Ordinary Differential Equations

Autumn 2018

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http://www.jamescannon.net/teaching/ordinary-differential-equations

http://raw.githubusercontent.com/NanoScaleDesign/OrdinaryDifferentialEquations/master/ode.pdf

Contents

Chapter 0

Course information

0.1 This course

This is the Autumn 2018 Ordinary Differential Equations course studied by 2nd-year undegraduate international students at Kyushu University.

0.1.1 How this works

- In contrast to the traditional lecture-homework model, in this course the learning is self-directed and active via publicly-available resources.
- Learning is guided through solving a series of carefully-developed challenges contained in this book, coupled with suggested resources that can be used to solve the challenges with instant feedback about the correctness of your answer.
- There are no lectures. Instead, there is discussion time. Here, you are encouraged to discuss any issues with your peers, teacher and any teaching assistants. Furthermore, you are encouraged to help your peers who are having trouble understanding something that you have understood; by doing so you actually increase your own understanding too.
- Discussion-time is from 08:40 to 10:10 on Mondays at room Centre Zone 1409.
- Peer discussion is encouraged, however, if you have help to solve a challenge, always make sure you do understand the details yourself. You will need to be able to do this in an exam environment. If you need additional challenges to solidify your understanding, then ask the teacher. The questions on the exam will be similar in nature to the challenges. If you can do all of the challenges, you can get 100% on the exam.
- Every challenge in the book typically contains a **Challenge** with suggested **Resources** which you are recommended to utilise in order to solve the challenge. Occasionally the teacher will provide extra **Comments** to help guide your thinking. A **Solution** is also made available for you to check your answer. Sometimes this solution will be given in encrypted form. For more information about encryption, see section 0.3.
- For deep understanding, it is recommended to study the suggested resources beyond the minimum required to complete the challenge.
- The challenge document has many pages and is continuously being developed. Therefore it is advised to view the document on an electronic device rather than print it. The date on the front page denotes the version of the document. You will be notified by email when the document is updated. The content may differ from last-year's document.
- A target challenge will be set each week. This will set the pace of the course and defines the examinable material. It's ok if you can't quite reach the target challenge for a given week, but then you will be expected to make it up the next week.
- You may work ahead, even beyond the target challenge, if you so wish. This can build greater flexibility into your personal schedule, especially as you become busier towards the end of the semester.
- Your contributions to the course are strongly welcomed. If you come across resources that you found useful that were not listed by the teacher or points of friction that made solving a challenge difficult, please let the teacher know about it!

0.1.2 Assessment

In order to prove to outside parties that you have learned something from the course, we must perform summative assessments. This will be in the form of a mid-term exam (weighted 30%), coursework (weighted 15%), a satisfactory challenge-log (weighted 5%) and a final exam (weighted 50%).

Your final score is calculated as Max(final exam score, weighted score), however you must pass the final exam to pass the course.

0.1.3 What you need to do

- Prepare a challenge-log in the form of a workbook or folder where you can clearly write the calculations you perform to solve each challenge. This will be a log of your progress during the course and will be occasionally reviewed by the teacher.
- You need to submit a brief report at `https://goo.gl/forms/AqTAZ6D1exFbH1PW2` by 4am on the day of the class. Here you can let the teacher know about any difficulties you are having and if you would like to discuss anything in particular.
- Please bring a wifi-capable internet device to class, as well as headphones if you need to access online components of the course during class. If you let me know in advance, I can lend computers and provide power extension cables for those who require them (limited number).

0.2 Timetable

	Discussion	Target	Note
1	9 Oct	-	Tuesday class
2	15 Oct	3.2	
3	22 Oct	3.11	
4	29 Oct		
5	5 Nov		
6	12 Nov		
7	19 Nov		
8	3 Dec		
9	10 Dec	Midterm exam	
10	17 Dec		
11	7 Jan		
12	15 Jan		Coursework assignment
13	21 Jan		
14	28 Jan	Coursework	Coursework submission
15	4 Feb	Final exam	

Example: To keep pace with the course, you should aim to complete challenge 2 of chapter 3 by the 13th of October.

