Ordinary Differential Equations

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0.1 This course

This course is the Ordinary Differential Equations course studied at Kyushu University by 2nd-year students.

0.1.1 How this works

- In contrast to the traditional lecture-homework model, in this course the learning is self-directed and active via publicly-available resources.

- Learning is guided through solving a series of carefully-developed challenges contained in this book (download from `http://www.jamescannon.net/teaching/ordinary-differential-equations`), coupled with suggested resources that can be used to solve the challenges with instant feedback about the correctness of your answer.

- There are no lectures. Instead, there is discussion time. Here, you are encoraged to discuss any issues with your peers, teacher and any teaching assistants. Furthermore, you are encoraged to help your peers who are having trouble understanding something that you have understood; by doing so you actually increase your own understanding too.

- Peer discussion is encoraged, however, if you have help to solve a challenge, always make sure you do understand the details yourself. You will need to be able to do this in an exam environment. If you need additional challenges to solidify your understanding, then ask the teacher. The questions on the exam will be similar in nature to the challenges. If you can do all of the challenges, you can get 100% on the exam.

- Discussion-time is from 10:30 to 12:00 on Wednesdays at room W4-862.

- Every challenge in the book typically contains a **Challenge** with suggested **Resources** which you are recommended to utilise in order to solve the challenge. A **Solution** is made available in encrypted form. If your encrypted solution matches the encrypted solution given, then you know you have the correct answer and can move on. For more information about encryption, see section 0.2. Occasionally the teacher will provide extra **Comments** to help guide your thinking.

- For deep understanding, it is recommended to study the suggested resources beyond the minimum required to complete the challenge.

- The challenge document has many pages and is continuously being developed. Therefore it is advised to view the document on an electronic device rather than print it. The date on the front page denotes the version of the document. You will be notified by email when the document is updated.

- A target challenge and minimum challenge will be set each week, to be achieved by the beginning of the discussion time.

  - Target challenge: You are expected to complete at least up to this challenge. This is because the group learning effect is strongest when everyone is roughly around the same level of understanding.

  - Minimum challenge: Due to personal (eg health) or professional (eg conference attendence) issues, or simply difficulties with the challenges, it may occasionally not be possible to reach the target challenge. In this case, you will still be considered to be progressing normally if you achieve the minimum challenge.

  - You may work ahead, even beyond the target challenge, if you so wish. This can build greater flexibility into your personal schedule, especially as you become busier towards the end of the semester.

- Your contributions to the course are strongly welcomed. If you come across resources that you found useful that were not listed by the teacher or points of friction that made solving a challenge

difficult (there's no such thing as "you should have learned it in high-school" - you're probably not the only one with that specific problem), please let the teacher know about it!

0.1.2 Assessment

In order to prove to outside parties that you have learned something from the course, we must perform summative assessments. This will be in the form of

- Exam(s): Final exam, and possibly a mid-term exam
- Group presentation towards the end of the course (details to be announced later)

0.1.3 What you need to do

- Prepare a challenge-log in the form of a workbook or folder where you can clearly write the calculations you perform to solve each challenge. This will be a log of your progress during the course and will be occasionally reviewed by the teacher.
- You will need to maintain a google spreadsheet detailing your work and progress. The purpose of this spreadsheet is to help the teacher optimise the discussion-time. Please ensure that it is up-to-date 24 hours before each discussion-time starts. It is fine for you to continue to work on challenges and update the spreadsheet after the 24-hour deadline.
- You also need to submit a brief report at `https://goo.gl/forms/Djl4FEZcJLMpipsY2` 24 hours before the discussion time starts. Here you can let the teacher know about any difficulties you are having and if you would like to discuss anything in particular.
- Please bring a wifi-capable internet device to class, as well as headphones if you need to access online components of the course during class. If you let me know in advance, I can lend computers and provide power extension cables for those who require them (limited number).

