

## Solution of $2 \times 2$ systems of first order linear equations

Consider a system of 2 simultaneous first order linear equations

$$\begin{aligned}x_1' &= ax_1 + bx_2 \\x_2' &= cx_1 + dx_2\end{aligned}$$

It has the alternate matrix-vector representation

$$\mathbf{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}.$$

Or, in shorthand  $\mathbf{x}' = \mathbf{A}\mathbf{x}$ , if  $\mathbf{A}$  is already known from context.

We know that the above system is equivalent to a second order homogeneous linear differential equation. As a result, we know that the general solution contains two linearly independent parts. As well, the solution will be consisted of some type of exponential functions. Therefore, assume that  $\mathbf{x} = \mathbf{k}e^{rt}$  is a solution of the system, where  $\mathbf{k}$  is a vector of coefficients (of  $x_1$  and  $x_2$ ). Substitute  $\mathbf{x}$  and  $\mathbf{x}' = r\mathbf{k}e^{rt}$  into the equation  $\mathbf{x}' = \mathbf{A}\mathbf{x}$ , and we have

$$r\mathbf{k}e^{rt} = \mathbf{A}\mathbf{k}e^{rt}.$$

Since  $e^{rt}$  is never zero, we can always divide both sides by  $e^{rt}$  and get

$$r\mathbf{k} = \mathbf{A}\mathbf{k}.$$

We see that this new equation is exactly the relation that defines eigenvalues and eigenvectors of the coefficient matrix  $\mathbf{A}$ . In other words, in order for a function  $\mathbf{x} = \mathbf{k}e^{rt}$  to satisfy our system of differential equations, the number  $r$  must be an eigenvalue of  $\mathbf{A}$ , and the vector  $\mathbf{k}$  must be an eigenvector of  $\mathbf{A}$  corresponding to  $r$ . Just like the solution of a second order homogeneous linear equation, there are three possibilities, depending on the number of distinct, and the type of, eigenvalues the coefficient matrix  $\mathbf{A}$  has.

The possibilities are that  $A$  has

- I. Two distinct real eigenvalues
- II. Complex conjugate eigenvalues
- III. A repeated eigenvalue

A related note, (from linear algebra,) we know that eigenvectors that each corresponds to a different eigenvalue are always linearly independent from each others. Consequently, if  $r_1$  and  $r_2$  are two different eigenvalues, then their respective eigenvectors  $\mathbf{k}_1$  and  $\mathbf{k}_2$ , and therefore the corresponding solutions, are always linearly independent.

**Case I**      Distinct real eigenvalues

If the coefficient matrix  $A$  has two distinct real eigenvalues  $r_1$  and  $r_2$ , and their respective eigenvectors are  $k_1$  and  $k_2$ . Then the  $2 \times 2$  system  $x' = Ax$  has a general solution

$$x = C_1 k_1 e^{r_1 t} + C_2 k_2 e^{r_2 t}.$$

*Example:* 
$$x' = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} x.$$

We have already found that the coefficient matrix has eigenvalues  $r = -1$  and  $6$ . And they each respectively has an eigenvector

$$k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad k_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Therefore, a general solution of this system of differential equations is

$$x = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} e^{6t}$$

*Example:*  $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The characteristic equation is  $r^2 - r - 2 = (r + 1)(r - 2) = 0$ . The eigenvalues are  $r = -1$  and  $2$ . They have, respectively, eigenvectors

For  $r = -1$ , the system is

$$(\mathbf{A} - r\mathbf{I})\mathbf{x} = (\mathbf{A} + \mathbf{I})\mathbf{x} = \begin{bmatrix} 3+1 & -2 \\ 2 & -2+1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the bottom equation of the system:  $2x_1 - x_2 = 0$ , we get the relation  $x_2 = 2x_1$ . Hence,

$$\mathbf{k}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

For  $r = 2$ , the system is

$$(\mathbf{A} - r\mathbf{I})\mathbf{x} = (\mathbf{A} - 2\mathbf{I})\mathbf{x} = \begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the first equation of the system:  $x_1 - 2x_2 = 0$ , we get the relation  $x_1 = 2x_2$ . Hence,

$$\mathbf{k}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore, a general solution is

$$x = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}.$$

Apply the initial values,

$$x(0) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0 + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^0 = \begin{bmatrix} C_1 + 2C_2 \\ 2C_1 + C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