0.3 Hash-generation

Some solutions to challenges are encrypted using MD5 hashes. In order to check your solution, you need to generate its MD5 hash and compare it to that provided. MD5 hashes can be generated at the following sites:

- Wolfram alpha: (For example: md5 hash of "q1.00") `http://www.wolframalpha.com/input/?i=md5+hash+of+%22q1.00%22`
- `www.md5hashgenerator.com`

Since MD5 hashes are very sensitive to even single-digit variation, you must enter the solution *exactly*. This means maintaining a sufficient level of accuracy when developing your solution, and then entering the solution according to the format suggested by the question. Some special input methods:

Solution	**Input**
5×10^{-476}	5.00e-476
5.0009×10^{-476}	5.00e-476
$-\infty$	-infinity (never "infinite")
2π	6.28
i	im(1.00)
2i	im(2.00)
1 + 2i	re(1.00)im(2.00)
-0.0002548 i	im(-2.55e-4)
1/i = i/-1 = -i	im(-1.00)
$e^{i2\pi}$ [$= cos(2\pi) + isin(2\pi) = 1 + i0 = 1$]	1.00
$e^{i\pi/3}$ [$= cos(\pi/3) + isin(\pi/3) = 0.5 + i0.87$]	re(0.50)im(0.87)
Choices in order A, B, C, D	abcd

The first 6 digits of the MD5 sum should match the first 6 digits of the given solution.

0.4 Questions about the final exam

Will the final exam cover the entire course or only the course content between the mid-term exam and the end of the course?

The final exam will cover the entire course.

Do we need to memorise formulae like that for the Runge-Kutta method?

The aim is to test understanding rather than the ability to memorise formulae. So if you need to use the Runge-Kutta method I will supply the forumula for it. That said, you will need to remember basic methods that are fundamental to the basis of solving ODE's, such as the characteristic solutions to 2nd-order differential equations.

Will it be stated if we should use method X or method Y to solve an ODE?

If it doesn't specifically state how to solve a problem, then you're welcome to use whatever method you find easiest. If it states how you should solve a problem then you should use the method indicated.

Chapter 1

Hash practise

1.1 Hash practise: Integer

X = 46.3847
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

hash of aX = e77fac

1.2 Hash practise: Decimal

X = 49
Form: Two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

hash of bX = 82c9e7

1.3 Hash practise: String

X = abcdef
Form: String.
Place the indicated letter in front of the number.
Example: aX where $X = abc$ is entered as aabc

hash of cX = 990ba0

1.4 Hash practise: Scientific form

X = 500,765.99
Form: Scientific notation with the mantissa in standard form to 2 decimal place and the exponent in integer form.
Place the indicated letter in front of the number.
Example: aX where $X = 4 \times 10^{-3}$ is entered as a4.00e-3

hash of dX = be8a0d

Chapter 2

Definitions

2.1 Order of a differential equation

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

What is the sum of the orders of the following equations?

$$\frac{dy}{dx}A = 5x^3 + 3 \tag{2.1}$$

$$cos(y)y'''(x) - y(x) = 25 \tag{2.2}$$

$$\frac{d}{dx}\frac{d^2y}{dx^2} = \frac{x^{-2}}{3} \tag{2.3}$$

Solution

X = Your solution
Form: Integer
Place the indicated letter in front of the number
Example: aX where $X = 46$ is entered as a46

hash of eX = 492585

2.2 Identifying linear and non-linear differential equations

Comment

Being able to identify linear and non-linear ODE's will help you understand how to approach different problems.

Generally speaking, the differential equation is linear if the functions and orders of the differentials are linear. For example,

$$y'' - 4yx = lnx - y$$

can be shown to be linear. Rearranging to collect all the y-terms together:

$$y'' - 4yx + y = lnx$$

the dependent variable y and its derivatives are each of the first degree and depend only on a constant or the independent variable.

