Example Discrete Fourier Transform (DFT)

Let $N = 4$ measurements (sample values) be given.

Then $$w = e^{-i2\pi/N} = e^{-i\pi/2}$$
$$= \cos \pi/2 - i \sin \pi/2$$
$$= -i$$

and thus $$w^{nk} = (-i)^{nk}$$

$$\hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}$$

$$\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f}$$

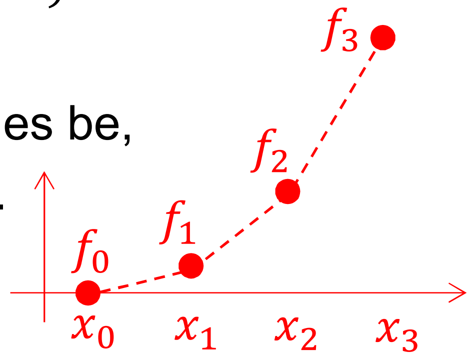
$$\mathbf{F}_\mathrm{N} = [e_{nk}]$$

$$e_{nk} = e^{-inx_k} = e^{-i2\pi nk/N} = w^{nk}$$

$$w = w_N = e^{-i2\pi/N}$$

$$e^{ix} = \cos x + i \sin x$$

Let the sample values be,
say $\mathbf{f} = [0\ \ 1\ \ 4\ \ 9]^\mathrm{T}$.



Then

$$\hat{\mathbf{f}} = \mathbf{F}_4 \mathbf{f} = \begin{array}{c} \\ n=0 \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} k=0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \end{array} \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}$$

e.g. $N = 4 \longrightarrow n = 0, 1, 2, 3$

$$\widehat{f_n} = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}$$

$$\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f}$$

$$\mathbf{F}_\mathrm{N} = [e_{nk}]$$

$$e_{nk} = e^{-inx_k} = e^{-i2\pi nk/N} = w^{nk}$$

$$w = w_N = e^{-i2\pi/N}$$

Discrete Fourier Transform

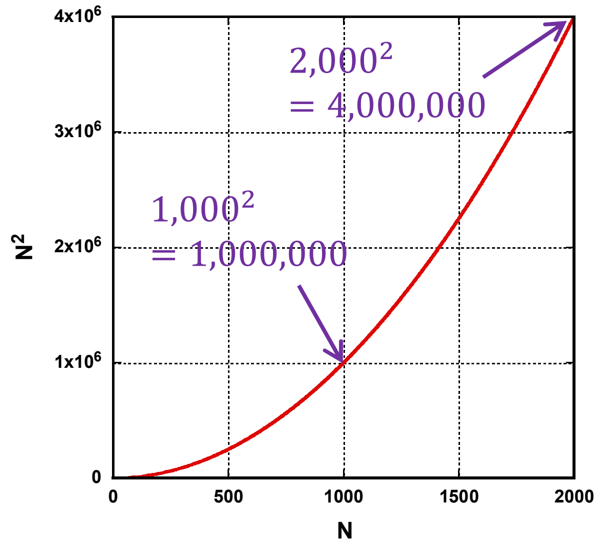
$$\widehat{f_n} = \sum_{k=0}^{N-1} f_k w^{nk}$$

$$\begin{aligned}
n = 0 \quad & \widehat{f_0} = f_0 w^{0\times0} + f_1 w^{0\times1} + f_2 w^{0\times2} + f_3 w^{0\times3} \\
n = 1 \quad & \widehat{f_1} = f_0 w^{1\times0} + f_1 w^{1\times1} + f_2 w^{1\times2} + f_3 w^{1\times3} \\
n = 2 \quad & \widehat{f_2} = f_0 w^{2\times0} + f_1 w^{2\times1} + f_2 w^{2\times2} + f_3 w^{2\times3} \\
n = 3 \quad & \widehat{f_3} = f_0 w^{3\times0} + f_1 w^{3\times1} + f_2 w^{3\times2} + f_3 w^{3\times3}
\end{aligned}$$

$$\hat{\mathbf{f}} = \mathbf{F}_4 \mathbf{f} = \begin{array}{c c} & \begin{array}{cccc} k=0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} n=0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \end{array} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

For $N = 6$ $\longrightarrow$ $n = 0, 1, 2, \cdots, 5$ Number of elements: $6^2 = 36$

$$\hat{\mathbf{f}} = \mathbf{F}_6\mathbf{f} = \begin{array}{cc} & \begin{array}{cccccc} k=0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} n=0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 & w^4 & w^5 \\ w^0 & w^2 & w^4 & w^6 & w^8 & w^{10} \\ w^0 & w^3 & w^6 & w^9 & w^{12} & w^{15} \\ w^0 & w^4 & w^8 & w^{12} & w^{16} & w^{20} \\ w^0 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} \end{bmatrix} \end{array} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}$$

