Ordinary Differential Equations

Autumn 2017

Last updated:
22nd December 2017 at 14:27

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http://www.jamescannon.net/teaching/ordinary-differential-equations

http://raw.githubusercontent.com/NanoScaleDesign/OrdinaryDifferentialEquations/master/ode.pdf

Contents

Chapter 0

Course information

0.1 This course

This is the Autumn 2017 Ordinary Differential Equations course studied by 2nd-year undegraduate international students at Kyushu University.

0.1.1 How this works

- In contrast to the traditional lecture-homework model, in this course the learning is self-directed and active via publicly-available resources.
- Learning is guided through solving a series of carefully-developed challenges contained in this book, coupled with suggested resources that can be used to solve the challenges with instant feedback about the correctness of your answer.
- There are no lectures. Instead, there is discussion time. Here, you are encouraged to discuss any issues with your peers, teacher and any teaching assistants. Furthermore, you are encouraged to help your peers who are having trouble understanding something that you have understood; by doing so you actually increase your own understanding too.
- Discussion-time is from 14:50 to 16:20 on Fridays at room Centre Zone 1409.
- Peer discussion is encouraged, however, if you have help to solve a challenge, always make sure you do understand the details yourself. You will need to be able to do this in an exam environment. If you need additional challenges to solidify your understanding, then ask the teacher. The questions on the exam will be similar in nature to the challenges. If you can do all of the challenges, you can get 100% on the exam.
- Every challenge in the book typically contains a **Challenge** with suggested **Resources** which you are recommended to utilise in order to solve the challenge. Occasionally the teacher will provide extra **Comments** to help guide your thinking. A **Solution** is also made available for you to check your answer. Sometimes this solution will be given in encrypted form. For more information about encryption, see section 0.3.
- For deep understanding, it is recommended to study the suggested resources beyond the minimum required to complete the challenge.
- The challenge document has many pages and is continuously being developed. Therefore it is advised to view the document on an electronic device rather than print it. The date on the front page denotes the version of the document. You will be notified by email when the document is updated. The content may differ from last-year's document.
- A target challenge will be set each week. This will set the pace of the course and defines the examinable material. It's ok if you can't quite reach the target challenge for a given week, but then you will be expected to make it up the next week.
- You may work ahead, even beyond the target challenge, if you so wish. This can build greater flexibility into your personal schedule, especially as you become busier towards the end of the semester.
- Your contributions to the course are strongly welcomed. If you come across resources that you found useful that were not listed by the teacher or points of friction that made solving a challenge difficult, please let the teacher know about it!

0.1.2 Assessment

In order to prove to outside parties that you have learned something from the course, we must perform summative assessments. This will be in the form of a mid-term exam (weighted 30%), coursework (weighted 20%), a satisfactory challenge-log (weighted 10%) and a final exam (weighted 40%).

Your final score is calculated as Max(final exam score, weighted score), however you must pass the final exam to pass the course.

0.1.3 What you need to do

- Prepare a challenge-log in the form of a workbook or folder where you can clearly write the calculations you perform to solve each challenge. This will be a log of your progress during the course and will be occasionally reviewed by the teacher.
- You need to submit a brief report at `https://goo.gl/forms/S69DuM4xCss0WtjH3` by 8am on the day of the class. Here you can let the teacher know about any difficulties you are having and if you would like to discuss anything in particular.
- Please bring a wifi-capable internet device to class, as well as headphones if you need to access online components of the course during class. If you let me know in advance, I can lend computers and provide power extension cables for those who require them (limited number).

0.2 Timetable

	Discussion	**Target**	**Note**
1	4 Oct	-	Wednesday class
2	13 Oct	3.2	
3	20 Oct	3.9	
4	27 Oct	4.7	
5	10 Nov	4.14	
6	17 Nov	4.18	
7	24 Nov	5.6	
8	1 Dec	5.11	
9	8 Dec	Midterm exam	
10	15 Dec	6.3	
11	22 Dec	6.7	
12	15 Jan	7.4	Monday class, Coursework assignment
13	19 Jan		
14	26 Jan	Coursework	Coursework submission
15	9 Feb	Final exam	

Example: To keep pace with the course, you should aim to complete challenge 2 of chapter 3 by the 13th of October.

0.3 Hash-generation

Some solutions to challenges are encrypted using MD5 hashes. In order to check your solution, you need to generate its MD5 hash and compare it to that provided. MD5 hashes can be generated at the following sites:

- Wolfram alpha: (For example: md5 hash of “q1.00”) `http://www.wolframalpha.com/input/?i=md5+hash+of+%22q1.00%22`
- `www.md5hashgenerator.com`

Since MD5 hashes are very sensitive to even single-digit variation, you must enter the solution *exactly*. This means maintaining a sufficient level of accuracy when developing your solution, and then entering the solution according to the format suggested by the question. Some special input methods:

Solution	**Input**
5×10^{-476}	5.00e-476
5.0009×10^{-476}	5.00e-476
$-\infty$	-infinity (never “infinite”)
2π	6.28
i	im(1.00)
2i	im(2.00)
1 + 2i	re(1.00)im(2.00)
-0.0002548 i	im(-2.55e-4)
1/i = i/-1 = -i	im(-1.00)
$e^{i2\pi}$ $[= cos(2\pi) + isin(2\pi) = 1 + i0 = 1]$	1.00
$e^{i\pi/3}$ $[= cos(\pi/3) + isin(\pi/3) = 0.5 + i0.87]$	re(0.50)im(0.87)
Choices in order A, B, C, D	abcd

The first 6 digits of the MD5 sum should match the first 6 digits of the given solution.

Chapter 1

Hash practise

1.1 Hash practise: Integer

X = 46.3847
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

hash of aX = e77fac

1.2 Hash practise: Decimal

X = 49
Form: Two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

hash of bX = 82c9e7

1.3 Hash practise: String

X = abcdef
Form: String.
Place the indicated letter in front of the number.
Example: aX where $X = abc$ is entered as aabc

hash of cX = 990ba0

1.4 Hash practise: Scientific form

X = 500,765.99
Form: Scientific notation with the mantissa in standard form to 2 decimal place and the exponent in integer form.
Place the indicated letter in front of the number.
Example: aX where $X = 4 \times 10^{-3}$ is entered as a4.00e-3

hash of dX = be8a0d

Chapter 2

Definitions

2.1 Order of a differential equation

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

What is the sum of the orders of the following equations?

$$\frac{dy}{dx}A = 5x^3 + 3 \tag{2.1}$$

$$cos(y)y'''(x) - y(x) = 25 \tag{2.2}$$

$$\frac{d}{dx}\frac{d^2y}{dx^2} = \frac{x^{-2}}{3} \tag{2.3}$$

Solution

X = Your solution
Form: Integer
Place the indicated letter in front of the number
Example: aX where $X = 46$ is entered as a46

hash of eX = 492585

2.2 Identifying linear and non-linear differential equations

Comment

Being able to identify linear and non-linear ODE's will help you understand how to approach different problems.

Generally speaking, the differential equation is linear if the functions and orders of the differentials are linear. For example,

$$y'' - 4yx = lnx - y$$

can be shown to be linear. Rearranging to collect all the y-terms together:

$$y'' - 4yx + y = lnx$$

the dependent variable y and its derivatives are each of the first degree and depend only on a constant or the independent variable.

An example of a non-linear equation however would be

$$5 + yy' = x - y$$

or

$$yy' + y = x - 5$$

The fact that y' is multiplied by y results in a non-linear equation in y.

Challenge

Sum the points corresponding to the equations that are linear. You may be able to judge some by eye, but you should prove mathematically that at least one of the equations are linear and at least one of the equations are non-linear.

1 point: $\dfrac{dy}{dt} = 5t^3 + 3$.

2 points: $cos(y)y'''(t) - y(t) = 25$.

4 points: $\dfrac{d}{dt}\dfrac{d^2y}{dt^2} = \dfrac{t^{-2}}{3}$.

8 points: $y'(t) - sin(y(t)) = 0$.

16 points: $y'(t) - y(t) = 0$.

32 points: $ty'(t) - y(t) = 0$.

Solution

X = Your solution
Form: Integer
Place the indicated letter in front of the number
Example: aX where $X = 46$ is entered as a46

hash of rX = f5d2c0

2.3 Linear differential equations vs non-linear differential equations

Resources

- Wikipedia: https://en.wikipedia.org/wiki/Nonlinear_system#Nonlinear_differential_equations
- Wikipedia: https://en.wikipedia.org/wiki/Linear_differential_equation

Challenge

Write no-more than 1 short paragraph describing in qualitative terms the difference between a linear and non-linear differential equation.

Solution

Please compare with your partner in class and discuss with the teacher if you are unsure.

2.4 Valid solutions

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Use substitution to prove that

$$y = \frac{5}{5+x} \tag{2.4}$$

is a solution to the equation

$$xy' + y = y^2 \tag{2.5}$$

and state the value of x for which the solution is undefined.

Solution

Value of x for which solution is undefined:

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of tX = 829f33

2.5 Range of valid solutions

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Use substitution to prove that

$$y = -\sqrt{100 - x^2} \tag{2.6}$$

is a solution to the equation

$$x + yy' = 0 \tag{2.7}$$

and state the range of x for which the solution is valid. Enter the value of the lower range as the solution below.

Solution

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of yX = d96920

Chapter 3

1st-order differential equations

3.1 Determining a simple DE from a description

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference with the ambient surroundings. (a) Write a differential equation describing this situation. (b) Assuming a proportionality constant of 0.2 /hour, what is the rate of temperature change when the object is at 30 °C and the ambient temperature is 20 °C?

Solution

(units: $^{\circ}\mathrm{C\,h^{-1}}$)

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of qX = 4aca8d

3.2 Direction (Slope) fields

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/DirectionFields.aspx`
- Video 1: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/creating-a-slope-field`
- Video 2: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/slope-field-to-visualize-solutions`

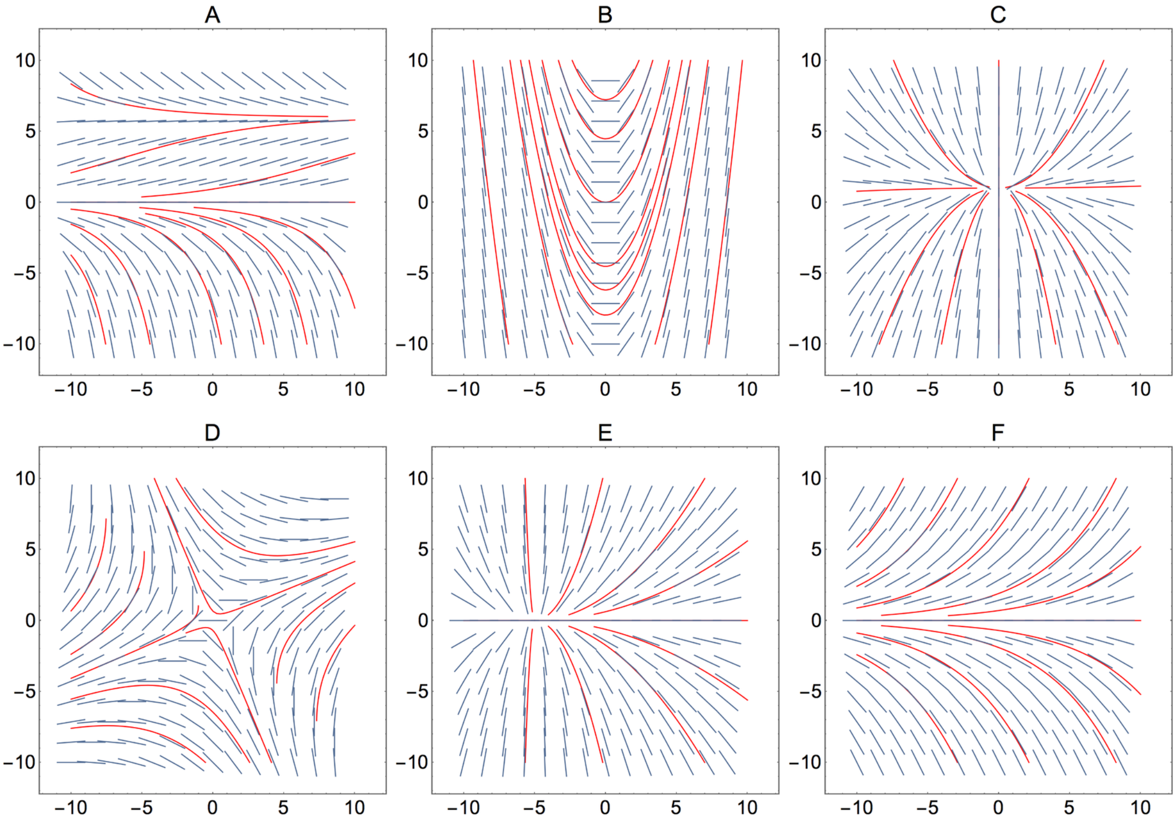
Comment

It is good practise to try drawing the below fields before looking at the next page. You need to be able to go in both directions (ie, drawing and recognising). You will not be given a glimps at the fields in the exam prior to being asked to draw them.

Question

Try drawing the slope field for at least 3 of the equations given below (your choice). Then, put the slope fields given on the next page in the same order as these equations.

1. $y' = x$
2. $y' = 0.2y$
3. $y' = 0.2y(1 - y/6)$
4. $y' = (x - y)/(x + y)$
5. $y' = 2(y - 1)/x$
6. $y' = 2y/(x + 5)$



Solution

X = Your solution
Form: String.
Place the indicated letter in front of the string.
Example: aX where X = abcdef is entered as aabcdef

Hash of qX = e93bfe

3.3 Separable equations I

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction`
- Video II: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example`
- Text:`http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx`

Comment

Let's start with a fundamental equation:

$$\frac{dy}{dt} = y \tag{3.1}$$

This is saying that the slope (the rate of change of y) linearly depends on y. That is, that as the value of y increases, the slope also increases; a positive feedback loop. In fact, you get an exponentially-increasing function.

So one aim of this course is to be able to solve such equations mathematically. But I also want you to understand the "physical" meaning of the relation between y and its slope, and how this leads to such a fundamental function such as an exponential.

Challenge

Considering the equation

$$\frac{dy}{dt} = y \tag{3.2}$$

solve for y.

Solution

To check your answer, solve for $y(5)$ given the initial condition $y(0) = 1$.

$y(5) = 148.413$

3.4 Separable equations II

Resources

- Video I: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction
- Video II: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example
- Text:http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx

Challenge

a) Now consider what is meant, physically speaking, by the relation:

$$\frac{dy}{dt} = -y \tag{3.3}$$

Why does it tend to zero for increasing t?

b) Solve for y.

Solution

a) Please compare your solution with your partner or discuss with the teacher.

b) To check your answer, solve for $y(5)$ given the initial condition $y(0) = 1$.

$y(5) = 0.00674$

3.5 Separable equations III

Resources

- Video I: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction
- Video II: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example
- Text:http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx

Challenge

a) Now consider when the slope of y not only depends on y but also on t:

$$\frac{dy}{dt} = ty \tag{3.4}$$

b) or on a constant a:

$$\frac{dy}{dt} = ay \tag{3.5}$$

See how the feedback is greater or lesser, depending on the constant or variable placed in front of y?

