Fourier Analysis

Autumn 2018

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23rd January 2019 at 10:13

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http://www.jamescannon.net/teaching/fourier-analysis

http://raw.githubusercontent.com/NanoScaleDesign/FourierAnalysis/master/fourier_analysis.pdf

Contents

Chapter 0

Course information

0.1 This course

This is the Autumn 2018 Fourier Analysis course studied by 3rd-year undegraduate international students at Kyushu University.

0.1.1 How this works

- In contrast to the traditional lecture-homework model, in this course the learning is self-directed and active via publicly-available resources.
- Learning is guided through solving a series of carefully-developed challenges contained in this book, coupled with suggested resources that can be used to solve the challenges with instant feedback about the correctness of your answer.
- There are no lectures. Instead, there is discussion time. Here, you are encouraged to discuss any issues with your peers, teacher and any teaching assistants. Furthermore, you are encouraged to help your peers who are having trouble understanding something that you have understood; by doing so you actually increase your own understanding too.
- Discussion-time is from 10:30 to 12:00 on Wednesdays at room W4-529.
- Peer discussion is encouraged, however, if you have help to solve a challenge, always make sure you do understand the details yourself. You will need to be able to do this in an exam environment. If you need additional challenges to solidify your understanding, then ask the teacher. The questions on the exam will be similar in nature to the challenges. If you can do all of the challenges, you can get 100% on the exam.
- Every challenge in the book typically contains a **Challenge** with suggested **Resources** which you are recommended to utilise in order to solve the challenge. Occasionally the teacher will provide extra **Comments** to help guide your thinking. A **Solution** is also made available for you to check your answer. Sometimes this solution will be given in encrypted form. For more information about encryption, see section 0.3.
- For deep understanding, it is recommended to study the suggested resources beyond the minimum required to complete the challenge.
- The challenge document has many pages and is continuously being developed. Therefore it is advised to view the document on an electronic device rather than print it. The date on the front page denotes the version of the document. You will be notified by email when the document is updated. The content may differ from last-year's document.
- A target challenge will be set each week. This will set the pace of the course and defines the examinable material. It's ok if you can't quite reach the target challenge for a given week, but then you will be expected to make it up the next week.
- You may work ahead, even beyond the target challenge, if you so wish. This can build greater flexibility into your personal schedule, especially as you become busier towards the end of the semester.
- Your contributions to the course are strongly welcomed. If you come across resources that you found useful that were not listed by the teacher or points of friction that made solving a challenge difficult, please let the teacher know about it!

0.1.2 Assessment

In order to prove to outside parties that you have learned something from the course, we must perform summative assessments. This will be in the form of a satisfactory challenge-log (weighted 5%), coursework (weighted 15%), a mid-term exam (weighted 30%) and a final exam (weighted 50%).

Your final score is calculated as Max(final exam score, weighted score), however you must pass the final exam to pass the course.

0.1.3 What you need to do

- Prepare a challenge-log in the form of a workbook or folder where you can clearly write the calculations you perform to solve each challenge. This will be a log of your progress during the course and will be occasionally reviewed by the teacher.
- You need to submit a brief report at `https://goo.gl/forms/vPPCYOZ6nW5ns1mm1` by 8am on the day of the class. Here you can let the teacher know about any difficulties you are having and if you would like to discuss anything in particular.
- Please bring a wifi-capable internet device to class, as well as headphones if you need to access online components of the course during class. If you let me know in advance, I can lend computers and provide power extension cables for those who require them (limited number).

0.1.4 Survey form

`https://goo.gl/forms/MS9EDVgmTOWPhy493`

0.2 Timetable

	Discussion	**Target**	**Note**
1	3 Oct	-	
2	17 Oct	2.11	
3	24 Oct	3.10	
4	31 Oct	3.17	
5	7 Nov	3.26	
6	14 Nov	4.4	
7	21 Nov	4.12	
8	28 Nov	5.1	
9	5 Dec	Mid-term exam	
10	12 Dec	-	
11	19 Dec	5.7	
12	9 Jan	5.8	Coursework information
13	16 Jan	5.10	
14	23 Jan	Coursework	
15	6 Feb	Final exam	10:30 Sogo plaza room 10

0.3 Hash-generation

Some solutions to challenges are encrypted using MD5 hashes. In order to check your solution, you need to generate its MD5 hash and compare it to that provided. MD5 hashes can be generated at the following sites:

- Wolfram alpha: (For example: md5 hash of "q1.00") `http://www.wolframalpha.com/input/?i=md5+hash+of+%22q1.00%22`
- `www.md5hashgenerator.com`

Since MD5 hashes are very sensitive to even single-digit variation, you must enter the solution *exactly*. This means maintaining a sufficient level of accuracy when developing your solution, and then entering the solution according to the format suggested by the question. Some special input methods:

Solution	**Input**
5×10^{-476}	5.00e-476
5.0009×10^{-476}	5.00e-476
$-\infty$	-infinity (never "infinite")
2π	6.28
i	im(1)
2i	im(2)
1 + 2i	re(1)im(2)
-0.0002548 i	im(-2.55e-4)
1/i = i/-1 = -i	im(-1)
$e^{i2\pi}$ $[= cos(2\pi) + isin(2\pi) = 1 + i0 = 1]$	1.00
$e^{i\pi/3}$ $[= cos(\pi/3) + isin(\pi/3) = 0.5 + i0.87]$	re(0.50)im(0.87)
Choices in order A, B, C, D	abcd

The first 6 digits of the MD5 sum should match the first 6 digits of the given solution.

0.4 Coursework

0.4.1 Task

There are many interesting things about Fourier analysis, whether in terms of the mathematics or applications. This coursework aims to give you the opportunity to follow areas related to your own interest in the subject.

Your task is to write a report or create a video exploring a topic related to Fourier analysis. This could involve:

- Exploring the characteristics of a real-world signal using application of *Fourier analysis and programming*, and drawing interesting insights into the signal, or
- Exploring an area of mathematics related to Fourier analysis that was not covered by the course, or
- Exploring an area of mathematics related to Fourier analysis that was touched on by the course, but going into further depth.

Points will be awarded on the basis of creativity, demonstration of knowledge, quality of explanation and accuracy.

0.4.2 Submission

Your submission should contain:

- Your report: Either text (guideline: 2 to 4 pages) or video (guideline: 10 minutes)
- A list of references (either included in the text or submitted with the video)
- Challenges with their fully-worked solutions (ie, not just the final answer) (guideline: 2 challenges)

Submission is electronic, and may be in text or video format. For text-based formats, submission may be in any format, including PDF, LibreOffice, MS Word, Google docs, Latex, etc... If you submit a PDF, please also submit the source-files used to generate the PDF. For video-based formats, please provide a link to the video either for viewing or download. Challenges and their solutions should be submitted in written form, even if a video is submitted.

For submission you will need to do both of the following things:

- Submit the materials by **email** to the teacher **before the class on 23 January 2019** with the subject "[Fourier Analysis] Coursework" and
- In addition, bring a paper copy to class that you can share with others in the class.

I will confirm in the class that I received your coursework. If you cannot attend the class, you must request confirmation of receipt when you send the email.

Late submission:
By 17:00 on 24 January 2019: 90% of the final mark.
By 17:00 on 30 January 2019: 50% of the final mark.
Later submissions cannot be considered.

0.4.3 Coursework links

`https://bit.ly/2AXeLgx`

Password is required and will be given in class.

Chapter 1

Hash practise

1.1 Hash practise: Integer

X = 46.3847
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

hash of aX = e77fac

1.2 Hash practise: Decimal

X = 49
Form: Two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

hash of bX = 82c9e7

1.3 Hash practise: String

X = abcdef
Form: String.
Place the indicated letter in front of the number.
Example: aX where $X = abc$ is entered as aabc

hash of cX = 990ba0

1.4 Hash practise: Scientific form

X = 500,765.99
Form: Scientific notation with the mantissa in standard form to 2 decimal place and the exponent in integer form.
Place the indicated letter in front of the number.
Example: aX where $X = 4 \times 10^{-3}$ is entered as a4.00e-3

hash of dX = be8a0d

1.5 Hash practise: Numbers with real and imaginary parts

X = $1 + 2i$
Form: Integer *(for imaginary numbers, "integer" means to write both the real and imaginary parts of the number as integers. If you were instructed to enter to two decimal places, then you would need to enter each of the real and imaginary parts to two decimal places. Refer to section 0.3 to see an example of how to handle input of imaginary numbers.)*
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

hash of eX = 4aa75a

1.6 π and imaginary numbers

X = $-2\pi i$
Form: Two decimal places. Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

hash of fX = ad3e8b

1.7 Imaginary exponentials

Note that you will need to understand how to expand exponentials in terms of their sines and cosines in order to do this. If you do not understand how to do this yet, skip this challenge and come back to it later.

X = $4e^{i3\pi/4}$
Form: Two decimal places. Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

hash of gX = 59a753

Chapter 2

Periods and frequencies

2.1 Period at 50 THz

Resources

- Video: `https://www.youtube.com/watch?v=v3CvAW8BDHI`

Challenge

A signal is oscillating at a frequency of 50 THz. What is the period?

Solution

X = Your solution
Form: Scientific notation with the mantissa in standard form to 2 decimal places and the exponent in integer form.
Place the indicated letter in front of the number.
Example: aX where $X = 4.0543 \times 10^{-3}$ is entered as a4.05e-3

Hash of qX = 3faf81

2.2 Frequency with k=1

Comment

Note that we're working in radians here. From now on a factor of 2π will be included in the oscillations so that $\sin(2\pi t)$ will complete 1 cycle in 1 second. (If you calculator defaults to degrees, be sure to change it to radians for this course.)

Challenge

What is the frequency of $\sin(2\pi kt)$, where t is time in seconds and $k = 1$?

Solution

(Hz)

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of eX = 720149

2.3 Frequency with k=2

Challenge

What is the frequency of $\sin(2\pi kt)$, where t is time in seconds and $k = 2$?

Solution

(Hz)

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of rX = 96ba66

2.4 The meaning of k

Challenge

Considering the previous two challenges, what does k physically represent in those challenges?

Solution

Please compare your answer with your partner in class or discuss with the teacher.

2.5 Smallest period with k=1

Resources

- Book: 1.3 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video (14m10s to 17m00s): `https://youtu.be/1rqJl7Rs6ps?t=14m10s`

Challenge

What is the smallest period of $sin(2\pi kt)$, where t is time in seconds and $k = 1$?

Solution

(s)

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of wX = 25c4fb

2.6 Smallest period with k=2

Resources

- Book: 1.3 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video (14m10s to 17m00s): `https://youtu.be/1rqJl7Rs6ps?t=14m10s`

Challenge

What is the smallest period of $sin(2\pi kt)$, where t is time in seconds and $k = 2$?

Solution

(s)

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of tX = bb995f

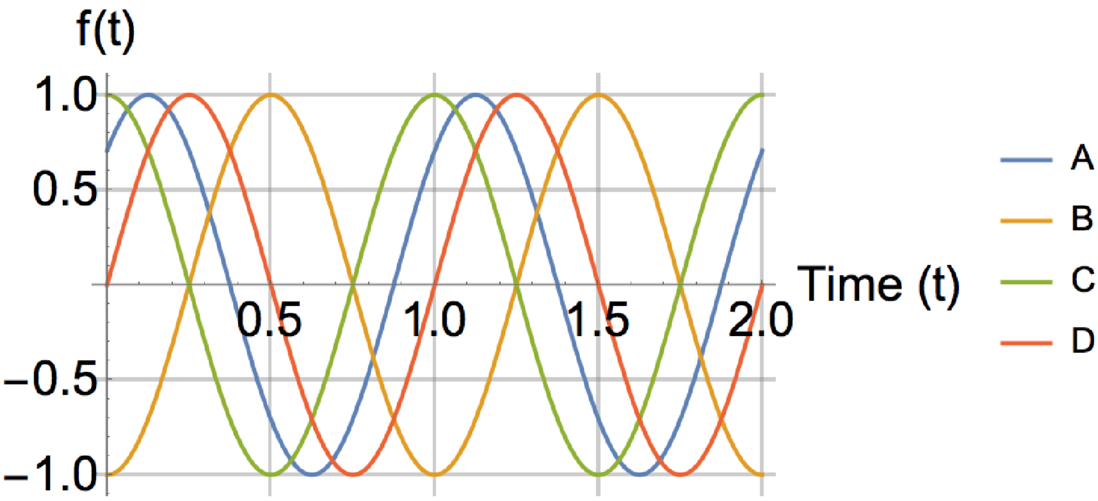
2.7 Phase

Comments

Another important concept is phase. For a simple sine signal $\theta(t) = sin(2\pi t)$, at $t = 0$ the angle θ is zero, but one can shift the phase (starting point) of the signal by effectively making the sine-curve non-zero at $t = 0$. Another way to think about it is to say the sine curve doesn't reach zero until a time $t - \phi$ where ϕ is the phase-shift added.

Challenge

Place the following four graphs in the following order:
$\sin(2\pi t + \pi/2)$
$\sin(2\pi t - \pi/2)$
$\sin(2\pi t + \pi/4)$
$\sin(2\pi t + 2\pi)$



Solution

X = Your solution
Form: String.
Place the indicated letter in front of the string.
Example: aX where X=abcd is entered as aabcd

Hash of iX = 5c0e8b

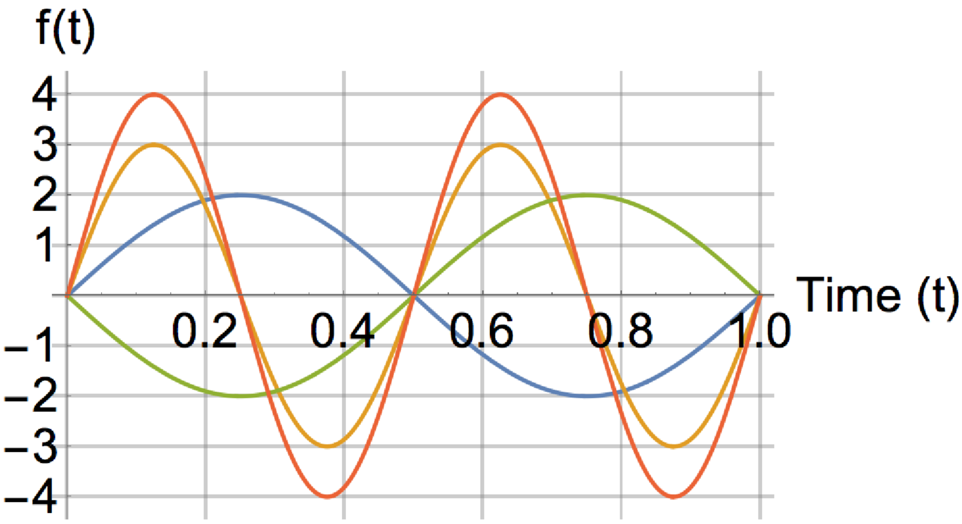
2.8 Amplitude

Comments

Another important concept is amplitude. $\sin(2\pi t)$ has an amplitude of 1, but this can be easily modified to go between $\pm A$ by multiplication with A.

