Fourier Analysis

Autumn 2018

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7th November 2018 at 13:14

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http://www.jamescannon.net/teaching/fourier-analysis

http://raw.githubusercontent.com/NanoScaleDesign/FourierAnalysis/master/fourier_analysis.pdf

Contents

Chapter 0

Course information

0.1 This course

This is the Autumn 2018 Fourier Analysis course studied by 3rd-year undegraduate international students at Kyushu University.

0.1.1 How this works

- In contrast to the traditional lecture-homework model, in this course the learning is self-directed and active via publicly-available resources.
- Learning is guided through solving a series of carefully-developed challenges contained in this book, coupled with suggested resources that can be used to solve the challenges with instant feedback about the correctness of your answer.
- There are no lectures. Instead, there is discussion time. Here, you are encouraged to discuss any issues with your peers, teacher and any teaching assistants. Furthermore, you are encouraged to help your peers who are having trouble understanding something that you have understood; by doing so you actually increase your own understanding too.
- Discussion-time is from 10:30 to 12:00 on Wednesdays at room W4-529.
- Peer discussion is encouraged, however, if you have help to solve a challenge, always make sure you do understand the details yourself. You will need to be able to do this in an exam environment. If you need additional challenges to solidify your understanding, then ask the teacher. The questions on the exam will be similar in nature to the challenges. If you can do all of the challenges, you can get 100% on the exam.
- Every challenge in the book typically contains a **Challenge** with suggested **Resources** which you are recommended to utilise in order to solve the challenge. Occasionally the teacher will provide extra **Comments** to help guide your thinking. A **Solution** is also made available for you to check your answer. Sometimes this solution will be given in encrypted form. For more information about encryption, see section 0.3.
- For deep understanding, it is recommended to study the suggested resources beyond the minimum required to complete the challenge.
- The challenge document has many pages and is continuously being developed. Therefore it is advised to view the document on an electronic device rather than print it. The date on the front page denotes the version of the document. You will be notified by email when the document is updated. The content may differ from last-year's document.
- A target challenge will be set each week. This will set the pace of the course and defines the examinable material. It's ok if you can't quite reach the target challenge for a given week, but then you will be expected to make it up the next week.
- You may work ahead, even beyond the target challenge, if you so wish. This can build greater flexibility into your personal schedule, especially as you become busier towards the end of the semester.
- Your contributions to the course are strongly welcomed. If you come across resources that you found useful that were not listed by the teacher or points of friction that made solving a challenge difficult, please let the teacher know about it!

0.1.2 Assessment

In order to prove to outside parties that you have learned something from the course, we must perform summative assessments. This will be in the form of a satisfactory challenge-log (weighted 5%), coursework (weighted 15%), a mid-term exam (weighted 30%) and a final exam (weighted 50%).

Your final score is calculated as Max(final exam score, weighted score), however you must pass the final exam to pass the course.

0.1.3 What you need to do

- Prepare a challenge-log in the form of a workbook or folder where you can clearly write the calculations you perform to solve each challenge. This will be a log of your progress during the course and will be occasionally reviewed by the teacher.
- You need to submit a brief report at `https://goo.gl/forms/vPPCYOZ6nW5ns1mm1` by 8am on the day of the class. Here you can let the teacher know about any difficulties you are having and if you would like to discuss anything in particular.
- Please bring a wifi-capable internet device to class, as well as headphones if you need to access online components of the course during class. If you let me know in advance, I can lend computers and provide power extension cables for those who require them (limited number).

0.2 Timetable

	Discussion	Target	Note
1	3 Oct	-	
2	17 Oct	2.11	
3	24 Oct	3.10	
4	31 Oct	3.17	
5	7 Nov	3.26	
6	14 Nov	4.4	
7	21 Nov		
8	28 Nov		
9	5 Dec	Mid-term exam	
10	12 Dec	-	
11	19 Dec		
12	9 Jan		
13	16 Jan		
14	23 Jan	Coursework	
15	6 Feb	Final exam	Tentative

0.3 Hash-generation

Some solutions to challenges are encrypted using MD5 hashes. In order to check your solution, you need to generate its MD5 hash and compare it to that provided. MD5 hashes can be generated at the following sites:

- Wolfram alpha: (For example: md5 hash of "q1.00") `http://www.wolframalpha.com/input/?i=md5+hash+of+%22q1.00%22`
- `www.md5hashgenerator.com`

Since MD5 hashes are very sensitive to even single-digit variation, you must enter the solution *exactly.* This means maintaining a sufficient level of accuracy when developing your solution, and then entering the solution according to the format suggested by the question. Some special input methods:

Solution	**Input**
5×10^{-476}	5.00e-476
5.0009×10^{-476}	5.00e-476
$-\infty$	-infinity (never "infinite")
2π	6.28
i	im(1)
2i	im(2)
1 + 2i	re(1)im(2)
-0.0002548 i	im(-2.55e-4)
1/i = i/-1 = -i	im(-1)
$e^{i2\pi}$ $[= cos(2\pi) + isin(2\pi) = 1 + i0 = 1]$	1.00
$e^{i\pi/3}$ $[= cos(\pi/3) + isin(\pi/3) = 0.5 + i0.87]$	re(0.50)im(0.87)
Choices in order A, B, C, D	abcd

The first 6 digits of the MD5 sum should match the first 6 digits of the given solution.

Chapter 1

Hash practise

1.1 Hash practise: Integer

X = 46.3847
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

hash of aX = e77fac

1.2 Hash practise: Decimal

X = 49
Form: Two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

hash of bX = 82c9e7

1.3 Hash practise: String

X = abcdef
Form: String.
Place the indicated letter in front of the number.
Example: aX where $X = abc$ is entered as aabc

hash of cX = 990ba0

1.4 Hash practise: Scientific form

X = 500,765.99
Form: Scientific notation with the mantissa in standard form to 2 decimal place and the exponent in integer form.
Place the indicated letter in front of the number.
Example: aX where $X = 4 \times 10^{-3}$ is entered as a4.00e-3

hash of dX = be8a0d

1.5 Hash practise: Numbers with real and imaginary parts

X = $1 + 2i$
Form: Integer *(for imaginary numbers, "integer" means to write both the real and imaginary parts of the number as integers. If you were instructed to enter to two decimal places, then you would need to enter each of the real and imaginary parts to two decimal places. Refer to section 0.3 to see an example of how to handle input of imaginary numbers.)*
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

hash of eX = 4aa75a

1.6 π and imaginary numbers

X $= -2\pi i$
Form: Two decimal places. Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

hash of fX = ad3e8b

1.7 Imaginary exponentials

Note that you will need to understand how to expand exponentials in terms of their sines and cosines in order to do this. If you do not understand how to do this yet, skip this challenge and come back to it later.

