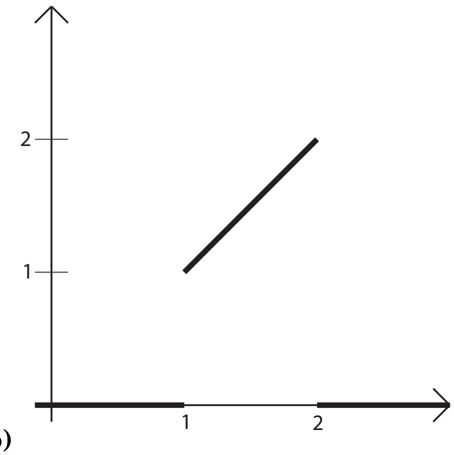



□

**EXERCISE 6:** Find formulas in terms of step functions for the functions whose graphs are shown in the following figures.




**(a)**




**(b)**

## Step Functions and Translations of Functions

Unit step functions are particularly useful when we try to work with Laplace Transforms of functions which have been translated. Observe that, for a function $f(t)$, the Laplace Transform of $f(t-a)$ is not necessarily very simple, as this calculation shows:

$$\begin{aligned} L[f(t-a)] &= \int_0^\infty f(t-a)e^{-st}\,dt \\ &= \int_{-a}^\infty f(u)e^{-s(u+a)}\,du \qquad (u = t-a,\ du = dt) \\ &= \int_{-a}^0 f(u)e^{-s(u+a)}\,du + \int_0^\infty f(u)e^{-su}e^{-as}\,du \\ &= \int_{-a}^0 f(u)e^{-s(u+a)}\,du + e^{-as}L[f(t)]. \end{aligned}$$

If the first integral in the last line is complicated, this may not be very useful. On the other hand, if the first integral in the last line were just zero, it wouldn't be very complicated at all! So to make sure that it is zero, when we translate a function $a$ units, as in $f(t-a)$, we will also multiply it by $\mathcal{U}_a$ (this is equivalent to imagining that $f(t) = 0$ for $t < 0$ before it is translated):

$$
\begin{aligned}
L[f(t-a)\mathcal{U}_a(t)] &= \int_0^\infty f(t-a)\mathcal{U}_a(t)e^{-st}\,dt \\
&= \int_0^a f(t-a)\cdot 0\cdot e^{-st}\,dt + \int_a^\infty f(t-a)\cdot 1\cdot e^{-st}\,dt \\
&= \int_a^\infty f(t-a)e^{-st}\,dt \\
&= \int_0^\infty f(u)e^{-s(u+a)}\,dt \qquad (u=t-a,\ du=dt) \\
&= e^{-as}\int_0^\infty f(u)e^{-su}\,du \\
&= e^{-as}L[f(t)].
\end{aligned}
$$

The calculation above says that $L[\mathcal{U}_a f(t-a)] = e^{-as}L[f(t)]$; however, this formula seems to be difficult for many students to remember and use correctly in this form. To simplify it, let's introduce a **shift-and-cutoff** operator, $S_a$, which acts on functions as follows: if $f$ is a function defined on $\mathbb{R}$, then $S_a(f)$ is another function defined on $\mathbb{R}$ according to the rule

$$
S_a(f)(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}.
$$

The effect of the operator $S_a$ is to shift the graph $a$ units to the right and then "cutoff" the function by setting it equal to zero for all $t < a$.

Thus, $S_a(f)(t)$ is also equal to $\mathcal{U}_a(t)f(t-a)$, which means that the rule we calculated above can be expressed as follows:

**Laplace Transform of a Shifted-and-Cutoff Function**

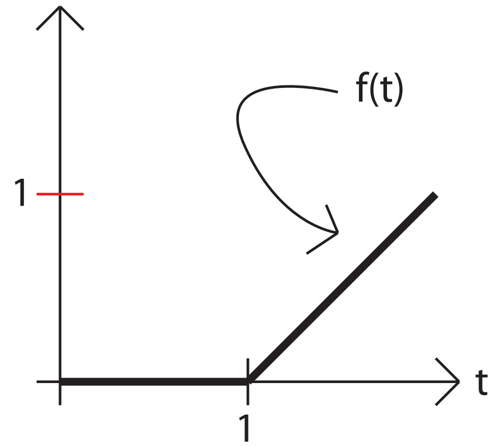
$$
L[S_a(f)] = e^{-as}L[f].
$$

Example 2: The Laplace Transform of $f(t) = (t-3)^2\mathcal{U}_3(t)$ is

$$
\begin{aligned}
L[(t-3)^2\mathcal{U}_3(t)] &= L[S_3\left(t^2\right)] \\
&= e^{-3s}L[t^2] \\
&= e^{-3s}\frac{2}{s^3}.
\end{aligned}
$$

□

EXERCISE 7: Find the Laplace Transform of the function in the figure below by expressing it as $S_a(f)$ where $f(t) = t$ (that is, express it as a shift-and-cutoff of the linear function $f(t) = t$) for some appropriate value of $a$.




The corresponding rule for the Inverse Laplace Transform can be stated as follows: If $L^{-1}[F(s)] = f(t)$, then $L^{-1}[e^{-as}F(s)] = f(t-a)\mathcal{U}_a(t)$. In terms of the shift-and-cutoff operator, we write the rule as follows:

**Inverse Laplace Transform of $e^{-as}F(s)$**

$$L^{-1}[e^{-as}F(s)] = S_a\left(L^{-1}[F(s)]\right) = S_a(f), \text{where} f = L^{-1}[F]$$

EXAMPLE 3: The Inverse Laplace Transform of $F(s) = \frac{e^{-2s}}{s-4}$ is $e^{4(t-2)}\mathcal{U}_2(t)$. We obtain this by recognizing that the Inverse Laplace Transform of $\frac{1}{s-4}$ is $e^{4t}$, which we then translate to the right by two units and multiply by the step function $\mathcal{U}_2$:

$$\begin{aligned} L^{-1}\left[e^{-2s}\frac{1}{s-4}\right] &= S_2\left(L^{-1}\left[\frac{1}{s-4}\right]\right) \\ &= S_2\left(e^{4t}\right) \\ &= e^{4(t-2)}\mathcal{U}_2(t). \end{aligned}$$

□

EXERCISE 8: Find the Inverse Laplace Transform of $F(s) = \frac{e^{-4s}}{s^3}$.

We are now ready to use these step functions as driving functions in differential equations.

EXAMPLE 4: Solve the differential equation $\dot{y} - y = \begin{cases} 0 & \text{if } t < 1 \\ 2 & \text{if } t > 1 \end{cases}$ subject to the initial condition $y(0) = 0$.

**Solution:** First, we rewrite the driving function as $2\mathcal{U}_1(t)$. Then we transform the differential equation:

$$\begin{aligned} L[\dot{y} - y] &= L[2\mathcal{U}_1(t)] \\ L[\dot{y}] - L[y] &= 2\frac{e^{-s}}{s} \\ sL[y] - y(0) - L[y] &= 2\frac{e^{-s}}{s} \end{aligned}$$

where we used the reduction formula for $L[\dot{y}]$ in the last line. Now plug in the initial condition $y(0) = 0$, collect like terms and isolate $L[y]$:

$$(s-1)L[y] = 2\frac{e^{-s}}{s}$$

so

$$L[y] = 2e^{-s}\left(\frac{1}{s(s-1)}\right).$$

We can use partial fractions to rewrite the expression in parentheses on the right:

$$L[y] = 2e^{-s}\left(\frac{1}{s-1} - \frac{1}{s}\right).$$

Therefore

$$\begin{aligned} y &= 2S_1\left(L^{-1}\left[\frac{1}{s-1} - \frac{1}{s}\right]\right) \\ &= 2S_1\left(e^t - 1\right) \\ &= 2\left(e^{t-1} - 1\right)\mathcal{U}_1(t). \end{aligned}$$

It is often more useful (and more pleasant) to express the result in piecewise notation, without the step function:

$$y = \begin{cases} 0 & \text{if } t < 1 \\ 2(e^{(t-1)} - 1) & \text{if } t > 1 \end{cases}.$$

□

EXAMPLE 5: Solve the initial value problem $\dot{y} + y = \begin{cases} 0 & \text{if } t < 5 \\ 2(t-5) & \text{if } t > 5 \end{cases}$, $y(0) = 0$.

**Solution:** The driving function can be written as $f(t) = 2(t-5)\mathcal{U}_5(t)$, or $f(t) = S_5(2t)$, so we have

$$\dot{y} + y = S_5(2t).$$

Taking Laplace Transforms of both sides gives us

$$L[\dot{y}] + L[y] = L[S_5(2t)],$$

or

$$sL[y] - y(0) + L[y] = e^{-5s}L[2t],$$

and thus

$$(s+1)L[y] = e^{-5s}\frac{2}{s^2}.$$

Isolating $L[y]$ gives us

$$L[y] = e^{-5s}\frac{2}{s^2(s+1)}.$$

A partial fraction decomposition for $\frac{2}{s^2(s+1)} = \frac{As+B}{s^2} + \frac{C}{s+1}$ gives us the coefficients $A = -2,\ B = 2,\ C = 2$, so we have

$$L[y] = e^{-5s}\left(\frac{-2s+2}{s^2} + \frac{2}{s+1}\right).$$

Splitting up the first fraction inside parentheses on the right side and simplifying yields

$$L[y] = e^{-5s}\left(\frac{-2}{s} + \frac{2}{s^2} + \frac{2}{s+1}\right).$$

Taking the Inverse Laplace Transform now gives us

$$y = S_5\left(-2 + 2t + 2e^{-t}\right).$$

In piecewise notation, this is

$$y = \begin{cases} 0 & \text{if } t < 5 \\ -2 + 2(t-5) + 2e^{-(t-5)} & \text{if } t > 5 \end{cases}.$$

$\square$

**EXERCISE 9:** Solve the following initial value problems using Laplace Transforms:

**(a)** $\dot{y} + 2y = \begin{cases} 0 & \text{if } t < 1 \\ 9 & \text{if } t > 1 \end{cases}, \ y(0) = 0$

**(b)** $\dot{y} - y = \begin{cases} 0 & \text{if } t < 1 \\ (t-1)^2 & \text{if } t > 1 \end{cases}, \ y(0) = 0.$

In typical applications, Laplace Transforms are frequently used to solve second-order problems. The process is generally the same.

EXAMPLE 6: Solve the initial value problem $\ddot{y} - y = f(t),\ y(0) = 0, \dot{y}(0) = 1$, where the driving function is

$$f(t) = \begin{cases} 0 & \text{for } t < 2 \\ 1 & \text{for } 2 < t < 5 \\ 0 & \text{for } t > 5 \end{cases}.$$

**Solution:** We rewrite the driving function as $f(t) = \mathcal{U}_2 - \mathcal{U}_5$. Then we transform the differential equation:

$$\begin{aligned} L[\ddot{y} - y] &= L[\mathcal{U}_2 - \mathcal{U}_5] \\ L[\ddot{y}] - L[y] &= L[\mathcal{U}_2] - L[\mathcal{U}_5] \\ sL[\dot{y}] - \dot{y}(0) - L[y] &= \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s} \\ s(sL[y] - y(0)) - \dot{y}(0) - L[y] &= \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s}. \end{aligned}$$

Then insert the initial conditions and solve for $L[y]$:

$$\begin{aligned} s(sL[y] - 0) - 1 - L[y] &= \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s} \\ (s^2 - 1)L[y] &= 1 + \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s} \\ L[y] &= \frac{1}{s^2 - 1} + (e^{-2s} - e^{-5s}) \left( \frac{1}{s(s^2 - 1)} \right). \end{aligned}$$

We will need two partial-fractions decompositions:

$$\frac{1}{s^2 - 1} = \frac{(1/2)}{s - 1} - \frac{(1/2)}{(s + 1)}$$

and

$$\frac{1}{s(s^2 - 1)} = -\frac{1}{s} + \frac{(1/2)}{s - 1} + \frac{(1/2)}{s + 1}.$$

Insert these into the formula for $L[y]$ to obtain

$$\begin{aligned} L[y] &= \frac{(1/2)}{s - 1} - \frac{(1/2)}{(s + 1)} + (e^{-2s} - e^{-5s}) \left( -\frac{1}{s} + \frac{(1/2)}{s - 1} + \frac{(1/2)}{s + 1} \right) \\ &= \frac{(1/2)}{s - 1} - \frac{(1/2)}{(s + 1)} + e^{-2s} \left( -\frac{1}{s} + \frac{(1/2)}{s - 1} + \frac{(1/2)}{s + 1} \right) \\ &\quad - e^{-5s} \left( -\frac{1}{s} + \frac{(1/2)}{s - 1} + \frac{(1/2)}{s + 1} \right). \end{aligned}$$

Consequently,

$$\begin{aligned}
y(t) &= \frac{1}{2}e^t - \frac{1}{2}e^{-t} + S_2\left(-1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t}\right) - S_5\left(-1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t}\right) \\
&= \frac{1}{2}e^t - \frac{1}{2}e^{-t} + \left(-1 + \frac{1}{2}e^{(t-2)} + \frac{1}{2}e^{-(t-2)}\right)\mathcal{U}_2(t) - \left(-1 + \frac{1}{2}e^{(t-5)} + \frac{1}{2}e^{-(t-5)}\right)\mathcal{U}_5(t) \\
&= \begin{cases} \frac{1}{2}e^t - \frac{1}{2}e^{-t} & \text{for } t < 2 \\ \frac{1}{2}e^t - \frac{1}{2}e^{-t} - 1 + \frac{1}{2}e^{(t-2)} + \frac{1}{2}e^{-(t-2)} & \text{for } 2 < t < 5 \\ \frac{1}{2}e^t - \frac{1}{2}e^{-t} + \frac{1}{2}e^{(t-2)} + \frac{1}{2}e^{-(t-2)} - \frac{1}{2}e^{(t-5)} - \frac{1}{2}e^{-(t-5)} & \text{for } t > 5 \end{cases} .
\end{aligned}$$

□

EXERCISE 10: Solve the following initial value problems using Laplace Transforms:

**(a)** $\ddot{y} + 4y = \begin{cases} 0 & \text{if } t < \pi \\ 1 & \text{if } t > \pi \end{cases}, \; y(0) = 0, \; \dot{y}(0) = 0$

**(b)** $\ddot{y} + 4y = \begin{cases} 1 & \text{if } t < \pi \\ 0 & \text{if } t > \pi \end{cases}, \; y(0) = 0, \; \dot{y}(0) = 0$

**(c)** $\ddot{y} + y = \begin{cases} 0 & \text{if } t < 2 \\ 3(t-2) & \text{if } t > 2 \end{cases}, \; y(0) = 0, \; \dot{y}(0) = 0.$

## DELTA (IMPULSE) FUNCTIONS

Step functions can be used to describe driving functions that 'start' or 'stop' at definite instants in time, such as when a switch is closed for a certain time interval allowing an external voltage source to drive the circuit. But we also sometimes want to model very short bursts of driving activity, such as a near-instantaneous jolt, and it turns out that the best means for this is with a so-called delta function.

The **delta function with pole at** $a$ is denoted by $\delta_a(x)$ and is defined by the following property:

$$\int_I f(x)\delta_a(x)\,dx = \begin{cases} f(a) & \text{if } a \in I \\ 0 & \text{if } a \notin I \end{cases}$$

for all continuous functions $f$ and for all intervals $I \subset \mathbb{R}$. We will sometimes write $\delta$ in place of $\delta_0$. Then we can also interpret $\delta_a(x)$ as $\delta(x - a)$.

An immediate consequence of this definition, if we use the constant function $f(x) = 1$, is that

$$\int_{-\infty}^{\infty} \delta(x)\ dx = 1,$$

however, on any interval $I$ that does not contain $0$,

$$\int_I \delta(x)\ dx = 0.$$

Thinking in terms of areas under the graph of $\delta$, it should not take the reader long to realize that this is impossible – that there is no *function* which can have both of these properties. Indeed, $\delta_a$ is actually a *distribution* (also called a *generalized function*). In contrast to functions which have a defined value at each point of their domains, often distributions can only be thought of as having average values over intervals.

Distributions are often studied in detail in an advanced course on Functional Analysis. Once defined, distributions can be multiplied by smooth functions, and the results can be integrated on intervals, but defining distributions carefully and illustrating just how all of this works in detail is well beyond the scope of this textbook. At this level, all we will need are the two properties described above and their consequences.

To illustrate the utility of this object as a driving function, let's consider the differential equation $\ddot{y} = \delta_2(t)$, with the initial conditions $y(0) = 0$ and $\dot{y}(0) = 0$. Because this is a fairly uncomplicated differential equation, we can solve it just by integrating. Integrate both sides over the interval $(0, t)$ (let's use $s$ as the variable of integration) to get

$$\int_0^t \ddot{y}(s)\ ds = \int_0^t \delta_2(s)\ ds.$$

The left side is just $\dot{y}(t) - \dot{y}(0)$, and the initial condition $\dot{y}(0) = 0$ allows us to just write the left side as $\dot{y}(t)$, so we have $\dot{y}(t) = \int_0^t \delta_2(s)\ ds$.