Challenge

The following 4 graphs correspond to the equation $A\sin(2\pi kt)$ with variation in the values of A and k. What is the sum of the values of A for the following graphs?



Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of uX = 6bce05

2.9 Periodic and non-periodic signals

Resources

- Video: https://www.youtube.com/watch?v=F_pdpbu8bgA
- Book: 1.3 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)

Challenge

The list below contains periodic and non-periodic signals. Sum the points of the signals below that are *periodic*.

1 point: $x(t) = t^2$
2 points: $x(t) = \sin(t)$
4 points: $x(t) = \sin(2\pi t)$
8 points: $x(t) = \sin(2\pi t + t)$
16 points: $x(t) = \sin(2\pi t) + \sin(t)$
32 points: $x(t) = \sin(5\pi t) + \sin(2\pi t)$
64 points: $x(t) = \sin(5\pi t) + \sin(37\pi t)$
128 points: $x(t) = \sin(5\pi t) + \sin(37.01\pi t)$
256 points: $x(t) = \sin(5\pi t) + \sin(\sqrt{2}\pi t)$
512 points: $x(t) = \sin(5\sqrt{2}\pi t) + \sin(\sqrt{2}\pi t)$

Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of uX = 5d906b

2.10 Making non-periodic signals from periodic signals

Challenge

It is not immediately intuitive that it is possible to make a non-periodic signal by simply adding two periodic signals. Referring to the previous challenge, in no-more than 1 paragraph, explain how this is possible.

Solution

Please compare your answer with your partner or ask the teacher in class.

2.11 Fundamental frequency

Challenge

Considering the periodic signals in challenge 2.9 in order of increasing point-score, calculate the fundamental frequency and period of the last periodic signal in the list.

Solution

Frequency (Hz) (be careful about rounding up or down to 2 decimal places):
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of aX = ac6698

Period (s):
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of bX = 6fdff9

Chapter 3

Fourier Series

3.1 Even and odd functions

Resources

- Wikipedia: `https://en.wikipedia.org/wiki/Even_and_odd_functions`

Challenge

Sum the points of all the following *true* statements:

1 point: $f(x) = \sin(x)$ is an odd function

2 points: $f(x) = \sin(x)$ is an even function

4 points: $f(x) = \cos(x)$ is an odd function

8 points: $f(x) = \cos(x)$ is an even function

16 points: $f(x) = x$ is an odd function

32 points: $f(x) = x$ is an even function

64 points: $f(x) = \sin(x) + \cos(x)$ is an odd function

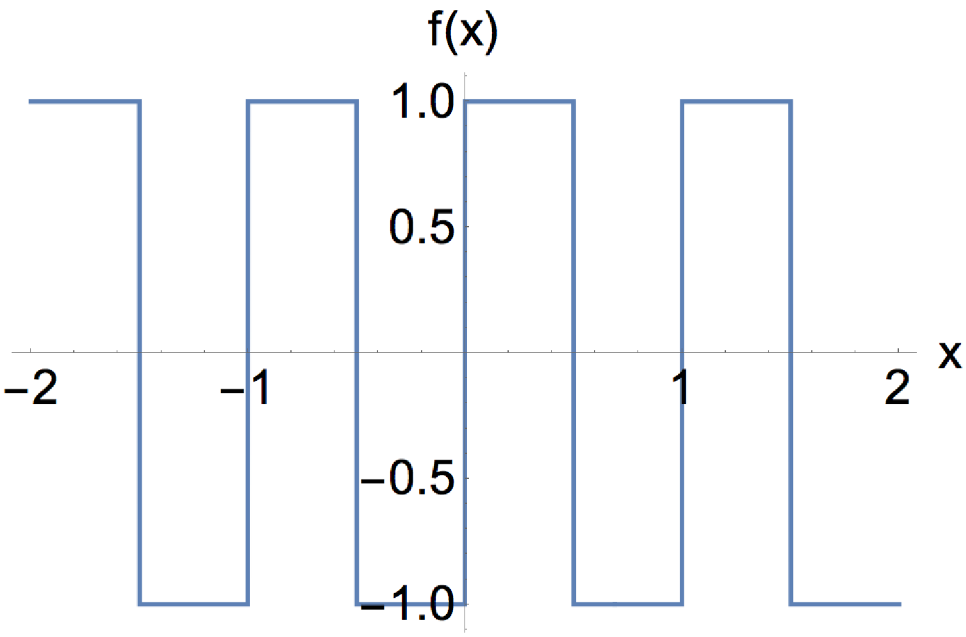
128 points: $f(x) = \sin(x) + \cos(x)$ is an even function

256 points: The infinitely repeating square wave (see figure below) where $f(x) = \begin{cases} 1 & \text{for } 0 < x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} < x < 1 \end{cases}$ is an odd function.

512 points: $f(x)$ above is an even function.

1024 points: The infinitely repeating square wave where $g(x) = \begin{cases} 1 & \text{for } 0 < x < 2 \\ 0 & \text{for } 2 < x < 4 \end{cases}$ is an odd function.

2048 points: $g(x)$ above is an even function.



Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of gX = 205d04

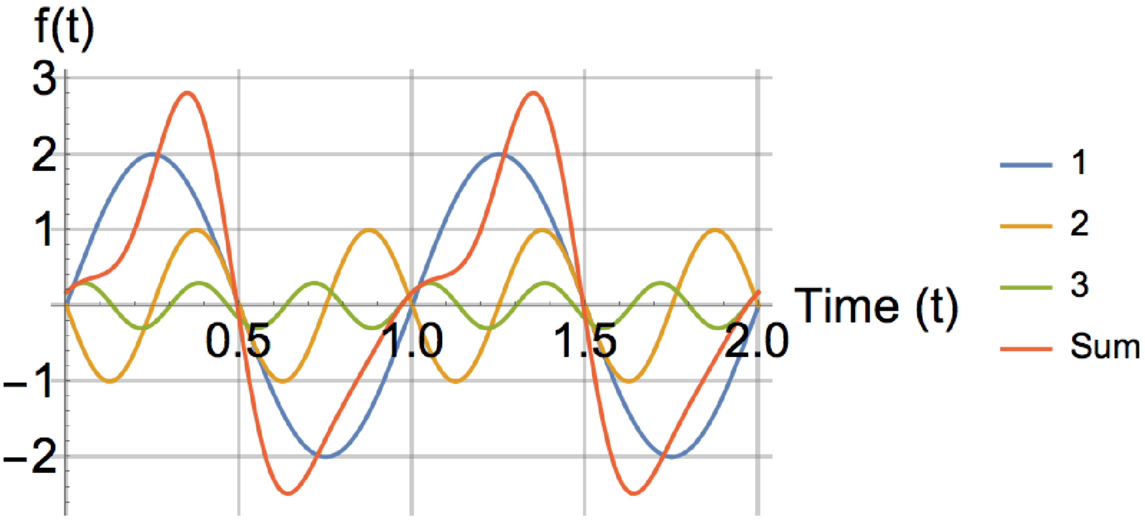
3.2 Introduction to sine and cosine Fourier coefficients

Comment

Fourier series involves the construction of a signal by summing multiple periodic signals and careful choice of each frequency's amplitude and phase. For example, considering the following 3 signals:

1. $f_1(t) = 2\sin(2\pi t)$
2. $f_2(t) = 1\sin(4\pi t + \pi)$
3. $f_3(t) = 0.3\sin(6\pi t + \pi/5)$

Their sum (k = 1 to 3) produces a much more complex shape:



Here we will consider Fourier sine and cosine series. We will go beyond this description later, but you should be aware of their existance and a method of their derivation.

Challenge

As shown above, a complex function (signal) can be made up of a sum of sine signals:

$$f(t) = A_0 + \sum_{k=1}^{N} A_k \sin(2\pi kt + \phi_k) \tag{3.1}$$

Using basic trigonometric relations, show that this can be re-written in terms of a sum of sines and cosines:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \tag{3.2}$$

I want you to see how the phase information is encoded in the representation of equation 3.2. a_k and b_k are referred to as Fourier coefficients. Write an expression for the coefficients a_k, b_k and a_0.

Solutions

To check your answer, calculate a_0, a_1 and b_1 given the following information:
$A_0 = -1/2$, $A_1 = 0.7$, $\phi_1 = \pi/5$.

(make sure your calculator is using radians)

$a_1 = 0.41145$, $b_1 = 0.566312$.

a_0: X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of fX = 75d06e

3.3 The significance of the Fourier coefficients

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-fourier-series-intro`

Comments

I recommend to view the suggested resource as this gives a nice introduction to the subject and challenges ahead. Note that the period used in the video is different to that we're using in our description, but you should become comfortable with moving between different notations.

Challenge

Write a few sentences describing in qualitative terms what the significance of the magnitude of the fourier coefficients a_k and b_k in challenge 3.2 is.

Solutions

Please compare your answer with your partner or discuss with the teacher in class.

3.4 Integral of $\sin(kt)$ and $\cos(kt)$

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-integral-of-sinmt-and-cosmt`

Challenge

Show that the following integrals evaluate to zero:

$$\int_0^1 \sin(2\pi kt)dt = 0 \tag{3.3}$$

$$\int_0^1 \cos(2\pi kt)dt = 0 \tag{3.4}$$

given that k is a non-zero positive integer.

Solutions

Please compare your answer with your partner or discuss with the teacher in class.

3.5 Integral of product of sines and cosines

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-integral-sine-times-sine`
- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-integral-cosine-times-cosine`

Challenge

Considering the following integrals, show under what conditions the following integrals evaluate to zero, and what condition they don't evaluate to zero. What is the non-zero evaluation result?

$$\int_0^1 \sin(2\pi mt)\sin(2\pi nt)dt \tag{3.5}$$

$$\int_0^1 \cos(2\pi mt)\cos(2\pi nt)dt \tag{3.6}$$

You may assume that m and n are non-zero positive integers.

Solutions

Please compare your answer with your partner or discuss with the teacher in class.

3.6 First term in a trigonometric Fourier series

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-first-term-fourier-series`

Challenge

Considering a fourier series represented by

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \tag{3.7}$$

What is a_0 for the following functions?

1. $f(t) = \sin(10\pi t)$
2. Square wave function $f(t) = \begin{cases} 2 & \text{for } 0 < t < \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} < t < 1 \end{cases}$
3. $f(t) = t^2$ considered only over the interval from $t = 0$ to $t = 1$

Solutions

1.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of dX = 705887

2.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of eX = e1509d

3.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of fX = 98680d

3.7 The Fourier coefficient a_k

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-fourier-coefficients-cosine`

Challenge

Derive an expression for the Fourier coefficient a_k in

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \tag{3.8}$$

Note the difference in period between the above equation and the suggested resource.

Solutions

Please compare your answer with your partner or discuss with the teacher in class.

3.8 The Fourier coefficient b_k *(optional challenge)*

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-fourier-coefficients-sine`

Comment

The aim is to have you understand how the Fourier series coefficients arise. If you have time, for extra practise I recommend that you try to calculate b_k. If you do not have time, please progress on to the Fourier Series calculations.

Challenge

Derive an expression for the Fourier coefficient b_k in

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \tag{3.9}$$

Note the difference in period between the above equation and the suggested resource.

Solutions

Please compare your answer with your partner or discuss with the teacher in class.

3.9 Trigonometric Fourier series of a square wave

Resources

- Video: `https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-fourier-coefficients-for-square-wave`

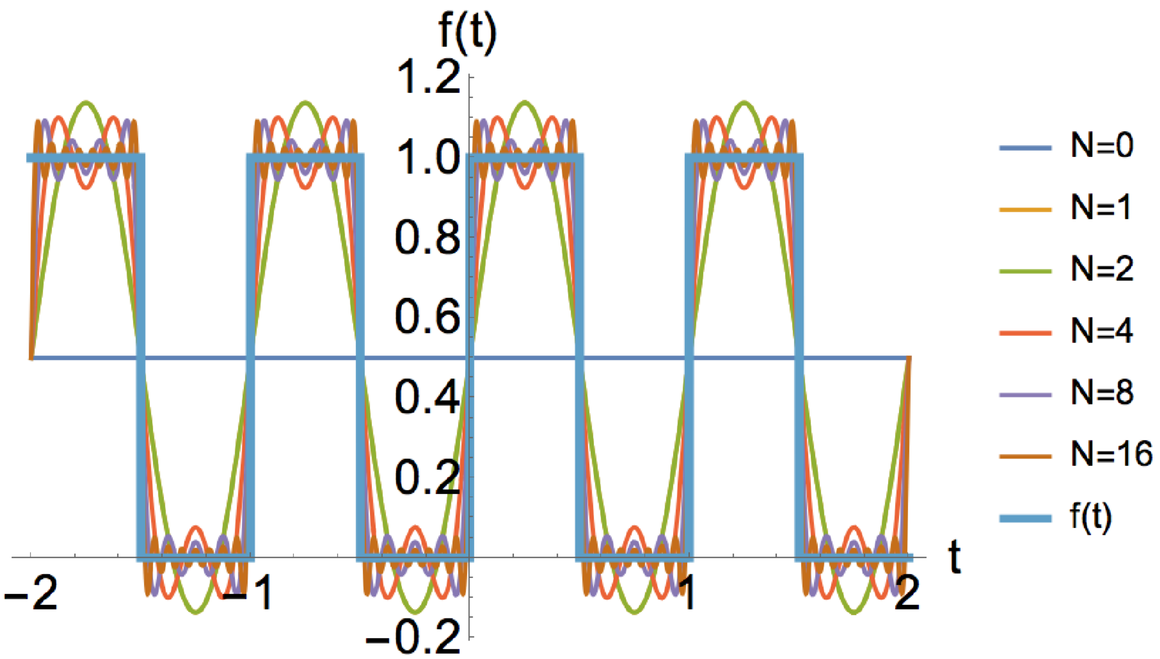
Comment

You may find `https://www.desmos.com/` useful for plotting equations of Fourier series that you calculate.