X $= 4e^{i3\pi/4}$
Form: Two decimal places. Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

hash of gX = 59a753

Chapter 2

Periods and frequencies

2.1 Period at 50 THz

Resources

- Video: `https://www.youtube.com/watch?v=v3CvAW8BDHI`

Challenge

A signal is oscillating at a frequency of 50 THz. What is the period?

Solution

X = Your solution
Form: Scientific notation with the mantissa in standard form to 2 decimal places and the exponent in integer form.
Place the indicated letter in front of the number.
Example: aX where $X = 4.0543 \times 10^{-3}$ is entered as a4.05e-3

Hash of qX = 3faf81

2.2 Frequency with k=1

Comment

Note that we're working in radians here. From now on a factor of 2π will be included in the oscillations so that $\sin(2\pi t)$ will complete 1 cycle in 1 second. (If you calculator defaults to degrees, be sure to change it to radians for this course.)

Challenge

What is the frequency of $\sin(2\pi kt)$, where t is time in seconds and $k = 1$?

Solution

(Hz)

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of eX = 720149

2.3 Frequency with k=2

Challenge

What is the frequency of $\sin(2\pi kt)$, where t is time in seconds and $k = 2$?

Solution

(Hz)

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of rX = 96ba66

2.4 The meaning of k

Challenge

Considering the previous two challenges, what does k physically represent in those challenges?

Solution

Please compare your answer with your partner in class or discuss with the teacher.

2.5 Smallest period with k=1

Resources

- Book: 1.3 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video (14m10s to 17m00s): `https://youtu.be/1rqJl7Rs6ps?t=14m10s`

Challenge

What is the smallest period of $sin(2\pi kt)$, where t is time in seconds and $k = 1$?

Solution

(s)

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of wX = 25c4fb

2.6 Smallest period with k=2

Resources

- Book: 1.3 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video (14m10s to 17m00s): `https://youtu.be/1rqJl7Rs6ps?t=14m10s`

Challenge

What is the smallest period of $sin(2\pi kt)$, where t is time in seconds and $k = 2$?

Solution

(s)

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of tX = bb995f

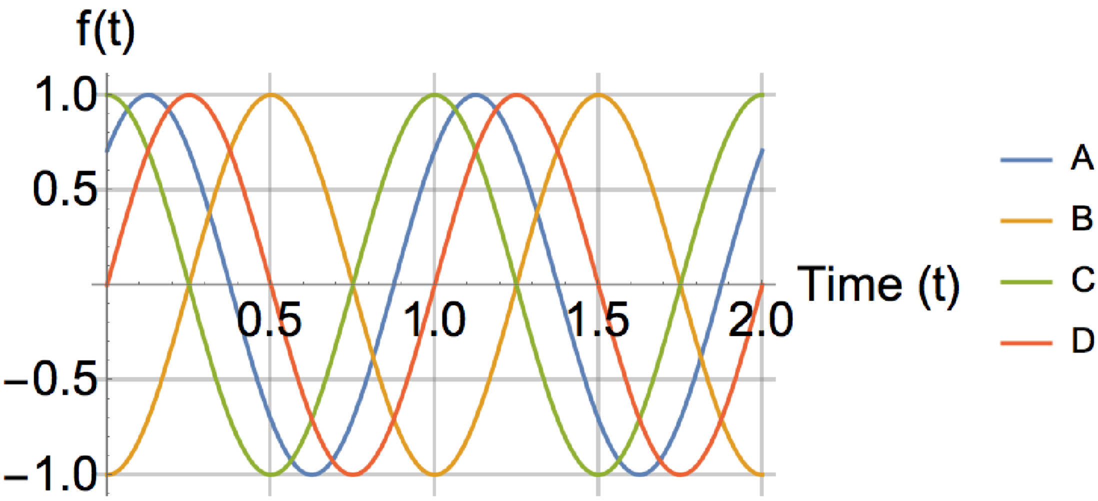
2.7 Phase

Comments

Another important concept is phase. For a simple sine signal $\theta(t) = sin(2\pi t)$, at $t = 0$ the angle θ is zero, but one can shift the phase (starting point) of the signal by effectively making the sine-curve non-zero at $t = 0$. Another way to think about it is to say the sine curve doesn't reach zero until a time $t - \phi$ where ϕ is the phase-shift added.

Challenge

Place the following four graphs in the following order:
$\sin(2\pi t + \pi/2)$
$\sin(2\pi t - \pi/2)$
$\sin(2\pi t + \pi/4)$
$\sin(2\pi t + 2\pi)$



Solution

X = Your solution
Form: String.
Place the indicated letter in front of the string.
Example: aX where X=abcd is entered as aabcd

Hash of iX = 5c0e8b

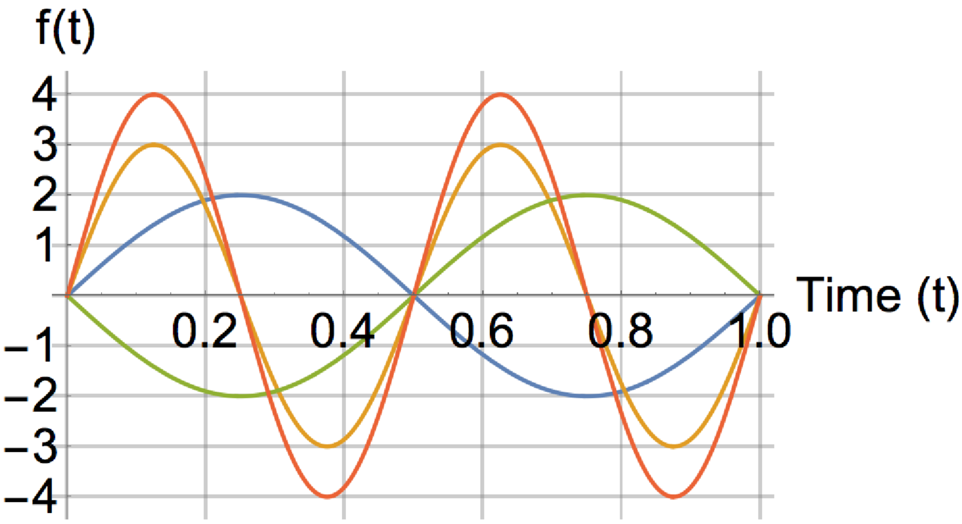
2.8 Amplitude

Comments

Another important concept is amplitude. $\sin(2\pi t)$ has an amplitude of 1, but this can be easily modified to go between $\pm A$ by multiplication with A.

