Fourier Analysis

Autumn 2016

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http://www.jamescannon.net/teaching/fourier-analysis

0.1 This course

This is the Autumn 2016 Fourier Analysis course studied by 3rd-year undegraduate international students at Kyushu University.

0.1.1 How this works

- In contrast to the traditional lecture-homework model, in this course the learning is self-directed and active via publicly-available resources.
- Learning is guided through solving a series of carefully-developed challenges contained in this book (download from `http://www.jamescannon.net/teaching/fourier-analysis`), coupled with suggested resources that can be used to solve the challenges with instant feedback about the correctness of your answer.
- There are no lectures. Instead, there is discussion time. Here, you are encouraged to discuss any issues with your peers, teacher and any teaching assistants. Furthermore, you are encouraged to help your peers who are having trouble understanding something that you have understood; by doing so you actually increase your own understanding too.
- Peer discussion is encouraged, however, if you have help to solve a challenge, always make sure you do understand the details yourself. You will need to be able to do this in an exam environment. If you need additional challenges to solidify your understanding, then ask the teacher. The questions on the exam will be similar in nature to the challenges. If you can do all of the challenges, you can get 100% on the exam.
- Discussion-time is from 10:30 to 12:00 on Wednesdays at room W4-529.
- Every challenge in the book typically contains a **Challenge** with suggested **Resources** which you are recommended to utilise in order to solve the challenge. A **Solution** is made available in encrypted form. If your encrypted solution matches the encrypted solution given, then you know you have the correct answer and can move on. For more information about encryption, see section 0.2. Occasionally the teacher will provide extra **Comments** to help guide your thinking.
- For deep understanding, it is recommended to study the suggested resources beyond the minimum required to complete the challenge.
- The challenge document has many pages and is continuously being developed. Therefore it is advised to view the document on an electronic device rather than print it. The date on the front page denotes the version of the document. You will be notified by email when the document is updated.
- A target challenge and minimum challenge will be set each week, to be achieved by the beginning of the discussion time.
  - Target challenge: You are expected to complete at least up to this challenge. This is because the group learning effect is strongest when everyone is roughly around the same level of understanding.
  - Minimum challenge: Due to personal (eg health) or professional (eg conference attendance) issues, or simply difficulties with the challenges, it may occasionally not be possible to reach the target challenge. In this case, you will still be considered to be progressing normally if you achieve the minimum challenge.
  - You may work ahead, even beyond the target challenge, if you so wish. This can build greater flexibility into your personal schedule, especially as you become busier towards the end of the semester.
- Your contributions to the course are strongly welcomed. If you come across resources that you found useful that were not listed by the teacher or points of friction that made solving a challenge

difficult (there's no such thing as "you should have learned it in high-school" - you're probably not the only one with that specific problem), please let the teacher know about it!

0.1.2 Assessment

In order to prove to outside parties that you have learned something from the course, we must perform summative assessments. This will be in the form of

- Exam(s): Final exam, and possibly a mid-term exam
- Group presentation towards the end of the course (details to be announced later)

0.1.3 What you need to do

- Prepare a challenge-log in the form of a workbook or folder where you can clearly write the calculations you perform to solve each challenge. This will be a log of your progress during the course and will be occasionally reviewed by the teacher.
- You will need to maintain a google spreadsheet detailing your work and progress. The purpose of this spreadsheet is to help the teacher optimise the discussion-time. Please ensure that it is up-to-date 24 hours before each discussion-time starts. It is fine for you to continue to work on challenges and update the spreadsheet after the 24-hour deadline.
- You also need to submit a brief report at `https://goo.gl/forms/Djl4FEZcJLMpipsY2` 24 hours before the discussion time starts. Here you can let the teacher know about any difficulties you are having and if you would like to discuss anything in particular.
- Please bring a wifi-capable internet device to class, as well as headphones if you need to access online components of the course during class. If you let me know in advance, I can lend computers and provide power extension cables for those who require them (limited number).