Challenge

Calculate the Fourier series for the following square-wave:

$$f(t) = \begin{cases} 1 & \text{for } 0 < t < \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < t < 1 \end{cases} \tag{3.10}$$



Solutions

To check your solution, write out the sum up to $k = 2$ and then evaluate $f(0.9)$

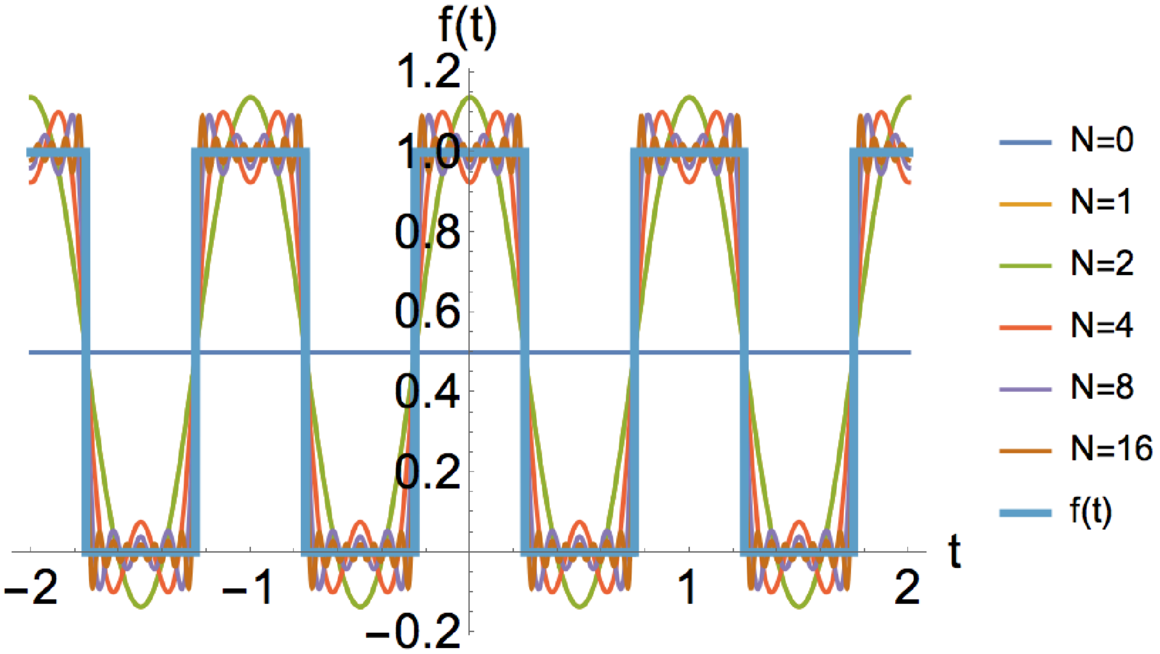
0.1258

3.10 Trigonometric Fourier series of another square wave

Challenge

Calculate the Fourier series for the following square-wave:

$$f(t) = \begin{cases} 1 & \text{for } -\frac{1}{4} < t < \frac{1}{4} \\ 0 & \text{for } \frac{1}{4} < t < \frac{3}{4} \end{cases} \tag{3.11}$$



Solutions

To check your solution, write out the sum up to $k = 2$ and then evaluate $f(0.7)$

0.30327

3.11 Fourier sine and cosine series

Challenge

Considering challenges 3.9 and 3.10, one corresponds to a Fourier sine series, and the other to a Fourier cosine series. Such series only contain sine and cosine terms, respectively. (1) State which challenge corresponds to which series, and (2) write a square-wave function that you think would involve both sine and cosine terms.

Solutions

If you are unsure about your answer, you can either try to solve it to prove it, or please discuss with your partner or ask the teacher.

3.12 Complex numbers

Resources

- Book: Appendex A starting at page 403 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Challenge

Considering $z = a + bi$ and $w = c + di$, determine:

1. $z + w$
2. zw
3. $z\bar{z}$
4. z/w
5. $|z|$

Solutions

To check your answers, substitute the following values: $a = 1$, $b = 2$, $c = 3$, $d = 4$.

1.
X = Your solution
Form: Imaginary form with integers.
Place the indicated letter in front of the number.
Example: aX where $X = 1 + 2i$ is entered as are(1)im(2)

Hash of hX = d8c7d5

2.
X = Your solution
Form: Imaginary form with integers.
Place the indicated letter in front of the number.
Example: aX where $X = 1 + 2i$ is entered as are(1)im(2)

Hash of iX = 3c520b

3.
X = Your solution
Form: Imaginary form with integers.
Place the indicated letter in front of the number.
Example: aX where $X = 1 + 2i$ is entered as are(1)im(2)

Hash of jX = 66248e

4.
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of kX = 39577d

5.
X = Your solution

Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of mX = b12a23

3.13 Complex form of sine and cosine

Challenge

Write sine and cosine in terms of complex exponentials.

Solutions

If you are unsure about your answer, please discuss in class.

3.14 An alternative way to write Fourier series

Comments

It turns out that a more mathematically-convenient and ultimately intuitive way to write Fourier series is in terms of complex exponentials in the form

$$f(t) = \sum_{k=-n}^{k=n} c_k e^{i2\pi kt} \tag{3.12}$$

A few things to note:

- The sum now runs from $-n$ to n.
- The $k = 0$ term that was originally excluded from the sum in equation 3.2 is now included.
- The c_k's, unlike the a_k's and b_k's, are complex.
- $c_{-k} = \bar{c_k}$.
- $c_0 = a_0/2$.

You see how we started with equation 3.1 and now end up with equation 3.12? Do you see more-or-less how the phase information is encoded in the c_k's?

Challenge

1. By comparing equation 3.2 to equation 3.12, write c_k in terms of a_k and b_k for $k < 0$ and $k > 0$.
2. Show that $c_{-k} = \bar{c_k}$
3. Show that c_0 must be $a_0/2$

Solutions

If you have trouble with the derivation, please discuss in class.

3.15 Integral of a complex exponential over a single period

Challenge

Integrate the following function over one period, assuming that k must be an non-zero integer:

$$g(t) = e^{i2\pi kt} \tag{3.13}$$

Solution

The numerical answer is given below. Be sure you understand why the answer comes out as this number.

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of nX = f979e9

3.16 Derivation of the complex Fourier coefficients

Resources

- Video starting at 41m 56s: `https://www.youtube.com/watch?v=1rqJl7Rs6ps&t=41m56s`

Comments

The video does a great job of showing the derivation. Try to become comfortable with manipulating complex exponentials and their periodicity.

Challenge

Derive an expression for the c_k's in the equation

$$f(t) = \sum_{k=-n}^{n} c_k e^{i2\pi kt} \tag{3.14}$$

in terms of the function $f(t)$.

Solution

Please discuss in class if you are unsure of your derivation.

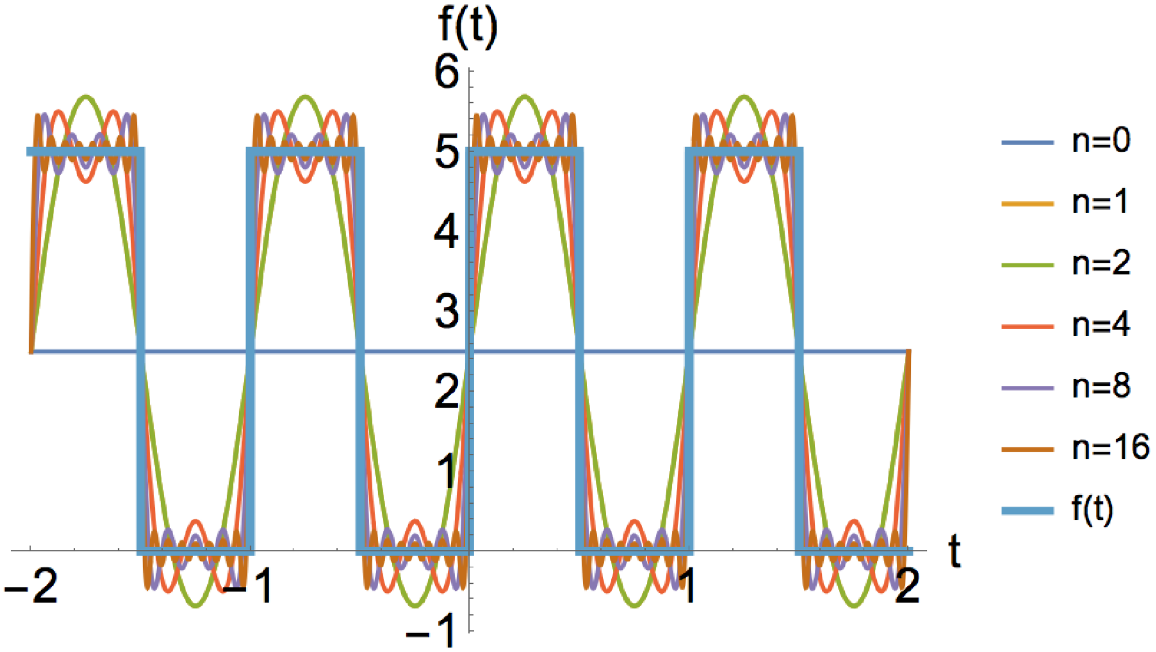
3.17 Complex Fourier series of a square wave

Challenge

1. Determine an expression for the complex Fourier coefficients for the following square wave:

$$f(t) = \begin{cases} 5 & \text{for } 0 < t < \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < t < 1 \end{cases} \tag{3.15}$$

2. Write the complex Fourier series for the square wave.



Solution

1.
You should find that $c_{-7} = i0.227$.

2.
If you perform the sum for $n = 1$ and evaluate at $t = 0.7$, you should obtain a value of -0.527.

3.18 Non-unit periods

Resources

- Book: Chapter 1.6 of the book (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Comment

Until now we considered signals with periodicity 1, but this will not always be the case, and in fact as we make the jump from Fourier series to Fourier transforms this will become more important. The resource gives the intuition behind the non-periodic case for complex Fourier series. For trigonometric Fourier series, the Fourier coefficients become:

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(2\pi kt/T)dt \tag{3.16}$$

$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(2\pi kt/T)dt \tag{3.17}$$

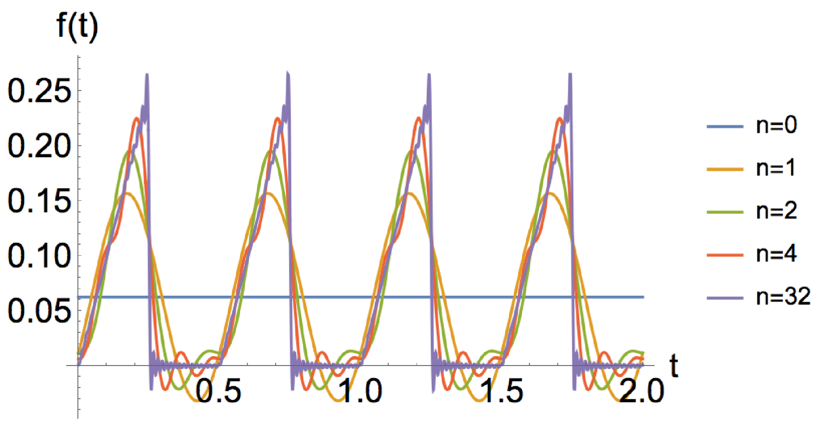
while for complex Fourier series the Fourier coefficients become:

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-i2\pi kt/T}dt \tag{3.18}$$

Challenge

Obtain an expression for the exponential Fourier series for the pulsing sawtooth function

$$f(t) = \begin{cases} t & \text{for } 0 < t < \frac{1}{4} \\ 0 & \text{for } \frac{1}{4} < t < \frac{1}{2} \end{cases} \tag{3.19}$$



Solution

You should find the c_k's for $k \neq 0$ are

$$\frac{i\pi ke^{-i\pi k} + e^{-i\pi k} - 1}{8\pi^2 k^2} \tag{3.20}$$

3.19 Gibb's phenomenon

Resources

- Book: 1.18 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Challenge

1. Qualitatively speaking, what is the Gibb's phenomenon?
2. Considering a square wave of the form

$$f(t) = \begin{cases} -1 & \text{for } -\frac{1}{2} < t < 0 \\ 1 & \text{for } 0 < t < \frac{1}{2} \end{cases} \tag{3.21}$$

as the sum of the Fourier series goes to infinity, what is the maximum value of the signal?

The full derivation is beyond the scope of this course, so it is not necessary to understand the derivation in the notes. The aim of this challenge is simply to enable you to be able to describe qualitatively what Gibb's phenomenon is using a few sentences, and know the amount of overshoot in the case of a standard ± 1 *square-wave.*

Solution

2.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of mX = d9c018

3.20 Fourier coefficients of sin(x)

Challenge

By writing $\sin(x)$ in exponential form, deduce the Fourier coefficients c_{-2}, c_{-1} and c_0. Try to do this by inspection rather than application of formulas.

Solutions

c_{-2}:
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of hX = 53126e

c_{-1}:
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of kX = 312a1a

c_0:
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of jX = b6ffa8

3.21 Fourier Coefficients of 1 + sin(x)

Challenge

Using the same approach as challenge 3.20, deduce for the function $1 + sin(x)$ the Fourier coefficients c_{-1}, c_0 and c_1.

Solutions

c_{-1}:
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of mX = 434572

c_0:
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of nX = 1444ea

c_1:
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of oX = 2e5a62

3.22 Relation of positive and negative Fourier coefficients for a real signal

Resources

- Challenge 3.14
- Book: 1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Challenge

If the Fourier coefficient C_1 is $4+6i$ for a real signal, what is the Fourier coefficient C_{-1}?

Solution

X = Your solution
Form: Imaginary form with integers.
Place the indicated letter in front of the number.
Example: aX where $X = 1 + 2i$ is entered as are(1)im(2)

Hash of pX = de57ee

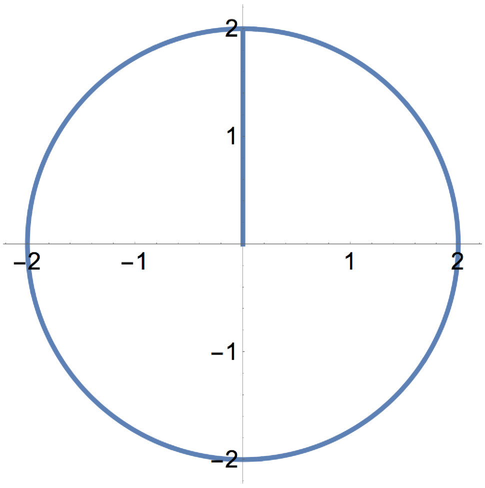
3.23 Circles and Fourier series

Resources

- Video 1: `https://www.youtube.com/watch?v=Y9pYHDSxc7g`
- Video 2: `https://www.youtube.com/watch?v=LznjC4Lo7lE`

Comment

In the first lecture we saw how it was possible to approximate any function given enough circles. Here we link what you have learned back to that first lecture. I strongly recommend viewing the fun and informative videos listed here under Resources. In summary, by building a Fourier series you are representing a function using a sum of periodic functions, where each component can be considered visually as a circle operating with individual radius, frequency and phase on the real-imaginary plane.



Challenge

Treating the x-axis as the real axis and the y-axis as the imaginary axis, arrange the equations below in the following order:

1. A point moving round on a circle with radius 2 units and frequency 2 Hz
2. A point moving round on a circle with radius 3 units and frequency 1 Hz
3. A point moving round on a circle with radius 2 units and a period of 1 second
4. A point moving round on a circle with radius 3 units and a period of 2 seconds

Equations:

$Ae^{2\pi ikt}$ where t is time in seconds and the values of A and k are as follows:

A: $A = 2$, $k = 2$

B: $A = 3$, $k = 1$

C: $A = 3$, $k = 0.5$

D: $A = 2$, $k = 1$

Solution

X = Your solution
Form: String.
Place the indicated letter in front of the string.
Example: aX where X=abcd is entered as aabcd

Hash of uX = d8a7e6

3.24 Partial derivatives

Challenge

Determine u_t and u_{xx} for the equation

$$u(x,t) = 5tx^2 + 3t - x \tag{3.22}$$

To check your answer, substitute $x = 3$ and $t = 2$ into your answers, as appropriate.

