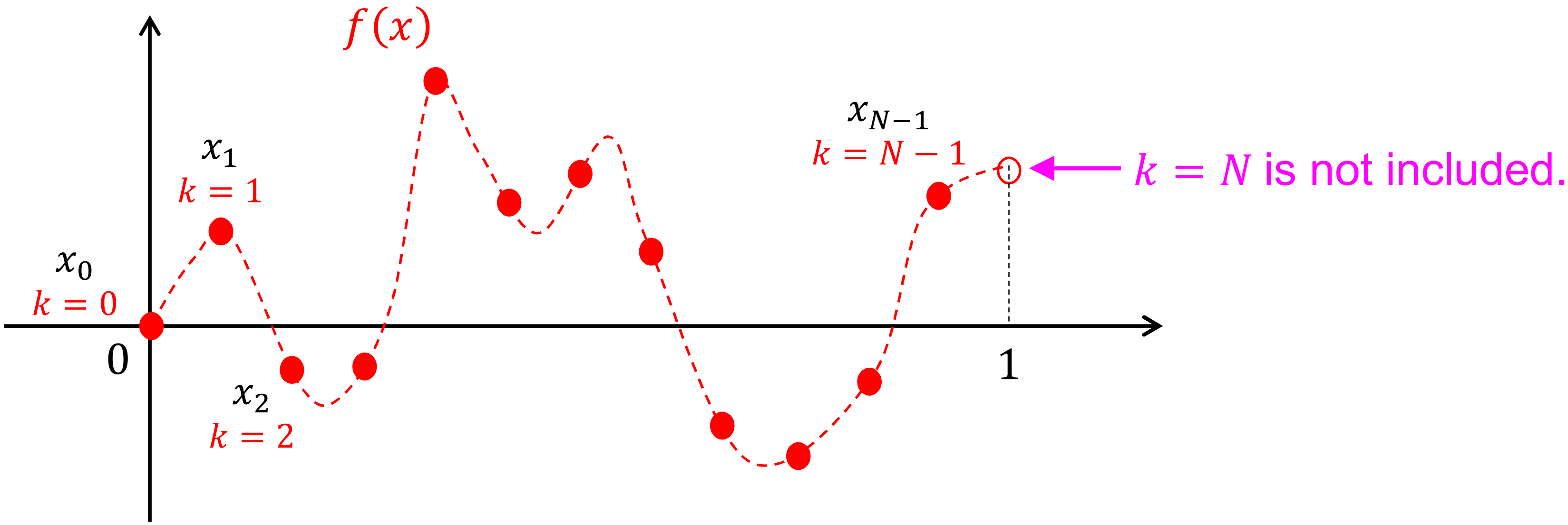
Discrete Fourier Transform (DFT)

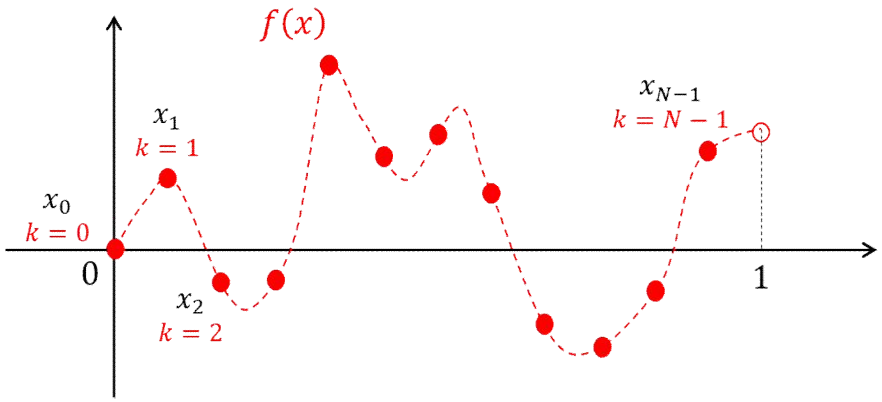


Let $f(x)$ be periodic, for simplicity of Period 1. We assume that N measurements of $f(x)$ are taken over the interval $0 \leq x \leq 1$ at regularly spaced points

$$x_k = \frac{k}{N} \qquad k = 0, 1, 2, \cdots, N-1$$

We now want to determine a complex trigonometric polynomial

$$q(x) = \sum_{n=0}^{N-1} c_n e^{i2\pi nx}$$



that interpolates $f(x)$ at the nodes x_k, that is, $q(x_k) = f(x_k)$.
Denoting $f(x_k)$ with f_k,

$$f_k = f(x_k) = q(x_k) = \sum_{n=0}^{N-1} c_n e^{i2\pi n x_k}$$

$$k = 0, 1, 2, \cdots, N-1$$

cf.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx}$$

$$e^{ix} = \cos x + i \sin x$$

The coefficients are determined by using the orthogonality of the trigonometric system.