The right side of the equation is now either equal to 1 (if the domain of integration includes 2) or 0 (if it does not). The domain of integration includes 2 if $t > 2$, so we can actually write the right side as $\mathcal{U}_2(t)$. Therefore

$$\dot{y}(t) = \mathcal{U}_2(t).$$

Let's integrate one more time to finish up:

$$\int_0^t \dot{y}(s)\ dt = \int_0^t \mathcal{U}_2(s)\ ds,$$

and the left side will simplify to just $y(t)$ (since $y(0) = 0$); the right side will simplify to 0 if $t < 2$, and if $t > 2$ then the right side will be

$$\int_0^t \mathcal{U}_2(s)\ ds = \int_0^2 \mathcal{U}_2(s)\ ds + \int_2^t \mathcal{U}_2(s)\ ds = 0 + \int_2^t 1\ ds = (t-2).$$

Therefore we have

$$y(t) = (t-2)\mathcal{U}_2(t) = \begin{cases} 0 & \text{if } t < 2 \\ (t-2) & \text{if } t > 2 \end{cases}.$$

This example illustrates how a delta function for a driving term provides an instantaneous change to the first derivative of the solution. (Notice how $\dot{y}$ above changes from 0 to 1 exactly at $t = 2$.) One way to visualize this is with a spring-mass system, and to think of the driving function provided by $\delta_a$ as representing the hitting of the mass with a hammer at time $t = a$, imparting a sudden change in the mass' momentum. In the language of physics, we would say that, as a driving function, $\delta_a$ imparts one unit of impulse to the system (in physics, **impulse** is a constant force multiplied by time, or a non-constant force integrated over an interval of time). Because of this physical interpretation, $\delta$ is also called an **impulse function**.

A unit of impulse could be imparted by a constant force over a given time interval. For example, the driving function $\mathcal{U}_1 - \mathcal{U}_2$ will impart one unit of impulse (such as $1\ N \cdot s$), over the time interval $1 < t < 2$. Over a smaller period of time, the same impulse could be delivered by a greater-magnitude force, such as that modeled by $2\,(\mathcal{U}_1 - \mathcal{U}_{1.5})$, and so on. The point of the delta function is that it models the transfer of impulse as happening instantaneously.

EXAMPLE 7: Consider a horizontal spring-mass system with $m = 2\ kg$, $b = 4\frac{N \cdot s}{m}$ and $k = 202\frac{N}{m}$. At time $t = 3$, an impulse of $5N \cdot s$ is delivered in a nearly-instantaneous collision with the mass, in the direction of compressing the spring. We could model this situation over a very short period of time with step functions (say, with a 0.001-second collision):

$$2\ddot{y} + 4\dot{y} + 202y = -5000\,(\mathcal{U}_3 - \mathcal{U}_{3.001})\,;$$

or we could imagine that the transfer of impulse happens instantaneously and model it with a delta function:

$$2\ddot{y} + 4\dot{y} + 202y = -5\delta_3(t).$$

□

Laplace transforms turn out to be a great tool for solving ordinary differential equations involving impulse functions. The key fact is:

**Laplace Transform of a Delta Function**

$$L[\delta_a] = e^{-sa} \quad \text{for } a > 0.$$

EXERCISE 11: Verify the formula $L[\delta_a] = e^{-as}$ for $a > 0$ using the defining properties of $\delta_a$.

EXAMPLE 8: Solve the differential equation $2\ddot{y} + 8y = 5\delta_4(t)$ together with the initial conditions $y(0) = 0.5,\ \dot{y}(0) = 0$ using the Laplace Transform.

**Solution:** Take the Laplace Transform of both sides to get

$$L[2\ddot{y} + 8y] = L[5\delta_4(t)]$$

which simplifies to

$$2s^2L[y] - 2sy(0) - 2\dot{y}(0) + 8L[y] = 5e^{-4s}.$$

Inserting the initial conditions and isolating $L[y]$ gives us

$$L[y] = 2.5\frac{e^{-4s}}{s^2+4} + 0.5\frac{s}{s^2+4}.$$

Take the Inverse Laplace Transform of both sides to obtain

$$y = 1.25\mathcal{U}_4(t)\sin(2(t-4)) + 0.5\cos(2t).$$

We can write this without step-function notation as

$$y = \begin{cases} 0.5\cos(2t) & \text{if } t < 4 \\ 0.5\cos(t) + 1.25\sin(2(t-4)) & \text{if } t > 4 \end{cases}.$$

□

EXERCISE 12: Solve the following initial value problems:

**(a)** $\frac{d^2y}{dx^2} + 9y = 3\delta_2(x),\ y(0) = 0,\ y'(0) = 0.$

**(b)** $y'' + 4y' + 4y = -\delta_3(x),\ y(0) = 1,\ y'(0) = 1.$

**(c)** $\ddot{y} + \dot{y} - 2y = -\delta_1(t),\ y(0) = 1,\ \dot{y}(0) = 1.$

# Additional Exercises

Write the function in piecewise notation. Simplify if possible.

**13** $3\mathcal{U}_2(t) - \mathcal{U}_4(t)$

**14** $t + (1-t)\mathcal{U}_1(t)$

**15** $2 - 2t\mathcal{U}_1(t)$

**16** $\mathcal{U}_\pi(t)\sin(2t)$

**17** $S_1\left(e^{2t}\right)$

**18** $S_3\left(\cos(4t)\right)$

Write the given function in terms of step functions.

**19** $f(t) = \begin{cases} 0 & \text{if } t < 2 \\ 3 & \text{if } t > 2 \end{cases}$

**20** $g(t) = \begin{cases} 3 & \text{if } 2 < t < 4 \\ 0 & \text{otherwise} \end{cases}$

**21** $h(t) = \begin{cases} 2t & \text{if } t < 4 \\ 8 & \text{if } t > 4 \end{cases}$

**22** $v(t) = \begin{cases} 0 & \text{if } t < 1 \\ t-1 & \text{if } 1 < t < 2 \\ 1 & \text{if } t > 2 \end{cases}$

Solve the initial value problem using the Laplace transform.

**23** $\ddot{y} + y = \begin{cases} 0 & \text{if } t < 3 \\ 12 & \text{if } t > 3 \end{cases}, y(0) = 0, \dot{y}(0) = 0$

**24** $\ddot{y} - y = \begin{cases} 0 & \text{if } t < 1 \\ 2 & \text{if } t > 1 \end{cases}, y(0) = 0, \dot{y}(0) = 1$

**25** $\ddot{y} + 4y = \begin{cases} 5 & \text{if } t < \pi \\ 0 & \text{if } t > \pi \end{cases}, y(0) = 0, \dot{y}(0) = 0$

**26** $\ddot{y} - 9y = \begin{cases} 1 & \text{if } 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}, y(0) = 0, \dot{y}(0) = 1$

**27** $\dot{y} - 2y = \delta_3(t), y(0) = 0$

**28** $\dot{y} + 3y = \delta_1(t), y(0) = 1$

**29** $\ddot{y} + \dot{y} = \delta_4(t), y(0) = 1$

**30** $\ddot{y} + y = \delta_{2\pi}(t), y(0) = 0, \dot{y}(0) = 2$

**31** $\ddot{y} + 4\dot{y} + 5y = \delta_1(t), y(0) = 0, \dot{y}(0) = 0$

**32** $\ddot{y} + 3\dot{y} + 2y = \delta_2(t), y(0) = 1, \dot{y}(0) = 0$

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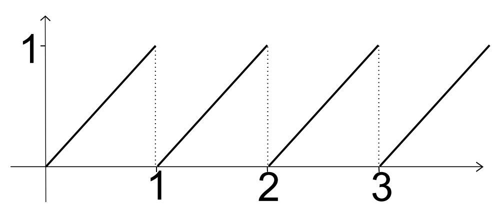
**33** Find a formula in terms of step functions for the periodic "sawtooth" function shown in the graph below. *(Hint: Your formula should involve an infinite sum; write it using $\sum$ notation.)*




**34** A mass of $3\ kg$ is attached to the end of a spring with spring constant $k = 48\frac{N}{m}$, and there is no damping. The mass is initially at rest with no outside forces acting on the spring-mass system (including no gravity).

At time $t = 4$ a hammer strikes the mass with $1N \cdot s$ of impulse in the direction which stretches the spring. Model this as a differential equation with a delta function, solve it, and graph the resulting solution.

**35** The delta function $\delta_a(t)$ can be thought of in some sense as a limit of the functions $\frac{1}{h}\left(\mathcal{U}_a(t) - \mathcal{U}_{a+h}(t)\right)$ as $h \searrow 0$. Illustrate this by **(a)** finding a function $y$ which solves $\ddot{y} = \frac{1}{h}\left(\mathcal{U}_a(t) - \mathcal{U}_{a+h}(t)\right)$ and writing it in piecewise notation, **(b)** taking the limit of the result of part (a) as $h \searrow 0$, and **(c)** comparing the result of (b) with the solution of $\ddot{y} = \delta_a(t)$.

**36** The problem above suggests that the delta function $\delta_a$ can be thought of as a derivative of a unit step function $\mathcal{U}_a$. Make this explicit by calculating the value of $\int_{-\infty}^{t} \delta_a(x)\, dx$ for $t \neq a$ and explaining how the results suggests a relationship between $\delta_a$ and $\mathcal{U}_a$.

**37** Find the Fourier Transform of the translated delta function, $F[\delta_a(t)]$. *(Refer to the definition of the Fourier Transform given in Problem 10.4, and use the defining properties of the delta function.)*

**38** Express the function $f(t) = \frac{\sqrt{t^2}+t}{2t}$ in terms of unit step functions. *(Hint: you can guess the answer by graphing $f(t)$ first; once you know what the answer should be, explain how to see this result from the formula itself.)* This question illustrates the fact that we don't really need to resort to piecewise notation to define step functions – that just happens to be an easier way to do it.

CHAPTER 12

# Representation Formulas and Convolutions

**Prototype Question:** Find a formula for the solution of $\ddot{y} - 4y = f(t)$, $y(0) = 0)$, $\dot{y}(0) = 0$ which can be evaluated to any desired accuracy for any given function $f(t)$.

In this section, we will write down several integral formulas for solutions of ODE. These formulas are especially useful when it is difficult or impossible to write down closed form anti-derivatives.

Let us begin by considering the general first-order linear equation in standard form:

$$\frac{dy}{dx} + p(x)y = q(x).$$

Suppose we seek a solution that satisfies the initial condition $y(x_0) = y_0$. On a domain $I$ containing $x_0$ and where $p(x)$ and $q(x)$ are continuous, we would normally introduce any integrating factor of the form $\exp\left(\int p(x)dx\right)$. However, let us now specify a particular anti-derivative as the argument of the exponential function (by taking advantage of the Fundamental Theorem of Calculus): we will use the integrating factor $\mu(x) = \exp\left(\int_{x_0}^{x} p(s)ds\right)$.

$$\frac{d}{dx}\left[\exp\left(\int_{x_0}^{x} p(s)ds\right) y\right] = q(x)\exp\left(\int_{x_0}^{x} p(s)ds\right).$$

And again, when we anti-differentiate both sides of this equation, we will use a particular anti-derivative on the right side:

$$\exp\left(\int_{x_0}^{x} p(s)ds\right) y = C + \int_{x_0}^{x} q(t)\exp\left(\int_{x_0}^{t} p(s)ds\right) dt.$$

(Note the presence of the constant of integration $C$ on the right side; also, we changed the variable from $x$ to $t$ on the right side before integrating to avoid conflict with the $x$ that appears in the upper limit of integration.) If we insert the initial condition at this point, notice that both definite integrals will be zero (since the upper and lower limits of integration will be identical), and we can see that $y_0 = C$, so now we have

$$\exp\left(\int_{x_0}^{x} p(s)ds\right) y = y_0 + \int_{x_0}^{x} q(t)\exp\left(\int_{x_0}^{t} p(s)ds\right) dt.$$

Isolating $y$ yields

**Representation Formula for First-Order Linear Initial Value Problems**

If $p(x)$ and $q(x)$ are continuous on an open interval $I$ containing $x_0$, then the unique solution of $y' + p(x)y = q(x)$ on $I$ is given by

$$y = \exp\left(-\int_{x_0}^{x} p(s)ds\right)\left(y_0 + \int_{x_0}^{x} q(t)\exp\left(\int_{x_0}^{t} p(s)ds\right)dt\right).$$

This is a representation formula that can be used for any first-order linear IVP in standard form. The integrals are guaranteed to be defined on any domain where $p$ and $q$ are both continuous. Even when we cannot write down an anti-derivative for the functions $p$ and $q$, we can often still write down approximate values of $y(x)$ by using a numerical method to approximate the integrals (such as the Trapezoid Rule, Simpson's Rule or another algorithm run by a calculator or computer).

EXAMPLE 1: Suppose $y$ satisfies the initial value problem $y' + 2xy = 1,\ y(0) = 2$. Find the approximate value of $y(1)$.

In theory the method of integrating factors will apply here, but we will run into some difficulty if we try to actually calculate the exact solution because we will end up trying to anti-differentiate $e^{(x^2)}$, and there is no closed-form anti-derivative for this function. However, we can apply the representation formula above (which is really just the method of integrating factors anyway) with $p(x) = 2x$ and $q(x) = 1$ to get

$$\begin{aligned} y(x) &= \exp\left(-\int_0^x 2s\ ds\right)\left(2 + \int_0^x 1\exp\left(\int_0^t 2s\ ds\right)dt\right) \\ &= e^{-(x^2)}\left(2 + \int_0^x e^{(t^2)}\ dt\right) \end{aligned}$$

In particular,

$$y(1) = e^{-1}\left(2 + \int_0^1 e^{(t^2)}\ dt\right).$$

The integral on the right side can be calculated to any desired accuracy. Simpson's Rule with $n = 10$ subdivisions gives us $\int_0^1 e^{(t^2)}dt \approx 1.46268$. Therefore $y(1) \approx 1.27385$. (Careful use of the error estimate for Simpson's rule and careful rounding would allow us to conclude that the accuracy of this answer is better than $10^{-4}$.) $\square$

EXERCISE 1: Use the above representation formula to write down a solution to $y' + xy = 1$, $y(0) = 1$. Then give an approximate value of $y(2)$ by using a numerical method or computer to evaluate the definite integrals involved.

Another representation formula can be obtained using the method of Laplace Transforms. The key idea necessary is an operation on functions which is called convolution, so we must take a brief excursion to define this operation and examine some of its properties.

In pre-calculus we learn about several operations that combine functions. The first few operations we explore are based on arithmetic: addition, subtraction, multiplication and division of functions. Then we introduce a new operation that is different from what one has studied before: composition of functions. Now will explore yet another way of combining functions which is of particular interest when working with Laplace Transforms. This operation is defined in terms of definite integrals.

The **convolution** of two integrable functions $f$ and $g$ defined on $[0, \infty)$ is written as $f * g$ and is defined by the formula

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau.$$

EXAMPLE 2: Let $f(t) = t$ and $g(t) = e^t$. Compute $f * g$.

$$\begin{aligned}
f * g(t) &= \int_0^t f(\tau)g(t-\tau)\, d\tau \\
&= \int_0^t \tau e^{t-\tau}\, d\tau \\
&= e^t \int_0^t \tau e^{-\tau}\, d\tau \\
&= e^t \left( -\tau e^{-\tau} + \int e^{-\tau}\, d\tau \right)\Big|_0^t \\
&= e^t \left( -\tau e^{-\tau} - e^{-\tau} \right)\Big|_0^t \\
&= e^t(-te^{-t} - e^{-t} + 0 + 1) \\
&= -t - 1 + e^t.
\end{aligned}$$

$\square$

The first fact we will prove about convolution is that it is commutative: $f * g = g * f$. Indeed,

$$\begin{aligned}
f * g &= \int_{\tau=0}^{\tau=t} f(\tau)g(t-\tau)\, d\tau \\
&= -\int_{u=t}^{u=0} f(t-u)g(u)\, du \quad \text{(substituting } u = t-\tau\text{)} \\
&= \int_{u=0}^{u=t} g(u)f(t-u)\, du \\
&= g * f.
\end{aligned}$$

Therefore we need not specify the order of the two functions in a convolution.

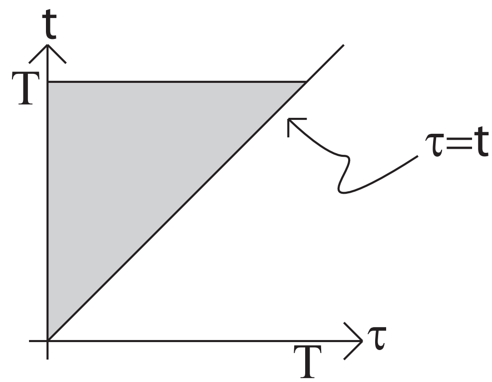
EXERCISE 2: Prove (by giving a counterexample) that the composition of functions $f \circ g$, defined by $(f \circ g)(x) = f(g(x))$, is not a commutative operation.

EXERCISE 3: Find the convolution of the functions $t$ and $t^2$.

Next, we examine what happens when we take the Laplace Transform of a convolution.

$$\begin{aligned}
L[f * g] &= \int_0^\infty f * g(t)e^{-st}\, dt \\
&= \lim_{T\to\infty} \int_0^T \int_0^t f(\tau)g(t-\tau)e^{-st}\, d\tau\, dt \\
&= \lim_{T\to\infty} \int_0^T \int_\tau^T f(\tau)g(t-\tau)e^{-st}\, dt\, d\tau \qquad (*) \\
&= \lim_{T\to\infty} \int_0^T \int_0^{T-\tau} f(\tau)g(u)e^{-s(u+\tau)}\, du\, d\tau \\
&= \int_0^\infty \int_0^\infty f(\tau)g(u)e^{-s\tau}e^{-su}\, du\, d\tau \\
&= \left(\int_0^\infty f(\tau)e^{-s\tau}\, d\tau\right)\left(\int_0^\infty g(u)e^{-su}\, du\right) \\
&= (L[f])(L[g]).
\end{aligned}$$

In the line marked (*) we changed the order of integration, and the following figure illustrates how we obtained the new limits of integration:




What this result shows is that the Laplace Transform of a convolution of two functions is just the product of their Laplace Transforms. This fact is valuable to us because it helps us to find more inverse transforms.

