Ordinary Differential Equations

Autumn 2016

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http://www.jamescannon.net/teaching/ordinary-differential-equations

http://raw.githubusercontent.com/NanoScaleDesign/OrdinaryDifferentialEquations/master/ode.pdf

Contents

Chapter 0

Course information

0.1 This course

This is the Autumn 2016 Ordinary Differential Equations course studied by 2nd-year undegraduate international students at Kyushu University.

0.1.1 How this works

- In contrast to the traditional lecture-homework model, in this course the learning is self-directed and active via publicly-available resources.
- Learning is guided through solving a series of carefully-developed challenges contained in this book, coupled with suggested resources that can be used to solve the challenges with instant feedback about the correctness of your answer.
- There are no lectures. Instead, there is discussion time. Here, you are encouraged to discuss any issues with your peers, teacher and any teaching assistants. Furthermore, you are encouraged to help your peers who are having trouble understanding something that you have understood; by doing so you actually increase your own understanding too.
- Discussion-time is from 10:30 to 12:00 on Fridays at room W4-529.
- Peer discussion is encouraged, however, if you have help to solve a challenge, always make sure you do understand the details yourself. You will need to be able to do this in an exam environment. If you need additional challenges to solidify your understanding, then ask the teacher. The questions on the exam will be similar in nature to the challenges. If you can do all of the challenges, you can get 100% on the exam.
- Every challenge in the book typically contains a **Challenge** with suggested **Resources** which you are recommended to utilise in order to solve the challenge. A **Solution** is made available in encrypted form. If your encrypted solution matches the encrypted solution given, then you know you have the correct answer and can move on. For more information about encryption, see section 0.3. Occasionally the teacher will provide extra **Comments** to help guide your thinking.
- For deep understanding, it is recommended to study the suggested resources beyond the minimum required to complete the challenge.
- The challenge document has many pages and is continuously being developed. Therefore it is advised to view the document on an electronic device rather than print it. The date on the front page denotes the version of the document. You will be notified by email when the document is updated.
- A target challenge will be set each week. This will set the pace of the course and defines the examinable material. It's ok if you can't quite reach the target challenge for a given week, but then you will be expected to make it up the next week.
- You may work ahead, even beyond the target challenge, if you so wish. This can build greater flexibility into your personal schedule, especially as you become busier towards the end of the semester.
- Your contributions to the course are strongly welcomed. If you come across resources that you found useful that were not listed by the teacher or points of friction that made solving a challenge difficult (there's no such thing as "you should have learned it in high-school" - you're probably not the only one with that specific problem), please let the teacher know about it!

0.1.2 Assessment

In order to prove to outside parties that you have learned something from the course, we must perform summative assessments. This will be in the form of

$$\text{overall score} = (0.5E_F + 0.3E_M + 0.2C)(0.9 + P/10)$$

Your final score is calculated as Max(E_F, overall score), however you must pass the final exam ($\geq 60\%$) to pass the course.

- E_F = % correct on final exam
- E_M = % correct on mid-term exam
- C = % grade on course-work
- P = participation calculated as $P = (F/N_D)(A/N_D)(L/N_L)$ where each terms is as follows
  - F = Number of weeks where feedback form is submitted 24 hours before discussion time (including spreadsheet update).
  - A = Number of discussion classes attended
  - N_D = Number of discussion classes held
  - L = Number of times that your collected challenge log is satisfactory. This means:
    * Available on request
    * Your calculations are clearly shown
    * It corresponds to your spreadsheet
    * It contains evidence of trying to more-or-less keep up with the target challenge, even if you don't quite reach completion of the target challenge due to difficulties in understanding.
  - N_L = Number of times that your challenge log is collected.

Note that P is only calculated from 21 October. If $N - F \leq 2$ then F is treated as being equal to N (ie, you can forget twice). You can be counted as attending the class even if you are not present if the reason for not attending is unavoidable (eg, health reasons) and you inform the teacher in advance.

Please also note that, since late arrivals disrupt the class by preventing intended pairing of students, attendance of a discussion class will be only counted as partial if you are more than a minute or two late (eg, 9 minutes late out of a 90-minute discussion class will count as attending only 90% of the class). Therefore, if you will be unavoidably late, you need to let the teacher know in advance. To allow for unexpected delays, for up to two late arrivals you will be considered to have attended 100% of the discussion time.

Grades will be assigned following the Kyushu University grading system:

- A: $90 \leq$ final score ≤ 100
- B: $80 \leq$ final score < 90
- C: $70 \leq$ final score < 80
- D: $60 \leq$ final score < 70
- F: final score < 60

0.1.3 What you need to do

- Prepare a challenge-log in the form of a workbook or folder where you can clearly write the calculations you perform to solve each challenge. This will be a log of your progress during the course and will be occasionally reviewed by the teacher.
- You will need to maintain a google spreadsheet detailing your work and progress. The purpose of this spreadsheet is to help the teacher optimise the discussion-time. Please ensure that it is

up-to-date 24 hours before each discussion-time starts. It is fine for you to continue to work on challenges and update the spreadsheet after the 24-hour deadline.

- You also need to submit a brief report at `https://goo.gl/forms/Djl4FEZcJLMpipsY2` 24 hours before the discussion time starts. Here you can let the teacher know about any difficulties you are having and if you would like to discuss anything in particular.
- Please bring a wifi-capable internet device to class, as well as headphones if you need to access online components of the course during class. If you let me know in advance, I can lend computers and provide power extension cables for those who require them (limited number).