Challenge

The following 4 graphs correspond to the equation $A\sin(2\pi kt)$ with variation in the values of A and k. What is the sum of the values of A for the following graphs?



Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of uX = 6bce05

2.9 Periodic and non-periodic signals

Resources

- Video: `https://www.youtube.com/watch?v=F_pdpbu8bgA`
- Book: 1.3 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Challenge

The list below contains periodic and non-periodic signals. Sum the points of the signals below that are *periodic*.

1 point: $x(t) = t^2$
2 points: $x(t) = \sin(t)$
4 points: $x(t) = \sin(2\pi t)$
8 points: $x(t) = \sin(2\pi t + t)$
16 points: $x(t) = \sin(2\pi t) + \sin(t)$
32 points: $x(t) = \sin(5\pi t) + \sin(2\pi t)$
64 points: $x(t) = \sin(5\pi t) + \sin(37\pi t)$
128 points: $x(t) = \sin(5\pi t) + \sin(37.01\pi t)$
256 points: $x(t) = \sin(5\pi t) + \sin(\sqrt{2}\pi t)$
512 points: $x(t) = \sin(5\sqrt{2}\pi t) + \sin(\sqrt{2}\pi t)$

Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of uX = 5d906b

2.10 Making non-periodic signals from periodic signals

Challenge

It is not immediately intuitive that it is possible to make a non-periodic signal by simply adding two periodic signals. Referring to the previous challenge, in no-more than 1 paragraph, explain how this is possible.

Solution

Please compare your answer with your partner or ask the teacher in class.

2.11 Fundamental frequency

Challenge

Considering the periodic signals in challenge 2.9 in order of increasing point-score, calculate the fundamental frequency and period of the last periodic signal in the list.

Solution

Frequency (Hz) (be careful about rounding up or down to 2 decimal places):
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of aX = ac6698

Period (s):
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of bX = 6fdff9

Chapter 3

Fourier Series

3.1 Even and odd functions

Resources

- Wikipedia: `https://en.wikipedia.org/wiki/Even_and_odd_functions`

Challenge

Sum the points of all the following *true* statements:

1 point: $f(x) = \sin(x)$ is an odd function

2 points: $f(x) = \sin(x)$ is an even function

4 points: $f(x) = \cos(x)$ is an odd function

8 points: $f(x) = \cos(x)$ is an even function

16 points: $f(x) = x$ is an odd function

32 points: $f(x) = x$ is an even function

64 points: $f(x) = \sin(x) + \cos(x)$ is an odd function

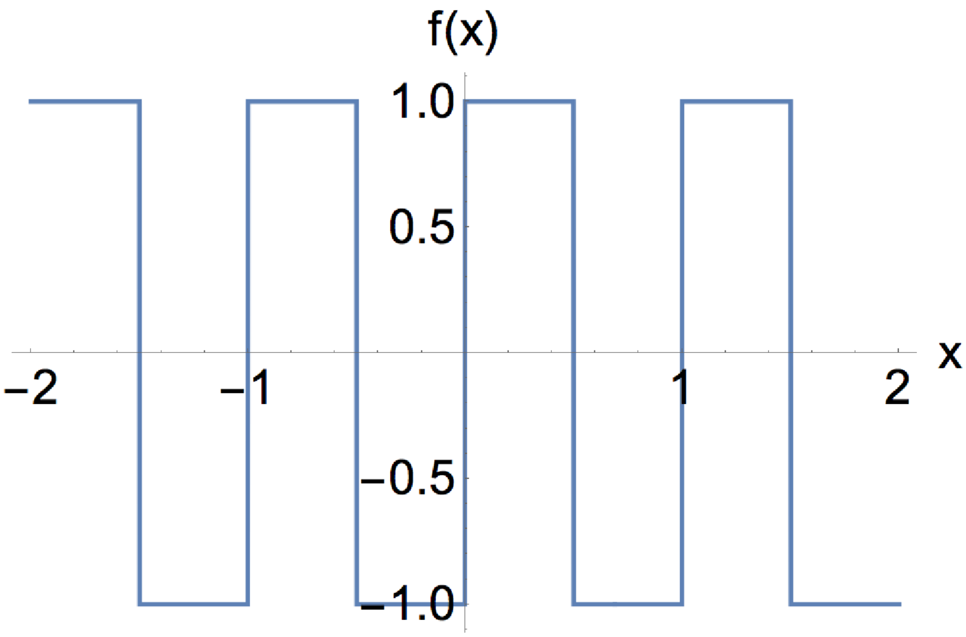
128 points: $f(x) = \sin(x) + \cos(x)$ is an even function

256 points: The infinitely repeating square wave (see figure below) where $f(x) = \begin{cases} 1 & \text{for } 0 < x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} < x < 1 \end{cases}$ is an odd function.

512 points: $f(x)$ above is an even function.

1024 points: The infinitely repeating square wave where $g(x) = \begin{cases} 1 & \text{for } 0 < x < 2 \\ 0 & \text{for } 2 < x < 4 \end{cases}$ is an odd function.

2048 points: $g(x)$ above is an even function.



Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of gX = 205d04

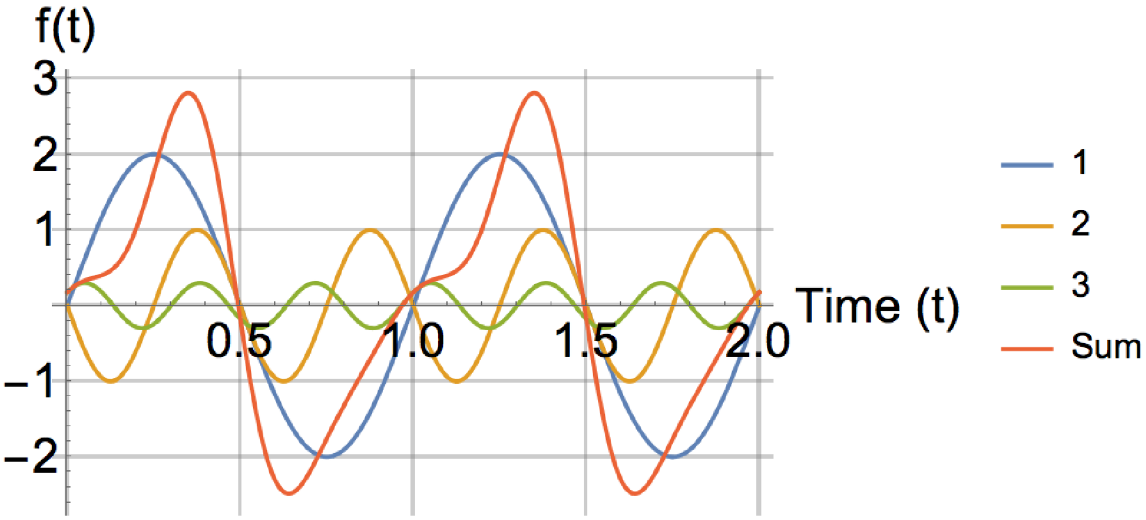
3.2 Introduction to sine and cosine Fourier coefficients

Comment

Fourier series involves the construction of a signal by summing multiple periodic signals and careful choice of each frequency's amplitude and phase. For example, considering the following 3 signals:

1. $f_1(t) = 2\sin(2\pi t)$
2. $f_2(t) = 1\sin(4\pi t + \pi)$
3. $f_3(t) = 0.3\sin(6\pi t + \pi/5)$

Their sum (k = 1 to 3) produces a much more complex shape:



Here we will consider Fourier sine and cosine series. We will go beyond this description later, but you should be aware of their existance and a method of their derivation.

Challenge

As shown above, a complex function (signal) can be made up of a sum of sine signals:

$$f(t) = A_0 + \sum_{k=1}^{N} A_k \sin(2\pi kt + \phi_k) \tag{3.1}$$

Using basic trigonometric relations, show that this can be re-written in terms of a sum of sines and cosines:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \tag{3.2}$$

I want you to see how the phase information is encoded in the representation of equation 3.2. a_k and b_k are referred to as Fourier coefficients. Write an expression for the coefficients a_k, b_k and a_0.

Solutions

To check your answer, calculate a_0, a_1 and b_1 given the following information:
$A_0 = -1/2$, $A_1 = 0.7$, $\phi_1 = \pi/5$.

(make sure your calculator is using radians)

$a_1 = 0.41145$, $b_1 = 0.566312$.

a_0: X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of fX = 75d06e

3.3 The significance of the Fourier coefficients

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-fourier-series-intro`

Comments

I recommend to view the suggested resource as this gives a nice introduction to the subject and challenges ahead. Note that the period used in the video is different to that we're using in our description, but you should become comfortable with moving between different notations.

Challenge

Write a few sentences describing in qualitative terms what the significance of the magnitude of the fourier coefficients a_k and b_k in challenge 3.2 is.

Solutions

Please compare your answer with your partner or discuss with the teacher in class.

3.4 Integral of $\sin(kt)$ and $\cos(kt)$

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-integral-of-sinmt-and-cosmt`

Challenge

Show that the following integrals evaluate to zero:

$$\int_0^1 \sin(2\pi kt)dt = 0 \tag{3.3}$$

$$\int_0^1 \cos(2\pi kt)dt = 0 \tag{3.4}$$

given that k is a non-zero positive integer.

Solutions

Please compare your answer with your partner or discuss with the teacher in class.

3.5 Integral of product of sines and cosines

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-integral-sine-times-sine`
- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-integral-cosine-times-cosine`

Challenge

Considering the following integrals, show under what conditions the following integrals evaluate to zero, and what condition they don't evaluate to zero. What is the non-zero evaluation result?

$$\int_0^1 \sin(2\pi mt)\sin(2\pi nt)dt \tag{3.5}$$

$$\int_0^1 \cos(2\pi mt)\cos(2\pi nt)dt \tag{3.6}$$

You may assume that m and n are non-zero positive integers.

Solutions

Please compare your answer with your partner or discuss with the teacher in class.

3.6 First term in a trigonometric Fourier series

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-first-term-fourier-series`

Challenge

Considering a fourier series represented by

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \tag{3.7}$$

What is a_0 for the following functions?

1. $f(t) = \sin(10\pi t)$
2. Square wave function $f(t) = \begin{cases} 2 & \text{for } 0 < t < \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} < t < 1 \end{cases}$
3. $f(t) = t^2$ considered only over the interval from $t = 0$ to $t = 1$

Solutions

1.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of dX = 705887

2.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of eX = e1509d

3.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of fX = 98680d

3.7 The Fourier coefficient a_k

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-fourier-coefficients-cosine`

Challenge

Derive an expression for the Fourier coefficient a_k in

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \tag{3.8}$$

Note the difference in period between the above equation and the suggested resource.

Solutions

Please compare your answer with your partner or discuss with the teacher in class.

3.8 The Fourier coefficient b_k *(optional challenge)*

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-fourier-coefficients-sine`

Comment

The aim is to have you understand how the Fourier series coefficients arise. If you have time, for extra practise I recommend that you try to calculate b_k. If you do not have time, please progress on to the Fourier Series calculations.

Challenge

Derive an expression for the Fourier coefficient b_k in

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \tag{3.9}$$

Note the difference in period between the above equation and the suggested resource.

Solutions

Please compare your answer with your partner or discuss with the teacher in class.

3.9 Trigonometric Fourier series of a square wave

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-fourier-coefficients-for-square-wave`