Solution

a) Solve for $y(5)$ under the initial condition $y(0) = 1$

268,337

b) Solve for $y(5)$ under the initial condition $y(0) = 1$ and with $a = 2$

22,026.5

3.6 Separable equations IV

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction`
- Video II: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example`
- Text:`http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx`

Challenge

Determine $y(t)$ for

$$\frac{dy}{dt} = e^t \tag{3.6}$$

Again, think about what is happening here. Do you see the link with challenge 3.3? There we wrote in terms of y. Here we write in terms of e^t. Do you see they're the same thing?

Solution

To check your answer, solve for $y(3)$ given the initial condition $y(0) = 1$.

$y(3) = 20.09$

3.7 Rate of growth

Resources

- Video: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/logistic-differential-equation/v/modeling-population-with-differential-equations`

Comment

One interesting application of 1st-order differential equations is that of population growth.

Challenge

Assuming there is no-limit on growth, a given bacteria would be able to reproduce at such a rate that the amount of bacteria measured in mg increases by 20% every 25 hours. Derive an expression for the rate of growth.

Solution

To check your answer, calculate the rate of growth when there are 20 mg of bacteria.

0.146 mg/hour

3.8 Logistic equation

Resources

- Videos: The 4 remaining logistic differential equation videos starting at: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/logistic-differential-equation/v/logistic-differential-equation-intuition`

Comment

We considered exponential growth, but in real life there is often a limit to this. This is where the logistic equation is useful.

Challenge

Assuming there is no-limit on growth, a given bacteria would be able to reproduce at such a rate that the amount of bacteria measured in mg increases by 20% every 25 hours. However, due to environmental factors the limiting (maximum) amount of bacteria that can exist in the system at any one time is 400 mg. Assuming an initial amount of bacteria of 20 mg, how much time, rounded to the nearest integer hours, must one wait to reach 100 mg of bacteria?

Solution

253.1 hours

3.9 Autonomous differential equations

Resources

- Wikipedia: https://en.wikipedia.org/wiki/Autonomous_system_(mathematics)

Challenge

The logistic equation is an example of an autonomous differential equation. Add the points of the autonomous differential equations in the following list:

1 point: $y' = cos(y) - 5$

2 points: $y' = cos(y)/x - 5$

4 points: $y' = cos(y)/x - 5/x$

8 points: $y^2 = y'y + 5$

16 points: $xy' = 5y$

32 points: $y' = 1$

Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of fX = 1227c7

3.10 The stability of solutions I

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`
- Text: `http://www.math.psu.edu/tseng/class/Math251/Notes-1st%20order%20ODE%20pt2.pdf`

Challenge

Considering the logistic equation $N' = 0.2N(1 - N/6)$, make 3 separate lists containing any equilibrium, semi-stable and unstable y-values.

To check your answer, sum the value of each list. If there are no values in a list, enter -999 to check the result.

Solution

Stable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of gX = 4a4314

Semi-stable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of hX = 9df203

Unstable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of jX = 17cb7f

3.11 The stability of solutions II

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`
- Text: `http://www.math.psu.edu/tseng/class/Math251/Notes-1st%20order%20ODE%20pt2.pdf`

Challenge

Considering the differential equation $y' = (y^2 - 16)(y+3)^2$, make 3 separate lists containing any equilibrium, semi-stable and unstable y-values.

To check your answer, sum the value of each list. If there are no values in a list, simply enter "none" to check the result.

Solution

Stable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of kX = ffc446

Semi-stable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of zX = f76cc4

Unstable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

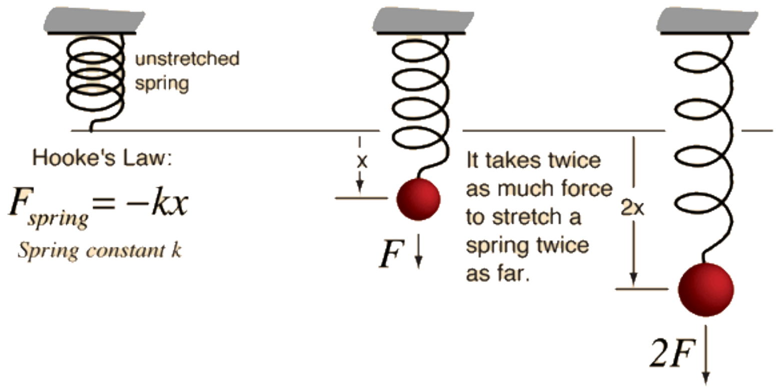
Hash of xX = bf947d

Chapter 4

2nd-order differential equations

4.1 Hooke's law

Comment



(Image from HyperPhysics by Rod Nave, Georgia State University)

Second-order differential equations deal with oscillations. Here we consider harmonic oscillation of a spring. The aim of this challenge is to give you the opportunity to think about how the terms of a 2nd-order ODE relate to force and stiffness in the context of a spring.

Equation 4.1 is a fundamental equation of mechanics describing oscillatory motion such as the spring here. Hooke's law states that the force leading to acceleration of the mass m is proportional to the stretching distance x. The proportionality constant is Hooke's constant, k.

$$mx'' + kx = 0 \tag{4.1}$$

or alternatively

$$mx'' = -kx \tag{4.2}$$

This leads to perfectly oscillating motion,

$$x(t) = \cos(\omega t) \tag{4.3}$$

which oscillates forever since there is no damping term.

Challenge

By considering the oscillatory motion (equation 4.3) as a solution of the 2nd-order differential equation given by Hooke's law (equations 4.1 and 4.2), determine the oscillation frequency ω in terms of the mass and spring constant.

Solution

To check your answer, calculate the oscillation frequency for a harmonic spring with a mass of 2 kg and spring-constant of 4 kg/s^2. Only enter numbers, without any units, in your answer.

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of gX = 9553fe

4.2 Exponentials and trigonometry

Resources

- Text: `https://www.phy.duke.edu/~rgb/Class/phy51/phy51/node15.html`
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Challenge

Write $\sin(x)$ and $\cos(x)$ in exponential form.

Solution

Please compare your solution with your partner or discuss with the teacher.

4.3 Characteristic equation: understanding

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 106.

Comment

It is possible to add a damping term B to Hooke's law that is proportional to the velocity of the movement. You could imagine this as a friction term, with the force from friction becoming stronger as the velocity increases.

Challenge

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0 \tag{4.4}$$

Show that, assuming that all solutions to a 2nd-order differential equation of the form above will have solutions $y(t) = e^{rt}$, the value of r can in principle be determined by solving the following a quadratic equation of the form

$$Ar^2 + Br + C = 0 \tag{4.5}$$

Solution

If you are unsure of your derivation, please ask someone.

4.4 Characteristic equation: roots

Resources

- Book (http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N) from page 106.
- Video (student suggestion): https://www.youtube.com/watch?v=gdr4dSmzZ8Q

Challenge

Sum the points of the differential equations that have characteristic equations with

- Real, distinct roots
- Complex roots
- Equal roots

1 point: $-3y'' - 5y' + 2y = 0$

2 points: $3y'' - 4y' + 3y = 0$

4 points: $3y'' - 6y' + 3y = 0$

8 points: $3y'' - 5y' + 2y = 0$

16 points: $3y'' - 5y' + 4y = 0$

32 points: $3y'' + 5y' + 2y = 0$

Solution

Real, distinct roots

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of iX = dc6ada

Complex roots

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of jX = 7c030b

Equal roots

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of kX = c90b44

4.5 Characteristic equation: real negative roots

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 108.
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Challenge

1. Solve the following 2nd-order differential equation that has real roots:

$$y'' + 3y' + 2y = 0 \tag{4.6}$$

2. What is the effect of the damping term here?

Solution

1. $y(1) = 1.14$ given initial conditions $y(0) = 5$ and $y'(0) = -8$.
2. Please compare your answer with your partner or discuss with the teacher in class.

4.6 Characteristic equation: real positive roots

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 108.
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Comment

Here we include a damping term again, but this time it is negative and this is enough to flip the sign of the roots.

Challenge

1. Solve the following 2nd-order differential equation that has real roots.

$$y'' - 3y' + 2y = 0 \tag{4.7}$$

2. What is the effect of the damping term here?

Solution

1. $y(1) = -47.13$ given initial conditions $y(0) = 5$ and $y'(0) = -8$.
2. Please compare your answer with your partner or discuss with the teacher in class.

4.8 Characteristic equation: equal roots

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 117.
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Comment

It is not necessary to follow the full derivation in the suggested resource.

Challenge

Solve the equation

$$y'' - 2y' + y = 0 \tag{4.8}$$

subject to the initial conditions $y(0) = 5$ and $y'(0) = 6$.

Solution

$y(1) = 16.310$

4.9 Characteristic equation: complex roots with B=0

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 112.
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Challenge

1. Assuming there is no damping term (ie, $B = 0$) show that the roots for the differential equation

$$Ay'' + Cy = 0 \tag{4.9}$$

are $\pm i\sqrt{C/A}$.

2. Solve the following ODE:

$$y'' + 4\pi^2 y = 0 \tag{4.10}$$

subject to the initial conditions $y(0) = 4$ and $y'(0) = 10\pi$.

Solution

$y(0.4) = -0.297$

4.10 Characteristic equation: complex roots with non-zero B I

Resources

- Book (http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N) from page 112.
- Text: http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf

Challenge

Solve the following ODE:

$$y'' + y' + y = 0 \tag{4.11}$$

subject to initial conditions $y(0) = 8$ and $y'(0) = 2$. One of the integration constants is $4\sqrt{3}$. You will need to find the other one.

Solution

$y(0.4) = 8.0867$

4.11 Characteristic equation: complex roots with non-zero B II

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 112.
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Challenge

Solve the following ODE:

$$y'' - y' + y = 0 \tag{4.12}$$

subject to initial conditions $y(0) = 1$ and $y'(0) = 2$. One of the integration constants is $\sqrt{3}$. You will need to find the other one.

Solution

$y(0.4) = 1.867$

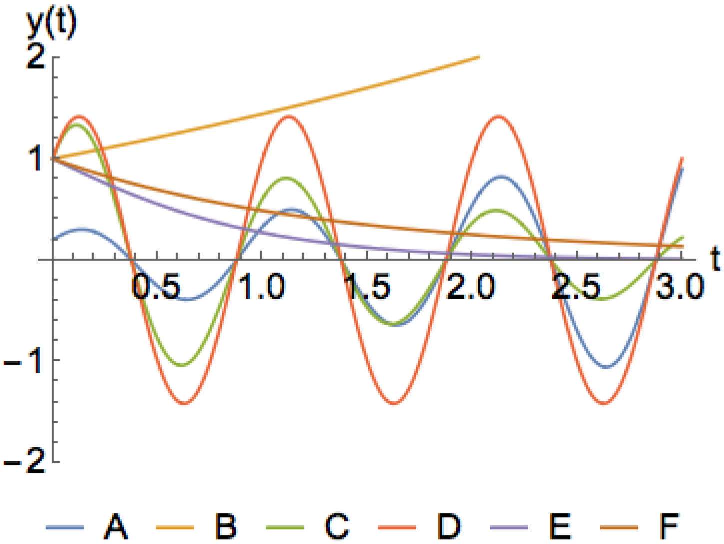
4.12 Damping

Resources

- Wikipedia: `https://en.wikipedia.org/wiki/Damping`
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Challenge

Of the 6 functions shown in the graph, place the 3 that correspond to over-damped, critically damped and under-damped in the order mentioned in this sentence.



Solution

(eg, “abc”)

X = Your solution
Form: String.
Place the indicated letter in front of the string.
Example: aX where X = abcdef is entered as aabcdef

Hash of tX = b3f888

4.13 Damping and 2nd-order differential equations

Challenge

1. The 6 functions shown in the graph in challenge 4.12 may represent solutions of a 2nd-order differential equation $Ay'' + By' + Cy = 0$. Assuming $A > 0$ and $C > 0$, place the solutions A-F in the order shown below.

I. Solution of a 2nd-order differential equation with real negative roots.

II. Solution of a 2nd-order differential equation with real positive roots.

III. Solution of a 2nd-order differential equation with equal roots.

IV. Solution of a 2nd-order differential equation with complex roots and B=0.

V. Solution of a 2nd-order differential equation with complex roots and positive damping.

VI. Solution of a 2nd-order differential equation with complex roots and negative damping.

Solution

(eg, "abcdef")

X = Your solution
Form: String.
Place the indicated letter in front of the string.
Example: aX where X = abcdef is entered as aabcdef

Hash of uX = 33db25

4.14 Characteristic equation: exercises

(Note that if you encounter a square-root during your calculations such as $\sqrt{7}$, it is best to work with $\sqrt{7}$ rather than 2.65 in order to maintain accuracy until the final step where you need to evaluate it. If the equation becomes too messy (eg $e^{(\sqrt{7}-1)/\sqrt{3}}$) you can always substitute $m = (\sqrt{7}-1)/\sqrt{3}$, etc, to make things clearer.)

Challenge

1. Determine $y(1)$ for the equation

$$2y'' + 8y' + y = 0 \tag{4.13}$$

given the initial conditions $y(0) = 4$ and $y'(0) = 3$.

2. Determine $y(0.2)$ for the equation

$$2y'' + 4y' + 2y = 0 \tag{4.14}$$

given the initial conditions $y(0) = 4$ and $y'(0) = 2$.

3. Determine $y(0.1)$ for the equation

$$4y'' + 3y' + y = 0 \tag{4.15}$$

given the initial conditions $y(0) = 6$ and $y'(0) = 2$.

Solution

1. 4.32
2. 4.26
3. 6.19

4.15 Non-homogeneous equations: Method of undetermined coefficients

Resources

- Video: All 4 Khan Academy videos starting at `https://www.khanacademy.org/math/differential-equations/second-order-differential-equations/undetermined-coefficients/v/undetermined-coefficients-1`

Comment

The 2nd-order equations we were considering until now were homogeneous equations (ie, the RHS was zero). We can now build upon this to expand our ability to solve non-homogeneous equations (ie, where the RHS of the equation is non-zero).

The Khan Academy videos give an excellent initial introduction to the subject, and so please do take the time to view and take notes about all four videos in the series.

In the 4th video Mr Kahn describes about how it is possible to add solutions if there are multiple terms on the right. This occasionally causes confusion. Consider for example:

$$y'' - 3y' - 4y = 2\sin x \tag{4.16}$$

This corresponds to the particular solution

$$y_p = A\sin x + B\cos x \tag{4.17}$$

A common point of confusion is about what to do in the case of something like

$$y'' - 3y' - 4y = 2\sin x + 2\cos x \tag{4.18}$$

Should you just write $y_p = (A\sin x + B\cos x) + (C\sin x + D\cos x)$? After all, you have two terms in equation 4.18 (ie, $2\sin x$ and $2\cos x$). You can note however that $A\sin x + C\sin x$ simplifies to $E\sin x$ where E is just another constant (in this case $A + B$) so in the end you will be left with $y_p = E\sin x + F\cos x$. So while it may be clearer to explicitly calculate coefficients for every term on the RHS, in many cases the terms will simplify.

Challenge

Find the general solution of the following non-homogeneous differential equations:

1. $y'' + 4y = 8$
2. $y'' + 4y = 8t^2 - 20t + 8$
3. $y'' + 4y = 5\sin 3t - 5\cos 3t$
4. $y'' + 4y = 24e^{-2t}$

Solution

The solutions are contained in the list on the next page in no particular order. Your answers should match one of the solutions given. Please try to not look at the solutions before completing the questions, since this will facilitate deep understanding and reproduce a real-life/exam environment.