**Laplace Transform of a Convolution**

$$L[f * g] = L[f]L[g].$$

Equivalently,

$$L^{-1}[F(s)G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)].$$

EXAMPLE 3: The inverse Laplace Transform of $\frac{1}{(s-a)^2}$ is $te^{at}$ since

$$\begin{aligned} L^{-1}\left[\frac{1}{s-a}\frac{1}{s-a}\right] &= L^{-1}\left[\frac{1}{s-a}\right] * L^{-1}\left[\frac{1}{s-a}\right] \\ &= e^{at} * e^{at} \\ &= \int_0^t e^{a\tau}e^{a(t-\tau)}\, d\tau \\ &= \int_0^t e^{at}\, d\tau \\ &= \tau e^{at}\Big|_0^t \\ &= te^{at}. \end{aligned}$$

□

EXERCISE 4: Use the result of Example 3 above to solve the IVP $y'' + 10y' + 25y = 0,\ y(0) = 1,\ y'(0) = 2$ via Laplace Transforms.

EXERCISE 5: Find $L^{-1}\left[\frac{1}{s(s-1)}\right]$ two ways: (a) using convolutions and (b) using partial fractions.

Now we have the necessary tool to develop more representation formulas.

EXAMPLE 4: Find a formula for the solution of the initial value problem $\dot{y} + 2y = f(t)$, $y(0) = 0$.

Taking the Laplace Transform of each side of the differential equation produces

$$L[\dot{y} + 2y] = L[f],$$

so that

$$sL[y] - y(0) + 2L[y] = L[f],$$

and using the initial condition then isolating $L[y]$ yields

$$L[y] = L[f]\frac{1}{s-1}.$$

If we know what $L[f]$ is, we might be able to evaluate this by hand, but only if we are able to look up the necessary inverse transforms in a table. However, that is not necessary, because we come to this battle armed with convolutions! Recall that $L[f * g] = L[f]L[g]$, and inverting that rule here with $g = L^{-1}\left[\frac{1}{s-1}\right] = e^t$ gives us

$$y = f(t) * e^t,$$

or

$$y = \int_0^t f(\tau)e^{t-\tau}\, d\tau.$$

□

This formula can be applied even if we do not know the Laplace Transform of $f$. For example, if $f(t) = \tan(t)$, and we want to know $y(0.5)$, then

$$y(0.5) = \int_0^{0.5} \tan(\tau)e^{0.5-\tau}\, d\tau = 0.155.$$

This approach gives us a numerical approximation, just like a technique such as Euler's Method would. The advantage here is that we can obtain any desired accuracy provided we know how to approximate the necessary integral within the prescribed level of error.

EXERCISE 6: Use Laplace Transforms and convolution to find an integral representation formula for $y(t)$ where $y$ satisfies the initial value problem $\dot{y} + 4y = \sec(t)$, $y(0) = 0$, and use it to find an approximate value of $y(0.2)$.

EXERCISE 7: Use Laplace Transforms and convolution to find an integral representation formula for $y(t)$ where $y$ satisfies the initial value problem $\ddot{y} + 4y = \tan(t)$, $y(0) = 0$, $\dot{y}(0) = 0$, and use it to find an approximate value of $y(0.3)$.

EXERCISE 8: Use Laplace Transforms and convolution to find an integral representation formula for $y(t)$ where $y$ satisfies the initial value problem $\ddot{y} - 4\dot{y} + 3y = e^{(t^2)}$, $y(0) = 1$, $\dot{y}(0) = 0$, and use it to find an approximate value of $y(0.2)$.

# Additional Exercises

Write down an integral representation formula for the solution of the given initial value problem. The use a graphing calculator or computer to evaluate the formula and approximate the value of $y(x_1)$.

**9** $y' + x^2 y = 1,\ y(0) = 0,\ x_1 = 2$

**10** $y' + \sin(x)y = x,\ y(0) = 0,\ x_1 = 1$

**11** $y' + \frac{y}{x} = e^{(x^4)},\ y(0) = 1,\ x_1 = 2$

**12** $y' - xy = x^2,\ y(0) = 2,\ x_1 = 1$

Calculate the given convolution of functions.

**13** $t^2 * t^2$

**14** $e^t * e^{2t}$

**15** $e^t * sin(t)$

**16** $cos(t) * cos(t)$

Use convolution to calculate the given inverse Laplace transform.

**17** $L^{-1}\left[\frac{1}{(s-1)(s+1)}\right]$

**18** $L^{-1}\left[\frac{1}{s(s+2)}\right]$

**19** $L^{-1}\left[\frac{1}{s(s^2+1)}\right]$

**20** $L^{-1}\left[\frac{1}{s^2(s^2+1)}\right]$

Use convolution to find an integral representation formula for the solution of the given initial value problem.

**21** $\ddot{y} = e^{(t^2)},\ y(0) = 0,\ \dot{y}(0) = 0$

**22** $\ddot{y} + y = \tan(t),\ y(0) = 0,\ \dot{y}(0) = 0$

**23** $\ddot{y} - y = \cos(t^3),\ y(0) = 0,\ \dot{y}(0) = 0$

**24** $\ddot{y} - 3\dot{y} + 2y = \sin(t^2),\ y(0) = 0,\ \dot{y}(0) = 0$

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**25** Use the method of integrating factors to find an integral representation formula for the solution of the following initial value problem with $b \neq 0$, and simplify your answer as much as possible:

$$\dot{y} + by = f(t),\ y(0) = y_0.$$

**26** Use Laplace Transforms to find an integral representation formula for the solution of the following initial value problem with $b \neq 0$:

$$\dot{y} + by = f(t),\ y(0) = y_0.$$

**27** Use Laplace Transforms to find an integral representation formula for the solution of the following initial value problem with $b \neq 0$:

$$\ddot{y} + 2b\dot{y} + b^2 y = f(t),\ y(0) = y_0, \dot{y}(0) = v_0.$$

**28** Use an integral representation formula to solve $\ddot{y} - y = f(t)$ with the initial conditions $y(0) = 0$ and $\dot{y}(0) = 0$. Then let $f(t) = \ln(t-1)$, and estimate the value of $y(1)$ by evaluating the necessary definite integrals using Simpson's Rule. Give an answer with an error less than $10^{-5}$.

# Part 4

# Systems of ODE

CHAPTER 13

# Systems of Differential Equations

**Prototype Question:** Model the size of two interacting populations – a predator species and its prey.

Up to this point, we have considered ODE in which there is one dependent variable, such as $y$ in the equation $\dot{y} = f(t, y)$. We will now turn our attention to **systems of ordinary differential equations** in which there are two or more dependent variables (the unknown functions for which we hope to solve). Here's one example of such a system:

$$\begin{cases} \dot{x} = 2x - 3y \\ \dot{y} = x - y \end{cases},$$

where $x(t)$ and $y(t)$are both unknown functions of the independent variable $t$. This is a system of two ordinary differential equations, and the system is **coupled** because we can't just solve for one variable and then the other – solving the differential equation for $x(t)$ would require us to know what $y(t)$ is, and solving the differential equation for $y(t)$ would require us to know what $x(t)$ is. It seems that, if we are going to be able to find solutions, we will have to find a way to solve for both $x$ and $y$ at the same time. Indeed, there is a way to do exactly that for this and many other problems. But let's begin by discussing some graphical and numerical methods for understanding systems before we try to solve any of them analytically.

In this chapter, we will deal exclusively with **first-order systems** of ODE, meaning the differential equations involve only first derivatives of the dependent variables. For a system with two unknown functions, the system is in **standard form** if it is written as:

$$\begin{cases} \dot{x} = f(x, y, t) \\ \dot{y} = g(x, y, t) \end{cases}.$$

Although we will emphasize systems of two equations, much of what we do will also apply to systems of three or more equations in the same number of unknowns. The standard form for a system of $n$ equations in $n$ dependent variables is

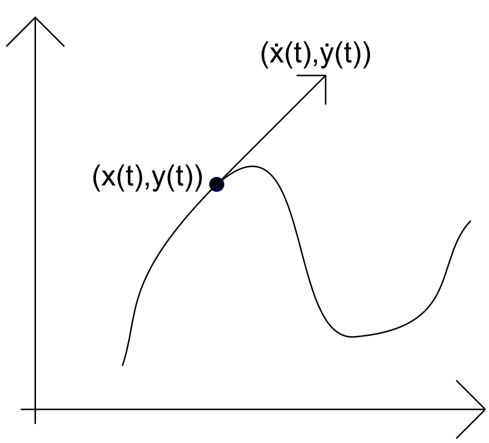
$$\begin{cases} \dot{y}_1 = f_1(y_1, y_2, ..., y_n, t) \\ \dot{y}_2 = f_2(y_1, y_2, ..., y_n, t) \\ \vdots \\ \dot{y}_n = f_n(y_1, y_2, ..., y_n, t) \end{cases}.$$

Fortunately, we will be able to develop the central ideas of this topic by sticking mostly to systems of two equations in two unknowns. Furthermore, we will also be able to focus on **autonomous systems** in which the independent variable does not appear in the differential equations:

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}. \tag{5}$$

(As we will see later, restricting our attention to autonomous systems actually does not require us to give anything up. It turns out that every system of ODE is equivalent to an autonomous system!)

A **solution** to the ODE system in equation 5 above would be a pair of functions $x(t),\ y(t)$, and we can think of these functions as parameterizing a curve in $\mathbb{R}^2$. In that case, the tangent vector to the curve at the point $(x(t), y(t))$ for any fixed value of $t$ would be the vector $(\dot{x}(t), \dot{y}(t))$. (Note: It will not be necessary for us to use different notations to distinguish between points and vectors. In fact, it would be cumbersome for us to try to do so, as we will tend to think of these as two different points of view for the same objects. Every vector in $\mathbb{R}^n$ corresponds to a point in $\mathbb{R}^n$ in an obvious way, and vice versa.)

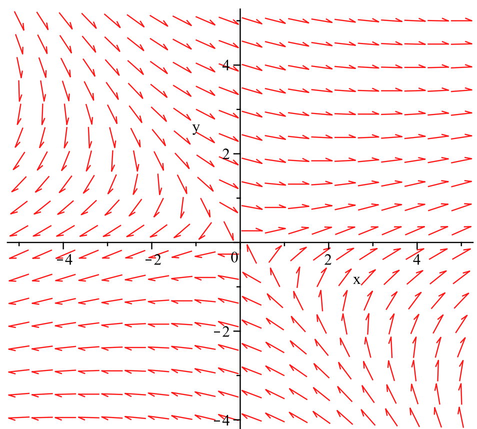



It turns out that we don't need the solutions in order to plot these tangent vectors, because the differential equation itself tells us everything we need.

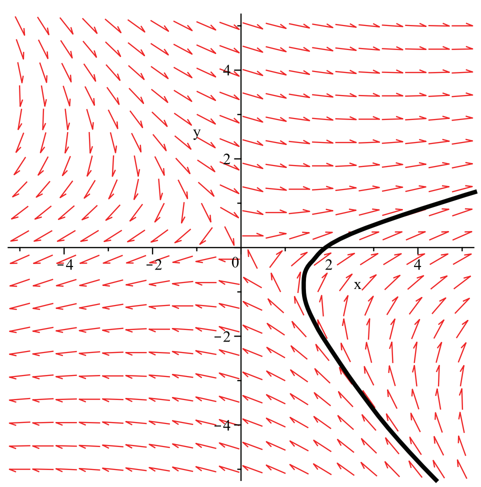
**EXAMPLE 1:** Consider the following system of ordinary differential equations:

$$\begin{cases} \dot{x} = 2x + 3y \\ \dot{y} = x - y \end{cases}.$$

If a solution to this system is a curve in $\mathbb{R}^2$ passing through the point $(x, y)$, then the tangent vector at that point is $(2x + 3y, x - y)$. At the point $(2, 0)$, this would be the vector $(4, 2)$. At the point $(2, 1)$, this would be the vector $(7, 1)$. Doing this for a lattice of points in the $xy$-plane and graphing the resulting vectors gives us a picture like the following:




This picture is a **slope field** (or **direction field**) for the ODE system. It shows us the paths that solution curves follow. For example, if $x(t),\ y(t)$ is a pair of functions satisfying the ODE system and the initial condition $x(0) = 2,\ y(0) = 0$, then the curve parameterized by $(x(t), y(t))$ should pass through the point $(2, 0)$ and remain tangent to the direction vectors; the graph below shows such a solution sketched on top of the direction field:



We will refer to the graph of $(x(t), y(t))$ as a **solution curve** or a **solution trajectory**.

EXAMPLE 2: Imagine a population of rabbits which, unchecked, would grow exponentially, but whose growth is controlled by a predator species - foxes. Let's write down a system of differential equations that will model the population growth of each of these interacting species.

We'll use our "rate-in minus rate-out" approach to come up with appropriate models, and we'll need to make some assumptions about how those rates are affected. Let $R$ denote the size of the rabbit population and $F$ the size of the fox population. We've already assumed that the rabbit population would grow exponentially if there were no foxes, so the "rate-in" component for $\dot{R}$ should be $aR$, where $a$ is some positive constant. The "rate-out" should depend on how fast rabbits are being killed by foxes. Let's assume that this rate is jointly proportional to both population sizes, so an increase in either the number of rabbits or the number of foxes should result in more rabbit-fox contacts and, consequently, more rabbit deaths. Thus the "rate-out' component could be modeled by the term $bRF$, where $b$ is another positive constant. This gives us

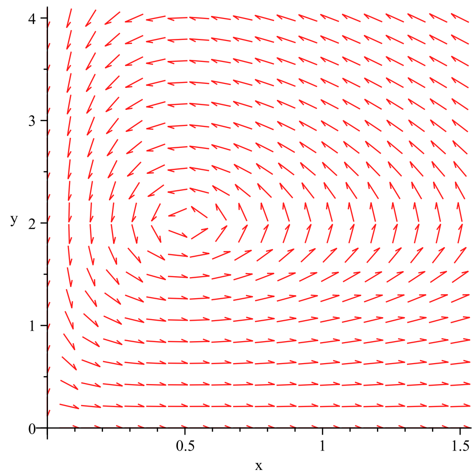
$$\dot{R} = aR - bRF$$

for the rabbit population. As for the foxes, let's assume that the growth of the fox population is proportional to the number of rabbit-fox contacts again (imagining that the growth of the fox population depends on the amount of food it obtains), so the "rate-in" would be $cRF$. Further, because foxes will tend to die off at a rate proportional to the size of the population, the "rate-out" will be $dF$. Again, $c$ and $d$ are positive constants. Therefore

$$\dot{F} = cRF - dF$$

for the fox population.

Here's a slope field for the above system of equations using the values $a = 2,\ b = 1,\ c = 1,\ d = 0.5$.




□

EXERCISE 1: On top of the slope field above, sketch a solution curve. Describe the physical interpretations of the curve you see – what does it means in terms of the population sizes of the two species? *(Hint for sketching: Every trajectory for this system should be a closed loop.)*

Next, let's turn our attention to numerical methods. For a system written in standard form, it turns out that we can apply a version of Euler's method to find approximate values of solutions. Recall that for Euler's Method, we considered a differential equation of the form $y' = f(t, y)$ with an initial condition $y(t_0) = y_0$. We selected a step size, $h$, we let $t_j = t_0 + jh$, and then used the recursive formula $y_{j+1} = y_j + f(t_j, y_j)h$ to obtain a sequence of $y$-values.

Almost exactly the same process can be used for a system. Let $h > 0$ be a fixed step size and let $t_j = t_0 + hj$ as before. For an initial-value problem such as

$$\begin{cases} \dot{x} = f(t, x, y) \\ \dot{y} = g(t, x, y) \\ x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases} \tag{6}$$

we will use two recursive formulas:

$$x_{j+1} = x_j + hf(t_j, x_j, y_j) \quad \text{and} \quad y_{j+1} = y_j + hg(t_j, x_j, y_j).$$

EXAMPLE 3: Consider the initial-value problem

$$\begin{cases} \dot{x} = x + y \\ \dot{y} = y - x \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

Let's approximate the value of $(x(1), y(1))$ using Euler's Method with a step size of $h = 0.5$. We'll organize our calculations in a table to make them easier to follow:

| $t_j$ | $x_j$ | $y_j$ | $x_{j+1} = x_j + (x_j + y_j)(0.5)$ | $y_{j+1} = y_j + (y_j - x_j)(0.5)$ |
|---|---|---|---|---|
| 0 | 1 | 0 | 1.5 | -0.5 |
| 0.5 | 1.5 | -0.5 | 2 | -1.5 |

This tells us that $x(1) \approx 2$ and $y(1) \approx -1.5$. □

EXERCISE 2: Redo Example 13.3 above with a step size of $h = 0.25$.