An example of a non-linear equation however would be

$$5 + yy' = x - y$$

or

$$yy' + y = x - 5$$

The fact that y' is multiplied by y results in a non-linear equation in y.

Challenge

Sum the points corresponding to the equations that are linear. You may be able to judge some by eye, but you should prove mathematically that at least one of the equations are linear and at least one of the equations are non-linear.

1 point: $\frac{dy}{dt} = 5t^3 + 3$.

2 points: $cos(y)y'''(t) - y(t) = 25$.

4 points: $\frac{d}{dt}\frac{d^2y}{dt^2} = \frac{t^{-2}}{3}$.

8 points: $y'(t) - sin(y(t)) = 0$.

16 points: $y'(t) - y(t) = 0$.

32 points: $ty'(t) - y(t) = 0$.

Solution

X = Your solution
Form: Integer
Place the indicated letter in front of the number
Example: aX where $X = 46$ is entered as a46

hash of rX = f5d2c0

2.3 Linear differential equations vs non-linear differential equations

Resources

- Wikipedia: `https://en.wikipedia.org/wiki/Nonlinear_system#Nonlinear_differential_equations`
- Wikipedia: `https://en.wikipedia.org/wiki/Linear_differential_equation`

Challenge

Write no-more than 1 short paragraph describing in qualitative terms the difference between a linear and non-linear differential equation.

Solution

Please compare with your partner in class and discuss with the teacher if you are unsure.

2.4 Valid solutions

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Use substitution to prove that

$$y = \frac{5}{5 + x} \tag{2.4}$$

is a solution to the equation

$$xy' + y = y^2 \tag{2.5}$$

and state the value of x for which the solution is undefined.

Solution

Value of x for which solution is undefined:

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of tX = 829f33

2.5 Range of valid solutions

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Use substitution to prove that

$$y = -\sqrt{100 - x^2} \tag{2.6}$$

is a solution to the equation

$$x + yy' = 0 \tag{2.7}$$

and state the range of x for which the solution is valid. Enter the value of the lower range as the solution below.

Solution

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of yX = d96920

Chapter 3

1st-order differential equations

3.1 Determining a simple DE from a description

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference with the ambient surroundings. (a) Write a differential equation describing this situation. (b) Assuming a proportionality constant of 0.2 /hour, what is the rate of temperature change when the object is at 30 °C and the ambient temperature is 20 °C?

Solution

(units: $^{\circ}\mathrm{C}\,\mathrm{h}^{-1}$)

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of qX = 4aca8d

3.2 Direction (Slope) fields

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/DirectionFields.aspx`
- Video 1: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/creating-a-slope-field`
- Video 2: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/slope-field-to-visualize-solutions`

