A (possible) glitch?

There is a complication that occurs under a certain circumstance…

Example: $$y'' - 2y' - 3y = 5e^{3t}$$

The old news is that $y_c = C_1 e^{-t} + C_2 e^{3t}$. Since $g(t) = 5e^{3t}$, we should be able to use the form $Y = A e^{3t}$, just like in the first example, right? But if we substitute Y, $Y' = 3A e^{3t}$, and $Y'' = 9A e^{3t}$ into the differential equation and simplify, we would get the equation

$$0 = 5e^{3t}.$$

That means there is no solution for A. Our method (that has worked well thus far) seems to have failed. The same outcome (an inability to find A) also happens when $g(t)$ is a multiple of e^{-t}. But, for any other exponent our choice of the form for Y works. What is so special about these two particular exponential functions, e^{3t} and e^{-t}, that causes our method to misfire? (Hint: What is the complementary solution of the nonhomgeneous equation?)

The answer is that those two functions are exactly the terms in y_c. Being a part of the complementary solution (the solution of the corresponding homogeneous equation) means that any constant multiple of either functions will ALWAYS results in zero on the right-hand side of the equation. Therefore, it is impossible to match the given $g(t)$.

The cure: The remedy is surprisingly simple: <u>multiply our usual choice by t</u>. In the above example, we should instead use the form $Y = A t e^{3t}$.

In general, whenever your initial choice of the form of Y has any term in common with the complementary solution, then you must alter it by multiplying your initial choice of Y by t, <u>as many times as necessary but no more than necessary</u>.

Example: $$y'' - 6y' + 9y = e^{3t}$$

The complementary solution is $y_c = C_1 e^{3t} + C_2 t e^{3t}$. $g(t) = e^{3t}$, therefore, the initial choice would be $Y = Ae^{3t}$. But wait, that is the same as the first term of y_c, so multiply Y by t to get $Y = Ate^{3t}$. However, the new Y is now in common with the second term of y_c. Multiply it by t again to get $Y = At^2 e^{3t}$. That is the final, correct choice of the general form of Y to use. (*Exercise*: Verify that neither $Y = Ae^{3t}$, nor $Y = Ate^{3t}$ would yield an answer to this problem.)

Once we have established that $Y = At^2 e^{3t}$, then $Y' = 2Ate^{3t} + 3At^2 e^{3t}$, and $Y'' = 2Ae^{3t} + 12Ate^{3t} + 9At^2 e^{3t}$. Substitute them back into the original equation:

$$(2Ae^{3t} + 12Ate^{3t} + 9At^2 e^{3t}) - 6(2Ate^{3t} + 3At^2 e^{3t}) + 9(At^2 e^{3t}) = e^{3t}$$

$$2Ae^{3t} + (12 - 12)Ate^{3t} + (9 - 18 + 9)At^2 e^{3t} = e^{3t}$$

$$2Ae^{3t} = e^{3t}$$

$$A = 1/2$$

Hence, $Y(t) = \frac{1}{2} t^2 e^{3t}$.

Therefore, $y = C_1 e^{3t} + C_2 te^{3t} + \frac{1}{2} t^2 e^{3t}$. Our "cure" has worked!

Since a second order linear equation's complementary solution only has two parts, there could be at most two shared terms with Y. Consequently we would only need to, at most, apply the cure twice (effectively multiplying by t^2) as the worst case scenario.

The lesson here is that you should always find the complementary solution first, since the correct choice of the form of Y depends on y_c. Therefore, you need to have y_c handy before you write down the form of Y. Before you finalize your choice, always compare it against y_c. And if there is anything those two have in common, multiplying your choice of form of Y by t. (However, you should do this ONLY when there actually exists something in common; you should never apply this cure unless you know for sure that a common term exists between Y and y_c, else you will not be able to find the correct answer!) Repeat until there is no shared term.

When $g(t)$ is a product of several functions

If $g(t)$ is a product of two or more simple functions, e.g. $g(t) = t^2e^{5t}\cos(3t)$, then our basic choice (before multiplying by t, if necessary) should be a product consist of the corresponding choices of the individual components of $g(t)$. One thing to keep in mind: <u>that there should be only as many undetermined coefficients in Y as there are distinct terms</u> (after expanding the expression and simplifying algebraically).

Example:

$$y'' - 2y' - 3y = t^3e^{5t}\cos(3t)$$

We have $g(t) = t^3e^{5t}\cos(3t)$. It is a product of a degree 3 polynomial†, an exponential function, and a cosine. Out choice of the form of Y therefore must be a product of their corresponding choices: a generic degree 3 polynomial, an exponential function, and both cosine and sine. Try

Correct form: $Y = (At^3 + Bt^2 + Ct + D)\,e^{5t}\cos(3t) +$
$(Et^3 + Ft^2 + Gt + H)\,e^{5t}\sin(3t)$

Wrong form: $Y = (At^3 + Bt^2 + Ct + D)\,E\,e^{5t}(F\cos(3t) + G\sin(3t))$

Note in the correct form above, each of the eight distinct terms has its own unique undetermined coefficient. Here is another thing to remember: that those coefficients should all be <u>independent</u> of each others, each uniquely associated with only one term.