$y = C_1 \cos 2t + C_2 \sin 2t + 3e^{-2t}$
$y = C_1 \cos 2t + C_2 \sin 2t + 8e^{-2t}$
$y = C_1 \cos 2t + C_2 \sin 2t + 2t^2 - 5t + 1$
$y = C_1 \cos 2t + C_2 \sin 2t + 3t^2 + t + 3$
$y = C_1 \cos 2t + C_2 \sin 2t + \cos 3t - \sin 3t$
$y = C_1 \cos 2t + C_2 \sin 2t + 2$
$y = C_1 \cos 2t + C_2 \sin 2t + 5$

4.16 Method of undetermined coefficients II

Comment

The following pages go into more detail than the videos, considering a greater range of cases. You may note that here the particular solution is denoted by Y while Sal Khan denoted it as y_p in the videos.

The following notes were developed by Zachary S. Tseng at Pennsylvania State University, USA (`http://www.math.psu.edu/tseng/`). Included here with kind permission.

A (possible) glitch?

There is a complication that occurs under a certain circumstance…

Example: $y'' - 2y' - 3y = 5e^{3t}$

The old news is that $y_c = C_1 e^{-t} + C_2 e^{3t}$. Since $g(t) = 5e^{3t}$, we should be able to use the form $Y = Ae^{3t}$, just like in the first example, right? But if we substitute Y, $Y' = 3Ae^{3t}$, and $Y'' = 9Ae^{3t}$ into the differential equation and simplify, we would get the equation

$$0 = 5e^{3t}.$$

That means there is no solution for A. Our method (that has worked well thus far) seems to have failed. The same outcome (an inability to find A) also happens when $g(t)$ is a multiple of e^{-t}. But, for any other exponent our choice of the form for Y works. What is so special about these two particular exponential functions, e^{3t} and e^{-t}, that causes our method to misfire? (Hint: What is the complementary solution of the nonhomgeneous equation?)

The answer is that those two functions are exactly the terms in y_c. Being a part of the complementary solution (the solution of the corresponding homogeneous equation) means that any constant multiple of either functions will ALWAYS results in zero on the right-hand side of the equation. Therefore, it is impossible to match the given $g(t)$.

The cure: The remedy is surprisingly simple: <u>multiply our usual choice by t</u>. In the above example, we should instead use the form $Y = Ate^{3t}$.

In general, whenever your initial choice of the form of Y has any term in common with the complementary solution, then you must alter it by multiplying your initial choice of Y by t, <u>as many times as necessary but no more than necessary</u>.

Example: $y'' - 6y' + 9y = e^{3t}$

The complementary solution is $y_c = C_1 e^{3t} + C_2 t e^{3t}$. $g(t) = e^{3t}$, therefore, the initial choice would be $Y = A e^{3t}$. But wait, that is the same as the first term of y_c, so multiply Y by t to get $Y = A t e^{3t}$. However, the new Y is now in common with the second term of y_c. Multiply it by t again to get $Y = A t^2 e^{3t}$. That is the final, correct choice of the general form of Y to use. (*Exercise*: Verify that neither $Y = A e^{3t}$, nor $Y = A t e^{3t}$ would yield an answer to this problem.)

Once we have established that $Y = A t^2 e^{3t}$, then $Y' = 2Ate^{3t} + 3At^2 e^{3t}$, and $Y'' = 2A e^{3t} + 12Ate^{3t} + 9At^2 e^{3t}$. Substitute them back into the original equation:

$$(2Ae^{3t} + 12Ate^{3t} + 9At^2 e^{3t}) - 6(2Ate^{3t} + 3At^2 e^{3t}) + 9(At^2 e^{3t}) = e^{3t}$$

$$2Ae^{3t} + (12 - 12)Ate^{3t} + (9 - 18 + 9)At^2 e^{3t} = e^{3t}$$

$$2Ae^{3t} = e^{3t}$$

$$A = 1/2$$

Hence, $Y(t) = \frac{1}{2} t^2 e^{3t}$.

Therefore, $y = C_1 e^{3t} + C_2 te^{3t} + \frac{1}{2} t^2 e^{3t}$. Our "cure" has worked!

Since a second order linear equation's complementary solution only has two parts, there could be at most two shared terms with Y. Consequently we would only need to, at most, apply the cure twice (effectively multiplying by t^2) as the worst case scenario.

The lesson here is that you should always find the complementary solution first, since the correct choice of the form of Y depends on y_c. Therefore, you need to have y_c handy before you write down the form of Y. Before you finalize your choice, always compare it against y_c. And if there is anything those two have in common, multiplying your choice of form of Y by t. (However, you should do this ONLY when there actually exists something in common; you should never apply this cure unless you know for sure that a common term exists between Y and y_c, else you will not be able to find the correct answer!) Repeat until there is no shared term.

When $g(t)$ is a product of several functions

If $g(t)$ is a product of two or more simple functions, e.g. $g(t) = t^2 e^{5t}\cos(3t)$, then our basic choice (before multiplying by t, if necessary) should be a product consist of the corresponding choices of the individual components of $g(t)$. One thing to keep in mind: <u>that there should be only as many undetermined coefficients in Y as there are distinct terms</u> (after expanding the expression and simplifying algebraically).

Example: $$y'' - 2y' - 3y = t^3 e^{5t}\cos(3t)$$

We have $g(t) = t^3 e^{5t}\cos(3t)$. It is a product of a degree 3 polynomial[†], an exponential function, and a cosine. Out choice of the form of Y therefore must be a product of their corresponding choices: a generic degree 3 polynomial, an exponential function, and both cosine and sine. Try

Correct form: $Y = (At^3 + Bt^2 + Ct + D)\, e^{5t}\cos(3t) +$
$(Et^3 + Ft^2 + Gt + H)\, e^{5t}\sin(3t)$

Wrong form: $Y = (At^3 + Bt^2 + Ct + D)\, E\, e^{5t}(F\cos(3t) + G\sin(3t))$

Note in the correct form above, each of the eight distinct terms has its own unique undetermined coefficient. Here is another thing to remember: that those coefficients should all be <u>independent</u> of each others, each uniquely associated with only one term.

In short, when $g(t)$ is a product of basic functions, $Y(t)$ is chosen based on:

i. $Y(t)$ is a product of the corresponding choices of all the parts of $g(t)$.
ii. There are as many coefficients as the number of distinct terms in $Y(t)$.
iii. Each distinct term must have its own coefficient, not shared with any other term.

[†] A power such as t^n is really just an n-th degree polynomial with only one (the n-th term's) nonzero coefficient.

Another way (longer, but less prone to mistakes) to come up with the correct form is to do the following.

Start with the basic forms of the corresponding functions that are to appear in the product, without assigning any coefficient. In the above example, they are $(t^3 + t^2 + t + 1)$, e^{5t}, and $\cos(3t) + \sin(3t)$.

Multiply them together to get all the distinct terms in the product:

$$\begin{aligned}&(t^3 + t^2 + t + 1)\, e^{5t}(\cos(3t) + \sin(3t))\\ &= t^3 e^{5t} \cos(3t) + t^2 e^{5t} \cos(3t) + t\, e^{5t} \cos(3t) + e^{5t} \cos(3t)\\ &+ t^3 e^{5t} \sin(3t) + t^2 e^{5t} \sin(3t) + t\, e^{5t} \sin(3t) + e^{5t} \sin(3t)\end{aligned}$$

Once we have expanded the product and identified the distinct terms in the product (8, in this example), then we insert the undetermined coefficients into the expression, one for each term:

$$\begin{aligned}Y &= A\, t^3 e^{5t} \cos(3t) + B\, t^2 e^{5t} \cos(3t) + C\, t\, e^{5t} \cos(3t)\\ &+ D\, e^{5t} \cos(3t) + E\, t^3 e^{5t} \sin(3t) + F\, t^2 e^{5t} \sin(3t) + G\, t\, e^{5t} \sin(3t)\\ &+ H\, e^{5t} \sin(3t)\end{aligned}$$

Which is the correct form of Y seen previously.

Therefore, whenever you have doubts as to what the correct form of Y for a product is, just first explicitly list all of terms you expect to see in the result. Then assign each term an undetermined coefficient.

Remember, however, the result obtained still needs to be compared against the complementary solution for shared term(s). If there is any term in common, then the entire complex of product that is the choice for Y must be multiplied by t. Repeat as necessary.

Example: $$y'' + 25y = 4t^3 \sin(5t) - 2e^{3t}\cos(5t)$$

The complementary solution is $y_c = C_1 \cos(5t) + C_2 \sin(5t)$. Let's break up $g(t)$ into 2 parts and work on them individually.

$g_1(t) = 4t^3 \sin(5t)$ is a product of a degree 3 polynomial and a sine function. Therefore, Y_1 should be a product of a generic degree 3 polynomial and both cosine and sine:

$$Y_1 = (At^3 + Bt^2 + Ct + D)\cos(5t) + (Et^3 + Ft^2 + Gt + H)\sin(5t)$$

The validity of the above choice of form can be verified by our second (longer) method. Note that the product of a degree 3 polynomial and both cosine and sine: $(t^3 + t^2 + t + 1)\times(\cos(5t) + \sin(5t))$ contains 8 distinct terms listed below.

$$\begin{array}{llll} t^3\cos(5t) & t^2\cos(5t) & t\cos(5t) & \cos(5t) \\ t^3\sin(5t) & t^2\sin(5t) & t\sin(5t) & \sin(5t) \end{array}$$

Now insert 8 independent undetermined coefficients, one for each:

$$\begin{aligned} Y_1 = {} & At^3\cos(5t) + Bt^2\cos(5t) + Ct\cos(5t) + D\cos(5t) + \\ & Et^3\sin(5t) + Ft^2\sin(5t) + Gt\sin(5t) + H\sin(5t) \end{aligned}$$

However, there is still one important detail to check before we could put the above expression down for Y_1. Is there anything in the expression that is shared with $y_c = C_1 \cos(5t) + C_2 \sin(5t)$? As we can see, there are – both the fourth and the eighth terms. Therefore, we need to multiply everything in this entire expression by t. Hence,

$$\begin{aligned} Y_1 &= t(At^3 + Bt^2 + Ct + D)\cos(5t) + \\ &\qquad t(Et^3 + Ft^2 + Gt + H)\sin(5t) \\ &= (At^4 + Bt^3 + Ct^2 + Dt)\cos(5t) + \\ &\qquad (Et^4 + Ft^3 + Gt^2 + Ht)\sin(5t). \end{aligned}$$

The second half of $g(t)$ is $g_2(t) = -2e^{3t}\cos(5t)$. It is a product of an exponential function and cosine. So our choice of form for Y_2 should be a product of an exponential function with both cosine and sine.

$$Y_2 = Ie^{3t}\cos(5t) + Je^{3t}\sin(5t).$$

There is no conflict with the complementary solution – even though both $\cos(5t)$ and $\sin(5t)$ are present within both y_c and Y_2, they appear alone in y_c, but in products with e^{3t} in Y_2, making them parts of completely different functions. Hence this is the correct choice.

Finally, the complete choice of Y is the sum of Y_1 and Y_2.

$$Y = Y_1 + Y_2 = (At^4 + Bt^3 + Ct^2 + Dt)\cos(5t) + (Et^4 + Ft^3 + Gt^2 + Ht)\sin(5t) + Ie^{3t}\cos(5t) + Je^{3t}\sin(5t).$$

Example: $y'' - 8y' + 12y = t^2 e^{6t} - 7t\sin(2t) + 4$

Complementary solution: $y_c = C_1 e^{2t} + C_2 e^{6t}$.

The form of particular solution is

$Y = (At^3 + Bt^2 + Ct)e^{6t} + (Dt + E)\cos(2t) + (Ft + G)\sin(2t) + H.$

Example: $y'' + 10y' + 25y = t e^{-5t} - 7t^2 e^{2t}\cos(4t) + 3t^2 - 2$

Complementary solution: $y_c = C_1 e^{-5t} + C_2\, t e^{-5t}$.

The form of particular solution is

$Y = (At^3 + Bt^2)e^{-5t} + (Ct^2 + Dt + E)\, e^{2t}\cos(4t) + (Ft^2 + Gt + H)\, e^{2t}\sin(4t)$
$+ It^2 + Jt + K.$

Example: Find a second order linear equation with constant coefficients whose general solution is

$$y = C_1 e^t + C_2 e^{-10t} + 4t^2.$$

The solution contains three parts, so it must come from a nonhomogeneous equation. The complementary part of the solution, $y_c = C_1 e^t + C_2 e^{-10t}$ suggests that $r = 1$ and $r = -10$ are the two roots of its characteristic equation. Hence, $r - 1$ and $r + 10$ are its two factors. Therefore, the characteristic equation is $(r - 1)(r + 10) = r^2 + 9r - 10$.

The corresponding homogeneous equation is, as a result,

$$y'' + 9y' - 10y = 0.$$

Hence, the nonhomogeneous equation is

$$y'' + 9y' - 10y = g(t).$$

The nonhomogeneous part $g(t)$ results in the particular solution $Y = 4t^2$. As well, $Y' = 8t$ and $Y'' = 8$. Therefore,

$$g(t) = Y'' + 9Y' - 10Y = 8 + 9(8t) - 10(4t^2) = 8 + 72\text{t} - 40\, t^2.$$

The equation with the given general solution is, therefore,

$$y'' + 9y' - 10y = 8 + 72\text{t} - 40\, t^2.$$

The 6 Rules-of-Thumb of the Method of Undetermined Coefficients

1. If an exponential function appears in $g(t)$, the starting choice for $Y(t)$ is an exponential function of the same exponent.

2. If a polynomial appears in $g(t)$, the starting choice for $Y(t)$ is a generic polynomial of the same degree.

3. If either cosine or sine appears in $g(t)$, the starting choice for $Y(t)$ needs to contain both cosine and sine of the same frequency.

4. If $g(t)$ is a sum of several functions, $g(t) = g_1(t) + g_2(t) + \ldots + g_n(t)$, separate it into n parts and solve them individually.

5. If $g(t)$ is a product of basic functions, the starting choice for $Y(t)$ is chosen based on:

   i. $Y(t)$ is a product of the corresponding choices of all the parts of $g(t)$.
   ii. There are as many coefficients as the number of distinct terms in $Y(t)$.
   iii. Each distinct term must have its own coefficient, not shared with any other term.

6. Before finalizing the choice of $Y(t)$, compare it against $y_c(t)$. If there is any shared term between the two, the present choice of $Y(t)$ needs to be multiplied by t. Repeat until there is no shared term.

Remember that, in order to use Rule 6 you always need to find the complementary solution first.

SUMMARY: Method of Undetermined Coefficients

Given $$a\,y'' + b\,y' + c\,y = g(t)$$

1. Find the complementary solution y_c.

2. Subdivide, if necessary, $g(t)$ into parts: $g(t) = g_1(t) + g_2(t) \ldots + g_k(t)$.

3. For each $g_i(t)$, choose the form of its corresponding particular solution $Y_i(t)$ according to:

$g_i(t)$	$Y_i(t)$
$P_n(t)$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)$
$P_n(t)\,e^{at}$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)\,e^{at}$
$P_n(t)\,e^{at}\cos\mu t$ and/or $P_n(t)\,e^{at}\sin\mu t$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_0)\,e^{at}\cos\mu t$ + $t^s(B_n t^n + B_{n-1} t^{n-1} + \ldots + B_0)\,e^{at}\sin\mu t$

Where s = 0, 1, or 2, is the **minimum** number of times the choice must be multiplied by t so that it shares no common terms with y_c.

$P_n(t)$ denotes a n-th degree polynomial. If there is no power of t present, then $n = 0$ and $P_0(t) = C_0$ is just the constant coefficient. If no exponential term is present, then set the exponent $a = 0$.