0.1.4 Details about the spreadsheet

To get started:

1. Log into google
2. Open `http://bit.ly/2cPYyQY`
3. File → Make a copy [→ rename] → ok
4. Click "Share" (top right)
5. Click "Get shareable link"
6. Set "Anyone with the link can edit"
7. Copy sharing address
8. Send an email to cannon@mech.kyushu-u.ac.jp containing
   (a) Subject: Ordinary Differential Equations registration
   (b) Your name
   (c) Student number
   (d) The link to your copy of the google sheet

Using the spreadsheet:

- Enter the appropriate challenge number. For example, for challenge 1.4, enter "1" in the **Section** column and "4" in the **Challenge** column.
- After successfully completing a challenge, please enter any particular friction points that you experienced (if any) so the course can be developed to reduce such friction in the future, as well as any extra resources you recommend (if any).
- Please also roughly estimate the amount of effort in required to complete the challenge (starting from when you completed the previous challenge, including any reading, watching videos, looking for resources, writing the answer to the challenge, discussing with peers, etc). This is not used for assessment in any way, but is very valuable in helping the teacher develop the course. Note: Although the column says **Hours**, please specify the time in terms of **minutes**.

Note: Please do not alter column names, ordering, etc. Just add section and challenge numbering and fill in the columns as appropriate. This is because spreadsheet data is downloaded and automatically analysed, and it breaks if anything is inconsistent.

0.2 Timetable

	Discussion	**Target**	**Note**
1	7 Oct	-	
2	14 Oct	2.2	
3	21 Oct	2.7	
4	28 Oct	2.12	
5	4 Nov	2.20	
6	11 Nov	3.8	
7	25 Nov	3.14	
8	2 Dec		
9	9 Dec	Midterm exam	Coursework instructions
10	16 Dec		
11	6 Jan		
12	12 Jan		Submission of coursework
13	20 Jan		
14	27 Jan		
15	10 Feb	Final exam	

Example: To keep pace with the course, you should aim to complete challenge 2 of chapter 2 by the 14th of October.

0.3 Hash-generation

Most solutions to challenges are encrypted using MD5 hashes. In order to check your solution, you need to generate its MD5 hash and compare it to that provided. MD5 hashes can be generated at the following sites:

- Wolfram alpha: (For example: md5 hash of "q_1.00") `http://www.wolframalpha.com/input/?i=md5+hash+of+%22q_1.00%22`
- `www.md5hashgenerator.com`

Since MD5 hashes are very sensitive to even single-digit variation, you must enter the solution exactly. This means maintaining a sufficient level of accuracy when developing your solution, and then entering the solution according to the format below:

Unless specified otherwise, any number from 0.00 to ±9999.99 should be represented as a normal number to two decimal places. All other numbers should be in scientific form. See the table below for examples.

Solution	**Input**
1	1.00
-3	-3.00
-3.5697	-3.57
0.05	0.05
0.005	5.00e-3
50	50.00
500	500.00
5000	5000.00
50,000	5.00e4
5×10^{-476}	5.00e-476
5.0009×10^{-476}	5.00e-476
$-\infty$	-infinity (never "infinite")
2π	6.28
i	im(1.00)
2i	im(2.00)
1 + 2i	re(1.00)im(2.00)
-0.0002548 i	im(-2.55e-4)
1/i = i/-1 = -i	im(-1.00)
$e^{i2\pi}$ [$= cos(2\pi) + isin(2\pi) = 1 + i0 = 1$]	1.00
$e^{i\pi/3}$ [$= cos(\pi/3) + isin(\pi/3) = 0.5 + i0.87$]	re(0.50)im(0.87)
Choices in order A, B, C, D	abcd

Entry format is given with the problem. So "q_X" means to enter "q_X" replacing "X" with your solution. The first 6 digits of the MD5 sum should match the given solution (MD5(q_X)= . . .).

Note that although some answers can usually only be integers (eg, number of elephants), for consistency, to generate the correct hash, the accuracy in terms of decimal places noted above is required.

Chapter 1

Definitions

1.1 Order of a differential equation

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

What is the sum of the orders of the following equations?

$$\frac{dy}{dx}A = 5x^3 + 3 \tag{1.1}$$

$$cos(y)y'''(x) - y(x) = 25 \tag{1.2}$$

$$\frac{d}{dx}\frac{d^2y}{dx^2} = \frac{x^{-2}}{3} \tag{1.3}$$

Solution

X

MD5(e_X) = adb5a9...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

1.2 Linear equations

Resources

- Video: https://www.youtube.com/watch?v=dwNMEMOGO_o
- Text: http://www.myphysicslab.com/classify_diff_eq.html
- Text: http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx

Comment

Using a function x dependent on time t as an example, a differential equation is defined as being linear if it can be written in the form

$$f_n(t)\frac{d^n x}{dt^n} + \cdots + f_1(t)\frac{dx}{dt} + f_0(t)x + f_c(t) = C \tag{1.4}$$

Here, $f_n(t)$ is a function of time only, such as $5t$ or $2/t^2$ or may even be constant with time (eg, 3). Any of the f_n's and the constant C may be zero. If it is possible to arrange an equation into the above form, then the equation must be linear. So for a linear equation, $x(t)$, $x'(t)$, $tx(t)$, $3t^2x'''(t)$ are linear terms in x, but $x(t)^2$, $x(t)x'(t)$ and $5tTan(x)$ are non-linear terms.

