Fourier Analysis

Autumn 2016

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http://www.jamescannon.net/teaching/fourier-analysis
http://raw.githubusercontent.com/NanoScaleDesign/FourierAnalysis/master/fourier_analysis.pdf

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Contents

Chapter 0

Course information

0.1 This course

This is the Autumn 2016 Fourier Analysis course studied by 3rd-year undegraduate international students at Kyushu University.

0.1.1 How this works

- In contrast to the traditional lecture-homework model, in this course the learning is self-directed and active via publicly-available resources.
- Learning is guided through solving a series of carefully-developed challenges contained in this book, coupled with suggested resources that can be used to solve the challenges with instant feedback about the correctness of your answer.
- There are no lectures. Instead, there is discussion time. Here, you are encouraged to discuss any issues with your peers, teacher and any teaching assistants. Furthermore, you are encouraged to help your peers who are having trouble understanding something that you have understood; by doing so you actually increase your own understanding too.
- Discussion-time is from 10:30 to 12:00 on Wednesdays at room W4-529.
- Peer discussion is encouraged, however, if you have help to solve a challenge, always make sure you do understand the details yourself. You will need to be able to do this in an exam environment. If you need additional challenges to solidify your understanding, then ask the teacher. The questions on the exam will be similar in nature to the challenges. If you can do all of the challenges, you can get 100% on the exam.
- Every challenge in the book typically contains a **Challenge** with suggested **Resources** which you are recommended to utilise in order to solve the challenge. A **Solution** is made available in encrypted form. If your encrypted solution matches the encrypted solution given, then you know you have the correct answer and can move on. For more information about encryption, see section 0.3. Occasionally the teacher will provide extra **Comments** to help guide your thinking.
- For deep understanding, it is recommended to study the suggested resources beyond the minimum required to complete the challenge.
- The challenge document has many pages and is continuously being developed. Therefore it is advised to view the document on an electronic device rather than print it. The date on the front page denotes the version of the document. You will be notified by email when the document is updated.
- A target challenge will be set each week. This will set the pace of the course and defines the examinable material. It's ok if you can't quite reach the target challenge for a given week, but then you will be expected to make it up the next week.
- You may work ahead, even beyond the target challenge, if you so wish. This can build greater flexibility into your personal schedule, especially as you become busier towards the end of the semester.
- Your contributions to the course are strongly welcomed. If you come across resources that you found useful that were not listed by the teacher or points of friction that made solving a challenge difficult, please let the teacher know about it!

0.1.2 Assessment

In order to prove to outside parties that you have learned something from the course, we must perform summative assessments. This will be in the form of

$$\text{overall score} = (0.5E_F + 0.3E_M + 0.2C)(0.9 + P/10)$$

Your final score is calculated as Max(E_F, overall score), however you must pass the final exam ($\geq 60\%$) to pass the course.

- E_F = % correct on final exam
- E_M = % correct on mid-term exam
- C = % grade on course-work
- P = participation calculated as $P = (F/N_D)(A/N_D)(L/N_L)$ where each terms is as follows
  - F = Number of weeks where feedback form is submitted 24 hours before discussion time.
  - A = Number of discussion classes attended
  - N_D = Number of discussion classes held
  - L = Number of times that your collected challenge log is satisfactory. This means:
    * Available on request.
    * Your calculations are clearly shown.
    * It contains evidence of trying to keep up with the target challenge. Short-term fluctuations in completing challenges are fine (eg, if you had trouble understanding material to overcome some challenges this week) but the long-term trend should be more-or-less to keep up with the target challenge.
  - N_L = Number of times that your challenge log is collected.

Note that P is only calculated from 26 October. If $N - F \leq 2$ then F is treated as being equal to N (ie, you can forget twice). You can be counted as attending the class even if you are not present if the reason for not attending is unavoidable (eg, health reasons) and you inform the teacher in advance.

Please also note that, since late arrivals disrupt the class by preventing intended pairing of students, attendance of a discussion class will be only counted as partial if you are more than a minute or two late (eg, 9 minutes late out of a 90-minute discussion class will count as attending only 90% of the class). Therefore, if you will be unavoidably late, you need to let the teacher know in advance. To allow for unexpected delays, for up to two late arrivals you will be considered to have attended 100% of the discussion time.

0.1.3 What you need to do

- Prepare a challenge-log in the form of a workbook or folder where you can clearly write the calculations you perform to solve each challenge. This will be a log of your progress during the course and will be occasionally reviewed by the teacher.
- You also need to submit feedback at `https://goo.gl/forms/Djl4FEZcJLMpipsY2` 24 hours before the discussion time starts. Here you can let the teacher know about any difficulties you are having and if you would like to discuss anything in particular.
- Please bring a wifi-capable internet device to class, as well as headphones if you need to access online components of the course during class. If you let me know in advance, I can lend computers and provide power extension cables for those who require them (limited number).

0.2 Timetable

	Discussion	**Target**	**Note**
1	5 Oct	-	
2	12 Oct	1.8	
3	19 Oct	2.9	
4	26 Oct	2.11	
5	2 Nov	2.15	
6	9 Nov	2.21	
7	16 Nov	2.25	
8	30 Nov	3.6	Coursework instructions
9	7 Dec	Midterm exam	
10	14 Dec	3.10	
11	21 Dec	3.10	
12	11 Jan	3.14	Submission of coursework
13	18 Jan	4.5	
14	25 Jan	4.7	
15	8 Feb	Final exam	Open Learning Plaza Lecture Room 5

0.3 Hash-generation

Most solutions to challenges are encrypted using MD5 hashes. In order to check your solution, you need to generate its MD5 hash and compare it to that provided. MD5 hashes can be generated at the following sites:

- Wolfram alpha: (For example: md5 hash of "q_1.00") `http://www.wolframalpha.com/input/?i=md5+hash+of+%22q_1.00%22`
- `www.md5hashgenerator.com`

Since MD5 hashes are very sensitive to even single-digit variation, you must enter the solution exactly. This means maintaining a sufficient level of accuracy when developing your solution, and then entering the solution according to the format below:

Unless specified otherwise, any number from 0.00 to ±9999.99 should be represented as a normal number to two decimal places. All other numbers should be in scientific form. See the table below for examples.

Solution	**Input**
1	1.00
-3	-3.00
-3.5697	-3.57
0.05	0.05
0.005	5.00e-3
50	50.00
500	500.00
5000	5000.00
50,000	5.00e4
5×10^{-476}	5.00e-476
5.0009×10^{-476}	5.00e-476
$-\infty$	-infinity (never "infinite")
2π	6.28
i	im(1.00)
2i	im(2.00)
1 + 2i	re(1.00)im(2.00)
-0.0002548 i	im(-2.55e-4)
1/i = i/-1 = -i	im(-1.00)
$e^{i2\pi}$ $[= cos(2\pi) + isin(2\pi) = 1 + i0 = 1]$	1.00
$e^{i\pi/3}$ $[= cos(\pi/3) + isin(\pi/3) = 0.5 + i0.87]$	re(0.50)im(0.87)
Choices in order A, B, C, D	abcd

Entry format is given with the problem. So "q_X" means to enter "q_X" replacing "X" with your solution. The first 6 digits of the MD5 sum should match the given solution (MD5(q_X)= ...).

Note that although some answers can usually only be integers (eg, number of elephants), for consistency, to generate the correct hash, the accuracy in terms of decimal places noted above is required.

0.4 Coursework

Fourier analysis is a large subject with a wide-range of applications. This coursework is designed to give you the opportunity to follow your personal interest and investigate in depth an area of Fourier analysis of your choice.

The task is as follows:

1) Create a document, explaining about any area of Fourier analysis that interests you. For example, you could consider how Fourier analysis is applied in the field you're pursuing for your degree, or you could explore some mathematics related to Fourier Analysis that you find particularly interesting. The document should be **at least 1 full page**, including any necessary figures, mathematics and references.

2) Create at least 1 challenge to accompany your report, so someone reading your document can test their knowledge.

3) Include **fully worked** solutions to challenges you make (ie, not only the final answer, but clearly show the steps involved in order to achieve the final answer).

Please choose a subject that does not directly repeat what is covered by the course up to the submission deadline. For example, we will be convering convolution, so if you choose to write about that subject, please make your content distinct from what is already covered by the course.

I may (or may not) choose to incorporate some aspects of the submissions into teaching of the final 1 or 2 classes.

0.4.1 Submission

You must submit **both a paper and electronic version**. Submit the materials by **email** to the teacher by **10:30 on 11 January 2017** with the subject "[Fourier Analysis] Coursework" and **bring a paper copy to the class on that day**.

The electronic version may be in any format, including LibreOffice, MS Word, Google docs, Latex, etc. . . If you submit a PDF, please also submit the source-files used to generate the PDF.

Late submission:
By 10:00 on 12 January 2017: 90% of the final mark.
By 10:00 on 16 January 2017: 50% of the final mark.
Later submissions cannot be considered.

0.4.2 Marking

Marks will be assigned based on the degree to the report fulfills the following criteria:

- Understanding: Clearly demonstrate your understanding of what you write about. You can do this by, for example, mathematically solving for a relevant case or explaining with words how it applies in different situations.
- Relevance: A subject that makes use of Fourier series or Fourier Transforms.
- Originality: It should be your own work. Also, you must **cite all references, as well as images and text taken from other sources**.
- Level: The subject should be pitched at a level whereby anyone else in the class could learn about the subject based on your report. Be sure to explain in reasonable depth.
- Accuracy: The explanation should be accurate and clear.

Chapter 1

Periods and frequencies

1.1 Period at 50 THz

Resources

- Book: 1.2 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Challenge

A signal is oscillating at a frequncy of 50 THz. What is the period?

Solution

X s

MD5(q_X) = 33efc2...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

1.2 Fundamental period with k=1

Resources

- Book: 1.3 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 2 (https://www.youtube.com/watch?v=1rqJl7Rs6ps)

Challenge

What is the fundamential period of $sin(2\pi kt)$, where t is time in seconds and $k = 1$?

Solution

X s

MD5(w_X) = 9ae2db...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

1.3 Frequency with k=1

Resources

- Book: 1.3 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 2 (https://www.youtube.com/watch?v=1rqJl7Rs6ps)

Challenge

What is the frequency of $sin(2\pi kt)$, where t is time in seconds and $k = 1$?

Solution

X Hz

MD5(e_X) = 453c99...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

1.4 Frequency with k=2

Resources

- Book: 1.3 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 2 (https://www.youtube.com/watch?v=1rqJl7Rs6ps)

Challenge

What is the frequency of $sin(2\pi kt)$, where t is time in seconds and $k = 2$?

Solution

X Hz

MD5(r_X) = 111802...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

1.5 Fundamental period with k=2

Resources

- Book: 1.3 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Challenge

What is the fundamental period of $sin(2\pi kt)$, where t is time in seconds and $k = 2$?

Solution

X s

MD5(t_X) = c4c8be...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

1.6 Fundamental period with multiple terms

Resources

- Book: 1.3-1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

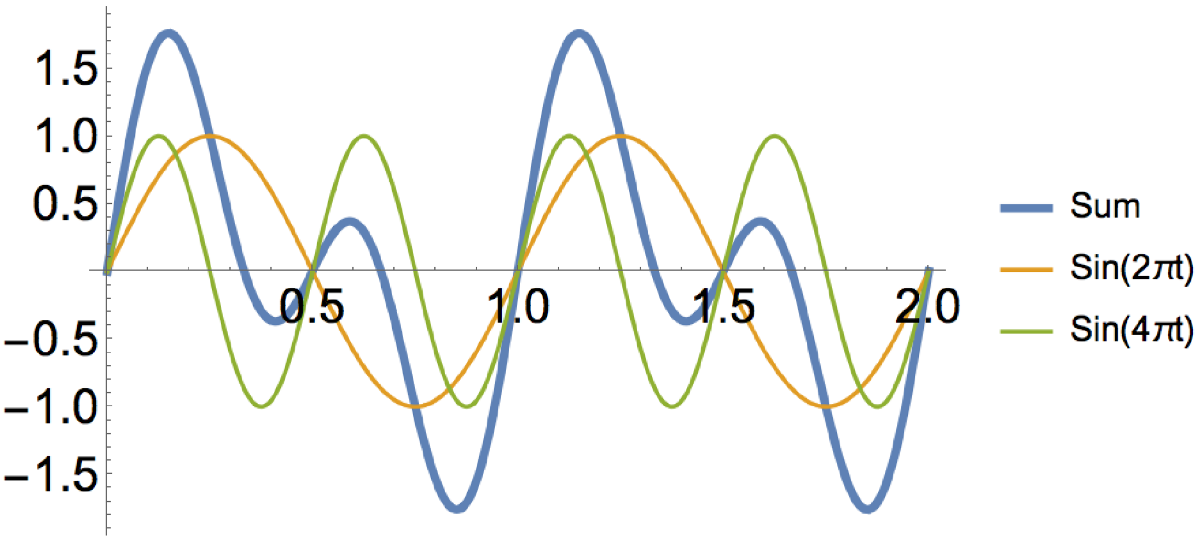
Comments

So k here is proportional to the frequency. Double k and the frequency doubles. Every "step" around the circle drawn by the sine curve becomes 2 steps when $k = 2$, so within half the time you are already one time around the circle for $k = 2$, and thus the number of times you go around the circle during one second (measured with t) is twice rather than once. k is also inversely proportional to the period. Since with $k = 2$ every "step" is now twice as large, so one completes the circle in half the time.

Even with multiple terms (frequencies), the period of the composite signal is always that of the highest (longest) period (lowest frequency), even if it is composed of multiple frequencies. You have to wait for every part of the signal to complete before a single period is complete. Ie, it is possible to add new frequencies to a signal without the period changing.

Challenge

What is the fundamental period of $sin(2\pi t) + sin(4\pi t)$, where t is time in seconds?



Solution

X s

MD5(y_X) = 80681b. . .