In short, when $g(t)$ is a product of basic functions, $Y(t)$ is chosen based on:

i. $Y(t)$ is a product of the corresponding choices of all the parts of $g(t)$.
ii. There are as many coefficients as the number of distinct terms in $Y(t)$.
iii. Each distinct term must have its own coefficient, not shared with any other term.

† A power such as t^n is really just an n-th degree polynomial with only one (the n-th term's) nonzero coefficient.

Another way (longer, but less prone to mistakes) to come up with the correct form is to do the following.

Start with the basic forms of the corresponding functions that are to appear in the product, without assigning any coefficient. In the above example, they are $(t^3 + t^2 + t + 1)$, e^{5t}, and $\cos(3t) + \sin(3t)$.

Multiply them together to get all the distinct terms in the product:

$$(t^3 + t^2 + t + 1)\, e^{5t}(\cos(3t) + \sin(3t))$$
$$= t^3 e^{5t} \cos(3t) + t^2 e^{5t} \cos(3t) + t\, e^{5t} \cos(3t) + e^{5t} \cos(3t)$$
$$+ t^3 e^{5t} \sin(3t) + t^2 e^{5t} \sin(3t) + t\, e^{5t} \sin(3t) + e^{5t} \sin(3t)$$

Once we have expanded the product and identified the distinct terms in the product (8, in this example), then we insert the undetermined coefficients into the expression, one for each term:

$$Y = A\, t^3 e^{5t} \cos(3t) + B\, t^2 e^{5t} \cos(3t) + C\, t\, e^{5t} \cos(3t)$$
$$+ D\, e^{5t} \cos(3t) + E\, t^3 e^{5t} \sin(3t) + F\, t^2 e^{5t} \sin(3t) + G\, t\, e^{5t} \sin(3t)$$
$$+ H\, e^{5t} \sin(3t)$$

Which is the correct form of Y seen previously.

Therefore, whenever you have doubts as to what the correct form of Y for a product is, just first explicitly list all of terms you expect to see in the result. Then assign each term an undetermined coefficient.

Remember, however, the result obtained still needs to be compared against the complementary solution for shared term(s). If there is any term in common, then the entire complex of product that is the choice for Y must be multiplied by t. Repeat as necessary.

Example: $$y'' + 25y = 4t^3 \sin(5t) - 2e^{3t} \cos(5t)$$

The complementary solution is $y_c = C_1 \cos(5t) + C_2 \sin(5t)$. Let's break up $g(t)$ into 2 parts and work on them individually.

$g_1(t) = 4t^3 \sin(5t)$ is a product of a degree 3 polynomial and a sine function. Therefore, Y_1 should be a product of a generic degree 3 polynomial and both cosine and sine:

$$Y_1 = (At^3 + Bt^2 + Ct + D) \cos(5t) + (Et^3 + Ft^2 + Gt + H) \sin(5t)$$

The validity of the above choice of form can be verified by our second (longer) method. Note that the product of a degree 3 polynomial and both cosine and sine: $(t^3 + t^2 + t + 1)\times(\cos(5t) + \sin(5t))$ contains 8 distinct terms listed below.

$$\begin{array}{llll} t^3\cos(5t) & t^2\cos(5t) & t\cos(5t) & \cos(5t) \\ t^3\sin(5t) & t^2\sin(5t) & t\sin(5t) & \sin(5t) \end{array}$$

Now insert 8 independent undetermined coefficients, one for each:

$$\begin{aligned} Y_1 = {} & At^3\cos(5t) + Bt^2\cos(5t) + Ct\cos(5t) + D\cos(5t) + \\ & Et^3\sin(5t) + Ft^2\sin(5t) + Gt\sin(5t) + H\sin(5t) \end{aligned}$$

However, there is still one important detail to check before we could put the above expression down for Y_1. Is there anything in the expression that is shared with $y_c = C_1 \cos(5t) + C_2 \sin(5t)$? As we can see, there are – both the fourth and the eighth terms. Therefore, we need to multiply everything in this entire expression by t. Hence,

$$\begin{aligned} Y_1 &= t(At^3 + Bt^2 + Ct + D) \cos(5t) + \\ &\qquad t(Et^3 + Ft^2 + Gt + H) \sin(5t) \\ &= (At^4 + Bt^3 + Ct^2 + Dt) \cos(5t) + \\ &\qquad (Et^4 + Ft^3 + Gt^2 + Ht) \sin(5t). \end{aligned}$$

The second half of $g(t)$ is $g_2(t) = -2e^{3t}\cos(5t)$. It is a product of an exponential function and cosine. So our choice of form for Y_2 should be a product of an exponential function with both cosine and sine.

$$Y_2 = Ie^{3t}\cos(5t) + Je^{3t}\sin(5t).$$

There is no conflict with the complementary solution – even though both $\cos(5t)$ and $\sin(5t)$ are present within both y_c and Y_2, they appear alone in y_c, but in products with e^{3t} in Y_2, making them parts of completely different functions. Hence this is the correct choice.