0.1.4 Details about the spreadsheet

To get started:

1. Log into google
2. Open `http://bit.ly/2cPYyQY`
3. File → Make a copy [→ rename] → ok
4. Click "Share" (top right)
5. Click "Get shareable link"
6. Set "Anyone with the link can view"
7. Copy sharing address
8. Send an email to cannon@mech.kyushu-u.ac.jp containing
   (a) Subject: [course name] registration
   (b) Your name
   (c) Student number
   (d) The link to your copy of the google sheet

Using the spreadsheet:

- Enter the appropriate challenge number. For example, for challenge 1.4, enter "1" in the **Section** column and "4" in the **Challenge** column.

- After successfully completing a challenge, please enter any particular friction points that you experienced (if any) so the course can be developed to reduce such friction in the future, as well as any extra resources you recommend (if any).

- Please also roughly estimate the amount of effort in **Hours** required to complete the challenge (starting from when you completed the previous challenge, including any reading, watching videos, looking for resources, writing the answer to the challenge, discussing with peers, etc). This is not used for assessment in any way, but is very valuable in helping the teacher develop the course.

Note: Please do not alter column names, ordering, etc. Just add section and challenge numbering and fill in the columns as appropriate. This is because spreadsheet data is downloaded and automatically analysed, and it breaks if anything is inconsistent.

0.2 Hash-generation

Most solutions to challenges are encrypted using MD5 hashes. In order to check your solution, you need to generate its MD5 hash and compare it to that provided. MD5 hashes can be generated at the following sites:

- Wolfram alpha: `http://www.wolframalpha.com/input/?i=md5+hash+of+%22q_1.00%22`

- `www.md5hashgenerator.com`

Since MD5 hashes are very sensitive to even single-digit variation, you must enter the solution exactly. This means maintaining a sufficient level of accuracy when developing your solution, and then entering the solution according to the format below:

Unless specified otherwise, any number from 0.00 to ± 9999.99 should be represented as a normal number to two decimal places. All other numbers should be in scientific form. See the table below for examples.

Solution	Input
1	1.00
-3	-3.00
-3.5697	-3.57
0.05	0.05
0.005	5.00e-3
50	50.00
500	500.00
5000	5000.00
50,000	5.00e4
5×10^{-476}	5.00e-476
5.0009×10^{-476}	5.00e-476
$-\infty$	-infinity (never "infinite")
2π	6.28
i	im(1.00)
2i	im(2.00)
1 + 2i	re(1.00)im(2.00)
1/i = i/-1 = -i	im(-1.00)
$e^{i2\pi}$ $[= cos(2\pi) + isin(2\pi) = 1 + i0 = 1]$	1.00
$e^{i\pi/3}$ $[= cos(\pi/3) + isin(\pi/3) = 0.5 + i0.87]$	re(0.50)im(0.87)
Choices in order A, B, C, D	abcd

Entry format is given with the problem. So "q_X" means to enter "q_X" replacing "X" with your solution. The first 6 digits of the MD5 sum should match the given solution (MD5(q_X)= ...).

Note that although some answers can usually only be integers (eg, number of elephants), for consistency, to generate the correct hash, the accuracy in terms of decimal places noted above is required.

Contents

Chapter 1

Definitions

1.1 Order of a differential equation

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

What is the sum of the orders of the following equations?

$$\frac{dy}{dx}A = 5x^3 + 3 \tag{1.1}$$

$$cos(y)y'''(x) - y(x) = 25 \tag{1.2}$$

$$\frac{d}{dx}\frac{d^2y}{dx^2} = \frac{x^{-2}}{3} \tag{1.3}$$

Solution

X

MD5(e_X) = adb5a9...

1.2 Linear equations

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Sum the points corresponding to the equations that are linear:

1 point: $\frac{dy}{dx} = 5x^3 + 3$.

2 points: $cos(y)y'''(x) - y(x) = 25$.