Study-time (from end of previous challenge to end of this challenge): ______ minutes

1.7 Amplitude

Resources

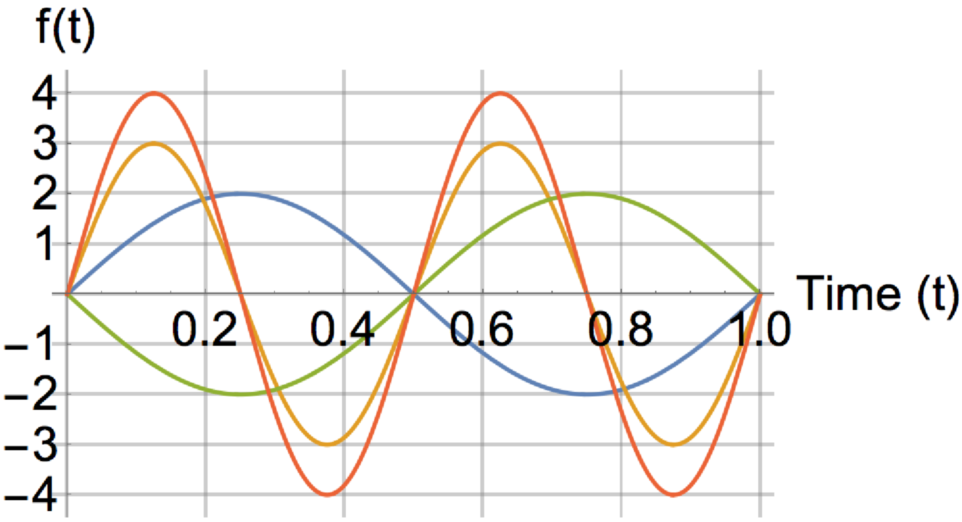
- Book: 1.3 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Comments

Another important concept is amplitude. $Sin(2\pi t)$ has an amplitude of 1, but this can be easily modified to go between $\pm A$ by multiplication with A.

Challenge

The following 4 graphs are of the form $ASin(2\pi kt)$ with variation in the values of A and k *only*. What is the sum of the values of A for the following graphs?



Solution

X

MD5(u_X) = 7bcfe4...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

1.8 Phase

Resources

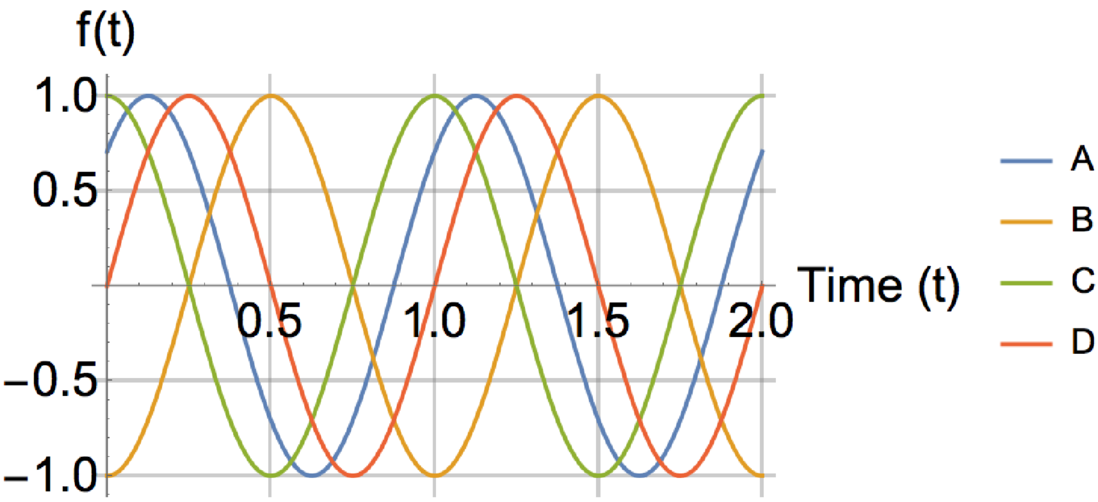
- Book: 1.3 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Comments

Another important concept is phase. For a simple Sine signal $\theta(t) = sin(2\pi t)$, at $t = 0$ the angle θ is zero, but one can shift the phase (starting point) of the signal by effectively making the sine-curve non-zero at $t = 0$. Another way to think about it is to say the sine curve doesn't reach zero until a time $t - \phi$ where ϕ is the phase-shift added.

Challenge

The following 4 graphs are of the form $Sin(2\pi t + \phi)$ where ϕ is the phase-shift. Put the graphs in order corresponding to the following order of phase-shifts: $\pi/2$, $-\pi/2$, $\pi/4$, 2π.



Solution

X

MD5(i_X) = 547005...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

Chapter 2

Fourier Series

2.1 Introduction to Fourier coefficients

Resources

- Book: 1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Challenge

Deduce in a simple way the Fourier coefficients a_1 and b_1 in the Fourier series

$$\sum_{k=1}^{N} a_k cos(2\pi kt) + b_k sin(2\pi kt) \tag{2.1}$$

for a signal made up of multiple sine signals

$$\sum_{k=1}^{N} A_k sin(2\pi kt + \phi_k) \tag{2.2}$$

for the following cases:

1. $N = 1$, $k = 1$, $A_1 = 1$, $\phi_1 = 0$
2. $N = 1$, $k = 1$, $A_1 = 1$, $\phi_1 = \pi/2$
3. $N = 1$, $k = 1$, $A_1 = 1$, $\phi_1 = \pi/5$

Hint: Using $sin(A+B) = sin(A)cos(B) + cos(A)sin(B)$ it should is possible to find the answer without resorting to complex calculation.

Solutions

1. MD5(o_a_k) = a2c1fe…, MD5(p_b_k) = de80c6…
2. MD5(a_a_k) = 718a6c…, MD5(s_b_k) = f86f0c…
3. MD5(d_a_k) = 93d647…, MD5(f_b_k) = 9a7b58…

2.2 Even and odd functions

Resources

- Wikipedia: `https://en.wikipedia.org/wiki/Even_and_odd_functions`

Challenge

Referring in part to the cases in challenge 2.1, sum the points of all the following TRUE statements:

1 point: Case 1 is an odd function

2 points: Case 1 is an even function

4 points: Case 2 is an odd function

8 points: Case 2 is an even function

16 point: Case 3 is an odd function

32 points: Case 3 is an even function

64 points: $f(x) = Sin(x)$ is an odd function

128 points: $f(x) = Sin(x)$ is an even function

256 points: $f(x) = Cos(x)$ is an odd function

512 points: $f(x) = Cos(x)$ is an even function

1024 points: $f(x) = x$ is an odd function

2048 points: $f(x) = x$ is an even function

Solution

X

MD5(g_X) = 6a18c0...

2.3 Fourier coefficients of sin(x)

Resources

- Book: 1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Comments

You should be able to follow the derivation of the formula for Fourier coefficients (C_k's) in the video. Feel free to seek help if you have trouble.

Challenge

1. Write $Sin(x)$ in exponential form.

2. Expand the Fourier series for the function $f(x)$ within the limit $k = -1$ to $k = 1$.

3. Deduce the Fourier coefficients (C_k's) for the function $Sin(x)$. *This should be possible by inspection, rather than any significant calculation.*

Solutions

C_{-1}: MD5(h_X) = 28f251...

C_0: MD5(j_X) = 4fd3f6...

C_1: MD5(k_X) = e82a2a...

2.4 Fourier Coefficients of 1 + sin(x)

Resources

- Book: 1.4 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 2 (https://www.youtube.com/watch?v=1rqJl7Rs6ps)

Comments

You should be able to follow the derivation of the formula for Fourier coefficients in the video. Feel free to seek help if you have trouble.

Challenge

Using the same approach as challenge 2.3, deduce for the function $1 + sin(x)$ the following Fourier coefficients:

Solutions

C_{-1}: MD5(z_X) = 39e026...

C_0: MD5(x_X) = 0ef183...

C_1: MD5(c_X) = 89b992...

2.5 Relation of positive and negative Fourier coefficients for a real signal

Resources

- Book: 1.4 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 2 (https://www.youtube.com/watch?v=1rqJl7Rs6ps)

Challenge

If the Fourier coefficient C_1 is $4 + 6i$, what is the Fourier coefficient C_{-1}?

Solution

X

MD5(m_X) = 36ab38...

2.7 The Fourier series of $f(t) = t$: C_0

Resources

- Book: 1.5, 1.7 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 3 (https://www.youtube.com/watch?v=BjBb5IlrNsQ)

Challenge

Considering the function $f(t) = t$ over the interval 0 to 1, calculate the Fourier coefficient C_0 using the derived formula for Fourier coefficients. Compare with the average over the interval.

Solution

X

MD5(aa_X) = 2708ad...

2.8 The Fourier series of $f(t) = t$: C_k

Resources

- Book: 1.5, 1.7 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 3 (https://www.youtube.com/watch?v=BjBb5IlrNsQ)

Challenge

Considering the function $f(t) = t$, calculate a general expression for the Fourier coefficients C_k where $k \neq 0$.

To check your answer, evaluate the Fourier coefficient for $k = -30$.

Solution

X

MD5(bb_X) = 8005c7...

2.9 The Fourier series of $f(t) = t$ in exponential form

Resources

- Book: 1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 3 (`https://www.youtube.com/watch?v=BjBb5IlrNsQ`)

Challenge

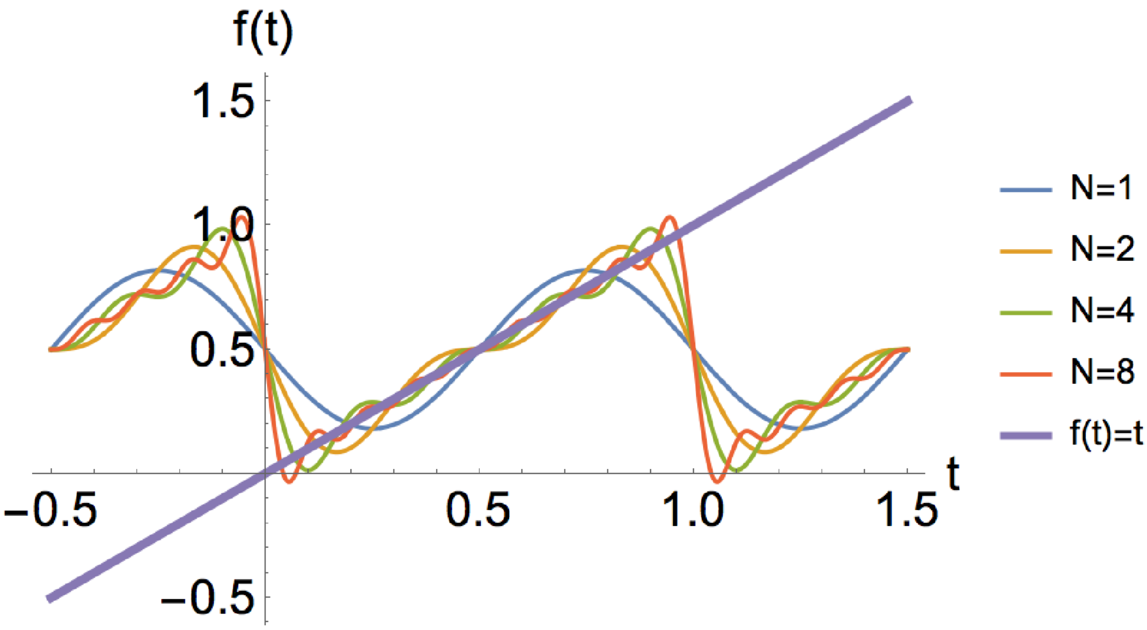
1. Write $e^{i2\pi k}$ in terms of cosines and sines.

2. Evaluate your expression obtained in (1) for $k = 0, 1, 2, 3, 4$

3. Write the function $f(t) = t$ evaluated between 0 and 1 in terms of its exponential Fourier series $f(t) = \sum_{k=-N}^{k=N} g(k, t)$ replacing $g(k, t)$ as appropriate.

To check your answer, evaluate the Fourier series up to $N = 1$ with $t = 0.8$.

The graph with increasing values of N looks like this:



Solution

X

MD5(cc_X) = caa033...

2.10 The Fourier series of $f(t) = t$ in trigonometric form

Resources

- Book: 1.5, 1.7 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Comment

Since cosine and sine can be written in terms of exponentials, it is possible to switch between Fourier series that are expressed in terms of exponentials and Fourier series that are expressed in terms of sines and cosines. Many textbooks will actually work in terms of these trigonometric forms. Where only sine terms are involved, it is called a "Fourier sine series" and where only cosines are involved it's termed a "Fourier cosine series". The series have coefficients a_0, a_k and b_k, in the following fashion:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{k=N} a_k cos(2\pi kt) + b_k sin(2\pi kt) \tag{2.3}$$

Note that the sum here goes from $k = 1$ to $k = N$, in contrast to the exponential form of Fourier series which goes between $k = \pm N$.

Challenge

1. Write $sin(2\pi t)$ in the form of an exponential sum:

$$sin(2\pi kt) = \sum_{k=-N}^{k=N} f(k)e^{g(k,t)} \tag{2.4}$$

Your answer in challenge 2.3 may help you.

What is $f(k)$, $g(k,t)$ and N? To check your answers for $f(k)$ and $g(k,t)$, substitute $k = 1$ and $t = 2$ as appropriate into your expressions for $f(k)$ and $g(k,t)$.

2. By converting from an exponential-form sum ($\sum_{k=-N}^{k=N}$) to a trigonometric-form sum ($\sum_{k=1}^{k=N}$), re-write the series obtained in challenge 2.9 in terms of a trigonometric infinite series (ie, using sines and cosines). To check your answer, evaluate the Fourier series with $N = 1$ with $t = 0.8$ and ensure that you get the same answer as you did for challenge 2.9.

3. You should find you are left with an expression only in terms of sine or cosine. Which is it? Is the function $f(t) = t$ is an odd function ($f(-t) = -f(t)$). Does an odd function result in a_k or b_k being zero? Note that for an even function ($f(-t) = f(t)$), the opposite is true.

Solution

(See the hash examples about entering imaginary numbers)

$f(k)$: MD5(iiif_X) = 3979fa...

$g(k)$: MD5(iiig_X) = 365dd7...

N: MD5(iiin_X) = e3f634...

2.11 Periods other than unity

Resources

- Book: 1.6.1 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Comment

Please read the resource. There, c_n is illustrated for an arbitrary period T going from 0 to T, but you do not need to start from 0. If instead you want to approximate a function from time T_0 to $T_0 + T$, you can simply swap the integration limits (ie, T_0 instead of 0 and $T_0 + T$ for T).

Challenge

Part I

1. By expanding the exponential out in terms of sine and cosine, determine the numerical value of $e^{i\pi k}$ for $k = 0, 1, 2, 3, 4$.