$$f_k = \sum_{n=0}^{N-1} c_n e^{i2\pi n x_k}$$

✓ Multiply f_k by $e^{-i2\pi m x_k}$ and sum over k from 0 to $N-1$
✓ Interchange the order of the two summations
✓ Replacing x_k with k/N

$$\sum_{k=0}^{N-1} f_k e^{-i2\pi m x_k} = \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} c_n e^{i2\pi(n-m)x_k} = \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} e^{i(n-m)2\pi k/N}$$

$$e^{i(n-m)2\pi k/N} = \left[e^{i(n-m)2\pi/N}\right]^k = r^k \qquad r = e^{i(n-m)2\pi/N}$$

For $n = m$

$$r^k = (e^0)^k = 1^k = 1 \qquad \sum_{k=0}^{N-1} r^k = N$$

For $n \neq m$

$$r \neq 1 \qquad \sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} = 0$$

$$r^N = e^{i(n-m)2\pi N/N} = e^{i(n-m)2\pi}$$
$$= \cos(n-m)2\pi + i\sin(n-m)2\pi = 1 + 0 = 1$$

Sum of a geometric series

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$= \frac{a(1-r^n)}{1-r} \qquad r \neq 1$$

$$\sum_{k=0}^{N-1} f_k e^{-i2\pi m x_k} = \underline{\sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} e^{i(n-m)2\pi k/N}} = c_0 0 + c_1 0 + \cdots + c_m N + \cdots c_{N-1} 0$$

$$= N c_m$$

$$c_m = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-i2\pi m x_k}$$

$$\sum_{k=0}^{N-1} r^k = N \qquad \text{For } n = m$$

$$\sum_{k=0}^{N-1} r^k = 0 \qquad \text{For } n \neq m$$

Replacing m with n,

$$c_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-i2\pi n x_k} \qquad f_k = f(x_k)$$

$$n = 0, 1, 2, \cdots, N-1$$

$$c_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i2\pi n x_k} \qquad f_k = f(x_k)$$

$$n = 0, 1, 2, \cdots, N-1$$

Discrete Fourier Transform

$$\widehat{f_n} = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k} \qquad f_k = f(x_k)$$

$$n = 0, 1, 2, \cdots, N-1$$

The Discrete Fourier Transform of the given signal $\mathbf{f} = [f_0 \cdots f_{N-1}]^{\mathsf{T}}$ to be the vector $\hat{\mathbf{f}} = [\hat{f}_0 \cdots \hat{f}_{N-1}]$ with components $\widehat{f_n}$

This is the frequency spectrum of the signal.

In vector notation, $\hat{\mathbf{f}} = \mathbf{F}_N\mathbf{f}$, where the $N \times N$ Fourier matrix $\mathbf{F}_{\mathrm{N}} = [e_{nk}]$ has the entries

$$e_{nk} = e^{-i2\pi n x_k} = e^{-i2\pi nk/N} = w^{nk}, \quad w = w_N = e^{-i2\pi/N} \qquad \left(x_k = \frac{k}{N} \right)$$

where $n,\ k = 0, \cdots, N-1$

Example Discrete Fourier Transform (DFT)

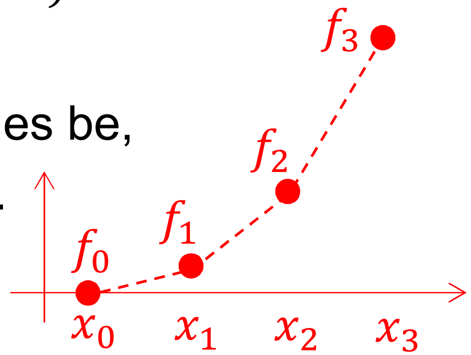
Let $N = 4$ measurements (sample values) be given.

Then
$$w = e^{-i2\pi/N} = e^{-i\pi/2}$$
$$= \cos\pi/2 - i\sin\pi/2$$
$$= -i$$

and thus
$$w^{nk} = (-i)^{nk}$$

$$\hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-i2\pi n x_k}$$
$$\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f}$$
$$\mathbf{F}_\mathrm{N} = [e_{nk}]$$
$$e_{nk} = e^{-i2\pi n x_k} = e^{-i2\pi nk/N} = w^{nk}$$
$$w = w_N = e^{-i2\pi/N}$$
$$e^{ix} = \cos x + i\sin x$$

Let the sample values be,
say $\mathbf{f} = [0 \;\; 1 \;\; 4 \;\; 9]^\mathsf{T}$.



Then

$$\hat{\mathbf{f}} = \mathbf{F}_4\mathbf{f} = \begin{array}{c} \\ n=0 \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} k=0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \end{array} \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}$$