Ordinary Differential Equations

Autumn 2016

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http://www.jamescannon.net/teaching/ordinary-differential-equations

http://raw.githubusercontent.com/NanoScaleDesign/OrdinaryDifferentialEquations/master/ode.pdf

Contents

Chapter 0

Course information

0.1 This course

This is the Autumn 2016 Ordinary Differential Equations course studied by 2nd-year undegraduate international students at Kyushu University.

0.1.1 How this works

- In contrast to the traditional lecture-homework model, in this course the learning is self-directed and active via publicly-available resources.
- Learning is guided through solving a series of carefully-developed challenges contained in this book, coupled with suggested resources that can be used to solve the challenges with instant feedback about the correctness of your answer.
- There are no lectures. Instead, there is discussion time. Here, you are encouraged to discuss any issues with your peers, teacher and any teaching assistants. Furthermore, you are encouraged to help your peers who are having trouble understanding something that you have understood; by doing so you actually increase your own understanding too.
- Discussion-time is from 10:30 to 12:00 on Fridays at room W4-529.
- Peer discussion is encouraged, however, if you have help to solve a challenge, always make sure you do understand the details yourself. You will need to be able to do this in an exam environment. If you need additional challenges to solidify your understanding, then ask the teacher. The questions on the exam will be similar in nature to the challenges. If you can do all of the challenges, you can get 100% on the exam.
- Every challenge in the book typically contains a **Challenge** with suggested **Resources** which you are recommended to utilise in order to solve the challenge. A **Solution** is made available in encrypted form. If your encrypted solution matches the encrypted solution given, then you know you have the correct answer and can move on. For more information about encryption, see section 0.3. Occasionally the teacher will provide extra **Comments** to help guide your thinking.
- For deep understanding, it is recommended to study the suggested resources beyond the minimum required to complete the challenge.
- The challenge document has many pages and is continuously being developed. Therefore it is advised to view the document on an electronic device rather than print it. The date on the front page denotes the version of the document. You will be notified by email when the document is updated.
- A target challenge will be set each week. This will set the pace of the course and defines the examinable material. It's ok if you can't quite reach the target challenge for a given week, but then you will be expected to make it up the next week.
- You may work ahead, even beyond the target challenge, if you so wish. This can build greater flexibility into your personal schedule, especially as you become busier towards the end of the semester.
- Your contributions to the course are strongly welcomed. If you come across resources that you found useful that were not listed by the teacher or points of friction that made solving a challenge difficult, please let the teacher know about it!

0.1.2 Assessment

In order to prove to outside parties that you have learned something from the course, we must perform summative assessments. This will be in the form of a mid-term exam (weighted 30%), coursework (weighted 20%) and a final exam (weighted 50%).

Your final score is calculated as Max(E_F, overall score), however you must pass the final exam to pass the course.

0.1.3 What you need to do

- Prepare a challenge-log in the form of a workbook or folder where you can clearly write the calculations you perform to solve each challenge. This will be a log of your progress during the course and will be occasionally reviewed by the teacher.
- You need to submit a brief report at `https://goo.gl/forms/Djl4FEZcJLMpipsY2` before the discussion time starts. Here you can let the teacher know about any difficulties you are having and if you would like to discuss anything in particular.
- Please bring a wifi-capable internet device to class, as well as headphones if you need to access online components of the course during class. If you let me know in advance, I can lend computers and provide power extension cables for those who require them (limited number).

0.2 Timetable

	Discussion	Target	Note
1	7 Oct	-	
2	14 Oct	2.2	
3	21 Oct	2.7	
4	28 Oct	2.12	
5	4 Nov	2.20	
6	11 Nov	3.8	
7	25 Nov	3.15	
8	2 Dec	3.18	Coursework instructions
9	9 Dec	Midterm exam	
10	16 Dec	4.6	
11	6 Jan	4.12	
12	12 Jan	5.4	Submission of coursework
13	20 Jan	5.5	
14	27 Jan	5.6	
15	10 Feb	Final exam	Open learning plaza room 4

Example: To keep pace with the course, you should aim to complete challenge 2 of chapter 2 by the 14th of October.

0.3 Hash-generation

Most solutions to challenges are encrypted using MD5 hashes. In order to check your solution, you need to generate its MD5 hash and compare it to that provided. MD5 hashes can be generated at the following sites:

- Wolfram alpha: (For example: md5 hash of "q_1.00") `http://www.wolframalpha.com/input/?i=md5+hash+of+%22q_1.00%22`
- `www.md5hashgenerator.com`

Since MD5 hashes are very sensitive to even single-digit variation, you must enter the solution exactly. This means maintaining a sufficient level of accuracy when developing your solution, and then entering the solution according to the format below:

Unless specified otherwise, any number from 0.00 to ±9999.99 should be represented as a normal number to two decimal places. All other numbers should be in scientific form. See the table below for examples.

Solution	**Input**
1	1.00
-3	-3.00
-3.5697	-3.57
0.05	0.05
0.005	5.00e-3
50	50.00
500	500.00
5000	5000.00
50,000	5.00e4
5×10^{-476}	5.00e-476
5.0009×10^{-476}	5.00e-476
$-\infty$	-infinity (never "infinite")
2π	6.28
i	im(1.00)
2i	im(2.00)
1 + 2i	re(1.00)im(2.00)
-0.0002548 i	im(-2.55e-4)
1/i = i/-1 = -i	im(-1.00)
$e^{i2\pi}$ $[= cos(2\pi) + isin(2\pi) = 1 + i0 = 1]$	1.00
$e^{i\pi/3}$ $[= cos(\pi/3) + isin(\pi/3) = 0.5 + i0.87]$	re(0.50)im(0.87)
Choices in order A, B, C, D	abcd

Entry format is given with the problem. So "q_X" means to enter "q_X" replacing "X" with your solution. The first 6 digits of the MD5 sum should match the given solution (MD5(q_X)= ...).

Note that although some answers can usually only be integers (eg, number of elephants), for consistency, to generate the correct hash, the accuracy in terms of decimal places noted above is required.

0.4 Coursework

Ordinary differential equations arise in a wide range of situations. This coursework is designed to give you the opportunity to investigate an application or phenomenon related to your field of interest that involves the use of ODE's.

The task is as follows:

1) Write a report at least 1 full page in length, explaining about an application or phenomenon which can be described in terms of Ordinary Differential Equations. Please include equations, figures and references.

2) Create at least 2 challenges to accompany your report, so someone reading your document can test their knowledge.

3) Include **fully worked** solutions to challenges you make (ie, not only the final answer, but clearly show the steps involved in order to achieve the final answer).

I may (or may not) choose to incorporate some aspects of the submissions into teaching of the final 1 or 2 classes.

0.4.1 Submission

You must submit **both a paper and electronic version**. Submit the materials by **email** to the teacher by **10:30 on 12 January 2017** with the subject "[ODE] Coursework" and **bring a paper copy to the class on that day**.

The electronic version may be in any format, including LibreOffice, MS Word, Google docs, Latex, etc... If you submit a PDF, please also submit the source-files used to generate the PDF.

Late submission:
By 10:00 on 13 January 2017 (electronic submission only): 90% of the final mark.
By 10:00 on 16 January 2017: 50% of the final mark.
Later submissions cannot be considered.

0.4.2 Marking

Marks will be assigned based on the degree to the report fulfills the following criteria:

- Understanding: Clearly demonstrate your understanding of what you write about. You can do this by, for example, solving the ODE for different cases or explaining with words how it applies in different situations.
- Relevance: An application or phenomenon that has a basis in ODE's.
- Originality: It should be your own work. Also, you must **cite all references, as well as images and text taken from other sources**.
- Level: The subject should be pitched at a level whereby anyone else in the class could learn about the subject based on your report. Be sure to explain in reasonable depth.
- Accuracy: The explanation should be accurate and clear.

Chapter 1

Definitions

1.1 Order of a differential equation

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

What is the sum of the orders of the following equations?

$$\frac{dy}{dx}A = 5x^3 + 3 \tag{1.1}$$

$$cos(y)y'''(x) - y(x) = 25 \tag{1.2}$$

$$\frac{d}{dx}\frac{d^2y}{dx^2} = \frac{x^{-2}}{3} \tag{1.3}$$

Solution

X

MD5(e_X) = adb5a9...

1.2 Linear equations

Resources

- Video: `https://www.youtube.com/watch?v=dwNMEMOGO_o`
- Text: `http://www.myphysicslab.com/classify_diff_eq.html`
- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Comment

Using a function x dependent on time t as an example, a differential equation is defined as being linear if it can be written in the form

$$f_n(t)\frac{d^n x}{dt^n} + \cdots + f_1(t)\frac{dx}{dt} + f_0(t)x + f_c(t) = C \tag{1.4}$$

Here, $f_n(t)$ is a function of time only, such as $5t$ or $2/t^2$ or may even be constant with time (eg, 3). Any of the f_n's and the constant C may be zero. If it is possible to arrange an equation into the above form, then the equation must be linear. So for a linear equation, $x(t)$, $x'(t)$, $tx(t)$, $3t^2x'''(t)$ are linear terms in x, but $x(t)^2$, $x(t)x'(t)$ and $5tTan(x)$ are non-linear terms.

Challenge

Sum the points corresponding to the equations that are linear:

1 point: $\frac{dx}{dt} = 5t^3 + 3$.

2 points: $cos(x)x'''(t) - x(t) = 25$.

4 points: $\frac{d}{dt}\frac{d^2x}{dt^2} = \frac{t^{-2}}{3}$.

8 points: $x'(t) - sin(x(t)) = 0$.