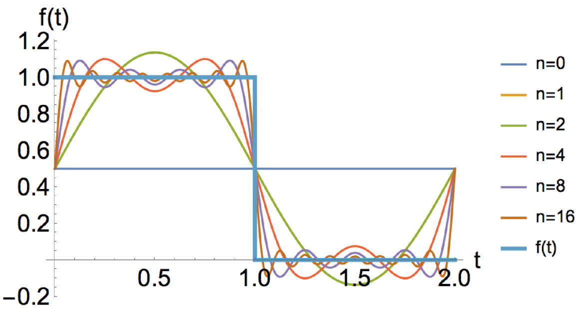
2. Assuming k can only be an integer, determine the value of N in the formula $e^{i\pi k} = N^k$ and the value of M in the formula $e^{i2\pi k} = M^k$.

Part II

Determine C_0 and C_k for the following square-wave functions using exponential fourier-series representation:

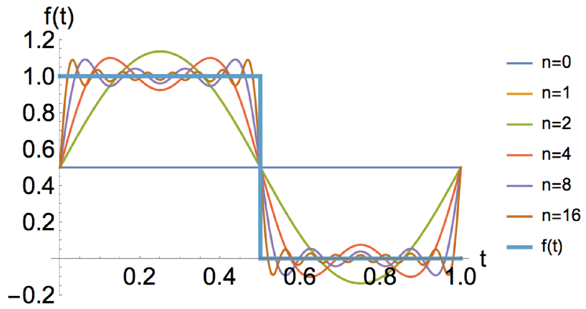
1.

$$f(t) = \begin{cases} 1 & \text{for } 0 < t < 1 \\ 0 & \text{for } 1 < t < 2 \end{cases} \tag{2.5}$$



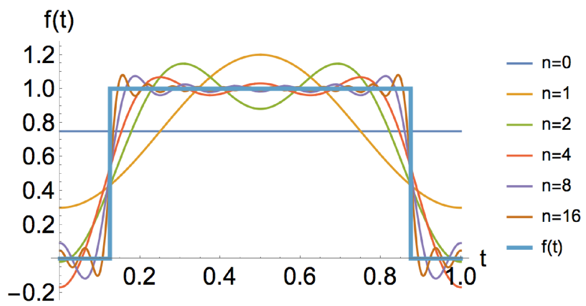
2.

$$f(t) = \begin{cases} 1 & \text{for } 0 < t < \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < t < 1 \end{cases} \tag{2.6}$$



3. *(Do not try to simplify the exponentials beyond cosines and sines)*

$$f(t) = \begin{cases} 1 & \text{for } \frac{1}{8} < t < \frac{7}{8} \\ 0 & \text{for } \frac{7}{8} < t < \frac{9}{8} \end{cases} \tag{2.7}$$



Part III

1. Express each of the square-waves above in terms of an exponential fourier series $f(t) = \sum_{k=-N}^{k=N} g(k,t)$, changing $g(k,t)$ for the exponential fourier series expression. To check your expressions, evaluate the series for $t = 1$ with $N = 1$.

Solution

Part I

1.
$k = 0$: MD5(ddk0_X) = 1b3914...
$k = 1$: MD5(ddk1_X) = da2480...
$k = 2$: MD5(ddk2_X) = 67efb2...
$k = 3$: MD5(ddk3_X) = 26a591...
$k = 4$: MD5(ddk4_X) = f09b1c...

2.
N: MD5(ddn_X) = f3550e...
M: MD5(ddm_X) = e84cb4...

Part II

1.
C_0: MD5(dd1c0_X) = aab052...
C_1: $\dfrac{-i}{\pi}$

2.
C_0: MD5(dd2c0_X) = c34c5d...
C_1: $\frac{-i}{\pi}$
3.
C_0: MD5(dd3c0_X) = ac102c...
C_1: $\frac{-1}{\sqrt{2}\pi}$

Part III

1. First square-wave: MD5(dd1sum_X) = 35788a...
2. Second square-wave: MD5(dd2sum_X) = 75b60f...
3. Third square-wave: 0.30 (in decimal form, but it can be written more neatly with fractions and roots)

2.12 Infinite series

Resources

- Book: 1.7 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)

Comment

It is important to understand why some series are infinite, while others are not (well, technically all series are infinite since they all involve sums to $n = \infty$, however for some series the Fourier coefficients are all zero above a certain value of n). Here

Challenge

1. In challenge 2.3 and 2.4 you determined the fourier coefficients for $sin(x)$ and $sin(x) + 1$. If you write the function in the form

$$sin(x) = \sum_{k=-N}^{k=N} C_k e^{ikx} \tag{2.8}$$

what is N here?

2. Expand e^{ikx} in terms of sine and cosine. Which has the higher frequency? $k = 1$ or $k = 100$?

3. Referring to the resource, do sharp corners in a function lead to higher or lower frequencies?

4. Does non-periodiciy lead to higher or lower frequencies? In challenge 2.9 you calculated the Fourier series for $f(t) = t$. If the series is written in a form similar to equation 2.8, what would N be in this case?

5. In general, the Fourier series is a sum to $\pm\infty$, however in some cases the coefficients (C_k's) are zero beyond a certain number of terms. Which of the functions below will have Fourier coefficients that are all zero after a certain number of terms? Sum the points of these functions.

1 point: x

2 points: x^2

4 points: $cos(2x) + 3sin(7x)$

8 points: $e^{2\pi ix}$

6. Briefly explain the characteristics of functions that lead to infinite Fourier series and finite Fourier series.

Solution

1. (enter as an integer, without ".00") MD5(ee1_X) = b6cadd...
2. ("1" or "100") MD5(ee2_X) = a0bebe...
3. ("lower" or "higher") MD5(ee3_X) = f0d1f9...
4. MD5(ee4_X) = 7cd9d2...
5. MD5(ee5_X) = 9d1559...

2.13 k-symmetry

Challenge

Determine what X and Y represent algebraically.

$$cos(k\pi t) = \frac{1}{2}e^{-ki\pi t} + \frac{1}{2}e^{\boldsymbol{X}i\pi x} \tag{2.9}$$

$$sin(k\pi t) = \frac{1}{2}ie^{\boldsymbol{Y}i\pi t} - \frac{1}{2}ie^{ki\pi t} \tag{2.10}$$

To check your answers you may substitute any appropriate values from the following list: $k = 2$, $t = 1$

Solution

X

MD5(ff_X) = 942d6f...

Y

MD5(gg_Y) = a379b8...

2.14 Direct trigonometric calculation of a Fourier series: the coefficients

Comment

This challenge introduces several key concepts at once, including decoupling of integral intervals and periodicity, the concept of a square wave and direct trigonometric evaluation of Fourier series. If you can master this you'll be in a really strong position.

It is hopefully clear now that for real signals, due to the symmetry of the positive and negative k's, one can fully compose Fourier series in terms of sine and cosine. In challenge 2.10 we saw the formula for the function in terms of Fourier coefficients a_0, a_n and b_n. While we will not use this approach, it is important to be able to utilise such a formulation since this is the way some books present it and some people have learnt it. Therefore, without proof, the coefficients can be calculated using

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)Cos(2\pi kt/T)dt \tag{2.11}$$

$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)Sin(2\pi kt/T)dt \tag{2.12}$$

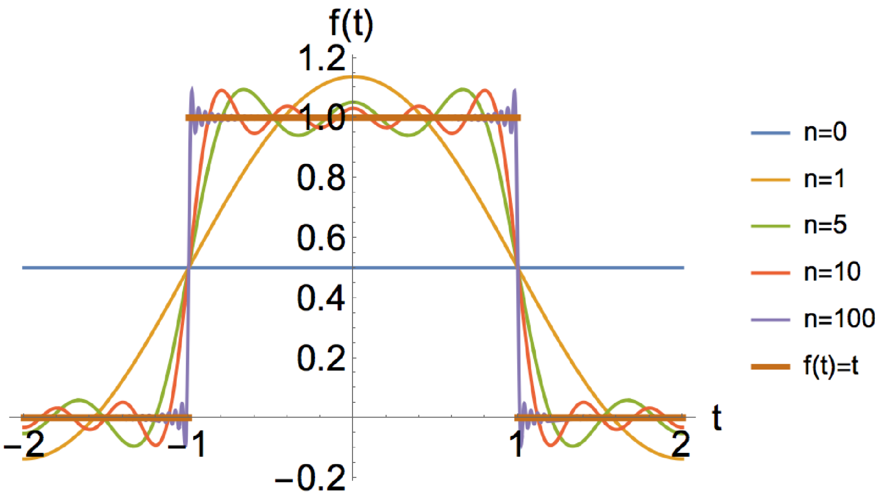
Challenge

Using the direct trigonometric Fourier series, obtain a general expression for the a_k and b_k coefficients for the square-wave signal with periodicity 4:

$$f(t) = \begin{cases} 1 & \text{for } -1 < t < 1 \\ 0 & \text{for } 1 < t < 3 \end{cases} \tag{2.13}$$

Note the symmetry of the problem. Can you see what terms will be zero? To check your solution, calculate a_k and b_k for $k = 0$, $k = 2$ and $k = 3$. Note that you will have to break the integrals into two parts and sum them in order to tackle this problem.

A graph of the function, including the solution for various values of n, is shown here:



Solution

k	a_k	b_k
0	MD5(hh_X)=e57c15...	MD5(ii_X)=377fe2...
2	MD5(jj_X)=54aaa1...	MD5(kk_X)=be063f...
3	MD5(mm_X)=b8fce7...	MD5(nn_X)=b6fbaf...

2.15 Direct trigonometric calculation of a Fourier series: the series

Comment

For a series with non-unit period, the Fourier series in trigonometric form given in equation 2.3 can be modified to read

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{k=N} a_k cos(2\pi kt/T) + b_k sin(2\pi kt/T) \tag{2.14}$$

Challenge

Calculate the Fourier series for the square wave introduced in challenge 2.14 using direct trigonometric calculation for up to $k = 3$. Check your solution by evaluating for $t = 0.1$.

Solution

0.9397

2.16 2D orthogonal vectors

Resources

- Book: 1.9 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Challenge

Sum the points of the vectors in 2D that are orthogonal:

1 point: (5, 4) and (-1, 1.25)

2 points: (2, -3) and (-6, 4)

4 points: (-2.25, 1.5) and (2, 3)

8 points: (4.5, 4) and (3, -3.375)

16 points: (6, 4) and (4, -6)

32 points: (5, 1) and (-2, 8.125)

64 points: (0, 1) and (1, 0)

128 points: (1, 1) and (1, 1)

Solution

X

MD5(pp_X) = 92843f...

2.17 Orthonormal basis

Resources

- Video: `https://www.khanacademy.org/math/linear-algebra/alternate-bases/orthonormal-basis/v/linear-algebra-introduction-to-orthonormal-bases`

Challenge

Sum the points of the following vectors that form an orthonormal basis:

1 point : $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ and $(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}})$

2 points: $(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ and $(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$

4 points: $(\frac{2}{\sqrt{2}}, \sqrt{\frac{7}{8}}, \frac{1}{\sqrt{6}})$, $(-\sqrt{\frac{2}{5}}, \frac{7}{\sqrt{14}}, -\frac{1}{\sqrt{6}})$ and $(\frac{1}{\sqrt{3}}, \frac{1}{5\sqrt{3}}, -\frac{7}{5\sqrt{3}})$

8 points: $(\frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{4}{\sqrt{21}})$, $(-\sqrt{\frac{2}{7}}, \frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}})$ and $(\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$

16 points: $(\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}})$, $(-\frac{1}{\sqrt{2}}, \frac{2\sqrt{2}}{5}, -\frac{3}{5\sqrt{2}})$ and $(\frac{1}{\sqrt{3}}, \frac{1}{5\sqrt{3}}, -\frac{7}{5\sqrt{3}})$

32 points: $(0, 2)$ and $(2, 0)$

64 points: $(0, 1)$ and $(1, 0)$

Solution

X

MD5(qq_X) = 097fd7...

2.18 Natural basis

Resources

- Book: 1.9 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Challenge

Sum the components of the following vectors of an orthonormal basis in $\mathbb{R}^{300}$ space:

- First component of the first vector
- first component of the second vector
- 200th component of the 100th vector
- 200th component of the 200th vector
- last component of the 299th vector
- last component of the last vector

Solution

X

MD5(rr_X) = 095c77...

2.19 Orthonormal basis for Fourier series

Resources

- Book: 1.9 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 4 (`https://www.youtube.com/watch?v=n5lBM7nn2eA`)

Comment

The previous challenges have focussed on the orthogonality and orthonormality of vectors. We now make the jump to functions. As chapter 1.9 explains, although its not perfect, the analogy between vectors and functions is a good way to help understand and visualise the role that the terms of a Fourier series play in defining a basis upon which to describe a function.

Challenge

Starting from the inner product of two terms ($e^{2\pi i k_1 t}$, $e^{2\pi i k_2 t}$) of a Fourier series, demonstrate that the terms of a Fourier series form an orthonormal basis. **Show a full derivation**.

To check your intuition, you may evaluate the following cases:

$X = (e^{2\pi i k_1 t}, e^{2\pi i k_1 t})$

$Y = (e^{2\pi i k_1 t}, e^{2\pi i k_2 t})$

Solution

If you are not confident about your derivation, please check with someone else. If there is any step that you do not fully understand, do not hesitate to ask. If you do not understand the connection between previous challenges on vectors and this challenge using functions, do not hesitate to ask someone.

X

MD5(ss_X) = 8f7f41...

Y

MD5(tt_Y) = 2c669b...

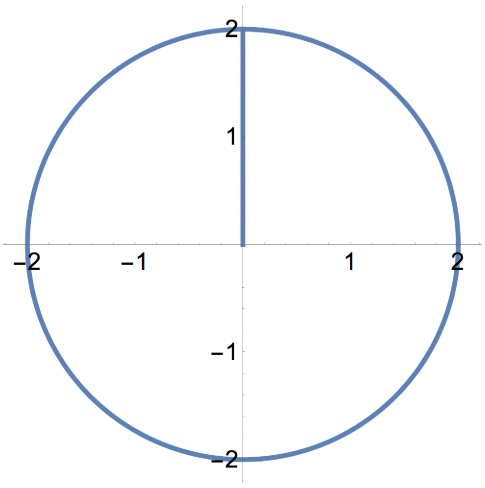
2.20 Circles and Fourier series

Resources

- Video 1: `https://www.youtube.com/watch?v=Y9pYHDSxc7g`
- Video 2: `https://www.youtube.com/watch?v=LznjC4Lo71E`

Comment

In the first lecture we saw how it was possible to approximate any function given enough circles. Here we link what you have learned back to that first lecture. I strongly recommend viewing the fun and informative videos listed here under Resources. In summary, by building a Fourier series you are representing a function using an orthonormal basis, where each component of the basis can be considered visually as a circle operating with individual radius and frequency on the real-imaginary plane. If, after completing this challenge, that last sentence makes sense to you, then you have achieved the first major goal of this course.