0.1.4 Details about the spreadsheet

To get started:

1. Log into google
2. Open `http://bit.ly/2cPYyQY`
3. File → Make a copy [→ rename] → ok
4. Click "Share" (top right)
5. Click "Get shareable link"
6. Set "Anyone with the link can edit"
7. Copy sharing address
8. Send an email to cannon@mech.kyushu-u.ac.jp containing
   (a) Subject: Fourier Analysis registration
   (b) Your name
   (c) Student number
   (d) The link to your copy of the google sheet

Using the spreadsheet:

- Enter the appropriate challenge number. For example, for challenge 1.4, enter "1" in the **Section** column and "4" in the **Challenge** column.

- After successfully completing a challenge, please enter any particular friction points that you experienced (if any) so the course can be developed to reduce such friction in the future, as well as any extra resources you recommend (if any).
- Please also roughly estimate the amount of effort in **Hours** required to complete the challenge (starting from when you completed the previous challenge, including any reading, watching videos, looking for resources, writing the answer to the challenge, discussing with peers, etc). This is not used for assessment in any way, but is very valuable in helping the teacher develop the course.

Note: Please do not alter column names, ordering, etc. Just add section and challenge numbering and fill in the columns as appropriate. This is because spreadsheet data is downloaded and automatically analysed, and it breaks if anything is inconsistent.

0.2 Hash-generation

Most solutions to challenges are encrypted using MD5 hashes. In order to check your solution, you need to generate its MD5 hash and compare it to that provided. MD5 hashes can be generated at the following sites:

- Wolfram alpha: `http://www.wolframalpha.com/input/?i=md5+hash+of+%22q_1.00%22`
- `www.md5hashgenerator.com`

Since MD5 hashes are very sensitive to even single-digit variation, you must enter the solution exactly. This means maintaining a sufficient level of accuracy when developing your solution, and then entering the solution according to the format below:

Unless specified otherwise, any number from 0.00 to ±9999.99 should be represented as a normal number to two decimal places. All other numbers should be in scientific form. See the table below for examples.

Solution	**Input**
1	1.00
-3	-3.00
-3.5697	-3.57
0.05	0.05
0.005	5.00e-3
50	50.00
500	500.00
5000	5000.00
50,000	5.00e4
5×10^{-476}	5.00e-476
5.0009×10^{-476}	5.00e-476
$-\infty$	-infinity (never “infinite”)
2π	6.28
i	im(1.00)
2i	im(2.00)
1 + 2i	re(1.00)im(2.00)
1/i = i/-1 = -i	im(-1.00)
$e^{i2\pi}$ $[= cos(2\pi) + isin(2\pi) = 1 + i0 = 1]$	1.00
$e^{i\pi/3}$ $[= cos(\pi/3) + isin(\pi/3) = 0.5 + i0.87]$	re(0.50)im(0.87)
Choices in order A, B, C, D	abcd

Entry format is given with the problem. So “q_X” means to enter “q_X” replacing “X” with your solution. The first 6 digits of the MD5 sum should match the given solution (MD5(q_X)= . . .).

Note that although some answers can usually only be integers (eg, number of elephants), for consistency, to generate the correct hash, the accuracy in terms of decimal places noted above is required.

Contents

Chapter 1

Periods and frequencies

1.1 Period at 50 THz

Resources

- Book: 1.2 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 2 (https://www.youtube.com/watch?v=1rqJl7Rs6ps)

Challenge

A signal is oscillating at a frequncy of 50 THz. What is the period?

Solution

X s

MD5(q_X) = 33efc2...

1.2 Fundamental period with k=1

Resources

- Book: 1.3 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Challenge

What is the fundamential period of $sin(2\pi kt)$, where t is time in seconds and $k = 1$?

Solution

X s

MD5(w_X) = 9ae2db...

1.3 Frequency with k=1

Resources

- Book: 1.3 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 2 (https://www.youtube.com/watch?v=1rqJl7Rs6ps)

Challenge

What is the frequency of $sin(2\pi kt)$, where t is time in seconds and $k = 1$?

Solution

X Hz

MD5(e_X) = 453c99...

1.4 Frequency with k=2

Resources

- Book: 1.3 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Challenge

What is the frequency of $sin(2\pi kt)$, where t is time in seconds and $k = 2$?

Solution

X Hz

MD5(r_X) = 111802...

1.5 Fundamental period with k=2

Resources

- Book: 1.3 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 2 (https://www.youtube.com/watch?v=1rqJl7Rs6ps)

Challenge

What is the fundamental period of $sin(2\pi kt)$, where t is time in seconds and $k = 2$?