Challenge

Sum the points corresponding to the equations that are linear:

1 point: $\frac{dx}{dt} = 5t^3 + 3.$

2 points: $cos(x)x'''(t) - x(t) = 25.$

4 points: $\frac{d}{dt}\frac{d^2x}{dt^2} = \frac{t^{-2}}{3}.$

8 points: $x'(t) - sin(x(t)) = 0.$

16 points: $x'(t) - x(t) = 0.$

32 points: $tx'(t) - x(t) = 0.$

Solution

X

MD5(r_X) = 9aea7d...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

1.3 Valid solutions

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Use substitution to prove that

$$y = \frac{5}{5+x} \tag{1.5}$$

is a solution to the equation

$$xy' + y = y^2 \tag{1.6}$$

and state the value of x for which the solution is undefined.

Solution

Value of x for which solution is undefined: X

MD5(t_X) = c69a20...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

1.4 Range of valid solutions

Resources

- Text: http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx

Challenge

Use substitution to prove that

$$y = -\sqrt{100 - x^2} \tag{1.7}$$

is a solution to the equation

$$x + yy' = 0 \tag{1.8}$$

and state the range of x for which the solution is valid. Enter the value of the lower range as the solution below.

Solution

X

MD5(y_X) = 8c8c08...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

Chapter 2

1st-order differential equations

2.1 Determining a simple DE from a description

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`
- Book: Chapter 1.2

Challenge

Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference with the ambient surroundings. (a) Write a differential equation describing this situation. (b) Assuming a proportionality constant of 0.2 /hour, what is the rate of temperature change when the object is at 30 °C and the ambient temperature is 20 °C?

Solution

X °C/hour

MD5(q_X) = 078383...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.2 Direction (Slope) fields

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/DirectionFields.aspx`
- Video 1: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/creating-a-slope-field`
- Video 2: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/slope-field-to-visualize-solutions`
- Book: Chapters 1.1, 1.2

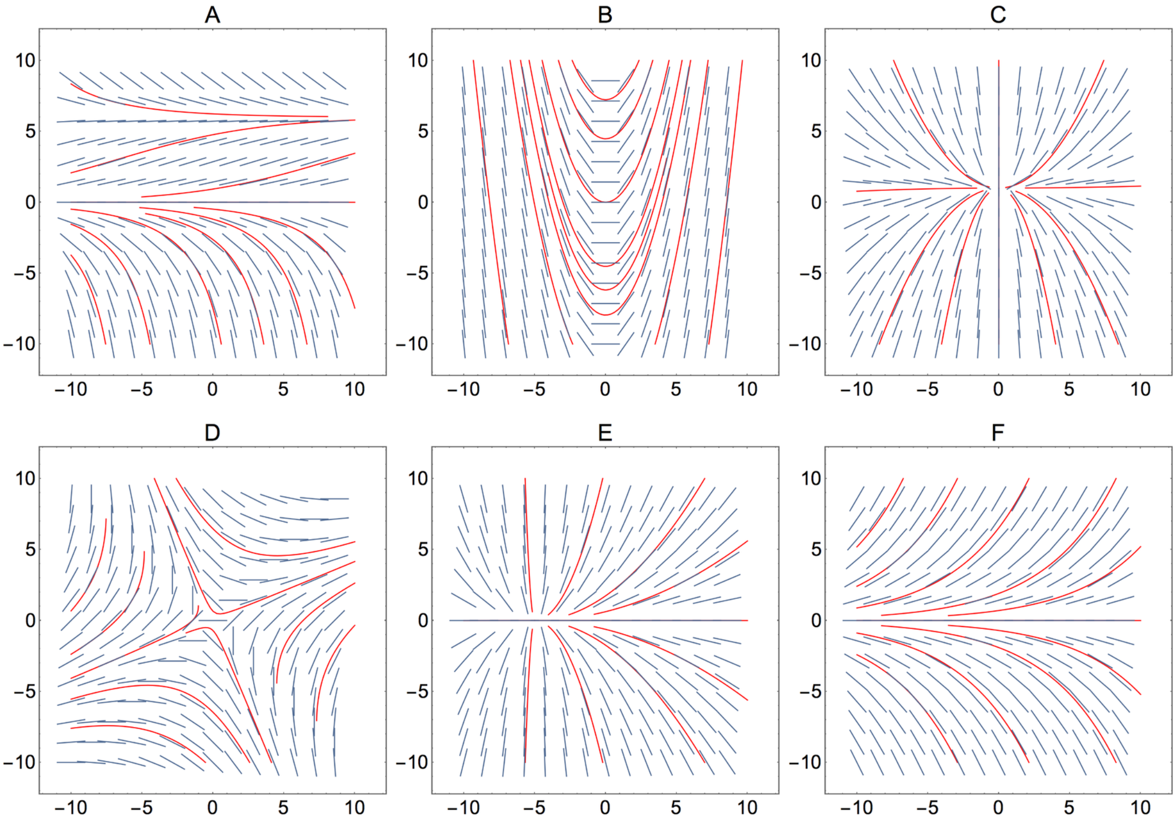
Comment

It is good practise to try drawing the below fields before looking at the next page. You need to be able to go in both directions (ie, drawing and recognising).