Solution

u_t:
X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of tX = 4fb068

u_{xx}:
X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of xX = 53502b

3.25 Heat equation: Periodicity

Resources

- Book: Section 1.13.1 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Lecture 4 from 37:00 onwards: (`https://www.youtube.com/watch?v=n5lBM7nn2eA`), continuing at the start of lecture 5 (`https://www.youtube.com/watch?v=X5qRpgfQld4`)

Comment

Here we can learn about an application of Fourier series to solve partial differential equations. This problem was one of the motivations for Fourier to develop the idea of Fourier series.

The motivation for the equation $u_t = \frac{1}{2}u_{xx}$ described in the notes is complicated somewhat by the interpretation in terms of equivalences between electrical and thermal capacitance. If this is not so clear then don't worry about it. At a minimum you should understand the following:

- Heat flow is proportional to the gradient of the temperature.
- Heat accumulates within a unit volume when the rate of heat flow into that volume is greater than the rate of heat flow out of that volume.

Challenge

The following statements concern a heated ring with circumference 1 and temperature distribution described by $u(x,t)$. Add the points of the statements that are defined by the system to be true:

1 point: $u(x,t) = u(x,t)$

2 points: $u(x,t) = u(x,t+1)$

4 points: $u(x,t) = u(x,t+2)$

8 points: $u(x,t) = u(x+1,t)$

16 points: $u(x,t) = u(x+1,t+1)$

32 points: $u(x,t) = u(x+1,t+2)$

64 points: $u(x,t) = u(x+2,t)$

128 points: $u(x,t) = u(x+2,t+1)$

256 points: $u(x,t) = u(x+2,t+2)$

512 points: The temperature distribution is periodic in space but not time

1024 points: The temperature distribution is periodic in time but not space

2048 points: The temperature distribution is periodic in both space and time

4096 points: The temperature distribution is periodic neither in space nor time

Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of yX = 2259d1

3.26 Heat equation on a ring: derivation

Resources

- Video: `https://www.youtube.com/watch?v=yAOCibHPgLA`
- Book: Section 1.13.1 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Lecture 4 from 37:00 onwards: (`https://www.youtube.com/watch?v=n5lBM7nn2eA`), continuing at the start of lecture 5 (`https://www.youtube.com/watch?v=X5qRpgfQld4`)

Challenge

Starting from the heat (diffusion) equation $u_t = u_{xx}/2$, show that the general solution to the heat equation on a ring is given by

$$u(x,t) = \sum_{n=-\infty}^{n=\infty} c_n(0) e^{-2\pi^2 n^2 t} e^{i2\pi n x} \tag{3.23}$$

and write an expression for $c_n(0)$ in terms of the initial temperature distribution $u(x,0)$.

Solution

Please compare with your peers during discussion time and ask if there is anything you do not understand.

3.27 Heat equation on a ring: calculation

Comment

Remember that any integer multiple of 2π in the complex exponential (eg, $e^{i4\pi}$ or $e^{i2\pi n}$ where n is an integer) is equivalent to having 2π in the exponential due to the periodic nature of complex exponentials (ie, that $e^{i2\pi} = \cos(2\pi) + i\sin(2\pi)$ and $\cos(2\pi) = \cos(4\pi) = \cos(6\pi)$ etc).

Challenge

Consider an initial heat distribution around a ring. Relative to ambient temperature, the initial temperature distribution follows a cosine distribution with the peak temperature of $u = 1$ at $x = 0$. Assume that the ring has a circumference length of 1 unit.

1. Write an expression for the initial relative temperature distribution, $u(x, 0)$.
2. Write an expression for the relative temperature distribution, $u(x, t)$, as a function of time.

Note: You may use a computer-algebra system such as Wolfram Alpha to help you do the necessary integrals.

You can see an animation of the solution here:
`https://raw.githubusercontent.com/NanoScaleDesign/FourierAnalysis/master/Images/cosrelax.gif`

Solution

1. You should find that your temperature distribution satisfies the result $u(0.2, 0) = 0.309$.
2. To check your answer you may substitute $x = 0.3$ and $t = 0.01$ into your solution: $u(0.3, 0.01) = -0.25$

Chapter 4

Fourier Transform

4.1 Limits of $\sin(x)$

Resources

- http://mathworld.wolfram.com/SeriesExpansion.html

Challenge

Considering the series expansion around 0, determine the limiting value of the following functions as $T \to \infty$:

1. $\sin(x/T)$

2. $T \sin(x/T)$

You may consider x to be any real-valued number.

Solution

To check your answer, substitute $x = 2$ as appropriate.

1.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of aX = 690969

2.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of bX = 2a5520

4.2 The transition to the Fourier Transform

Resources

- 3B1B video: https://www.youtube.com/watch?v=spUNpyF58BY
- Stanford video (part I): Lecture 5 from 27:00 (https://youtu.be/X5qRpgfQld4?t=27m)
- Stanford video (part II): Lecture 6 up to 20:00 (https://www.youtube.com/watch?v=4lcvROAtN_Q)
- Book: Chapter 2 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)

Comment

Until now we have been considering Fourier series. This is however limited to describing periodic phenomena since it is assumed that the signal repeats outside the region of integration. The Fourier transform can be thought of as an extension of Fourier series which allows the analysis of non-periodic phenomena.

The suggested resources provide an excellent intuitive path of the connection between Fourier series and Fourier transforms, and this challenge is designed to give you the opportunity to take some time to try to understand the main concepts behind the transition.

Challenge

Using the resources above, show how to make the transition from Fourier series to the Fourier transform.

Solution

Please compare your working with your partner.

4.3 L'Hôpital's rule

Challenge

Use L'Hôpital's rule to determine the limit of

$$\frac{\sin(x)}{x} \tag{4.1}$$

as $x \to 0$.

Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of dX = 9948c6

4.4 Fourier transform of a window function

Resources

- Book: Chapter 2.1 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 6 starting from 29:26 (`https://youtu.be/4lcvROAtN_Q?t=29m26s`)

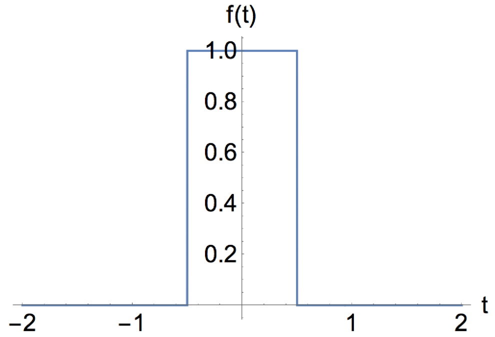
Challenge

Calculate the Fourier Transform for the window function

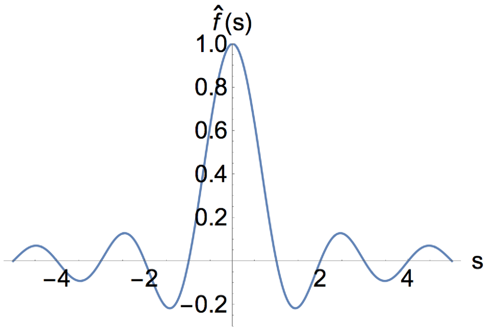
$$f(t) = \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{for } |t| > 1/2 \end{cases} \tag{4.2}$$

In graph-form the function and its transform appear as follows:

Function



Fourier transform



Solution

You should find that your solution is consistent with $\hat{f}(s = 1.5) = -0.21$.

4.5 Fourier transform of sine and cosine

Comments

With Fourier series we understood the integer k in $\sin(2\pi kt)$ as the frequency of the signal, and by building up the sum of sines and cosines (using exponential notation) we could reproduce very well an arbitrary periodic signal. In the case of challenges 3.20 and 3.21 we determined the Fourier coefficients for a single-frequency $k = 1$ sine signal.

Can we still understand the s in the Fourier transform $\int_{-\infty}^{\infty} f(t)e^{i2\pi st}$ as analogous to k somehow? What happens when you take the Fourier transform of a single-frequency signal? How is phase-information encoded? This is difficult to visualise but this challenge, dealing with simple single-frequency signals, should give you an initial insight.

Challenge

1. Calculate the Fourier transforms of
(I) $\sin(2\pi t)$
(II) $\cos(2\pi t)$

You may find the following identity useful:

$$\int_{-\infty}^{\infty} e^{-i2\pi(s+c)t}dt = \delta(s+c) \tag{4.3}$$

2. Write $\sin(2\pi t)$ and $\cos(2\pi t)$ in terms of complex exponentials. Can you see an analogy between the Fourier transforms you calculated and the complex exponential forms of sine and cosine?

Solution

1. In both cases, to check your answers calculate $\int_0^2 f(s)ds$.

(I)
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of fX = c68bbf

(II)
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of eX = 923c59

2. Please discuss with your partner in class, and ask the teacher if you are unsure.

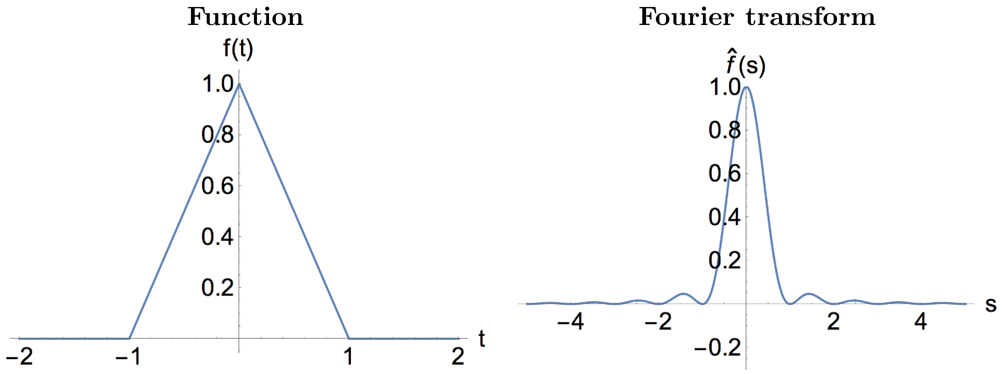
4.6 Fourier transform of a triangle function

Resources

- Book: Chapter 2.2.1 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 6 (`https://www.youtube.com/watch?v=4lcvROAtN_Q`)

Challenge

What is the Fourier transform of a triangle function, as shown below, in terms of the window function of challenge 4.4? You may just state the function; calculation is not required unless you would like to try and prove it.



Solution

Your answer should be consistent with $F(s = 1.5) = 0.045$

4.7 Fourier transform of a Gaussian

Resources

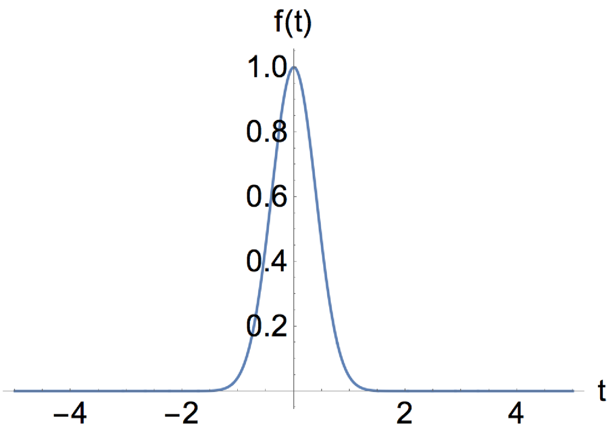
- Book: Chapter 2.2.2 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 7 (https://www.youtube.com/watch?v=mdFETbe1n5Q)

Challenge

Starting from the general formula for Fourier transform, calculate the Fourier transform for the Gaussian function:

$$f(t) = e^{-\pi t^2} \tag{4.4}$$

The Gaussian function looks like



Solution

Your answer should be consistent with $F(s = 1.5) = 8.51 \times 10^{-4}$

4.8 Fourier transform of a rocket function

Resources

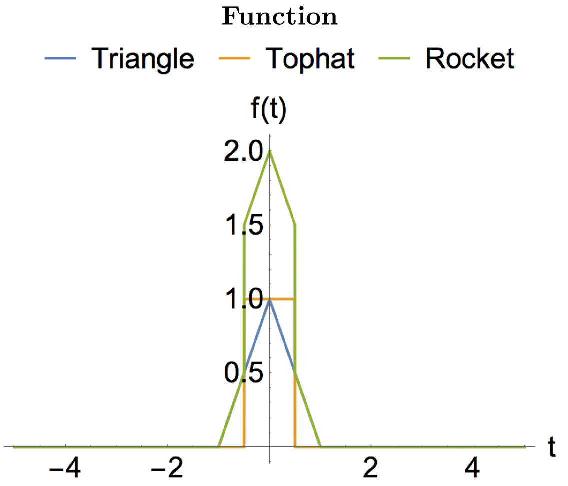
- Book: Chapter 2.2.6 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

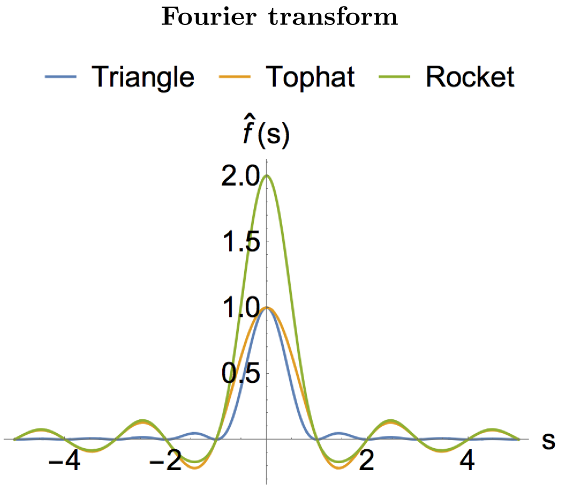
Challenge

Using the results from previous challenges, calculate the Fourier Transform for the rocket function:

$$f(t) = \begin{cases} 2 - |t| & \text{for } 0 \leq |t| < \frac{1}{2} \\ 1 - |t| & \text{for } \frac{1}{2} \leq |t| \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

shown below. In graph-form the function and its transform appear as follows:





Solution

Your answer should be consistent with $F(s = 1.5) = -0.17$

4.9 Fourier transform of a shifted window function

Resources

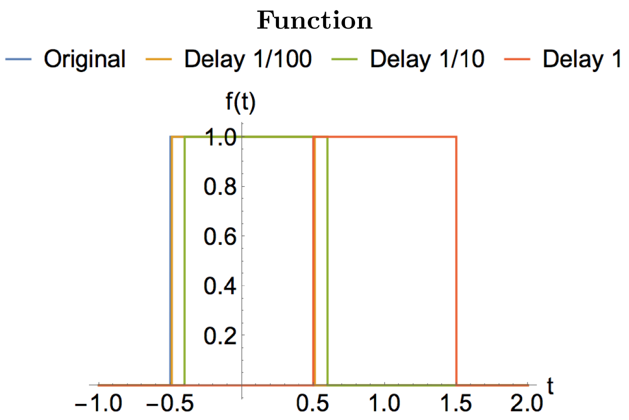
- Book: Chapter 2.2.7 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video lecture 8 from 1:56 to 11:20 (`https://youtu.be/wUT1huREHJM?t=1m56s`)

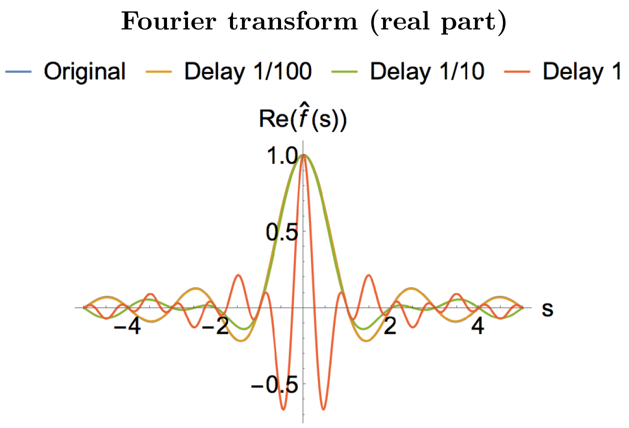
Challenge

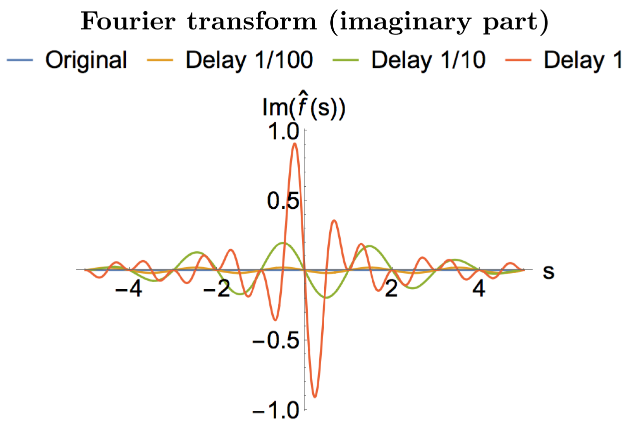
1. What is meant by a "phase-shift" of a signal in time? What does it mean in terms of the window-function here?