4 points: $\frac{d}{dx}\frac{d^2y}{dx^2} = \frac{x^{-2}}{3}$.

8 points: $y'(x) - sin(y(x)) = 0$.

16 points: $y'(x) - y(x) = 0$.

32 points: $xy'(x) - y(x) = 0$.

Solution

X

$MD5(r\_X) = 9aea7d\ldots$

1.3   Valid solutions

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Use substitution to prove that

$$y = \frac{5}{5 + x} \tag{1.4}$$

is a solution to the equation

$$xy' + y = y^2 \tag{1.5}$$

and state the value of x for which the solution is undefined.

Solution

Value of x for which solution is undefined: X

$MD5(t\_X) = c69a20\ldots$

1.4 Range of valid solutions

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Use substitution to prove that

$$y = -\sqrt{100 - x^2} \tag{1.6}$$

is a solution to the equation

$$x + yy' = 0 \tag{1.7}$$

and state the range of x for which the solution is valid. Enter the value of the lower range as the solution below.

Solution

X

MD5(y_X) = ef0eb9...

Chapter 2

1st-order differential equations

2.1 Determining a simple DE from a description

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`
- Book: Chapter 1.2

Challenge

Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference with the ambient surroundings. (a) Write a differential equation describing this situation. (b) Assuming a proportionality constant of 0.2 /hour, what is the rate of temperature change when the object is at $30\,°\mathrm{C}$?

Solution

X °C/hour

MD5(q_X) = 078383...

2.2 Direction (Slope) fields

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/DirectionFields.aspx`
- Video 1: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/creating-a-slope-field`
- Video 2: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/slope-field-to-visualize-solutions`
- Book: Chapters 1.1, 1.2

Comment

It is good practise to try drawing the below fields before looking at the next page. You need to be able to go in both directions (ie, drawing and recognising).

Question

Try drawing the slope field for at least 3 of the equations given below (your choice). Then, put the slope fields given on the next page in the same order as these equations.

1. $y' = x$
2. $y' = 0.2y$
3. $y' = 0.2y(1 - y/6)$
4. $y' = (x - y)/(x + y)$
5. $y' = 2(y - 1)/x$
6. $y' = 2y/(x + 5)$
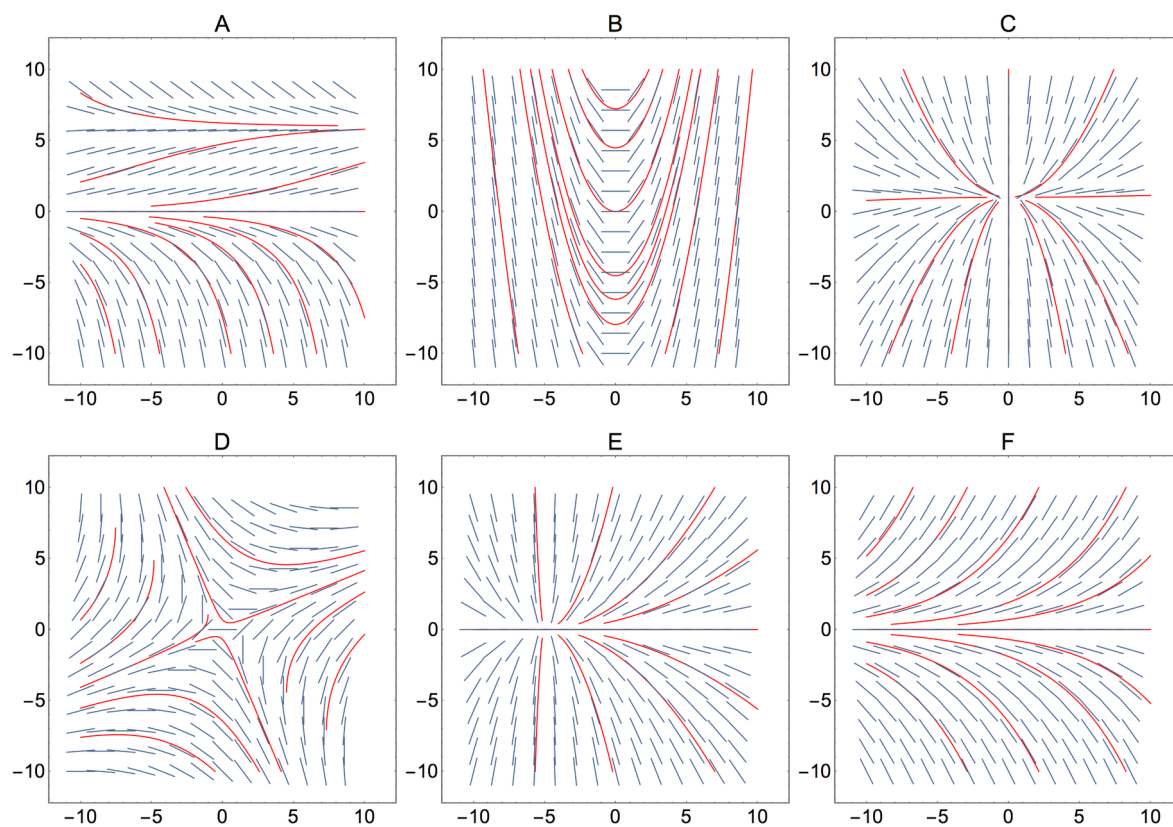
Solution