16 points: $x'(t) - x(t) = 0$.

32 points: $tx'(t) - x(t) = 0$.

Solution

X

MD5(r_X) = 9aea7d...

1.3 Valid solutions

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Use substitution to prove that

$$y = \frac{5}{5+x} \tag{1.5}$$

is a solution to the equation

$$xy' + y = y^2 \tag{1.6}$$

and state the value of x for which the solution is undefined.

Solution

Value of x for which solution is undefined: X

MD5(t_X) = c69a20...

1.4 Range of valid solutions

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Use substitution to prove that

$$y = -\sqrt{100 - x^2} \tag{1.7}$$

is a solution to the equation

$$x + yy' = 0 \tag{1.8}$$

and state the range of x for which the solution is valid. Enter the value of the lower range as the solution below.

Solution

X

MD5(y_X) = 8c8c08...

Chapter 2

1st-order differential equations

2.1 Determining a simple DE from a description

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`
- Book: Chapter 1.2

Challenge

Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference with the ambient surroundings. (a) Write a differential equation describing this situation. (b) Assuming a proportionality constant of 0.2 /hour, what is the rate of temperature change when the object is at 30 °C and the ambient temperature is 20 °C?

Solution

X °C/hour

MD5(q_X) = 078383...

2.2 Direction (Slope) fields

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/DirectionFields.aspx`
- Video 1: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/creating-a-slope-field`
- Video 2: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/slope-field-to-visualize-solutions`
- Book: Chapters 1.1, 1.2

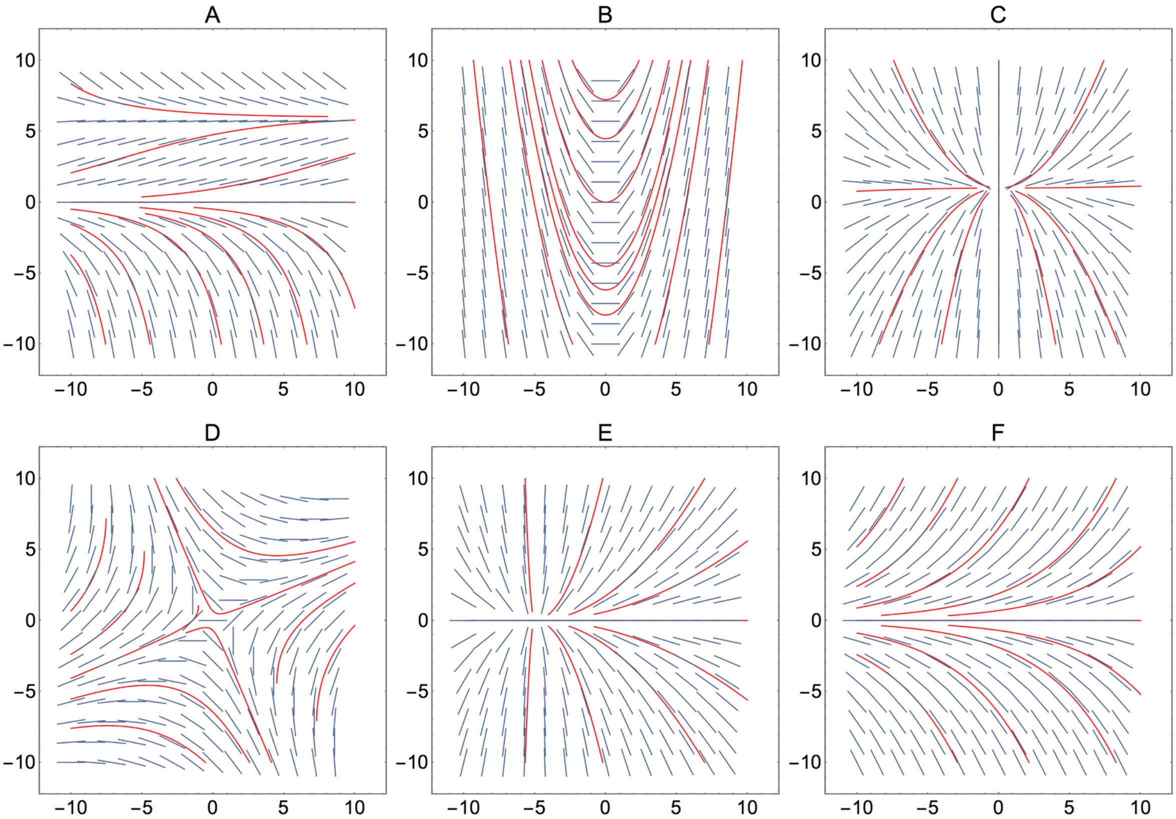
Comment

It is good practise to try drawing the below fields before looking at the next page. You need to be able to go in both directions (ie, drawing and recognising).

Question

Try drawing the slope field for at least 3 of the equations given below (your choice). Then, put the slope fields given on the next page in the same order as these equations.

1. $y' = x$
2. $y' = 0.2y$
3. $y' = 0.2y(1 - y/6)$
4. $y' = (x - y)/(x + y)$
5. $y' = 2(y - 1)/x$
6. $y' = 2y/(x + 5)$



Solution

X (eg, “abcdef”)

MD5(q_X) = 743eb9...

2.3 Solving a simple 1st-order linear equation

Resources

- Video: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/finding-particular-linear-solution-to-differential-equation`

Comment

It's not easy to see when equations can be solved simply, like in the challenge below, and when they can not. But for a 1st-order linear differential equation, this method is often good to try first. If it's not 1st order and not linear, you know to try a different approach.

Challenge

Determine the value of $y(x = 1)$ for the following equation:

$$y' = 2y - 5x + 2 \tag{2.1}$$

Solution

X

MD5(u_X) = 505c7b...

2.4 Separable equations I

Resources

- Video I: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction
- Video II: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example
- Text:http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx

Challenge

Given the following equation:

$$r' = -Sin(\theta) \tag{2.2}$$

Determine the function $r(\theta)$ that passes through the point (0,1) in $\theta - r$ space, and then solve for $\theta = \pi/4$.

Solution

X

MD5(i_X) = 286117...

2.5 Separable equations II

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction`
- Video II: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example`
- Text:`http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx`

Challenge

Given the following equation:

$$r'cot(\theta) + r = 2 \tag{2.3}$$

Determine the function $r(\theta)$ that passes through the point (0,1) in $\theta - r$ space, and then solve for $\theta = \pi/4$.

Solution

X

MD5(o_X) = 87ee92...

2.6 Separable equations III

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction`
- Video II: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example`
- Text:`http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx`

Challenge

Given the following equation:

$$y'x^2 - y = 0 \tag{2.4}$$

Determine the function $y(x)$ that passes through the point $x = 2, y = 1$ and then solve for the given x value. State the value of x where the solution is undefined.

Solution

Solve for $x = 4$:

X

MD5(p_X) = 7cb08e...

Value of x where solution is undefined:

X

MD5(a_X) = b30fe7...

2.8 Logistic equation

Resources

- Videos: The 5 videos on the logistic differential equation and function starting at: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/logistic-differential-equation/v/modeling-population-with-differential-equations`

Comment

The rate of growth can be calculated considering the equation $dN/dt = rN$. This was not clear in an earlier version of the challenge. The question has also been adjusted (100 mg instead of 300 mg) to minimise the chance of rounding errors effecting final answer. The hash-code has been updated to reflect this.

Challenge

Assuming there is no-limit on growth, a given bacteria would be able to reproduce at such a rate that the amount of bacteria measured in mg increases by 20% every 25 hours. However, due to environmental factors the limiting (maximum) amount of bacteria that can exist in the system at any one time is 400 mg. Assuming an initial amount of bacteria of 20 mg, how much time, rounded to the nearest integer hours, must one wait to reach 100 mg of bacteria?

Note: Be sure to maintain a sufficient number of significant figures in your numbers while performing the calculation.

Solution

X hours (expressed as an integer - do not enter ".00".)

MD5(d_X) = 0eba84...

2.9 Autonomous differential equations

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`
- Wikipedia: `https://en.wikipedia.org/wiki/Autonomous_system_(mathematics)`

Challenge

Add the points of the autonomous differential equations in the following list:

1 point: $y' = cos(y) - 5$

2 points: $y' = cos(y)/x - 5$

4 points: $y' = cos(y)/x - 5/x$

8 points: $y^2 = y'y + 5$

16 points: $xy' = 5y$

32 points: $y' = 1$

Solution

X

MD5(f_X) = 9bf043...

2.10 The stability of solutions I

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`
- Text: `http://www.math.psu.edu/tseng/class/Math251/Notes-1st%20order%20ODE%20pt2.pdf`

Challenge

Considering the logistic equation $N' = 0.2N(1 - N/6)$, make 3 separate lists containing any equilibrium, semi-stable and unstable y-values.

To check your answer, sum the value of each list. If there are no values in a list, simply enter "none" to check the result.

Solution

Stable

X

MD5(g_X) = 2c32d8...

Semi-stable

X

MD5(h_X) = 8b595d...

Unstable

X

MD5(j_X) = 4fd3f6...

2.11 The stability of solutions II

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`
- Text: `http://www.math.psu.edu/tseng/class/Math251/Notes-1st%20order%20ODE%20pt2.pdf`

Challenge

Considering the differential equation $y' = (y^2 - 16)(y + 3)^2$, make 3 separate lists containing any equilibrium, semi-stable and unstable y-values.

To check your answer, sum the value of each list. If there are no values in a list, simply enter "none" to check the result.

Solution

Stable

X

MD5(k_X) = 667798...

Semi-stable

X

MD5(z_X) = 200aa3...