EXERCISE 3: Use Euler's Method with a step size of $h = 0.5$ to find approximate values of $x(2)$ and $y(2)$ where $x$ and $y$ satisfy

$$\begin{cases} \dot{x} = 2x + y \\ \dot{y} = -xy \\ x(1) = 3 \\ y(1) = 2 \end{cases}.$$

The following is an extremely important fact about systems of ODE: any $n^{th}$-order ODE written in standard form

$$y^{(n)} = f(t, y, y', y'', ..., y^{(n-1)})$$

can be written as an equivalent system of $n$ first-order equations by making the substitutions $u_j = y^{(j-1)}$:

$$\begin{cases} u_1' = u_2 \\ u_2' = u_3 \\ \vdots \\ u_{n-1}' = u_n \\ u_n' = f(t, u_1, u_2, ..., u_n) \end{cases}.$$

EXAMPLE 4: Consider the second-order initial-value problem

$$y'' = y^2 + y', \quad y(0) = 1, \quad y'(0) = 0.$$

Suppose we wish to know the (approximate) value of $y(0.75)$. This is not a linear system, so we cannot solve it analytically using methods previously discussed. But we can convert it to a first-order system and then apply Euler's Method. Let $u_j = y^{(j-1)}$. Then we have

$$\begin{cases} u_1' = u_2 \\ u_2' = u_1^2 + u_2 \\ u_1(0) = 1 \\ u_2(0) = 0 \end{cases}.$$

We apply Euler's method with a step size of $h = 0.25$:

| $t_j$ | $(u_1)_j$ | $(u_2)_j$ | $(u_1)_{j+1} = (u_1)_j + (u_2)_j(0.25)$ | $(u_2)_{j+1} = (u_2)_j + ((u_1)_j^2 + (u_2)_j)(0.25)$ |
|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0.25 |
| 0.25 | 1 | 0.25 | 1.0625 | 0.5625 |
| 0.5 | 1.0625 | 0.5625 | 1.22 | 0.9853515625 |

The last line tells us that $u_1(0.75) \approx 1.22$, and therefore $y(0.75) \approx 1.22$. □

EXERCISE 4: Consider the second-order initial-value problem $y'' = y^2$, $y(0) = 1$, $y'(0) = 0$. Convert this to an equivalent first-order system, and then use Euler's Method with a step size of $h = 0.5$ to approximate the value of $y(1)$.

Another application of the same idea is to change a non-autonomous ODE into an autonomous system. We say that a system such as in line (6) is **autonomous** if the right side functions do not depend explicitly on the independent variable, in which case the system of ODE can be written as

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}.$$

Otherwise, if $f$ and or $g$ depends on $t$, then the system is **non-autonomous**, as in

$$\begin{cases} \dot{x} = f(t, x, y) \\ \dot{y} = g(t, x, y) \end{cases}.$$

Let's introduce another independent variable, say $\tau$, which satisfies the differential equation $\dot{\tau} = 1$ and the initial condition $\tau(t_0) = t_0$. The solution of this simple initial-value problem is $\tau = t$. Let's now augment our non-autonomous system by adding this differential equation:

$$\begin{cases} \dot{x} = f(t, x, y) \\ \dot{y} = g(t, x, y) \\ \dot{\tau} = 1 \end{cases},$$

and then let's replace every occurrence of $t$ with $\tau$ on the right sides:

$$\begin{cases} \dot{x} = f(\tau, x, y) \\ \dot{y} = g(\tau, x, y) \\ \dot{\tau} = 1 \end{cases}.$$

This is an autonomous system of three ordinary differential equations - the independent variable, $t$, does not appear anywhere on the right sides of the equations.

EXAMPLE 5: Consider the non-autonomous initial-value problem

$$\frac{du}{dv} = u + v, \quad u(0) = 1.$$

Here, $v$ is the independent variable. Let's introduce a new variable, $w$, which satisfies $w = v$ and therefore $\frac{dw}{dv} = 1$. Replacing the occurrence of $v$ on the right side of our original ODE with $w$, we obtain the following autonomous initial-value problem for a system of two unknowns functions:

$$\begin{cases} \frac{du}{dv} = u + w \\ \frac{dw}{dv} = 1 \\ u(0) = 1 \\ w(0) = 0 \end{cases}.$$

□

To end this chapter, let's address existence and uniqueness for systems. To do so, it will be useful to introduce more efficient notation. Let $Y = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ be a vector-valued function of $t$, so that $Y' = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$, and let $F$ be a vector-valued function with a vector-valued input:

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}.$$

With this notation, a system such as

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

can be written as

$$Y' = F(Y),$$

and we see that adopting vector notation allows us to write the system in a form that closely parallels the form for a single ordinary differential equation. If we further define $Y_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, then the initial conditions $x(t_0) = x_0,\ y(t_0) = y_0$ can be written as

$$Y(t_0) = Y_0.$$

Similar constructions can be made for systems of 3 or more differential equations.

An existence and uniqueness theorem can now be stated succinctly:

### Existence and Uniqueness for Systems

Suppose that $F(Y)$ and $F'(Y)$ are defined and continuous on an open set containing $Y_0$. Then there is an open interval $I$ containing $t_0$ such that the initial value problem

$$Y' = F(Y), \quad Y(t_0) = Y_0$$

has a unique solution $Y(t)$ defined on $I$.

Because $F$ is a vector-valued function of several variables, the notation $F'$ actually represents a matrix of partial derivatives:

$$F'\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix}.$$

EXERCISE 5: Suppose that $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ x^2+y^2 \end{bmatrix}$. Write down $F'$.

# Additional Exercises

Use Euler's method with a step size of $\Delta t = 0.5$ to approximate the value of $(x(1), y(1))$, where $(x(t), y(t))$ satisfies the given initial value problem.

**6** $\begin{cases} \dot{x} = x + y \\ \dot{y} = x - 2y \\ x(0) = 1 \\ y(0) = 0 \end{cases}$

**7** $\begin{cases} \dot{x} = xy \\ \dot{y} = y - x \\ x(0) = 2 \\ y(0) = 1 \end{cases}$

---

**8** The `dfieldplot` command in Maple can be used to generate a slope field for a system of two autonomous first-order differential equations. Load the package for this algorithm into Maple by executing the command `with(DEtools)`. The command to generate the plot in Example 13.1 is:

```
dfieldplot([x'(t)=2x(t)+3y(t),
            y'(t)=x(t)-y(t)],
            [x(t),y(t)],t=0..1,
            x=-5..5,y=-5..5)
```

(Note that Maple requires a range for the independent variable $t$ to be specified, even though it has no effect on the graph. Therefore, the range specified here is really arbitrary.) Modify this command to generate a direction field for the system $\dot{x} = -y,\ \dot{y} = x$ on the domain $-2 \le x \le 2,\ -2 \le y \le 2$, and discuss the behavior of a trajectory satisfying the initial condition $x(0) = 0,\ y(0) = 1$.

**9** Consider the second-order initial-value problem

$$\ddot{y} + y = 1, \quad y(0) = 1, \quad \dot{y}(0) = 0.$$

Convert this into an equivalent first-order system by introducing $u = \dot{y}$. Then use a slope-field to analyze the qualitative behavior of the solution $y(t)$. Compare this with the analytic solution which can be found explicitly using techniques from earlier chapters.

**10** Convert the non-autonomous initial-value problem $y' = x - y^2,\ y(0) = 0$ into an autonomous system in two unknown functions. Generate a slope field to analyze the behavior of the solution $y(x)$.

**11** Consider the system $\dot{x} = -y,\ \dot{y} = x$ on the region $-2 \le x \le 2$ and $-2 \le y \le 2$. A slope field for this system (see Problem 1 above) suggests that the trajectories appear to be circles centered at the origin. Prove this as follows. **(a)** Introduce two new variables, $r = \sqrt{x^2 + y^2}$ and

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$. (These are just polar coordinates!) Differentiate these formulas to verify that $\dot{r} = \frac{x\dot{x}+y\dot{y}}{\sqrt{x^2+y^2}}$ and $\dot{\theta} = \frac{x\dot{y}-y\dot{x}}{x^2+y^2}$. **(b)** Replace $\dot{x}$ with $-y$ and replace $\dot{y}$ with $x$ to simplify these equations for $\dot{r}$ and $\dot{\theta}$ as much as possible. **(c)** Convert the initial conditions $x(0) = x_0,\ y(0) = y_0$ to corresponding initial conditions for $r$ and $\theta$ (in terms of $x_0$ and $y_0$). Then solve the differential equations for $r$ and $\theta$ subject to those initial conditions. **(d)** Use the formulas you find for $r(t)$ and $\theta(t)$ to write down formulas for $x(t)$ and $y(t)$. Verify directly that these functions satisfy the given system of differential equations. What shapes do these functions parametrize in $\mathbb{R}^2$? *(Keep in mind that $x_0$ and $y_0$ are constants.)*

12 Here is an alternative approach to solving the differential equations in the previous problem. Thanks to the particular structure of the system in Exercise 11, we can convert it into a single, second-order ODE. Differentiating $\dot{x} = -y$ gives us $\ddot{x} = -\dot{y}$, and then substituting $\dot{y} = x$ gives us $\ddot{x} = -x$, or $\ddot{x} + x = 0$. Solve this second-order equation subject to the initial conditions $x(0) = x_0$ and $\dot{x}(0) = -y_0$. Then use the result to find a formula for $y(t)$. Compare with the conclusions in Exercise 11.

13 Let $R(t)$ represent Romeo's affection for Juliet at time $t$, and let $J(t)$ represent Juliet's affection for Romeo at time $t$. Positive values of $R$ and $J$ represent love, and negative values represent hate. (Let's call the units of these quantities 'cupids'.) Juliet becomes more attracted to Romeo when he doesn't like her, and she becomes more repulsed by him when he does like her. Romeo, on the other hand, becomes more attracted to Juliet when she is attracted to him. Therefore their feelings for one another are modeled by the system of equations:

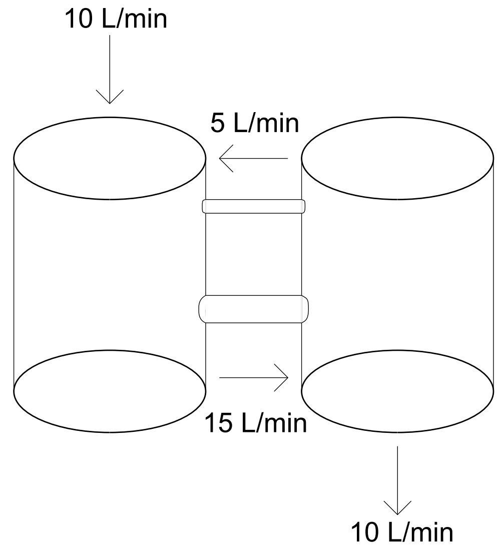
$$\dot{R} = aJ$$
$$\dot{J} = -bR,$$

where $a$ and $b$ are positive constants. Determine the behavior of $R$ and $J$ over time. Will Romeo and Juliet find happiness together? Explain. Draw some trajectories on direction fields to illustrate.

14 Juliet, from the previous question, undergoes a sudden change in personality and ends up more like Romeo – when he is attracted to her, she grows more attracted to him, and when he is repulsed by her, she grows more repulsed by him. Write down a system of equations to model this new behavior and analyze it. What happens to their long-term attraction for each other? What does it depend upon?

15 Consider two interconnected tanks used to mix saltwater. Both tanks begin with 100 liters of pure water. A mixture of 50 grams of salt per liter of water is pumped into the first tank at a rate of 10 liters per minute.

Also, liquid from the second tank is pumped into the first tank at a rate of 5 liters per minute. Liquid in the first tank is thoroughly mixed and pumped into the second tank at a rate of 15 liters per minute. Liquid in the second tank is kept thoroughly mixed and drains out 10 liters per minute. (Therefore, the total volume of liquid in both tanks remains constant.) Set up an initial-value problem for two functions which described the rate of change of the masses of salt in each tank over time.



CHAPTER 14

# Systems of Two Linear Equations

**Prototype Question:** Find an explicit solution of the following initial-value problem::

$$\begin{cases} \dot{x} = -x + 5y \\ \dot{y} = 5x - y \\ x(0) = 2 \\ y(0) = 4 \end{cases}.$$

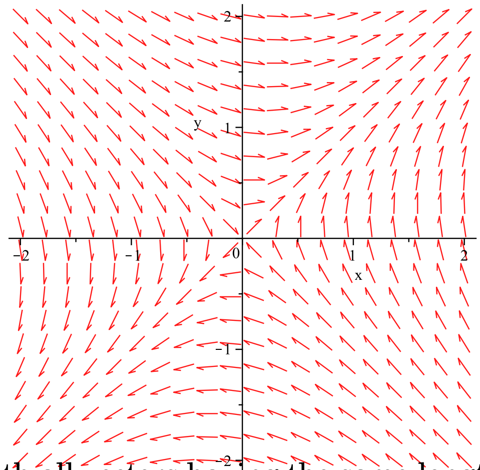
In this chapter we will study systems of the form

(7)
$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases},$$

where $a,\ b,\ c$ and $d$ are constants. These are **linear, constant coefficient, homogeneous** systems of two ordinary differential equations. If we add a driving term to the right side of one of the equations (e.g. $\dot{x} = ax + by + f(t)$), we would have a **non-homogeneous** system.

We will focus now on finding explicit solutions for homogeneous systems, and to do so we will rely on some basic techniques from linear algebra. Readers may wish to examine Appendix D before reading this chapter.

Let's look at a slope field for the system in our prototype question:




This graph was drawn with all vectors having the same length, so that it would be easier to see their directions. Notice how there appear to be some straight-line trajectories – two pointing directly toward the origin, and two pointing directly away.

Let's assume for the moment that this interpretation of the graph is correct and see if we can use that to discover what explicit solutions of the ODE system might look like. If a trajectory of (7) follows a straight line through the origin, then the $x$ and $y$ coordinates must satisfy an equation for such a line: $\alpha x + \beta y = 0$ for some constants $\alpha, \beta \in \mathbb{R}$, not both equal to zero. If $\beta \neq 0$, then we can write $y = -\frac{\alpha}{\beta} x$ and insert this into the first equation of (7) to get $\dot{x} = \left(a - \frac{b\alpha}{\beta}\right) x$, any solution of which is some exponential function, $x(t) = C_1 e^{\lambda t}$, and inserting this into the relationship $y = -\frac{\alpha}{\beta} x$ gives us another exponential function, $y(t) = C_2 e^{\lambda t}$. (If instead we assumed that $\alpha \neq 0$, we would end up with the same result.) This reasoning leads us to the following important conclusion:

### Straight Line Solutions of Linear Systems

Any trajectory $(x(t), y(t))$ that satisfies

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

and that follows a straight line through the origin can be written in the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

for some constants $\lambda,\ C_1,\ C_2$.

Before we use this fact, notice first that the system of equations can be written as a single matrix equation:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

Furthermore we can write the right side of this equation as a product of matrices:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we now write $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, it follows that $\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$, and the matrix equation above can be expressed as

$$\dot{X} = AX.$$

In this form, the vector-valued function $X(t)$ is the unknown, and $A$ is a constant coefficient matrix.

In this form, our lemma tells us that any straight-line solution can be written in the form

$$X(t) = e^{\lambda t} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

EXAMPLE 1: Consider the system in our prototype question,

$$\begin{cases} \dot{x} = -x + 5y \\ \dot{y} = 5x - y \end{cases}. \tag{8}$$

Using matrix notation, we can write this as

$$\dot{X} = \begin{bmatrix} -1 & 5 \\ 5 & -1 \end{bmatrix} X. \tag{9}$$

The lemma tells us that any straight-line solution must be of the form $X = e^{\lambda t} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, and therefore $\dot{X} = \lambda e^{\lambda t} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$. Inserting these into (9) gives us

$$\lambda e^{\lambda t} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = e^{\lambda t} \begin{bmatrix} -1 & 5 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Divide both sides by the non-zero scalar function $e^{\lambda t}$ to obtain

$$\lambda \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

This is precisely the statement that $\lambda$ is an eigenvalue of $\begin{bmatrix} -1 & 5 \\ 5 & -1 \end{bmatrix}$ and that $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ is a corresponding eigenvector. The eigenvalues of $\begin{bmatrix} -1 & 5 \\ 5 & -1 \end{bmatrix}$ are $\lambda_1 = 4$ and $\lambda_2 = -6$. Eigenvectors corresponding to $\lambda_1$ are nonzero multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and eigenvectors corresponding to $\lambda_2$ are nonzero multiples of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. In the first case, we get

$$X(t) = Ce^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and in the second case,

$$X(t) = Ce^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

That is to say, straight-line solutions of this system of ordinary differential equations have one of two possible forms:

$$\begin{cases} x(t) = Ce^{4t} \\ y(t) = Ce^{-6t} \end{cases} \quad \text{or} \quad \begin{cases} x(t) = Ce^{4t} \\ y(t) = -Ce^{-6t} \end{cases}.$$

(In each case, the value of $C$ is the same in both lines.) $\square$

EXERCISE 1: For the system of equations in line (8): **(a)** find a straight-line solution satisfying $x(0) = 2,\ y(0) = 2$; **(b)** find a straight-line solution satisfying $x(0) = 3, y(0) = -3$; is there a straight-line solution satisfying $x(0) = 2,\ y(0) = 4$? Explain.

EXERCISE 2: Fill in the details of finding the eigenvalues and eigenvectors in the previous example. *(There is a review of the relevant linear algebra in the appendix.)*

The reasoning in the last example can be applied to any constant-coefficient, linear, homogeneous system. Remarkably, knowing how to find straight-line solutions will actually help us to find *all* solutions, thanks to the fact that our systems are linear. The following exercises will provide us with the main ingredients we need for constructing more general solutions.

EXERCISE 3: Suppose $X(t) = e^{\lambda t}\xi$, where $\lambda$ is an eigenvalue of $A$ and $\xi$ is an associated eigenvector. Prove that $X(t)$ satisfies $\dot{X} = AX$.

