

Ordinary Differential Equations

Autumn 2017

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<http://www.jamescannon.net/teaching/ordinary-differential-equations>
<http://raw.githubusercontent.com/NanoScaleDesign/OrdinaryDifferentialEquations/master/ode.pdf>

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Chapter 0

Course information

0.1 This course

This is the Autumn 2017 Ordinary Differential Equations course studied by 2nd-year undergraduate international students at Kyushu University.

0.1.1 How this works

- In contrast to the traditional lecture-homework model, in this course the learning is self-directed and active via publicly-available resources.
- Learning is guided through solving a series of carefully-developed challenges contained in this book, coupled with suggested resources that can be used to solve the challenges with instant feedback about the correctness of your answer.
- There are no lectures. Instead, there is discussion time. Here, you are encouraged to discuss any issues with your peers, teacher and any teaching assistants. Furthermore, you are encouraged to help your peers who are having trouble understanding something that you have understood; by doing so you actually increase your own understanding too.
- Discussion-time is from 14:50 to 16:20 on Fridays at room Centre Zone 1409.
- Peer discussion is encouraged, however, if you have help to solve a challenge, always make sure you do understand the details yourself. You will need to be able to do this in an exam environment. If you need additional challenges to solidify your understanding, then ask the teacher. The questions on the exam will be similar in nature to the challenges. If you can do all of the challenges, you can get 100% on the exam.
- Every challenge in the book typically contains a **Challenge** with suggested **Resources** which you are recommended to utilise in order to solve the challenge. Occasionally the teacher will provide extra **Comments** to help guide your thinking. A **Solution** is also made available for you to check your answer. Sometimes this solution will be given in encrypted form. For more information about encryption, see section 0.3.
- For deep understanding, it is recommended to study the suggested resources beyond the minimum required to complete the challenge.
- The challenge document has many pages and is continuously being developed. Therefore it is advised to view the document on an electronic device rather than print it. The date on the front page denotes the version of the document. You will be notified by email when the document is updated. The content may differ from last-year's document.
- A target challenge will be set each week. This will set the pace of the course and defines the examinable material. It's ok if you can't quite reach the target challenge for a given week, but then you will be expected to make it up the next week.
- You may work ahead, even beyond the target challenge, if you so wish. This can build greater flexibility into your personal schedule, especially as you become busier towards the end of the semester.
- Your contributions to the course are strongly welcomed. If you come across resources that you found useful that were not listed by the teacher or points of friction that made solving a challenge difficult, please let the teacher know about it!

0.1.2 Assessment

In order to prove to outside parties that you have learned something from the course, we must perform summative assessments. This will be in the form of a mid-term exam (weighted 30%), coursework (weighted 20%), a satisfactory challenge-log (weighted 10%) and a final exam (weighted 40%).

Your final score is calculated as $\text{Max}(\text{final exam score}, \text{weighted score})$, however you must pass the final exam to pass the course.

0.1.3 What you need to do

- Prepare a challenge-log in the form of a workbook or folder where you can clearly write the calculations you perform to solve each challenge. This will be a log of your progress during the course and will be occasionally reviewed by the teacher.
- You need to submit a brief report at <https://goo.gl/forms/S69DuM4xCss0WtjH3> by 8am on the day of the class. Here you can let the teacher know about any difficulties you are having and if you would like to discuss anything in particular.
- Please bring a wifi-capable internet device to class, as well as headphones if you need to access online components of the course during class. If you let me know in advance, I can lend computers and provide power extension cables for those who require them (limited number).

0.2 Timetable

	Discussion	Target	Note
1	4 Oct	-	Wednesday class
2	13 Oct	3.2	
3	20 Oct		
4	27 Oct		
5	10 Nov		
6	17 Nov		
7	24 Nov		
8	1 Dec		Coursework instructions
9	8 Dec	Midterm exam	
10	15 Dec		
11	22 Dec		
12	15 Jan		Monday class
13	19 Jan		Submission of coursework
14	26 Jan		
15	2 Feb		
16	9 Feb	Final exam	

Example: To keep pace with the course, you should aim to complete challenge 2 of chapter 3 by the 13th of October.