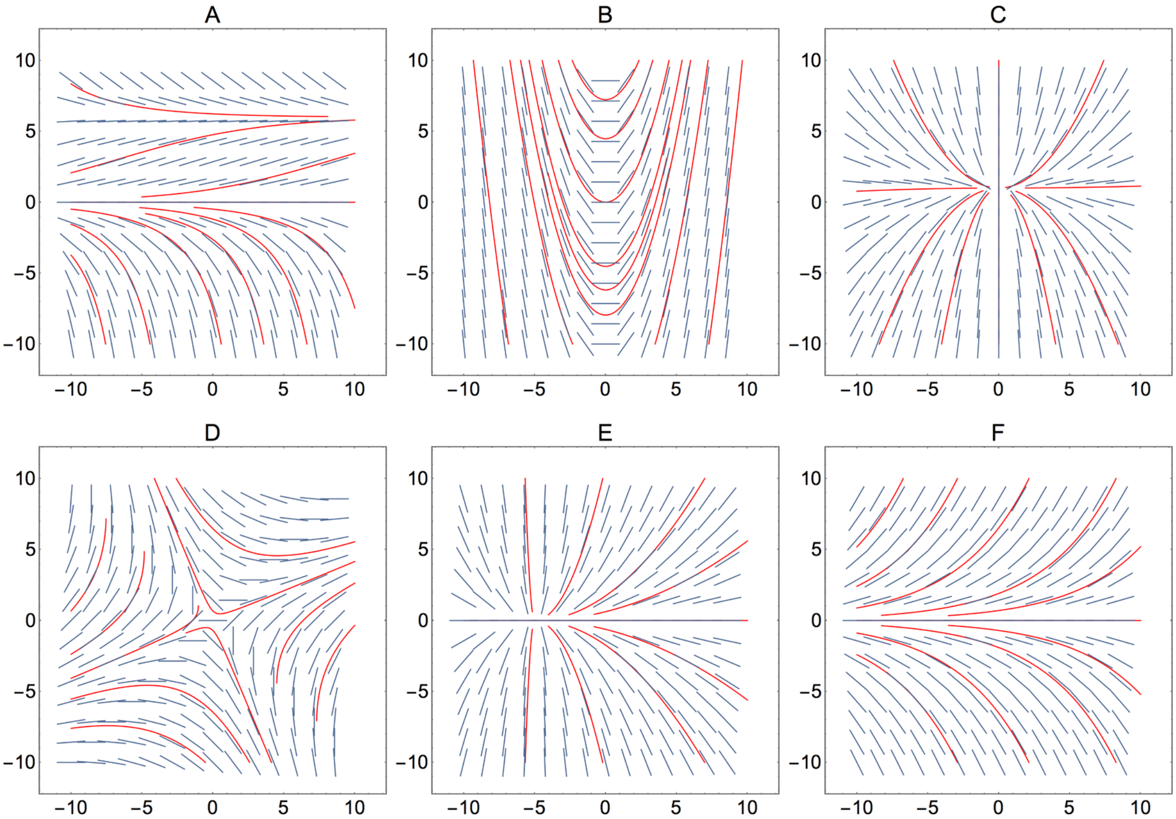
Comment

It is good practise to try drawing the below fields before looking at the next page. You need to be able to go in both directions (ie, drawing and recognising). You will not be given a glimps at the fields in the exam prior to being asked to draw them.

Question

Try drawing the slope field for at least 3 of the equations given below (your choice). Then, put the slope fields given on the next page in the same order as these equations.

1. $y' = x$
2. $y' = 0.2y$
3. $y' = 0.2y(1 - y/6)$
4. $y' = (x - y)/(x + y)$
5. $y' = 2(y - 1)/x$
6. $y' = 2y/(x + 5)$



Solution

X = Your solution
Form: String.
Place the indicated letter in front of the string.
Example: aX where X = abcdef is entered as aabcdef

Hash of qX = e93bfe

3.3 Separable equations I

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction`
- Video II: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example`
- Text:`http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx`

Comment

Let's start with a fundamental equation:

$$\frac{dy}{dt} = y \tag{3.1}$$

This is saying that the slope (the rate of change of y) linearly depends on y. That is, that as the value of y increases, the slope also increases; a positive feedback loop. In fact, you get an exponentially-increasing function.

So one aim of this course is to be able to solve such equations mathematically. But I also want you to understand the "physical" meaning of the relation between y and its slope, and how this leads to such a fundamental function such as an exponential.

Challenge

Considering the equation

$$\frac{dy}{dt} = y \tag{3.2}$$

solve for y.

Solution

To check your answer, solve for $y(5)$ given the initial condition $y(0) = 1$.

$y(5) = 148.413$

3.4 Separable equations II

Resources

- Video I: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction
- Video II: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example
- Text:http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx

Challenge

a) Now consider what is meant, physically speaking, by the relation:

$$\frac{dy}{dt} = -y \tag{3.3}$$

Why does it tend to zero for increasing t?

b) Solve for y.

Solution

a) Please compare your solution with your partner or discuss with the teacher.

b) To check your answer, solve for $y(5)$ given the initial condition $y(0) = 1$.