Question

Try drawing the slope field for at least 3 of the equations given below (your choice). Then, put the slope fields given on the next page in the same order as these equations.

1. $y' = x$
2. $y' = 0.2y$
3. $y' = 0.2y(1 - y/6)$
4. $y' = (x - y)/(x + y)$
5. $y' = 2(y - 1)/x$
6. $y' = 2y/(x + 5)$



Solution

X (eg, “abcdef”)

MD5(q_X) = 743eb9...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.3 Solving a simple 1st-order linear equation

Resources

- Video: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/finding-particular-linear-solution-to-differential-equation`

Comment

It's not easy to see when equations can be solved simply, like in the challenge below, and when they can not. But for a 1st-order linear differential equation, this method is often good to try first. If it's not 1st order and not linear, you know to try a different approach.

Challenge

Determine the value of $y(x = 1)$ for the following equation:

$$y' = 2y - 5x + 2 \tag{2.1}$$

Solution

X

MD5(u_X) = 505c7b...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.4 Separable equations I

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction`
- Video II: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example`
- Text:`http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx`

Challenge

Given the following equation:

$$r' = -Sin(\theta) \tag{2.2}$$

Determine the function $r(\theta)$ that passes through the point (0,1) in $\theta - r$ space, and then solve for $\theta = \pi/4$.

Solution

X

MD5(i_X) = 286117...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.5 Separable equations II

Resources

- Video I: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction
- Video II: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example
- Text:http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx

Challenge

Given the following equation:

$$r'cot(\theta) + r = 2 \tag{2.3}$$

Determine the function $r(\theta)$ that passes through the point (0,1) in $\theta - r$ space, and then solve for $\theta = \pi/4$.

Solution

X

MD5(o_X) = 87ee92...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.6 Separable equations III

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction`
- Video II: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example`
- Text:`http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx`

Challenge

Given the following equation:

$$y'x^2 - y = 0 \tag{2.4}$$

Determine the function $y(x)$ that passes through the point $x = 2, y = 1$ and then solve for the given x value. State the value of x where the solution is undefined.

Solution

Solve for $x = 4$:

X

MD5(p_X) = 7cb08e...

Value of x where solution is undefined:

X

MD5(a_X) = b30fe7...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.7 Compound increase

Resources

- Videos: `https://www.khanacademy.org/economics-finance-domain/core-finance/interest-tutorial/compound-interest-tutorial/v/introduction-to-compound-interest`

Challenge

In an unconstrained situation (ie, no limit to the amount of bacteria so it is able to increase exponentially with time, forever), if the amount of bacteria on a surface increases by 20% every 25 hours, how much does the amount grow per hour in percentage terms? Ie, if one observes the amount of bacteria one hour from now, by what percentage should one expect the amount of bacteria to have increased?

Solution

X %

MD5(s_X) = 4288d8...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.8 Logistic equation

Resources

- Videos: The 5 videos on the logistic differential equation and function starting at: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/logistic-differential-equation/v/modeling-population-with-differential-equations`

Challenge

In an unconstrained situation, a given bacteria is able to reproduce at such a rate that the amount of bacteria measured in mg increases by 20% every 25 hours. However, due to environmental factors the limiting (maximum) amount of bacteria that can exist in the system at any one time is 400 mg. Assuming an initial amount of bacteria of 20 mg, how much time, rounded to the nearest integer hours, must one wait to reach 300 mg of bacteria?

Note: Be sure to maintain a sufficient number of significant figures in your numbers while performing the calculation.

Solution

X hours (expressed as an integer - do not enter ".00".)

MD5(d_X) = b0d406...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.9 Autonomous differential equations

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`
- Wikipedia: `https://en.wikipedia.org/wiki/Autonomous_system_(mathematics)`

Challenge

Add the points of the autonomous differential equations in the following list:

1 point: $y' = cos(y) - 5$

2 points: $y' = cos(y)/x - 5$

4 points: $y' = cos(y)/x - 5/x$

8 points: $y^2 = y'y + 5$

16 points: $xy' = 5y$

32 points: $y' = 1$

Solution

X

MD5(f_X) = 9bf043...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.10 The stability of solutions I

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`

Challenge

Considering the logistic equation $N' = 0.2N(1 - N/6)$, make 3 separate lists containing any equilibrium, semi-stable and unstable y-values.

To check your answer, sum the value of each list. If there are no values in a list, simply enter "none" to check the result.

Solution

Stable

X

MD5(g_X) = 2c32d8...

Semi-stable

X

MD5(h_X) = 8b595d...

Unstable

X

MD5(j_X) = 4fd3f6...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.11 The stability of solutions II

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`

Challenge

Considering the differential equation $y' = (y^2 - 16)(y+3)^2$, make 3 separate lists containing any equilibrium, semi-stable and unstable y-values.

To check your answer, sum the value of each list. If there are no values in a list, simply enter "none" to check the result.

Solution

Stable

X

MD5(k_X) = 667798...

Semi-stable

X

MD5(z_X) = 200aa3...