Solution

X s

MD5(t_X) = c4c8be...

1.6 Fundamental period with multiple terms

Resources

- Book: 1.3-1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

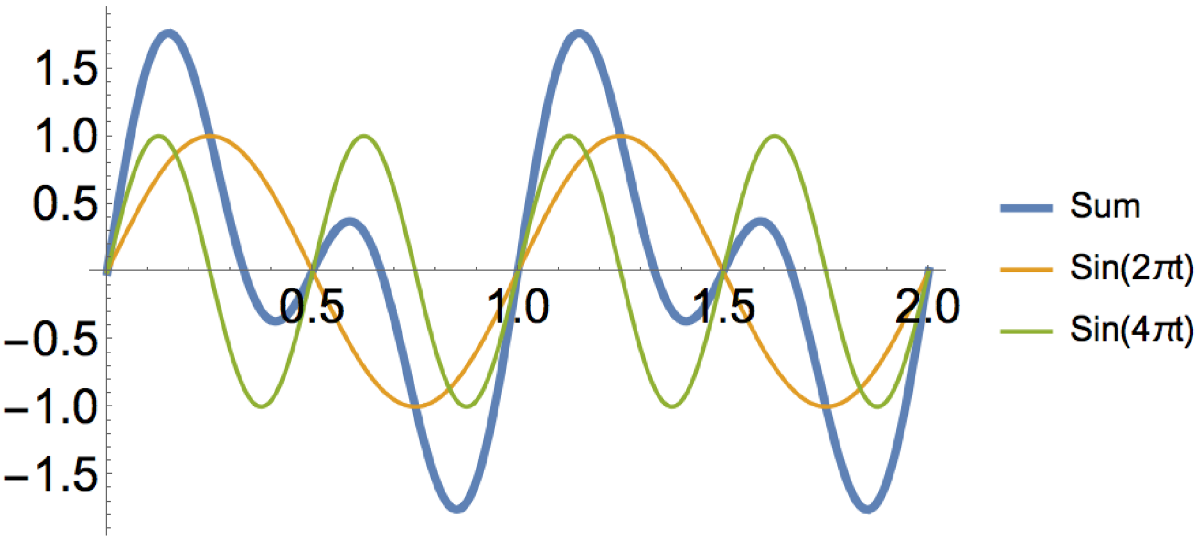
Comments

So k here is proportional to the frequency. Double k and the frequency doubles. Every "step" around the circle drawn by the sine curve becomes 2 steps when $k = 2$, so within half the time you are already one time around the circle for $k = 2$, and thus the number of times you go around the circle during one second (measured with t) is twice rather than once. k is also inversely proportional to the period. Since with $k = 2$ every "step" is now twice as large, so one completes the circle in half the time.

Even with multiple terms (frequencies), the period of the composite signal is always that of the highest (longest) period (lowest frequency), even if it is composed of multiple frequencies. You have to wait for every part of the signal to complete before a single period is complete. Ie, it is possible to add new frequencies to a signal without the period changing.

Challenge

What is the fundamental period of $sin(2\pi t) + sin(4\pi t)$, where t is time in seconds?



Solution

X s

MD5(y_X) = 80681b...

1.7 Amplitude

Resources

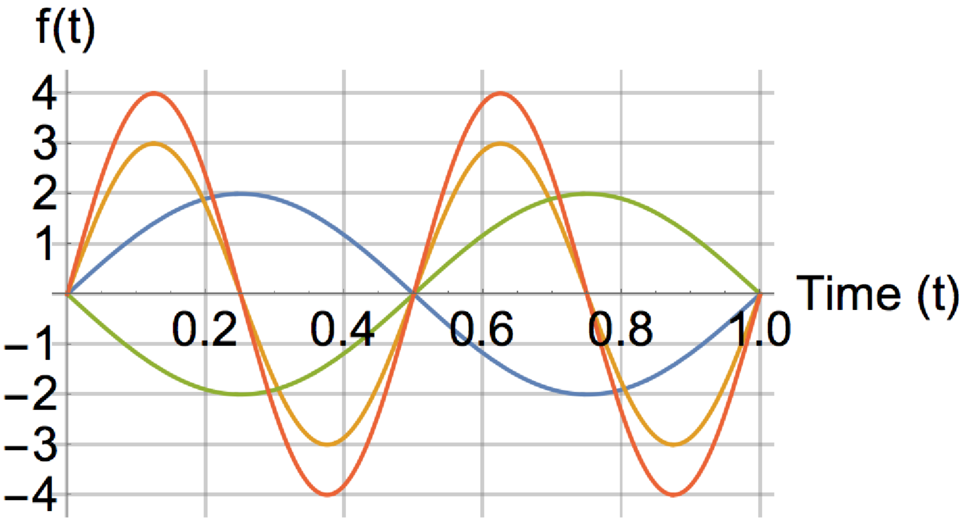
- Book: 1.3 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Comments