X (eg, "abcdef")

MD5(q_X) = 743eb9...

2.3 Solving a simple 1st-order linear equation

Resources

- Video: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/finding-particular-linear-solution-to-differential-equation`

Comment

It's not easy to see when equations can be solved simply, like in the challenge below, and when they can not. But for a 1st-order linear differential equation, this method is often good to try first. If it's not 1st order and not linear, you know to try a different approach.

Challenge

Determine the value of $y(x = 1)$ for the following equation:

$$y' = 2y - 5x + 2 \tag{2.1}$$

Solution

X

$\text{MD5}(\text{u\_X}) = 505\text{c}7\text{b}\dots$

2.4 Separable equations I

Resources

- Video I: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction
- Video II: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example
- Text:http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx

Challenge

Given the following equation:

$$r' = -Sin(\theta) \tag{2.2}$$

Determine the function $r(\theta)$ that passes through the point (0,1) in $\theta-r$ space, and then solve for $\theta = \pi/4$.

Solution

X

MD5(i_X) = 7b35e2...

2.5   Separable equations II

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction`

- Video II: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example`

- Text:`http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx`

Challenge

Given the following equation:

$$r' cot(\theta) + r = 2 \tag{2.3}$$

Determine the function $r(\theta)$ that passes through the point (0,1) in $\theta - r$ space, and then solve for $\theta = \pi/4$.

Solution

X

MD5(o_X) = 87ee92. . .

2.6 Separable equations III

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction`
- Video II: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example`
- Text:`http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx`

Challenge

Given the following equation:

$$y'x^2 - y = 0 \tag{2.4}$$

Determine the function $y(x)$ that passes through the point $x = 2, y = 1$ and then solve for the given x value. State the value of x where the solution is undefined.

Solution

Solve for $x = 4$:

X

MD5(p_X) = 09bb0e...

Value of x where solution is undefined:

X

MD5(a_X) = b30fe7...

2.7   Compound increase

Challenge

If the amount of bacteria on a surface increases by 20% every 25 days, how much does the amount grow per day in percentage terms? Ie, if one observes the amount of bacteria one hour from now, by what percentage should one expect the amount of bacteria to have increased?

Solution

X %

MD5(s_X) = 4288d8. . .