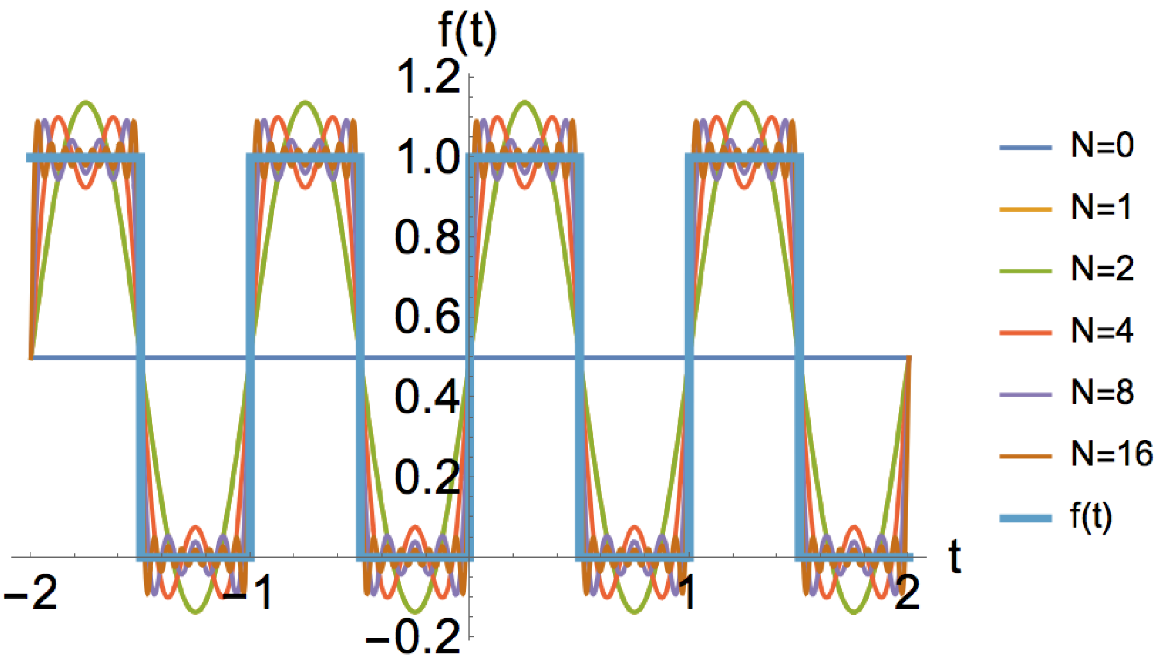
Comment

You may find `https://www.desmos.com/` useful for plotting equations of Fourier series that you calculate.

Challenge

Calculate the Fourier series for the following square-wave:

$$f(t) = \begin{cases} 1 & \text{for } 0 < t < \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < t < 1 \end{cases} \tag{3.10}$$



Solutions

To check your solution, write out the sum up to $k = 2$ and then evaluate $f(0.9)$

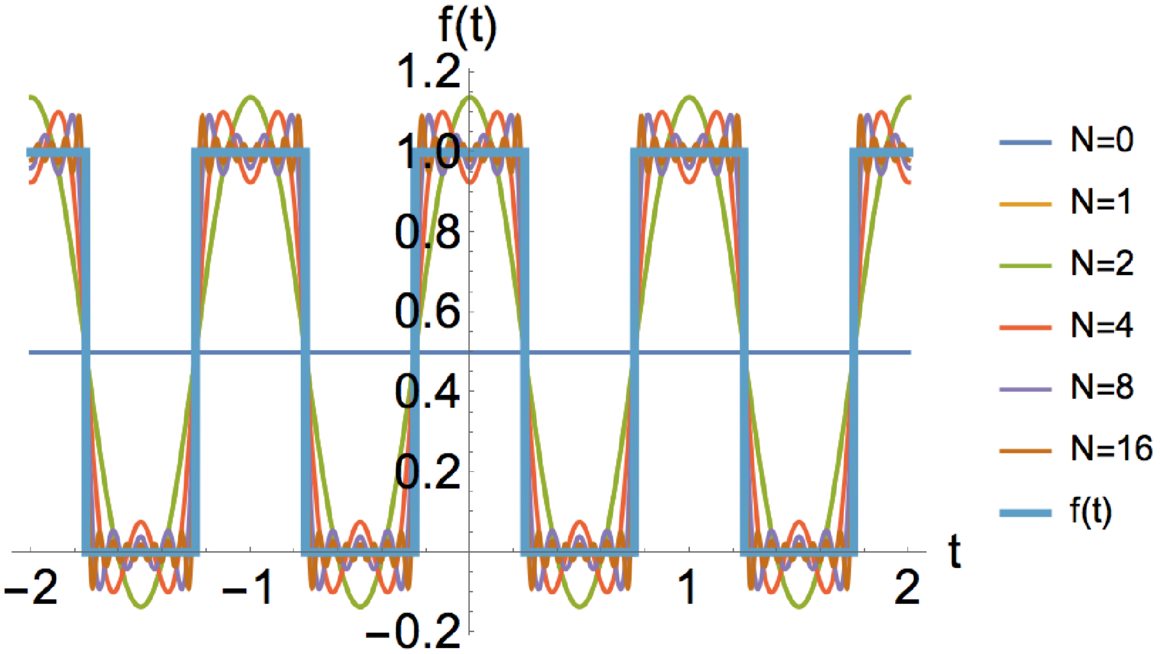
0.1258

3.10 Trigonometric Fourier series of another square wave

Challenge

Calculate the Fourier series for the following square-wave:

$$f(t) = \begin{cases} 1 & \text{for } -\frac{1}{4} < t < \frac{1}{4} \\ 0 & \text{for } \frac{1}{4} < t < \frac{3}{4} \end{cases} \tag{3.11}$$



Solutions

To check your solution, write out the sum up to $k = 2$ and then evaluate $f(0.7)$

0.30327

3.11 Fourier sine and cosine series

Challenge

Considering challenges 3.9 and 3.10, one corresponds to a Fourier sine series, and the other to a Fourier cosine series. Such series only contain sine and cosine terms, respectively. (1) State which challenge corresponds to which series, and (2) write a square-wave function that you think would involve both sine and cosine terms.

Solutions

If you are unsure about your answer, you can either try to solve it to prove it, or please discuss with your partner or ask the teacher.

3.12 Complex numbers

Resources

- Book: Appendex A starting at page 403 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Challenge

Considering $z = a + bi$ and $w = c + di$, determine:

1. $z + w$
2. zw
3. $z\bar{z}$
4. z/w
5. $|z|$

Solutions

To check your answers, substitute the following values: $a = 1$, $b = 2$, $c = 3$, $d = 4$.

1.
X = Your solution
Form: Imaginary form with integers.
Place the indicated letter in front of the number.
Example: aX where $X = 1 + 2i$ is entered as are(1)im(2)

Hash of hX = d8c7d5

2.
X = Your solution
Form: Imaginary form with integers.
Place the indicated letter in front of the number.
Example: aX where $X = 1 + 2i$ is entered as are(1)im(2)

Hash of iX = 3c520b

3.
X = Your solution
Form: Imaginary form with integers.
Place the indicated letter in front of the number.
Example: aX where $X = 1 + 2i$ is entered as are(1)im(2)

Hash of jX = 66248e

4.
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of kX = 39577d

5.
X = Your solution

Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of mX = b12a23

3.13 Complex form of sine and cosine

Challenge

Write sine and cosine in terms of complex exponentials.