Unstable

X

MD5(x_X) = d1ee5b...

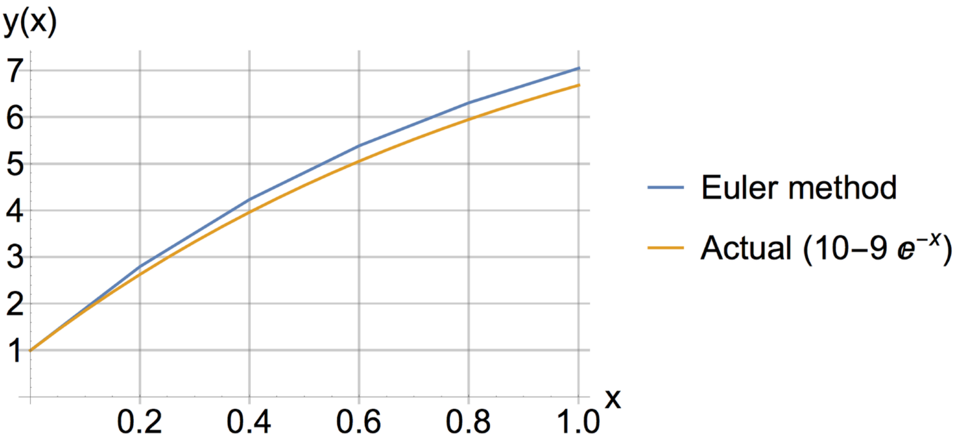
2.12 Euler's method

Resources

- Videos and exersizes in the "Euler's Method" section of Khan academy: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/eulers-method-tutorial/v/eulers-method`
- Text: `http://tutorial.math.lamar.edu/Classes/DE/EulersMethod.aspx`

Challenge

Considering the differential equation $y' = 10 - y$, an initial value of $y(0) = 1$ and a step size of $\Delta x = 0.2$, use Euler's method to estimate the value of $y(x = 1)$. The actual solution, $y(x) = 10 - 9e^{-x}$, is shown below.



Solution

X

MD5(c_X) = 1f90fa...

2.13 Exact differential equations: derivation

Resources

- Videos: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/exact-equations/v/exact-equations-intuition-1-proofy`

Challenge

Please follow the two videos on derivation and intuition regarding exact differential equations starting at the video listed above.

If

$$\frac{d\psi(x,y)}{dx} = 2xy + x^2y' - (x+y)/100 \tag{2.5}$$

what is $\frac{\partial\psi}{\partial x}$?

To check your answer, substitute $x = 3.1$ and $y = -2$ into the resulting equation.

Solution

X

MD5(v_X) = f7f178...

2.14 Exact differential equations: possible solutions given ψ_x

Resources

- Videos: Exact equations intuition 1,2 and examples 1,2,3 starting from `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/exact-equations/v/exact-equations-intuition-1-proofy`
- Text: `http://tutorial.math.lamar.edu/Classes/DE/Exact.aspx`

Challenge

Sum the points of all the possible solutions to the integral of the partial-differential equation:

$$\psi_x = 6x - 3e^x sin(y) \tag{2.6}$$

1 point: $\psi(x, y) = 3x^2 - 3e^x sin(y) + 4$

2 points: $\psi(x, y) = 3x^2 - 3e^x sin(y) + x$

4 points: $\psi(x, y) = 3x^2 - 3e^x sin(y) + y$

8 points: $\psi(x, y) = 3x^2 - 3e^x sin(y) + yx$

16 points: $\psi(x, y) = 3x^2 - 3e^x sin(y) + y^2$

32 points: $\psi(x, y) = 3x^2 - 3e^x sin(y) + 5sin(y)$

64 points: $\psi(x, y) = 3x^2 - 3e^x sin(y) + 5sin(y)cos(x)$

Solution

X

MD5(b_X) = 408993...

2.15 Exact differential equations: identification

Resources

- Videos: Exact equations intuition 1,2 starting from `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/exact-equations/v/exact-equations-intuition-1-proofy`
- Text: `http://tutorial.math.lamar.edu/Classes/DE/Exact.aspx`

Challenge

Sum the points of the equations below that are exact differential equations:

1 point: $(3x^2y + 8xy^2)dx + (x^3 + 8x^2y + 12y^2)dy = 0$

2 points: $sin(x)cos(y)dx + cos(x)sin(y)dy = 0$

4 points: $sin(x)cos(y)dx + sin(x)sin(y)dy = 0$

8 points: $\frac{dx}{x} + \frac{dy}{y} = 0$

16 points: $-\frac{ydx + xdy}{x^2} = 0$

32 points: $-\frac{ydx - xdy}{x^2} = 0$

Solution

X

MD5(n_X) = 868f48...

2.16 Exact differential equations: solving

Resources

- Videos: Exact equations examples 1,2,3 starting from `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/exact-equations/v/exact-equations-example-1`
- Text: `http://tutorial.math.lamar.edu/Classes/DE/Exact.aspx`

Challenge

In challenge 2.15 you should have identified 4 exact differential equations. Considering each of the 4 EDE's in order, try to solve the EDE's applying the following conditions:

1st EDE

Do not try to solve this one.

2nd EDE

Use the condition $y(\pi/4) = \pi/4$ to find an explicit solution for the equation and then evaluate y at $x = \pi$.

3rd EDE

Use the condition $y(1) = 3$ to find an explicit solution for the equation and then evaluate y at $x = 4$.

4th EDE

Use the condition $y(1) = 2$ to find an explicit solution for the equation and then evaluate y at $x = 1$.

Solution

2nd EDE

X

MD5(m_X) = af87e2...

3rd EDE

X

MD5(aa_X) = d01c3d...

4th EDE

X

MD5(bb_X) = 5e1074...

2.17 Exact differential equations: a useful integration method

Challenge

Obtain an expression for $g(x)$ in terms of $f(x)$ in the following integral:

$$\int \frac{f'(x)}{f(x)} dx = g(x) \tag{2.7}$$

ie, you should be able to re-write $g(x)$ in terms of a simple (non-integral) function of $f(x)$, in the form $g(x) = \cdots$.

Solution

You can check your answer by putting a function of x into $f(x)$.

2.18 Exact differential equations: integrating factors

Resources

- Videos: Integrating factors 1,2 starting from `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/exact-equations/v/integrating-factors-1`

Comment

Note that in the videos, Sal Khan does an example considering an integrating factor of $\mu(x)$, but in some cases $\mu(y)$ leads to a solution more easily. You may need to try both to determine an answer.

Challenge

Solve the exact differential equations below using integrating factors.

1. Solve the equation below using an integrating factor. Place the solution in the form $f(x, y) = C$, then calculate the value of C when substituting $x = 2$ and $y = 1$ into the equation. Do not try to solve the equation to get it in the form $y(x) = \cdots$.

$$ydx + (2xy - e^{-2y})dy = 0 \tag{2.8}$$

2. Calculate the integrating factor for the following equation. To check your answer, substitute $x = 1$ or $y = 1$ into any final expression, assuming an integration constant of zero.

$$y(3x - y)dx + x(x - y)dy = 0 \tag{2.9}$$

3. Show that $1/(x^y + y^2)$ is an integrating factor for the equation

$$xdx + ydy + 4y^3(x^2 + y^2)dy = 0 \tag{2.10}$$

Solution

Challenge related to equation 2.8: MD5(cc_X) = bb15d6...

Challenge related to equation 2.9: MD5(dd_X) = 6a8742...

2.19 Exact differential equations: integrating factor derivation

Challenge

1. Starting from the equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0 \tag{2.11}$$

show that if the integrating factor μ is only a function of x, then

$$\mu_x = \mu \left(\frac{M_y - N_x}{N} \right) \tag{2.12}$$

2. Do the same, assuming that μ is only a function of y.

2.20 Exact differential equations: integrating factor calculation

Comment

Without proof, we can use equation 2.12 to gain information about the existance of an integration factor. If $\left(\frac{M_y - N_x}{N}\right)$ is a function of x only, then we know that the integration factor is only a function of x, and it can be solved for by integration of equation 2.12. The same can be said for $\mu(y)$ that you derived an expression for in challenge 2.19.

Challenge

Use equations from section 2.19 and information provided in the comment here to determine the integrating factor for

$$e^x dx + (e^x Cot(y) + 2yCsc(y))dy = 0 \tag{2.13}$$

and

$$(x - y^2)dx + 2xydy = 0 \tag{2.14}$$

To check your answer, for both cases substitute $x = \pi$ or $y = \pi$ into the integrating factor, and assume an integration constant of 1.

Solution

Equation 2.13: MD5(ee_X) = 51a0ae. . .

Equation 2.14: MD5(ff_X) = a56bce. . .

2.21 Summary of 1st-order differential equations

Challenge

1. Create a flowchart describing how you will approach solving a general 1st-order differential equation.
2. Solve the following 1st-order differential equations:

$$y' - 4y = 8x + 3 \tag{2.15}$$

evaluated at $x = 1$.

$$4yy' = 8x + 3 \tag{2.16}$$

assuming an integration constant of zero and evaluating the final equation at $x = 2$.

$$y' + 4y = e^{-8x} \tag{2.17}$$

assuming an integration constant of zero and evaluating the final equation at $x = 1/8$.

Solution

Equation 2.15: MD5(qq_X) = a43ab2...

Equation 2.16: MD5(rr_X) = 990bfa...