Another important concept is amplitude. $Sin(2\pi t)$ has an amplitude of 1, but this can be easily modified to go between $\pm A$ by multiplication with A.

Challenge

The following 4 graphs are of the form $ASin(2\pi kt)$ with variation in the values of A and k *only*. What is the sum of the values of A for the following graphs?



Solution

X

MD5(u_X) = 7bcfe4...

1.8 Phase

Resources

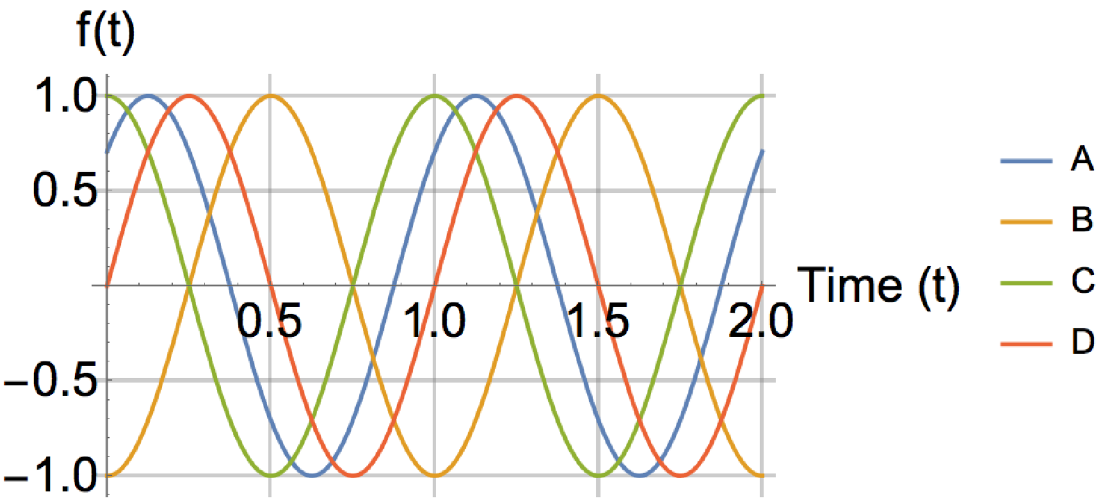
- Book: 1.3 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)
- Video: Lecture 2 (https://www.youtube.com/watch?v=1rqJl7Rs6ps)

Comments

Another important concept is phase. For a simple Sine signal $\theta(t) = sin(2\pi t)$, at $t = 0$ the angle θ is zero, but one can shift the phase (starting point) of the signal by effectively making the sine-curve non-zero at $t = 0$. Another way to think about it is to say the sine curve doesn't reach zero until a time $t - \phi$ where ϕ is the phase-shift added.

Challenge

The following 4 graphs are of the form $Sin(2\pi t + \phi)$ where ϕ is the phase-shift. Put the graphs in order corresponding to the following order of phase-shifts: $\pi/2$, $-\pi/2$, $\pi/4$, 2π.



Solution

X

MD5(i_X) = 0efd66...

Chapter 2

Fourier Series

2.1 Introduction to Fourier coefficients

Resources

- Book: 1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Challenge

Deduce in a simple way the Fourier coefficients a_1 and b_1 in the Fourier series

$$\sum_{k=1}^{N} a_k cos(2\pi kt) + b_k sin(2\pi kt) \tag{2.1}$$

for a signal made up of multiple sine signals

$$\sum_{k=1}^{N} A_k sin(2\pi kt + \phi_k) \tag{2.2}$$

for the following cases:

1. $N = 1$, $k = 1$, $A_1 = 1$, $\phi_1 = 0$
2. $N = 1$, $k = 1$, $A_1 = 1$, $\phi_1 = \pi/2$
3. $N = 1$, $k = 1$, $A_1 = 1$, $\phi_1 = \pi/5$

Hint: Using $sin(A + B) = sin(A)cos(B) + cos(A)sin(B)$ it should is possible to find the answer without resorting to complex calculation.