Challenge

Treating the x-axis as the real axis and the y-axis as the imaginary axis, arrange the equations below in the following order:

1. A point moving round on a circle with radius 2 units and frequency 2 Hz
2. A point moving round on a circle with radius 3 units and frequency 1 Hz
3. A point moving round on a circle with radius 2 units and a period of 1 second
4. A point moving round on a circle with radius 3 units and a period of 2 seconds

Equations:

$Ae^{2\pi ikt}$ where t is time in seconds and the values of A and k are as follows:

A: $A = 2$, $k = 2$

B: $A = 3$, $k = 1$

C: $A = 3$, $k = 0.5$

D: $A = 2$, $k = 1$

Solution

X

MD5(uu_X) = cb7845...

2.21 Gibb's phenomenon

Resources

- Wikipedia: `https://en.wikipedia.org/wiki/Gibbs_phenomenon`
- Book: 1.18 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Challenge

Write a few sentences summarising your understanding of what Gibb's phenomenon is. By what percentage does overshoot of a discontinuity of a square-wave function occur?

The full derivation is beyond the scope of this course, so it is not necessary to understand the (rather complex) derivation in the notes. The aim of this challenge is simply to enable you to be able to describe qualitatively what Gibb's phenomenon is using a few sentences, and know the amount of overshoot in the case of a standard square-wave. No maths expected.

Solution

X %

MD5(vv_X) = aa19f2...

2.22 Partial derivatives

Challenge

Determine u_t and u_{xx} for the equation

$$u(x,t) = 5tx^2 + 3t - x \tag{2.15}$$

To check your answer, substitute $x = 3$ and $t = 2$ into your answers, as appropriate.

Solution

u_t: MD5(ww_X) = 7901cb...

u_{xx}: MD5(xx_X) = 6aba1c...

2.23 Heat equation: Periodicity

Resources

- Lecture 4 from 37:00 onwards: (`https://www.youtube.com/watch?v=n5lBM7nn2eA`), continuing at the start of lecture 5 (`https://www.youtube.com/watch?v=X5qRpgfQld4`)

Comment

Please watch and makes notes following the derivation of the heat equation shown in the videos in lectures 4-5.

This use of Fourier series is a great example of how the relatively abstract mathematical concepts covered by the course so-far can have real physical applications in science and engineering.

Challenge

Add the points of the following true statements concerning a heated ring with circumference 1 and temperature distribution described by $u(x,t)$:

1 point: $u(x,t) = u(x,t)$

2 points: $u(x,t) = u(x,t+1)$

4 points: $u(x,t) = u(x,t+2)$

8 points: $u(x,t) = u(x+1,t)$

16 points: $u(x,t) = u(x+1,t+1)$

32 points: $u(x,t) = u(x+1,t+2)$

64 points: $u(x,t) = u(x+2,t)$

128 points: $u(x,t) = u(x+2,t+1)$

256 points: $u(x,t) = u(x+2,t+2)$

512 points: The temperature distribution is periodic in space but not time

1024 points: The temperature distribution is periodic in time but not space

2048 points: The temperature distribution is periodic in both space and time

4096 points: The temperature distribution is periodic neither in space nor time

Solution

X

MD5(yy_X) = ed4019...

2.24 Heat equation: Fourier coefficients

Resources

- Lecture 4 from 37:00 onwards: (`https://www.youtube.com/watch?v=n5lBM7nn2eA`), continuing at the start of lecture 5 (`https://www.youtube.com/watch?v=X5qRpgfQld4`)

Challenge

Starting from the heat (diffusion) equation $u_t = u_{xx}/2$, show that the Fourier coefficient at time t is given by

$$C_k(t) = C_k(0)e^{-2\pi^2 k^2 t} \tag{2.16}$$

Solution

Compare with your peers during discussion time and please ask if there is anything you do not understand.

2.25 Heat equation

Resources

- Lecture 4 from 37:00 onwards: (`https://www.youtube.com/watch?v=n5lBM7nn2eA`), continuing at the start of lecture 5 (`https://www.youtube.com/watch?v=X5qRpgfQld4`)

Comment

Note that the exponential in the answer for $B(t)$ is negative, leading to a decay to mean ambient temperature over time (you can think of the temperature $u(x,t)$ as the temperature relative to the surrounding environment rather than absolute temperature measured in Kelvin).

Challenge

Write the heat equation and its Fourier coefficient in the form below. Identify A(x), B(t) and C(x). To check your answers, substitute $k = 1$, $x = 2$ and $t = 3$ into the expressions (purely for checking the solution; these numbers have no physical basis).

$$\hat{f}(k) = \int_0^1 A(x) f(x) dx \tag{2.17}$$

$$u(x,t) = \sum_{k=-\infty}^{k=\infty} \hat{f}(k) B(t) C(x) \tag{2.18}$$

Solution

A: MD5(zz_X) = ffef92...

B: MD5(aaa_X) = c04067...

C: MD5(bbb_X) = ae8c65...

Chapter 3

Fourier Transform

3.1 The transition to the Fourier Transform

Resources

- Video I: Lecture 5 from 27:00 (`https://youtu.be/X5qRpgfQld4?t=27m`)
- Video II: Lecture 6 up to 20:00 (`https://www.youtube.com/watch?v=4lcvROAtN_Q`)
- Book: Chapter 2 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Comment

Until now we have been considering Fourier series which allows us to describe periodic phenomena. This can however cause issues, such as in challenge 2.9 when the function isn't actually periodic, since the derived fourier series results in periodicity outside the region of integration. The Fourier transform can help overcome this limitation and accurately describe signals which are non-periodic.

The suggested resources provide an excellent intuetive path to the connection between Fourier Series and Fourier Transforms, and this challenge is designed to give you the opportunity to take some time to try to understand the main concepts behind the transition.

Challenge

Watch the two videos in the Resources section above, taking notes about the transition from Fourier Series to Fourier Analysis.

3.2 Fourier transform notation

Resources

- Book: Chapter 2.1, page 75 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Challenge

Add the points of the legitimate forms of Fourier notation:

1 point: $\mathcal{F}f(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwt} f(t)dt$

2 points: $\mathcal{F}f(w) = \int_{-\infty}^{\infty} e^{-i2\pi wt} f(t)dt$

4 points: $\mathcal{F}f(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwt} f(t)dt$

8 points: $\mathcal{F}f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\pi st} f(t)dt$

16 points: $\mathcal{F}f(w) = \int_{-\infty}^{\infty} e^{i2\pi wt} f(t)dt$

32 points: $\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-ist} f(t)dt$

Solution

MD5(eee_X) = 58851c...

3.3 L'Hôpital's rule

Challenge

Use L'Hôpital's rule to determine the limit of

$$\frac{Sin(x)}{x} \tag{3.1}$$

as $x \to 0$.

Solution

MD5(ccc_X) = 9cc92f...

3.4 Fourier transform of a window function

Resources

- Book: Chapter 2.1, page 68, particularly at the bottom of the page (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 6 starting from 29:26 (`https://youtu.be/4lcvROAtN_Q?t=29m26s`)

Comments

This challenge, and the next few, represent very common functions that can be evaluated using the Fourier transform. Try to understand the mathematics used in their evaluation (I will not expect you to be able to derive it starting Fourier series however, like Prof. Osgood does from page 65 onwards). The main learning concept is to understand how to use the Fourier Transform to convert between real-space and frequency-space.

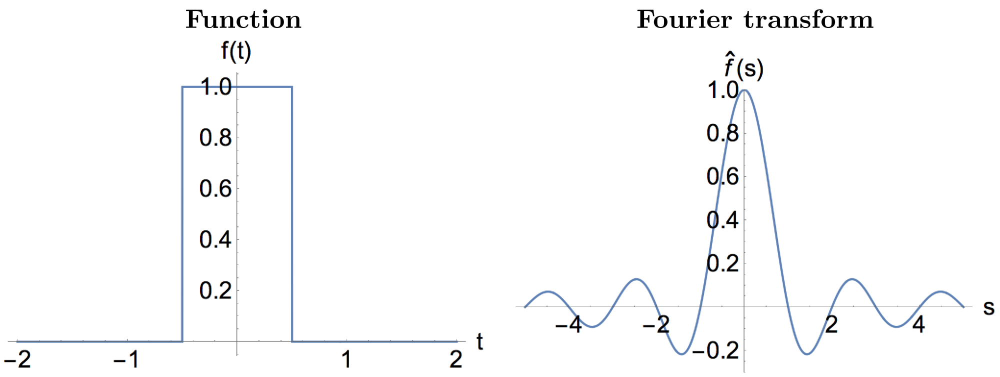
In this and subsequent challenges, I list both reading resources and video. For this challenge up to and including challenge 3.10, I would suggest starting with the reading resources, and then if you have trouble understanding, to use the video resource for further support.

Challenge

Calculate the Fourier Transform for the window function

$$f(t) = \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{for } |t| > 1/2 \end{cases} \tag{3.2}$$

In graph-form the function and its transform appear as follows:



To check your answer, calculate $\hat{f}(s = 1.5)$.

Solution

MD5(ddd_X) = b78cd8...

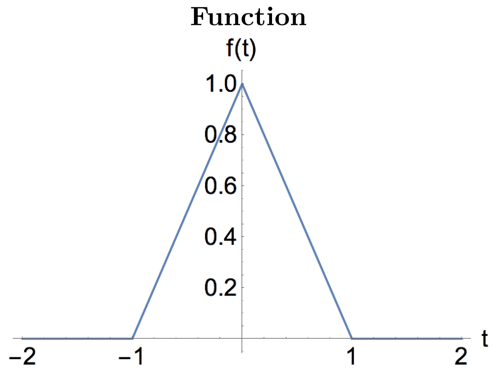
3.5 Fourier transform of a triangle function

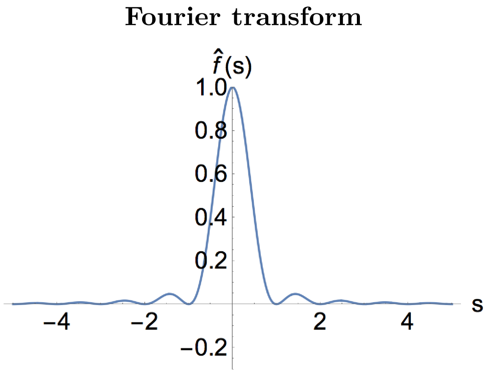
Resources

- Book: Chapter 2.2.1 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 6 (`https://www.youtube.com/watch?v=4lcvROAtN_Q`)

Challenge

Calculate the Fourier Transform for the triangle function shown below. In graph-form the function and its transform appear as follows:





To check your answer, calculate $\hat{f}(s = 1.5)$.

Solution

MD5(fff_X) = 55e1c1...

3.6 Fourier transform of a gaussian

Resources

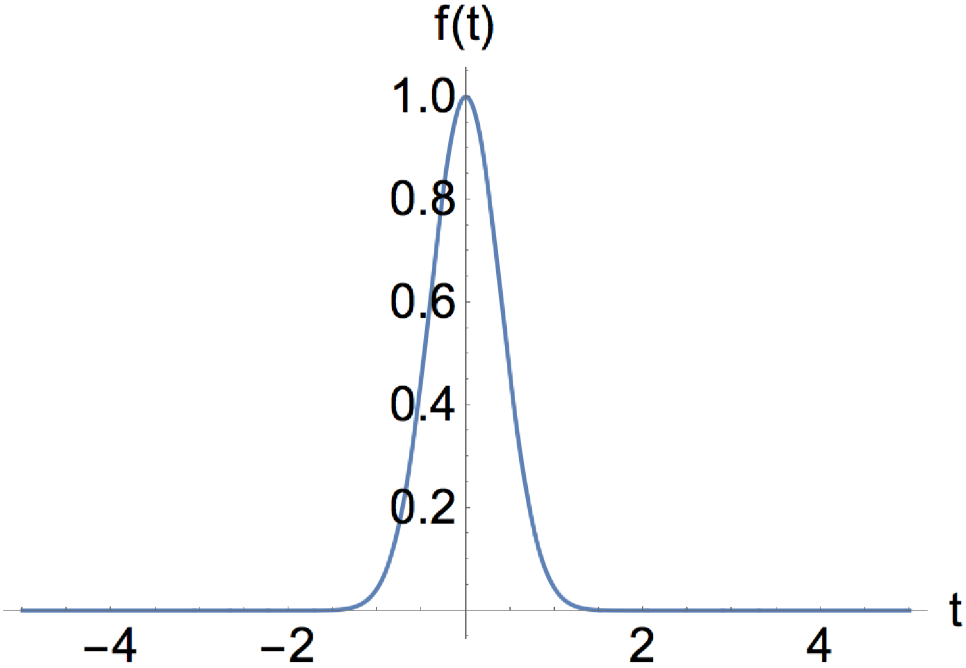
- Book: Chapter 2.2.2 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 7 (https://www.youtube.com/watch?v=mdFETbe1n5Q)

Challenge

Calculate the Fourier Transform for the gaussian function:

$$f(t) = e^{-\pi t^2} \tag{3.3}$$

The gaussian function looks like



To check your answer, calculate $\hat{f}(s = 1.5)$.

Solution

MD5(ggg_X) = 341b23...

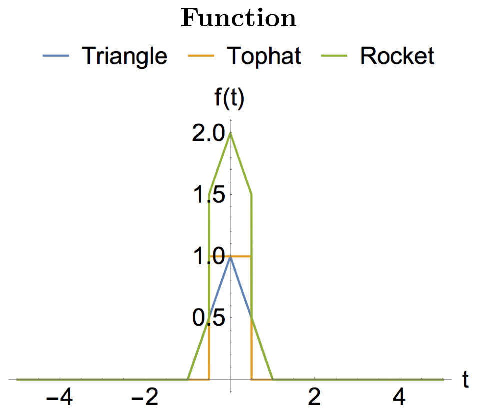
3.7 Fourier transform of a rocket function

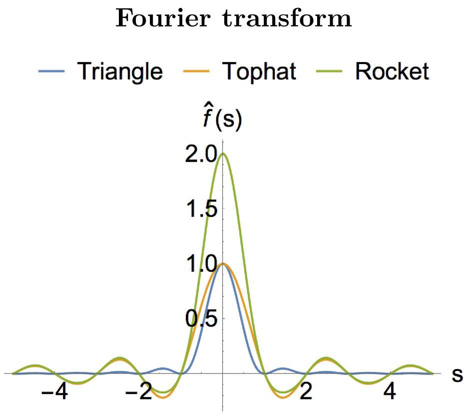
Resources

- Book: Chapter 2.2.6 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Challenge

Calculate the Fourier Transform for the rocket function shown below. In graph-form the function and its transform appear as follows:





To check your answer, calculate $\hat{f}(s = 1.5)$.