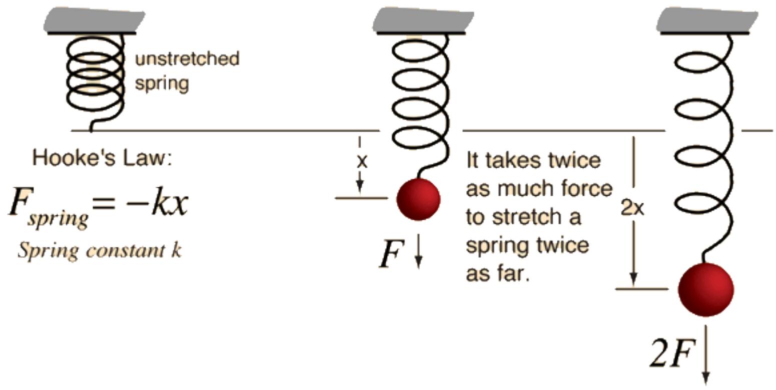
Equation 2.17: MD5(ss_X) = 91989d...

Chapter 3

2nd-order differential equations

3.1 Hooke's law

Resources



(Image from HyperPhysics by Rod Nave, Georgia State University)

Challenge

2nd-order differential equations deal with oscillations.

Considering Hooke's law, what are A and C in the following equation?

$$Ax'' + Cx = 0 \tag{3.1}$$

To check your answer, substitute a mass of 2 kg and spring-constant of 3 kg/s^2 as appropriate.

Solution

Enter only numerical values without units such as kg.

A: MD5(gg_X) = 4e5fe6...

C: MD5(hh_X) = 6a7015...

3.2 Exponentials and trigonometry

Resources

- Text: `https://www.phy.duke.edu/~rgb/Class/phy51/phy51/node15.html`

Challenge

Write $sin(x)$ and $cos(x)$ in exponential form.

Solution

Check your answer with someone if you are unsure.

3.3 Characteristic equation: understanding

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 111.

Comment

A homogeneous (ie, equal to zero) second-order differential equation typically takes the form:

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0 \tag{3.2}$$

The first (A) term describes acceleration, while the third (C) term is the force-constant term (something like the "stiffness" of the spring). The second (B) term could describe a frictional force that is proportional to the velocity (dy/dt). Due to its relation with oscillation (and by extension, sines and cosines which can be expressed in terms of exponentials) we can typically assume an exponential-form solution to the differential equation.

Challenge

Show that, assuming that all solutions to a 2nd-order differential equation of the form above will have solutions $y(t) = e^{rt}$, the value of r can in principle be determined by solving the following a quadratic equation of the form

$$Ar^2 + Br + C = 0 \tag{3.3}$$

Solution

If you are unsure of your derivation, please ask someone.

3.4 Characteristic equation: roots

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 111.

Challenge

Sum the points of the differential equations that have characteristic equations with

- Real, distinct roots
- Complex roots
- Equal roots

1 point: $-3y'' - 5y' + 2y = 0$

2 points: $3y'' - 4y' + 3y = 0$

4 points: $3y'' - 6y' + 3y = 0$

8 points: $3y'' - 5y' + 2y = 0$

16 points: $3y'' - 5y' + 4y = 0$

32 points: $3y'' + 5y' + 2y = 0$

Solution

- Real, distinct roots: MD5(ii_X) = 064a6e...
- Complex roots: MD5(jj_X) = 5cdb6c...
- Equal roots: MD5(kk_X) = 70cd8f...

3.5 Characteristic equation: real roots with positive B

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 116.

Challenge

Solve the following 2nd-order differential equation that has real roots:

$$y'' + 3y' + 2y = 0 \tag{3.4}$$

with initial conditions $y(0) = 5$ and $y'(0) = -8$.

To check your answer, substitute $t = 1$ into the final expression.

Solution

1.14

3.6 Characteristic equation: real roots with negative B

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 116.

Challenge

Solve the following 2nd-order differential equation that has real roots.

$$y'' - 3y' + 2y = 0 \tag{3.5}$$

with initial conditions $y(0) = 5$ and $y'(0) = -8$. Substitute $t = 1$ into the final expression to check your answer.

Note that this equation is the same as equation 3.4, but simply the dampening (friction) term B has been changed from positive to negative.

Solution

-47.13

3.7 Characteristic equation: B in equations with real roots

Challenge

(Note that there are two parts to this challenge.)

1. Considering real root, sum the points of the following true statements:

Considering the equation

$$Ay'' + By' + Cy = 0 \tag{3.6}$$

1 point: Positive damping (positive B) leads to solutions with exponentials with positive exponents.

2 points: Positive damping (positive B) leads to solutions with exponentials with negative exponents.

4 points: Negative damping (negative B) leads to solutions with exponentials with positive exponents.

8 points: Negative damping (negative B) leads to solutions with exponentials with negative exponents.

16 points: Exponentials with positive exponents (eg, e^t) lead to exponential growth (instability).

32 points: Exponentials with negative exponents (eg, e^{-t}) lead to exponential growth (instability).

64 points: Exponentials with positive exponents (eg, e^t) lead to a damped signal (stability).

128 points: Exponentials with negative exponents (eg, e^{-t}) lead to damped signal (stability).

2. Write a sentence summarising your understanding of the significance of having a positive or negative coefficient of B when the roots are real.

Solution

MD5(oo_X) = fa6adf...

3.8 Characteristic equation: equal roots

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 125.

Comment

It is not necessary to follow the full derivation in the suggested resource.

Challenge

Solve the equation

$$y'' - 2y' + y = 0 \tag{3.7}$$

To check your solution, substitute $t = 1$ into the equation and assume $c_1 = c_2 = 1$.

Solution

5.44

3.9 Characteristic equation: complex roots with B=0

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 120.

Challenge

1. Assuming there is no damping term (ie, $B = 0$) show that the roots for the differential equation

$$Ay'' + Cy = 0 \tag{3.8}$$

are $\pm i\sqrt{C/A}$.

2. Solve the following ODE:

$$y'' + 4\pi^2 y = 0 \tag{3.9}$$

To check your answer, assume integration constants of 1 and calculate $y(\pi/2)$.

Solution

2: -1.33

3.10 Characteristic equation: complex roots with positive B

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 120.

Challenge

Solve the following ODE:

$$y'' + y' + y = 0 \tag{3.10}$$

To check your answer, assume integration constants of 1 and calculate $y(\pi/2)$.

Solution

0.54

3.11 Characteristic equation: complex roots with negative B

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 120.

Challenge

Solve the following ODE:

$$y'' - y' + y = 0 \tag{3.11}$$

To check your answer, assume integration constants of 1 and calculate $y(\pi/2)$.

Solution

2.60

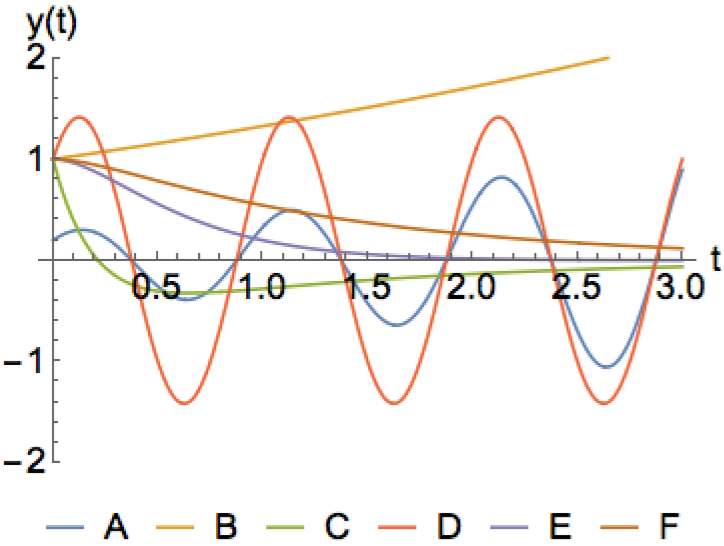
3.12 Damping

Resources

- Wikipedia: `https://en.wikipedia.org/wiki/Damping`

Challenge

Of the 6 functions shown in the graph, place the 3 that correspond to over-damped, critically damped and under-damped in the order mentioned in this sentence.



Solution

(eg, "abc")

MD5(tt_X) = cc92af...

3.13 Damping and 2nd-order differential equations

Challenge

1. The 6 functions shown in the graph in challenge 3.12 may represent solutions of a 2nd-order differential equation $Ay'' + By' + C = 0$. Assuming $A > 0$ and $C > 0$, place the solutions A-F in the order shown below.

I. Solution of a 2nd-order differential equation with real roots and positive B.

II. Solution of a 2nd-order differential equation with real roots and negative B.

III. Solution of a 2nd-order differential equation with equal roots.

IV. Solution of a 2nd-order differential equation with complex roots and B=0.

V. Solution of a 2nd-order differential equation with complex roots and positive B.

VI. Solution of a 2nd-order differential equation with complex roots and negative B.

Solution

(eg, “abcdef”)

MD5(uu_X) = a96870…

3.14 The Wronskian

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 125 and page 130.

Challenge

Please write the following answers clearly and in a manner that can be easily shared with others in the class.

1. What is meant by a "fundamental set of solutions"?
2. Why is the final solution for real and complex roots always a sum of two terms?
3. What is the "Wronskian", and what is the formula for its calculation?
4. Considering $C_1y_1(t) + C_2y_2(t) = 0$, how is linear dependence and independence defined?
5. How can the Wronskian be used to determine linear independence?

Solution

Please read at least 1 other peer's solution and discuss any differences. The teacher will also help check your understanding.

