Systems of First Order Linear Differential Equations

We will now turn our attention to solving systems of simultaneous homogeneous first order linear differential equations. The solutions of such systems require much linear algebra (Math 220). But since it is not a prerequisite for this course, we have to limit ourselves to the simplest instances: those systems of two equations and two unknowns only. But first, we shall have a brief overview and learn some notations and terminology.

A system of n linear first order differential equations in n unknowns (an $n \times n$ system of linear equations) has the general form:

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + g_1 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + g_2 \\
x_3' &= a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n + g_3 \qquad (*) \\
&\;\vdots \qquad\quad \vdots \qquad\quad \vdots \\
&\;\vdots \qquad\quad \vdots \qquad\quad \vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + g_n
\end{aligned}
$$

Where the coefficients a_{ij}'s, and g_i's are arbitrary functions of t. If every term g_i is constant zero, then the system is said to be homogeneous. Otherwise, it is a nonhomogeneous system if even one of the g's is nonzero.

The system (*) is most often given in a shorthand format as a matrix-vector equation, in the form:

$$\boxed{\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}}$$

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ \vdots \\ x_n' \end{bmatrix}}_{\boldsymbol{x}'} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ \vdots \\ g_n \end{bmatrix}}_{\boldsymbol{g}}$$

Where the matrix of coefficients, $\boldsymbol{A}$, is called the *coefficient matrix* of the system. The vectors $\boldsymbol{x}'$, $\boldsymbol{x}$, and $\boldsymbol{g}$ are

$$\boldsymbol{x}' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{bmatrix}.$$

For a homogeneous system, $\boldsymbol{g}$ is the zero vector. Hence it has the form

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}.$$