Solution

MD5(hhh_X) = a344fd...

3.8 Fourier transform of a shifted window function

Resources

- Book: Chapter 2.2.7 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video lecture 8 from 1:56 to 11:20 (`https://youtu.be/wUT1huREHJM?t=1m56s`)

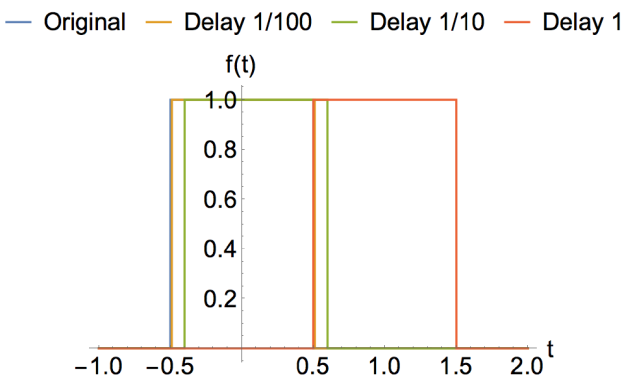
Challenge

1. By delaying the window function slightly as shown below, we are introducing a phase-shift. Write a few sentences explaining what is meant by a phase-shift and how it relates to the current situation.

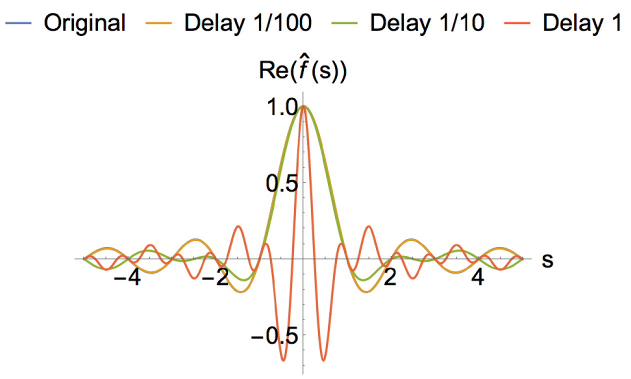
2. Calculate the Fourier Transform for the window function delayed by 1/100 s, 1/10 s and 1 s.

3. Write a short summary of how the real and imaginary parts of the signal are changing as the delay increases. You considered the original window function in challenge 3.4.

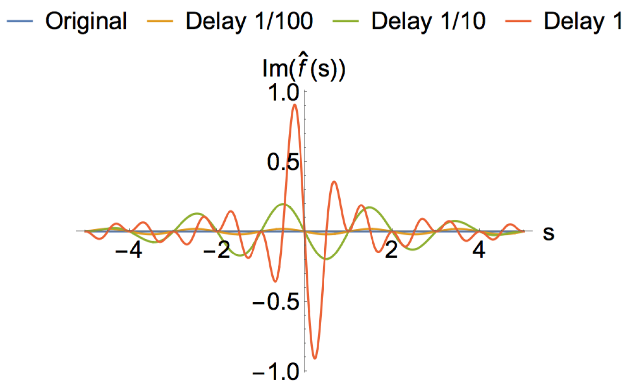
Function



Fourier transform (real part)



Fourier transform (imaginary part)



To check your answer, calculate $\hat{f}(s = 2.9)$ for each case. By doing this, you will also gain insight into the new nature of the real and imaginary parts of the function with shifting.

Solution

1/100 s delay: $0.0333568 - 0.0061462i$

1/10 s delay: $-0.00843515 - 0.0328527i$

1 s delay: $0.0274405 + 0.0199367i$

3.9 Fourier transform of a stretched triangle function

Resources

- Book: Chapter 2.2.8 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video lecture 8 from 12:12 to 28:50 (`https://youtu.be/wUT1huREHJM?t=12m12s`)

Challenge

Since this can be a little confusing, please be sure to fully read the listed resource and make sure you understand the reasoning and derivation.

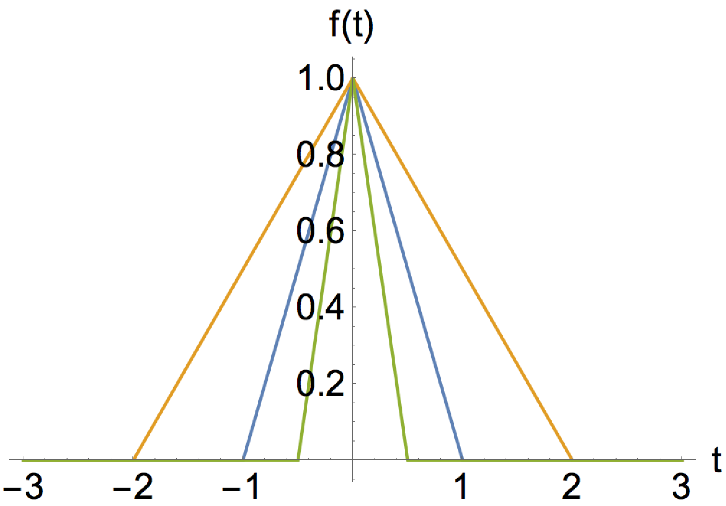
A triangle function of general width can be defined as

$$f(at) = \begin{cases} 1 - a|t| & \text{for } a|t| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

1. In challenge 3.5 the base-width of the triangle was 2 (ie, $1-(-1)$) and the half-base-width was 1 (ie, $1-0$). What is the half-base-width for a triangle when $a = 2$ and $a = 1/2$?

2. Calculate the Fourier Transform $\hat{f}(s)$ for the case where the width of the triangle-function is doubled. To check your answer, evaluate the transform at $s = 0.1$.

3. Write a few sentences explaining how stretching and squeezing in the time-domain is related to stretching and squeezing in the frequency domain.



Solution

1.
$a = 2$: MD5(iii_X) = e8cb88...
$a = 1/2$: MD5(jjj_X) = eb8c63...

2. $\hat{f}(0.1) = 1.75$

3.10 Fourier transform of a shifted-stretched function

Resources

- Book: Chapter 2.2.9 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Comment

During the past few challenges we have been considering various transforms of common functions. The ability to utilise such transforms; and generate your own manipulations of them; can be very useful for applications related to filtering.

Challenge

Show that the Fourier Transform of the function $f(at \pm b)$ can be given by

$$\frac{1}{|a|} e^{\pm i2\pi sb/a} \hat{f}(s/a) \tag{3.5}$$

3.11 Convolution introduction

Resources

- Video lecture 8 starting at 32:15: `https://youtu.be/wUT1huREHJM?t=32m15s`

Comment

Here we expand our study to convolution; a powerful function for processing signals. Convolution is not immediately intuetive, but Prof. Osgood provides an excellent introduction.

Challenge

Make notes following the above video deriving the formula relating convolution in the frequency and time domains.

3.12 Filtering

Resources

- Video lecture 9 until 24:15: `https://www.youtube.com/watch?v=NrOR2qMVWOs`

Comment

The above video includes an excellent example of using fourier analysis in a scientific context, including its application in filters.

The challenge below asks you to watch to 24:15, however after this point he goes on to describe the futility of trying to visualise convolution in the time-domain. Nevertheless, the graphics on convolution on Wikipedia [1] (especially these: [2a,b]) I think go some way to visualising what's happening in the time-domain.

1: `https://en.wikipedia.org/wiki/Convolution`
2a: `https://en.wikipedia.org/wiki/Convolution#/media/File:Convolution_of_box_signal_with_itself2.gif`
2b: `https://en.wikipedia.org/wiki/Convolution#/media/File:Convolution_of_spiky_function_with_box2.gif`

For your reference, turbidity standards of 5, 50, and 500 Nephelometric Turbidity Units (left to right respectively) are shown here:

Source: *`https://en.wikipedia.org/wiki/Turbidity`*

Challenge

1. Take notes following the above video until 24:15. There is a lot of useful information here. The questions below highlight the key points that I want you to understand, however they do not cover everything, so please be sure to follow the video.

2. Using a few sentences and diagrams, describe an ideal low-pass, high-pass and band-pass filter. How can they be applied in the frequency domain to influence the signal in the time-domain?

3. Write a brief summary of how convolution in time is expressed in the frequency domain.

3.13 Convolution with a window function

Comment

The "signal" here is $g(t-\tau)$ and the "window function" is $f(\tau)$.

Challenge

1. Consider that you have a (somewhat unrealistic but mathematically manageable) input signal that varies as $g(t-\tau) = (t-\tau)^2$ with time.

Obtain the convolution of the signal $(f \star g)(t)$ with a window function:

$$f(\tau) = \begin{cases} 1 & \text{for } -1/2 < \tau < 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

2. If this window-function were a filter, what sort of band-pass filter could you consider this to be?

Solution

1. Substitute $t = 0$ into your final answer: MD5(ppp_X) = 290253...

3.14 Convolution of two window functions

Challenge

In challenge 3.4 you calculated the spectrum of a window function. Imagine here you have two window functions $f(t)$ and $g(\tau)$.

$$f(t) = \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{for } |t| > 1/2 \end{cases} \tag{3.7}$$

$$g(\tau) = \begin{cases} 1 & \text{for } |\tau| < 1/2 \\ 0 & \text{for } |\tau| > 1/2 \end{cases} \tag{3.8}$$

1. Use your knowledge of the window-function in the frequency domain from challenge 3.4 to calculate the frequency-domain spectrum of the convolution of the two window functions.

2. Compare your answer to that obtained in challenge 3.5. What is the resulting function in the time-domain? *This should be possible by comparison, without calculation.*

Solution

1. To check your answer, substitute $s = 1/2$ into your answer: 0.41

2. To check your answer, substitute $t = 1/2$ into your answer: 0.50

Chapter 4

Discrete Fourier Transform

4.1 Introduction to digital signals

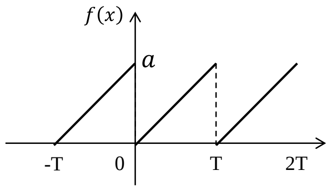
Resources

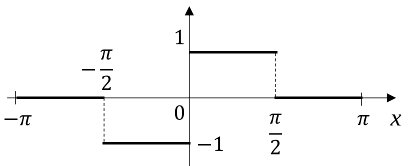
The following pages are based on slides developed by Prof. Maraso Yamashiro for the 2015 class on Fourier Analysis. Please work through the slides on the following pages and view the video afterwards.

We know how to calculate Fourier series and transforms from given functions.
But what if we don`t know what the underlying function is?

$f(x)$: A given function

e.g.





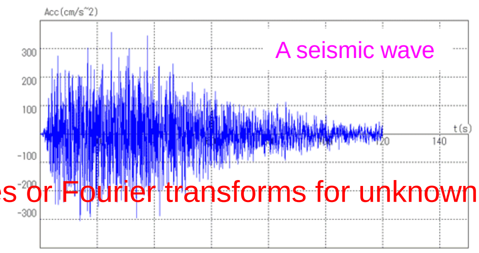
$$f(x) = \frac{ax}{T} \quad 0 \le x < T, \quad f(x+T) = f(x)$$

$$f(x) = \begin{cases} 1 & if \quad 0 \le x < \pi/2 \\ -1 & if \quad -\pi/2 \le x \le 0 \end{cases}$$

$f(x)$: Unknown

e.g.

Natural phenomena



How can we find Fourier series or Fourier transforms for unknown functions?

For an unknown function, we need to move from integration to numerical calculation

Integration → Numerical calculation

Analog data → Digital data

Continuous functions

Discretization

Another viewpoint

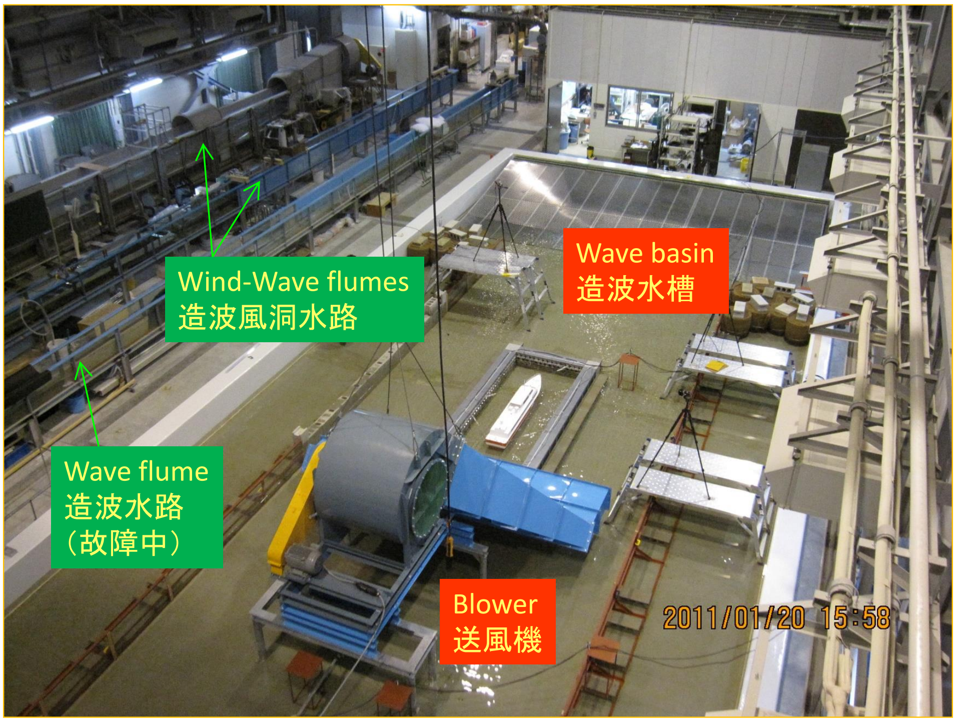
- ✓ Data are often obtained by field observations, model experiments, etc.
- ✓ Data is usually analyzed using PC.