2. Calculate the Fourier Transform for the window function delayed by 1 s.

3. A graph with different delays is shown in the figure below. Think about what is happening to the real and imaginary parts of the signal as the delay increases. Briefly qualitatively summarise what you see happening. You considered the original window function in challenge 4.4.







Solution

1. Please compare your answer with that of your partner.

2. You should find that $\hat{f}(s = 2.9) = 0.0274405 + 0.0199367i$

3. Please compare your answer with that of your partner.

4.10 Fourier transform of a stretched triangle function

Resources

- Book: Chapter 2.2.8 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video lecture 8 from 12:12 to 28:50 (`https://youtu.be/wUT1huREHJM?t=12m12s`)

Challenge

Since this can be a little confusing, please be sure to fully read the listed resource and make sure you understand the reasoning and derivation.

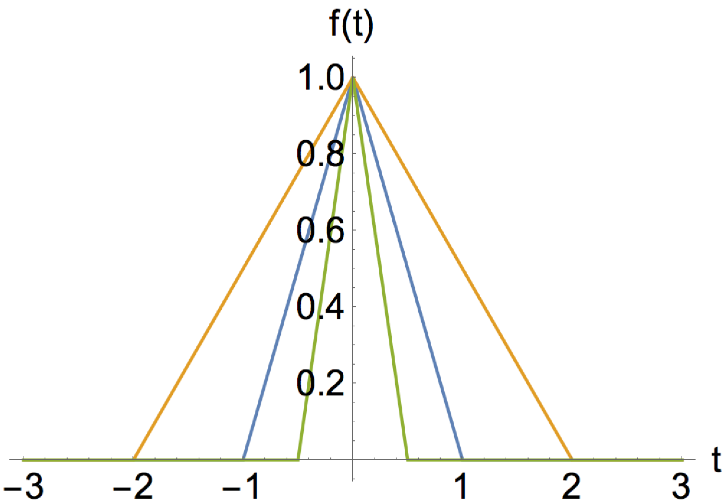
A triangle function of general width can be defined as

$$f(at) = \begin{cases} 1 - a|t| & \text{for } a|t| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

1. In challenge 4.6 the base-width of the triangle was 2 (ie, $1 - (-1)$) and the half-base-width was 1 (ie, $1 - 0$). What is the half-base-width for a triangle when $a = 2$ and $a = 1/2$?

2. Calculate the Fourier Transform $\hat{f}(s)$ for the case where the width of the triangle-function is doubled.

3. Write a few sentences explaining how stretching and squeezing in the time-domain is related to stretching and squeezing in the frequency domain.



Solution

1.
$\boldsymbol{a = 2}$
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of gX = 3cd169

$\boldsymbol{a = 1/2}$
X = Your solution

Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of hX = 3bb5c8

2.
To check your answer, evaluate the transform at $s = 0.1$.
$\hat{f}(0.1) = 1.75$

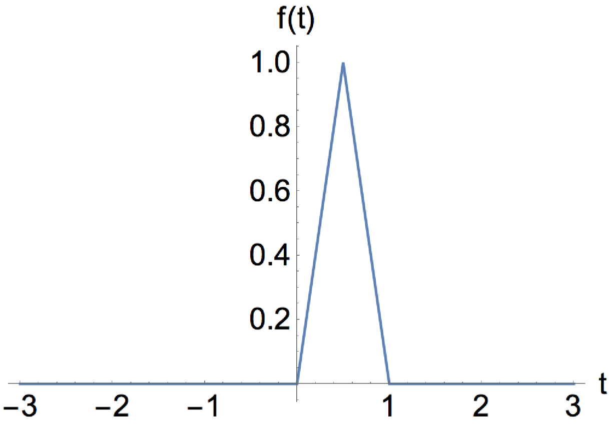
4.11 Fourier transform of a shifted-stretched function

Resources

- Book: Chapter 2.2.9 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Challenge

Calculate the Fourier transform of the signal shown below.



Solution

Your answer should be consistent with $\hat{f}(s = 1.1) = -0.155 + 0.050i$.

4.12 Fourier transform notation

Comment

Just so that you are aware, you should know that there are various equivalent notations in use, and if you use the Fourier transform in your later career you will probably come across notations different to the ones used in this course. This is partly because Fourier analysis is applicable to a wide range of fields and each field tends to have its own conventions based on how the Fourier transform is interpreted physically (if at all) in that field.

In general, the Fourier transform is given by

$$\mathcal{F}f(\omega) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} f(t) e^{ib\omega t} dt \tag{4.7}$$

where a and b typically take on values such as

Case	**a**	**b**
1	0	1
2	1	-1
3	-1	1
4	0	-2π

Challenge

1. Write the Fourier transform using the four possible notations.
2. Add the points of the legitimate forms of Fourier notation:

1 point: $\mathcal{F}f(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwt} f(t) dt$

2 points: $\mathcal{F}f(w) = \int_{-\infty}^{\infty} e^{-i2\pi wt} f(t) dt$

4 points: $\mathcal{F}f(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwt} f(t) dt$

8 points: $\mathcal{F}f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\pi st} f(t) dt$

16 points: $\mathcal{F}f(w) = \int_{-\infty}^{\infty} e^{i2\pi wt} f(t) dt$

32 points: $\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-ist} f(t) dt$

Solution

2.
X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of cX = 8cecbb

4.13 Convolution introduction

Resources

- Video lecture 8 starting at 32:15: `https://youtu.be/wUT1huREHJM?t=32m15s`
- Book: Chapter 3.1 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Comment

Here we expand our study to convolution; a powerful function for processing signals. This introduction by Prof. Osgood provides an excellent introduction. What you will learn here is that convolution in real space is related to multiplication in frequency space. Visualisation of what convolution in real-space means however, is rather difficult, and I would recommend to focus purely on the mathematical meaning here of how it relates functions in real space and frequency space.

Challenge

Briefly describe what convolution of two functions means, and how it can relate two functions of time in real space to their equivalent functions in frequency space.

Show that convolution in real-space ($\mathscr{F}(g * f)(s)$) corresponds to multiplication in frequency space ($\mathscr{F}g(s)\mathscr{F}f(s)$).

Solution

Please compare your answer with your partner or discuss with the teacher in class.

4.14 Filtering

Resources

- Video lecture 9 until 24:15: `https://www.youtube.com/watch?v=NrOR2qMVWOs`

Comment

The above video includes an excellent example of using fourier analysis in a scientific context, including its application in filters. I encorage you to watch until 24:15. After this point Prof. Osgood goes on to describe the futility of trying to visualise convolution in the time-domain, however the graphics on convolution on Wikipedia [1] (especially these: [2a,b]) I think go a little way to visualising what's happening in the time-domain.

1: `https://en.wikipedia.org/wiki/Convolution`
2a: `https://en.wikipedia.org/wiki/Convolution#/media/File:Convolution_of_box_signal_with_itself2.gif`
2b: `https://en.wikipedia.org/wiki/Convolution#/media/File:Convolution_of_spiky_function_with_box2.gif`

For your reference, turbidity standards of 5, 50, and 500 Nephelometric Turbidity Units (left to right respectively) are shown here:

Source: *`https://en.wikipedia.org/wiki/Turbidity`*

Challenge

Using a few sentences and diagrams, describe an ideal low-pass, high-pass and band-pass filter. How can they be applied in the frequency domain to influence the signal in the time-domain?

Solution

Please compare your answer with your partner or discuss with the teacher in class.

4.15 Convolution with a window function

Comment

The “signal” here is $g(t-\tau)$ and the “window function” is $f(\tau)$.

Challenge

Consider that you have a (somewhat unrealistic but mathematically manageable) input signal that varies as $g(t) = t^2$ with time.

Obtain the convolution of the signal $(f \star g)(t)$ with a window function:

$$f(t) = \begin{cases} 1 & \text{for } -1/2 < t < 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

Solution

Your answer should be consistent with $(f \star g)(2) = 49/12$

4.16 Convolution with a continuous function

Challenge

Calculate the convolution of $f(t) = t$ with $g(t) = e^{-|t|}$.

Hint 1: $\int_{-\infty}^{\infty}$ *(even function)* $= 2\int_{0}^{\infty}$ *(even function).*

Hint 2: It does not matter which function you make "f" or "g", but one way is easier than the other to compute.

Solution

To check your answer, substitute $t = 1$ into your final answer and you should obtain $(f \star g)(1) = 2$.

4.17 Convolution of two window functions

Challenge

In challenge 4.4 you calculated the spectrum of a window function. Imagine here you have two window functions $f(t)$ and $g(t)$.

$$f(t) = \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{for } |t| > 1/2 \end{cases} \tag{4.9}$$

$$g(t) = \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{for } |t| > 1/2 \end{cases} \tag{4.10}$$

1. Use your knowledge of the window-function in the frequency domain from challenge 4.4 to calculate the frequency-domain spectrum of the convolution of the two window functions.

2. What is the resulting function in the time-domain? *This should be possible by deduction based on previous challenges, without calculation.*

Solution

1. Substitute $s = 1/2$ into your final answer and you should obtain 0.405.

2. Substitute $t = 1/2$ into your final answer and you should obtain 0.50.

Chapter 5

Discrete Fourier Transform

5.1 Introduction to digital signals

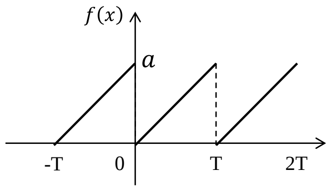
Resources

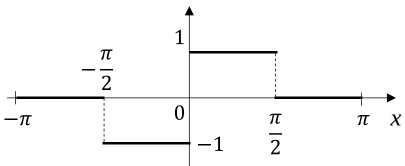
The following pages are based on slides developed by Prof. Maraso Yamashiro for the 2015 class on Fourier Analysis. Please work through the slides on the following pages and view the video afterwards.

We know how to calculate Fourier series and transforms from given functions.
But what if we don`t know what the underlying function is?

$f(x)$: A given function

e.g.





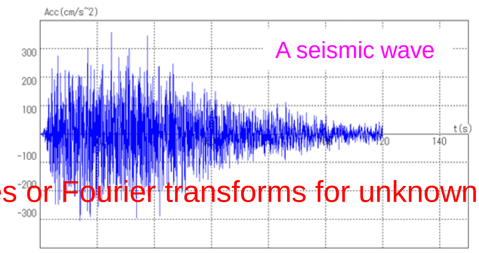
$$f(x) = \frac{ax}{T} \quad 0 \le x < T, \quad f(x+T) = f(x)$$

$$f(x) = \begin{cases} 1 & if \quad 0 \le x < \pi/2 \\ -1 & if \quad -\pi/2 \le x \le 0 \end{cases}$$

$f(x)$: Unknown

e.g.

Natural phenomena



How can we find Fourier series or Fourier transforms for unknown functions?

For an unknown function, we need to move from integration to numerical calculation

Integration → Numerical calculation

Analog data → Digital data

Continuous functions

Discretization

Another viewpoint

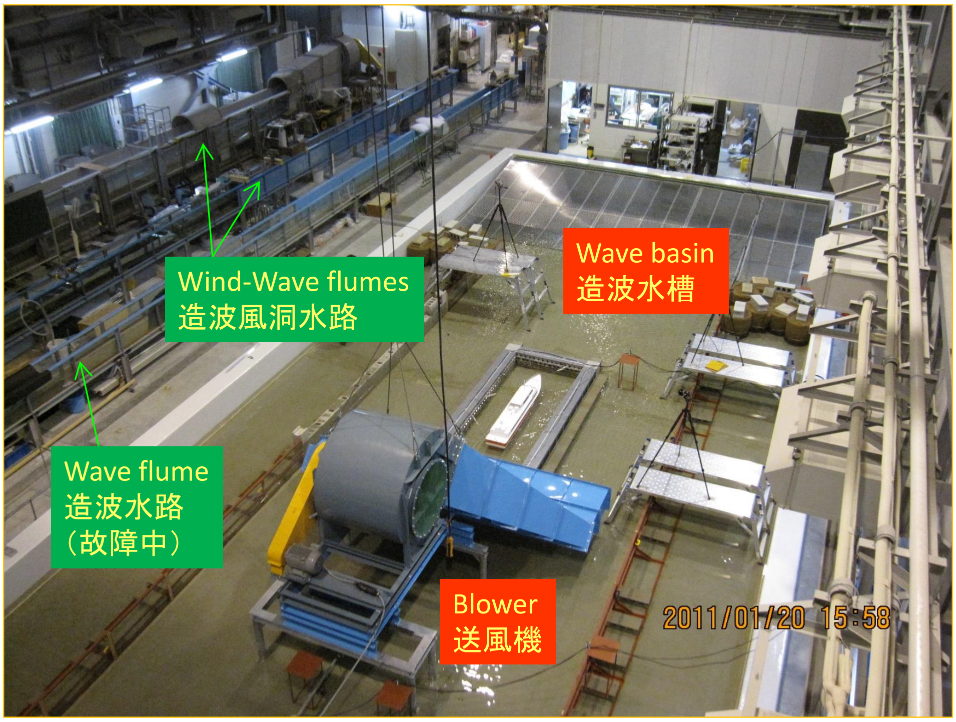
- ✓ Data are often obtained by field observations, model experiments, etc.
- ✓ Data is usually analyzed using PC.