Solutions

1. MD5(o_a_k) = a2c1fe..., MD5(p_b_k) = de80c6...
2. MD5(a_a_k) = 718a6c..., MD5(s_b_k) = f86f0c...
3. MD5(d_a_k) = 93d647..., MD5(f_b_k) = 9a7b58...

2.2 Even and odd functions

Resources

- Wikipedia: `https://en.wikipedia.org/wiki/Even_and_odd_functions`

Challenge

Referring in part to the cases in challenge 2.1, sum the points of all the following TRUE statements:

1 point: Case 1 is an odd function

2 points: Case 1 is an even function

4 points: Case 2 is an odd function

8 points: Case 2 is an even function

16 point: Case 3 is an odd function

32 points: Case 3 is an even function

64 points: $f(x) = Sin(x)$ is an odd function

128 points: $f(x) = Sin(x)$ is an even function

256 points: $f(x) = Cos(x)$ is an odd function

512 points: $f(x) = Cos(x)$ is an even function

1024 points: $f(x) = x$ is an odd function

2048 points: $f(x) = x$ is an even function

Solution

X

MD5(g_X) = 6a18c0...

2.3 Fourier Coefficients of sin(x)

Resources

- Book: 1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Comments

You should be able to follow the derivation of the formula for Fourier coefficients (C_k's) in the video. Feel free to seek help if you have trouble.

Challenge

By writing $sin(x)$ in exponential form, deduce the Fourier coefficients (C_k's) for the function $sin(x)$, for the following cases:

1. k = -1
2. k = 0
3. k = 1

Solutions

k=-1

X

MD5(h_X) = 28f251...

k=0

X

MD5(j_X) = 4fd3f6...

k=1

X

MD5(k_X) = e82a2a...

2.4 Fourier Coefficients of 1 + sin(x)

Resources

- Book: 1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Comments

You should be able to follow the derivation of the formula for Fourier coefficients in the video. Feel free to seek help if you have trouble.

Challenge

Continuing from challenge 2.3, deduce the Fourier coefficients (C_k's) for the function $1 + sin(x)$, for the following cases:

1. k = -1
2. k = 0
3. k = 1

Solutions

k=-1

X

MD5(z_X) = 39e026...

k=0

X

MD5(x_X) = 0ef183...

k=1

X

MD5(c_X) = fda796...

2.5 Relation of positive and negative Fourier coefficients for a real signal

Resources

- Book: 1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Challenge

If the Fourier coefficient C_1 is $4 + 6i$, what is the Fourier coefficient C_{-1}?

Solution

X

MD5(m_X) = 36ab38...

2.6 Converting between trigonometric and exponential forms

Resources

- Book: 1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Challenge

Derive an expression for C_k in terms of the a_n's and b_n's in the expression

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{k=N} a_k cos(2\pi kt) + b_n sin(2\pi kt) = \sum_{k=-N}^{k=N} C_k e^{2\pi ikt} \tag{2.3}$$

To check your answer, substitute $a_0 = 1$, $a_1 = 3$ and $b_1 = 5$ as required to calculate C_k for k = -1, 0 and 1.

Solutions

k=-1

X

MD5(v_X) = 052df3...

k=0

X

MD5(b_X) = fb29ff...

k=1

X

MD5(n_X) = 284b53...

2.7 The Fourier series of $f(t) = t$: C_0

Resources

- Book: 1.5, 1.7 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 3 (`https://www.youtube.com/watch?v=BjBb5IlrNsQ`)

Challenge

Considering the function $f(t) = t$ over the interval 0 to 1, calculate the Fourier coefficient C_0 using the derived formula for Fourier coefficients. Compare with the average over the interval.

Solution

X

MD5(aa_X) = 2708ad...

2.8 The Fourier series of $f(t) = t$: C_k

Resources

- Book: 1.5, 1.7 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 3 (`https://www.youtube.com/watch?v=BjBb5IlrNsQ`)

Challenge

Considering the function $f(t) = t$, calculate a general expression for the Fourier coefficients C_k where $k \neq 0$.