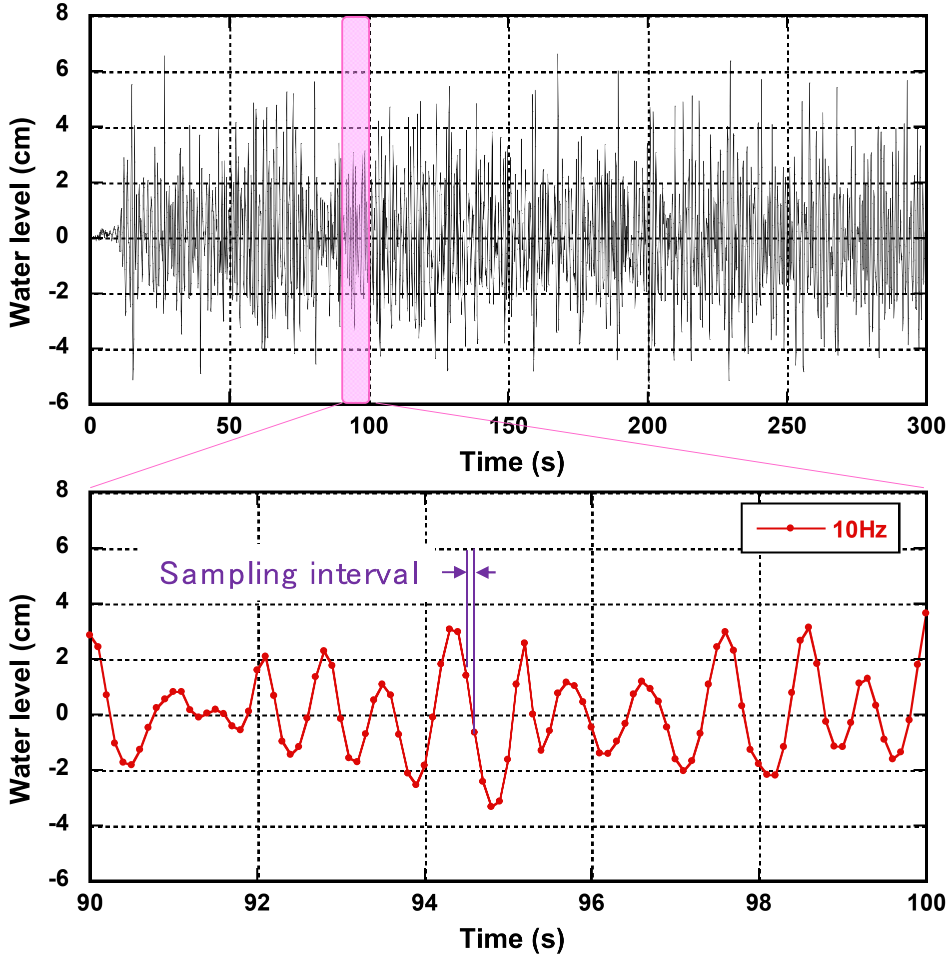


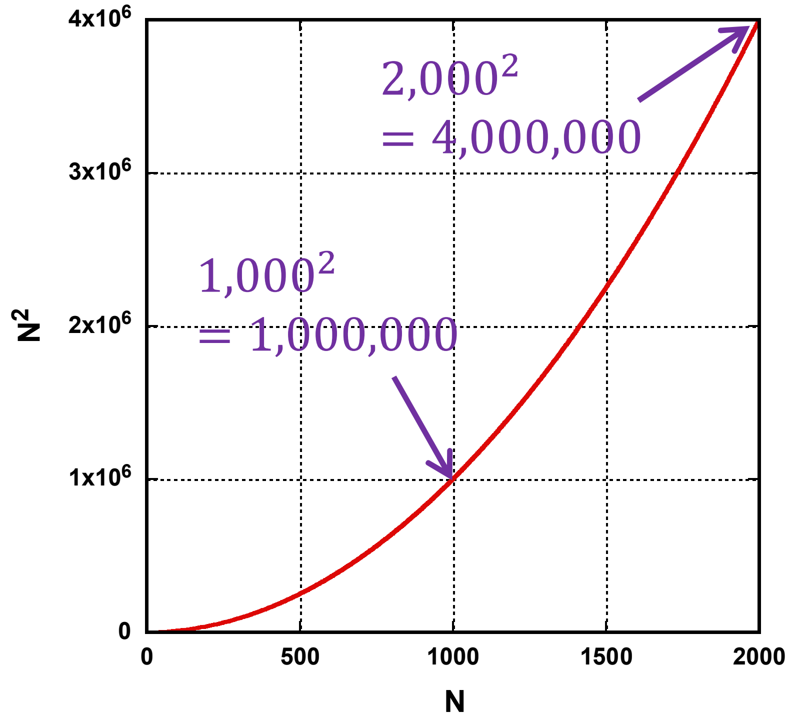
For $N = 8$ $\longrightarrow$ $n = 0, 1, 2, \cdots, 7$ Number of elements: $8^2 = 64$

Number of multiplications

$$\hat{\mathbf{f}} = \mathbf{F}_8\mathbf{f} = \begin{array}{cc} & \begin{array}{cccccccc} k=0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} n=0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ w^0 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & w^{14} \\ w^0 & w^3 & w^6 & w^9 & w^{12} & w^{15} & w^{18} & w^{21} \\ w^0 & w^4 & w^8 & w^{12} & w^{16} & w^{20} & w^{24} & w^{28} \\ w^0 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} & w^{30} & w^{35} \\ w^0 & w^6 & w^{12} & w^{18} & w^{24} & w^{30} & w^{36} & w^{42} \\ w^0 & w^7 & w^{14} & w^{21} & w^{28} & w^{35} & w^{42} & w^{49} \end{bmatrix} \end{array} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

Examples of Wave data





The number of data:
Thousands or more

Enormous calculation amount

Solution

Fast Fourier Transform (FFT)

J. W. Cooley and J. W. Tukey (1965)