3.15 Characteristic equation: exercises

(Note that if you encounter a square-root during your calculations such as $\sqrt{7}$, it is best to work with $\sqrt{7}$ rather than 2.65 in order to maintain accuracy until the final step where you need to evaluate it. If the equation becomes too messy (eg $e^{(\sqrt{7}-1)/\sqrt{3}}$) you can always substitute $m = (\sqrt{7}-1)/\sqrt{3}$, etc, to make things clearer.)

Challenge

1. Determine $y(1)$ for the equation

$$2y'' + 8y' + y = 0 \tag{3.12}$$

given the initial conditions $y(0) = 4$ and $y'(0) = 3$.

2. Determine $y(0.2)$ for the equation

$$2y'' + 4y' + 2y = 0 \tag{3.13}$$

given the initial conditions $y(0) = 4$ and $y'(0) = 2$.

3. Determine $y(0.1)$ for the equation

$$4y'' + 3y' + y = 0 \tag{3.14}$$

given the initial conditions $y(0) = 6$ and $y'(0) = 2$.

Solution

1. 4.32
2. 4.26
3. 6.19

3.16 Non-homogeneous equations: Method of undetermined coefficients

Resources

- Video: All 4 Khan Academy videos starting at https://www.khanacademy.org/math/differential-equations/second-order-differential-equations/undetermined-coefficients/v/undetermined-coefficients-1
- PDF: http://www.math.psu.edu/tseng/class/Math251/Notes-2nd%20order%20ODE%20pt2.pdf

Comment

The 2nd-order equations we were considering until now were homogeneous equations (ie, the RHS was zero). We can now build upon this to expand our ability to solve non-homogeneous equations (ie, where the RHS of the equation is non-zero). This will put you in a really strong position in terms of solving certain classes of 2nd-order ODE's.

The Khan Academy videos give an excellent initial introduction to the subject, and so please do take the time to view and take notes about all four videos in the series. The PDF listed above then allows us to develop our repertoire much further and explains very clearly about cases more-complicated than the videos. Please therefore make notes covering the material in the PDF from page 11 to onwards. Prior pages are largely covered by the videos.

You may note that in the PDF, the particular solution is denoted by Y while Sal Khan denotes it as y_p in the videos.

Challenge

Complete questions 1-4 on page 22 in the PDF. These first challenges cover the fundamental basic cases upon which all subsequent cases are built.

Solution

The questions in this challenge are taken from the PDF and the answers can be found on the last page. Perhaps obviously, since you will not have the answers in a real-life/exam environment, please don't review each answer until completion. If you get stuck, be sure to review your notes (especially the worked-examples in the PDF) rather than the answers, to facilitate deep learning.

3.17 Method of undetermined coefficients II

Resources

- PDF: `http://www.math.psu.edu/tseng/class/Math251/Notes-2nd%20order%20ODE%20pt2.pdf`

Challenge

Continuing from the previous challenge, complete questions 5-10 on page 22 in the PDF. For at least 1 of these, make up some initial conditions (eg, $y(0) = 1$, $y'(0) = 2$) and try to solve for those conditions. These challenges introduce a range of more complex situations and thus provide excellent practise of the concepts covered.

Solution

The questions in this challenge are taken from the PDF and the answers can be found on the last page. Perhaps obviously, since you will not have the answers in a real-life/exam environment, please don't review each answer until completion. If you get stuck, be sure to review your notes (especially the worked-examples in the PDF) rather than the answers, to facilitate deep learning.

3.18 Method of undetermined coefficients III

Resources

- PDF: `http://www.math.psu.edu/tseng/class/Math251/Notes-2nd%20order%20ODE%20pt2.pdf`

Comment

This challenge gives you useful practise of going the other way; determining a differential equation that describes a given solution. This can be a little confusing at first, so take time to understand where things originate from.

Challenge

Complete challenges 19 and 20 from page 23 of the PDF.

Solution

The questions in this challenge are taken from the PDF and the answers can be found on the last page. Perhaps obviously, since you will not have the answers in a real-life/exam environment, please don't review each answer until completion. If you get stuck, be sure to review your notes (especially the worked-examples in the PDF) rather than the answers, to facilitate deep learning.

Chapter 4

Laplace transformation

4.1 Your first Laplace Transform calculations

Resources

- Videos: The **four** Khan-academy videos starting at `https://www.khanacademy.org/math/differential-equations/laplace-transform/laplace-transform-tutorial/v/laplace-transform-1`

Comment

The Laplace Transform is a powerful technique that has many uses beyond solving ODE's. It can however appear a bit abstract at first. Becoming comfortable with controlling and manipulating the transform will help provide confidence when using it to solve ODE's. The four videos in the resources above provide an excellent starting point for getting you comfortable with this powerful technique.

Challenge

1. Calculate $\mathcal{L}\{1\}$
2. Calculate $\mathcal{L}\{at\}$
3. Calculate $\mathcal{L}\{Cos(at)\}$

Solution

To check your answer, substitute $s = 1$ and $a = 2$ into your final solution.

1. 1
2. 2
3. $\frac{1}{5}$

4.2 Laplace transform of a 3rd derivative

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/laplace-transform-5`
- Video II: `https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/laplace-transform-6`

Challenge

1. Calculate $\frac{d^3}{dt^3}\left(te^{at}\right)$
2. Given

$$\mathcal{L}\{te^{at}\} = \frac{1}{(a-s)^2} \tag{4.1}$$

determine $\mathcal{L}\{3a^2e^{at} + a^3te^{at}\}$

Solution

To check your answer, substitute $s = 1$ and $a = 2$ into your final solution.

-4

4.3 Shifting a transform

Resources

- Video: `https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/more-laplace-transform-tools`

Challenge

Given

$$\mathcal{L}\{Cosh(at)\} = \frac{s}{s^2 - a^2} \tag{4.2}$$

1. What is $\mathcal{L}\{e^{3t}Cosh(5t)\}$?
2. What is $f(t)$ in the equation $\mathcal{L}\{f(t)\} = \frac{s-4}{(s-4)^2-100}$?

Solution

To check your answer, substitute $s = 2$ and $t = 2$ as appropriate:

1. 0.0417
2. 7.23×10^{11}

4.4 L'Hôpital's rule

Resources

- Wikipedia: `https://en.wikipedia.org/wiki/L%27H%C3%B4pital%27s_rule`

Challenge

1. Use L'Hôpital's rule to determine the limit of

$$te^{-st} \tag{4.3}$$

as $t \to 0$.

2. Considering the case of

$$\frac{t^n}{e^{st}} \tag{4.4}$$

if we apply L'Hôpital's rule n times with respect to t, what is the power of t in the numerator? Note that e^{st} is always constant, so by repeated differentiation we can apply L'Hôpital's rule even for t^n.

Solution

1. MD5(ww_X) = 76c8d4...
2. MD5(xx_X) = 1592d7...

4.5 Laplace Transformation of the unit step function

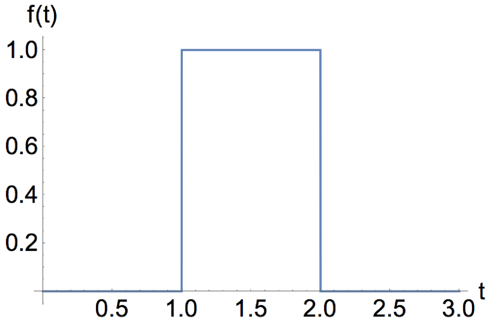
Resources

- Video: `https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/laplace-transform-of-the-unit-step-function`

Challenge

Considering U_c as the unit step-function at c, calculate the following Laplace transformations:

1. $\mathcal{L}\{U_0\}$
2. $\mathcal{L}\{U_c\}$
3. A 1-second pulse function starting at time $t = 1$ with value $f(y) = 1$ as shown in the graph below:



4. $\mathcal{L}\{U_\pi(t)cos(t - \pi)\}$

Solution

To check your answers, substitute $c = 1$ and $s = 2$ as appropriate.

1. MD5(yy_X) = 39574c...
2. 0.0677
3. 0.0585
4. 7.470×10^{-4}

4.6 Inverse Laplace Transform

Resources

- Video: `https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/inverse-laplace-examples`

Comment

Being able to reversing the Laplace transform is a crucial skill required for applying it to solving ODE's. It can be a little confusing at first however, so I recommend to take your time to understand the essential steps involved thoroughly, as this will then give you greater confidence when you come to apply this to solving ODE's. To this end, the video listed in the resource is a fantastic introduction to this.

Challenge

Determine the function $f(t)$ by finding the inverse of the following Laplace transforms:

1. $F(s) = \dfrac{1}{(s-1)^2}$
2. $F(s) = \dfrac{1-s}{s^2}$
3. $F(s) = \dfrac{2e^{-2s}}{s^2-2s+2}$
4. $F(s) = \dfrac{6}{(2+s)^4}$
5. $F(s) = \dfrac{120+6s^3}{s^6}$
6. $F(s) = \dfrac{e^{12-3s}}{s-4}$

Solution

To check your answers, substitute $t = 2$ into your final answer. If there is a unit-step in your solution, precede your numerical answer with "u(c)" where "c" is the position of the unit step. So for example, an answer of $U_5 t^2$ would be entered as "u(5.00)4.00" (all numbers to two decimal places). An answer without a unit-step would just be entered to two decimal places (eg, "4.00" in the previous example).