Unstable

X

MD5(x_X) = d1ee5b...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

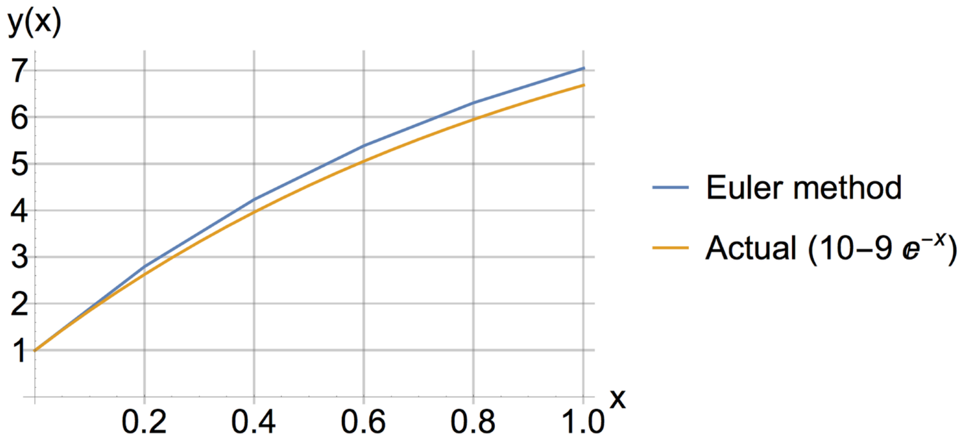
2.12 Euler's method

Resources

- Videos and exersizes in the "Euler's Method" section of Khan academy: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/eulers-method-tutorial/v/eulers-method`
- Text: `http://tutorial.math.lamar.edu/Classes/DE/EulersMethod.aspx`

Challenge

Considering the differential equation $y' = 10 - y$, an initial value of $y(0) = 1$ and a step size of $\Delta x = 0.2$, use Euler's method to estimate the value of $y(x = 1)$. The actual solution, $y(x) = 10 - 9e^{-x}$, is shown below.



Solution

X

MD5(c_X) = 1f90fa...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.13 Exact differential equations: derivation

Resources

- Videos: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/exact-equations/v/exact-equations-intuition-1-proofy`

Challenge

Please follow the two videos on derivation and intuition regarding exact differential equations starting at the video listed above.

If

$$\frac{d\psi(x,y)}{dx} = cos(x)sin(y) + x^2y' - (x+y)/100 \tag{2.5}$$

what is $\dfrac{\partial\psi}{\partial x}$?

To check your answer, substitute $x = 3.1$ and $y = -2$ into the resulting equation.

Solution

X

MD5(v_X) = 705027...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.14 Exact differential equations: possible solutions given ψ_x

Resources

- Videos: Exact equations intuition 1,2 and examples 1,2,3 starting from `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/exact-equations/v/exact-equations-intuition-1-proofy`
- Text: `http://tutorial.math.lamar.edu/Classes/DE/Exact.aspx`

Challenge

Sum the points of all the possible solutions to the integral of the partial-differential equation:

$$\psi_x = 6x - 3e^x sin(y) \tag{2.6}$$

1 point: $\psi(x, y) = 3x^2 - 3e^x sin(y) + 4$

2 points: $\psi(x, y) = 3x^2 - 3e^x sin(y) + x$

4 points: $\psi(x, y) = 3x^2 - 3e^x sin(y) + y$

8 points: $\psi(x, y) = 3x^2 - 3e^x sin(y) + yx$

16 points: $\psi(x, y) = 3x^2 - 3e^x sin(y) + y^2$

32 points: $\psi(x, y) = 3x^2 - 3e^x sin(y) + 5sin(y)$

64 points: $\psi(x, y) = 3x^2 - 3e^x sin(y) + 5sin(y)cos(x)$

Solution

X

MD5(b_X) = 408993...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.15 Exact differential equations: identification

Resources

- Videos: Exact equations intuition 1,2 starting from `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/exact-equations/v/exact-equations-intuition-1-proofy`
- Text: `http://tutorial.math.lamar.edu/Classes/DE/Exact.aspx`

Challenge

Sum the points of the equations below that are exact differential equations:

1 point: $(3x^2y + 8xy^2)dx + (x^3 + 8x^2y + 12y^2)dy = 0$

2 points: $sin(x)cos(y)dx + cos(x)sin(y)dy = 0$

4 points: $sin(x)cos(y)dx + sin(x)sin(y)dy = 0$

8 points: $\dfrac{dx}{x} + \dfrac{dy}{y} = 0$

16 points: $-\dfrac{ydx + xdy}{x^2} = 0$

32 points: $-\dfrac{ydx - xdy}{x^2} = 0$

Solution

X

MD5(n_X) = 868f48...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.16 Exact differential equations: solving

Resources

- Videos: Exact equations examples 1,2,3 starting from `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/exact-equations/v/exact-equations-example-1`
- Text: `http://tutorial.math.lamar.edu/Classes/DE/Exact.aspx`

Challenge

In challenge 2.15 you should have identified 4 exact differential equations. Considering each of the 4 EDE's in order, try to solve the EDE's applying the following conditions:

1st EDE

Do not try to solve this one.

2nd EDE

Use the condition $y(\pi/4) = \pi/4$ to find an explicit solution for the equation and then evaluate y at $x = \pi$.

3rd EDE

Use the condition $y(1) = 3$ to find an explicit solution for the equation and then evaluate y at $x = 4$.

4th EDE

Use the condition $y(1) = 2$ to find an explicit solution for the equation and then evaluate y at $x = 1$.

Solution

2nd EDE

X

MD5(m_X) = af87e2...

3rd EDE

X

MD5(aa_X) = d01c3d...