EXERCISE 4: Suppose that $X_1(t)$ and $X_2(t)$ both satisfy the system of differential equations represented by $\dot{X} = AX$. Prove that for any constant scalar coefficients $c_1$ and $c_2$ the function $X = c_1X_1 + c_2X_2$ is also a solution.

It is proven in linear algebra that if a $2\times 2$ matrix $A$ has two distinct eigenvalues $\lambda_1$ and $\lambda_2$ (the word 'distinct' meaning that $\lambda_1 \neq \lambda_2$), then any eigenvectors $\xi_1$ and $\xi_2$ associated to $\lambda_1$ and $\lambda_2$ must be linearly independent. According to the two previous exercises, the function

$$X(t) = c_1e^{\lambda_1 t}\xi_1 + c_2e^{\lambda_2 t}\xi_2$$

is a solution of $\dot{X} = AX$. Notice that $X(0) = c_1\xi_1 + c_2\xi_2$, and because the eigenvectors are linearly independent, it follows that any initial value of $X(0)$ can be satisfied by an appropriate choice of $c_1$ and $c_2$. The existence and uniqueness theorem tells us that solutions of a linear system are unique for $t$ close to $t_0$ (actually, it turns out that they are unique *for all* $t \in \mathbb{R}$). We can now state the following formula for finding general solutions of matrix differential equations.

**Solutions of Homogeneous Linear Systems with Distinct Eigenvalues**

Suppose that $A$ is a $2 \times 2$ matrix with distinct eigenvalues $\lambda_1$ and $\lambda_2$, and suppose that $\xi_1$ and $\xi_2$ are corresponding eigenvectors. Then the general solution of $\dot{X} = AX$ is given by

$$X(t) = c_1e^{\lambda_1 t}\xi_1 + c_2e^{\lambda_2 t}\xi_2.$$

We will also give a complete description of the general solution when the coefficient matrix $A$ has only one eigenvalue. However, we will postpone that until we explore a few examples of this first result.

EXAMPLE 2: Consider the initial-value problem in the prototype question for this chapter. The general solution of this system is obtained from linear combinations of the straight-line solutions:

$$X(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

or

$$\begin{cases} x(t) = c_1 e^{4t} + c_2 e^{-6t} \\ y(t) = c_1 e^{4t} - c_2 e^{-6t} \end{cases}.$$

Inserting $t = 0$ and the initial conditions for $x$ and $y$ gives us

$$2 = c_1 + c_2 \quad \text{and} \quad 4 = c_1 - c_2.$$

Thus the solution of this algebraic system of equations is $c_1 = 3,\ c_2 = -1$. Therefore

$$\begin{cases} x(t) = 3e^{4t} - e^{-6t} \\ y(t) = 3e^{4t} + e^{6t} \end{cases}.$$

□

EXAMPLE 3: Solve the initial-value problem:

$$\begin{cases} \dot{x} = x + y \\ \dot{y} = 3x - y \\ x(0) = 1 \\ y(0) = 2 \end{cases}$$

If we write this as a matrix differential equation $\dot{X} = AX$, the coefficient matrix is $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$. The eigenvalues of $A$ are $\lambda_1 = 2$ and $\lambda_2 = -2$. An eigenvector corresponding to $\lambda_1 = 2$ is $\xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and an eigenvector corresponding to $\lambda_2 = -2$ is $\xi_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Therefore the general solution is given by

$$X(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

Combining the right side into a single vector and writing the matrix $X$ in terms of its component functions yields

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1e^{2t} + c_2e^{-2t} \\ c_1e^{2t} - 3c_2e^{-2t} \end{bmatrix}.$$

This gives us general solutions for the scalar functions: $x(t) = c_1e^{2t} + c_2e^{-2t}$ and $y(t) = c_1e^{2t} - 3c_2e^{-2t}$. Inserting the initial conditions $x(0) = 1$ and $y(0) = 2$ gives us a system of equations we can use to solve for the coefficients $c_1$ and $c_2$:

$$\begin{cases} 1 = c_1 + c_2 \\ 2 = c_1 - 3c_2 \end{cases} \implies c_1 = \frac{5}{4} \text{ and } c_2 = -\frac{1}{4}.$$

Now we have

$$x(t) = \frac{5}{4}e^{2t} - \frac{1}{4}e^{-2t} \quad \text{and} \quad y(t) = \frac{5}{4}e^{2t} + \frac{3}{4}e^{-2t}.$$

□

EXERCISE 5: Verify directly that the functions $x$ and $y$ found in the previous example satisfy the system of differential equations there.

EXERCISE 6: Solve the initial-value problem

$$\begin{cases} \dot{x} = 2x + y \\ \dot{y} = 4x + 3y \\ x(0) = 0 \\ y(0) = 4 \end{cases}$$

EXAMPLE 4: Consider the initial-value problem

$$\begin{cases} \dot{x} = 2x - 2y \\ \dot{y} = x \\ x(0) = 0 \\ y(0) = 1 \end{cases}$$

Written as $\dot{X} = AX$, the coefficient matrix will be $A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$. The characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$, and this has complex roots: $\lambda_1 = 1+i$ and $\lambda_2 = 1-i$. Inserting the first eigenvalue into the equation $(A - \lambda I)\xi_1 = 0$ gives us

$$\begin{bmatrix} 2-(1+i) & -2 \\ 1 & -(1+i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $\xi_1 = \begin{bmatrix} a \\ b \end{bmatrix}$. Hence

$$\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The first row of this matrix equation is equivalent to the scalar equation

$$(1-i)a - 2b = 0,$$

from which we can conclude $b = \frac{(1-i)a}{2}$. A simple choice for the eigenvector would be $\xi_1 = \begin{bmatrix} 1 \\ \frac{1+i}{2} \end{bmatrix}$. A similar analysis will lead us to the eigenvector $\xi_2 = \begin{bmatrix} 1 \\ \frac{-i-1}{2} \end{bmatrix}$ corresponding to $\lambda_2 = 1-i$. Now we can write the general solution as

$$X(t) = c_1 e^{(1+i)t} \begin{bmatrix} 1 \\ \frac{i-1}{2} \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} 1 \\ \frac{-i-1}{2} \end{bmatrix}.$$

We write the matrix in component form to obtain

$$\begin{aligned} x(t) &= c_1 e^{(1+i)t} + c_2 e^{(1-i)t} \\ y(t) &= c_1 \frac{(i-1)}{2} e^{(1+i)t} + c_2 \frac{-i-1}{2} e^{(1-i)t}. \end{aligned}$$

The initial conditions imply

$$\begin{aligned} 0 &= c_1 + c_2 \\ 1 &= c_1 \frac{(i-1)}{2} + c_2 \frac{(-i-1)}{2}, \end{aligned}$$

which has solutions $c_1 = -i$ and $c_2 = i$. We insert these values and use Euler's identity to simplify the formulas for $x(t)$ and $y(t)$:

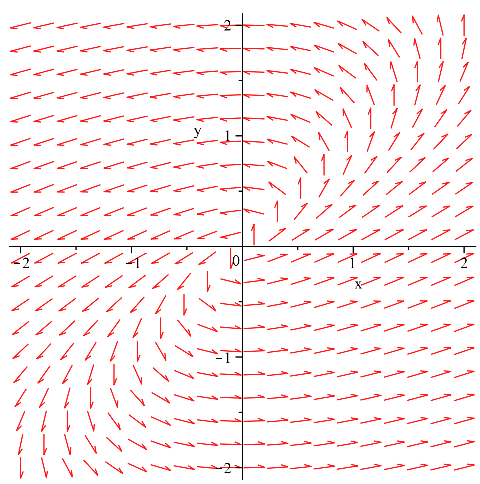
$$\begin{aligned} x(t) &= -ie^{(1+i)t} + ie^{(1-i)t} \\ &= -ie^t\left(\cos(t) + i\sin(t)\right) + ie^t\left(\cos(t) - i\sin(t)\right) \\ &= 2e^t\sin(t) \end{aligned}$$

and

$$\begin{aligned} y(t) &= -i\frac{i-1}{2}e^{(1+i)t} + i\frac{-i-1}{2}e^{(1-i)t} \\ &= \frac{1+i}{2}e^t\left(\cos(t) + i\sin(t)\right) + \frac{1-i}{2}e^t\left(cos(t) - i\sin(t)\right) \\ &= e^t\cos(t) - e^t\sin(t). \end{aligned}$$

$\square$

The solution in the previous example seems to 'spiral outward' from the origin as $t$ increases. A direction field for this system will help us to visualize why:




**EXERCISE 7:** Solve the initial value problem:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ x(0) = 1 \\ y(0) = 2 \end{cases}$$

We need to explore what to do if there is only one eigenvalue for the coefficient matrix $A$. There are really two cases here: either it is possible to find two linearly independent eigenvectors for the same eigenvalue $\lambda$, or it is not. The first case turns out to be uninteresting, because if a $2 \times 2$ matrix $A$ has two linearly independent eigenvectors for the eigenvalue $\lambda$, then $A$ is a multiple of the identity matrix $A = \lambda I$. The reason this is uninteresting is that it implies that the corresponding system of differential equations is

$$\begin{cases} \dot{x} = \lambda x \\ \dot{y} = \lambda y \end{cases},$$

and this system is **uncoupled**, meaning that the equations for $x$ and $y$ can each be solved separately. Nothing besides separation of variables or the method of integrating factors is necessary for each equation. Thus we will focus our attention on the case when all the eigenvectors for $A$ are scalar multiples of a single eigenvector $\xi$.

To this end, we introduce another idea from linear algebra. If $\lambda$ is an eigenvalue for $A$ and $\xi$ is an associated eigenvector, then a **generalized eigenvector** $\eta$ is a vector that satisfies the matrix equation

$$(A - \lambda I)\eta = \xi.$$

(Note the similarity of this with the matrix equation that defines an eigenvector, $(A - \lambda I)\xi = 0$.)

EXAMPLE 5: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The characteristic equation is $(\lambda - 1)^2 = 0$, so the only eigenvalue is $\lambda = 1$. A corresponding eigenvector is $\xi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We want to find a generalized eigenvector $\eta$, so we need to solve the matrix equation $(A - I)\eta = \xi$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The first row of this matrix equation implies $l = 1$; meanwhile $k$ can be anything. Therefore, any generalized eigenvector can be written as $\begin{bmatrix} k \\ 1 \end{bmatrix}$, and a simple choice would be $\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. □

EXERCISE 8: The matrix $A = \begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix}$ has only one eigenvalue, $\lambda = 3$, and $\xi = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a corresponding eigenvector. Verify these statements, and then find a generalized eigenvector.

With generalized eigenvectors at our disposal, we can now state the necessary formula for a general solution.

**Solutions of Homogeneous Linear Systems with Repeated Eigenvalues**

Suppose that $A$ is a $2\times 2$ matrix with only one eigenvalue, $\lambda$. Suppose also that $A$ is not a scalar multiple of the identity matrix. Let $\xi$ be an eigenvector of $A$ and let $\eta$ be a generalized eigenvector. Then the general solution of $\dot{X} = AX$ is

$$X(t) = c_1 e^{\lambda t}\xi + c_2 \left(te^{\lambda t}\xi + e^{\lambda t}\eta\right).$$

We already know that the function $X_1(t) = e^{\lambda t}\xi$ satisfies the differential equation, and we know that linear combinations of solutions are solutions, so we need to prove that the function $X_2(t) = te^{\lambda t}\xi + e^{\lambda t}\eta$ is a solution of $\dot{X} = AX$. Observe that

$$\dot{X}_2 = e^{\lambda t}\xi + \lambda te^{\lambda t}\xi + e^{\lambda t}\eta$$

by the product rule, and then

$$\begin{aligned} AX_2 &= A\left(te^{\lambda t}\xi + e^{\lambda t}\eta\right) \\ &= te^{\lambda t}A\xi + e^{\lambda t}A\eta \\ &= te^{\lambda t}\lambda\xi + e^{\lambda t}\left(\xi + \lambda\eta\right) \\ &= \dot{X}_2, \end{aligned}$$

as desired. In the second to last line above, we used the facts that eigenvectors satisfy $A\xi = \lambda\xi$ and that generalized eigenvectors satisfy $A\eta = \lambda\eta + \xi$. That last fact also implies that $\eta$ cannot be an eigenvector, since $\xi \neq 0$, and consequently $\eta$ is not a scalar multiple of $\xi$. Therefore the collection $\{\xi,\ \eta\}$ is linearly independent. That tells us that an appropriate selection of $c_1$ and $c_2$ in the general formula will allow us to satisfy any initial condition for $X(0)$.

EXAMPLE 6: Consider the initial value problem

$$\begin{cases} \dot{x} = 2x + 3y \\ \dot{y} = -3x + 8y \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

The characteristic equation is $(\lambda-5)^2=0$, so $\lambda=5$ is the only eigenvalue. An eigenvector to go with this eigenvalue is $\xi=\begin{bmatrix}1\\1\end{bmatrix}$. The coefficient matrix is not a mulitple of the identity matrix, so we next seek a generalized eigenvector $\eta$ satisfying $(A-5I)\eta=\xi$:

$$\begin{bmatrix}-3 & 3\\ -3 & 3\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix},$$

which implies $-3a+3b=1$, and we can select the solution $a=1,\ b=\frac{4}{3}$, so that $\eta=\begin{bmatrix}1\\ \frac{4}{3}\end{bmatrix}$. Now we can write down the general solution of the matrix differential equation,

$$X(t)=c_1e^{5t}\begin{bmatrix}1\\1\end{bmatrix}+c_2\left(te^{5t}\begin{bmatrix}1\\1\end{bmatrix}+e^{5t}\begin{bmatrix}1\\ \frac{4}{3}\end{bmatrix}\right).$$

The component functions are therefore

$$x(t)=(c_1+c_2)e^{5t}+c_2te^{5t}$$
$$y(t)=(c_1+\frac{4}{3}c_2)e^{5t}+c_1te^{5t}$$

Inserting the initial conditions gives us

$$1=c_1+c_2$$
$$0=c_1+\frac{4}{3}c_2,$$

and this algebraic system has solutions $c_1=4,\ c_2=-3$. Therefore the solutions we need are

$$x(t)=e^{5t}-3te^{5t}$$
$$y(t)=4te^{5t}$$

$\square$

EXERCISE 9: Solve the initial value problem:

$$\begin{cases}\dot{x}=2x+4y\\ \dot{y}=-x+6y\\ x(0)=0\\ y(0)=2\end{cases}$$

# Additional Exercises

Find all straight-line solutions of the given system of differential equations.

**10** $\begin{cases} \dot{x} = 2x + 4y \\ \dot{y} = 8x + 6y \end{cases}$

**11** $\begin{cases} \dot{x} = x + 2y \\ \dot{y} = 3x + 2y \end{cases}$

Find the general solution of the system of differential equations.

**12** $\begin{cases} \dot{x} = 3x + y \\ \dot{y} = 2x + y \end{cases}$

**13** $\begin{cases} \dot{x} = 2x + 3y \\ \dot{y} = 2x + y \end{cases}$

**14** $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$

**15** $\begin{cases} \dot{x} = x + y \\ \dot{y} = -x + y \end{cases}$

**16** $\begin{cases} \dot{x} = 4x - y \\ \dot{y} = 8x - 2y \end{cases}$

**17** $\begin{cases} \dot{x} = -9x + 2y \\ \dot{y} = -18x + 3y \end{cases}$

Solve the initial value problem.

**18** $\begin{cases} \dot{x} = 6x + 2y \\ \dot{y} = 4x + 2y \\ x(0) = 1 \\ y(0) = 2 \end{cases}$

**19** $\begin{cases} \dot{x} = 2x + 3y \\ \dot{y} = 2x + y \\ x(0) = 0 \\ y(0) = 1 \end{cases}$

**20** $\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ x(0) = 3 \\ y(0) = 4 \end{cases}$

**21** $\begin{cases} \dot{x} = x + y \\ \dot{y} = -x + y \\ x(0) = 1 \\ y(0) = -1 \end{cases}$

**22** $\begin{cases} \dot{x} = 4x - y \\ \dot{y} = 8x - 2y \\ x(0) = 1 \\ y(0) = 2 \end{cases}$

**23** $\begin{cases} \dot{x} = -9x + 2y \\ \dot{y} = -18x + 3y \\ x(0) = 0 \\ y(0) = 1 \end{cases}$

**24** Consider the second order equation $\ddot{u} + 2\dot{u} - 3u = 0$. If we introduce a function $v$ satisfying $\dot{u} = v$, then it also follows that $\dot{v} = \ddot{u} = 3u - 2\dot{u} = 3u - 2v$. Now for these functions $u$ and $v$ we have a first order system of equations:

$$\begin{cases} \dot{u} = v \\ \dot{v} = 3u - 2v \end{cases}.$$

Find the general solution for this system. In particular, verify that it gives the same general solution for $u$ that is obtained using other means.

**25** Following the approach illustrated in Exercise 24, find the general solution of $\ddot{x} + 2\dot{x} + x = 0$ by letting $y = \dot{x}$ and then solving a first order linear system of differential equations for $(x(t), y(t))$. Verify that the solution you obtain for $x(t)$ is the same you as you would find by solving the second order equation for $x$ using other means.

**26** Find general solutions for the following systems of ODE:

$$\begin{cases} \dot{R} = aJ \\ \dot{J} = bR \end{cases} \quad \text{and} \quad \begin{cases} \dot{R} = aJ \\ \dot{J} = -bR \end{cases}.$$

Compare your results with your answers for Problems 13.6 and 13.7.