That is

$$\begin{aligned} C_1 + 2C_2 &= 1 \\ 2C_1 + C_2 &= -1. \end{aligned}$$

We find  $C_1 = -1$  and  $C_2 = 1$ , hence we have the particular solution

$$x = - \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} -e^{-t} + 2e^{2t} \\ -2e^{-t} + e^{2t} \end{bmatrix}.$$

## Case II      Complex conjugate eigenvalues

If the coefficient matrix  $A$  has two distinct complex conjugate eigenvalues  $\lambda \pm \mu i$ . Also suppose  $\mathbf{k} = \mathbf{a} + \mathbf{b}i$  is an eigenvector (necessarily has complex-valued entries) of the eigenvalue  $\lambda + \mu i$ . Then the  $2 \times 2$  system  $\mathbf{x}' = A\mathbf{x}$  has a real-valued general solution

$$\mathbf{x} = C_1 e^{\lambda t} (a \cos(\mu t) - b \sin(\mu t)) + C_2 e^{\lambda t} (a \sin(\mu t) + b \cos(\mu t))$$

*A little detail:* Similar to what we have done before, first there was the complex-valued general solution in the form

$$\mathbf{x} = C_1 k_1 e^{(\lambda + \mu i)t} + C_2 k_2 e^{(\lambda - \mu i)t}.$$

We “filter out” the imaginary parts by carefully choosing two sets of coefficients to obtain two corresponding real-valued solutions that are also linearly independent:

$$\begin{aligned} \mathbf{u} &= e^{\lambda t} (a \cos(\mu t) - b \sin(\mu t)) \\ \mathbf{v} &= e^{\lambda t} (a \sin(\mu t) + b \cos(\mu t)) \end{aligned}$$

The real-valued general solution above is just  $\mathbf{x} = C_1 \mathbf{u} + C_2 \mathbf{v}$ . In particular, it might be useful to know how  $\mathbf{u}$  and  $\mathbf{v}$  could be derived by expanding the following complex-valued expression (the front half of the complex-valued general solution):

$$\begin{aligned} k_1 e^{(\lambda + \mu i)t} &= (a + bi)e^{\lambda t} e^{(\mu t)i} = e^{\lambda t} (a + bi)(\cos(\mu t) + i \sin(\mu t)) \\ &= e^{\lambda t} (a \cos(\mu t) + ia \sin(\mu t) + ib \cos(\mu t) + i^2 b \sin(\mu t)) \\ &= e^{\lambda t} (a \cos(\mu t) - b \sin(\mu t)) + i e^{\lambda t} (a \sin(\mu t) + b \cos(\mu t)) \end{aligned}$$

Then,  $\mathbf{u}$  is just the real part of this complex-valued function, and  $\mathbf{v}$  is its imaginary part.

Example:

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$$

The characteristic equation is  $r^2 + 1 = 0$ , giving eigenvalues  $r = \pm i$ .  
That is,  $\lambda = 0$  and  $\mu = 1$ .

Take the first (the one with positive imaginary part) eigenvalue  $r = i$ ,  
and find one of its eigenvectors:

$$(\mathbf{A} - r\mathbf{I})\mathbf{x} = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the first equation of the system:  $(2 - i)x_1 - 5x_2 = 0$ , we get  
the relation  $(2 - i)x_1 = 5x_2$ . Hence,

$$k = \begin{bmatrix} 5 \\ 2-i \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}_{\mathbf{a}} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{\mathbf{b}} i = \mathbf{a} + bi$$

Therefore, a general solution is

$$\begin{aligned} \mathbf{x} &= C_1 e^{0t} \left( \begin{bmatrix} 5 \\ 2 \end{bmatrix} \cos(t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin(t) \right) + C_2 e^{0t} \left( \begin{bmatrix} 5 \\ 2 \end{bmatrix} \sin(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos(t) \right) \\ &= C_1 \begin{pmatrix} 5 \cos(t) \\ 2 \cos(t) + \sin(t) \end{pmatrix} + C_2 \begin{pmatrix} 5 \sin(t) \\ 2 \sin(t) - \cos(t) \end{pmatrix} \end{aligned}$$

*Example:*  $\mathbf{x}' = \begin{bmatrix} -1 & -6 \\ 3 & 5 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$

The characteristic equation is  $r^2 - 4r + 13 = 0$ , giving eigenvalues  $r = 2 \pm 3i$ . Thus,  $\lambda = 2$  and  $\mu = 3$ .