4. $Y = Y_1 + Y_2 + \ldots + Y_k$.

5. The general solution is $y = y_c + Y$.

6. Finally, apply any initial conditions to determine the as yet unknown coefficients C_1 and C_2 in y_c.

Challenge

The following challenges expand the range of problems to give you practise in a range of situations.

1. $y'' + 4y = 8\cos 2t$
2. $y'' + 2y' = 2te^{-t}$
3. $y'' + 2y' = 6e^{-2t}$
4. $y'' + 2y' = 12t^2$
5. $y'' - 6y' - 7y = 13\cos 2t + 34\sin 2t$
6. $y'' - 6y' - 7y = 8e^{-t} - 7t - 6$

Solution

The solutions are contained in the list on the next page in no particular order. Your answers should match one of the solutions given. Please try to not look at the solutions before completing the questions, since this will facilitate deep understanding and reproduce a real-life/exam environment.

$y = C_1 \cos 2t + C_2 \sin 2t + 2t \sin 2t$
$y = C_1 \cos 2t + C_2 \sin 2t + 8t \sin 2t$
$y = C_1 e^{-t} + C_2 e^{7t} + \cos 2t - 2 \sin 2t$
$y = C_1 e^{-2t} + C_2 + 2t^3 - 3t^2 + 3t$
$y = C_1 e^{-2t} + C_2 - 2te^{-t}$
$y = C_1 e^{-2t} + C_2 + 5te^{-t}$
$y = C_1 e^{-2t} + C_2 - 3te^{-2t}$
$y = C_1 e^{-t} + C_2 e^{7t} + t - te^{-t}$

4.17 Method of undetermined coefficients: Determining the ODE I

Comment

This challenge gives you useful practise of going the other way; determining a differential equation that describes a given solution. This can be a little confusing at first, so take time to understand where things originate from.

Challenge

Determine the 2nd-order linear differential equation which has the general solution

$$y = C_1 \cos 4t + C_2 \sin 4t - e^t \sin 2t \tag{4.19}$$

Solution

The solution is given on the next page. Please try to not look at the solution before completing the questions, since this will facilitate deep understanding and reproduce a real-life/exam environment.

$$y'' + 16y = -4e^t \cos 2t - 13e^t \sin 2t$$

4.18 Method of undetermined coefficients: Determining the ODE II

Comment

This challenge gives you useful practise of going the other way; determining a differential equation that describes a given solution. This can be a little confusing at first, so take time to understand where things originate from.

Challenge

Determine the 2nd-order linear differential equation which has the general solution

$$y = C_1 e^{-2t} + C_2 t e^{-2t} + t^3 - 3t \tag{4.20}$$

Solution

The solution is given on the next page. Please try to not look at the solution before completing the questions, since this will facilitate deep understanding and reproduce a real-life/exam environment.

$y'' + 4y' + 4y = 4t^3 + 12t^2 - 6t - 12$

Chapter 5

Laplace transformation

5.1 Your first Laplace Transform calculations

Resources

- Videos: The **four** Khan-academy videos starting at https://www.khanacademy.org/math/differential-equations/laplace-transform/laplace-transform-tutorial/v/laplace-transform-1
- Text (student suggestion, Thai language): http://facstaff.swu.ac.th/surachap/documents/mathematic_i_chapter_3.pdf

Comment

The Laplace Transform is a powerful technique that has many uses beyond solving ODE's. It can however appear a bit abstract at first. Becoming comfortable with controlling and manipulating the transform will help provide confidence when using it to solve ODE's. The four videos in the resources above provide an excellent starting point for getting you comfortable with this powerful technique.

Challenge

1. Calculate $\mathcal{L}\{1\}$

(2. *Moved to challenge 5.2*)

3. Calculate $\mathcal{L}\{Cos(at)\}$

Solution

To check your answer, substitute $s = 1$ and $a = 2$ into your final solution.

1. 1

3. $\frac{1}{5}$

5.2 Laplace transform and derivatives

Resources

- Video I: https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/laplace-transform-5
- Video II: https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/laplace-transform-6

Challenge

1. Calculate $\frac{d^3}{dt^3}\left(te^{at}\right)$

2. Given

$$\mathcal{L}\{te^{at}\} = \frac{1}{(a-s)^2} \tag{5.1}$$

determine $\mathcal{L}\{3a^2e^{at} + a^3te^{at}\}$

3. Calculate $\mathcal{L}\{at\}$
Notes:

- *L'Hôpital's rule cannot be applied to this question - if you don't understand why, please ask your partner or the teacher in class.*
- *You may be able to solve this considering the rate of increase of e^t vs the rate of increase of t as t approaches ∞, however there is another way using the relation of derivatives of the Laplace transform, and I would encorage you to understand how to solve it this way.*

Solution

2.
If you substitute $s = 3$ and $a = 2$ into your final solution you should obtain 20.

3.
If you substitute $s = 1$ and $a = 2$ into your final solution you should obtain 2.

5.3 Shifting a transform

Resources

- Video: `https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/more-laplace-transform-tools`

Challenge

Given

$$\mathcal{L}\{Cosh(at)\} = \frac{s}{s^2 - a^2} \tag{5.2}$$

1. What is $\mathcal{L}\{e^{3t}Cosh(5t)\}$?
2. What is $f(t)$ in the equation $\mathcal{L}\{f(t)\} = \frac{s-4}{(s-4)^2-100}$?

Solution

To check your answer, substitute $s = 2$ and $t = 1/2$ as appropriate:

1. 0.0417
2. 548

5.5 Laplace Transformation of the unit step function

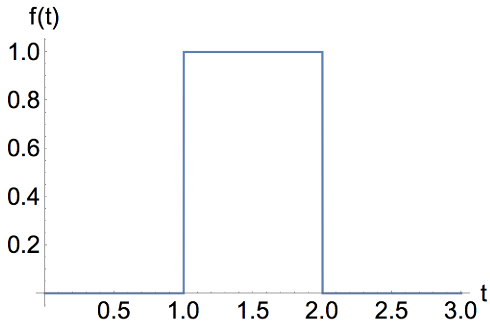
Resources

- Video: https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/laplace-transform-of-the-unit-step-function

Challenge

Considering U_c as the unit step-function at c, calculate the following Laplace transformations:

1. $\mathcal{L}\{U_0\}$
2. $\mathcal{L}\{U_c\}$
3. A 1-second pulse function starting at time $t = 1$ with value $f(t) = 1$ as shown in the graph below:



4. $\mathcal{L}\{U_\pi(t)cos(t-\pi)\}$

Solution

To check your answers, substitute $c = 1$ and $s = 2$ as appropriate.

1.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of yX = 4f9ac8

2.
0.0677

3.
0.0585

4.
7.470×10^{-4}

5.6 Inverse Laplace Transform

Resources

- Video: `https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/inverse-laplace-examples`

Comment

Being able to reversing the Laplace transform is a crucial skill required for applying it to solving ODE's. It can be a little confusing at first however, so I recommend to take your time to understand the essential steps involved thoroughly, as this will then give you greater confidence when you come to apply this to solving ODE's. To this end, the video listed in the resource is a fantastic introduction to this.

Also, it can be helpful to remember that if you're shifting the transform, perform the transform and *then* apply the shift.

Challenge

Determine the function $f(t)$ by finding the inverse of the following Laplace transforms:

1. $F(s) = \dfrac{1}{(s-1)^2}$
2. $F(s) = \dfrac{1}{s^2} - \dfrac{1}{s}$
3. $F(s) = \dfrac{5-5s}{s^2}$
4. $F(s) = \dfrac{6}{(2+s)^4}$
5. $F(s) = \dfrac{2e^{-2s}}{s^2-2s+2}$
6. $F(s) = \dfrac{e^{12-3s}}{s-4}$

Solution

To check your answers, substitute $t = 2$ into your final answer. If there is a unit-step in your solution, precede your numerical answer with "u(c)" where "c" is the position of the unit step. So for example, an answer of $U_5 t^2$ with a hash key of "a" would be entered as "au(5.00)4.00" (all numbers to two decimal places). An answer without a unit-step would just be entered to two decimal places (eg, "a4.00" in the previous example).

1.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of zX = de950a

2.
X = Your solution
Form: Decimal to 2 decimal places.

Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of aX = a1e88c

3.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of cX = ae77f2

4.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of cX = af01f2

5.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of bX = b27f51

6.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of eX = cbc74a

5.7 The Dirac delta function and its Laplace transform

Resources

- Video I: https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/dirac-delta-function
- Video II: https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/laplace-transform-of-the-dirac-delta-function

Challenge

Calculate the following Laplace transforms (treat c as a positive constant):

1. $\mathcal{L}\{\delta(t)\}$
2. $\mathcal{L}\{\delta(t-c)\}$
3. $\mathcal{L}\{\delta(t-2)\cos(4t)\}$
4. $\mathcal{L}\{\delta(t)(t^2+10)\}$

Solution

To check your solution, set $s = 1$ and $c = 2$ as appropriate to check your answers.

1.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of zX = a78955

2. 0.135

3. -0.020

4.
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of cX = c181ec

5.8 The Dirac delta function and its inverse Laplace transform

Challenge

1. Calculate the following Laplace transform:

$\delta(t-2)\sin(2t)$

2. Calculate the following inverse Laplace transforms assuming they contain a Dirac delta function:

I. $e^{-2s}\sin(2)$

II. $e^{-2s}\sin(4)$

Solution

2. To check your answer, substitute $t = 1$ into the final expression and evaluate the part inside and outside of the Dirac delta function separately. So for example, if your answer is $\delta(t-2)(t^2+1)$, the expression inside the delta-function is $t-2$ and will evaluate to -1.00 while the expression outside of the delta-function is t^2+1 and will evaluate to 2.00.

I. Inside delta function:
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of dX = 4881ba

I. Outside delta function
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of eX = 724c45

II. Inside delta function:
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of fX = 75d06e

II. Outside delta function:
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of gX = fca71c

5.9 A forced spring

- The **four** videos starting at `https://www.khanacademy.org/math/differential-equations/laplace-transform/laplace-transform-to-solve-differential-equation/v/laplace-transform-to-solve-an-equation`
- A useful table of Laplace transforms: `http://tutorial.math.lamar.edu/pdf/Laplace_Table.pdf`

Comment

Here you finally get the opportunity to practise solving ODE's using the powerful method of Laplace transformations. Please takes notes from all four videos listed in the resources section; they provide very useful examples of how to use this method, including related algebraic techniques that are commonly required to solve such challenges.

Challenge

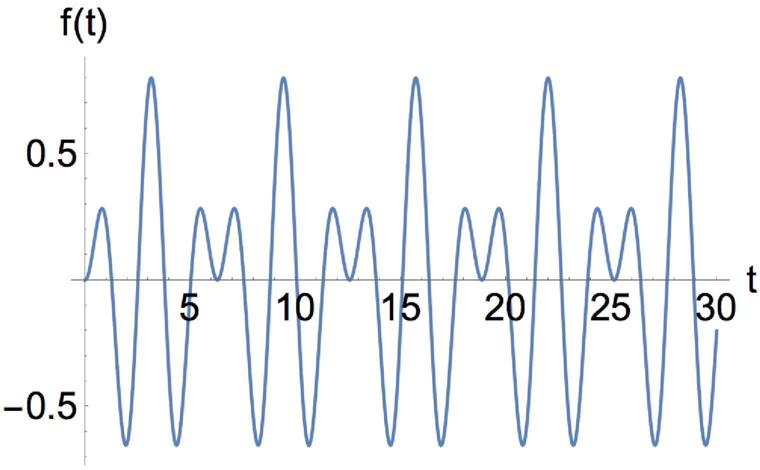
The spring equation you encountered in challenge 4.1 introduced you to the concept of oscillation of a mass on a spring. There, the equation to determine the displacement of the spring y from its equilibrium position was $y'' + y = 0$, which yields a solution $y = C_1 \cos(t) + C_2 \sin(t)$. This is free oscillation without external damping or driving, and it will oscillate according to the cosine and sine sum for all time (t). It is also possible to add a forcing term to the equation by making it non-homogeneous, such as in the form

$$y'' + 4y = 2\cos(3t) \tag{5.3}$$

Here the forcing varies with time t in the form of a cosine wave.

Use the Laplace transform method to solve the ODE in the above equation given a starting displacement of zero and an initial velocity of zero. You may use the table of Laplace transforms in the resources to help you.

A graph of the resulting oscillation is shown here:



Solution

Your final expression should satisfy $y(t = 1) = 0.2295$.

5.10 An exponential function

Challenge

1. Solve

$$y'' + 5y' + 4y = 100e^{-2t} \tag{5.4}$$

for y, given initial conditions $y(0) = -1$ and $y'(0) = 0$. Since the algebra gets very messy, you may use the following equation to help you:

$$\frac{-s^2 - 7s + 90}{(s+1)(s+2)(s+4)} = \frac{32}{s+1} - \frac{50}{s+2} + \frac{17}{s+4} \tag{5.5}$$

2. The homogeneous solution to this equation is

$$y(t) = \frac{1}{3}\left(e^{-4t} - 4e^{-t}\right) \tag{5.6}$$

Show how the two graphs of the homogeneous and non-homogenous solutions differ, and explain how the forcing leads to the different shape observed.

Solution

1. Your final expression should satisfy $y(1) = 5.32$.
2. Please compare your answer with your partner in class or discuss with the teacher.

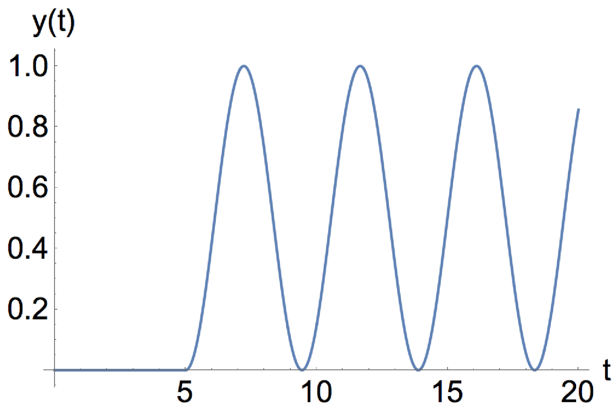
5.11 A unit step

Comment

There are two things to watch out for in this challenge:

- We have defined the Laplace transform for a step function in the sense of $U_c f(t-c)$. So if $f(t) = t$ we must evaluate for $f(t-c) = t-c$. If the function does not contain t then we simply do not need to subtract c.
- The "$t-c$" is essentially substituted in place of any t in the equation. So "at" will become $a(t-c)$ and not $at-c$.

This challenge is interesting because unlike previous challenges, it is the first challenge where we really have no other option but to use the Laplace transform method, and so you can appreciate its power. In this challenge, we have a 2nd-order homogeneous equation (unforced oscillation) until $t = 5$ when we apply a constant force. You will find your answer leads to a constant oscillation. But how can it lead to a constant oscillation if we are constantly applying a force? Shouldn't the oscillation slowly increase in magnitude due to the energy that is being added to the system from the constant force being applied? The answer is of course no: we take just as much energy out of the system when the velocity is in the opposite direction to the force as we add to the system when the velocity is in the same direction as the applied force.



Challenge

Solve

$$y'' + 2y = U_5 \tag{5.7}$$

for y, given initial conditions $y(0) = 0$ and $y'(0) = 0$.

Solution

Your final expression should satisfy $y(t = 6) = 0.42$.