To check your answer, evaluate the Fourier coefficient for $k = -30$.

Solution

X

MD5(bb_X) = e904e9...

2.9 The Fourier series of $f(t) = t$ in exponential form

Resources

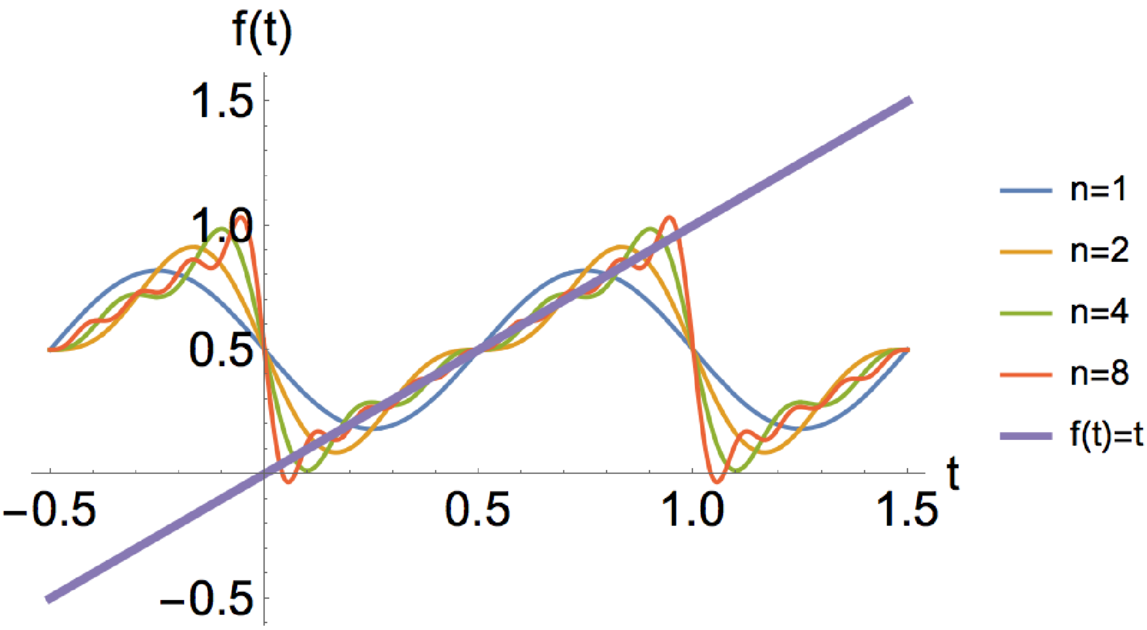
- Book: 1.4 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 3 (`https://www.youtube.com/watch?v=BjBb5IlrNsQ`)

Challenge

Write the function $f(t) = t$ in terms of its infinite exponential Fourier series.

To check your answer, evaluate the Fourier series up to $n = 2$ with $t = 0.8$.

The graph with increasing values of n looks like this:



Solution

X

MD5(cc_X) = d21e19...

2.10 The Fourier series of $f(t) = t$ in trigonometric form

Resources

- Book: 1.5, 1.7 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)
- Video: Lecture 2 (`https://www.youtube.com/watch?v=1rqJl7Rs6ps`)

Comment

Many textbooks will work in terms of "Fourier sine series" and "Fourier cosine series". For a function that is perfectly even or odd, it is possible to write a Fourier series using one of these two forms. There are direct approaches to calculating sine and cosine Fourier series, in contrast to the route taken here via the exponential form. We will continue to work in the exponential form however, since not only does this provide a deeper understanding, but you can always easily switch between exponential and trigonometric forms if you really want to.

Challenge

Re-write the series obtained in challenge 2.9 in terms of a trigonometric infinite series.

To check your answer, evaluate the Fourier series up to $n = 2$ with $t = 0.8$ and ensure that you get the same answer as you did for challenge 2.9.