0.3 Hash-generation

Some solutions to challenges are encrypted using MD5 hashes. In order to check your solution, you need to generate its MD5 hash and compare it to that provided. MD5 hashes can be generated at the following sites:

- Wolfram alpha: (For example: md5 hash of “q1.00”) <http://www.wolframalpha.com/input/?i=md5+hash+of+%22q1.00%22>
- www.md5hashgenerator.com

Since MD5 hashes are very sensitive to even single-digit variation, you must enter the solution *exactly*. This means maintaining a sufficient level of accuracy when developing your solution, and then entering the solution according to the format suggested by the question. Some special input methods:

Solution	Input
5×10^{-476}	5.00e-476
5.0009×10^{-476}	5.00e-476
$-\infty$	-infinity (never “infinite”)
2π	6.28
i	im(1.00)
2i	im(2.00)
$1 + 2i$	re(1.00)im(2.00)
$-0.0002548 i$	im(-2.55e-4)
$1/i = i/-1 = -i$	im(-1.00)
$e^{i2\pi} [= \cos(2\pi) + i\sin(2\pi) = 1 + i0 = 1]$	1.00
$e^{i\pi/3} [= \cos(\pi/3) + i\sin(\pi/3) = 0.5 + i0.87]$	re(0.50)im(0.87)
Choices in order A, B, C, D	abcd

The first 6 digits of the MD5 sum should match the first 6 digits of the given solution.

Chapter 1

Hash practise

1.1 Hash practise: Integer

$X = 46.3847$

Form: Integer.

Place the indicated letter in front of the number.

Example: aX where $X = 46$ is entered as a46

hash of aX = e77fac

1.2 Hash practise: Decimal

$X = 49$

Form: Two decimal places.

Place the indicated letter in front of the number.

Example: aX where $X = 46.00$ is entered as a46.00

hash of bX = 82c9e7

1.3 Hash practise: String

$X = abcdef$

Form: String.

Place the indicated letter in front of the number.

Example: aX where $X = abc$ is entered as aabc

hash of cX = 990ba0

1.4 Hash practise: Scientific form

$X = 500,765.99$

Form: Scientific notation with the mantissa in standard form to 2 decimal place and the exponent in integer form.

Place the indicated letter in front of the number.

Example: aX where $X = 4 \times 10^{-3}$ is entered as a4.00e-3

hash of dX = be8a0d

Chapter 2

Definitions

2.1 Order of a differential equation

Resources

- Text: <http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx>

Challenge

What is the sum of the orders of the following equations?

$$\frac{dy}{dx}A = 5x^3 + 3 \quad (2.1)$$

$$\cos(y)y'''(x) - y(x) = 25 \quad (2.2)$$

$$\frac{d}{dx} \frac{d^2y}{dx^2} = \frac{x^{-2}}{3} \quad (2.3)$$

Solution

X = Your solution

Form: Integer

Place the indicated letter in front of the number

Example: aX where $X = 46$ is entered as a46

hash of eX = 492585

2.2 Identifying linear and non-linear differential equations

Comment

Being able to identify linear and non-linear ODE's will help you understand how to approach different problems.

A function is linear if it satisfies the following condition for all y_1 and y_2 :

$$f(y_1 + y_2) = f(y_1) + f(y_2) \quad (2.4)$$

For example,

$$y = 5x^2 + 4$$

is trivially linear, because setting $f(y)$ to be equal to the LHS (which is only a function of y),

$$f(y_1 + y_2) = (y_1 + y_2) = y_1 + y_2 = f(y_1) + f(y_2)$$

The function on the left side does not have to be so simple. It just has to be a function of y . For example, the equation

$$y'' - 4yx = \ln x - y$$

is more complex. However it is still linear. Rearranging to collect all the y -terms together:

$$f(y) = y'' - 4yx + y = \ln x$$

linearity can be proved:

$$\begin{aligned} f(y_1 + y_2) &= (y_1 + y_2)'' - 4(y_1 + y_2)x + (y_1 + y_2) \\ &= y_1'' + y_2'' - 4y_1x + 4y_2x + y_1 + y_2 \\ &= y_1 + 3y_1x + y_1 + y_2 + 3y_2x + y_2 \\ &= f(y_1) + f(y_2) \end{aligned}$$