The output is Digital data

Discrete Fourier Transform (DFT)
Fast Fourier Transform (FFT)

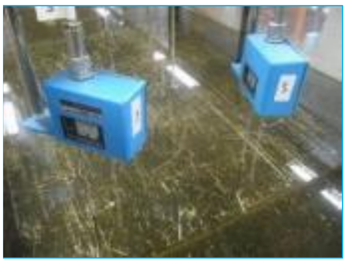
Examples of digital data

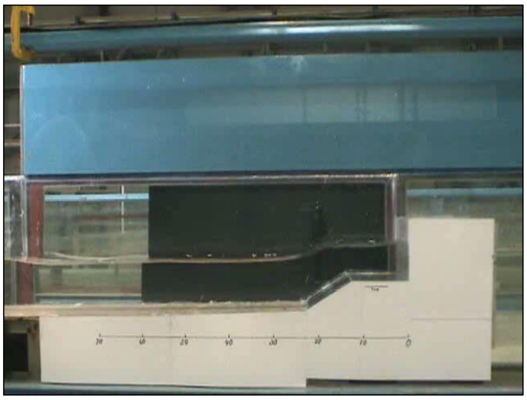
Experimental facilities of coastal and ocean engineering laboratory



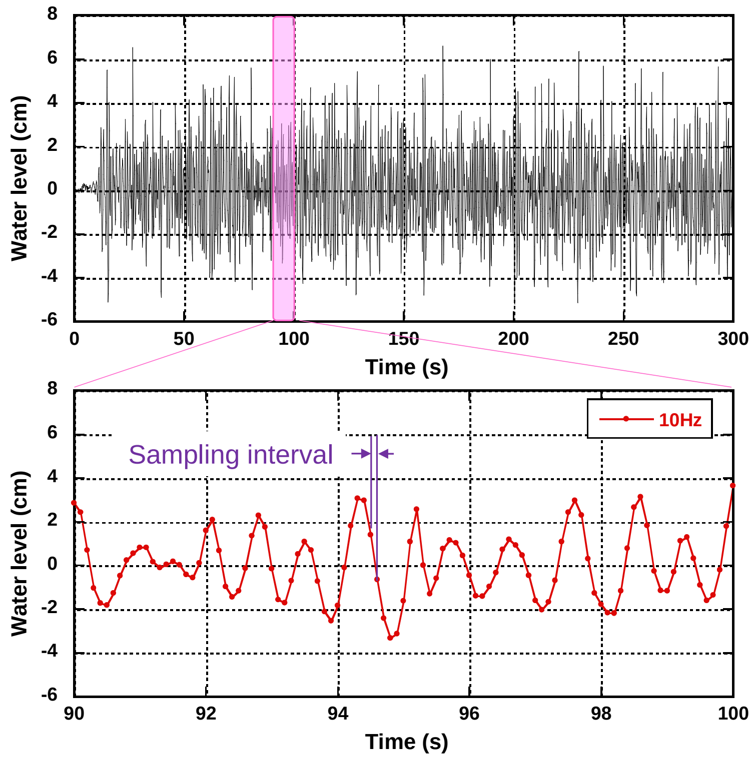
Model experiments

Wave gauge





Examples of Wave data



- Video: `https://www.youtube.com/watch?v=yWqrx08UeUs&feature=youtu.be&t=47s`

Challenge

1. An audio CD can record frequencies up to 44.1 kHz. Consider how often you would need to sample a sound in order to reproduce frequencies up to 44.1 kHz. What is the highest sampling frequency that experiences aliasing?

2. Write a sentence or two summarising what aliasing is and the Nyquist sampling theorem.

Solutions

1.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of xX = 43ef8f

2. Check your answer with your partner and discuss any differences. Ask the teacher if you are unsure.

5.2 Discrete Fourier Transform: Coefficients

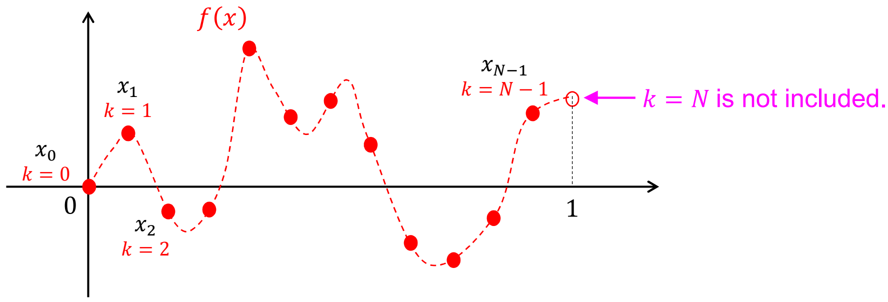
Resources

The following pages are based on slides developed by Prof. Maraso Yamashiro for the 2015 class on Fourier Analysis. The slides have been lightly modified for notation differences. Note that the equation for the discrete Fourier Transform on the 5th slide should read

$$\hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-i2\pi n x_k} \tag{5.1}$$

and not e^{-inx_k} (my mistake).

Discrete Fourier Transform (DFT)

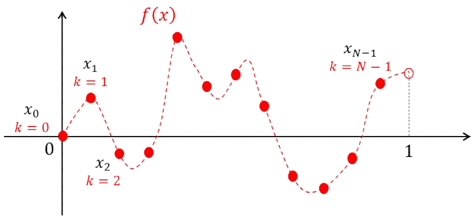


Let $f(x)$ be periodic, for simplicity of Period 1. We assume that N measurements of $f(x)$ are taken over the interval $0 \leq x \leq 1$ at regularly spaced points

$$x_k = \frac{k}{N} \qquad k = 0, 1, 2, \cdots, N - 1$$

We now want to determine a complex trigonometric polynomial

$$q(x) = \sum_{n=0}^{N-1} c_n e^{i2\pi nx}$$



that interpolates $f(x)$ at the nodes x_k, that is, $q(x_k) = f(x_k)$.
Denoting $f(x_k)$ with f_k,

$$f_k = f(x_k) = q(x_k) = \sum_{n=0}^{N-1} c_n e^{i2\pi n x_k}$$

$$k = 0, 1, 2, \cdots, N - 1$$

cf.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx}$$

$$e^{ix} = \cos x + i \sin x$$

The coefficients are determined by using the orthogonality of the trigonometric system.

$$f_k = \sum_{n=0}^{N-1} c_n e^{i2\pi n x_k}$$

- ✓ Multiply f_k by $e^{-i2\pi m x_k}$ and sum over k from 0 to $N-1$
- ✓ Interchange the order of the two summations
- ✓ Replacing x_k with k/N

$$\sum_{k=0}^{N-1} f_k e^{-i2\pi m x_k} = \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} c_n e^{i2\pi(n-m)x_k} = \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} e^{i(n-m)2\pi k/N}$$

$$e^{i(n-m)2\pi k/N} = \left[e^{i(n-m)2\pi/N}\right]^k = r^k \qquad r = e^{i(n-m)2\pi/N}$$

For $n = m$

$$r^k = (e^0)^k = 1^k = 1 \qquad \sum_{k=0}^{N-1} r^k = N$$

For $n \neq m$

$$r \neq 1 \qquad \sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} = 0$$

$$r^N = e^{i(n-m)2\pi N/N} = e^{i(n-m)2\pi}$$
$$= \cos(n-m)2\pi + i\sin(n-m)2\pi = 1 + 0 = 1$$

Sum of a geometric series

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$= \frac{a(1-r^n)}{1-r} \qquad r \neq 1$$

$$\sum_{k=0}^{N-1} f_k e^{-i2\pi m x_k} = \underline{\sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} e^{i(n-m)2\pi k/N}} = c_0 0 + c_1 0 + \cdots + c_m N + \cdots c_{N-1} 0$$

$$= N c_m$$

$$c_m = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i2\pi m x_k}$$

$$\sum_{k=0}^{N-1} r^k = N \qquad \text{For } n = m$$

$$\sum_{k=0}^{N-1} r^k = 0 \qquad \text{For } n \neq m$$

Replacing m with n,

$$c_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i2\pi n x_k} \qquad f_k = f(x_k)$$

$$n = 0, 1, 2, \cdots, N-1$$

$$c_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i2\pi n x_k} \qquad f_k = f(x_k)$$

$$n = 0, 1, 2, \cdots, N-1$$

Discrete Fourier Transform

$$\hat{f}_n = N c_n = \sum_{k=0}^{N-1} f_k e^{-inx_k} \qquad f_k = f(x_k)$$

$$n = 0, 1, 2, \cdots, N-1$$

The Discrete Fourier Transform of the given signal $\mathbf{f} = [f_0 \cdots f_{N-1}]^{\mathsf{T}}$ to be the vector $\hat{\mathbf{f}} = [\hat{f}_0 \cdots \hat{f}_{N-1}]$ with components $\hat{f}_n$

This is the frequency spectrum of the signal.

In vector notation, $\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f}$, where the $N \times N$ Fourier matrix $\mathbf{F}_\mathrm{N} = [e_{nk}]$ has the entries

$$e_{nk} = e^{-i2\pi n x_k} = e^{-i2\pi nk/N} = w^{nk}, \quad w = w_N = e^{-i2\pi/N} \qquad x_k = \frac{k}{N}$$

where $n,\ k = 0, \cdots, N-1$

Example Discrete Fourier Transform (DFT)

Let $N = 4$ measurements (sample values) be given.

Then
$$\begin{aligned} w &= e^{-i2\pi/N} = e^{-i\pi/2} \\ &= \cos\pi/2 - i\sin\pi/2 \\ &= -i \end{aligned}$$

and thus
$$w^{nk} = (-i)^{nk}$$

$$\hat{f}_n = N c_n = \sum_{k=0}^{N-1} f_k e^{-i2\pi n x_k}$$

$$\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f}$$

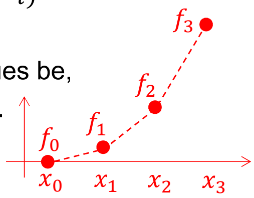
$$\mathbf{F}_\mathrm{N} = [e_{nk}]$$

$$e_{nk} = e^{-i2\pi n x_k} = e^{-i2\pi nk/N} = w^{nk}$$

$$w = w_N = e^{-i2\pi/N}$$

$$e^{ix} = \cos x + i \sin x$$

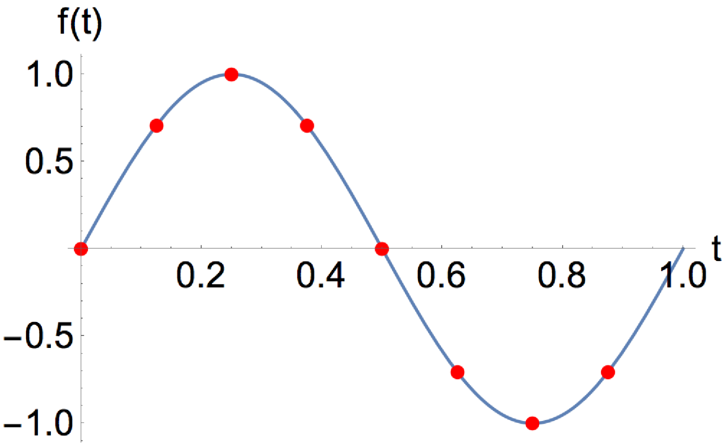
Let the sample values be, say $\mathbf{f} = [0\ \ 1\ \ 4\ \ 9]^{\mathsf{T}}$.



Then

$$\hat{\mathbf{f}} = \mathbf{F}_4\mathbf{f} = \begin{array}{c} \\ n=0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} k=0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \end{array} \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}$$

The challenge below considers sampling of a 1 Hz sine-wave 8 times per second, sampling at time 0, $\frac{1}{8}$, $\ldots$, $\frac{7}{8}$:



This yields sample values of

$$f = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ -1 \\ -1/\sqrt{2} \end{pmatrix} \tag{5.2}$$

Challenge

Calculate the missing values A-H in the following calculation of the frequency spectrum. To a certain extent you may be able to pattern-match to guess the values, but please be sure that you practise the calculation method too, since that is the main learning objective here.

$$F_8 = \begin{pmatrix} 1 & \mathbf{A} & \mathbf{B} & 1 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{C} & \mathbf{D} & -\frac{1+i}{\sqrt{2}} & -1 & -\frac{1-i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \\ 1 & \mathbf{E} & \mathbf{F} & i & 1 & -i & -1 & i \\ 1 & -\frac{1+i}{\sqrt{2}} & i & \frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & -i & -\frac{1-i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1-i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & \frac{1-i}{\sqrt{2}} & i & -\frac{1+i}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{1+i}{\sqrt{2}} & i & -\frac{1-i}{\sqrt{2}} & -1 & -\frac{1+i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} \end{pmatrix} \tag{5.3}$$

$$\hat{f} = \begin{pmatrix} \mathbf{G} \\ \mathbf{H} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4i \end{pmatrix} \tag{5.4}$$

Solutions

Enter imaginary numbers as indicated below. For example: $-1/\sqrt{2}-i/\sqrt{2}$ with a prefix of "a" would be entered as "are(-0.71)im(-0.71)", $-i$ would be entered as "are(0.00)im(-1.00)" and -1 would be "are(-1.00)im(0.00)".

A
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of aX = 95c9e4

B
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of bX = ec0ffd

C
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of cX = 13085b

D
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of dX = d01771

E
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of eX = 332be1

F
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of fX = 27b237

G
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of gX = cbd79b

H
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of hX = 293fff

5.3 Discrete Fourier Transform: Analysis

Resources

Having identified the frequency spectrum, it is possible then to analyse the frequencies of the signal. The video listed below provides a nice introduction into understanding our frequency spectrum. Note that it uses the "j" representation of imaginary numbers (instead of "i").

- Video: `https://www.youtube.com/watch?v=mkGsMWi_j4Q`

Comment

The video discusses the phase of the signal as the angle formed by rotating anti-clockwise from an arrow pointing in the positive direction along the real axis. You saw in challenge 4.5 how the continuous Fourier transform of a cosine function yields real values and the Fourier transform of a sine function yields imaginary values. Here you can see how phase-shift of a cosine signal leads to real and imaginary solutions and how this can be represented in a real-complex plane.

Challenge

What is the frequency, magnitude and phase of the sine-wave as determined through our discrete Fourier transform analysis?