The output is Digital data

Discrete Fourier Transform (DFT)
Fast Fourier Transform (FFT)

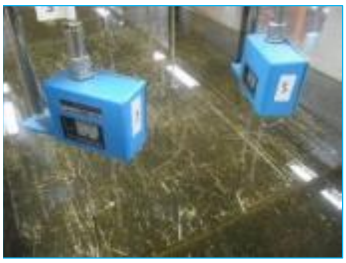
Examples of digital data

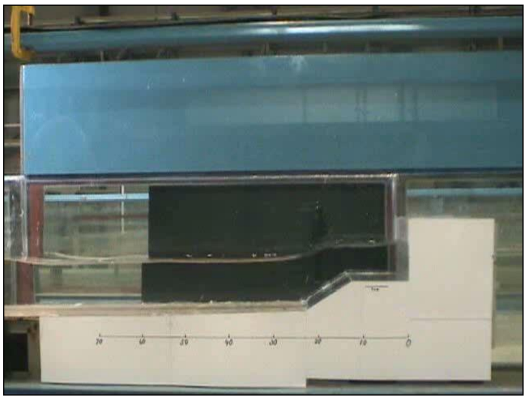
Experimental facilities of coastal and ocean engineering laboratory



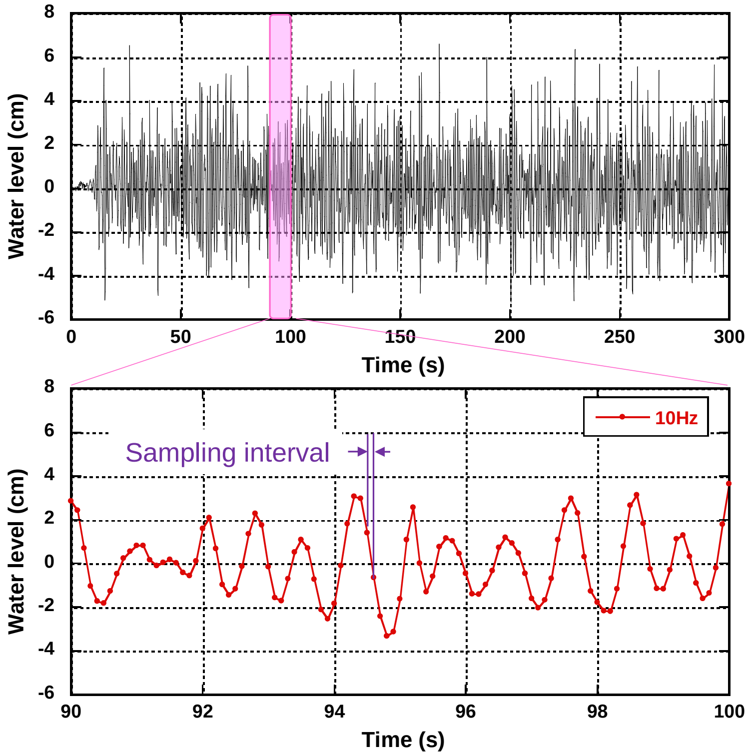
Model experiments

Wave gauge





Examples of Wave data



- Video: `https://www.youtube.com/watch?v=yWqrx08UeUs&feature=youtu.be&t=47s`

Challenge

1. An audio CD has a 44.1 kHz sampling rate. What is the highest frequency that experiences aliasing?
2. Write a sentence or two summarising what aliasing is and the Nyquist sampling theorem.

Solutions

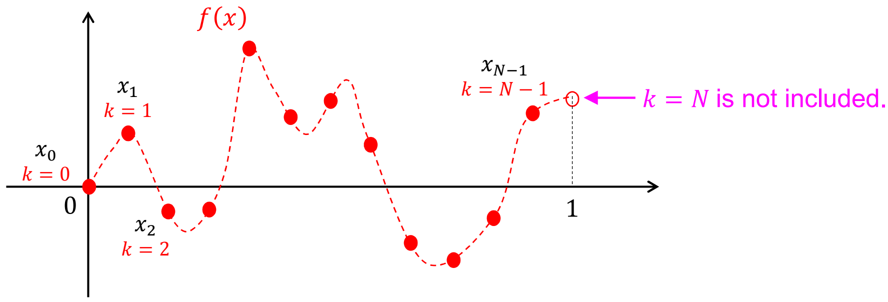
1. Write your solution X to 2 decimal places in units of kHz: MD5(xxx_X) = 8a6fe0...
2. Check your answer with your partner and discuss any differences. Ask the teacher if you are unsure.

4.2 Discrete Fourier Transform: Coefficients

Resources

The following pages are based on slides developed by Prof. Maraso Yamashiro for the 2015 class on Fourier Analysis. Please follow the derivation on the following pages.

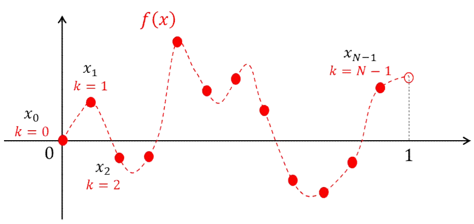
Discrete Fourier Transform (DFT)



Let $f(x)$ be periodic, for simplicity of Period 1. We assume that N measurements of $f(x)$ are taken over the interval $0 \leq x \leq 1$ at regularly spaced points

$$x_k = \frac{k}{N} \qquad k = 0, 1, 2, \cdots, N-1$$

We now want to determine a complex trigonometric polynomial



$$q(x) = \sum_{n=0}^{N-1} c_n e^{i2\pi nx}$$

that interpolates $f(x)$ at the nodes x_k, that is, $q(x_k) = f(x_k)$.
Denoting $f(x_k)$ with f_k,

$$f_k = f(x_k) = q(x_k) = \sum_{n=0}^{N-1} c_n e^{i2\pi n x_k}$$

$$k = 0, 1, 2, \cdots, N-1$$

cf.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx}$$

$$e^{ix} = \cos x + i \sin x$$

The coefficients are determined by using the orthogonality of the trigonometric system.

$$f_k = \sum_{n=0}^{N-1} c_n e^{i2\pi n x_k}$$

- ✓ Multiply f_k by $e^{-i2\pi m x_k}$ and sum over k from 0 to $N-1$
- ✓ Interchange the order of the two summations
- ✓ Replacing x_k with k/N

$$\sum_{k=0}^{N-1} f_k e^{-i2\pi m x_k} = \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} c_n e^{i2\pi(n-m)x_k} = \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} e^{i(n-m)2\pi k/N}$$

$$e^{i(n-m)2\pi k/N} = \left[e^{i(n-m)2\pi/N}\right]^k = r^k \qquad r = e^{i(n-m)2\pi/N}$$

For $n = m$

$$r^k = (e^0)^k = 1^k = 1 \qquad \sum_{k=0}^{N-1} r^k = N$$

For $n \neq m$

$$r \neq 1 \qquad \sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} = 0$$

$$r^N = e^{i(n-m)2\pi N/N} = e^{i(n-m)2\pi}$$
$$= \cos(n-m)2\pi + i\sin(n-m)2\pi = 1 + 0 = 1$$

Sum of a geometric series

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$= \frac{a(1-r^n)}{1-r} \qquad r \neq 1$$

$$\sum_{k=0}^{N-1} f_k e^{-i2\pi m x_k} = \underbrace{\sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} e^{i(n-m)2\pi k/N}}_{= Nc_m} = c_0 0 + c_1 0 + \cdots + c_m N + \cdots c_{N-1} 0$$

$$c_m = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i2\pi m x_k}$$

$$\sum_{k=0}^{N-1} r^k = N \qquad \text{For } n = m$$

$$\sum_{k=0}^{N-1} r^k = 0 \qquad \text{For } n \neq m$$

Replacing m with n,

$$c_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i2\pi n x_k} \qquad f_k = f(x_k)$$

$$n = 0, 1, 2, \cdots, N-1$$

$$c_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i2\pi n x_k} \qquad f_k = f(x_k)$$

$$n = 0, 1, 2, \cdots, N-1$$

Discrete Fourier Transform

$$\hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k} \qquad f_k = f(x_k)$$

$$n = 0, 1, 2, \cdots, N-1$$

The Discrete Fourier Transform of the given signal $\mathbf{f} = [f_0 \cdots f_{N-1}]^{\mathsf{T}}$ to be the vector $\hat{\mathbf{f}} = [\hat{f}_0 \cdots \hat{f}_{N-1}]$ with components $\hat{f}_n$

This is the frequency spectrum of the signal.

In vector notation, $\hat{\mathbf{f}} = \mathbf{F}_N\mathbf{f}$, where the $N \times N$ Fourier matrix $\mathbf{F}_{\mathrm{N}} = [e_{nk}]$ has the entries

$$e_{nk} = e^{-i2\pi n x_k} = e^{-i2\pi nk/N} = w^{nk}, \quad w = w_N = e^{-i2\pi/N} \qquad x_k = \frac{k}{N}$$

where $n,\ k = 0, \cdots, N-1$

Example Discrete Fourier Transform (DFT)

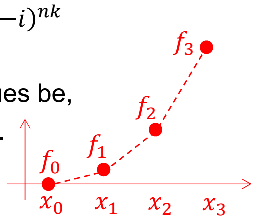
Let $N = 4$ measurements (sample values) be given.

Then
$$w = e^{-i2\pi/N} = e^{-i\pi/2}$$
$$= \cos\pi/2 - i\sin\pi/2$$
$$= -i$$

and thus
$$w^{nk} = (-i)^{nk}$$

$$\hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-i2\pi n x_k}$$
$$\hat{\mathbf{f}} = \mathbf{F}_N\mathbf{f}$$
$$\mathbf{F}_{\mathrm{N}} = [e_{nk}]$$
$$e_{nk} = e^{-i2\pi n x_k} = e^{-i2\pi nk/N} = w^{nk}$$
$$w = w_N = e^{-i2\pi/N}$$
$$e^{ix} = \cos x + i\sin x$$

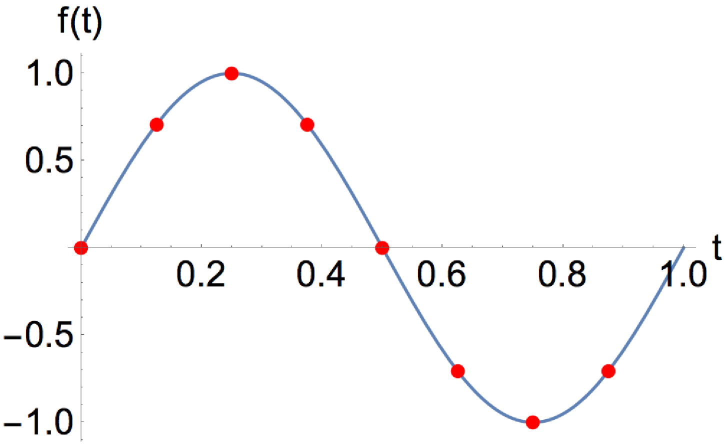
Let the sample values be, say $\mathbf{f} = [0\ \ 1\ \ 4\ \ 9]^{\mathsf{T}}$.



Then

$$\hat{\mathbf{f}} = \mathbf{F}_4\mathbf{f} = \begin{array}{c} \\ n=0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{matrix} k=0 & 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \end{array} \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}$$

The challenge below considers sampling of a 1 Hz sine-wave 8 times per second, sampling at time 0, $\frac{1}{8}$, $\ldots$, $\frac{7}{8}$:



This yields sample values of

$$f = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ -1 \\ -1/\sqrt{2} \end{pmatrix} \tag{4.1}$$

Challenge

Calculate the missing values A-H in the following calculation of the frequency spectrum:

$$F_8 = \begin{pmatrix} 1 & \mathbf{A} & \mathbf{B} & 1 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{C} & \mathbf{D} & -\frac{1+i}{\sqrt{2}} & -1 & -\frac{1-i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \\ 1 & \mathbf{E} & \mathbf{F} & i & 1 & -i & -1 & i \\ 1 & -\frac{1+i}{\sqrt{2}} & i & -\frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & -i & -\frac{1-i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1-i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & -\frac{1-i}{\sqrt{2}} & i & -\frac{1+i}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{1+i}{\sqrt{2}} & i & -\frac{1-i}{\sqrt{2}} & -1 & -\frac{1+i}{\sqrt{2}} & -i & -\frac{1-i}{\sqrt{2}} \end{pmatrix} \tag{4.2}$$

$$\hat{f} = \begin{pmatrix} \mathbf{G} \\ \mathbf{H} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4i \end{pmatrix} \tag{4.3}$$

Solutions

Enter imaginary numbers as shown in the table in section 0.3. For example: $-1/\sqrt{2} - i/\sqrt{2}$ would be entered for MD5(abc_X) = a1b2c3... as "abc_re(-0.71)im(-0.71)", $-i$ would be entered simply as "abc_im(-1.00)" and -1 would just be "abc_-1.00".

A	MD5(yyya_X) = b3a860...
B	MD5(yyyb_X) = 1058f4...
C	MD5(yyyc_X) = e6bc8f...
D	MD5(yyyd_X) = da2637...
E	MD5(yyye_X) = 696976...
F	MD5(yyyf_X) = 850784...
G	MD5(yyyg_X) = dca1dd...
H	MD5(yyyh_X) = cb48e5...

4.3 Discrete Fourier Transform: Analysis

Resources

Having identified the frequency spectrum, it is possible then to analyse the frequencies of the signal. The video listed below provides a nice introduction into understanding our frequency spectrum. Note that it uses the “j” representation of imaginary numbers (instead of “i”).

- Video: `https://www.youtube.com/watch?v=mkGsMWi_j4Q`

Challenge

What is the frequency, magnitude and phase of the sine-wave as determined through our discrete Fourier transform analysis?

Solutions

Frequency: MD5(zzz_X) = 874a2c...

Magnitude: MD5(aaaa_X) = 2b946c...

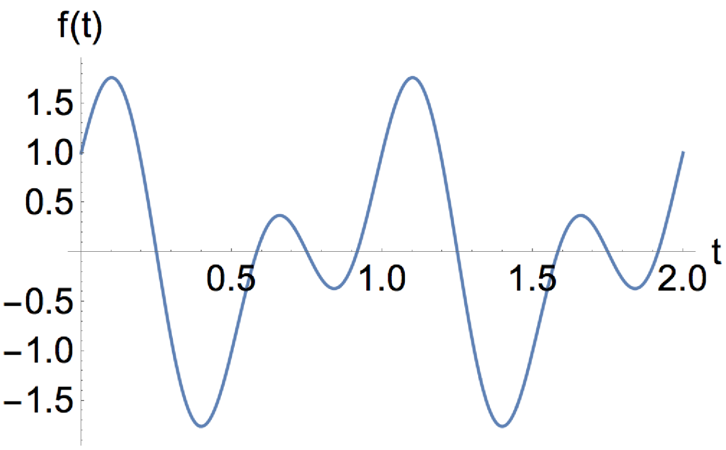
Phase: MD5(bbbb_X) = 132b51...