For non-linear equations however, equation 2.4 does not hold true. For example,

$$5 + yy' = x - y$$

can be rearranged to isolate all the y -terms on the LHS as

$$f(y) = yy' + y = x - 5$$

however it can easily be seen that 2.4 does not hold true:

$$\begin{aligned} f(y_1 + y_2) &= (y_1 + y_2)(y_1 + y_2)' + (y_1 + y_2) \\ &= y_1y_1' + y_2y_2' + y_1y_2' + y_2y_1' + y_1 + y_2 \\ &= y_1y_1' + y_1 + y_2y_2' + y_2 + y_1y_2' + y_2y_1' \\ &= f(y_1) + f(y_2) + y_1y_2' + y_2y_1' \end{aligned}$$

which does not satisfy equation 2.4 for all y_1 and y_2 .

Challenge

Sum the points corresponding to the equations that are linear. You may be able to judge some by eye, but you should prove mathematically that at least one of the equations are linear and at least one of the equations are non-linear.

1 point: $\frac{dx}{dt} = 5t^3 + 3$.

2 points: $\cos(x)x'''(t) - x(t) = 25$.

4 points: $\frac{d}{dt} \frac{d^2x}{dt^2} = \frac{t^{-2}}{3}$.

8 points: $x'(t) - \sin(x(t)) = 0$.

16 points: $x'(t) - x(t) = 0$.

32 points: $tx'(t) - x(t) = 0$.

Solution

X = Your solution

Form: Integer

Place the indicated letter in front of the number

Example: aX where $X = 46$ is entered as a46

hash of rX = f5d2c0

2.3 Linear differential equations vs non-linear differential equations

Resources

- Wikipedia: https://en.wikipedia.org/wiki/Nonlinear_system#Nonlinear_differential_equations
- Wikipedia: https://en.wikipedia.org/wiki/Linear_differential_equation

Challenge

Write no-more than 1 short paragraph describing in qualitative terms the difference between a linear and non-linear differential equation.

Solution

Please compare with your partner in class and discuss with the teacher if you are unsure.

2.4 Valid solutions

Resources

- Text: <http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx>

Challenge

Use substitution to prove that

$$y = \frac{5}{5+x} \tag{2.5}$$

is a solution to the equation

$$xy' + y = y^2 \tag{2.6}$$

and state the value of x for which the solution is undefined.

Solution

Value of x for which solution is undefined:

X = Your solution

Form: Decimal to 2 decimal places.

Place the indicated letter in front of the number.

Example: aX where $X = 46.00$ is entered as a46.00

Hash of tX = 829f33

2.5 Range of valid solutions

Resources

- Text: <http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx>

Challenge

Use substitution to prove that

$$y = -\sqrt{100 - x^2} \tag{2.7}$$

is a solution to the equation

$$x + yy' = 0 \tag{2.8}$$

and state the range of x for which the solution is valid. Enter the value of the lower range as the solution below.

Solution

X = Your solution

Form: Decimal to 2 decimal places.

Place the indicated letter in front of the number.

Example: aX where $X = 46.00$ is entered as a46.00

Hash of yX = d96920

Chapter 3

1st-order differential equations

3.1 Determining a simple DE from a description

Resources

- Text: <http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx>

Challenge

Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference with the ambient surroundings. (a) Write a differential equation describing this situation. (b) Assuming a proportionality constant of 0.2 /hour, what is the rate of temperature change when the object is at 30 °C and the ambient temperature is 20 °C?

Solution

(units: °C h⁻¹)

X = Your solution

Form: Decimal to 2 decimal places.

Place the indicated letter in front of the number.