Note that for $t < 5$, the solution is zero. This is because there was no initial velocity and no initial acceleration, so there was no motion until a forcing was applied in terms of a constant force of "1" from $t = 5$. If either of these had been non-zero, we would have had a non-zero value for $t < 5$!

Optionally, you can try setting the initial conditions to non-zero values to see the effect this has on the final solution.

5.12 A sudden impulse

Comment

Here the system is stationary until $t = 5$ when, instead of applying a constant force, we "kick" the system to start the oscillation. Thus you should expect your answer to reflect physics such as this.

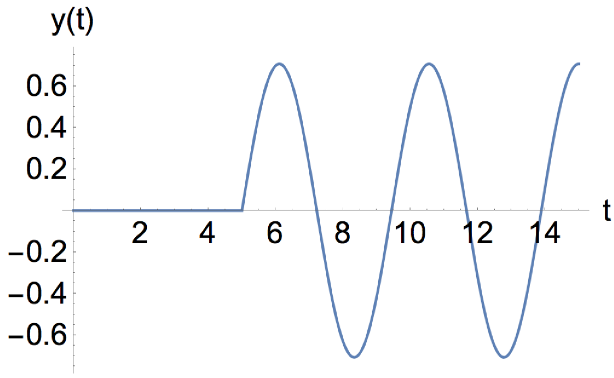
A hint for doing the inverse Laplace transform in this challenge: you will need to multiply the transform by a constant in order to obtain the inverse transform. Also, the graph below should give you a strong indication as to the form of the solution.

Challenge

Solve

$$y'' + 2y = \delta(t - 5) \tag{5.8}$$

for y, given initial conditions $y(0) = 0$ and $y'(0) = 0$.



Solution

Your answer should satisfy $y(6) = 0.698$.

Note how this gives rise to a simple oscillation after $t = 5$, and nothing before it.

5.13 Introduction to convolution

Resources

- Video https://www.khanacademy.org/math/differential-equations/laplace-transform/convolution-integral/v/introduction-to-the-convolution

Comment

In our final study of the application of Laplace transform to solving ODE's, we will look at convolution. This is partly because this is a useful technique for solving ODE's, but also because it will be good for you to have a first-encounter with convolution. So while we are focusing on using Laplace transform and convolution to solve ODE's, my hope is that your familiarity with these topics in this context will make future related topics in other contexts more accessable.

Challenge

Evaluate the following expressions:

1. $y(t) = \sin t * \cos t$
2. $y(t) = t * \cos t$
3. $y(t) = 1 * \cos t$

Solutions

1. Your answer should satisfy $y(0.9) = 0.352$
2. Your answer should satisfy $y(0.9) = 0.378$
3. Your answer should satisfy $y(0.9) = 0.783$

5.14 Convolution and the Laplace transform

Resources

- Video `https://www.khanacademy.org/math/differential-equations/laplace-transform/convolution-integral/v/the-convolution-and-the-laplace-transform`

Challenge

Obtain the inverse Laplace transform of the following functions using convolution.

2. $\dfrac{1}{s^2 - 2s}$
3. $\dfrac{1}{s^4 + s^2}$

Solutions

2. Your solution should satisfy $y(0.8) = 1.977$
3. Your solution should satisfy $y(0.8) = 0.083$

5.15 Using convolution to solve ODE's

Resources

- Video https://www.khanacademy.org/math/differential-equations/laplace-transform/convolution-integral/v/the-convolution-and-the-laplace-transform

Challenge

Using the Laplace transform and convolution, obtain the function $y(t)$ in the form $y(t) = C(A(t) * B(t))$ where C is a constant and A and B are functions of t. Do not attempt to perform the convolution.

1. $y'' - 4y = \sin t$, with the initial conditions $y(0) = 0$ and $y'(0) = 0$.

2. $y''' - 3ay'' + 3a^2y' - a^3y = 2t$, with the initial conditions $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 0$.

3. $y'' - 10y' + 26y = \cos t$, with the initial conditions $y(0) = 0$ and $y'(0) = 0$.

Solutions

1. Your solution should be consistent with $A(0.8) = 0.717$ and $B(0.8) = 2.376$

2. Substituting $a = 4$ into the final solution, your solution should be consistent with $A(0.8) = 0.8$ and $B(0.8) = 15.70$.

3. Your solution should be consistent with $A(0.8) = 0.697$ and $B(0.8) = 39.17$

Chapter 6

Systems of ODE's

6.1 Homogeneous vs non-homogeneous

Comment

The following notes were developed by Zachary S. Tseng at Pennsylvania State University, USA (`http://www.math.psu.edu/tseng/`). Included here with kind permission.

Systems of First Order Linear Differential Equations

We will now turn our attention to solving systems of simultaneous homogeneous first order linear differential equations. The solutions of such systems require much linear algebra (Math 220). But since it is not a prerequisite for this course, we have to limit ourselves to the simplest instances: those systems of two equations and two unknowns only. But first, we shall have a brief overview and learn some notations and terminology.

A system of n linear first order differential equations in n unknowns (an $n \times n$ system of linear equations) has the general form:

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + g_1 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + g_2 \\
x_3' &= a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n + g_3 \qquad\qquad (*) \\
&\;\;\vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + g_n
\end{aligned}
$$

Where the coefficients a_{ij}'s, and g_i's are arbitrary functions of t. If every term g_i is constant zero, then the system is said to be homogeneous. Otherwise, it is a nonhomogeneous system if even one of the g's is nonzero.

The system (*) is most often given in a shorthand format as a matrix-vector equation, in the form:

$$\boxed{\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}}$$

$$\underset{\boldsymbol{x}'}{\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ : \\ : \\ x_n' \end{bmatrix}} = \underset{\boldsymbol{A}}{\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{31} & a_{32} & \ldots & a_{3n} \\ : & : & : & : \\ : & : & : & : \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}} \underset{\boldsymbol{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ : \\ x_n \end{bmatrix}} + \underset{\boldsymbol{g}}{\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ : \\ : \\ g_n \end{bmatrix}}$$

Where the matrix of coefficients, $\boldsymbol{A}$, is called the *coefficient matrix* of the system. The vectors $\boldsymbol{x}'$, $\boldsymbol{x}$, and $\boldsymbol{g}$ are

$$\boldsymbol{x}' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ : \\ x_n' \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix}, \qquad \boldsymbol{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ : \\ g_n \end{bmatrix}.$$

For a homogeneous system, $\boldsymbol{g}$ is the zero vector. Hence it has the form

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}.$$

Challenge

Separately add the points of the following *homogeneous* and *non-homogeneous* ODE systems:

1 point: $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

2 points: $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

4 points: $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ \sin(t) \\ 0 \end{pmatrix}$

8 points: $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ \sin(t) \\ \tan(t) \end{pmatrix}$

Solution

Homogeneous
X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of hX = 346b81

Non-homogeneous
X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of iX = 773ffc

6.2 Basis for creating a system of equations from a single homogeneous ODE

Resources

- Pages 1-4 of the PDF `http://www.math.psu.edu/tseng/class/Math251/Notes-LinearSystems.pdf`

Comment

Note that the notation $y^{(2)}$ means "the 2nd differential of y" while the notation y^2 (without the brackets around the 2) means "y-squared".

Considering the general form of an nth-order linear equation,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = g(t) \tag{6.1}$$

we substitute $x_1 = y$, $x_2 = y'$, ..., $x_n = y^{(n-1)}$ and $x_n' = y^{(n)}$.

When replacing a y-term by an x term, the n in x_n corresponds to one more than the number of times y is differentiated. So x_{n+1} corresponds to y being differentiated n times and similarly x_n corresponds to y being differentiated $n-1$ times.

Note that x_n' is one more differential than x_n, so x_n' corresponds to $(y^{(n-1)})' = y^{(n)}$. So the n in x_n' corresponds to the number of times y is differentiated (ie, $y^{(n)}$).

To understand how this helps us write high-order ODE's as a system of equations, consider the equation

$$y''' - 2y'' + 3y' - 4y = 0 \tag{6.2}$$

First re-write the ODE in terms of x and x'. Note that there is no "x_0'" so we just write it as x_1 in both equations.

$$x_4 - 2x_3 + 3x_2 - 4x_1 = 0 \tag{6.3}$$
$$x_3' - 2x_2' + 3x_1' - 4x_1 = 0 \tag{6.4}$$

Our aim is to write the system of equations in the form $\boldsymbol{x'} = \boldsymbol{Ax}$. Note that there is no "x_4'" in our equations, so the largest value of n in x_n' will be 3 (ie, x_3'). So we can write

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{6.5}$$

where the question marks are values that we have to find.

By direct comparison of equations 6.3 and 6.4 we know that $x_1' = x_2$ which can be written as $x_1' = 0x_1 + 1x_2 + 0x_3$ yielding the first line in the matrix $\boldsymbol{A}$:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{6.6}$$

We can then proceed to do x_2 in a similar fashion:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{6.7}$$

In order to express x_3' in the above matrix form, we need it in terms of x_1, x_2 and x_3 rather than x_4, so instead of direct comparison, we swap x_4 for x_3' in equation 6.3 to read

$$x_3' - 2x_3 + 3x_2 - 4x_1 = 0 \tag{6.8}$$

and then isolate x_3' to read $x_3' = 4x_1 - 3x_2 + 2x_3$ yielding the final form of our systems of equations

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{6.9}$$

Note that this is only considering a homogeneous equation. If it is non-homogeneous, you will have an extra term in the final step and will need a matrix of the form $\boldsymbol{x'} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}$ as shown in the answer to exercise 4(b) on page 5 of the PDF.

So why do we want to do this? Well, notice that in this example we started with a complicated 3rd-order ODE and reduced it into 3 1st-order ODE's. Similarly, if we started with a 2nd-order ODE, we could reduce the equation to 2 1st-order ODE's. In general, for an nth-order ODE we can reduce it to n 1st-order ODE's. If we can then learn how to solve simultanious sets of 1st-order ODE's, we have a powerful method of increasing our understanding (and even solving) difficult higher-order ODE's.

Similarly, if you are given a system of 2 1st-order ODE's, you can know that it can form a single 2nd-order ODE.

Challenge

Write the following ODE's in matrix form:

1) $2y'' + 4y' - 6y = 0$

2) $y'' - 4y' + 5y = 0$

Solutions

To check your answers, sum the values of all the terms in your matrix $\boldsymbol{A}$.

1) 2

2) 0

6.3 Systems of equations from a non-homogeneous ODE

Comment

The approach is very similar when you have a non-homogeneous ODE, but you just have to remember to add the forcing term at the end. For example, considering an equation similar to before, but this time with a cosine forcing term:

$$y''' - 2y'' + 3y' - 4y = \cos t \tag{6.10}$$

We have

$$x_4 - 2x_3 + 3x_2 - 4x_1 = \cos t \tag{6.11}$$
$$x_3' - 2x_2' + 3x_1' - 4x_1 = \cos t \tag{6.12}$$

By comparison we still have $x_1' = x_2$ and $x_2' = x_3$ but now when we replace x_4 with x_3' and re-arrange we have

$$x_3' = \cos t + 4x_1 - 3x_2 + 2x_3 \tag{6.13}$$

leading to a final system of equations of the form

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cos t \end{pmatrix} \tag{6.14}$$

Challenge

Re-write the following differential equations as a system of first-order differential equations:

1) $y'' + y' + y = \cos t$

2) $y'' - 5y' + 9y = t \cos 2t$

Solutions

Please compare your solution with that of your partner or ask the teacher.

6.4 Matrices

Resources

- PDF: Pages 6-17 of the PDF `http://www.math.psu.edu/tseng/class/Math251/Notes-LinearSystems.pdf`

Comment

It is worth spending some time getting comfortable with manipulating matrices, since this is an indispensable basis for the work that is about to follow. The PDF gives a quick introduction to matrices. For a more thorough introduction, the Khan Academy playlist on linear algebra [1] is excellent, although beyond the scope of this course.

One note to deal with any confusion arising with regard to eigenvectors with matrices with zeros. For $(A - rI)$ equal to something like

$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \tag{6.15}$$

the top row can be ignored since any x_1 and x_2 will satisfy the top row.

Similarly, for a case such as

$$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \tag{6.16}$$

you will have

$$2x_1 + 0x_2 = 0 \tag{6.17}$$
$$2x_1 = 0 \tag{6.18}$$
$$x_1 = 0 \tag{6.19}$$

which is satisfied by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.20}$$

(where the 1 could in principle be any number, but is the minimum integer that satisfies the condition.)

Finally, note that $(A - rI) = ((a, b), (c, d))$ will give you two equivalent formulas $ax_1 + bx_2 = 0$ and $cx_1 + bx_2 = 0$, even if they may appear different on first glance. If you want, you can prove to yourself that they are the same by multiplying the bottom row by a/c.

[1] `https://www.khanacademy.org/math/linear-algebra/alternate-bases`

The following notes were developed by Zachary S. Tseng at Pennsylvania State University, USA (http://www.math.psu.edu/tseng/). Included here with kind permission.

A Crash Course in (2×2) Matrices

Several weeks worth of matrix algebra in an hour… (Relax, we will only study the simplest case, that of 2×2 matrices.)

Review topics:

1. What is a matrix (*pl*. matrices)?

> A matrix is a rectangular array of objects (called *entries*). Those entries are usually numbers, but they can also include functions, vectors, or even other matrices. Each entry's position is addressed by the row and column (in that order) where it is located. For example, a_{52} represents the entry positioned at the 5th row and the 2nd column of the matrix $\boldsymbol{A}$.

2. The size of a matrix

> The size of a matrix is specified by 2 numbers
>
> [*number of rows*] × [*number of columns*].
>
> Therefore, an $m \times n$ matrix is a matrix that contains m rows and n columns. A matrix that has equal number of rows and columns is called a *square matrix*. A square matrix of size $n \times n$ is usually referred to simply as a square matrix of size (or order) n.

Notice that if the number of rows or columns is 1, the result (respectively, a $1 \times n$, or an $m \times 1$ matrix) is just a vector. A $1 \times n$ matrix is called a *row vector*, and an $m \times 1$ matrix is called a *column vector*. Therefore, vectors are really just special types of matrices. Hence, you will probably notice the similarities between many of the matrix operations defined below and vector operations that you might be familiar with.

3. Two special types of matrices

Identity matrices (square matrices only)

The $n \times n$ identity matrix is often denoted by $\boldsymbol{I_n}$.

$$\boldsymbol{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ etc.}$$

Properties (assume $\boldsymbol{A}$ and $\boldsymbol{I}$ are of the same size):

$$\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$$
$$\boldsymbol{I_n\, x} = \boldsymbol{x}, \qquad \boldsymbol{x} = \text{any } n \times 1 \text{ vector}$$

Zero matrices – matrices that contain all-zero entries.

Properties:

$$\boldsymbol{A} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{A} = \boldsymbol{A}$$
$$\boldsymbol{A\,0} = \boldsymbol{0} = \boldsymbol{0\,A}$$

4. Arithmetic operations of matrices

(i) Addition / subtraction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \pm \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{bmatrix}$$

(ii) Scalar Multiplication

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}, \qquad \text{for any scalar } k.$$

(iii) Matrix multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

The matrix multiplication $\boldsymbol{AB} = \boldsymbol{C}$ is defined only if there are as many rows in $\boldsymbol{B}$ as there are columns in $\boldsymbol{A}$. For example, when $\boldsymbol{A}$ is $m \times k$ and $\boldsymbol{B}$ is $k \times n$. The product matrix $\boldsymbol{C}$ is going to be of size $m \times n$, and whose ij-th entry, c_{ij}, is equal to the vector dot product between the i-th row of $\boldsymbol{A}$ and the j-th column of $\boldsymbol{B}$. Since vectors are matrices, we can also multiply together a matrix and a vector, assuming the above restriction on their sizes is met. The product of a 2×2 matrix and a 2-entry column vector is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}.$$

Note 1: Two square matrices of the same size can always be multiplied together. Because, obviously, having the same number of rows and columns, they satisfy the size requirement outlined above.