4th EDE

X

MD5(bb_X) = 5e1074...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.17 Exact differential equations: a useful integration method

Challenge

Obtain an expression for $g(x)$ in terms of $f(x)$ in the following integral:

$$\int \frac{f'(x)}{f(x)} dx = g(x) \tag{2.7}$$

ie, you should be able to re-write $g(x)$ in terms of a simple (non-integral) function of $f(x)$, in the form $g(x) = \cdots$.

Solution

You can check your answer by putting a function of x into $f(x)$.

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.18 Exact differential equations: integrating factors

Resources

- Videos: Integrating factors 1,2 starting from `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/exact-equations/v/integrating-factors-1`

Comment

Note that in the videos, Sal Khan does an example considering an integrating factor of $\mu(x)$, but in some cases $\mu(y)$ leads to a solution more easily. You may need to try both to determine an answer.

Challenge

Solve the exact differential equations below using integrating factors.

1. Solve the equation below using an integrating factor. Place the solution in the form $f(x, y) = C$, then calculate the value of C when substituting $x = 2$ and $y = 1$ into the equation. Do not try to solve the equation to get it in the form $y(x) = \cdots$.

$$ydx + (2xy - e^{-2y})dy = 0 \tag{2.8}$$

2. Calculate the integrating factor for the following equation. To check your answer, substitute $x = 1$ or $y = 1$ into any final expression, assuming an integration constant of zero.

$$y(3x - y)dx + x(x - y)dy = 0 \tag{2.9}$$

3. Show that $1/(x^y + y^2)$ is an integrating factor for the equation

$$xdx + ydy + 4y^3(x^2 + y^2) = 0 \tag{2.10}$$

Solution

Challenge related to equation 2.8: MD5(cc_X) = bb15d6...

Challenge related to equation 2.9: MD5(dd_X) = 6a8742...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.19 Exact differential equations: integrating factor derivation

Challenge

1. Starting from the equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0 \tag{2.11}$$

show that if the integrating factor μ is only a function of x, then

$$\mu_x = \mu\left(\frac{M_y - N_x}{N}\right) \tag{2.12}$$

2. Do the same, assuming that μ is only a function of y.

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.20 Exact differential equations: integrating factor calculation

Comment

Without proof, we can use equation 2.12 to gain information about the existance of an integration factor. If $\left(\frac{M_y - N_x}{N}\right)$ is a function of x only, then we know that the integration factor is only a function of x, and it can be solved for by integration of equation 2.12. The same can be said for $\mu(y)$ that you derived an expression for in challenge 2.19.

Challenge

Use equations from section 2.19 and information provided in the comment here to determine the integrating factor for

$$e^x dx + (e^x Cot(y) + 2yCsc(y))dy = 0 \tag{2.13}$$

and

$$(y - x^2)dx + 2xydy = 0 \tag{2.14}$$

To check your answer, for both cases substitute $x = \pi$ or $y = \pi$ into the integrating factor, and assume an integration constant of 1.

Solution

Equation 2.13: MD5(ee_X) = 51a0ae...

Equation 2.14: MD5(ff_X) = a56bce...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

2.21 Summary of 1st-order differential equations

Challenge

1. Create a flowchart describing how you will approach solving a general 1st-order differential equation.
2. Solve the following 1st-order differential equations:

$$y' - 4y = 8x + 3 \tag{2.15}$$

evaluated at $x = 1$.

$$4yy' = 8x + 3 \tag{2.16}$$

assuming an integration constant of zero and evaluating the final equation at $x = 2$.

$$y' + 4y = e^{-8x} \tag{2.17}$$

assuming an integration constant of zero and evaluating the final equation at $x = 1/8$.

Solution

Equation 2.15: MD5(qq_X) = a43ab2...

Equation 2.16: MD5(rr_X) = 990bfa...

Equation 2.17: MD5(ss_X) = 91989d...

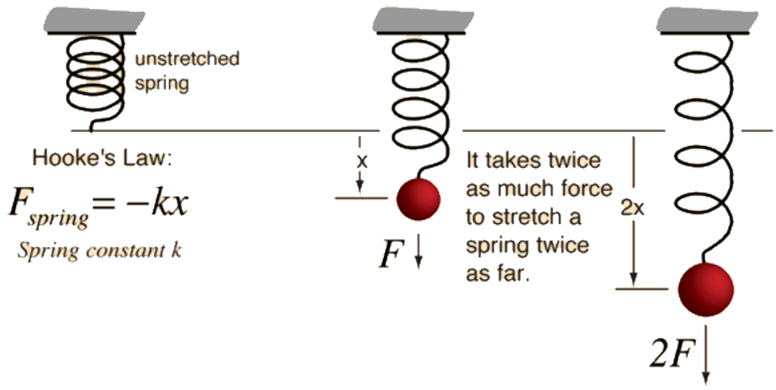
Study-time (from end of previous challenge to end of this challenge): ______ minutes

Chapter 3

2nd-order differential equations

3.1 Hooke's law

Resources



(Image from HyperPhysics by Rod Nave, Georgia State University)

Challenge

2nd-order differential equations deal with oscillations.

Considering Hooke's law, what are A and C in the following equation?

$$Ax'' + Cx = 0 \tag{3.1}$$

To check your answer, substitute a mass of 2 kg and spring-constant of $3\,\mathrm{kg/s^2}$ as appropriate.

Solution

Enter only numberical values without units such as kg.