1. Hash = 5cacdb...
2. Hash = 41cf26...
3. Hash = 45c11e...
4. Hash = 9ffc7a...
5. Hash = 766fd0...
6. Hash = e60079...

4.7 The Dirac delta function and its Laplace transform

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/dirac-delta-function`
- Video II: `https://www.khanacademy.org/math/differential-equations/laplace-transform/properties-of-laplace-transform/v/laplace-transform-of-the-dirac-delta-function`

Challenge

Calculate the following Laplace transforms (treat c as a positive constant):

1. $\mathcal{L}\{\delta(t)\}$
2. $\mathcal{L}\{\delta(t-c)\}$
3. $\mathcal{L}\{\delta(t-2)Cos(4t)\}$
4. $\mathcal{L}\{\delta(t)(t^2+10)\}$

Solution

To check your solution, set $s=1$, $c=2$ and $t=1$ as appropriate to check your answers.

1. MD5(zz_X) = ffef92...
2. MD5(aaa_X) = 826784...
3. MD5(bbb_X) = f44448...
4. MD5(ccc_X) = 4ca484...

4.8 The Dirac delta function and its inverse Laplace transform

Challenge

Calculate the following Laplace transform:

$\delta(t-2)Sin(2t)$

Calculate the following inverse Laplace transforms:

1. $e^{-2s}Sin(2)$
2. $e^{-2s}Sin(4)$

Solution

To check your answer, substitute $t = 1$ into the final expression and evaluate the part inside and outside of the Dirac delta function separately. So for example, if your answer is $\delta(t-2)(t^2+1)$, the expression inside the delta-function is $t-2$ and will evaluate to -1.00 while the expression outside of the delta-function is t^2+1 and will evaluate to 2.00.

1. Inside delta function: MD5(ddd_X) = 7cec9e...; Outside delta function: MD5(eee_X) = 8147e6...
2. Inside delta function: MD5(fff_X) = 033c55...; Outside delta function: MD5(ggg_X) = b1643a...

4.9 A forced spring

- The **four** videos starting at `https://www.khanacademy.org/math/differential-equations/laplace-transform/laplace-transform-to-solve-differential-equation/v/laplace-transform-to-solve-an-equation`
- A useful table of Laplace transforms: `http://tutorial.math.lamar.edu/pdf/Laplace_Table.pdf`

Comment

Here you finally get the opportunity to practise solving ODE's using the powerful method of Laplace transformations. Please takes notes from all four videos listed in the resources section; they provide very useful examples of how to use this method, including related algebraic techniques that are commonly required to solve such challenges.

Challenge

The spring equation you encountered in challenge 3.1 introduced you to the concept of oscillation of a mass on a spring. There, the equation to determine the displacement of the spring y from its equilibrium position was $y'' + y = 0$, which yields a solution $y = C_1 Cos(t) + C_2 Sin(t)$. This is free oscillation without external damping or driving, and it will oscillate according to the cosine and sine sum for all time (t). It is also possible to add a forcing term to the equation by making it non-homogeneous, such as in the form

$$y'' + 4y = 2Cos(3t) \tag{4.5}$$

Here the forcing varies with time t in the form of a cosine wave.

Use the Laplace transform method to solve the ODE in the above equation given a starting displacement of zero and an initial velocity of zero. You may use the table of Laplace transforms in the resources to help you.

Solution

Substitute $t = 1$ to check your final solution: $y(t = 1) = 0.2295$.

4.10 An exponential function

Challenge

Solve

$$y'' + 5y' + 4y = 100e^{-2t} \tag{4.6}$$

for y, given initial conditions $y(0) = -1$ and $y'(0) = 0$. Since the algebra gets very messy, you may use the following equation to help you:

$$\frac{-s^2 - 7s + 90}{(s+1)(s+2)(s+4)} = \frac{32}{s+1} - \frac{50}{s+2} + \frac{17}{s+4} \tag{4.7}$$

Solution

Substitute $t = 1$ to check your final solution: $y(1) = 5.32$.

4.11 A unit step

Comment

In past challenges we studied the Laplace transform for $U_c f(t-c)$. So if $f(t) = t$ we must evaluate for $f(t-c) = t-c$. In the challenge here, we effectively have $f(t) = 1$ and since "1" doesn't depend on t, $t-c$ doesn't do anything to the "function".

This challenge is interesting because unlike previous challenges, it is the first challenge where we really have no other option but to use the Laplace transform method, and so you can appreciate its power. In this challenge, we have a 2nd-order homogeneous equation (unforced oscillation) until $t = 5$ when we apply a constant force. You will find your answer leads to a constant oscillation. But how can it lead to a constant oscillation if we are constantly applying a force? Shouldn't the oscillation slowly increase in magnitude due to the energy that is being added to the system from the constant force being applied? The answer is of course no: we take just as much energy out of the system when the velocity is in the opposite direction to the force as we add to the system when the velocity is in the same direction as the applied force.

One important point to note is that the inverse Laplace transform of $e^{-cs}s/(s^2+a^2)$ is $\mathcal{L}\{U_c Cos(a[t-c])\}$ (not $\mathcal{L}\{U_c Cos([at-c])\}$).

Challenge

Solve

$$y'' + 2y = U_5 \tag{4.8}$$

for y, given initial conditions $y(0) = 0$ and $y'(0) = 0$.

Solution

$y(t = 6) = 0.42$

Note that for $t < 5$, the solution is zero. This is because there was no initial velocity and no initial acceleration, so there was no motion until a forcing was applied in terms of a constant force of "1" from $t = 5$. If either of these had been non-zero, we would have had a non-zero value for $t < 5$!

Optionally, you can try setting the initial conditions to non-zero values to see the effect this has on the final solution.

4.12 A sudden impulse

Comment

Here the system is stationary until $t = 5$ when, instead of applying a constant force, we “kick” the system to start the oscillation. Thus you should expect your answer to reflect physics such as this.

Challenge

Solve

$$y'' + 2y = \delta(t - 5) \tag{4.9}$$

for y, given initial conditions $y(0) = 0$ and $y'(0) = 0$.

Solution

$y(6) = 0.698$

Note how we have a simple oscillation after $t = 5$, and nothing before it.

Chapter 5

Systems of ODE's

5.1 Homogeneous vs non-homogeneous

Resources

- Page 1 of the PDF `http://www.math.psu.edu/tseng/class/Math251/Notes-LinearSystems.pdf`

Challenge

Separately add the points of the following *homogeneous* and *non-homogeneous* ODE systems:

1 point: $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

2 points: $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} Cos(t) \\ 0 \\ 0 \end{pmatrix}$

4 points: $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} Cos(t) \\ Sin(t) \\ 0 \end{pmatrix}$

8 points: $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} Cos(t) \\ Sin(t) \\ Tan(t) \end{pmatrix}$

Solution

Homogeneous: MD5(hhh_X) = 106c67...
Non-homogeneous: MD5(iii_X) = d51f57...

5.2 Basis for creating a system of equations from a single ODE

Resources

- Pages 1-4 of the PDF `http://www.math.psu.edu/tseng/class/Math251/Notes-LinearSystems.pdf`

Comment

Note that the notation $y^{(2)}$ means "the 2nd differential of y" while the notation y^2 (without the brackets around the 2) means "y-squared".

Considering the general form of an nth-order linear equation,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = g(t) \tag{5.1}$$

we substitute $x_1 = y$, $x_2 = y'$, ..., $x_n = y^{(n-1)}$ and $x_n' = y^{(n)}$.

When replacing a y-term by an x term, the n in x_n corresponds to one more than the number of times y is differentiated. So x_{n+1} corresponds to y being differentiated n times and x_n corresponds to y being differentiated $n-1$ times. So x_2 corresponds to $y^{(1)}$ (differentiated 1 time) and x_1 corresponds to y (differentiated 0 times).

Note that x_n' is one more differential than x_n, so x_n' corresponds to $(y^{(n-1)})' = y^{(n)}$. So the n in x_n' corresponds to the number of times y is differentiated (ie, $y^{(n)}$).

The examples on page 3 are clearer after reading page 4, so I encourage you to read page 4 before considering the examples.

Considering example (II) on page 3, you are given the equation

$$y''' - 2y'' + 3y' - 4y = 0 \tag{5.2}$$

To add a more detailed explanation to that found in the PDF: First re-write the ODE in terms of x and x'. Note that there is no "x_0'" so we just write it as x_1 in both equations.

$$x_4 - 2x_3 + 3x_2 - 4x_1 = 0 \tag{5.3}$$

$$x_3' - 2x_2' + 3x_1' - 4x_1 = 0 \tag{5.4}$$

Our aim is to write the system of equations in the form $\boldsymbol{x'} = \boldsymbol{A}\boldsymbol{x}$. Note that there is no "x_4'" in our equations, so the largest value of n in x_n' will be 3 (ie, x_3').

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{5.5}$$

where the question marks are values that we have to find.

By direct comparison of equations 5.3 and 5.4 we know that $x_1' = x_2$ which can be written as $x_1' = 0x_1 + 1x_2 + 0x_3$ yielding the first line in the matrix $\boldsymbol{A}$:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{5.6}$$

We can then proceed to do x_2 in a similar fashion:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{5.7}$$

In order to express x_3' in the above matrix form, we need it in terms of x_1, x_2 and x_3 rather than x_4, so instead of direct comparison, we swap x_4 for x_3' in equation 5.3 to read

$$x_3' - 2x_3 + 3x_2 - 4x_1 = 0 \tag{5.8}$$

and then isolate x_3' to read $x_3' = 4x_1 - 3x_2 + 2x_3$ yielding the final form of our systems of equations

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{5.9}$$

Note that this is only considering a homogeneous equation. If it is non-homogeneous, you will have an extra term in the final step and will need a matrix of the form $\boldsymbol{x'} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}$ as shown in the answer to exercise 4(b) on page 5 of the PDF.