Note 2: In general, $\boldsymbol{AB} \neq \boldsymbol{BA}$. Indeed, depending on the sizes of $\boldsymbol{A}$ and $\boldsymbol{B}$, one product might not even be defined while the other product is.

5. Determinant (square matrices only)

For a 2 × 2 matrix, its determinant is given by the formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Note: The determinant is a function whose domain is the set of all square matrices of a certain size, and whose range is the set of all real (or complex) numbers.

6. Inverse matrix (of a square matrix)

Given an $n \times n$ square matrix $\boldsymbol{A}$, if there exists a matrix $\boldsymbol{B}$ (necessarily of the same size) such that

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}_n,$$

then the matrix $\boldsymbol{B}$ is called the *inverse matrix* of $\boldsymbol{A}$, denoted $\boldsymbol{A}^{-1}$. The inverse matrix, if it exists, is unique for each $\boldsymbol{A}$. A matrix is called *invertible* if it has an inverse matrix.

Theorem: For any 2 × 2 matrix $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,
its inverse, if exists, is given by

$$\boldsymbol{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Theorem: A square matrix is invertible if and only if its determinant is nonzero.

Examples: Let $\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ 5 & 2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$.

(i) $2\boldsymbol{A} - \boldsymbol{B} = 2\begin{bmatrix} 1 & -2 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2-2 & -4-(-3) \\ 10-(-1) & 4-4 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 11 & 0 \end{bmatrix}$

(ii) $\boldsymbol{AB} = \begin{bmatrix} 1 & -2 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2+2 & -3-8 \\ 10-2 & -15+8 \end{bmatrix} = \begin{bmatrix} 4 & -11 \\ 8 & -7 \end{bmatrix}$

On the other hand:

$\boldsymbol{BA} = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2-15 & -4-6 \\ -1+20 & 2+8 \end{bmatrix} = \begin{bmatrix} -13 & -10 \\ 19 & 10 \end{bmatrix}$

(iii) $\det(\boldsymbol{A}) = 2 - (-10) = 12$, $\det(\boldsymbol{B}) = 8 - 3 = 5$.

Since neither is zero, as a result, they are both invertible matrices.

(iv) $\boldsymbol{A}^{-1} = \dfrac{1}{2-(-10)}\begin{bmatrix} 2 & 2 \\ -5 & 1 \end{bmatrix} = \dfrac{1}{12}\begin{bmatrix} 2 & 2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/6 \\ -5/12 & 1/12 \end{bmatrix}$

7. Systems of linear equations (also known as *linear systems*)

A system of linear (algebraic) equations, $\boldsymbol{Ax} = \boldsymbol{b}$, could have zero, exactly one, or infinitely many solutions. (Recall that each linear equation has a line as its graph. A solution of a linear system is a common intersection point of all the equations' graphs − and there are only 3 ways a set of lines could intersect.)

If the vector $\boldsymbol{b}$ on the right-hand side is the zero vector, then the system is called homogeneous. A homogeneous linear system always has a solution, namely the all-zero solution (that is, the origin). This solution is called the *trivial solution* of the system. Therefore, a homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ could have either exactly one solution, or infinitely many solutions. There is no other possibility, since it always has, at least, the trivial solution. If such a system has n equations and exactly the same number of unknowns, then the number of solution(s) the system has can be determined, without having to solve the system, by the determinant of its coefficient matrix:

> *Theorem*: If $\boldsymbol{A}$ is an $n \times n$ matrix, then the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has exactly one solution (the trivial solution) if and only if $\boldsymbol{A}$ is invertible (that is, it has a nonzero determinant). It will have infinitely many solutions (the trivial solution, plus infinitely many nonzero solutions) if $\boldsymbol{A}$ is not invertible (equivalently, has zero determinant).

8. Eigenvalues and Eigenvectors

Given a square matrix $\boldsymbol{A}$, suppose there are a constant r and a nonzero vector $\boldsymbol{x}$ such that

$$\boldsymbol{Ax} = r\boldsymbol{x},$$

then r is called an *Eigenvalue* of $\boldsymbol{A}$, and $\boldsymbol{x}$ is an *Eigenvector* of $\boldsymbol{A}$ corresponding to r.

Do eigenvalues/vectors always exist for any given square matrix? The answer is yes. How do we find them, then?

Rewrite the above equation, we get $\boldsymbol{Ax} - r\boldsymbol{x} = \boldsymbol{0}$. The next step would be to factor out $\boldsymbol{x}$. But doing so would give the expression

$$(\boldsymbol{A} - r)\boldsymbol{x} = \boldsymbol{0}.$$

Notice that it requires us to subtract a number from an $n \times n$ matrix. That's an undefined operation. Hence, we need to further refined it by rewriting the term $r\boldsymbol{x} = r\boldsymbol{I}\boldsymbol{x}$, and then factoring out $\boldsymbol{x}$, obtaining

$$(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}.$$

This is an $n \times n$ system of homogeneous linear (algebraic) equations, where the coefficient matrix is $(\boldsymbol{A} - r\boldsymbol{I})$. We are looking for a nonzero solution $\boldsymbol{x}$ of this system. Hence, by the theorem we have just seen, the necessary and sufficient condition for the existence of such a nonzero solution, which will become an eigenvector of $\boldsymbol{A}$, is that the coefficient matrix $(\boldsymbol{A} - r\boldsymbol{I})$ must have zero determinant. Set its determinant to zero and what we get is a degree n polynomial equation in terms of r. The case of a 2×2 matrix is as follow:

$$\boldsymbol{A} - r\boldsymbol{I} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a-r & b \\ c & d-r \end{bmatrix}.$$

Its determinant, set to 0, yields the equation

$$\det\begin{bmatrix} a-r & b \\ c & d-r \end{bmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + (ad-bc) = 0$$

It is a degree 2 polynomial equation of r, as you can see.

This polynomial on the left is called the *characteristic polynomial* of the (original) matrix $\boldsymbol{A}$, and the equation is the *characteristic equation* of $\boldsymbol{A}$. The root(s) of the characteristic polynomial are the eigenvalues of $\boldsymbol{A}$. Since any degree n polynomial always has n roots (real and/or complex; not necessarily distinct), any $n \times n$ matrix always has at least one, and up to n different eigenvalues.

Once we have found the eigenvalue(s) of the given matrix, we put each specific eigenvalue back into the linear system $(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$ to find the corresponding eigenvectors.

Examples: $\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$

$$\boldsymbol{A} - r\boldsymbol{I} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} - r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-r & 3 \\ 4 & 3-r \end{bmatrix}.$$

Its characteristic equation is

$$\det\begin{bmatrix} 2-r & 3 \\ 4 & 3-r \end{bmatrix} = (2-r)(3-r)-12 = r^2 - 5r - 6 = (r+1)(r-6) = 0$$

The eigenvalues are, therefore, $r = -1$ and 6.

Next, we will substitute each of the 2 eigenvalues into the matrix equation $(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = \boldsymbol{0}$.

For $r = -1$, the system of linear equations is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = (\boldsymbol{A} + \boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 2+1 & 3 \\ 4 & 3+1 \end{bmatrix} x = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Notice that the matrix equation represents a degenerated system of 2 linear equations. Both equations are constant multiples of the equation $x_1 + x_2 = 0$. There is now only 1 equation for the 2 unknowns, therefore, there are infinitely many possible solutions. <u>This is always the case when solving for eigenvectors</u>. Necessarily, there are infinitely many eigenvectors corresponding to each eigenvalue.

Solving the equation $x_1 + x_2 = 0$, we get the relation $x_2 = -x_1$. Hence, the eigenvectors corresponding to $r = -1$ are all nonzero multiples of

$$k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Similarly, for $r = 6$, the system of equations is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = (\boldsymbol{A} - 6\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 2-6 & 3 \\ 4 & 3-6 \end{bmatrix} x = \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Both equations in this second linear system are equivalent to $4x_1 - 3x_2 = 0$. Its solutions are given by the relation $4x_1 = 3x_2$. Hence, the eigenvectors corresponding to $r = 6$ are all nonzero multiples of

$$k_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Note: Every nonzero multiple of an eigenvector is also an eigenvector.

Two short-cuts to find eigenvalues:

1. If $\boldsymbol{A}$ is a diagonal or triangular matrix, that is, if it has the form

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \text{ or } \qquad \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \text{ or } \qquad \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}.$$

Then the eigenvalues are just the main diagonal entries, $r = a$ and d in all 3 examples above.

2. If $\boldsymbol{A}$ is any 2×2 matrix, then its characteristic equation is

$$\det \begin{bmatrix} a-r & b \\ c & d-r \end{bmatrix} = r^2 - (a+d)r + (ad - bc) = 0$$

If you are familiar with terminology of linear algebra, the characteristic equation can be memorized rather easily as

$$\boxed{r^2 - \text{Trace}(\boldsymbol{A})\, r + \det(\boldsymbol{A}) = 0.}$$

Note: For any square matrix $\boldsymbol{A}$, Trace($\boldsymbol{A}$) = [*sum of all entries on the main diagonal (running from top-left to bottom-right)*]. For a 2×2 matrix $\boldsymbol{A}$, Trace($\boldsymbol{A}$) = $a + d$.

A short-cut to find eigenvectors (of a 2 × 2 matrix):

Similarly, there is a trick that enables us to find the eigenvectors of any 2 × 2 matrix without having to go through the whole process of solving systems of linear equations. This short-cut is especially handy when the eigenvalues are complex numbers, since it avoids the need to solve the linear equations which will have complex number coefficients. (*Warning*: *This method does not work for any matrix of size larger than 2 × 2.*)

We first find the eigenvalue(s) and then write down, for each eigenvalue, the matrix $(\boldsymbol{A} - r\boldsymbol{I})$ as usual. Then we take *any* row of $(\boldsymbol{A} - r\boldsymbol{I})$ that is not consisted of entirely zero entries, say it is the row vector (α, β). We put a minus sign in front of one of the entries, for example, $(\alpha, -\beta)$. Then an eigenvector of the matrix A is found by switching the two entries in the above vector, that is, $\boldsymbol{k} = (-\beta, \alpha)$.

Example: Previously, we have seen $\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$.

The characteristic equation is

$$r^2 - \text{Trace}(\boldsymbol{A})\, r + \det(\boldsymbol{A}) = r^2 - 5r - 6 = (r + 1)(r - 6) = 0,$$

which has roots $r = -1$ and 6. For $r = -1$, the matrix $(\boldsymbol{A} - r\boldsymbol{I})$ is $\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$.

Take the first row, (3, 3), which is a non-zero vector; put a minus sign to the first entry to get (−3, 3); then switch the entry, we now have $\boldsymbol{k}_1 = (3, -3)$. It is indeed an eigenvector, since it is a nonzero constant multiple of the vector we found earlier.

On very rare occasions, both rows of the matrix $(\boldsymbol{A} - r\boldsymbol{I})$ have all zero entries. If so, the above algorithm will not be able to find an eigenvector. Instead, under this circumstance any non-zero vector will be an eigenvector.

Challenges

Considering the matrices $\mathbf{C} = \begin{pmatrix} -5 & -1 \\ 7 & 3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ -2 & -1 \end{pmatrix}$ calculate:

1. $\mathbf{C} + 2\mathbf{D}$
2. $\mathbf{CD}$
3. $\mathbf{DC}$
4. $\det(\mathbf{CD})$
5. $(\mathbf{CD})^{-1}$
6. Find the eigenvectors and eigenvalues of $\mathbf{C}$
7. Find the eigenvectors and eigenvalues of $\mathbf{D}$

Solutions

1. The sum of the terms in the matrix should be 2.
2. The sum of the terms in the matrix should be -2.
3. The sum of the terms in the matrix should be -10.
4. 16
5. The sum of the terms in the matrix should be $-5/4$.
6. Eigenvalues 2 and -4 will have eigenvectors $(s, -7s)$ and $(s, -s)$ respectively where s is any non-zero number.
7. Eigenvalues 2 and -1 will have eigenvectors $(s, -2s/3)$ and $(0, s)$ respectively where s is any non-zero number.

6.5 Eigenvector equivalence

Comment

Considering the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \tag{6.21}$$

The eigenvalues are -2 and -1. Considering the eigenvalue -2,

$$A - Ir = \begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix} \tag{6.22}$$

To determine the eigenvector we can either take the top or bottom row in the calculation $(A - Ir)x = 0$. The top and bottom row appear with different numbers but it is easy to see that they yield multiples of the same eigenvector and are therefore equivalent.

Complex eigenvectors are no different, but it can sometimes be hard to see that they are indeed equivalent.

Challenge

Show that the equation $(A - Ir)\boldsymbol{x} = 0$, where

$$A - Ir = \begin{pmatrix} -3 - 3i & 6 \\ -3 & 3 - 3i \end{pmatrix} \tag{6.23}$$

yields the same eigenvector, irrespective of whether you calculate the eigenvector using the top or bottom row of $(A - Ir)$. You may find that one of the representations of the eigenvectors looks like $(1 - i, 1)$.

Solutions

You should be able to generate two eigenvectors by using the top and bottom rows of the $A - Ir$ matrix, and show that they are in fact the same eigenvector by multiplying by an equivalent (imaginary) number. Please discuss with your partner or the teacher in class if you have trouble.

6.6 Solving systems of ODE's

Comment

When decomposing a higher-order ODE into a system of 1st-order ODE's, we are actually dealing with a special form of 1st-order ODE system that looks like

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_5/a_1 & -a_4/a_1 & -a_3/a_1 & -a_2/a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(y) \end{pmatrix} \tag{6.24}$$

which naturally arise from a higher-order ODE like

$$a_1 y'''' + a_2 y''' + a_3 y'' + a_4 y' + a_5 y = f(y) \tag{6.25}$$

This can be considered in terms of other phenomena however whereby the system of ODE's cannot be expressed as a higher-order ODE. In this challenge you can learn about such systems.

The following notes were developed by Zachary S. Tseng at Pennsylvania State University, USA (`http://www.math.psu.edu/tseng/`). Included here with kind permission.

Solution of 2 × 2 systems of first order linear equations

Consider a system of 2 simultaneous first order linear equations

$$x_1' = a x_1 + b x_2$$
$$x_2' = c x_1 + d x_2$$

It has the alternate matrix-vector representation

$$\mathbf{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}.$$

Or, in shorthand $\mathbf{x}' = \mathbf{A}\mathbf{x}$, if $\mathbf{A}$ is already known from context.

We know that the above system is equivalent to a second order homogeneous linear differential equation. As a result, we know that the general solution contains two linearly independent parts. As well, the solution will be consisted of some type of exponential functions. Therefore, assume that $\mathbf{x} = \mathbf{k}\, e^{rt}$ is a solution of the system, where $\mathbf{k}$ is a vector of coefficients (of x_1 and x_2). Substitute $\mathbf{x}$ and $\mathbf{x}' = r\,\mathbf{k}\, e^{rt}$ into the equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and we have

$$r\,\mathbf{k}\, e^{rt} = \mathbf{A}\,\mathbf{k}\, e^{rt}.$$

Since e^{rt} is never zero, we can always divide both sides by e^{rt} and get

$$r\,\mathbf{k} = \mathbf{A}\,\mathbf{k}.$$

We see that this new equation is exactly the relation that defines eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A}$. In other words, in order for a function $\mathbf{x} = \mathbf{k}\, e^{rt}$ to satisfy our system of differential equations, the number r must be an eigenvalue of $\mathbf{A}$, and the vector $\mathbf{k}$ must be an eigenvector of $\mathbf{A}$ corresponding to r. Just like the solution of a second order homogeneous linear equation, there are three possibilities, depending on the number of distinct, and the type of, eigenvalues the coefficient matrix $\mathbf{A}$ has.