Solutions

If you are unsure about your answer, please discuss in class.

3.14 An alternative way to write Fourier series

Comments

It turns out that a more mathematically-convenient and ultimately intuitive way to write Fourier series is in terms of complex exponentials in the form

$$f(t) = \sum_{k=-n}^{k=n} c_k e^{i2\pi kt} \tag{3.12}$$

A few things to note:

- The sum now runs from $-n$ to n.
- The $k = 0$ term that was originally excluded from the sum in equation 3.2 is now included.
- The c_k's, unlike the a_k's and b_k's, are complex.
- $c_{-k} = \bar{c_k}$.
- $c_0 = a_0/2$.

You see how we started with equation 3.1 and now end up with equation 3.12? Do you see more-or-less how the phase information is encoded in the c_k's?

Challenge

1. By comparing equation 3.2 to equation 3.12, write c_k in terms of a_k and b_k for $k < 0$ and $k > 0$.
2. Show that $c_{-k} = \bar{c_k}$
3. Show that c_0 must be $a_0/2$

Solutions

If you have trouble with the derivation, please discuss in class.

3.15 Integral of a complex exponential over a single period

Challenge

Integrate the following function over one period, assuming that k must be an non-zero integer:

$$g(t) = e^{i2\pi kt} \tag{3.13}$$

Solution

The numerical answer is given below. Be sure you understand why the answer comes out as this number.

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of nX = f979e9

3.16 Derivation of the complex Fourier coefficients

Resources

- Video starting at 41m 56s: `https://www.youtube.com/watch?v=1rqJl7Rs6ps&t=41m56s`

Comments

The video does a great job of showing the derivation. Try to become comfortable with manipulating complex exponentials and their periodicity.

Challenge

Derive an expression for the c_k's in the equation

$$f(t) = \sum_{k=-n}^{n} c_k e^{i2\pi kt} \tag{3.14}$$

in terms of the function $f(t)$.

Solution

Please discuss in class if you are unsure of your derivation.

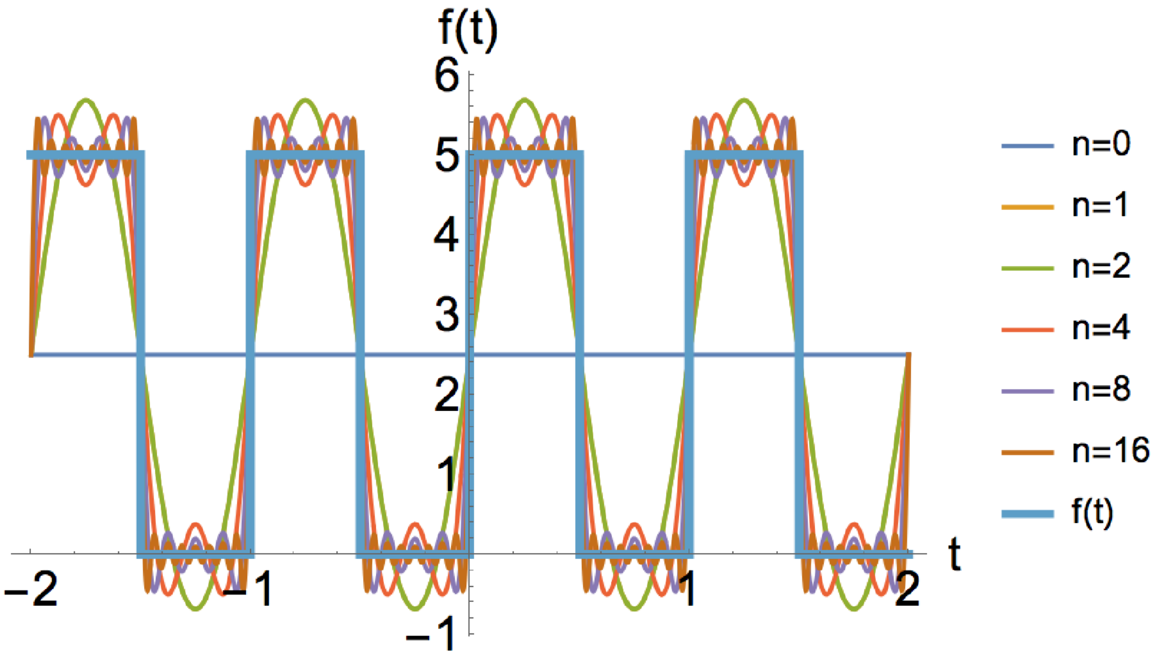
3.17 Complex Fourier series of a square wave

Challenge

1. Determine an expression for the complex Fourier coefficients for the following square wave:

$$f(t) = \begin{cases} 5 & \text{for } 0 < t < \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < t < 1 \end{cases} \tag{3.15}$$

2. Write the complex Fourier series for the square wave.



Solution

1.
You should find that $c_{-7} = i0.227$.

2.
If you perform the sum for $n = 1$ and evaluate at $t = 0.7$, you should obtain a value of -0.527.

3.18 Non-unit periods

Resources

- Book: Chapter 1.6 of the book (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Comment

Until now we considered signals with periodicity 1, but this will not always be the case, and in fact as we make the jump from Fourier series to Fourier transforms this will become more important. The resource gives the intuition behind the non-periodic case for complex Fourier series. For trigonometric Fourier series, the Fourier coefficients become:

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(2\pi kt/T)dt \tag{3.16}$$

$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(2\pi kt/T)dt \tag{3.17}$$

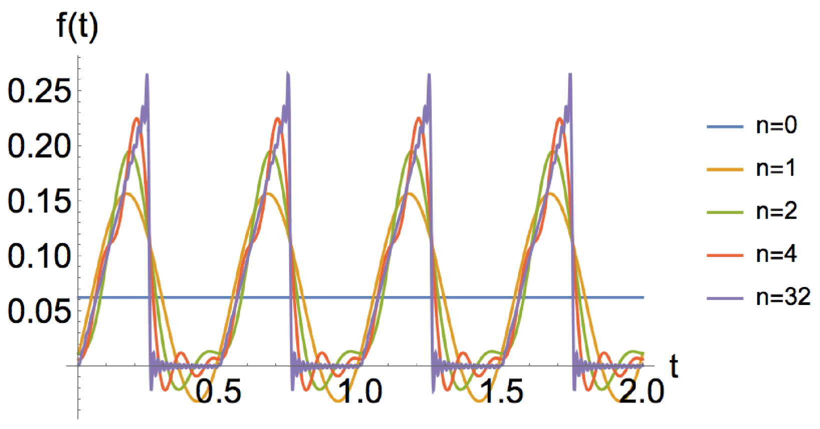
while for complex Fourier series the Fourier coefficients become:

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-i2\pi kt/T}dt \tag{3.18}$$

Challenge

Obtain an expression for the exponential Fourier series for the pulsing sawtooth function

$$f(t) = \begin{cases} t & \text{for } 0 < t < \frac{1}{4} \\ 0 & \text{for } \frac{1}{4} < t < \frac{1}{2} \end{cases} \tag{3.19}$$



Solution

You should find the c_k's for $k \neq 0$ are

$$\frac{i\pi k e^{-i\pi k} + e^{-i\pi k} - 1}{8\pi^2 k^2} \tag{3.20}$$

3.19 Gibb's phenomenon

Resources

- Book: 1.18 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Challenge

1. Qualitatively speaking, what is the Gibb's phenomenon?
2. Considering a square wave of the form

$$f(t) = \begin{cases} -1 & \text{for } -\frac{1}{2} < t < 0 \\ 1 & \text{for } 0 < t < \frac{1}{2} \end{cases} \tag{3.21}$$

as the sum of the Fourier series goes to infinity, what is the maximum value of the signal?

The full derivation is beyond the scope of this course, so it is not necessary to understand the derivation in the notes. The aim of this challenge is simply to enable you to be able to describe qualitatively what Gibb's phenomenon is using a few sentences, and know the amount of overshoot in the case of a standard ± 1 *square-wave.*

Solution

2.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of mX = d9c018

3.20 Fourier coefficients of sin(x)

Challenge

By writing $\sin(x)$ in exponential form, deduce the Fourier coefficients c_{-2}, c_{-1} and c_0. Try to do this by inspection rather than application of formulas.

Solutions

c_{-2}:
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of hX = 53126e

c_{-1}:
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of kX = 312a1a

c_0:
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of jX = b6ffa8

3.21 Fourier Coefficients of 1 + sin(x)

Challenge

Using the same approach as challenge 3.20, deduce for the function $1 + sin(x)$ the Fourier coefficients c_{-1}, c_0 and c_1.

Solutions

c_{-1}:
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of mX = 434572

c_0:
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of nX = 1444ea

c_1:
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of oX = 2e5a62

3.22 Relation of positive and negative Fourier coefficients for a real signal

Resources

- Challenge 3.14
- Book: 1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Challenge

If the Fourier coefficient C_1 is $4+6i$ for a real signal, what is the Fourier coefficient C_{-1}?

Solution

X = Your solution
Form: Imaginary form with integers.
Place the indicated letter in front of the number.
Example: aX where $X = 1 + 2i$ is entered as are(1)im(2)

Hash of pX = de57ee

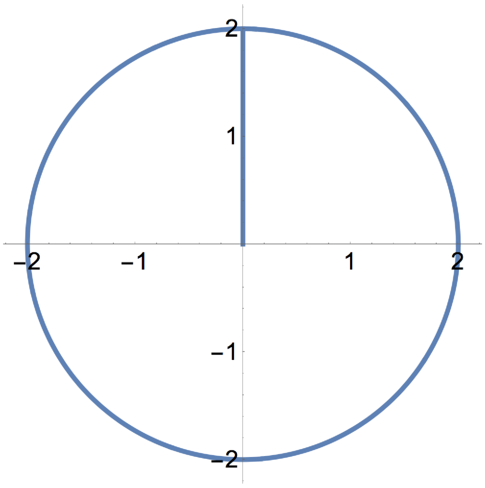
3.23 Circles and Fourier series

Resources

- Video 1: `https://www.youtube.com/watch?v=Y9pYHDSxc7g`
- Video 2: `https://www.youtube.com/watch?v=LznjC4Lo71E`

Comment

In the first lecture we saw how it was possible to approximate any function given enough circles. Here we link what you have learned back to that first lecture. I strongly recommend viewing the fun and informative videos listed here under Resources. In summary, by building a Fourier series you are representing a function using a sum of periodic functions, where each component can be considered visually as a circle operating with individual radius, frequency and phase on the real-imaginary plane.



Challenge

Treating the x-axis as the real axis and the y-axis as the imaginary axis, arrange the equations below in the following order:

1. A point moving round on a circle with radius 2 units and frequency 2 Hz
2. A point moving round on a circle with radius 3 units and frequency 1 Hz
3. A point moving round on a circle with radius 2 units and a period of 1 second
4. A point moving round on a circle with radius 3 units and a period of 2 seconds

Equations:

$Ae^{2\pi ikt}$ where t is time in seconds and the values of A and k are as follows:

A: $A = 2$, $k = 2$

B: $A = 3$, $k = 1$

C: $A = 3$, $k = 0.5$

D: $A = 2$, $k = 1$

Solution

X = Your solution
Form: String.
Place the indicated letter in front of the string.
Example: aX where X=abcd is entered as aabcd

Hash of uX = d8a7e6

3.24 Partial derivatives

Challenge

Determine u_t and u_{xx} for the equation

$$u(x,t) = 5tx^2 + 3t - x \tag{3.22}$$

To check your answer, substitute $x = 3$ and $t = 2$ into your answers, as appropriate.

Solution

u_t:
X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of tX = 4fb068

u_{xx}:
X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of xX = 53502b

3.25 Heat equation: Periodicity

Resources

- Book: Section 1.13.1 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Lecture 4 from 37:00 onwards: (`https://www.youtube.com/watch?v=n5lBM7nn2eA`), continuing at the start of lecture 5 (`https://www.youtube.com/watch?v=X5qRpgfQld4`)

Comment

Here we can learn about an application of Fourier series to solve partial differential equations. This problem was one of the motivations for Fourier to develop the idea of Fourier series.

The motivation for the equation $u_t = \frac{1}{2}u_{xx}$ described in the notes is complicated somewhat by the interpretation in terms of equivalences between electrical and thermal capacitance. If this is not so clear then don't worry about it. At a minimum you should understand the following:

- Heat flow is proportional to the gradient of the temperature.
- Heat accumulates within a unit volume when the rate of heat flow into that volume is greater than the rate of heat flow out of that volume.