Take  $r = 2 + 3i$  and find one of its eigenvectors:

$$(\mathbf{A} - r\mathbf{I})\mathbf{x} = \begin{bmatrix} -1 - (2 + 3i) & -6 \\ 3 & 5 - (2 + 3i) \end{bmatrix} \mathbf{x} = \begin{bmatrix} -3 - 3i & -6 \\ 3 & 3 - 3i \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the second equation of the system:  $3x_1 + (3 - 3i)x_2 = 0$ , we get the relation  $x_1 = (-1 + i)x_2$ . Hence,

$$\mathbf{k} = \begin{bmatrix} -1 + i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i = \mathbf{a} + bi$$

The general solution is

$$\begin{aligned} \mathbf{x} &= C_1 e^{2t} \left( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(3t) \right) + C_2 e^{2t} \left( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin(3t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(3t) \right) \\ &= C_1 e^{2t} \begin{pmatrix} -\cos(3t) - \sin(3t) \\ \cos(3t) \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} \cos(3t) - \sin(3t) \\ \sin(3t) \end{pmatrix} \end{aligned}$$



Apply the initial values to find  $C_1$  and  $C_2$ :

$$\begin{aligned}x(0) &= C_1 e^0 \left( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos(0) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(0) \right) + C_2 e^0 \left( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin(0) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(0) \right) \\&= C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -C_1 + C_2 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}\end{aligned}$$

Therefore,  $C_1 = 2$  and  $C_2 = 2$ . Consequently, the particular solution is

$$\begin{aligned}x &= 2e^{2t} \begin{pmatrix} -\cos(3t) - \sin(3t) \\ \cos(3t) \end{pmatrix} + 2e^{2t} \begin{pmatrix} \cos(3t) - \sin(3t) \\ \sin(3t) \end{pmatrix} \\&= e^{2t} \begin{pmatrix} -4\sin(3t) \\ 2\cos(3t) + 2\sin(3t) \end{pmatrix}\end{aligned}$$

### Case III    Repeated real eigenvalue

Suppose the coefficient matrix  $A$  has a repeated real eigenvalues  $r$ , there are 2 sub-cases.

(i) If  $r$  has two linearly independent eigenvectors  $k_1$  and  $k_2$ . Then the  $2 \times 2$  system  $\mathbf{x}' = A\mathbf{x}$  has a general solution

$$\mathbf{x} = C_1 \mathbf{k}_1 e^{rt} + C_2 \mathbf{k}_2 e^{rt}.$$

*Note:* For  $2 \times 2$  matrices, this possibility only occurs when the coefficient matrix  $A$  is a scalar multiple of the identity matrix. That is,  $A$  has the form

$$\alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad \text{for any constant } \alpha.$$

*Example:*

$$\mathbf{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}.$$

The eigenvalue is  $r = 2$  (repeated). There are 2 sets of linearly independent eigenvectors, which could be represented by any 2 nonzero vectors that are not constant multiples of each other. For example

$$\mathbf{k}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{k}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, a general solution is

$$\mathbf{x} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}.$$

(ii) If  $r$ , as it usually does, only has one linearly independent eigenvector  $\mathbf{k}$ . Then the  $2 \times 2$  system  $\mathbf{x}' = \mathbf{A}\mathbf{x}$  has a general solution

$$\mathbf{x} = C_1 \mathbf{k} e^{rt} + C_2 (\mathbf{k} t e^{rt} + \boldsymbol{\eta} e^{rt}).$$

Where the second vector  $\boldsymbol{\eta}$  is any solution of the nonhomogeneous linear system of algebraic equations

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\eta} = \mathbf{k}.$$

*Example:*  $\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$

The eigenvalue is  $r = -3$  (repeated). The corresponding system is

$$(\mathbf{A} - r\mathbf{I})\mathbf{x} = \begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Both equations of the system are  $4x_1 - 4x_2 = 0$ , we get the same relation  $x_1 = x_2$ . Hence, there is only one linearly independent eigenvector:

$$\mathbf{k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Next, solve for  $\boldsymbol{\eta}$ :

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \boldsymbol{\eta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It has solution in the form  $\boldsymbol{\eta} = \begin{bmatrix} \frac{1}{4} + \eta_2 \\ \eta_2 \end{bmatrix}$ .