Finally, the complete choice of Y is the sum of Y_1 and Y_2.

$$Y = Y_1 + Y_2 = (At^4 + Bt^3 + Ct^2 + Dt)\cos(5t) + (Et^4 + Ft^3 + Gt^2 + Ht)\sin(5t) + Ie^{3t}\cos(5t) + Je^{3t}\sin(5t).$$

Example: $y'' - 8y' + 12y = t^2 e^{6t} - 7t\sin(2t) + 4$

Complementary solution: $y_c = C_1 e^{2t} + C_2 e^{6t}$.

The form of particular solution is

$Y = (At^3 + Bt^2 + Ct)e^{6t} + (Dt + E)\cos(2t) + (Ft + G)\sin(2t) + H.$

Example: $y'' + 10y' + 25y = t e^{-5t} - 7t^2 e^{2t}\cos(4t) + 3t^2 - 2$

Complementary solution: $y_c = C_1 e^{-5t} + C_2 t e^{-5t}$.

The form of particular solution is

$Y = (At^3 + Bt^2)e^{-5t} + (Ct^2 + Dt + E)e^{2t}\cos(4t) + (Ft^2 + Gt + H)e^{2t}\sin(4t)$
$+ It^2 + Jt + K.$

Example: Find a second order linear equation with constant coefficients whose general solution is

$$y = C_1 e^t + C_2 e^{-10t} + 4t^2.$$

The solution contains three parts, so it must come from a nonhomogeneous equation. The complementary part of the solution, $y_c = C_1 e^t + C_2 e^{-10t}$ suggests that $r = 1$ and $r = -10$ are the two roots of its characteristic equation. Hence, $r - 1$ and $r + 10$ are its two factors. Therefore, the characteristic equation is $(r - 1)(r + 10) = r^2 + 9r - 10$.

The corresponding homogeneous equation is, as a result,

$$y'' + 9y' - 10y = 0.$$

Hence, the nonhomogeneous equation is

$$y'' + 9y' - 10y = g(t).$$

The nonhomogeneous part $g(t)$ results in the particular solution $Y = 4t^2$. As well, $Y' = 8t$ and $Y'' = 8$. Therefore,

$$g(t) = Y'' + 9Y' - 10Y = 8 + 9(8t) - 10(4t^2) = 8 + 72\text{t} - 40\,t^2.$$

The equation with the given general solution is, therefore,

$$y'' + 9y' - 10y = 8 + 72\text{t} - 40\,t^2.$$

The 6 Rules-of-Thumb of the Method of Undetermined Coefficients

1. If an exponential function appears in $g(t)$, the starting choice for $Y(t)$ is an exponential function of the same exponent.

2. If a polynomial appears in $g(t)$, the starting choice for $Y(t)$ is a generic polynomial of the same degree.

3. If either cosine or sine appears in $g(t)$, the starting choice for $Y(t)$ needs to contain both cosine and sine of the same frequency.

4. If $g(t)$ is a sum of several functions, $g(t) = g_1(t) + g_2(t) + \ldots + g_n(t)$, separate it into n parts and solve them individually.

5. If $g(t)$ is a product of basic functions, the starting choice for $Y(t)$ is chosen based on:

   i. $Y(t)$ is a product of the corresponding choices of all the parts of $g(t)$.
   ii. There are as many coefficients as the number of distinct terms in $Y(t)$.
   iii. Each distinct term must have its own coefficient, not shared with any other term.

6. Before finalizing the choice of $Y(t)$, compare it against $y_c(t)$. If there is any shared term between the two, the present choice of $Y(t)$ needs to be multiplied by t. Repeat until there is no shared term.

Remember that, in order to use Rule 6 you always need to find the complementary solution first.

SUMMARY: Method of Undetermined Coefficients

Given $$a\,y'' + b\,y' + c\,y = g(t)$$

1. Find the complementary solution y_c.

2. Subdivide, if necessary, $g(t)$ into parts: $g(t) = g_1(t) + g_2(t) \ldots + g_k(t)$.

3. For each $g_i(t)$, choose the form of its corresponding particular solution $Y_i(t)$ according to:

$g_i(t)$	$Y_i(t)$
$P_n(t)$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)$
$P_n(t)\,e^{at}$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)\,e^{at}$
$P_n(t)\,e^{at}\cos\mu t$ and/or $P_n(t)\,e^{at}\sin\mu t$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_0)\,e^{at}\cos\mu t$ + $t^s(B_n t^n + B_{n-1} t^{n-1} + \ldots + B_0)\,e^{at}\sin\mu t$

Where $s = 0$, 1, or 2, is the **minimum** number of times the choice must be multiplied by t so that it shares no common terms with y_c.

$P_n(t)$ denotes a n-th degree polynomial. If there is no power of t present, then $n = 0$ and $P_0(t) = C_0$ is just the constant coefficient. If no exponential term is present, then set the exponent $a = 0$.

4. $Y = Y_1 + Y_2 + \ldots + Y_k$.

5. The general solution is $y = y_c + Y$.

6. Finally, apply any initial conditions to determine the as yet unknown coefficients C_1 and C_2 in y_c.