4.4 Analysing a more complex function: part I

Challenge

The following signal shown in the graph below has a fundamental frequency of 1 Hz:

$$f(t) = Cos(2\pi t) + Sin(4\pi t) \tag{4.4}$$



1. What are the two frequencies that make up the signal?

2. As described earlier, if you sample too infrequently, aliasing will occur. What is the maximum sampling frequency at which aliasing will occur for this signal?

Solutions

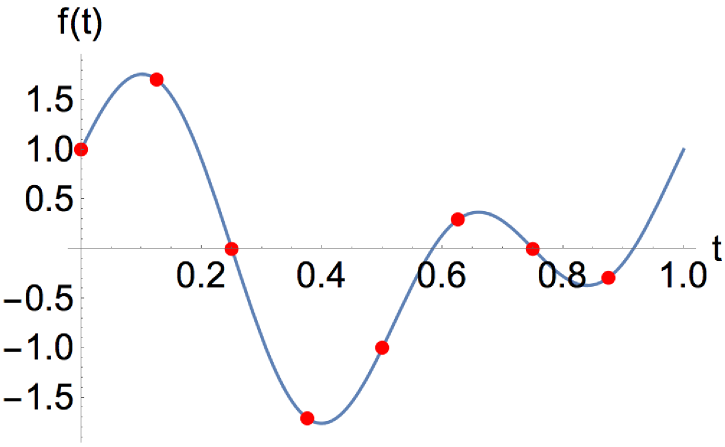
1.
Lower frequency in Hz: MD5(cccc_X) = 5a14e7...
Higher frequency in Hz: MD5(dddd_X) = 1125b4...

2. MD5(eeee_X) = 6ee287... times per second (or Hz)

4.5 Analysing a more complex function: part II

Challenge

Sampling the signal in the previous challenge 8 times per second:



yields values of

$$\begin{pmatrix} 1. \\ 1.71 \\ 0. \\ -1.71 \\ -1. \\ 0.29 \\ 0. \\ -0.29 \end{pmatrix} \tag{4.5}$$

Calculating the matrix F yields

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & (1-i)/\sqrt{2} & \boldsymbol{A} & -(1+i)/\sqrt{2} & -1 & (-1+i)/\sqrt{2} & i & (1+i)/\sqrt{2} \\ 1 & \boldsymbol{B} & -1 & i & 1 & -i & -1 & i \\ 1 & -(1+i)/\sqrt{2} & i & (1-i)/\sqrt{2} & -1 & (1+i)/\sqrt{2} & -i & (-1+i)/\sqrt{2} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & (-1+i)/\sqrt{2} & -i & (1+i)/\sqrt{2} & -1 & (1-i)/\sqrt{2} & i & -(1+i)/\sqrt{2} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & (1+i)/\sqrt{2} & i & (-1+i)/\sqrt{2} & -1 & -(1+i)/\sqrt{2} & -i & (1-i)/\sqrt{2} \end{pmatrix} \tag{4.6}$$

1. Determine the missing values **A** and **B** by calculation.

2. Determine, by calculation, the frequencies with their magnitudes and phases of this signal. In this case, you can know the constituent frequencies, their magnitudes and phases because you know how the signal is made up of two signals. Check that your calculation aligns with your intuition.

Solutions

1.
A = MD5(a_X) = d90da1...
B = MD5(b_X) = 23ae42...

2. If you are not sure what your intuition should be, or if your answer does not match your intuition, please discuss with your partner or the teacher in class.

4.6 The limits of DFT calculation

Resources

Practically, the DFT is calculated using the Fast Fourier Transform (FFT). The reason is that the number of operations grows with the square of the number of samples which is unsustainable for typical signals.

The following pages are based on slides developed by Prof. Maraso Yamashiro for the 2015 class on Fourier Analysis, and highlight this problem quantitatively.

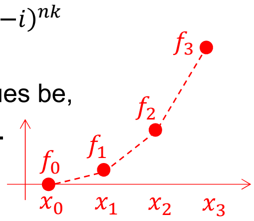
Example Discrete Fourier Transform (DFT)

Let $N = 4$ measurements (sample values) be given.

Then
$$w = e^{-i2\pi/N} = e^{-i\pi/2} = \cos \pi/2 - i \sin \pi/2 = -i$$

and thus
$$w^{nk} = (-i)^{nk}$$

Let the sample values be, say $\mathbf{f} = [0\ \ 1\ \ 4\ \ 9]^{\mathsf{T}}$.



Then

$$\hat{\mathbf{f}} = \mathbf{F}_4\mathbf{f} = \begin{array}{c|cccc} & k=0 & 1 & 2 & 3 \\ \hline n=0 & w^0 & w^0 & w^0 & w^0 \\ 1 & w^0 & w^1 & w^2 & w^3 \\ 2 & w^0 & w^2 & w^4 & w^6 \\ 3 & w^0 & w^3 & w^6 & w^9 \end{array}\ \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}$$

$$\hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}$$
$$\hat{\mathbf{f}} = \mathbf{F}_N\mathbf{f}$$
$$\mathbf{F}_{\mathrm{N}} = [e_{nk}]$$
$$e_{nk} = e^{-inx_k} = e^{-i2\pi nk/N} = w^{nk}$$
$$w = w_N = e^{-i2\pi/N}$$
$$e^{ix} = \cos x + i \sin x$$

---

e.g. $N = 4 \longrightarrow n = 0, 1, 2, 3$

Discrete Fourier Transform

$$\hat{f}_n = \sum_{k=0}^{N-1} f_k w^{nk}$$

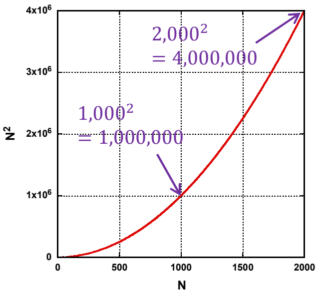
$$\hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}$$
$$\hat{\mathbf{f}} = \mathbf{F}_N\mathbf{f}$$
$$\mathbf{F}_{\mathrm{N}} = [e_{nk}]$$
$$e_{nk} = e^{-inx_k} = e^{-i2\pi nk/N} = w^{nk}$$
$$w = w_N = e^{-i2\pi/N}$$

$$\begin{aligned} n = 0 \quad & \hat{f}_0 = f_0 w^{0\times 0} + f_1 w^{0\times 1} + f_2 w^{0\times 2} + f_3 w^{0\times 3} \\ n = 1 \quad & \hat{f}_1 = f_0 w^{1\times 0} + f_1 w^{1\times 1} + f_2 w^{1\times 2} + f_3 w^{1\times 3} \\ n = 2 \quad & \hat{f}_2 = f_0 w^{2\times 0} + f_1 w^{2\times 1} + f_2 w^{2\times 2} + f_3 w^{2\times 3} \\ n = 3 \quad & \hat{f}_3 = f_0 w^{3\times 0} + f_1 w^{3\times 1} + f_2 w^{3\times 2} + f_3 w^{3\times 3} \end{aligned}$$

$$\hat{\mathbf{f}} = \mathbf{F}_4\mathbf{f} = \begin{array}{c|cccc} & k=0 & 1 & 2 & 3 \\ \hline n=0 & w^0 & w^0 & w^0 & w^0 \\ 1 & w^0 & w^1 & w^2 & w^3 \\ 2 & w^0 & w^2 & w^4 & w^6 \\ 3 & w^0 & w^3 & w^6 & w^9 \end{array} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

For $N = 6$ $\longrightarrow$ $n = 0, 1, 2, \cdots, 5$ Number of elements: $6^2 = 36$

$$\hat{\mathbf{f}} = \mathbf{F}_6\mathbf{f} = \begin{array}{cc} & \begin{array}{cccccc} k=0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} n=0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 & w^4 & w^5 \\ w^0 & w^2 & w^4 & w^6 & w^8 & w^{10} \\ w^0 & w^3 & w^6 & w^9 & w^{12} & w^{15} \\ w^0 & w^4 & w^8 & w^{12} & w^{16} & w^{20} \\ w^0 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} \end{bmatrix} \end{array} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}$$

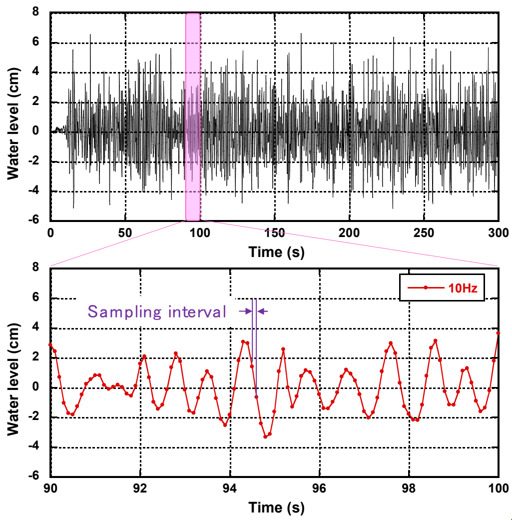


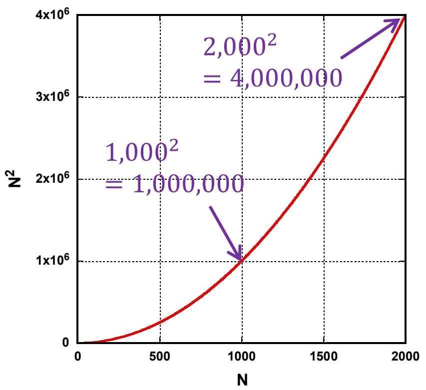
For $N = 8$ $\longrightarrow$ $n = 0, 1, 2, \cdots, 7$ Number of elements: $8^2 = 64$

$$\hat{\mathbf{f}} = \mathbf{F}_8\mathbf{f} = \begin{array}{cc} & \begin{array}{cccccccc} k=0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} n=0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ w^0 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & w^{14} \\ w^0 & w^3 & w^6 & w^9 & w^{12} & w^{15} & w^{18} & w^{21} \\ w^0 & w^4 & w^8 & w^{12} & w^{16} & w^{20} & w^{24} & w^{28} \\ w^0 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} & w^{30} & w^{35} \\ w^0 & w^6 & w^{12} & w^{18} & w^{24} & w^{30} & w^{36} & w^{42} \\ w^0 & w^7 & w^{14} & w^{21} & w^{28} & w^{35} & w^{42} & w^{49} \end{bmatrix} \end{array} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

Number of multiplications

Examples of Wave data





The number of data:
Thousands or more

Enormous calculation amount

Solution

Fast Fourier Transform (FFT)

J. W. Cooley and J. W. Tukey (1965)

Challenge

Considering a signal sampled at 22 kHz, how many multiplications are required to calculate F in the calculation $\hat{\boldsymbol{f}} = F\boldsymbol{f}$?

Solutions

(Note: a number like 5,232,500,000 is entered as "5.23e9")

MD5(c_X) = 5bd29d...

4.7 The concepts of the FFT

Resources

The Cooley and Tukey algorithm for calculating the FFT is not that complicated, but can get very messy and it takes some time to really understand it. Therefore, we will limit ourselves to understanding some basic concepts upon which it is built.

The main concept is that many of the elements in the Fourier matrix are repetitive. For example, you can notice the symmetry about the diagonal in the F_4 matrix:

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \tag{4.7}$$

One key concept therefore is that once we have calculated an element of the matrix, we don't need to calculate it again for elements that will turn out to be the same. The key question then becomes: how do we determine which elements will be repetitions of which other elements?

The "twiddle factor" shows how elements of matrix F_N (ie, w^{nk}) are repeated. This can be visualised as points on a circle on the real-imaginary plane as the following slides from Prof. Maraso Yamashiro show.

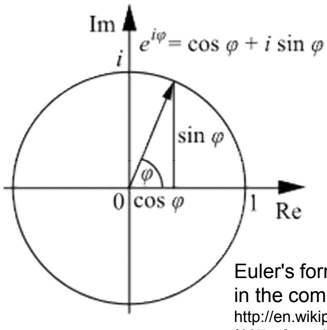
e.g. $N = 4$ $\qquad$ w: the twiddle factor

$$w^{nk} = \left(e^{-i2\pi/N}\right)^{nk}$$

$$w = e^{-i2\pi/N} = e^{-i\pi/2}$$

$$w^{nk} = \left(e^{-i\pi/2}\right)^{nk} = e^{-ink\pi/2}$$
$$= \cos nk\pi/2 - i \sin nk\pi/2$$

$$\left.\begin{array}{l} n = 0, 1, 2, 3 \\ k = 0, 1, 2, 3 \end{array}\right\} \quad nk = 0, 1, 2, 3, 4, 6, 9$$



Euler's formula illustrated in the complex plane
http://en.wikipedia.org/wiki/Euler%27s_formula

$$w^0 = \cos 0 - i \sin 0 = 1$$

$$w^1 = \cos\pi/2 - i\sin\pi/2 = -i$$

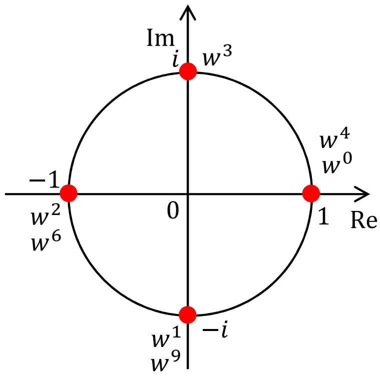
$$w^2 = \cos\pi - i\sin\pi = -1$$

$$w^3 = \cos 3\pi/2 - i\sin 3\pi/2 = i$$

$$w^4 = \cos 2\pi - i\sin 2\pi = 1$$

$$w^6 = \cos 3\pi - i\sin 3\pi = -1$$

$$w^9 = \cos 9\pi/2 - i\sin 9\pi/2 = -i$$



e.g. $N = 8$

$$\left.\begin{array}{l} n = 0, 1, 2, \cdots, 7 \\ k = 0, 1, 2, \cdots, 7 \end{array}\right\} \quad nk = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 28, 30, 35, 36, 42, 49$$

$$w^{nk} = \left(e^{-i2\pi/N}\right)^{nk} = e^{-ink\pi/4}$$
$$= \cos nk\pi/4 - i\sin nk\pi/4$$

$$w^0 = \cos 0 - i\sin 0 = 1$$

$$w^1 = \cos\pi/4 - i\sin\pi/4 = \frac{1-i}{\sqrt{2}}$$

$$w^2 = \cos\pi/2 - i\sin\pi/2 = -i$$

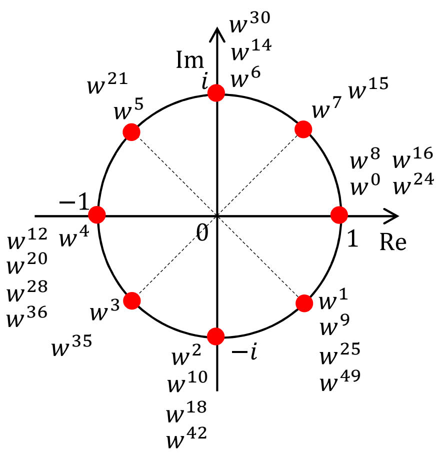
$$w^3 = \cos 3\pi/4 - i\sin 3\pi/4 = \frac{-1-i}{\sqrt{2}}$$

$$w^4 = \cos\pi - i\sin\pi = -1$$

$$w^5 = \cos 5\pi/4 - i\sin 5\pi/4 = \frac{-1+i}{\sqrt{2}}$$

$$w^6 = \cos 3\pi/2 - i\sin 3\pi/2 = i$$

$$w^7 = \cos 7\pi/4 - i\sin 7\pi/4 = \frac{1+i}{\sqrt{2}}$$



$$w^8 = w^{16} = w^{24} = w^0 \qquad w^{35} = w^3$$

$$w^9 = w^{25} = w^{49} = w^1 \qquad w^{12} = w^{20} = w^{28} = w^{36} = w^4$$

$$w^{10} = w^{18} = w^{42} = w^2 \qquad w^{21} = w^5 \qquad w^{14} = w^{30} = w^6 \qquad w^{15} = w^7$$

First one can notice that repeated terms are consistently even or odd. For example, for $N = 8$ it was seen that w^0, w^8, w^{16} and w^{24} all share the same value ($= 1$) while w_7 and w_{15} share the same (off-axis) value. You will notice however that simply breaking the functions into odd and even terms is not enough to unambiguously define the values of the twiddle factors, since even and odd terms can each still have multiple values.