Solutions

Frequency
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of zX = a78955

Magnitude
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of aX = a1e88c

Phase X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

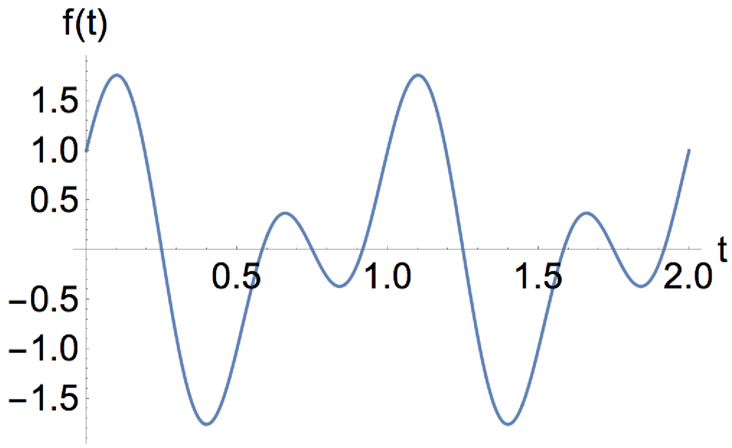
Hash of bX = fd9a84

5.4 Analysing a more complex function: part I

Challenge

Considering the signal:

$$f(t) = \cos(2\pi t) + \sin(4\pi t) \tag{5.5}$$



1. What are the two frequencies that make up the signal?
2. What is the fundamental frequency of this signal?
3. What is the maximum sampling frequency at which aliasing will occur for this signal?

Solutions

1.
(Lower of the two frequencies in Hz)
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of cX = 166765

(Higher of the two frequencies in Hz)
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of dX = c22bee

2.
(units: Hz)
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of fX = 6786da

3.
(units: Hz)

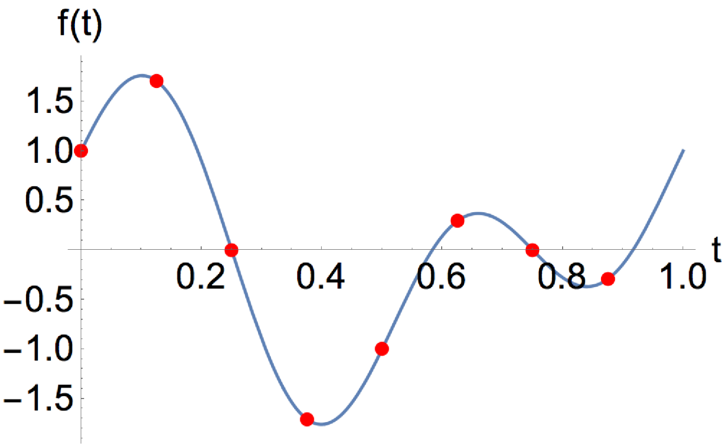
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of eX = 0a9397

5.5 Analysing a more complex function: part II

Challenge

Sampling the signal in the previous challenge 8 times per second:



yields values of

$$\begin{pmatrix} 1. \\ 1.71 \\ 0. \\ -1.71 \\ -1. \\ 0.29 \\ 0. \\ -0.29 \end{pmatrix} \tag{5.6}$$

Calculating the matrix F yields

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & (1-i)/\sqrt{2} & \boldsymbol{A} & -(1+i)/\sqrt{2} & -1 & (-1+i)/\sqrt{2} & i & (1+i)/\sqrt{2} \\ 1 & \boldsymbol{B} & -1 & i & 1 & -i & -1 & i \\ 1 & -(1+i)/\sqrt{2} & i & (1-i)/\sqrt{2} & -1 & (1+i)/\sqrt{2} & -i & (-1+i)/\sqrt{2} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & (-1+i)/\sqrt{2} & -i & (1+i)/\sqrt{2} & -1 & (1-i)/\sqrt{2} & i & -(1+i)/\sqrt{2} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & (1+i)/\sqrt{2} & i & (-1+i)/\sqrt{2} & -1 & -(1+i)/\sqrt{2} & -i & (1-i)/\sqrt{2} \end{pmatrix} \tag{5.7}$$

1. Determine the missing values **A** and **B** by calculation.

2. Determine, by calculation, the frequencies with their magnitudes and phases of this signal. In this case, you can know the constituent frequencies, their magnitudes and phases because you know how the signal is made up of two signals. Check that your calculation aligns with your intuition.

Solutions

1. A
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of aX = 223b8b

1. B
X = Your solution
Form: Imaginary form with numbers to two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 1.23 + 4.56i$ is entered as are(1.23)im(4.56)

Hash of bX = df600a

2. If you are not sure what your intuition should be, or if your answer does not match your intuition, please discuss with your partner or the teacher in class.

5.6 The limits of DFT calculation

Resources

Practically, the DFT is calculated using the Fast Fourier Transform (FFT). The reason is that the number of operations grows with the square of the number of samples which is unsustainable for typical signals.

The following pages are based on slides developed by Prof. Maraso Yamashiro for the 2015 class on Fourier Analysis, and highlight this problem quantitatively.

Example Discrete Fourier Transform (DFT)

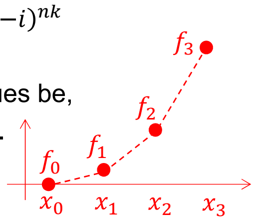
Let $N = 4$ measurements (sample values) be given.

Then $w = e^{-i2\pi/N} = e^{-i\pi/2}$
$= \cos\pi/2 - i\sin\pi/2$
$= -i$

and thus $w^{nk} = (-i)^{nk}$

$$\hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}$$
$$\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f}$$
$$\mathbf{F}_{\mathrm{N}} = [e_{nk}]$$
$$e_{nk} = e^{-inx_k} = e^{-i2\pi nk/N} = w^{nk}$$
$$w = w_N = e^{-i2\pi/N}$$
$$e^{ix} = \cos x + i\sin x$$

Let the sample values be, say $\mathbf{f} = [0\ \ 1\ \ 4\ \ 9]^{\mathsf{T}}$.



Then

$$\hat{\mathbf{f}} = \mathbf{F}_4\mathbf{f} = \begin{array}{c} \\ n=0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} k=0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \end{array} \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}$$

---

e.g. $N = 4 \longrightarrow n = 0, 1, 2, 3$

$$\hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}$$
$$\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f}$$
$$\mathbf{F}_{\mathrm{N}} = [e_{nk}]$$
$$e_{nk} = e^{-inx_k} = e^{-i2\pi nk/N} = w^{nk}$$
$$w = w_N = e^{-i2\pi/N}$$

Discrete Fourier Transform

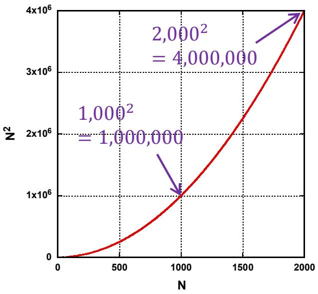
$$\hat{f}_n = \sum_{k=0}^{N-1} f_k w^{nk}$$

$$n=0 \quad \hat{f}_0 = f_0 w^{0\times 0} + f_1 w^{0\times 1} + f_2 w^{0\times 2} + f_3 w^{0\times 3}$$
$$n=1 \quad \hat{f}_1 = f_0 w^{1\times 0} + f_1 w^{1\times 1} + f_2 w^{1\times 2} + f_3 w^{1\times 3}$$
$$n=2 \quad \hat{f}_2 = f_0 w^{2\times 0} + f_1 w^{2\times 1} + f_2 w^{2\times 2} + f_3 w^{2\times 3}$$
$$n=3 \quad \hat{f}_3 = f_0 w^{3\times 0} + f_1 w^{3\times 1} + f_2 w^{3\times 2} + f_3 w^{3\times 3}$$

$$\hat{\mathbf{f}} = \mathbf{F}_4\mathbf{f} = \begin{array}{c} \\ n=0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} k=0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \end{array} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

For $N = 6$ $\longrightarrow$ $n = 0, 1, 2, \cdots, 5$ Number of elements: $6^2 = 36$

$$\hat{\mathbf{f}} = \mathbf{F}_6 \mathbf{f} = \begin{array}{c|cccccc} & k=0 & 1 & 2 & 3 & 4 & 5 \\ \hline n=0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ 1 & w^0 & w^1 & w^2 & w^3 & w^4 & w^5 \\ 2 & w^0 & w^2 & w^4 & w^6 & w^8 & w^{10} \\ 3 & w^0 & w^3 & w^6 & w^9 & w^{12} & w^{15} \\ 4 & w^0 & w^4 & w^8 & w^{12} & w^{16} & w^{20} \\ 5 & w^0 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} \end{array} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}$$

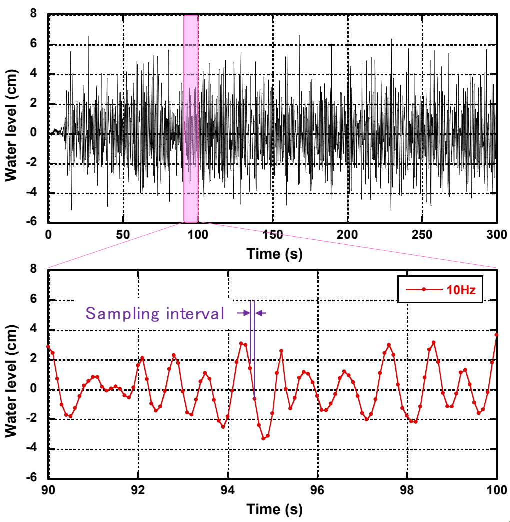


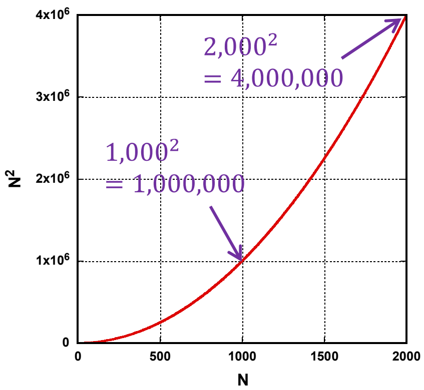
For $N = 8$ $\longrightarrow$ $n = 0, 1, 2, \cdots, 7$ Number of elements: $8^2 = 64$

$$\hat{\mathbf{f}} = \mathbf{F}_8 \mathbf{f} = \begin{array}{c|cccccccc} & k=0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline n=0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ 1 & w^0 & w^1 & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ 2 & w^0 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & w^{14} \\ 3 & w^0 & w^3 & w^6 & w^9 & w^{12} & w^{15} & w^{18} & w^{21} \\ 4 & w^0 & w^4 & w^8 & w^{12} & w^{16} & w^{20} & w^{24} & w^{28} \\ 5 & w^0 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} & w^{30} & w^{35} \\ 6 & w^0 & w^6 & w^{12} & w^{18} & w^{24} & w^{30} & w^{36} & w^{42} \\ 7 & w^0 & w^7 & w^{14} & w^{21} & w^{28} & w^{35} & w^{42} & w^{49} \end{array} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

Number of multiplications

Examples of Wave data





The number of data:
Thousands or more

Enormous calculation amount

Solution

Fast Fourier Transform (FFT)

J. W. Cooley and J. W. Tukey (1965)

Challenge

Considering a signal sampled at 22 kHz, how many multiplications are required to calculate F in the calculation $\hat{\boldsymbol{f}} = F\boldsymbol{f}$?

Solutions

X = Your solution
Form: Scientific notation with the mantissa in standard form to 2 decimal places and the exponent in integer form.
Place the indicated letter in front of the number.
Example: aX where $X = 4.0543 \times 10^{-3}$ is entered as a4.05e-3

Hash of cX = eb9210

5.7 The concepts of the FFT

Resources

The Cooley and Tukey algorithm for calculating the FFT is not that complicated, but can get very messy and it takes some time to really understand it. Therefore, we will limit ourselves to understanding some basic concepts upon which it is built.

The main concept is that many of the elements in the Fourier matrix are repetitive. For example, you can notice the symmetry about the diagonal in the F_4 matrix:

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \tag{5.8}$$

One key concept therefore is that once we have calculated an element of the matrix, we don't need to calculate it again for elements that will turn out to be the same. The key question then becomes: how do we determine which elements will be repetitions of which other elements?

The "twiddle factor" shows how elements of matrix F_N (ie, w^{nk}) are repeated. This can be visualised as points on a circle on the real-imaginary plane as the following slides from Prof. Maraso Yamashiro show.

e.g. $N = 4$ $\qquad$ w: the twiddle factor

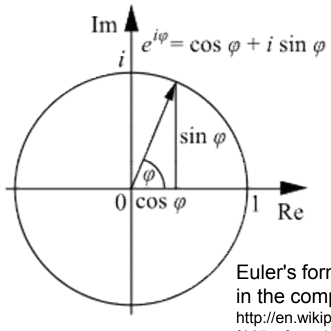
$$w^{nk} = \left(e^{-i2\pi/N}\right)^{nk}$$

$$w = e^{-i2\pi/N} = e^{-i\pi/2}$$

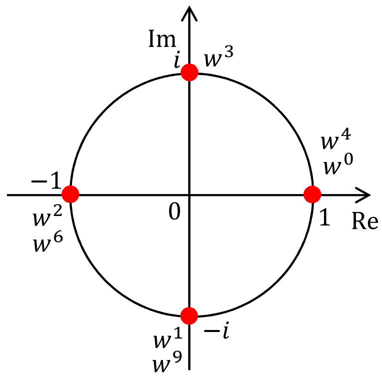
$$w^{nk} = \left(e^{-i\pi/2}\right)^{nk} = e^{-ink\pi/2}$$
$$= \cos nk\pi/2 - i\sin nk\pi/2$$

$$\left.\begin{array}{l} n = 0, 1, 2{,}3 \\ k = 0, 1, 2{,}3 \end{array}\right\} \quad nk = 0, 1, 2{,}3{,}4{,}6{,}9$$

$$w^0 = \cos 0 - i\sin 0 = 1$$
$$w^1 = \cos\pi/2 - i\sin\pi/2 = -i$$
$$w^2 = \cos\pi - i\sin\pi = -1$$
$$w^3 = \cos 3\pi/2 - i\sin 3\pi/2 = i$$
$$w^4 = \cos 2\pi - i\sin 2\pi = 1$$
$$w^6 = \cos 3\pi - i\sin 3\pi = -1$$
$$w^9 = \cos 9\pi/2 - i\sin 9\pi/2 = -i$$



Euler's formula illustrated in the complex plane
http://en.wikipedia.org/wiki/Euler%27s_formula

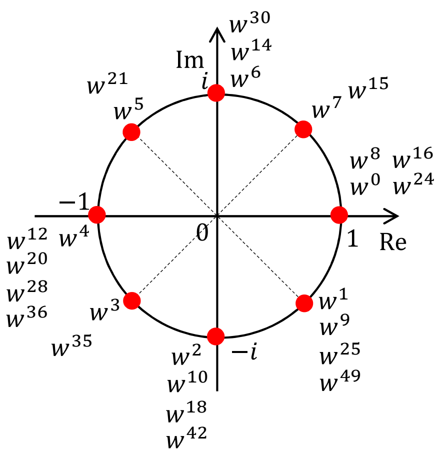


e.g. $N = 8$

$$\left.\begin{array}{l} n = 0, 1, 2, \cdots, 7 \\ k = 0, 1, 2, \cdots, 7 \end{array}\right\} \quad nk = 0, 1, 2{,}3{,}4{,}5{,}6{,}7{,}8{,}9{,}10{,}12{,}14{,}15{,}16{,}18{,}20{,}21{,}24{,}25{,}28{,}30{,}35{,}36{,}42{,}49$$

$$w^{nk} = \left(e^{-i2\pi/N}\right)^{nk} = e^{-ink\pi/4}$$
$$= \cos nk\pi/4 - i\sin nk\pi/4$$

$$w^0 = \cos 0 - i\sin 0 = 1$$
$$w^1 = \cos\pi/4 - i\sin\pi/4 = \frac{1-i}{\sqrt{2}}$$
$$w^2 = \cos\pi/2 - i\sin\pi/2 = -i$$
$$w^3 = \cos 3\pi/4 - i\sin 3\pi/4 = \frac{-1-i}{\sqrt{2}}$$
$$w^4 = \cos\pi - i\sin\pi = -1$$
$$w^5 = \cos 5\pi/4 - i\sin 5\pi/4 = \frac{-1+i}{\sqrt{2}}$$
$$w^6 = \cos 3\pi/2 - i\sin 3\pi/2 = i$$
$$w^7 = \cos 7\pi/4 - i\sin 7\pi/4 = \frac{1+i}{\sqrt{2}}$$



$$w^8 = w^{16} = w^{24} = w^0 \qquad w^{35} = w^3$$
$$w^9 = w^{25} = w^{49} = w^1 \qquad w^{12} = w^{20} = w^{28} = w^{36} = w^4$$
$$w^{10} = w^{18} = w^{42} = w^2 \qquad w^{21} = w^5 \qquad w^{14} = w^{30} = w^6 \qquad w^{15} = w^7$$

First one can notice that repeated terms are consistently even or odd. For example, for $N = 8$ it was seen that w^0, w^8, w^{16} and w^{24} all share the same value ($= 1$) while w_7 and w_{15} share the same (off-axis) value. You will notice however that simply breaking the functions into odd and even terms is not enough to unambiguously define the values of the twiddle factors, since even and odd terms can each still have multiple values.