So why do we want to do this? Well, notice that in this example we started with a complicated 3rd-order ODE and reduced it into 3 1st-order ODE's. Similarly, if we started with a 2nd-order ODE, we could reduce the equation to 2 1st-order ODE's. In general, for an nth-order ODE we can reduce it to n 1st-order ODE's. If we can then learn how to solve simultanious sets of 1st-order ODE's, we have a powerful method of increasing our understanding (and even solving) difficult higher-order ODE's.

Similarly, if you are given a system of 2 1st-order ODE's, you can know that it can form a single 2nd-order ODE.

Challenge

Write the following ODE's in matrix form:

1) $2y'' + 4y' - 6y = 0$

2) $y'' + y = Cos(t)$

Complete exercises 1 and 2 on page 5 of the PDF.

Solutions

To check your answers, sum the values of all the terms in your matrix $\boldsymbol{A}$.

1) 2

2) 0 (remember there is also a $+\boldsymbol{g}$ column-vector added to $\boldsymbol{A}\boldsymbol{x}$ too)

The answers to the PDF exercises are shown on page 5 of the PDF. Perhaps obviously, since you will not have the answers in a real-life/exam environment, please don't review each answer until completion. If you get stuck, be sure to review your notes (especially the worked-examples in the PDF) rather than the answers, to facilitate deep learning.

5.3 Matricies

Resources

- PDF: Pages 6-17 of the PDF `http://www.math.psu.edu/tseng/class/Math251/Notes-LinearSystems.pdf`

Comment

It is worth spending some time getting comfortable with manipulating matricies, since this is an indispensible basis for the work that is about to follow. The PDF gives a quick introduction to matricies. For a more thorough introduction, the Khan Academy playlist on linear algebra [1] is excellent, although beyond the scope of this course.

One note to deal with any confusion arising with regard to eigenvectors with matricies with zeros. For $(A - rI)$ equal to something like

$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \tag{5.10}$$

the top row can be ignored since any x_1 and x_2 will satisfy the top row.

Similarly, for a case such as

$$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \tag{5.11}$$

you will have

$$2x_1 + 0x_2 = 0 \tag{5.12}$$
$$2x_1 = 0 \tag{5.13}$$
$$x_1 = 0 \tag{5.14}$$

which is satisfied by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.15}$$

(where the 1 could in principle be any number, but is the minimum integer that satisfies the condition.)

Finally, note that $(A - rI) = ((a, b), (c, d))$ will give you two equivalent formulas $ax_1 + bx_2 = 0$ and $cx_1 + bx_2 = 0$, even if they may appear different on first glance. If you want, you can prove to yourself that they are the same by multiplying the bottom row by a/c.

[1] `https://www.khanacademy.org/math/linear-algebra/alternate-bases`

Challenges

Complete exersizes 1, 2, 3, 4 (I and II only) and 5 on page 18 of the PDF.

Solutions

The answers to the PDF exercises are shown on page 18 of the PDF. Perhaps obviously, since you will not have the answers in a real-life/exam environment, please don't review each answer until completion. If you get stuck, be sure to review your notes (especially the worked-examples in the PDF) rather than the answers, to facilitate deep learning.

The solution to question 1 above can be found on the next page.

5.4 Eigenvector equivalence

Comment

Considering the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \tag{5.16}$$

The eigvenvalues are -2 and -1. Considering the eigenvalue -2,

$$A - Ir = \begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix} \tag{5.17}$$

To determine the eigenvector we can either take the top or bottom row in the calculation $(A - Ir)x = 0$. The top and bottom row appear with different numbers but it is easy to see that they yield multiples of the same eigenvector and are therefore equivalent.

Complex eigenvectors are no different, but it can sometimes be hard to see that they are indeed equivalent.

Challenge

Show that the equation $(A - Ir)\boldsymbol{x} = 0$, where

$$A - Ir = \begin{pmatrix} -3 - 3i & 6 \\ -3 & 3 - 3i \end{pmatrix} \tag{5.18}$$

yields the same eigenvector, irrespective of whether you calculate the eigenvector using the top or bottom row of $(A - Ir)$. You may find that one of the representations of the eigenvectors looks like $(i - 1, 1)$.

Solutions

You should be able to generate two eigenvectors by using the top and bottom rows of the $A - Ir$ matrix, and show that they are infact the same eigenvector by multiplying by an equivalent (imaginary) number. Please discuss with your partner or the teacher in class if you have trouble.

5.5 Solving systems of ODE's

Resources

- Pages 6-31 of the PDF `http://www.math.psu.edu/tseng/class/Math251/Notes-LinearSystems.pdf`

Challenge

Complete at least exercises 1-10 on page 32-33 of the PDF.

Solutions

It might not be clear to you why solutions involve vectors and what this means physically, but for now, please just get used to solving equations in this fashion.

The answers are shown on page 33-34 of the PDF. Perhaps obviously, since you will not have the answers in a real-life/exam environment, please don't review each answer until completion. If you get stuck, be sure to review your notes (especially the worked-examples in the PDF) rather than the answers, to facilitate deep learning.

5.6 Graphs of system solutions

Resources

In the previous challenge you determined x_1 and x_2 with solutions such as

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 6 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} \tag{5.19}$$

or written another way:

$$x_1 = -c_1 e^{-6t} + c_2 e^t \tag{5.20}$$
$$x_2 = 6c_1 e^{-6t} + c_2 e^t \tag{5.21}$$

This particular system arose from a 2nd-order differential equation:

$$y'' + 5y' - 6y = 0 \tag{5.22}$$

we have learned in challenge 5.2 that this 2nd-order equation can be written in terms of x:

$$x_3 + 5x_2 - 6x_1 = 0 \tag{5.23}$$

Thus we remember that $x_1 = y$ and $x_2 = y'$, allowing equations 5.20 and 5.21 to be written as

$$y = -c_1 e^{-6t} + c_2 e^t \tag{5.24}$$
$$y' = 6c_1 e^{-6t} + c_2 e^t \tag{5.25}$$

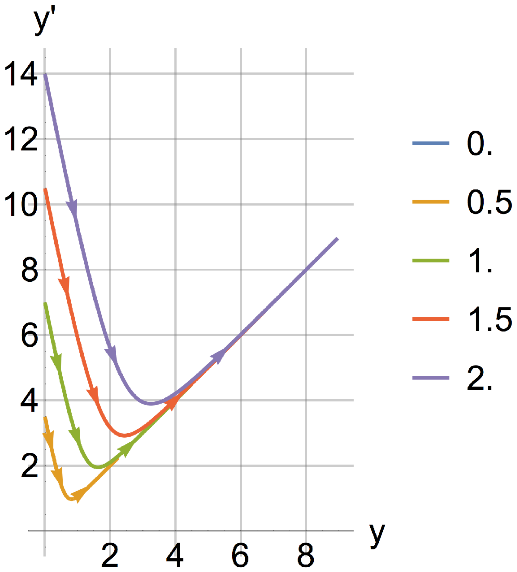
Perhaps, for example, the original 2nd-order ODE (equation 5.22) represented the position of an atom on an axis with respect to time. Then equation 5.24 represents position at time t while equation 5.25 represents the velocity (or more commonly, when multiplied by the mass, represents the momentum).

Thus the graph represents the variation of momentum (velocity) with position, called the "phase-space" of the system. A specific trajectory can be followed given boundary conditions that determine the starting condition. For example, if the particle at time $t = 0$ is known to have position $y = 1$ and velocity $y' = 2$ we can impose the boundary condition

$$\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{5.26}$$

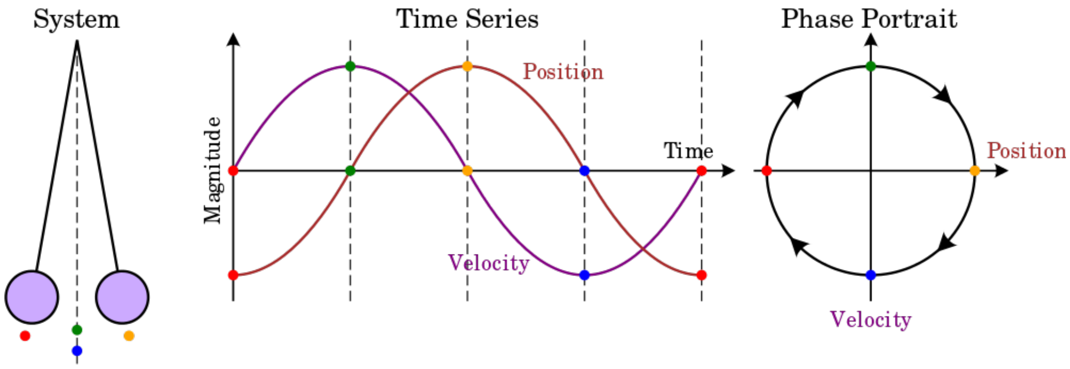
to determine the coefficients c_1 and c_2 and obtain a unique trajectory.