One can take advantage of this symmetry (in ways not explained here) by taking the even-numbered terms and odd-numbered terms of the signal-vector $\boldsymbol{f}$ to create two new vectors, each of half the length of the original signal-vector:

$$\boldsymbol{f_{even}} = [f_0, f_2, \ldots, f_{N-2}]^T \tag{4.8}$$

$$\boldsymbol{f_{odd}} = [f_1, f_3, \ldots, f_{N-1}]^T \tag{4.9}$$

Then renumbering the terms back to $0, 1, 2, \ldots, N/2 - 1$:

$$\boldsymbol{f_{even}} = [f_{ev,0}, f_{ev,1}, \ldots, f_{ev,N/2-1}]^T \tag{4.10}$$

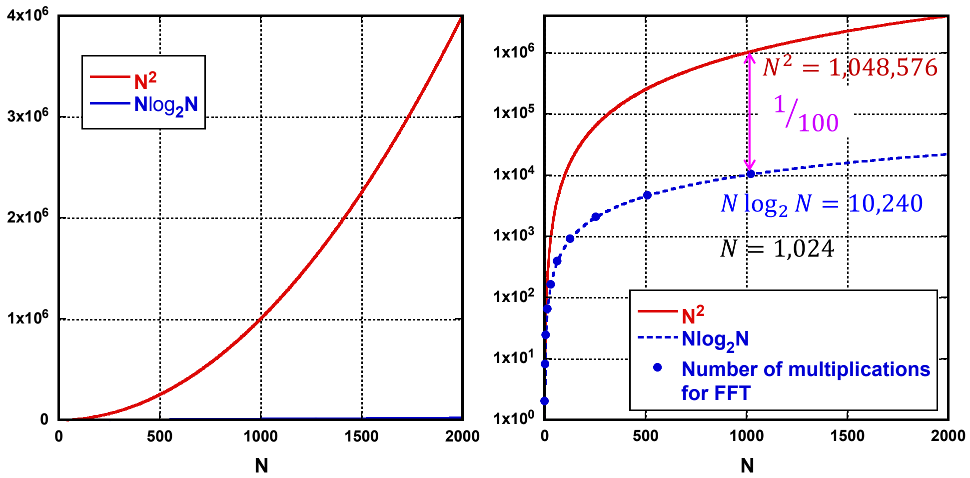
$$\boldsymbol{f_{odd}} = [f_{od,0}, f_{od,1}, \ldots, f_{od,N/2-1}]^T \tag{4.11}$$

$$\tag{4.12}$$

This can then be repeated, taking the even- and odd-numbered terms of $\boldsymbol{f_{even}}$ to make two more vectors of half-length, and doing the same for $\boldsymbol{f_{odd}}$ so we now have 4 vectors each 1/4 the length of the original signal-vector $\boldsymbol{f}$. This is repeated until there are $N/2$ vectors each of length 2.

Although beyond the scope of the explanation here, this ultimately allows one to take advantage of the repetitive nature of the twiddle factor, reducing the scaling of the number of multiplications drastically from N^2 to $Nlog_2N$, as shown in the following slide from Prof. Maraso Yamashiro:

The number of multiplications DFT vs FFT



A further point to note is that, in order to continuously divide the signal in half until each vector is length 2, the original signal must be of length 2^p where p is some integer. For example, a signal of length 8 such as $\mathbf{[1,2,3,4,5,6,7,8]}$ can be halved once into two vectors of length four ($\mathbf{[1,3,5,7]}$ and $\mathbf{[2,4,6,8]}$) and then again into four vectors of length two ($\mathbf{[1,5]}$, $\mathbf{[3,7]}$, $\mathbf{[2,6]}$, $\mathbf{[4,8]}$). If the vector is of length 9 however, such as $\mathbf{[1,2,3,4,5,6,7,8,9]}$, the signal cannot be divided into two signals of *equal length*.

This problem is overcome through the use of *zero-padding*. In the example in the previous paragraph, the signal of length 9 would simply have seven zeros added to it at the end to create a signal of total length 16 ($\mathbf{[1,2,3,4,5,6,7,8,9,0,0,0,0,0,0,0]}$), which could then be halved three times into eight vectors of length 2.

Challenge

1. Considering the signal that generated the spectrum in challenge 4.6, how many zeros must be added to the end of the sampled signal to permit FFT to be performed?

2. How many times can the padded signal be halved?

3. How many multiplications does this signal require under FFT?

4. Considering the original unpadded signal, how many times more multiplications are required under standard DFT compared to processing under FFT?

Solutions

1. MD5(d_X) = 65464b...

2. MD5(e_X) = 166b99...

3. 491,520

4. 984.70

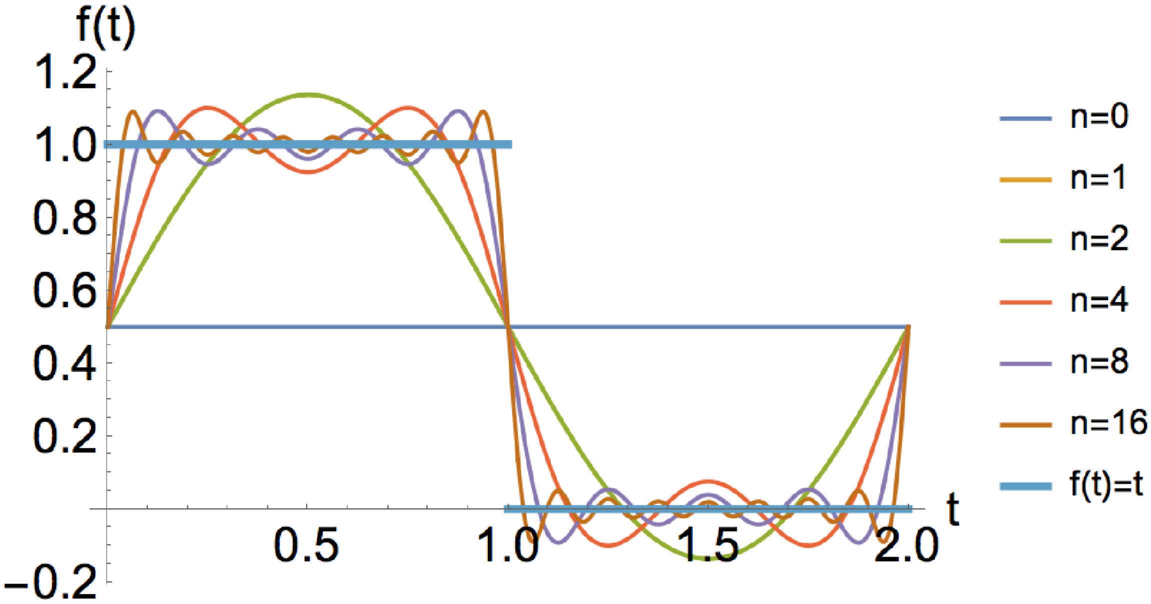
Appendix A

Practise challenges

A.1 Shifting squarewave

Challenge

Determine the Fourier series in terms of exponentials and trigonometric terms for the square-wave below and compare it to the square-wave you calculated earlier in challenge 2.14. Notice how the presence of the terms are related to the even/odd nature of the function.

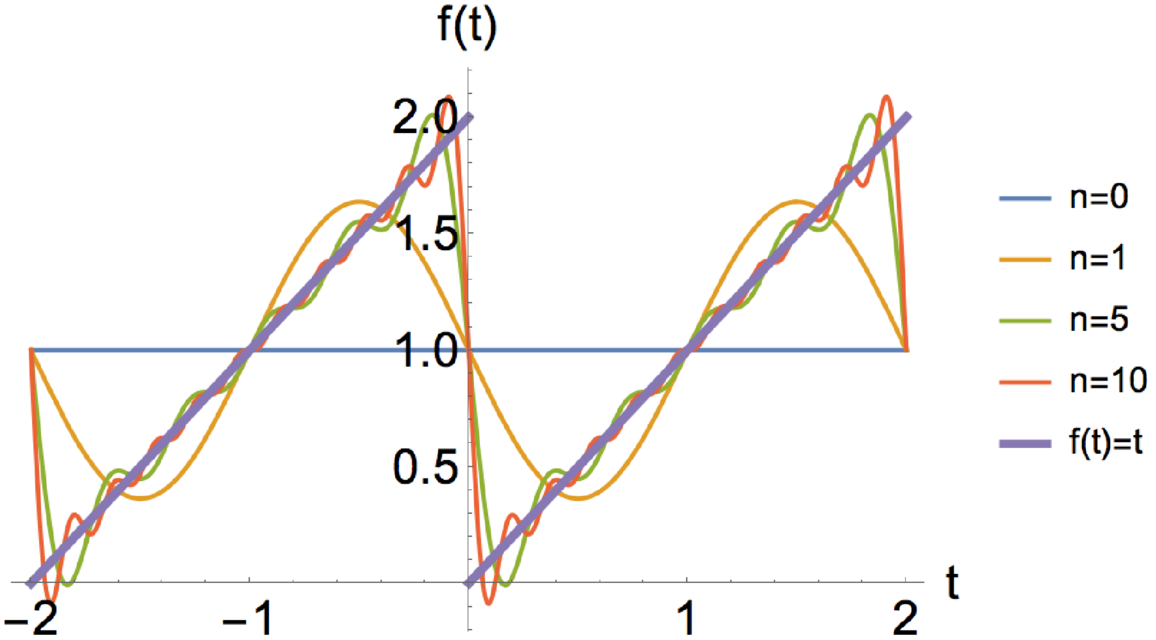


A.2 Sawtooth wave

Challenge

1. Determine the expression for a sawtooth wave of the form shown in the graph below, in terms of a trigonometric Fourier series.

2. Write a sentence explaining how the symmetry of the problem effects the final expression.



To check your answer, evaluate the function at $t = 1.1$ including only the first 3 terms of the Fourier series.

Solution

MD5(apaa_X) = 603043...

Appendix B

Mid-term exam questions

B.1

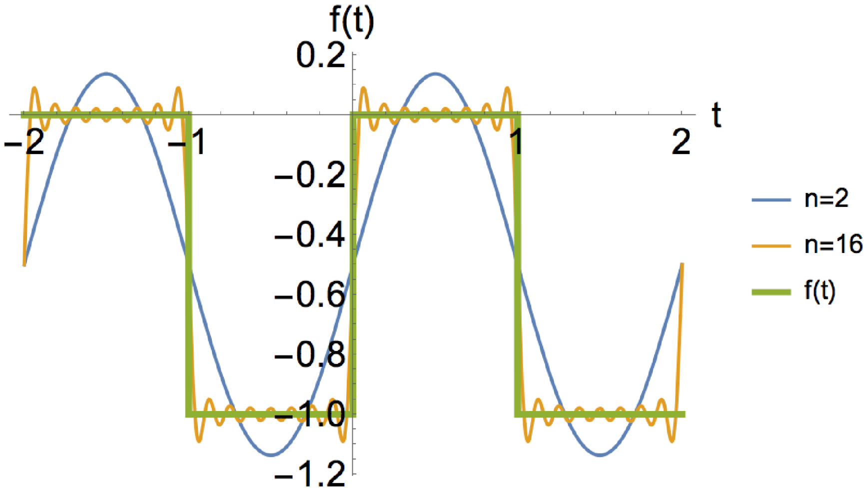
What are the Fourier coefficients C_{-1}, C_0 and C_1 of the exponential Fourier series for the function below?

$$f(x) = 2 + cos(2\pi x) \tag{B.1}$$

B.2

Considering the periodic squarewave described by the function

$$f(t) = \begin{cases} 0 & \text{for } 0 < t < 1 \\ -1 & \text{for } 1 < t < 2 \end{cases} \tag{B.2}$$



1. Calculate the Fourier coefficient C_0.

2. Obtain an expression for the Fourier coefficients C_k where $k \neq 0$. Write your final answer so that it does not contain any exponentials, sines or cosines.

3. What is Gibb's phenomenon? Under what situations does it occur?

B.3

In the first lecture we saw that it was possible to make any function (such as the "Homer Simpson" function) by taking a point and rotating it around a circle that itself was rotating around a larger circle, that itself was rotating around a larger circle, and so on. This is also how the ancient greeks tried to approximate the orbit of the planets and sun around the earth.

1. How is the ability to approximate an arbitrary function using sums of circles related to Fourier series? You may refer to the equation

$$f(t) = \sum_{k=-N}^{k=N} C_k e^{2\pi i k t} \tag{B.3}$$

during your discussion.

2. By what variables are frequencies and radii of the circles defined in the Fourier series in the above equation?

3. The terms of the Fourier series are orthogonal. Describe in a few sentences what this means. You may refer to the below expression in your answer.

$$(e^{2\pi i k_1 t}, e^{2\pi i k_2 t}) \tag{B.4}$$