$y(5) = 0.00674$

3.5 Separable equations III

Resources

- Video I: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction
- Video II: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example
- Text:http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx

Challenge

a) Now consider when the slope of y not only depends on y but also on t:

$$\frac{dy}{dt} = ty \tag{3.4}$$

b) or on a constant a:

$$\frac{dy}{dt} = ay \tag{3.5}$$

See how the feedback is greater or lesser, depending on the constant or variable placed in front of y?

Solution

a) Solve for $y(5)$ under the initial condition $y(0) = 1$

268,337

b) Solve for $y(5)$ under the initial condition $y(0) = 1$ and with $a = 2$

22,026.5

3.6 Separable equations IV

Resources

- Video I: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction
- Video II: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example
- Text:http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx

Challenge

Determine $y(t)$ for

$$\frac{dy}{dt} = e^t \tag{3.6}$$

Again, think about what is happening here. Do you see the link with challenge 3.3? There we wrote in terms of y. Here we write in terms of e^t. Do you see they're the same thing?

Solution

To check your answer, solve for $y(3)$ given the initial condition $y(0) = 1$.

$y(3) = 20.09$

3.7 Rate of growth

Resources

- Video: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/logistic-differential-equation/v/modeling-population-with-differential-equations`

Comment

One interesting application of 1st-order differential equations is that of population growth.

Challenge

Assuming there is no-limit on growth, a given bacteria would be able to reproduce at such a rate that the amount of bacteria measured in mg increases by 20% every 25 hours. Derive an expression for the rate of growth.

Solution

To check your answer, calculate the rate of growth when there are 20 mg of bacteria. *To ensure accuracy, you will need to maintain a large degree of precision during your calculations.*

0.146 mg/hour

3.8 Logistic equation

Resources

- Videos: The 4 remaining logistic differential equation videos starting at: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/logistic-differential-equation/v/logistic-differential-equation-intuition`

Comment

We considered exponential growth, but in real life there is often a limit to this. This is where the logistic equation is useful.

Challenge

Assuming there is no-limit on growth, a given bacteria would be able to reproduce at such a rate that the amount of bacteria measured in mg increases by 20% every 25 hours. However, due to environmental factors the limiting (maximum) amount of bacteria that can exist in the system at any one time is 400 mg. Assuming an initial amount of bacteria of 20 mg, how much time must one wait to reach 100 mg of bacteria?

Solution

253 hours

3.9 Autonomous differential equations

Resources

- Wikipedia: `https://en.wikipedia.org/wiki/Autonomous_system_(mathematics)`

Challenge

The logistic equation is an example of an autonomous differential equation. Add the points of the autonomous differential equations in the following list:

1 point: $y' = cos(y) - 5$

2 points: $y' = cos(y)/x - 5$

4 points: $y' = cos(y)/x - 5/x$

8 points: $y^2 = y'y + 5$

16 points: $xy' = 5y$

32 points: $y' = 1$

Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of fX = 1227c7

3.10 The stability of solutions I

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`
- Text: `http://www.math.psu.edu/tseng/class/Math251/Notes-1st%20order%20ODE%20pt2.pdf`

Challenge

Considering the logistic equation $N' = 0.2N(1 - N/6)$, make 3 separate lists containing any equilibrium, semi-stable and unstable y-values.

To check your answer, sum the value of each list. If there are no values in a list, enter -999 to check the result.

Solution

Stable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of gX = 4a4314

Semi-stable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of hX = 9df203

Unstable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of jX = 17cb7f

3.11 The stability of solutions II

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`
- Text: `http://www.math.psu.edu/tseng/class/Math251/Notes-1st%20order%20ODE%20pt2.pdf`

Challenge

Considering the differential equation $y' = (y^2 - 16)(y + 3)^2$, make 3 separate lists containing any equilibrium, semi-stable and unstable y-values.

To check your answer, sum the value of each list. If there are no values in a list, simply enter "none" to check the result.

Solution

Stable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of kX = ffc446

Semi-stable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of zX = f76cc4

Unstable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of xX = bf947d