The possibilities are that $\boldsymbol{A}$ has

I. Two distinct real eigenvalues
II. Complex conjugate eigenvalues
III. A repeated eigenvalue

A related note, (from linear algebra,) we know that eigenvectors that each corresponds to a different eigenvalue are always linearly independent from each others. Consequently, if r_1 and r_2 are two different eigenvalues, then their respective eigenvectors $\boldsymbol{k}_1$ anf $\boldsymbol{k}_2$, and therefore the corresponding solutions, are always linearly independent.

<u>Case I</u> Distinct real eigenvalues

If the coefficient matrix $\boldsymbol{A}$ has two distinct real eigenvalues r_1 and r_2, and their respective eigenvectors are $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Then the 2×2 system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a general solution

$$\boxed{x = C_1\, k_1\, e^{r_1 t} + C_2\, k_2\, e^{r_2 t}\,.}$$

Example: $$\boldsymbol{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}\boldsymbol{x}.$$

We have already found that the coefficient matrix has eigenvalues $r = -1$ and 6. And they each respectively has an eigenvector

$$k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad k_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Therefore, a general solution of this system of differential equations is

$$x = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} e^{6t}$$

Example: $\boldsymbol{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\boldsymbol{x}$, $\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The characteristic equation is $r^2 - r - 2 = (r + 1)(r - 2) = 0$. The eigenvalues are $r = -1$ and 2. They have, respectively, eigenvectors

For $r = -1$, the system is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = (\boldsymbol{A} + \boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 3+1 & -2 \\ 2 & -2+1 \end{bmatrix} x = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the bottom equation of the system: $2x_1 - x_2 = 0$, we get the relation $x_2 = 2x_1$. Hence,

$$k_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

For $r = 2$, the system is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = (\boldsymbol{A} - 2\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} x = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the first equation of the system: $x_1 - 2x_2 = 0$, we get the relation $x_1 = 2x_2$. Hence,

$$k_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore, a general solution is

$$x = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}.$$

Apply the initial values,

$$x(0) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{0} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{0} = \begin{bmatrix} C_1 + 2C_2 \\ 2C_1 + C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

That is

$$\begin{array}{ccccc} C_1 & + & 2C_2 & = & 1 \\ 2C_1 & + & C_2 & = & -1 \end{array}.$$

We find $C_1 = -1$ and $C_2 = 1$, hence we have the particular solution

$$x = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} -e^{-t} + 2e^{2t} \\ -2e^{-t} + e^{2t} \end{bmatrix}.$$

Case II Complex conjugate eigenvalues

If the coefficient matrix $\boldsymbol{A}$ has two distinct complex conjugate eigenvalues $\lambda \pm \mu i$. Also suppose $\boldsymbol{k} = \boldsymbol{a} + \boldsymbol{b}\, i$ is an eigenvector (necessarily has complex-valued entries) of the eigenvalue $\lambda + \mu i$. Then the 2×2 system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a <u>real-valued</u> general solution

$$\boxed{x = C_1 e^{\lambda t}\big(a\cos(\mu t) - b\sin(\mu t)\big) + C_2 e^{\lambda t}\big(a\sin(\mu t) + b\cos(\mu t)\big)}$$

A little detail: Similar to what we have done before, first there was the complex-valued general solution in the form

$$x = C_1 k_1 e^{(\lambda + \mu i)t} + C_2 k_2 e^{(\lambda - \mu i)t}.$$

We "filter out" the imaginary parts by carefully choosing two sets of coefficients to obtain two corresponding real-valued solutions that are also linearly independent:

$$u = e^{\lambda t}\big(a\cos(\mu t) - b\sin(\mu t)\big)$$
$$v = e^{\lambda t}\big(a\sin(\mu t) + b\cos(\mu t)\big)$$

The real-valued general solution above is just $\boldsymbol{x} = C_1\, \boldsymbol{u} + C_2\, \boldsymbol{v}$. In particular, it might be useful to know how $\boldsymbol{u}$ and $\boldsymbol{v}$ could be derived by expanding the following complex-valued expression (the front half of the complex-valued general solution):

$$k_1 e^{(\lambda + \mu i)t} = (a + bi)e^{\lambda t} e^{(\mu t)i} = e^{\lambda t}(a + bi)(\cos(\mu t) + i\sin(\mu t))$$
$$= e^{\lambda t}(a\cos(\mu t) + ia\sin(\mu t) + ib\cos(\mu t) + i^2 b\sin(\mu t))$$
$$= e^{\lambda t}(a\cos(\mu t) - b\sin(\mu t)) + i\, e^{\lambda t}(a\sin(\mu t) + b\cos(\mu t))$$

Then, $\boldsymbol{u}$ is just the real part of this complex-valued function, and $\boldsymbol{v}$ is its imaginary part.

Example: $$\boldsymbol{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \boldsymbol{x}$$

The characteristic equation is $r^2 + 1 = 0$, giving eigenvalues $r = \pm\, i$. That is, $\lambda = 0$ and $\mu = 1$.

Take the first (the one with positive imaginary part) eigenvalue $r = i$, and find one of its eigenvectors:

$$(\boldsymbol{A} - r\,\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the first equation of the system: $(2 - i)x_1 - 5x_2 = 0$, we get the relation $(2 - i)x_1 = 5x_2$. Hence,

$$k = \begin{bmatrix} 5 \\ 2-i \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}_{\boldsymbol{a}} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{\boldsymbol{b}} i = a + bi$$

Therefore, a general solution is

$$x = C_1 e^{0t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\cos(t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin(t)\right) + C_2 e^{0t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\sin(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos(t)\right)$$

$$= C_1\begin{pmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} + C_2\begin{pmatrix} 5\sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix}$$

Example: $\quad \boldsymbol{x}' = \begin{bmatrix} -1 & -6 \\ 3 & 5 \end{bmatrix}\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

The characteristic equation is $r^2 - 4r + 13 = 0$, giving eigenvalues $r = 2 \pm 3i$. Thus, $\lambda = 2$ and $\mu = 3$.

Take $r = 2 + 3i$ and find one of its eigenvectors:

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} -1-(2+3i) & -6 \\ 3 & 5-(2+3i) \end{bmatrix} x = \begin{bmatrix} -3-3i & -6 \\ 3 & 3-3i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the second equation of the system: $3x_1 + (3 - 3i)x_2 = 0$, we get the relation $x_1 = (-1 + i)x_2$. Hence,

$$k = \begin{bmatrix} -1+i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i = a + bi$$

The general solution is

$$x = C_1 e^{2t}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\cos(3t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(3t)\right) + C_2 e^{2t}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\sin(3t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(3t)\right)$$

$$= C_1 e^{2t}\begin{pmatrix} -\cos(3t) - \sin(3t) \\ \cos(3t) \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} \cos(3t) - \sin(3t) \\ \sin(3t) \end{pmatrix}$$

Apply the initial values to find C_1 and C_2:

$$x(0) = C_1 e^0\left(\begin{bmatrix}-1\\1\end{bmatrix}\cos(0) - \begin{bmatrix}1\\0\end{bmatrix}\sin(0)\right) + C_2 e^0\left(\begin{bmatrix}-1\\1\end{bmatrix}\sin(0) + \begin{bmatrix}1\\0\end{bmatrix}\cos(0)\right)$$

$$= C_1\begin{bmatrix}-1\\1\end{bmatrix} + C_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-C_1 + C_2\\C_1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$$

Therefore, $C_1 = 2$ and $C_2 = 2$. Consequently, the particular solution is

$$x = 2e^{2t}\begin{pmatrix}-\cos(3t) - \sin(3t)\\ \cos(3t)\end{pmatrix} + 2e^{2t}\begin{pmatrix}\cos(3t) - \sin(3t)\\ \sin(3t)\end{pmatrix}$$

$$= e^{2t}\begin{pmatrix}-4\sin(3t)\\ 2\cos(3t) + 2\sin(3t)\end{pmatrix}$$

<u>Case III</u> Repeated real eigenvalue

Suppose the coefficient matrix $\boldsymbol{A}$ has a repeated real eigenvalues r, there are 2 sub-cases.

(i) If r has two linearly independent eigenvectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Then the 2×2 system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a general solution

$$\boxed{\boldsymbol{x} = C_1 \boldsymbol{k}_1 e^{rt} + C_2 \boldsymbol{k}_2 e^{rt}.}$$

Note: For 2×2 matrices, this possibility <u>only occurs when the coefficient matrix $\boldsymbol{A}$ is a scalar multiple of the identity matrix</u>. That is, $\boldsymbol{A}$ has the form

$$\alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \qquad \text{for any constant } \alpha.$$

Example: $$\boldsymbol{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \boldsymbol{x}.$$

The eigenvalue is $r = 2$ (repeated). There are 2 sets of linearly independent eigenvectors, which could be represented by any 2 nonzero vectors that are not constant multiples of each other. For example

$$k_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad k_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, a general solution is

$$x = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}.$$

(ii) If r, as it usually does, only has one linearly independent eigenvector $\boldsymbol{k}$. Then the 2 × 2 system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a general solution

$$\boxed{\boldsymbol{x} = C_1 \boldsymbol{k} e^{rt} + C_2(\boldsymbol{k} t e^{rt} + \boldsymbol{\eta} e^{rt}).}$$

Where the second vector $\boldsymbol{\eta}$ is any solution of the nonhomogeneous linear system of algebraic equations

$$\boxed{(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{k}.}$$

Example: $\boldsymbol{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}\boldsymbol{x}$, $\boldsymbol{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

The eigenvalue is $r = -3$ (repeated). The corresponding system is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} x = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Both equations of the system are $4x_1 - 4x_2 = 0$, we get the same relation $x_1 = x_2$. Hence, there is only one linearly independent eigenvector:

$$k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Next, solve for $\boldsymbol{\eta}$:

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \boldsymbol{\eta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It has solution in the form $\boldsymbol{\eta} = \begin{bmatrix} \frac{1}{4} + \eta_2 \\ \eta_2 \end{bmatrix}$.

Choose $\eta_2 = 0$, we get $\boldsymbol{\eta} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$.

A general solution is, therefore,

$$x = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + C_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} e^{-3t} \right)$$

Apply the initial values to find $C_1 = 1$ and $C_2 = -12$. The particular solution is

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} - 12 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} e^{-3t} \right) = \begin{bmatrix} -12t - 2 \\ -12t + 1 \end{bmatrix} e^{-3t}$$

<u>Summary</u>: Solving a Homogeneous System of Two Linear First Order Equations in Two Unknowns

Given:

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}.$$

First find the two eigenvalues, r, and their respective corresponding eigenvectors, $\boldsymbol{k}$, of the coefficient matrix $\boldsymbol{A}$. Depending on the eigenvalues and eigenvectors, the general solution is:

I. Two distinct real eigenvalues r_1 and r_2:

$$x = C_1 k_1 e^{r_1 t} + C_2 k_2 e^{r_2 t}.$$

II. Two complex conjugate eigenvalues $\lambda \pm \mu i$, where $\lambda + \mu i$ has as an eigenvector $\boldsymbol{k} = \boldsymbol{a} + \boldsymbol{b}\,i$:

$$x = C_1 e^{\lambda t}\big(a\cos(\mu t) - b\sin(\mu t)\big) + C_2 e^{\lambda t}\big(a\sin(\mu t) + b\cos(\mu t)\big)$$

III. A repeated real eigenvalue r:

(i) When two linearly independent eigenvectors exist –

$$\boldsymbol{x} = C_1 \boldsymbol{k}_1 e^{rt} + C_2 \boldsymbol{k}_2 e^{rt}.$$

(ii) When only one linearly independent eigenvector exist –

$$\boldsymbol{x} = C_1 \boldsymbol{k} e^{rt} + C_2 (\boldsymbol{k}\,t\,e^{rt} + \boldsymbol{\eta}\,e^{rt}).$$

Note: Solve the system $(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{\eta} = \boldsymbol{k}$ to find the vector $\boldsymbol{\eta}$.

Exercises D-1.3:

1. Rewrite the following second order linear equation into a system of two equations.

$$y'' + 5y' - 6y = 0$$

Then: (a) show that both the given equation and the new system have the same characteristic equation. (b) Find the system's general solution.

2 – 7 Find the general solution of each system below.

2. $\boldsymbol{x}' = \begin{bmatrix} 2 & 7 \\ -5 & -10 \end{bmatrix}\boldsymbol{x}.$ 3. $\boldsymbol{x}' = \begin{bmatrix} -3 & 6 \\ -3 & 3 \end{bmatrix}\boldsymbol{x}.$

4. $\boldsymbol{x}' = \begin{bmatrix} 8 & -4 \\ 1 & 4 \end{bmatrix}\boldsymbol{x}.$ 5. $\boldsymbol{x}' = \begin{bmatrix} -3 & 2 \\ -1 & -5 \end{bmatrix}\boldsymbol{x}.$

6. $\boldsymbol{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\boldsymbol{x}.$ 7. $\boldsymbol{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -4 \end{bmatrix}\boldsymbol{x}.$

8 – 15 Solve the following initial value problems.

8. $\boldsymbol{x}' = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\boldsymbol{x},$ $\boldsymbol{x}(0) = \begin{bmatrix} -4 \\ 2 \end{bmatrix}.$

9. $\boldsymbol{x}' = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}\boldsymbol{x},$ $\boldsymbol{x}(3) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$

10. $\boldsymbol{x}' = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\boldsymbol{x},$ $\boldsymbol{x}(1) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$

11. $\boldsymbol{x}' = \begin{bmatrix} 6 & 8 \\ 2 & 6 \end{bmatrix}\boldsymbol{x},$ $\boldsymbol{x}(0) = \begin{bmatrix} 8 \\ 0 \end{bmatrix}.$

12. $\boldsymbol{x}' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\boldsymbol{x},$ $\boldsymbol{x}(0) = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$

Challenge

Answer questions 2, 3 and 4 in the last page in the notes above.

Solutions

It might not be clear to you why solutions involve vectors and what this means physically, but for now, please just get used to solving equations in this fashion.

Solutions can be found in the Appendix A.1.

6.7 Solving systems of ODE's with initial conditions

Challenge

Continuing from the previous challenge, solve the problems 8, 9 and 10 in the PDF resource above, using initial conditions as given.