**27** Solve the initial-value problem

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

using the techniques from this chapter. Compare your results with those of Exercises 13.11 and 13.12. Are they equivalent?

APPENDIX A

# Separation of Variables

Solving ordinary differential equations is usually introduced in integral calculus. Indeed, finding an anti-derivative is really solving a differential equation of the form $y' = f(x)$. But one also learns how to solve certain differential equations in which the dependent variable also appears: so-called separable differential equations.

A first order ODE is called **separable** if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y). \tag{10}$$

The name comes from the fact that we will try to find solutions of this differential equation by separating the dependent and independent variables to opposite sides of the equation:

$$\frac{dy}{g(y)} = f(x)\ dx. \tag{11}$$

This equation can be given an independent meaning if one studies differentials in a rigorous way, but we will instead think of it as a shorthand for the following. Assume that (10) holds and that $g(y_0) \neq 0$. Then dividing both sides of (10) by $g(y)$ gives us

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x).$$

Write $y$ out as $y(x)$ and integrate both sides of the equation from $x_0$ to $x$:

$$\int_{x_0}^{x} \frac{1}{g(y(x))}\frac{dy}{dx}\ dx = \int_{x_0}^{x} f(x)\ dx.$$

Make a substitution $u = y(x)$ in the integral on the left side, with $du = \frac{dy}{dx}dx$, to obtain

$$\int_{y_0}^{y} \frac{1}{g(u)}\ du = \int_{x_0}^{x} f(x)dx.$$

Now if $G$ is any anti-derivative of $\frac{1}{g}$, and if $F$ is any anti-derivative of $f$, we have

$$G(y) - G(y_0) = F(x) - F(x_0). \tag{12}$$

And if $G$ is an invertible function, we can solve for $y$:

$$y = G^{-1}\left(F(x) - F(x_0) + G(y_0)\right).$$

The process above is valid if $f$ and $g$ are both continuous and if $x$ is sufficiently close to $x_0$. (In fact, continuity and the assumption $g(y_0) \neq 0$ are enough to guarantee that $G$ will be invertible where necessary.)

However, this is not usually how the process is used. Instead of using definite integrals, we will typically write indefinite integrals. Beginning with (11), we anti-differentiate both sides to obtain

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

which leads to

$$G(y) + C_1 = F(x) + C_2.$$

This is equivalent to (12) if appropriate values are chosen for $C_1$ and $C_2$. However, we can also write this in a simpler form if we let $C = C_2 - C_1$:

$$G(y) = F(x) + C.$$

The value of $C$ can be obtained from a given initial condition, or it may be treated as a free parameter. Isolating $y$ gives an explicit formula for $y$ in terms of $x$. The process of solving an ODE in this manner is called **separation of variables**.

EXAMPLE A.1: Use separation of variables to solve the IVP $y' = \frac{x}{y^2},\ y(0) = 1$.

Separating the variables in $\frac{dy}{dx} = \frac{x}{y^2}$ and integrating produces

$$\int y^2\, dy = \int x\, dx.$$

Finding the general anti-derivatives on each side, we write

$$\frac{y^3}{3} + c_1 = \frac{x^2}{2} + c_2.$$

Isolating $y$ now gives us

$$y = \sqrt[3]{\frac{3}{2}x^2 + 3(c_2 - c_1)}.$$

Because $c_1$ and $c_2$ are both just constants, we can replace the expression $3(c_2 - c_1)$ with a single constant, say, $C$:

$$y = \sqrt[3]{\frac{3}{2}x^2 + C}.$$

By selecting the appropriate value for $C$, we can now solve the initial value problem. Inserting $x = 0$ and $y = 1$ produces

$$1 = \sqrt[3]{\frac{3}{2}(0)^2 + C},$$

and consequently, we see that $C = 1$. Therefore

$$y = \sqrt[3]{\frac{3}{2}x^2 + 1},$$

and it is easy to verify that this is indeed a solution of the initial value problem. □

It is typical to combine the constants of integration as in the previous example right away without ever writing them separately; when we evaluate the anti-derivatives, we just write one constant of integration on one side of the equation (usually the side with the independent variable). Also, if the algebra we encounter forces us to mutiply that constant by another constant, we typically consume the constants into a single symbol, and it is common to reuse a symbol from the previous line. It is understood in this context that a constant such as $C$ may differ in value from line to line, though in each individual line it is known to be a constant.

EXAMPLE A.2: Solve the IVP $y' = xe^x y^4,\ y(0) = 2$.

Separating variables gives us

$$\int \frac{1}{y^4}\, dy = \int xe^x\, dx.$$

We integrate both sides, using integration-by-parts on the right:

$$\begin{aligned} \frac{-1}{3y^3} &= xe^x - \int e^x\, dx \\ &= xe^x - e^x + C. \end{aligned}$$

Multiply both sides by $-3$:

$$\frac{1}{y^3} = -3xe^x + 3e^x + C.$$

*(Notice that $C$ has changed value from the previous line!)*. Let's find the value of the unknown constant in this line by inserting the initial condition, $y = 2$ when $x = 0$:

$$\frac{1}{8} = 3 + C, \quad \text{so } C = -\frac{23}{8}.$$

Insert this for $C$, and then isolate $y$ by taking reciprocals and cube roots of both sides:

$$y = \frac{1}{\sqrt[3]{-3xe^x + 3e^x - \frac{23}{8}}}.$$

□

As the next example shows, the independent variable need not explicitly appear in the ODE.

EXAMPLE A.3: Solve $y' = y^3,\ y(0) = 3$.

Separating variables produces

$$\int \frac{1}{y^3}\,dy = \int 1\,dx.$$

Therefore

$$\frac{-1}{2y^2} = x + C,$$

and multiplying both sides by $-2$ produces:

$$\frac{1}{y^2} = -2x + C.$$

*(Notice that $C$ here has a different value from the previous line!)* Taking reciprocals produces

$$y^2 = \frac{1}{C - 2x},$$

and trying to isolate $y$ results in the relation

$$y = \pm\frac{1}{\sqrt{C - 2x}}.$$

Because the $y$ we seek is a function of $x$, and a continuous one at that, we have to go with either $+$ or $-$ in our solution (if we leave a $\pm$ symbol in place, it suggests two outputs for each input, violating the definition of 'function'.) Because the initial condition $y(0) = 3$ gives a positive output, we can settle on the choice of $+$:

$$y = \frac{1}{\sqrt{C - 2x}}.$$

Finally, using the initial condition to solve for the unknown constant will give us $C = \frac{1}{9}$, so

$$y = \frac{1}{\sqrt{\frac{1}{9} - 2x}}.$$

□

EXERCISE A.1: Solve $y' = \frac{y}{x},\ y(1) = 2$.

EXERCISE A.2: Solve $y' = 1 + y^2,\ y(0) = 0$.

For an ODE to be separable, we must be able to use only multiplication and division to separate the variables. This is because we treat $\frac{dy}{dx}$ as if it were a fraction of two quantities, $dy$ and $dx$, and we use multiplication by $dx$ to separate these across the equal sign. If we try to mix this operation with addition or subtraction, we wil not be able to "get all of the x's on one side and all of the y's on the other."

EXERCISE A.3: Determine which of each of the following equations is separable.

(1) $y' = x + y$
(2) $y' = x^2y + xy$
(3) $y' = \ln(xy)$
(4) $y' = e^{x+y}$

Recall that we required $g(y_0)$ be nonzero when we deduced this method. If $g(y_0) = 0$, then this process may or may not produce a solution – there are no guarantees. However, when that does happen, the constant function $y(x) = y_0$ for all $x$ *is* a solution of $\frac{dy}{dx} = f(x)g(y)$ because both sides of the equation will be zero (the left, because $y$ is a constant function, the right, because $g(y_0) = 0$). Constant solutions of differential equations are called **equilibrium solutions**.

EXERCISE A.4: Solve the IVP $y' = y^2 - 6y + 8,\ y(0) = 4$.

EXAMPLE A.4: Find a solution of the initial-value problem $y' = y(1 - y)$, $y(0) = y_0$.

We first observe that $y = 0$ and $y = 1$ are equilibrium solutions. To find other solutions, we begin by separating variables to obtain

$$\int \frac{1}{y(1-y)}\,dy = \int\,dx.$$

The left side can be rewritten using a partial fractions decomposition: $\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-u}$ when we select $A = 1$ and $B = 1$, so

$$\int \frac{1}{y} + \frac{1}{1-y}\,dy = \int\,dx.$$

Anti-differentiation gives us

$$\ln|y| - \ln|1-y| = x + C.$$

The left side can be simplified using the property of logarithms that $\ln a - \ln b = \ln\frac{a}{b}$:

$$\ln\left|\frac{y}{1-y}\right| = x + C.$$

Exponentiate both sides to obtain

$$\left|\frac{y}{1-y}\right| = e^{x+C}.$$

Therefore

$$\frac{y}{1-y} = \pm e^C e^X.$$

The value of $C$ and the choice of $\pm$ are determined by initial conditions:

$$\frac{y_0}{1-y_0} = \pm e^C.$$

So

$$\frac{y}{1-y} = \frac{y_0}{1-y_0} e^x.$$

Multiplying both sides by $1-y$, expanding, and then isolating $y$ gives us

$$y = \frac{\frac{y_0}{1-y_0} e^x}{1 + \frac{y_0}{1-y_0} e^x}.$$

Observe that this formula captures the equilibrium solution $y = 0$ because, when $y_0 = 0$, the whole function will be zero. On the other hand, this expression is not defined when $y_0 = 1$. However, notice that this can be rectified if we multiply the top and bottom of the expression by $1 - y_0$:

$$y = \frac{y_0 e^x}{1 - y_0 + y_0 e^x}.$$

This formulation gives us a valid solution for all possible values of the initial condition $y_0$. □

The final form of the answer in the last example is what we call a **general solution**. It represents all possible solutions of the differential equation and can be used to satisfy any initial condition.

EXAMPLE A.5: Find a general solution for the differential equation $y' = 6xy^2$.

**Solution:** Observe that $y = 0$ is an equilibrium solution. For nonzero initial values, we find solutions by separating variables:

$$\int \frac{dy}{y^2} = \int 6x dx$$

Hence

$$-\frac{1}{y} = 3x^2 + C, \tag{13}$$

where $C$ can be any real number. Therefore

$$y = -\frac{1}{3x^2 + C}.$$

By selecting an appropriate value of $C$, we can solve any initial value problem of the form $y(x_0) = y_0$ as long as $y_0 \neq 0$. (There's no value $C$ could take which would produce an output of $0$ from this function.) Therefore, our general solution is

$$y = \begin{cases} 0 & \text{if } y_0 = 0 \\ -\frac{1}{3x^2+C} & \text{otherwise} \end{cases}.$$

Note that $C$ can be any constant other than $-3x_0^2$ (in order to avoid dividing by zero). To satisfy $y(x_0) = y_0$, we isolate $C$ in equation (13) to obtain $C = -\frac{1}{y_0} - 3x_0^2$. □

Notice how we were able to use line (13) to identify the value of $C$ necessary to satisfy any initial condition. When we need to find a general solution, it is a good idea to avoid changing the values of unknown constants from line to line, just to make this easier. On the other hand, sometimes it is easier to display the general solution if we allow ourselves to change the meaning of the parameter, as the next example illustrates.

EXAMPLE A.6: Find a general solution of $y' = ky$.

**Solution:** The constant function $y = 0$ is an equilibrium solution. Separating variables gives us

$$\int \frac{dy}{y} = \int k\, dx,$$

so that

$$\ln|y| = kx + C.$$

Exponentiate both sides to get

$$|y| = e^{kx+C},$$

and thus

$$y = \pm e^C e^{kx}.$$

$C$ can be any real number, but then $e^C$ will be any *positive* number. The ability to select plus or minus means that the expression $\pm e^C$ can be any *nonzero* number. So a simpler way to display the solution would be to write

$$y = Ae^{kx} \quad \text{where } A \text{ is any nonzero number.}$$

The parameter $A$ would never be zero as a result of how it is derived from separating variables. So we can express the general solution in the form

$$y = \begin{cases} Ae^{kx} & \text{if } y_0 \neq 0 \\ 0 & \text{if } y_0 = 0 \end{cases}.$$

However, notice that this can be simplified if we allow $A$ to take the value zero: we can just say the the general solution is

$$y = Ae^{kx}, \quad \text{where } A \text{ is any real number.}$$

To satisfy an initial condition of the form $y(x_0) = y_0$, we let $A = y_0 e^{-kx_0}$. □

EXERCISE A.5: Find a general solution of $y' = 1 - y$. What value should your parameter take in order to satisfy the initial condition $y(x_0) = y_0$?

EXERCISE A.6: Find a general solution of $y' = e^{x+y}$. What value should your parameter take in order to satisfy the initial condition $y(x_0) = y_0$?

EXERCISE A.7: Solve the initial value problem $y' = y(3 - y),\ y(0) = 1$.

EXERCISE A.8: Solve the initial value problem $y' = y^2 + 2y + 2,\ y(0) = 1$. *(Hint: After you separate variables, you will need to write the quadratic denominator you find in 'vertex form'* $(y - h)^2 + k$ *before you can integrate.)*

APPENDIX B

# Complex Numbers

When we solve characteristic equations, we are often faced with complex numbers. For example, the solutions of a quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

and if the discriminant $b^2 - 4ac$ is negative, then we are looking at square roots of negative numbers, so the roots of the original equation are complex numbers. These can be written in the form

$$z = \alpha + \beta i,$$

where $\alpha$ and $\beta$ are both real, and $i$ satisfies $i^2 = -1$. If $z = \alpha + \beta i$ in this way, then we call $\alpha$ the **real part** of $z$, and we call $\beta$ the **imaginary part**. A complex-valued function $f(x)$ can also be written in the form $f(x) = u(x) + iv(x)$, where $u$ and $v$ are real-valued functions. In this case, $u$ is the real part of $f$ and $v$ is the imaginary part of $f$.

We can usually understand the arithmetic operations on complex numbers by writing numbers in terms of their real and imaginary parts.

EXAMPLE B.1: Let $z = 1 + 3i$ and $w = 3 - 2i$. Then:

- $z + w = (1 + 3i) + (3 - 2i) = (1 + 3) + (3 - 2)i = 4 + i$
- $zw = (1 + 3i)(3 - 2i) = (1)(3) + (3i)(3) + (1)(-2i) + (3i)(-2i) = 3 - 2i + 3i - 6i^2 = 3 - 2i + 3i - 6(-1) = 9 + i$

EXERCISE B.1: Let $u = 2 + 4i$ and $v = 1 - 2i$. Find $2u + 3v$ and $2uv$.

The **complex conjugate** of $z = \alpha + \beta i$ is the complex number $\overline{z} = \alpha - \beta i$. For example, $\overline{1 + 3i} = 1 - 3i$.

EXERCISE B.2: Prove that for any complex numbers $z$ and $w$, $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = (\overline{z})(\overline{w})$.

EXERCISE B.3: Prove that for any complex number $z$, the product $z \cdot \overline{z}$ is a real number. *(Hint: Start by writing $z = \alpha + \beta i$.)* Can you say anything more about the value of $z\overline{z}$?

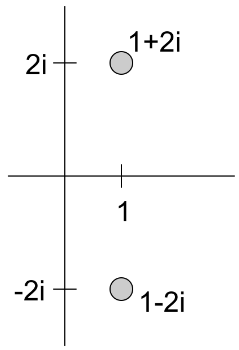
The conjugate is especially useful for simplifying division with complex numbers. For any complex numbers $z$ and $w$, with $w \neq 0$, we see that $\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}}$, and the latter denominator is a real number.

EXAMPLE B.2: Let $z = 1 + 2i$ and $w = 2 - 4i$. Then

$$\begin{aligned} \frac{z}{w} &= \frac{1+2i}{2-4i} \\ &= \frac{(1+2i)}{(2-4i)}\frac{(2+4i)}{(2+4i)} \\ &= \frac{2+4i+4i+8i^2}{4-16i^2} \\ &= \frac{-6+8i}{20} \\ &= -\frac{3}{10} + \frac{2}{5}i. \end{aligned}$$

EXERCISE B.4: Let $u = 2 + 4i$ and $v = 1 - i$. Simplify the expressions $\frac{u}{v}$ and $\frac{v}{u}$. Write the answers in the form $\alpha + \beta i$.

Real numbers are usually visualized as points on a line. Complex numbers can be visualized similarly as points in a plane. We let the horizontal axis represent the real numbers, and the vertical axis the imaginary numbers. Then the coordinates of a point in the plane represent the real and imaginary parts of the corresponding complex number:




In the study of ordinary differential equations, we will often see complex numbers arise in exponential functions. Therefore we now turn our attention to finding a better understanding of exponentials.

First of all, we need to say what we mean by $e^z$ when $z$ is complex. To answer this, we turn to the power series representation of the exponential function:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

(Here, we use the standard convention when working with power series that $z^0 = 1$, even when $z = 0$.) This series has an infinite radius of convergence and therefore converges for all complex numbers $z$.

To work with a complex exponent, we usually write it in terms of its real and imaginary parts, and then use a law of exponents[1] to separate these:

$$e^z = e^{\alpha+\beta i} = e^{\alpha}e^{\beta i}.$$

[1]The law $e^{x+y} = e^x e^y$ is true for real as well as for complex exponents. The proof uses the series representation for $e^z$ and rearranges terms in the sum. It involves a careful use of the Binomial Theorem. See [**4**].