Choose  $\eta_2 = 0$ , we get  $\boldsymbol{\eta} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$ .

A general solution is, therefore,

$$\mathbf{x} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + C_2 \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} e^{-3t} \right)$$

Apply the initial values to find  $C_1 = 1$  and  $C_2 = -12$ . The particular solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} - 12 \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} e^{-3t} \right) = \begin{bmatrix} -12t - 2 \\ -12t + 1 \end{bmatrix} e^{-3t}$$

## **Summary: Solving a Homogeneous System of Two Linear First Order Equations in Two Unknowns**

Given:

$$\mathbf{x}' = A\mathbf{x}.$$

First find the two eigenvalues,  $r$ , and their respective corresponding eigenvectors,  $\mathbf{k}$ , of the coefficient matrix  $A$ . Depending on the eigenvalues and eigenvectors, the general solution is:

I. Two distinct real eigenvalues  $r_1$  and  $r_2$ :

$$\mathbf{x} = C_1 \mathbf{k}_1 e^{r_1 t} + C_2 \mathbf{k}_2 e^{r_2 t}.$$

II. Two complex conjugate eigenvalues  $\lambda \pm \mu i$ , where  $\lambda + \mu i$  has as an eigenvector  $\mathbf{k} = \mathbf{a} + \mu i \mathbf{b}$ :

$$\mathbf{x} = C_1 e^{\lambda t} (a \cos(\mu t) - b \sin(\mu t)) + C_2 e^{\lambda t} (a \sin(\mu t) + b \cos(\mu t))$$

III. A repeated real eigenvalue  $r$ :

(i) When two linearly independent eigenvectors exist –

$$\mathbf{x} = C_1 \mathbf{k}_1 e^{rt} + C_2 \mathbf{k}_2 e^{rt}.$$

(ii) When only one linearly independent eigenvector exist –

$$\mathbf{x} = C_1 \mathbf{k} e^{rt} + C_2 (\mathbf{k} t e^{rt} + \boldsymbol{\eta} e^{rt}).$$

*Note:* Solve the system  $(A - rI)\boldsymbol{\eta} = \mathbf{k}$  to find the vector  $\boldsymbol{\eta}$ .

*Exercises D-1.3:*

1. Rewrite the following second order linear equation into a system of two equations.

$$y'' + 5y' - 6y = 0$$

Then: (a) show that both the given equation and the new system have the same characteristic equation. (b) Find the system's general solution.

2 – 7 Find the general solution of each system below.

$$2. \quad \mathbf{x}' = \begin{bmatrix} 2 & 7 \\ -5 & -10 \end{bmatrix} \mathbf{x}. \quad 3. \quad \mathbf{x}' = \begin{bmatrix} -3 & 6 \\ -3 & 3 \end{bmatrix} \mathbf{x}.$$

$$4. \quad \mathbf{x}' = \begin{bmatrix} 8 & -4 \\ 1 & 4 \end{bmatrix} \mathbf{x}. \quad 5. \quad \mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -1 & -5 \end{bmatrix} \mathbf{x}.$$

$$6. \quad \mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}. \quad 7. \quad \mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -4 \end{bmatrix} \mathbf{x}.$$

8 – 15 Solve the following initial value problems.

$$8. \quad \mathbf{x}' = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} -4 \\ 2 \end{bmatrix}.$$

$$9. \quad \mathbf{x}' = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(3) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$

$$10. \quad \mathbf{x}' = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(1) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

$$11. \quad \mathbf{x}' = \begin{bmatrix} 6 & 8 \\ 2 & 6 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 8 \\ 0 \end{bmatrix}.$$

$$12. \quad \mathbf{x}' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$