One can take advantage of this symmetry (in ways not explained here) by taking the even-numbered terms and odd-numbered terms of the signal-vector $\boldsymbol{f}$ to create two new vectors, each of half the length of the original signal-vector:

$$\boldsymbol{f_{even}} = [f_0, f_2, \ldots, f_{N-2}]^T \tag{5.9}$$

$$\boldsymbol{f_{odd}} = [f_1, f_3, \ldots, f_{N-1}]^T \tag{5.10}$$

Then renumbering the terms back to $0, 1, 2, \ldots, N/2 - 1$:

$$\boldsymbol{f_{even}} = [f_{ev,0}, f_{ev,1}, \ldots, f_{ev,N/2-1}]^T \tag{5.11}$$

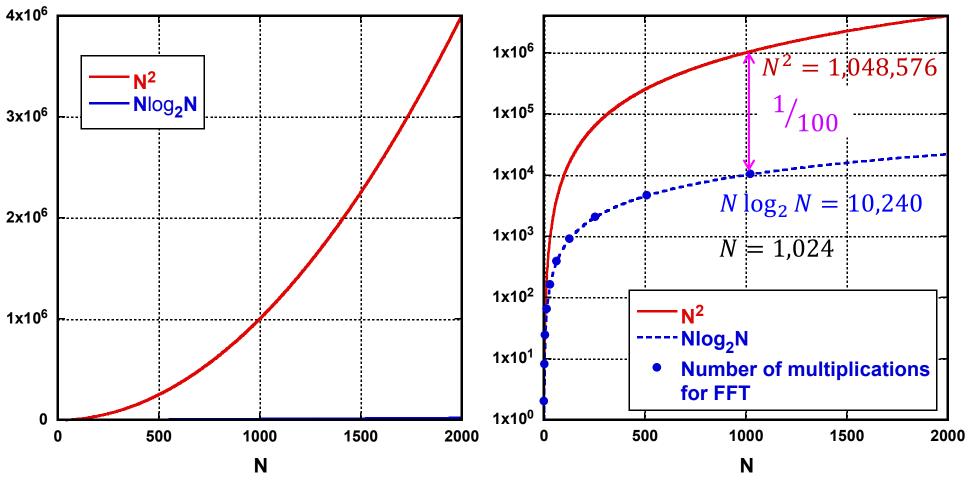
$$\boldsymbol{f_{odd}} = [f_{od,0}, f_{od,1}, \ldots, f_{od,N/2-1}]^T \tag{5.12}$$

$$\tag{5.13}$$

This can then be repeated, taking the even- and odd-numbered terms of $\boldsymbol{f_{even}}$ to make two more vectors of half-length, and doing the same for $\boldsymbol{f_{odd}}$ so we now have 4 vectors each 1/4 the length of the original signal-vector $\boldsymbol{f}$. This is repeated until there are $N/2$ vectors each of length 2.

Although beyond the scope of the explanation here, this ultimately allows one to take advantage of the repetitive nature of the twiddle factor, reducing the scaling of the number of multiplications drastically from N^2 to $Nlog_2N$, as shown in the following slide from Prof. Maraso Yamashiro:

The number of multiplications DFT vs FFT



A further point to note is that, in order to continuously divide the signal in half until each vector is length 2, the original signal must be of length 2^p where p is some integer. For example, a signal of length 8 such as $[\mathbf{1,2,3,4,5,6,7,8}]$ can be halved once into two vectors of length four ($[\mathbf{1,3,5,7}]$ and $[\mathbf{2,4,6,8}]$) and then again into four vectors of length two ($[\mathbf{1,5}]$, $[\mathbf{3,7}]$, $[\mathbf{2,6}]$, $[\mathbf{4,8}]$). If the vector is of length 9 however, such as $[\mathbf{1,2,3,4,5,6,7,8,9}]$, the signal cannot be divided into two signals of *equal length*.

This problem is overcome through the use of *zero-padding*. In the example in the previous paragraph, the signal of length 9 would simply have seven zeros added to it at the end to create a signal of total length 16 ($[\mathbf{1,2,3,4,5,6,7,8,9,0,0,0,0,0,0,0}]$), which could then be halved three times into eight vectors of length 2.

Challenge

1. Considering the signal that generated the spectrum in challenge 5.6, how many zeros must be added to the end of the sampled signal to permit FFT to be performed?

2. How many times can the padded signal be halved?

3. How many multiplications does this signal require under FFT?

4. Considering the original unpadded signal, how many times more multiplications are required under standard DFT compared to processing under FFT?

Solutions

1
X = Your solution
Form: Scientific notation with the mantissa in standard form to 2 decimal places and the exponent in integer form.
Place the indicated letter in front of the number.
Example: aX where $X = 4.0543 \times 10^{-3}$ is entered as a4.05e-3

Hash of dX = 293b72

2
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of eX = f5df85

3
491,520

4
984.70

5.8 Signal processing with Python

Comment

Here you can put your understanding to practical use by cleaning the noise from an audio signal. This gives you experience of a very common application of Fourier analysis.

Note that running the python program will result in sound being played, which can be loud. PLEASE BE VERY CAREFUL WHEN PLAYING THE SOUND FOR THE FIRST TIME, ESPECIALLY IF YOU ARE WEARING HEADPHONES.

Graphs will also be displayed in the default browser on your computer.

For Mac users: I recommend downloading the terminal software iTerm2 (`https://www.iterm2.com`). It is much easier to use than the default terminal on Macs.

When you download the files below, I recommend that you create a new directory beforehand (for example, on your desktop) and save all the files to that new directory.

Resources

- Python outline code: `http://raw.githubusercontent.com/NanoScaleDesign/FourierAnalysis/master/Audio/audio.py` *
- Noisy audio (to be read by the python code): `http://raw.githubusercontent.com/NanoScaleDesign/FourierAnalysis/master/Audio/bbcnews181210.noisy.wav` *
- Noisy audio (media-player format**): `http://raw.githubusercontent.com/NanoScaleDesign/FourierAnalysis/master/Audio/bbcnews181210.mediaformat.noisy.wav`

* If this brings up a webpage, you might need to right-click on the page and download it, ensuring the name is correct.
** Can be played in a media player such as iTunes or VLC.

Challenge

You are trying to listen to the BBC news but there is background noise that makes it difficult to hear. Your challenge is to remove the noise from the signal.

You may use the outline python code above to help you, or you may use your own code from scratch. The code above relies on the following libraries:

- sounddevice
- numpy
- scipy
- plotly

Generally, to install new libraries you can use *pip3 install sounddevice* etc.

Solutions

There will be a final output file called *bbcnews181210.mediaformat.clean.wav*. You can play this in a media player such as iTunes or VLC, and should be a clean sound without the background noise.

5.9 2D Fourier Transform

Comment

2D fourier transforms primarily come up in relation to images. The extention from 1D to 2D is fairly straightforward, and this challenge is designed to give you a basic understanding of the transition to multiple dimensions.

Note: Discrete-Time Fourier Transforms are mentioned briefly in the video resource below. This is not something we have time to cover in this course so you may ignore that part of the video. Also, we have not really discussed the inverse DFT and we will not cover this in this course apart from a programming perspective.

Resources

- Video: `https://youtu.be/NbQY1x8H6QQ?t=137` from 2:17 until 18:25.

Challenge

1. Relate the 2D Discrete Fourier Transform shown in the video to the 1D DFT you are already familiar with. What has changed?

2. What is meant by a two-dimensional basis function? Draw an example of a 2D basis function and describe its relation to the 2D DFT.

Solution

Please compare your solutions with your partner and discuss in class.

5.10 Edge detection and blurring with Fourier analysis and Python

Comment

In the 1D case, after performing DFT, low frequencies are considered to be on the left and high frequencies on the right. In 2D, the low frequencies are considered to be in the top-left corner. Due to symmetry however, modification of frequencies typically requires both altering the intended frequency as well as its symmetrical counterpart. As a consequence, the spectrum in 2D is usually shifted so that the lowest frequencies are in the centre of the image. This is done using the *fftshift* function of numpy.

Resources

- Python outline code: `http://raw.githubusercontent.com/NanoScaleDesign/FourierAnalysis/master/Image_processing/image_processing.py`
- Kyudai image: `http://raw.githubusercontent.com/NanoScaleDesign/FourierAnalysis/master/Image_processing/kyudai.png`

Challenge

Here you are going to use your Fourier Analysis knowledge and python skills to do image-processing.

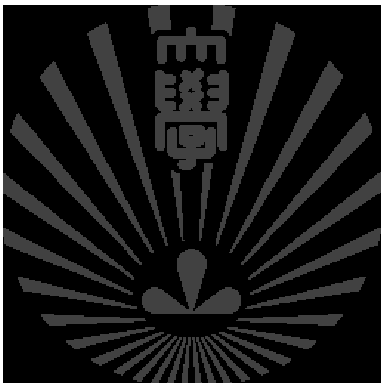
1. Convert the original image of the Kyudai logo so that only the edges of the image are shown.
2. Convert the original image of the Kyudai logo so that the image becomes blurred.
3. Write a paragraph summarising the reasoning of your approach.
4. (optional) Try with another image (requirements for pre-processing the image can be found in the python file above).

You may use the outline python code above to help you, or you may use your own code from scratch. The code above relies on the following libraries:

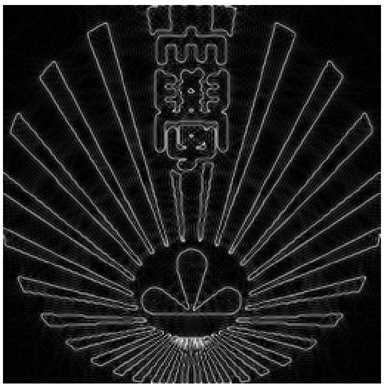
- numpy
- PIL

Generally, to install new libraries you can use *pip3 install numpy* etc.

The code above will output the shifted DFT of the image to *kyudai.fft.png* and the inverse DFT of the image to *kyudai.fft.ifft.png.*



Original image



Edge-only image

Blurred image

Solutions

You should be able to generate images that are similar to those shown above. Please compare your answer for part (3) with your partner and consider sharing any interesting images generated in part (4) with others in the class!

Appendix A

Mid-term exam questions

A.1 2017

A.1.1

Considering the equation

$$f(t) = \sin(6\pi t) + \sin(10\pi t) + \sin(30\pi t) + \sin(A\pi t) \tag{A.1}$$

1. For the case of $A = 0$, determine the fundamental *period* of $f(t)$.

2. Suggest a value of A that would make the signal non-periodic and state why your choice makes it non-periodic.

A.1.2

Considering the Fourier coefficients a_k, b_k, c_k and c_{-k} where k is a positive integer greater than 0,

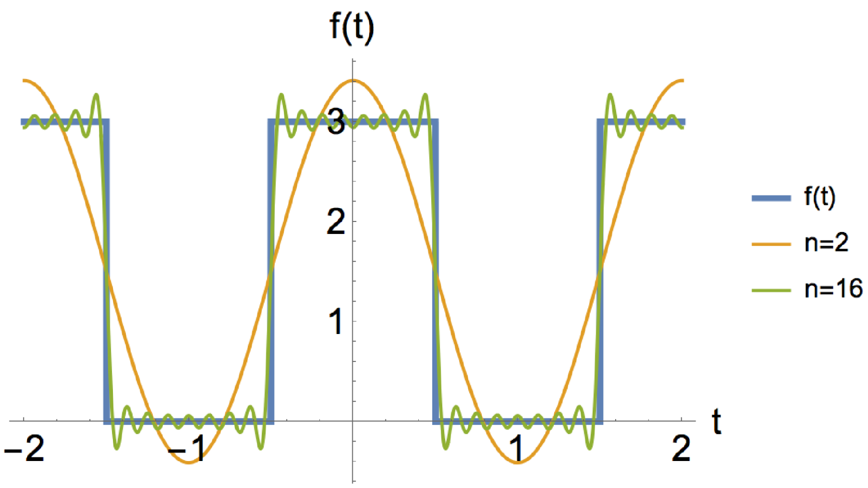
1. What is the relation between c_k and c_{-k}?

2. Which of the Fourier coefficients a_k, b_k and c_k can be complex, if any?

A.1.3

Considering the periodic squarewave described by the function

$$f(t) = \begin{cases} 3 & \text{for } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < t < \frac{3}{2} \end{cases} \tag{A.2}$$



1. Obtain the *trigonometric* Fourier series of the above function.

2. What is the common name for a series like the one you have obtained? Why do you obtain a series like this in this situation?

3. What is Gibb's phenomenon? Under what situations does it occur?

A.1.4

Considering heat on a ring of circumference 1 unit:

1. Write any two possible functions denoting initial temperature distribution on the ring. Please ensure the functional form of each of the two functions is different from the other (ie, you can't just multiply

one function by a constant and get the other function). State why you think these functions are valid descriptions of initial temperature distribution.

2. Using any function of initial temperature distribution of your choice (other than the case where the initial temperature is zero everywhere on the ring), obtain an expression for the temperature on the ring at any location and at any time. You will not lose any points for using the simplest possible initial temperature distribution you can imagine (except for zero temperature everywhere on the ring).