Therefore it will be profitable for us if we now focus our attention on expressions of the form $e^{\beta i}$, and that's where the power series representation becomes helpful:

$$\begin{aligned}
e^{\beta i} &= \sum_{n=0}^{\infty} \frac{(\beta i)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\beta^n i^n}{n!} \\
&= \sum_{n=0,\ n\ even}^{\infty} \frac{\beta^n i^n}{n!} + \sum_{n=0,\ n\ odd}^{\infty} \frac{\beta^n i^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\beta^{2n} i^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\beta^{2n+1} i^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{\beta^{2n}(-1)^n}{(2n)!} + \sum_{n=0}^{\infty} \frac{\beta^{2n+1} i(-1)^n}{(2n+1)!} \\
&= \cos(\beta) + i\sin(\beta).
\end{aligned}$$

Combining this with the previous result gives us **Euler's formula**:

**Euler's Formula**

$$e^{\alpha+\beta i} = e^{\alpha}(\cos(\beta) + i\sin(\beta))$$

Note that in the calculation above, we made use of the power series for sine and cosine:

$$\sin(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}(-1)^n}{(2n+1)!} \quad \text{and} \quad \cos(z) = \sum_{n=0}^{\infty} \frac{z^{2n}(-1)^n}{(2n)!}$$

These series also allow us to define sine and cosine for complex arguments, and this will be explored briefly in the exercises below.

EXERCISE B.5: Find the values of $e^{\pi i}$, $e^{2\pi i}$ and $e^{\frac{1}{2}\ln(2)-\frac{\pi i}{4}}$. Sketch these as points in the complex plane.

EXERCISE B.6: Prove that, if the solutions of the quadratic equation $ax^2+bx+c=0$ are complex numbers, and if the coefficients $a$, $b$ and $c$ are all real numbers, then the solutions are complex conjugates of one another. *(Hint: Taking the complex conjugate of both sides of the equation gives us* $\overline{ax^2+bx+c}=0$*; now take advantage of the result in Exercise B.2.)*

**Exercise B.7:** Use power series to prove that $\sin(z) = \frac{e^{-iz}-e^{iz}}{2}$. Use this formula to evaluate $\sin(i)$.

**Exercise B.8:** Find a representation formula for $\cos(z)$ (similar to the one for sine above) and use it to evaluate $\cos(2i)$.

APPENDIX C

# Reduction of Order

Sometimes it is not too hard to find one nontrivial solution of a second order differential equation, and the method we explore next can often provide us with a means of moving from just one solution to a general solution.

EXAMPLE C.1: Consider the second order differential equation $\ddot{y}-y=0$. We observe that the function $y_1(t)=e^t$ is a solution on the interval $\mathbb{R}$, and that this solution is non-zero for all $t\in\mathbb{R}$. If $y(t)$ is *any* solution of the ODE, let $u(t)$ be defined by $u(t)=\frac{y(t)}{y_1(t)}$, or $y=uy_1$. We substitute this into the ODE to see that

$$\begin{aligned} 0 &= \ddot{y}-y \\ &= \frac{d^2}{dt^2}\left[ue^t\right]-\left(ue^t\right) \\ &= \left(\ddot{u}e^t+2\dot{u}e^t+ue^t\right)-\left(ue^t\right) \\ &= \ddot{u}e^t+2\dot{u}e^t. \end{aligned}$$

Dividing by $e^t$, which is never zero, gives us the following differential equation for $u$:

$$\ddot{u}+2\dot{u}=0.$$

Make the substitution $v=\dot{u}$ to obtain

$$\dot{v}+2v=0.$$

This equation can be solved using the method of integrating factors (the integrating factor is $e^{2t}$):

$$\begin{aligned} \frac{d}{dt}\left[e^{2t}v\right] &= 0 \\ e^{2t}v &= C \\ v &= Ce^{-2t} \end{aligned}$$

Integrating this shows that $u = Ce^{-2t} + D$, and inserting this into the equation $y = uy_1$ we see that

$$y(t) = (Ce^{-2t} + D)e^t = Ce^{-t} + De^t$$

is the general solution of the ODE. □

The procedure used in this example is called **reduction of order**, and the general process is as follows.

### Reduction of Order

- Find a solution $y_1$ of the ODE
- set $y = uy_1$, and apply this substitution for $y$ in the ODE
- simplify to find a differential equation for $u$
- find a general solution for $u$
- the product $y = uy_1$ gives the general solution for the ODE on the set where $y_1 \neq 0$

The last point is an important one: because we define $u$ by $u = \frac{y}{y_1}$, this process is only guaranteed to give a formula for a general solution on the set where $y_1 \neq 0$. One might get lucky and obtain a general solution on a larger domain, but there is no guarantee that will happen in general.

EXERCISE C.1: Verify that $y_1(x) = e^{-x}$ is a solution of the differential equation $y'' + 3y' + 2y = 0$. Then use reduction of order to find a general solution of this ODE defined on $\mathbb{R}$.

EXERCISE C.2: Verify that $y_1(x) = e^{2x}$ is a solution of $y'' - 2y' = 0$. The use reduction of order to find a general solution on $\mathbb{R}$.

EXERCISE C.3: Verify that $y_1(t) = t$ is a solution of the ODE $t^2\ddot{y} + 2t\dot{y} - 2y = 0$. Then use reduction of order to find the general solution of this ODE defined on the interval $(0, \infty)$.

EXERCISE C.4: Verify that $y_1(t) = \sin(2t)$ is a solution of the ODE $\ddot{y} + 4y = 0$. Then use reduction of order to find a general solution on $\mathbb{R}$. *(Note: Because $y_1(t) = 0$ for $t = \frac{k\pi}{2}$, the method only guarantees a solution on an interval of the form $\left(\frac{k\pi}{2}, \frac{(k+1)\pi}{2}\right)$; therefore you will need to verify directly that the formula you obtain is a solution on $\mathbb{R}$.)*

Now we will explore the theory of this method – that is to say, we will discuss why it works. Before getting into the details, let's point out that although the idea of finding a general solution of the form $y = uy_1$, where $y_1$ is a known solution, might seem like a big intuitive leap, it is related to an idea which the reader has certainly seen before. For example, if we know that $x_1$ is a root of some polynomial equation $p(x) = 0$, we can try to find other roots by writing $p(x)$ as $(x-x_1)q(x)$ and trying to find the roots of $q(x)$. This idea is where polynomial division comes from. Finding one particular solution as a stepping stone to finding more solutions is a deep and powerful idea in mathematics.

Now, on to the details of the theory. Begin with an ODE of the form

$$a(x)y'' + b(x)y' + c(x)y = 0,$$

and a function $y_1(x)$ which is a solution of this equation. Let $I$ be an open interval where $y_1 \neq 0$. Then if $y(x)$ is *any* solution of this ODE on $I$, we can define $u = \frac{y}{y_1}$ on $I$; thus $y = uy_1$. The product rule gives us $y' = u'y_1 + uy_1'$ and $y'' = u''y_1 + 2u'y_1' + uy_1''$. Inserting these into the ODE yields

$$\begin{aligned} 0 =& a(x)y'' + b(x)y' + c(x)y \\ &= a(x)\left(u''y_1 + 2u'y_1' + uy_1''\right) + b(x)\left(u'y_1 + uy_1'\right) + c(x)(uy_1) \\ &= a(x)y_1u'' + (2a(x)y_1' + b(x)y_1)u' + (a(x)y_1'' + b(x)y_1' + c(x)y_1)u, \end{aligned}$$

and the last term in the last line is zero on $I$ since $y_1$ satisfies the ODE there. We are left with

$$a(x)y_1u'' + (2a(x)y_1' + b(x)y_1)u' = 0.$$

If we make the subsitution $v = u'$, we get

$$a(x)y_1(x)v' + (2a(x)y_1'(x) + b(x)y_1(x)v = 0.$$

This is a first-order equation which can often be solved to find a general formula for $v$, and integrating that solution gives us a general formula for $u$; inserting that formula for $u$ into the equation $y = uy_1$ gives us a general formula for $y$ on $I$.

It is because of the fact we always obtain an equation of the form $\tilde{a}(x)u'' + \tilde{b}(x)u' = 0$, which can be reduced to a first order equation $\tilde{a}(x)v' + \tilde{b}(x)v = 0$ via the substitution $v = u'$, that this method gets it name.

**EXERCISE C.5:** Verify that the function $y_1(t) = e^t$ is a solution of the third order ODE $y''' - y = 0$. Then let $y$ be any other solution of the ODE, and use reduction of order to show that $y = ue^x$, where

$u$ is a solution of the ODE $u''' + 3u'' + 3u' = 0$. *(You may try to solve this ODE after finishing Chapter 7. Start by making a substitution.)*

EXERCISE C.6: Find a power function $y_1(x) = x^n$ that solves the differential equation $x^2y'' - 3xy' + 4y = 0$. Then use reduction of order to find a general solution on the interval $(0, \infty)$. *(Hint: Plug $y_1$ into the differential equation to obtain an algebraic equation for $n$.)*

EXERCISE C.7: Find a power function $y_1(x) = x^n$ that solves the differential equation $x^2y'' - 7xy' - 6y = 0$. Then use reduction of order to find a general solution on the interval $(0, \infty)$.

EXERCISE C.8: Consider the ODE $y'' - 2\alpha y' + \alpha^2 y = 0$, where $\alpha$ is a constant. **(a)** Find a value of $r$ such that $y_1(x) = e^{rx}$ is a solution of this ODE. **(b)** Use the solution you found in part (a) and reduction of order to find a general solution of the ODE.

EXERCISE C.9: Consider the ODE $y'' - (\alpha + \beta)y' + \alpha\beta y = 0$, where $\alpha,\ \beta$ are constants and $\alpha \neq \beta$. **(a)** Prove that the only values of $r$ such that $y_1(x) = e^{rx}$ solves the ODE are $r = \alpha$ and $r = \beta$. **(b)** Use the solution $y_1(x) = e^{\alpha x}$ and reduction of order to find the general solution of the ODE. Simplify your solution.

APPENDIX D

# Matrix Algebra

In this appendix we introduce some basic terminology and notation used in linear algebra.

Our first goal here is to develop an efficient means of representing a system of $m$ linear algebraic equations in $n$ unknowns, $x_1, x_2, ..., x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &= b_m \end{aligned} \quad . \tag{14}$$

MATRICES

To achieve this end, our main objects of discussion will be **matrices** (which is the plural form of the word *matrix*). A **matrix** is a collection of numbers $a_{ij}$, where $i$ and $j$ are independent **indices**: $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$; the symbol $i$ here is an **index**, as is the symbol $j$ (*indices* is the plural form of *index*). Each number $a_{ij}$ is called an **entry** of the matrix. We often display a matrix as a rectangular array in the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \tag{15}$$

and we can use a single symbol, such as $A$, to denote the entire matrix. It is common to denote a matrix by a capital letter, such as $B$, and its entries by the corresponding lower-case letter with indices, such as $b_{ij}$. Alternatively, we can indicate the entries of $B$ by using the functions $ent_{ij}$, which 'extract' the entries: $ent_{ij}(B) = b_{ij}$.

EXAMPLE D.1: For the matrix $B = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$, we have $b_{22} = -1$ and $ent_{21}(B) = 3$. □

Horizontal subsets of (15) are called **rows**, and vertical subsets are called **columns**. Therefore the matrix in (15) has $m$ rows, and it has $n$ columns. These numbers are the

**dimensions** of the matrix; we can also say the the **dimension** of the matrix is $m \times n$. (This is read aloud as " m by n".) If the entries of the matrix $A$ are real numbers, we abbreviate all this by writing $A \in \mathbb{R}^{m\times n}$ *('A is in the set of $m \times n$ real matrices')*. But we can also consider matrices with complex-valued entries, in which case we would write $A \in \mathbb{C}^{m\times n}$ *('A is in the set of $m \times n$ complex matrices')*. We say that two matrices are **equal** if their corresponding coefficients are equal: $A = B$ if and only if $a_{ij} = b_{ij}$ for all indices $i, j$. Two matrices can only be equal if they have the same dimension.

EXAMPLE D.2: Consider the matrix

$$C = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -1 & 3 \end{bmatrix}.$$

The matrix $C$ has 2 rows and 3 columns, so the dimension of this matrix is $2 \times 3$. All the entries are real numbers, so $C \in \mathbb{R}^{2\times 3}$. The entries of the matrix are

$$c_{11} = 2,\ c_{12} = -1,\ c_{13} = 3,\ c_{21} = 4,\ c_{22} = -1,\ c_{23} = 3.$$

□

Note from the previous example that, even though the second and third columns are identical, we still say that the matrix has 3 columns.

EXERCISE D.1: Consider the matrix

$$D = \begin{bmatrix} 3 & 5 \\ 2 & -1 \\ -3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Identify the dimension of $D$, and list all of its entries.

If a matrix has only one row or one column, we refer to it as a **vector**. An $m \times 1$ matrix is called a **column vector**, and a $1 \times n$ matrix is called a **row vector**.

Now let us start defining algebraic operations on matrices. When two matrices have the same dimension, we can define **addition** of the matrices by just adding corresponding entries in the same positions. That is, the **sum** of $A$ and $B$ is denoted by $A + B$, and it's entries are $ent_{ij}(A + B) = ent_{ij}(A) + ent_{ij}(B)$. **Subtraction** of matrices is performed similarly, with the **difference** $A - B$ defined by $ent_{ij}(A - B) = ent_{ij}(A) - ent_{ij}(B)$. Any matrix can be multiplied by a **scalar**, meaning a real-or complex number, according to the rule $ent_{ij}(sA) = sent_{ij}(A)$. We follow the conventional order or operations for real and complex numbers; so, for example, $2A + 3B$ requires that both scalar multiplications be performed before the addition.

EXAMPLE D.3: Let $A = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$. Then

$$A + B = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}, \quad B - A = \begin{bmatrix} -3 & -2 \\ 1 & 3 \end{bmatrix}, \quad 3A = \begin{bmatrix} 6 & 9 \\ -3 & -3 \end{bmatrix} \quad \text{and} \quad 3A + B = \begin{bmatrix} 5 & 10 \\ -3 & -1 \end{bmatrix}.$$

□

All this suggests the following natural question: can we multiply two matrices together? Of course, we could try to define multiplication of matrices in a similar way – by multiplying corresponding entries – but that turns out not to be very useful for our purposes. Instead, we define matrix **multiplication** as follows: if $A$ is $m \times n$ and $B$ is $n \times l$, then the **product** $AB$ is defined by $ent_{ij}(AB) = \sum_{k=1}^{n} ent_{ik}(A)ent_{kj}(B)$. This definition may not seem intuitive at first, but we will see shortly how useful it is. It is important to note that we do not require the matrices to have the same dimensions for multiplication; instead, we can only compute the product $AB$ if *the number of columns of $A$ is equal to the number of rows of $B$*. The result is a matrix $AB$ with the same number of rows as $A$ and the same number of columns as $B$.

EXAMPLE D.4: Let $A = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -2 & 0 & 1 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$. Then $A$ is $2 \times 3$ and $B$ is $3 \times 4$. Therefore $AB$ is defined and is a $2 \times 4$ matrix. (But note that $BA$ is not defined!) The first entry of the product matrix $AB$ is

$$ent_{11}(AB) = \sum_{k=1}^{3} a_{1k}b_{k1} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = (2)(1) + (3)(-2) + (-1)(3) = -7.$$

Doing this for all the entries of $AB$ gives us

$$AB = \begin{bmatrix} -7 & -2 & 6 & 1 \\ 2 & 1 & -1 & 2 \end{bmatrix}$$

□

EXERCISE D.2: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$. Compute both $AB$ and $BA$.

The previous examples shows that *matrix multiplication is not commutative*: even when both products are defined, and even when both products have the same dimensions, $AB$ and $BA$ may not be equal.

Now we are ready explain why this is a useful way to define matrix multiplication. Consider the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then $AX$ is an $m \times 1$ matrix (the same a $B$):

$$AX = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n \end{bmatrix}.$$

Thus the statement $AX = B$ *is equivalent to* the system of linear equations in (14). It is a compact, efficient way of expressing the same information: a system of linear equations can be written as a single matrix equation. The matrix $A$ is called the **coefficient matrix** for the system, as it contains the coefficients of the unknowns $x_j$ in the corresponding system.

EXAMPLE D.5: The system of algebraic equations

$$2x + 3y = 7$$
$$x - 3y = 0$$

is equivalent to the matrix equation

$$\begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}.$$

□

EXERCISE D.3: Express the system of equations

$$2x + 4y - z = 0$$
$$x + y + z = 1$$
$$4z - 2x = 2$$

as a matrix equation. *(Hint: Rewrite the last equation in the system so that the terms line up better with the lines above, and remember that there is a hidden $0y$ there.)*

One very important matrix in linear algebra is the $n \times n$ **identity matrix** defined by

$$ent_{ij}(I) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

In full display, the identity matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

It is probably easiest to remember the description of $I$ in words: $I$ has 1's along the **main diagonal** which goes from the top left to the bottom right of the matrix, and all the other entries of $I$ are 0.

In matrix multiplication, the identity matrix plays the same role as the number 1 does in the multiplication of numbers: for any matrix $A$,

$$AI = IA = A. \tag{16}$$

$I$ is always assumed to have the necessary dimensions to make any matrix multiplications valid, even if this varies from one occurrence to the next. Therefore, if $A$ is $m \times n$, then the first $I$ in (16) is $n \times n$ and the second $I$ is $m \times m$. If it is ever important to specify that the identity matrix has a certain size, that can be done with subscripts: the $n \times n$ identity matrix can be written as $I_{n \times n}$, or even just $I_n$.

EXERCISE D.4: Verify equation (16) for $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \end{bmatrix}$.