2.11 Periods other than unity

Resources

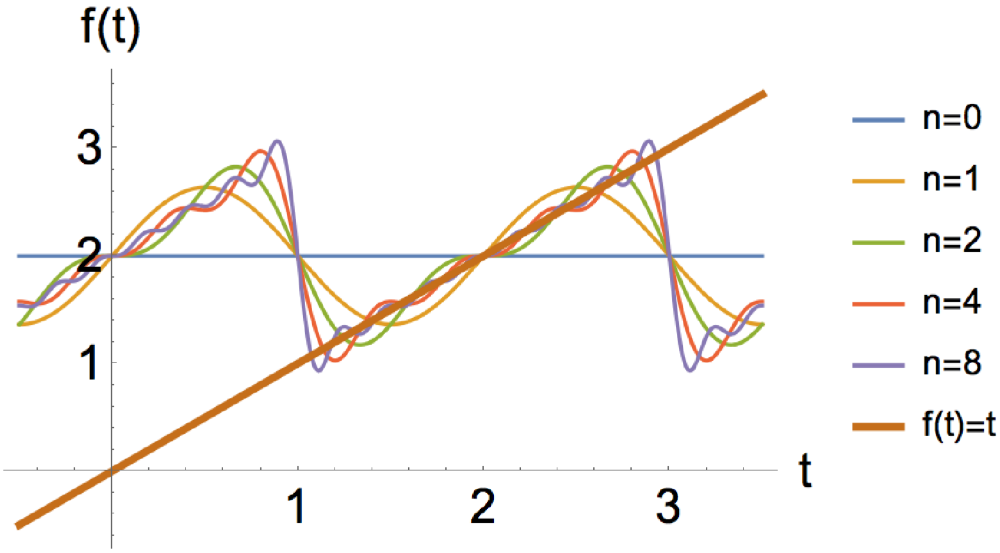
- Book: 1.6 (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)

Challenge

Determine the Fourier series for the same function as in 2.9 ($f(t) = t$), except approximate the function over the region $1 < x < 3$ instead of $0 < x < 1$.

To check your answer, evaluate the Fourier series up to $n = 2$ with $t = 1.8$.

The graph with increasing values of n looks like this:



Solution

X

MD5(dd_X) = 83e943...

2.12 Infinite series

Comment

It is important to understand why some series are infinite, while others are not (well, technically all series are infinite since they all involve sums to $n = \infty$, however for some series the Fourier coefficients are all zero above a certain value of n). Therefore, make sure you understand why the answer is as it is, below. If you don't, be sure to discuss with others.

Challenge

The Fourier series is a sum to $\pm\infty$, however in some cases the coefficients (C_k's) are zero beyond a certain number of terms. Which of the terms below will have Fourier coefficients that are all zero after a certain number of terms? Sum the points of these functions.

1 point: x

2 points: x^2

4 points: $cos(2x) + 3sin(7x)$

8 points: $e^{2\pi ix}$

Solution

X

MD5(ee_X) = 8e05a4...

2.13 k-symmetry

Challenge

Determine what X and Y represent algebraically.

$$cos(k\pi t) = \frac{1}{2}e^{-ki\pi t} + \frac{1}{2}e^{\boldsymbol{X}i\pi x} \tag{2.4}$$

$$sin(k\pi t) = \frac{1}{2}ie^{\boldsymbol{Y}i\pi t} - \frac{1}{2}ie^{ki\pi t} \tag{2.5}$$

To check your answers you may substitute any appropriate values from the following list: $k = 2$, $t = 1$

Solution

X

MD5(ff_X) = 942d6f...

Y

MD5(gg_Y) = a379b8...

2.14 Direct trigonometric calculation of a Fourier series: the coefficients

Comment

This challenge introduces several key concepts at once, including decoupling of integral intervals and periodicity, the concept of a square wave and direct trigonometric evaluation of Fourier series. If you can master this you'll be in a really strong position.

It is hopefully clear now that for real signals, due to the symmetry of the positive and negative k's, one can fully compose Fourier series in terms of sine and cosine. In challenge 2.6 we saw the formula for the function in terms of Fourier coefficients a_0, a_n and b_n. While we will not use this approach, it is important to be able to utilise such a formulation since this is the way some books present it and some people have learnt it. Therefore, without proof, the coefficients can be calculated using