Solutions

You should find that you solution is consistent with

8. $\boldsymbol{x}(0.8) = \begin{pmatrix} -1.90 \\ -0.66 \end{pmatrix}$

9. $\boldsymbol{x}(0.8) = \begin{pmatrix} 33100 \\ -13300 \end{pmatrix}$

10. $\boldsymbol{x}(0.8) = \begin{pmatrix} -1.71 \\ 1.96 \end{pmatrix}$

6.8 Graphs of system solutions

Resources

In the previous challenge you determined x_1 and x_2 with solutions such as

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 6 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} \tag{6.26}$$

or written another way:

$$x_1 = -c_1 e^{-6t} + c_2 e^t \tag{6.27}$$
$$x_2 = 6c_1 e^{-6t} + c_2 e^t \tag{6.28}$$

This particular system arose from a 2nd-order differential equation:

$$y'' + 5y' - 6y = 0 \tag{6.29}$$

we have learned in challenge 6.2 that this 2nd-order equation can be written in terms of x:

$$x_3 + 5x_2 - 6x_1 = 0 \tag{6.30}$$

Thus we remember that $x_1 = y$ and $x_2 = y'$, allowing equations 6.27 and 6.28 to be written as

$$y = -c_1 e^{-6t} + c_2 e^t \tag{6.31}$$
$$y' = 6c_1 e^{-6t} + c_2 e^t \tag{6.32}$$

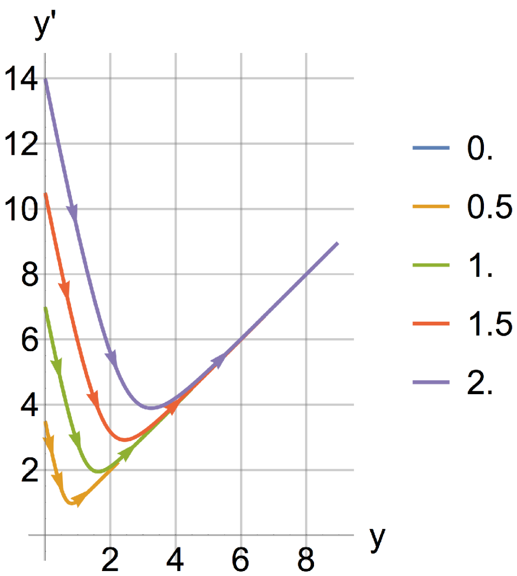
Perhaps, for example, the original 2nd-order ODE (equation 6.29) represented the position of an atom on an axis with respect to time. Then equation 6.31 represents position at time t while equation 6.32 represents the velocity (or more commonly, when multiplied by the mass, represents the momentum).

Thus the graph represents the variation of momentum (velocity) with position, called the "phase-space" of the system. A specific trajectory can be followed given boundary conditions that determine the starting condition. For example, if the particle at time $t = 0$ is known to have position $y = 1$ and velocity $y' = 2$ we can impose the boundary condition

$$\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{6.33}$$

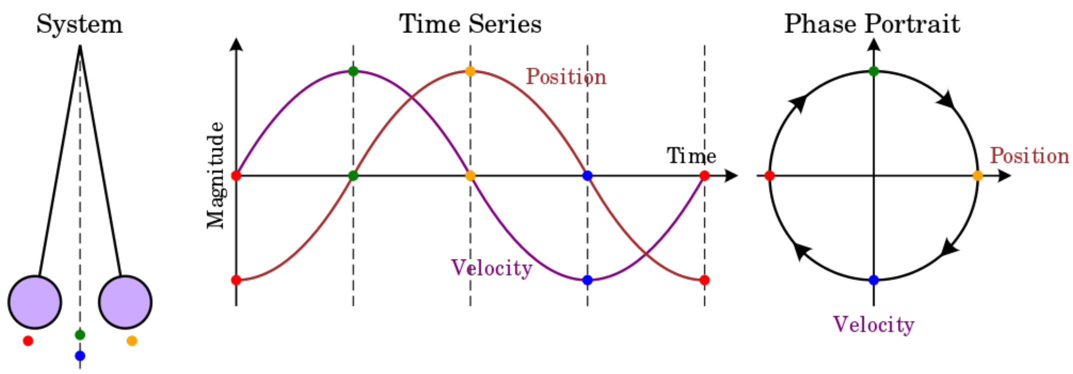
to determine the coefficients c_1 and c_2 and obtain a unique trajectory.

We can then plot the phase-space for various boundary conditions. In the graph below, we show examples where $c_1 = c_2 = \{0, 0.5, 1, 1.5, 2\}$:



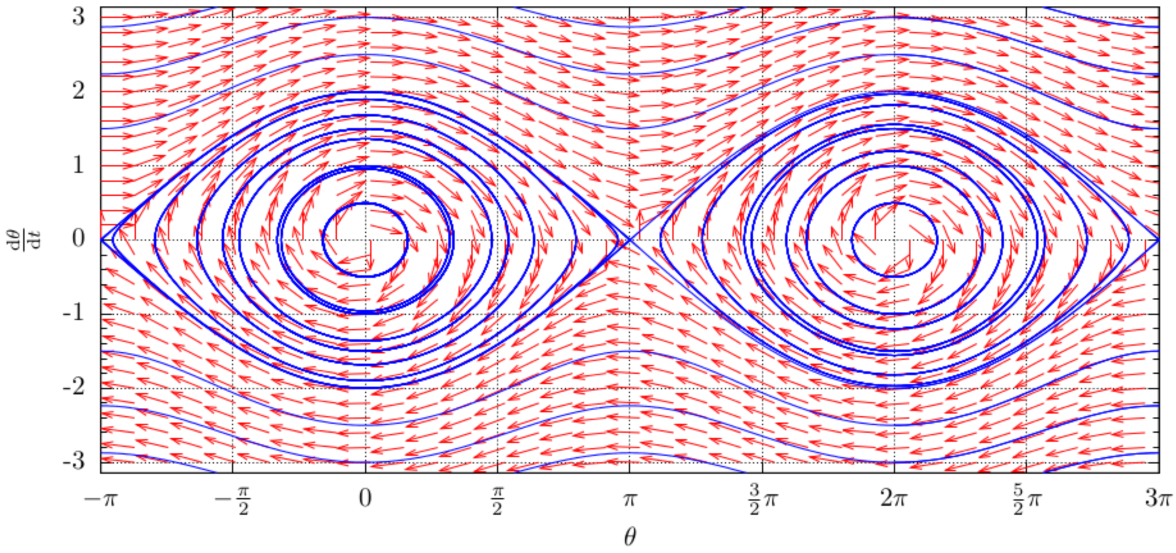
You can note that as t increases, the term e^{-6t} goes to zero leaving the e^{t} dominant, and since this features in both y and y', you get $y \propto y'$ for large t.

The examples we are considering here are relatively simple, however this can be used to identify complex and chaotic phenomena visually. For example, considering a pendulum gently swinging backwards and forwards, it is possible to trace out the phase-space as shown here:



Source: `https://commons.wikimedia.org/wiki/File:Pendulum_phase_portrait_illustration.svg`*, Wikipedia user Krishnavedala*

If you increase the speed of the pendulum, at some critical point, instead of swinging back to the original position it will start whirring round and round. Expressed in terms of vertical angle and angular velocity, the graph becomes:



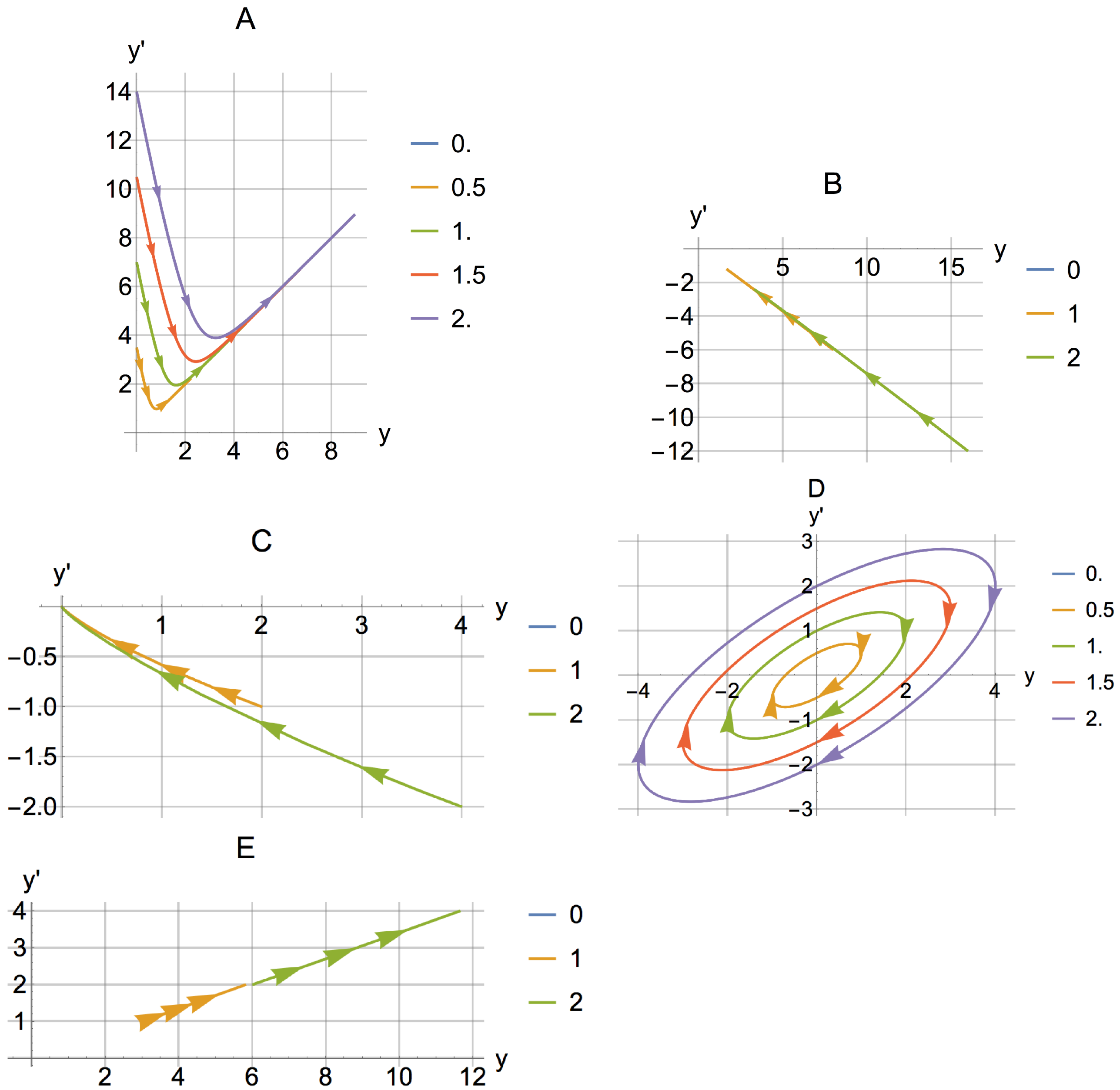
Source: *https://commons.wikimedia.org/wiki/File:Pendulumphase.png*

At low velocities the pendulum swings back and forth (blue circles, angular velocity both positive and negative), but at high velocities, the angular velocity stays positive (or negative) and the pendulum whirs round and round in one direction (blue wavy lines). Note that position $\theta = \pi$ is when the rigid pendulum is pointing exactly upwards. So with no momentum it is stationary here, albeit unstable, because with a tiny velocity it will perform a full loop, slowing (but not stopping) as it reaches the top again.

Challenge

1.
In challenge 6.6 you determined general solutions to exercises 2, 3 and 4. The example earlier in this challenge corresponds to exercise 1 of that same page.

The graphs below correspond to systems arising from exercises 1 to 5. Place the graphs below in the same order those exercises. Note that in order to maintain clarity, the graphs are not necessarily plotted over the same time interval t.



2.
Considering the graph shown earlier of angular momentum vs angle for a rigid pendulum, add the points of the following true statements:

1 point An initial angular velocity of 1 unit results in whirring circular motion irrespective of the starting angle.

2 points An initial angular velocity of -2.5 units results in whirring circular motion irrespective of the starting angle.

4 points An initial angle of $\pi/2$ combined with an angular velocity of 1 unit results in periodic swinging motion.

8 points An initial angle of $\pi/2$ combined with an angular velocity of 1 unit results in circular whirring motion.

16 points An initial angle of 0 combined with an angular velocity of 0 units results in periodic swinging motion.

32 points An initial angle of 0 combined with an angular velocity of 0 units results in a stationary system.

64 points An initial angle of π combined with an angular velocity of 0 units results in a stationary system.

128 points An initial angle of $\pi/2$ combined with an angular velocity of 0 units results in a stationary system.

256 points An initial angular velocity of 3 units results in whirring circular motion in the same direction as an initial angular velocity of -3 units.

512 points An initial angular velocity of 3 units results in whirring circular motion in the opposite direction as an initial angular velocity of -3 units.

Solutions

1.
X = Your solution
Form: String.
Place the indicated letter in front of the string.
Example: aX where X = abcdef is entered as aabcdef

Hash of jX = b63166

2.
X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of kX = 5f8194

Chapter 7

Numerical methods

7.1 Sketching the system

Comment

Numerical methods of solving ODE's are common. Here we will consider Euler and Runge-Kutta approaches.

Challenge

Considering the following ODE:

$$\dot{v} = g - v^2 \tag{7.1}$$

1. Sketch a direction field to show the behaviour of solutions to this ODE. In particular, what values of v will lead to stable solutions?

2. What type of ODE is this?

Solutions

1. (Stable)
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of aX = 4ac0c3

1. (Unstable)
X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of bX = 9ed192

2. Please compare your solution with your partner or ask the teacher.

7.2 Tangent lines

Resource

- Chapter 3: `https://raw.githubusercontent.com/kriskissel/ConceptsODE/master/main.pdf`

Challenge

1. Given that $y(t)$ satisfies the equation $y' = y^3 + 3t$ subject to $y(1) = 2$, find $y'(1)$ without solving the differential equation and obtain the equation of the tangent to the curve $y(t)$ at the point (1,2).

2. Use the tangent line to estimte the value at $t = 1.5$.

Solution

2. 7.5

7.3 Euler's method

Resource

- Chapter 3: `https://raw.githubusercontent.com/kriskissel/ConceptsODE/master/main.pdf`

Challenge

Given that $v(t)$ satisfies the relation $v' = g - v^2$, assuming an initial value of $v(0) = 0$, using Euler's method estimate $v(1)$ using step sizes of

1. $\Delta t = 1/2$
2. $\Delta t = 1/4$

Referring to the slope-field you drew in challenge 7.1, explain the difference in the behaviour with the different step sizes. It may be helpful to draw a graph.

Solution

1. $v(1) = -2.22$
2. $v(1) = 3.22$

7.4 4th-order Runge-Kutta

Resource

- Chapter 3: https://raw.githubusercontent.com/kriskissel/ConceptsODE/master/main.pdf

Comment

Derivation of the Runge-Kutta method is beyond this course, however there are many resources online going into more detail. This video-series is nice, although optional:

1. https://www.youtube.com/watch?v=b-OSyxOpxKc
2. https://www.youtube.com/watch?v=JySrVHRmqfU
3. https://www.youtube.com/watch?v=iS3hsHGY1Ok
4. https://www.youtube.com/watch?v=wr3-dWoxiY4

Challenge

1. Using the same function as challenge 7.3, estimate $v(1/2)$ using the Runge-Kutta method and a step-size of $\Delta t = \frac{1}{4}$.

2. Compare your answer to $v(1/2)$ obtained using the Euler method with the same step-size. How does the behaviour differ?

Solution

1. $v(1/2) = 2.99$

2. Please compare your answer with your partner or check with the teacher.

Appendix A

Solutions

A.1 Challenge 6.6

2. $x = C_1 \begin{pmatrix} 7 \\ -5 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t}$

3. $x = C_1 \begin{pmatrix} \cos 3t + \sin 3t \\ \cos 3t \end{pmatrix} + C_2 \begin{pmatrix} -\cos 3t + \sin 3t \\ \sin 3t \end{pmatrix}$

4. $x = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{6t} + C_2 \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{6t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{6t} \right)$