2.8 Logistic equation

Resources

- Videos: The 5 videos on the logistic differential equation and function starting at: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/logistic-differential-equation/v/modeling-population-with-differential-equations`

Challenge

Using the logistic equation, assuming an increase in the amount of bacteria measured in mg of 20% every 25 hours, an initial amount of bacteria of 20 mg and a limiting amount of 400 mg, how much time, rounded to the nearest integer hours, must one wait to reach 300 mg of bacteria?

Solution

X hours (expressed as an integer)

MD5(d_X) = b0d406. . .

2.9   Autonomous differential equations

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`
- Wikipedia: `https://en.wikipedia.org/wiki/Autonomous_system_(mathematics)`

Challenge

Add the points of the autonomous differential equations in the following list:

1 point: $y' = cos(y) - 5$

2 points: $y' = cos(y)/x - 5$

4 points: $y' = cos(y)/x - 5/x$

8 points: $y^2 = y'y + 5$

16 points: $xy' = 5y$

32 points: $y' = 1$

Solution

X

MD5(f_X) = 9bf043...

2.10 The stability of solutions I

Resources

- Text: http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx

Challenge

Considering the logistic equation $N' = 0.2t(1 - N/6)$, make 3 separate lists containing any equilibrium, semi-stable and unstable y-values.

To check your answer, sum the value of each list. If there are no values in a list, simply enter "none" to check the result.

Solution

Equilibrium

X

MD5(g_X) = 2c32d8...

Semi-stable

X

MD5(h_X) = 8b595d...

Unstable

X

MD5(j_X) = 4fd3f6...

2.11 The stability of solutions II

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`

Challenge

Considering the differential equation $y' = (y^2 - 16)(y + 3)^2$, make 3 separate lists containing any equilibrium, semi-stable and unstable y-values.

To check your answer, sum the value of each list. If there are no values in a list, simply enter "none" to check the result.

Solution

Equilibrium

X

MD5(k_X) = 667798. . .

Semi-stable

X

MD5(z_X) = 200aa3. . .

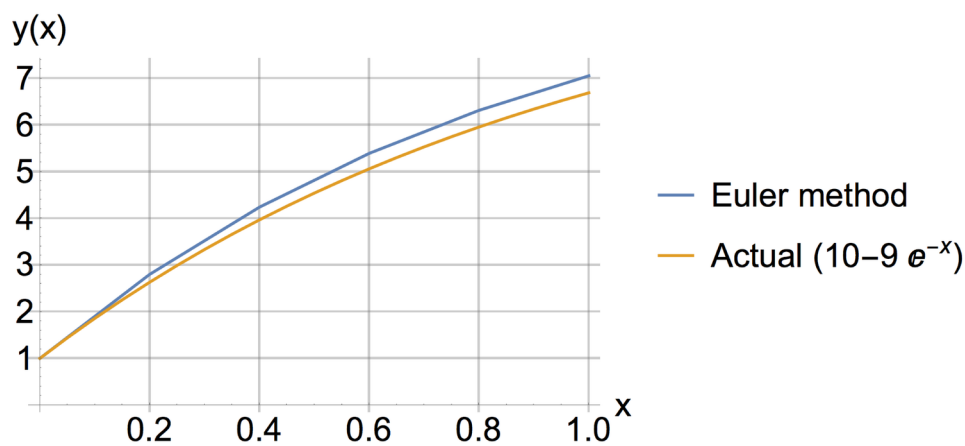**Unstable**

X

MD5(x_X) = d1ee5b. . .

2.12 Euler's method

Resources

- Videos and exersizes in the "Euler's Method" section of Khan academy: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/eulers-method-tutorial/v/eulers-method`

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EulersMethod.aspx`

Challenge

Considering the the differential equation $y' = 10 - y$, an initial value of $y(0) = 1$ and a step size of $\Delta x = 0.2$, use Euler's method to estimate the value of $y(x = 1)$. The actual solution, $y(x) = 10 - 9e^{-x}$, is shown below.

Solution

X

MD5(c_X) = 1f90fa...