Challenge

The following statements concern a heated ring with circumference 1 and temperature distribution described by $u(x,t)$. Add the points of the statements that are defined by the system to be true:

1 point: $u(x,t) = u(x,t)$

2 points: $u(x,t) = u(x,t+1)$

4 points: $u(x,t) = u(x,t+2)$

8 points: $u(x,t) = u(x+1,t)$

16 points: $u(x,t) = u(x+1,t+1)$

32 points: $u(x,t) = u(x+1,t+2)$

64 points: $u(x,t) = u(x+2,t)$

128 points: $u(x,t) = u(x+2,t+1)$

256 points: $u(x,t) = u(x+2,t+2)$

512 points: The temperature distribution is periodic in space but not time

1024 points: The temperature distribution is periodic in time but not space

2048 points: The temperature distribution is periodic in both space and time

4096 points: The temperature distribution is periodic neither in space nor time

Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of yX = 2259d1

3.26 Heat equation on a ring: derivation

Resources

- Video: `https://www.youtube.com/watch?v=yAOCibHPgLA`
- Book: Section 1.13.1 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Lecture 4 from 37:00 onwards: (`https://www.youtube.com/watch?v=n5lBM7nn2eA`), continuing at the start of lecture 5 (`https://www.youtube.com/watch?v=X5qRpgfQld4`)

Challenge

Starting from the heat (diffusion) equation $u_t = u_{xx}/2$, show that the general solution to the heat equation on a ring is given by

$$u(x,t) = \sum_{n=-\infty}^{n=\infty} c_n(0) e^{-2\pi^2 n^2 t} e^{i2\pi n x} \tag{3.23}$$

and write an expression for $c_n(0)$ in terms of the initial temperature distribution $u(x, 0)$.

Solution

Please compare with your peers during discussion time and ask if there is anything you do not understand.

3.27 Heat equation on a ring: calculation

Comment

Remember that any integer multiple of 2π in the complex exponential (eg, $e^{i4\pi}$ or $e^{i2\pi n}$ where n is an integer) is equivalent to having 2π in the exponential due to the periodic nature of complex exponentials (ie, that $e^{i2\pi} = \cos(2\pi) + i\sin(2\pi)$ and $\cos(2\pi) = \cos(4\pi) = \cos(6\pi)$ etc).

Challenge

Consider an initial heat distribution around a ring. Relative to ambient temperature, the initial temperature distribution follows a cosine distribution with the peak temperature of $u = 1$ at $x = 0$. Assume that the ring has a circumference length of 1 unit.

1. Write an expression for the initial relative temperature distribution, $u(x, 0)$.
2. Write an expression for the relative temperature distribution, $u(x, t)$, as a function of time.

Note: You may use a computer-algebra system such as Wolfram Alpha to help you do the necessary integrals.

You can see an animation of the solution here:
`https://raw.githubusercontent.com/NanoScaleDesign/FourierAnalysis/master/Images/cosrelax.gif`

Solution

1. You should find that your temperature distribution satisfies the result $u(0.2, 0) = 0.309$.
2. To check your answer you may substitute $x = 0.3$ and $t = 0.01$ into your solution: $u(0.3, 0.01) = -0.25$

Chapter 4

Fourier Transform

4.1 Limits of $\sin(x)$

Resources

- http://mathworld.wolfram.com/SeriesExpansion.html

Challenge

Considering the series expansion around 0, determine the limiting value of the following functions as $T \to \infty$:

1. $\sin(x/T)$

2. $T \sin(x/T)$

You may consider x to be any real-valued number.

Solution

To check your answer, substitute $x = 2$ as appropriate.

1.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of aX = 690969

2.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of bX = 2a5520

4.2 The transition to the Fourier Transform

Resources

- Video I: Lecture 5 from 27:00 (`https://youtu.be/X5qRpgfQld4?t=27m`)
- Video II: Lecture 6 up to 20:00 (`https://www.youtube.com/watch?v=4lcvROAtN_Q`)
- Book: Chapter 2 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Comment

Until now we have been considering Fourier series. This is however limited to describing periodic phenomena since it is assumed that the signal repeats outside the region of integration. The Fourier transform can be thought of as an extension of Fourier series which allows the analysis of non-periodic phenomena.

The suggested resources provide an excellent intuitive path of the connection between Fourier series and Fourier transforms, and this challenge is designed to give you the opportunity to take some time to try to understand the main concepts behind the transition.

Challenge

Using the resources above, show how to make the transition from Fourier series to the Fourier transform.

Solution

Please compare your working with your partner.

4.3 L'Hôpital's rule

Challenge

Use L'Hôpital's rule to determine the limit of

$$\frac{\sin(x)}{x} \tag{4.1}$$

as $x \to 0$.

Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of dX = 9948c6

4.4 Fourier transform of a window function

Resources

- Book: Chapter 2.1 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 6 starting from 29:26 (`https://youtu.be/4lcvROAtN_Q?t=29m26s`)

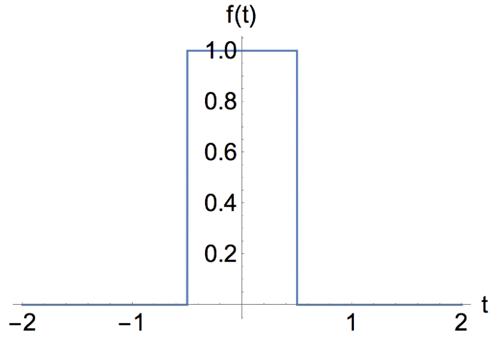
Challenge

Calculate the Fourier Transform for the window function

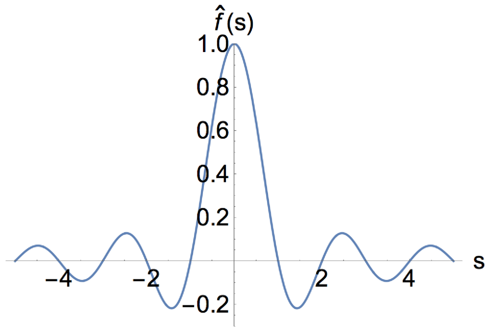
$$f(t) = \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{for } |t| > 1/2 \end{cases} \tag{4.2}$$

In graph-form the function and its transform appear as follows:

Function



Fourier transform



Solution

You should find that your solution is consistent with $\hat{f}(s = 1.5) = -0.21$.