A: MD5(gg_X) = 4e5fe6...

C: MD5(hh_X) = 6a7015...

3.2 Exponentials and trigonometry

Resources

- Text: `https://www.phy.duke.edu/~rgb/Class/phy51/phy51/node15.html`

Challenge

Write $sin(x)$ and $cos(x)$ in exponential form.

Solution

Check your answer with someone if you are unsure.

Study-time (from end of previous challenge to end of this challenge): ______ minutes

3.3 Characteristic equation: understanding

Resources

- Text: `http://tutorial.math.lamar.edu/getfile.aspx?file=S,88,N`

Comment

A homogeneous (ie, equal to zero) second-order differential equation typically takes the form:

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0 \tag{3.2}$$

The first (A) term describes acceleration, while the third (C) term is the force-constant term (something like the "stiffness" of the spring). The second (B) term could describe a frictional force that is proportional to the velocity (dy/dt). Due to its relation with oscillation (and by extension, sines and cosines which can be expressed in terms of exponentials) we can typically assume an exponential-form solution to the differential equation.

Challenge

Show that, assuming that all solutions to a 2nd-order differential equation of the form above will have solutions $y(t) = e^{rt}$, the value of r can in principle be determined by solving the following a quadratic equation of the form

$$Ar^2 + Br + C = 0 \tag{3.3}$$

Solution

If you are unsure of your derivation, please ask someone.

Study-time (from end of previous challenge to end of this challenge): ______ minutes

3.4 Characteristic equation: roots

Resources

- Text: `http://tutorial.math.lamar.edu/getfile.aspx?file=S,88,N`

Challenge

Sum the points of the differential equations that have characteristic equations with

- Real, distinct roots
- Complex roots
- Equal roots

1 point: $-3y'' - 5y' + 2y = 0$

2 points: $3y'' - 4y' + 3y = 0$

4 points: $3y'' - 6y' + 3y = 0$

8 points: $3y'' - 5y' + 2y = 0$

16 points: $3y'' - 5y' + 4y = 0$

32 points: $3y'' + 5y' + 2y = 0$

Solution

- Real, distinct roots: MD5(ii_X) = 064a6e...
- Complex roots: MD5(jj_X) = 50385e...
- Equal roots: MD5(kk_X) = 70cd8f...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

3.5 Characteristic equation: real roots with positive B

Resources

- Text: `http://tutorial.math.lamar.edu/getfile.aspx?file=S,94,N`

Comment

Challenge

Solve the following 2nd-order differential equation that has real roots:

$$y'' + 3y' + 2y = 0 \tag{3.4}$$

with initial conditions $y(0) = 5$ and $y'(0) = -8$.

To check your answer, substitute $t = 1$ into the final expression.

Solution

MD5(mm_X) = 9b9be5...

3.6 Characteristic equation: real roots with negative B

Resources

- Text: `http://tutorial.math.lamar.edu/getfile.aspx?file=S,94,N`

Challenge

Solve the following 2nd-order differential equation that has real roots.

$$y'' - 3y' + 2y = 0 \tag{3.5}$$

with initial conditions $y(0) = 5$ and $y'(0) = 8$. Substitue $t = 1$ into the final expression to check your answer.

Note that this equation is the same as equation 3.4, but simply the dampening (friction) term B has been changed from positive to negative.

Solution

MD5(nn_X) = 473835...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

3.7 Characteristic equation: B in equations with real roots

Challenge

(Note that there are two parts to this challenge.)

1. Considering real root, sum the points of the following true statements:

Considering the equation

$$Ay'' + By' + Cy = 0 \tag{3.6}$$

1 point: Positive damping (positive B) leads to solutions with exponentials with positive exponents.

2 points: Positive damping (positive B) leads to solutions with exponentials with negative exponents.

4 points: Negative damping (negative B) leads to solutions with exponentials with positive exponents.

8 points: Negative damping (negative B) leads to solutions with exponentials with negative exponents.

16 points: Exponentials with positive exponents (eg, e^t) lead to exponential growth (instability).

32 points: Exponentials with negative exponents (eg, e^{-t}) lead to exponential growth (instability).

64 points: Exponentials with positive exponents (eg, e^t) lead to a damped signal (stability).

128 points: Exponentials with negative exponents (eg, e^{-t}) lead to damped signal (stability).

2. Write a sentence summarising your understanding of the significance of having a positive or negative coefficient of B when the roots are real.

Solution

MD5(oo_X) = fa6adf...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

3.8 Characteristic equation: equal roots

Resources

- Text: `http://tutorial.math.lamar.edu/getfile.aspx?file=S,96,N`

Comment

It is not necessary to follow the full derivation in the suggested resource.

Challenge

Solve the equation

$$y'' - 2y' + y = 0 \tag{3.7}$$

To check your solution, substitute $t = 1$ into the equation and assume $c_1 = c_2 = 1$.

Solution

MD5(pp_X) = ff7ca2...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

3.9 Characteristic equation: complex roots with B=0

Resources

- Text: `http://tutorial.math.lamar.edu/getfile.aspx?file=S,96,N`

Challenge

1. Assuming there is no damping term (ie, $B = 0$) show that the roots for the differential equation

$$Ay'' + Cy = 0 \tag{3.8}$$

are $\pm i\sqrt{C/A}$.