We can then plot the phase-space for various boundary conditions. In the graph below, we show examples where $c_1 = c_2 = \{0, 0.5, 1, 1.5, 2\}$:



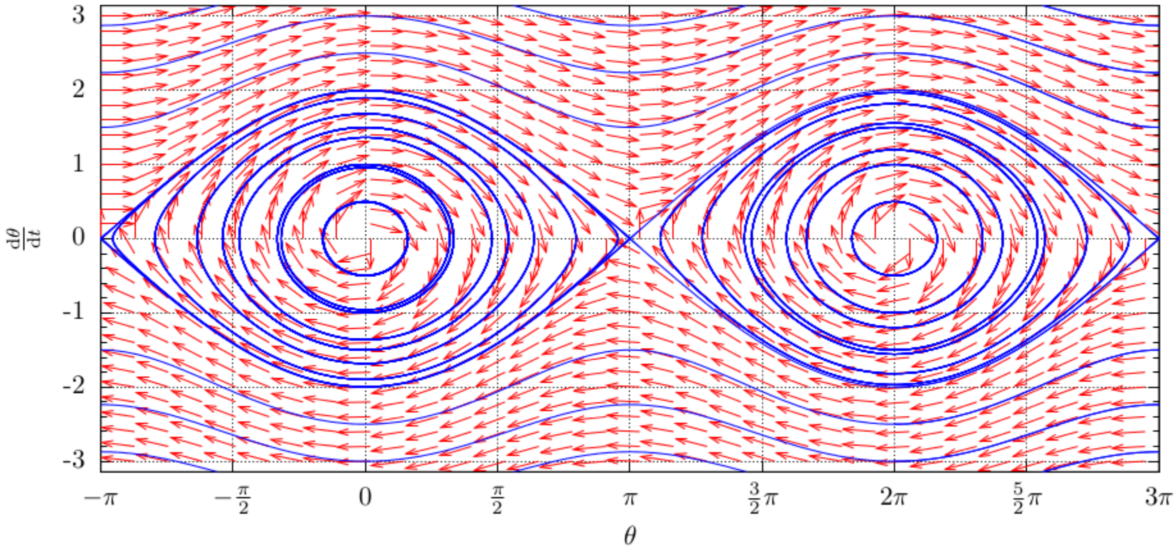
You can note that as t increases, the term e^{-6t} goes to zero leaving the e^t dominant, and since this features in both y and y', you get $y \propto y'$ for large t.

The examples we are considering here are relatively simple, however this can be used to identify complex and chaotic phenomena visually. For example, considering a pendulum gently swinging backwards and forwards, it is possible to trace out the phase-space as shown here:



Source: `https://commons.wikimedia.org/wiki/File:Pendulum_phase_portrait_illustration.svg`, *Wikipedia user Krishnavedala*

If you increase the speed of the pendulum, at some critical point, instead of swinging back to the original position it will start whirring round and round. Expressed in terms of vertical angle and angular velocity, the graph becomes:

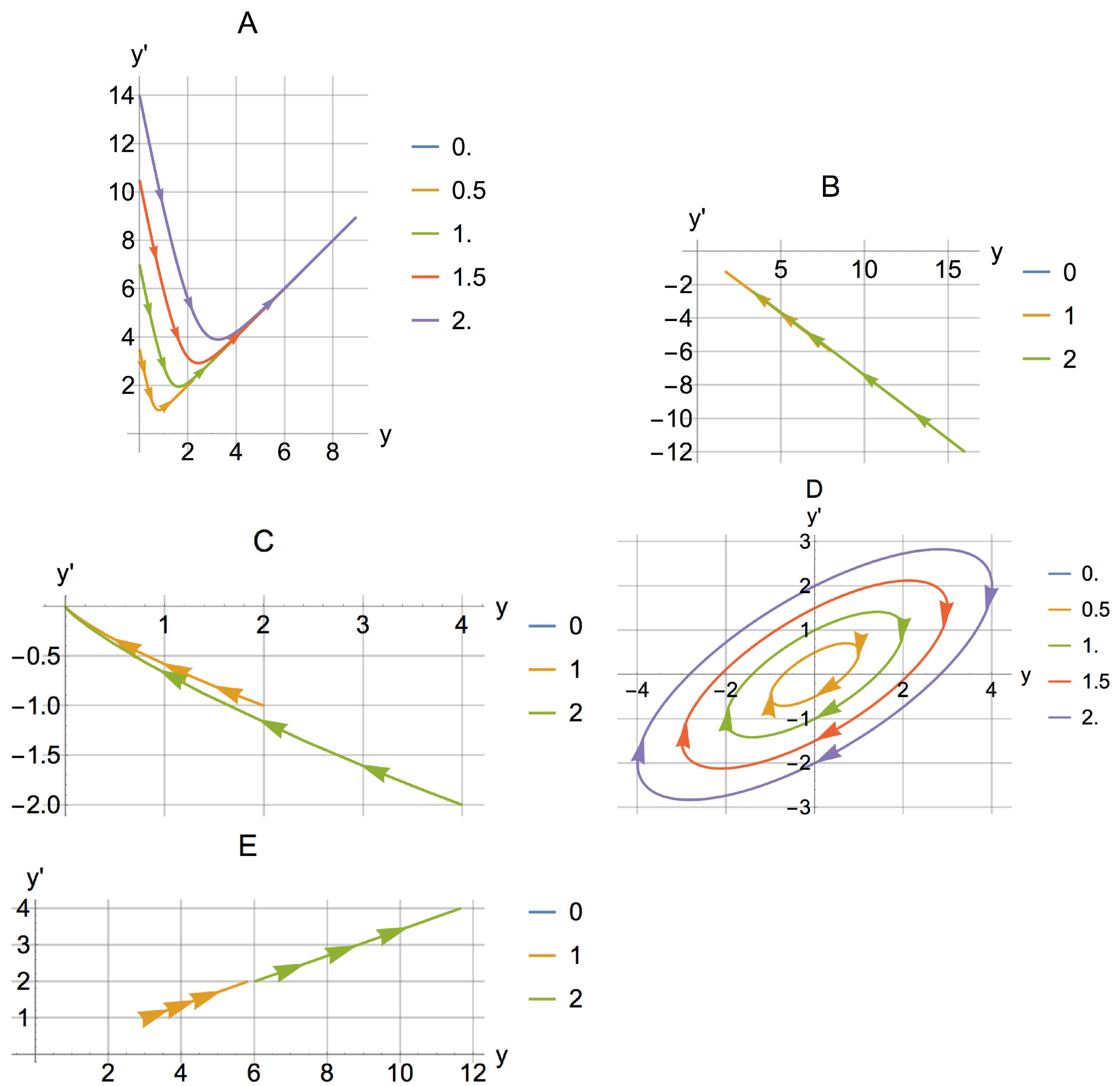


Source: *https://commons.wikimedia.org/wiki/File:Pendulumphase.png*

At low velocities the pendulum swings back and forth (blue circles, angular velocity both positive and negative), but at high velocities, the angular velocity stays positive (or negative) and the pendulum whirs round and round in one direction (blue wavy lines). Note that position $\theta = \pi$ is when the rigid pendulum is pointing exactly upwards. So with no momentum it is stationary here, albeit unstable, because with a tiny velocity it will perform a full loop, slowing (but not stopping) as it reaches the top again.

Challenge

1. The graphs below represent the solutions to the exercises 1-5 in challenge 5.5. Place the graphs below in the same order as exercises 1-5. Note that in order to maintain clarity, the graphs are not necessarily plotted over the same time interval t.



2. Considering the graph shown earlier of angular momentum vs angle for a rigid pendulum, add the points of the following true statements:

1 point An initial angular velocity of 1 unit results in whirring circular motion irrespective of the starting angle.

2 points An initial angular velocity of -2.5 units results in whirring circular motion irrespective of the starting angle.

4 points An initial angle of $\pi/2$ combined with an angular velocity of 1 unit results in periodic swinging motion.

8 points An initial angle of $\pi/2$ combined with an angular velocity of 1 unit results in circular whirring motion.

16 points An initial angle of 0 combined with an angular velocity of 0 units results in periodic swinging motion.

32 points An initial angle of 0 combined with an angular velocity of 0 units results in a stationary system.

64 points An initial angle of π combined with an angular velocity of 0 units results in a stationary system.

128 points An initial angle of $\pi/2$ combined with an angular velocity of 0 units results in a stationary system.

256 points An initial angular velocity of 3 units results in whirring circular motion in the same direction as an initial angular velocity of -3 units.

512 points An initial angular velocity of 3 units results in whirring circular motion in the opposite direction as an initial angular velocity of -3 units.

Solutions

1. (eg, "abcde") MD5(jjj_X) = 778bbb...
2. (enter number to 2 decimal places, as usual) MD5(kkk_X) = 7febe0...

Appendix A

Mid-term exam questions

A.1

Solve the following ODE for y given the condition $y(3) = 9e^9$.

$$\frac{x}{y}\frac{dy}{dx} - 1 = x^3 \tag{A.1}$$

A.2

The following equation is an autonamous equation:

$$y' = \frac{y^2}{5}(1 - \frac{y}{5}) \tag{A.2}$$

1. What key property does an autonamous equation have?
2. Determine the points of equilibrium and their stabilities.

A.3

Solve the following 2nd-order ODE's for y, and state what sort of damping they correspond to:

$$y'' + 5y' + 4y = 0 \tag{A.3}$$

$$y'' + 4y' + 4y = 0 \tag{A.4}$$

$$y'' + 3y' + 4y = 0 \tag{A.5}$$

A.4

Solve the following differential equation for y:

$$3x^2y + 2xy + y^3 + (x^2 + y^2)y' = 0 \tag{A.6}$$

Solutions can be found on the following page.

A.5 Solutions

Question 1

$y = 3xe^{x^3/3}$

Question 2

1. $y' = f(y)$
2. $y = 0$ (semi-stable), $y = 5$ (stable)

Question 3

$y(t) = C_1e^{-t} + C_2e^{-4t}$, Overdamped

$y(t) = C_1e^{-2t} + C_2te^{-2t}$, Critically-damped

$y(t) = C_1e^{-3t/2}Cos(\sqrt{7}t/2) + C_2e^{-3t/2}Sin(\sqrt{7}t/2)$, Under-damped

Question 4

$C = yx^2e^{3x} + \frac{1}{3}y^3e^{3x}$