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)Cos(2\pi kt/T) \tag{2.6}$$

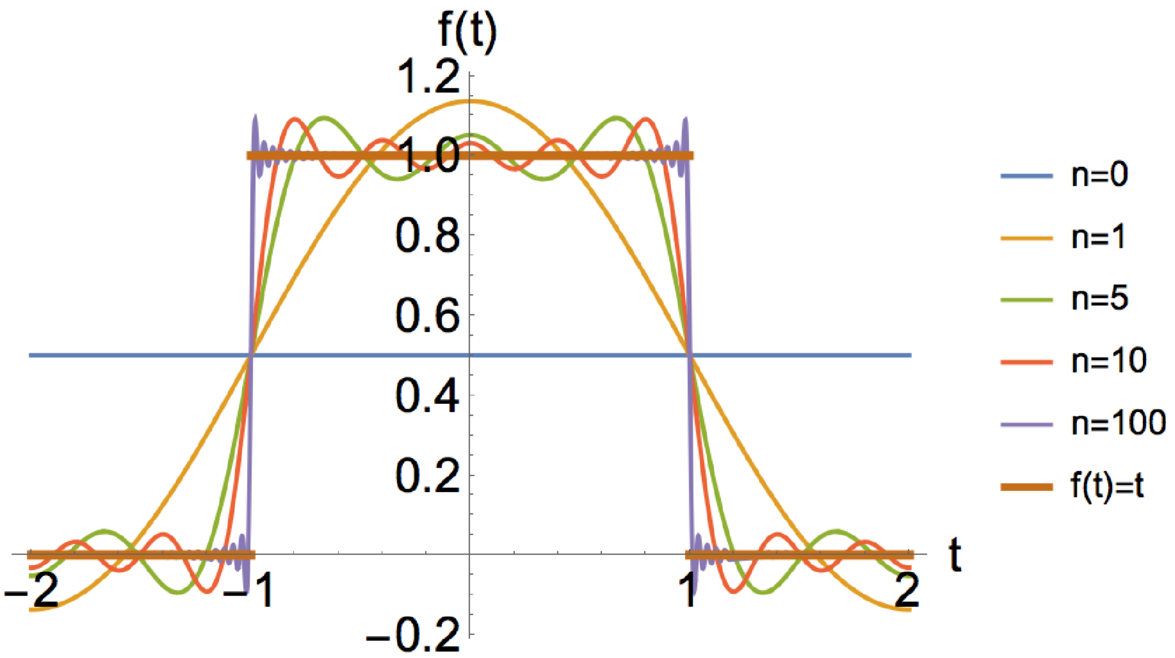
$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)Sin(2\pi kt/T) \tag{2.7}$$

Challenge

Using the direct trigonometric Fourier series, obtain a general expression for the a_k and b_k coefficients for the square-wave signal with periodicity 4

$$f(t) = \begin{cases} 1 & \text{for } -1 < t < 1 \\ 0 & \text{for } 1 < t < 3 \end{cases} \tag{2.8}$$

A graph of the function, including the solution for various values of n, is shown here:



Note the symmetry of the problem. Can you see what terms will be zero? To check your solution, calculate a_k and b_k for $k = 0$, $k = 2$ and $k = 3$. Also note that you will have to break the integrals into two parts and sum them in order to tackle this problem.

Solution

k	a_k	b_k
0	MD5(hh_X)=e57c15...	MD5(ii_X)=377fe2...
2	MD5(jj_X)=54aaa1...	MD5(kk_X)=be063f...
3	MD5(mm_X)=b8fce7...	MD5(nn_X)=b6fbaf...

2.15 Direct trigonometric calculation of a Fourier series: the series

Challenge

Calculate the Fourier series for the square wave introduced in challenge 2.14 using direct trigonometric calculation for up to $n = 3$. Check your solution by evaluating for $t = 0.1$.

Solution

X

MD5(oo_X) = bd7e5e...

2.16 2D orthogonal vectors

Resources

- Book: 1.9 (`https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf`)

Challenge

Sum the points of the vectors in 2D that are orthogonal:

1 point: (5, 4) and (-1, 1.25)

2 points: (2, -3) and (-6, 4)

4 points: (-2.25, 1.5) and (2, 3)

8 points: (4.5, 4) and (3, -3.375)

16 points: (6, 4) and (4, -6)

32 points: (5, 1) and (-2, 8.125)

64 points: (0, 1) and (1, 0)

128 points: (1, 1) and (1, 1)

Solution

X

MD5(pp_X) = 92843f…