2. Solve the following ODE:

$$y'' + 4\pi^2 y = 0 \tag{3.9}$$

To check your answer, assume integration constants of 1 and calculate $y(\pi/2)$.

Solution

Solution to part 2: MD5(qq_X) = 7eb2c9...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

3.10 Characteristic equation: complex roots with positive B

Resources

- Text: `http://tutorial.math.lamar.edu/getfile.aspx?file=S,96,N`

Challenge

Solve the following ODE:

$$y'' + y' + y = 0 \tag{3.10}$$

To check your answer, assume integration constants of 1 and calculate $y(\pi/2)$.

Solution

MD5(rr_X) = 1d0cb5...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

3.11 Characteristic equation: complex roots with negative B

Resources

- Text: `http://tutorial.math.lamar.edu/getfile.aspx?file=S,96,N`

Challenge

Solve the following ODE:

$$y'' - y' + y = 0 \tag{3.11}$$

To check your answer, assume integration constants of 1 and calculate $y(\pi/2)$.

Solution

MD5(ss_X) = caf35b...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

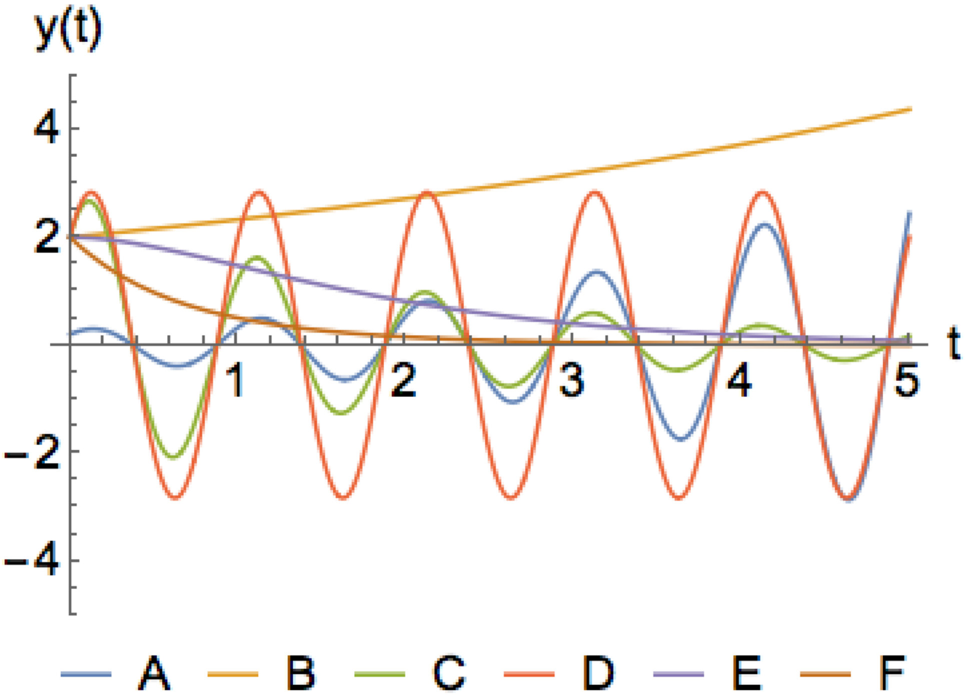
3.12 Damping

Resources

- Wikipedia: https://en.wikipedia.org/wiki/Damping

Challenge

Of the 6 functions shown in the graph, place the 3 that correspond to overdamped, critially damped and underdamped in the order mentioned in this sentence.



Solution

(eg, "abc")

MD5(tt_X) = 060b2a...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

3.13 Damping and 2nd-order differential equations

Challenge

1. The 6 functions shown in the graph in challenge 3.12 may represent solutions of a 2nd-order differential equation. Place the solutions A-F in the order shown below. Note that one of the descriptions below is impossible, and you should ignore that one.

I. Solution of a 2nd-order differential equation with real roots and positive B.

II. Solution of a 2nd-order differential equation with real roots and negative B.

III. Solution of a 2nd-order differential equation with real roots and B=0.

IV. Solution of a 2nd-order differential equation with equal roots.

V. Solution of a 2nd-order differential equation with complex roots and B=0.

VI. Solution of a 2nd-order differential equation with complex roots and positive B.

VII. Solution of a 2nd-order differential equation with complex roots and negative B.

2. Write one sentence stating why one of the above solutions is impossible.

Solution

(eg, "abcdef")

MD5(uu_X) = a96870...

Study-time (from end of previous challenge to end of this challenge): ______ minutes

3.14 The Wronskian

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 133.

Challenge

Imagine you need to write a letter to a student, explaining what the Wronskian is, where it comes from, and how it is useful in determining the validity of fundamental sets of solutions. You may assume the student knows the formulas for solving different forms (complex, real and equal-roots) of 2nd-order ODE's with the characteristic equation method, but does not know what is meant by a "fundamental solution", doesn't understand why such fundamental solutions are sums of two terms, and does not know about the Wronskian nor how it is calculated. You may use the suggested resource to help formulate your letter. The student also would like to know about the connection of the Wronskian to linear independence and how this is related to the fundamental solutions (ie, why it matters that the two terms of the fundamental solutions are linearly independent).

Suggested length including all text and equations: around 3/4 to 1 A4 sheet.

Solution

Please give the letter to the teacher for posting. The teacher will check the depth of your understanding prior to posting.

Study-time (from end of previous challenge to end of this challenge): ______ minutes