DETERMINANTS

A system of linear equations with the same number of equations as unknowns would be represented using a **square** coefficient matrix $A$, meaning it would have the same number of rows as columns. (Every identity matrix is a square matrix.) Such a system could have either no solution, one solution, or infinitely many solutions, depending in part on the right sides of the equations. But when a certain condition is satisfied by a square coefficient matrix $A$, we can be guaranteed that there will always be exactly one solution $X$ of the matrix equation $AX = B$, *regardless of what $B$ is*. We will state this condition presently.

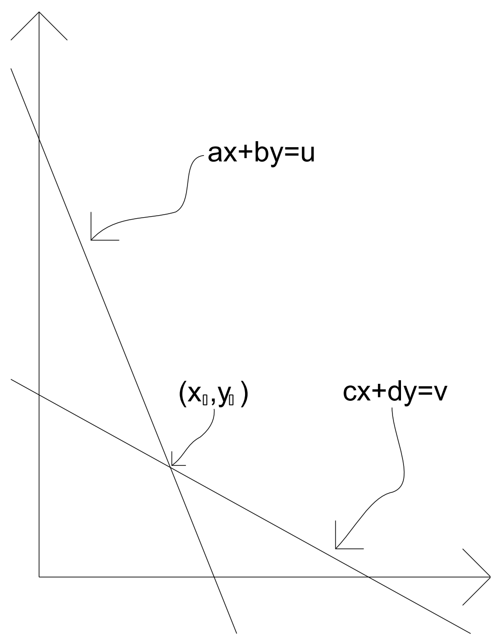
First, to gain some insight, let's restrict our attention to the situation with two equations in two unknowns:

$$\begin{aligned} ax + by &= u \\ cx + dy &= v \end{aligned}. \tag{17}$$

This can be represented by the matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

We can also visualize it geometrically as a pair of lines in the $xy$-plane, with a solution of the equations corresponding to the coordinates of the point of intersection, if there is one. The figure below shows a possible graph of this pair of lines.




We'll know for sure that these lines will have a unique intersection of the lines are *not parallel*, which is the case if they have *different slopes*. If the two lines do have the same slope, then either (a) they have no intersection (in which case the system in (17) has no solution), or (b) the two equations both represent the same line (in which case (17) actually has infinitely many solutions).

EXERCISE D.5: Consider the pair of lines defined by the equations in (17). **(a)** Prove that, if both lines are vertical, then $ad - bc = 0$. **(b)** Prove that, if the lines are not vertical but are still parallel, then $ad - bc = 0$. **(c)** Prove that, if $ad - bc = 0$, then the lines are parallel.

We define the **determinant** of a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be

$$det(A) = ad - bc.$$

The determinant function can also be defined for larger square matrices (see any textbook on linear algebra, for example, [**5**].). The solution of the previous exercise proves the following important theorem when $n = 2$.

THEOREM 1. *Suppose $A$ is a square $n \times n$ matrix and $X,\ B$ are $n \times 1$ vectors.*

*(1) If $\det(A) \neq 0$, then $AX = B$ has a unique solution $X$ for every given $B$;*

*(2) If $\det(A) = 0$, then $AX = B$ has either infinitely many solutions $X$ or no solutions, depending on $B$.*

EXAMPLE D.6: The system of equations

$$\begin{aligned} 2099x - y &= u \\ 1010x + 2y &= v \end{aligned}$$

has a unique solution for each choice of $u,\ v$ because the determinant of the coefficient matrix is not zero:

$$det \begin{bmatrix} 2099 & -1 \\ 1010 & 2 \end{bmatrix} = (2099)(2) - (-1)(1010) = 5208 \neq 0.$$

On the other hand, if there is a solution $(x, y)$ to the system

$$\begin{aligned} 44x - 99y &= u \\ -12x + 27y &= v \end{aligned},$$

then there must be infinitely many solutions because the determinant of the coefficient matrix is zero:

$$det \begin{bmatrix} 44 & -99 \\ -12 & 27 \end{bmatrix} = (44)(27) - (-99)(-12) = 0.$$

□

When $det(A) = 0$, it can still be hard to determine whether there are infinitely many solutions of $AX = B$ or no solutions at all, because that usually depends on what $B$ is. However, there is one situation for which we can always give a complete description.

THEOREM 2. *If* $det(A) \neq 0$*, then the only solution of* $AX = 0$ *is* $X = 0$*. If* $det(A) = 0$*, then the matrix equation* $AX = 0$ *has infinitely many solution vectors* $X$*.*

In the matrix equation $AX = 0$, it is understood that $0$ does not represent a scalar – instead, $0$ represents a vector whose entries are all zeros and whose dimension matches that of $AX$.

## EIGENVALUES AND EIGENVECTORS

An common matrix equation we need to solve is $AX = \lambda X$. Here, $A$ is a given square matrix and both $X$ and $\lambda$ are unknowns: $X$ is a vector, and $\lambda$ is a scalar. Finding solutions of this equation means finding vectors $X$ for which $AX$ is just a scalar multiple of $X$.

EXERCISE D.6: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. **(a)** Verify that if $X = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, then $AX$ is a scalar multiple of $X$. What is the scalar? **(b)** Show that if $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then $AX$ *is not* a multiple of $X$.

Observe that if $X = 0$, then the equation $AX = \lambda X$ is always true, for any $A$ and for any $\lambda$. This is not very interesting or useful. We will only be concerned with non-zero vectors $X$ which satisfy the equation (though it is perfectly permissible for $\lambda$ to be the zero scalar, because that is not a trivial case). When there is a non-zero vector $X$ and a scalar $\lambda$ satsifying $AX = \lambda X$, we call $\lambda$ an **eigenvalue** of $A$, and we call $X$ a corresponding **eigenvector** of $A$.

There is a straightforward method of finding eigenvalues and eigenvectors for square matrices $A$ (and it is fairly efficient, provided the matrix $A$ is not too big). Notice that $\lambda X$ is the same as $\lambda IX$, and that $AX = \lambda IX$ if and only if $\lambda IX - AX = 0$ (a matrix whose entries are all zeros). Factoring out $X$ allows us to write $(\lambda I - A)X = 0$. According to Theorem (2), this system has infinitely many solutions (and therefore non-zero solutions) $X$ precisely when $det(\lambda I - A) = 0$. This is the key insight we will use to find eigenvalues. What makes it useful is the fact that $\det(\lambda I - A) = 0$ is a scalar equation which we can always solve (if the dimension of $A$ is not too large). We call this equation the **characteristic equation** of the matrix $A$.

### Finding Eigenvalues of a Matrix

The eigenvalues of a square matrix $A$ are precisely the solutions of the characteristic equation

$$\det(\lambda I - A) = 0.$$

Once we know the eigenvalues, we can insert them into the equation $(\lambda I - A)X = 0$ to find corresponding eigenvectors.

EXAMPLE D.7: Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} -8 & 10 \\ -5 & 7 \end{bmatrix}$.

The characteristic equation is

$$0 = det \begin{bmatrix} \lambda + 8 & -10 \\ 5 & \lambda - 7 \end{bmatrix} = (\lambda + 8)(\lambda - 7) - (-10)(5) = \lambda^2 + \lambda - 6,$$

or $0 = (\lambda + 3)(\lambda - 2)$, which has solutions $\lambda = -3$ and $\lambda = 2$. These are the eigenvalues of $A$.

For the eigenvalue $\lambda = 2$, we get the equation $(2I - A)X = 0$, which would be written out as

$$\begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This is equivalent to the system of equations

$$\begin{aligned} 10x_1 - 10x_2 &= 0 \\ 5x_1 - 5x_2 &= 0 \end{aligned}.$$

Notice that these equations are algebraically equivalent, so any solution of one is also a solution of the other. We can see that the solutions must satisfy $x_1 = x_2$; therefore any scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution of the equation $(2I - A)X = 0$. In particular, any *non-zero* scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda = 2$.

Similarly, inserting $\lambda = -3$ into $(\lambda I - A)X = 0$ gives us

$$\begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and the solutions of this satisfy $x_1 = 2x_2$. Thus *non-zero* multiples of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are eigenvectors of $A$ corresponding to the eigenvalue $\lambda = -3$. $\square$

It will always be the case that, if $X$ is an eigenvector of $A$, then so is any non-zero scalar multiple of $X$, because $AX = \lambda X$ implies that, for any scalar $s$, $A(sX) = sAX = s\lambda X = \lambda(sX)$.

EXERCISE D.7: Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} -2 & \frac{3}{4} \\ 0 & 1 \end{bmatrix}$.

EXERCISE D.8: Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

EXERCISE D.9: Find the eigenvalues and corresponding eigenvectors of $4I$, where $I$ is the $2 \times 2$ identity matrix.

EXERCISE D.10: Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

The last two exercises illustrate what can happen when a $2 \times 2$ matrix $A$ has only one eigenvalue: either $A$ is a scalar multiple of the identity matrix, or the set of eigenvectors of $A$ is merely the set of non-zero scalar multiples of a single vector $X$.

APPENDIX E

# Linear Operators

In advanced mathematics, we say that a real-valued function defined on $\mathbb{R}^n$ is **linear** if it satisfies the following two properties:

(1) $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$; and

(2) $f(cx) = cf(x)$ for all $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

This definition *is not* consistent with what most students are taught in beginning algebra – at that level, we call a function defined on $\mathbb{R}$ linear if its graph is a straight line; however, the properties listed above imply that $f(0) = 0$, which means, according to this definition, a function on $\mathbb{R}$ is linear only if *its graph is a straight line through the origin.* Functions whose graphs are straight lines but do not pass through the origin are instead called **affine**.

EXERCISE E.1: Prove that if $f$ is a linear function on $\mathbb{R}^n$, then $f(0) = 0$. (Here, $0$ indicates the origin of $\mathbb{R}^n$ when it appears as the input of the function, and it represents the real number zero when it appears as the output of the function.)

We can extend the definition of linear to other mathematical objects as well. For example, the same definition works equally well when applied to functions defined on $\mathbb{C}^n$. We will be interested in using the idea of linearity in the context of operators.

A function $F$ whose input and output are both functions defined on the same domain is called an **operator**. For example, let $D$ represent the **differentiation operator** defined on the set of differentiable functions on $\mathbb{R}$, so that

$$Df = f'.$$

Observe that

$$D(f+g) = (f+g)' = f' + g' = Df + Dg$$

and, for any constant $c$,

$$D(cf) = (cf)' = cf' = cDf.$$

These are the same properties listed at the beginning of this section for linear functions; we therefore say that $D$ is an example of a **linear operator**. We use the notation $D^2$ to denote the second-derivative operator, defined by $D^2 f = f''$. The symbol $D^3$ denotes the third derivative operator satisfying $D^3 f = f^{(3)}$, and so on. These are all linear operators.

EXAMPLE E.1: Consider the operator $L = D^2 + 3$, defined by $Lf = (D^2 + 3)f = f'' + 3f$. Then $L$ is linear because, for any twice-differentiable functions $f$ and $g$ we have

$$\begin{aligned} L(f+g) &= (D^2+3)(f+g) \\ &= (f+g)'' + 3(f+g) \\ &= f'' + g'' + 3f + 3g \\ &= f'' + 3f + g'' + 3g \\ &= (D^2+3)f + (D^2+3)g \\ &= Lf + Lg, \end{aligned}$$

and, if $c$ is any scalar,

$$\begin{aligned} L(cf) &= (D^2+3)(cf) \\ &= (cf)'' + 3(cf) \\ &= cf'' + c3f \\ &= c(f'' + 3f) \\ &= c(D^2+3)f \\ &= cLf. \end{aligned}$$

. □

Notice from the previous example that we usually don't use parentheses to surround the input of an operator; instead, we just write the symbol for the operator to the left of the function on which it acts. This should not lead us to confuse the operation with multiplication, since it wouldn't make any sense to multiply a function and an operator together.

Suppose that $p(x)$ is a polynomial, $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$; we use the notation $p(D)$ to denote the linear operator

$$p(D) = a_0 + a_1 D + a_2 D^2 + \cdots + a_k D^k.$$

EXAMPLE E.2: Let $p(x) = x^2 + 1$, and suppose $f(x) = e^{2x} + x$ Then

$$\begin{aligned}
P(D)f(x) &= (D^2 + 1)f(x) \\
&= D^2 f(x) + f(x) \\
&= f''(x) + f(x) \\
&= 4e^{2x} + e^{2x} + x \\
&= 5e^{2x} + x.
\end{aligned}$$

EXERCISE E.2: Prove that, for any polynomial $p(x)$, the operator $p(D)$ is linear.

With this notation, we can efficiently summarize the key idea of Chapter 7 and extend it to higher-order differential equations. The notation $\Pi_{n=1}^{k} a_k = a_1 * a_2 * \cdots * a_k$ is used to denote products (similar to how $\Sigma$ notation is used to denote sums). The following result is a consequence of the observation that, for a differential equation $p(D)y = 0$, the corresponding characteristic equation is $p(r) = 0$.

**General Solutions of $p(D)y = 0$**

Suppose

$$p(x) = \Pi_{k=1}^{n} a_k (x - r_k)^{m_k}$$

is a polynomial. Here, each $r_k \in \mathbb{C}$ is a distinct root of the polynomial, and $m_k$ is the multiplicity of the root $r_k$. Then the general solution of the differential equation

$$p(D)y = 0$$

is

$$y(t) = \sum_{k=1}^{n} \sum_{l=1}^{m_k} c_{k,l} t^{l-1} e^{r_k t},$$

where the $c_{k,l}$ are arbitrary coefficients.

EXAMPLE E.3: Consider the differential equation $y^{(4)} + 2y^{(3)} + y'' + 2y' + y = 0$. This can be written as $p(D)y = 0$, where

$$\begin{aligned}
p(x) &= x^4 + 2x^3 + x^2 + 2x + 1 \\
&= (x + 1)^2 (x - i)(x + i).
\end{aligned}$$

The roots are $x_1 = -1$ (with multiplicity 2) and $x_2 = i$, $x_3 = -i$ (each with multiplicity 1). Therefore, the general solution of this differential equation is

$$y(t) = c_{1,1}e^{-t} + c_{1,2}te^{-t} + c_{2,1}e^{it} + c_{3,1}e^{-it},$$

where $c_{1,1},\ c_{1,2},\ c_{2,1}$ and $c_{3,1}$ are arbitrary constants. □

We can also use this notation to simplify parts of the method of Laplace Transforms.

Consider a differential equation of the form $a\ddot{y} + b\dot{y} + cy = f(t)$, with rest initial conditions $y(0) = 0,\ \dot{y}(0) = 0$. Taking the Laplace Transform of both sides gives us

$$aL[\ddot{y}] + bL[\dot{y}] + cL[y] = L[f],$$

and the reduction formula implies

$$a(s(sL[y] - y(0)) - \dot{y}(0)) + b(sL[y] - y(0)) + cL[y] = L[f].$$

We can simplify this using the rest initial conditions to obtain

$$(as^2 + bs + c)L[y] = L[f],$$

or

$$L[y] = \frac{L[f]}{as^2 + bs + c}.$$

This analysis applies in general to higher-order, constant-coefficient linear differential equations. For an $n^{th}$ order equation, we would use the phrase **rest initial conditions** to specify the $n$ initial values $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$.

**Laplace Transform of $p(D)y = f$ with Rest Initial Conditions**

If $p(x)$ is a polynomial (not the zero function) and $p(D)y = f$, where $y$ satisfies rest initial conditions, then

$$L[y] = \frac{L[f]}{p(s)}.$$

Let's end this section by explaining the origin of the term 'linear differential equation'. In general, an $n^{th}$ order differential equation can be written in the form

$$f(x, y, y', \cdots, y^{(n)}) = g(x),$$

where $y(x)$ is the unknown function. The function $f$ has $n+2$ inputs. We say that the differential equation is **linear** if $f$ is a linear function of the vector $(y, y', \cdots, y^{(n)})$.

This will be easier to understand if we begin by concentrating on first-order equations, which have the form

$$f(x, y, y') = g(x).$$

If $f$ is linear in the vector $(y, y')$, then it satisfies conditions (1) and (2) at the beginning of this section in the following way: for any $(y, y')$ and $(z, z')$ in $\mathbb{R}^2$,

$$f(x, y + z, y' + z') = f(x, y, y') + f(x, z, z')$$

and, for any scalar $c$,

$$f(x, cy, cy') = cf(x, y, y').$$

Using these two conditions with the vectors $(y, 0)$ and $(0, y')$, we can write

$$\begin{aligned} f(x, y, y') &= f(x, 0 + y, y' + 0) \\ &= f(x, 0, y') + f(x, y, 0) \\ &= f(x, y' * 0, y' * 1) + f(x, y * 1, y * 0) \\ &= y' f(x, 0, 1) + y f(x, 1, 0). \end{aligned}$$

Therefore the differential equation can be written as

$$y' f(x, 0, 1) + y f(x, 1, 0) = g(x).$$

If we let $a(x) = f(x, 0, 1)$ and $b(x) = f(x, 1, 0)$, we then have

$$a(x)y' + b(x)y = g(x),$$

which is exactly how we defined first-order linear equations in Chapter 4. At the time, the reason for describing an equation of the form $a(x)y' + b(x)y = g(x)$ as linear may not have been obvious, but we see now that the term comes from the fact that the differential equation itself involves a linear function of the unknown $y$ and its derivative. Similar results can be obtained for higher order differential equations. If the coefficient functions $a(x), b(x)$ (etc.) are constant, then the equation can be written in the form $p(D)y = g$ for some polynomial $p(x)$.

# Index

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