Example: aX where $X = 46.00$ is entered as a46.00

Hash of qX = 4aca8d

3.2 Direction (Slope) fields

Resources

- Text: <http://tutorial.math.lamar.edu/Classes/DE/DirectionFields.aspx>
- Video 1: <https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/creating-a-slope-field>
- Video 2: <https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/slope-field-to-visualize-solutions>

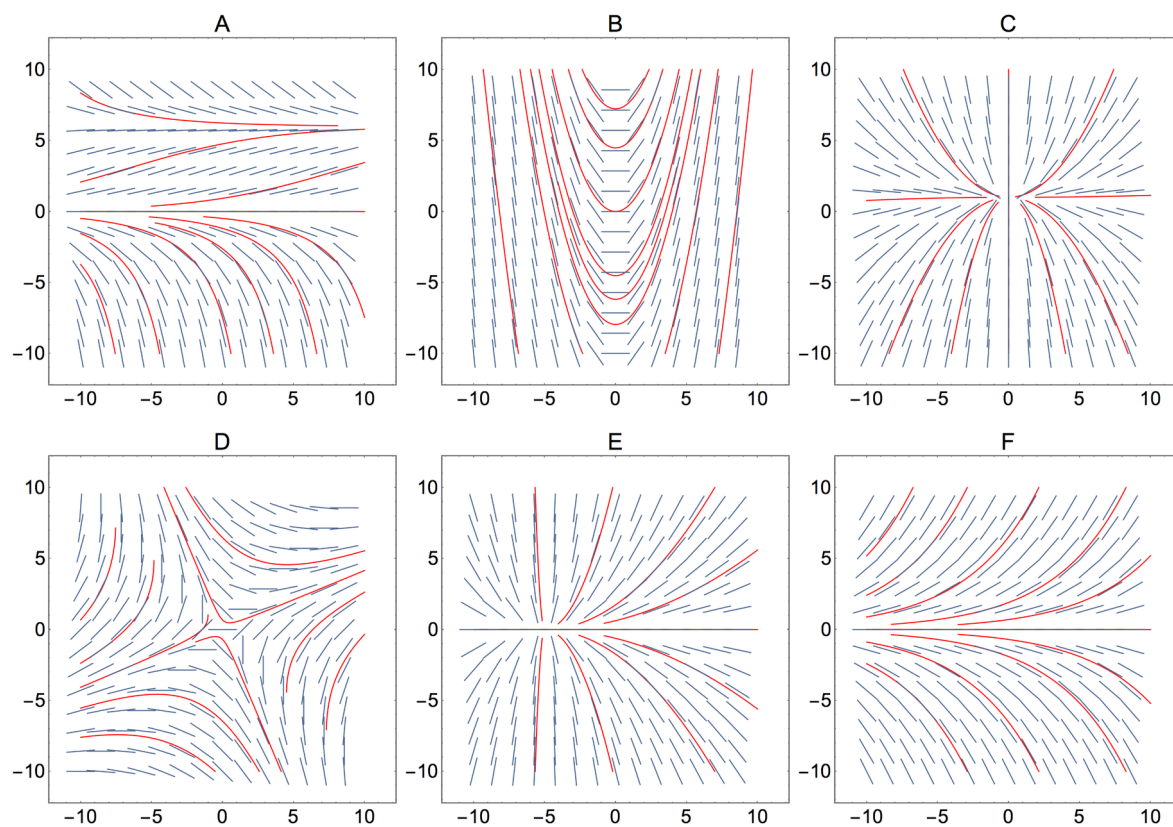
Comment

It is good practise to try drawing the below fields before looking at the next page. You need to be able to go in both directions (ie, drawing and recognising). You will not be given a glimpse at the fields in the exam prior to being asked to draw them.

Question

Try drawing the slope field for at least 3 of the equations given below (your choice). Then, put the slope fields given on the next page in the same order as these equations.

1. $y' = x$
2. $y' = 0.2y$
3. $y' = 0.2y(1 - y/6)$
4. $y' = (x - y)/(x + y)$
5. $y' = 2(y - 1)/x$
6. $y' = 2y/(x + 5)$



Solution

X = Your solution

Form: String.

Place the indicated letter in front of the string.

Example: aX where $X = abcdef$ is entered as aabcdef

Hash of qX = e93bfe