Reflector imaging from ambient noise signals

The goal is to show numerically that the cross correlations of signals emitted by noise sources and recorded by a receiver array can be processed to localize reflectors.

1) Geometric set-up.

We consider the solution u of the wave equation in a three-dimensional medium:

$$\frac{1}{c(\boldsymbol{x})^2}\frac{\partial^2 u}{\partial t^2} - \Delta_{\boldsymbol{x}} u = n(t, \boldsymbol{x}). \tag{1}$$

The term $n(t, \boldsymbol{x})$ models a random field of noise sources. It is a zero-mean stationary (in time) random process with autocorrelation function

$$\langle n(t_1, \boldsymbol{y}_1) n(t_2, \boldsymbol{y}_2)\rangle = F(t_2 - t_1)\delta(\boldsymbol{y}_1 - \boldsymbol{y}_2)K(\boldsymbol{y}_1). \tag{2}$$

Here $\langle\cdot\rangle$ stands for statistical average with respect to the distribution of the noise sources.

K is the probability density function of the uniform distribution $[-50, 50] \times [-10, 10] \times [185, 200]$. It can be approximated by 200 point noise sources randomly distributed in the domain $[-50, 50] \times [-10, 10] \times [185, 200]$.

The power spectral density of the noise sources is Gaussian $\hat{F}(\omega) = \exp(-\omega^2)$.

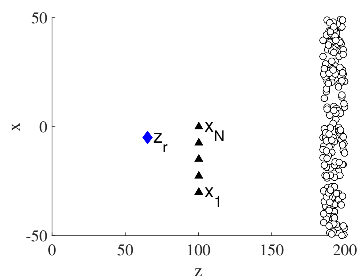

Figure 1: Geometric set-up: receivers (triangles), reflector (diamond), noise sources (circles).

Numerically, this can be simulated by sampling N points $(\boldsymbol{y}_s)_{s=1,\dots,N}$ independently according to the prescribed uniform distribution and each point emits independent and identically distributed signals $n_s(t)$ with stationary Gaussian statistics, mean zero, and covariance function $\langle n_s(t)n_s(t')\rangle = F(t - t')$:

$$n(t, \boldsymbol{x}) = \frac{1}{\sqrt{N}}\sum_{s=1}^{N} n_s(t)\delta(\boldsymbol{x} - \boldsymbol{y}_s) \tag{3}$$

There are five receivers: $\boldsymbol{x}_j = (-30 + 7.5(j-1), 0, 100)$, $j = 1, \dots, 5$ (see Figure 1).

2) Preliminaries.

The homogeneous three-dimensional Green's function $\hat{G}_0(\omega, \boldsymbol{x}, \boldsymbol{y})$ is solution of

$$\Delta_{\boldsymbol{x}} \hat{G}_0 + \frac{\omega^2}{c_0^2} \hat{G}_0 = -\delta(\boldsymbol{x} - \boldsymbol{y}), \qquad \boldsymbol{x} \in \mathbb{R}^2 \tag{4}$$

with the Sommerfeld radiation condition. It is given by

$$\hat{G}_0(\omega, \boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} \exp\Big(i \frac{\omega}{c_0} |\boldsymbol{x} - \boldsymbol{y}| \Big) \tag{5}$$

In order to generate the data in the presence of a point-like reflector at $\boldsymbol{z}_{\mathrm{r}}$, we consider the Born approximation:

$$\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{y}) = \hat{G}_0(\omega, \boldsymbol{x}, \boldsymbol{y}) + \sigma_{\mathrm{r}} \omega^2 \hat{G}(\omega, \boldsymbol{x}, \boldsymbol{z}_{\mathrm{r}}) \hat{G}(\omega, \boldsymbol{z}_{\mathrm{r}}, \boldsymbol{y})$$

Take $c_0 = 1$, $\boldsymbol{z}_{\mathrm{r}} = (x_{\mathrm{r}} = 65, y_{\mathrm{r}} = 0, z_{\mathrm{r}} = 65)$, and $\sigma_{\mathrm{r}} = 10^{-3}$.

3) Empirical cross correlation. The recorded signals are $(u(t, \boldsymbol{x}_j))_{j=1,\dots,5,\, t \in [0,T]}$ with

$$\begin{aligned} u(t, \boldsymbol{x}_j) &= \frac{1}{2\pi} \iint \hat{G}(\omega, \boldsymbol{x}_j, \boldsymbol{y}) \hat{n}(\omega, \boldsymbol{y}) \exp(-i\omega t) d\boldsymbol{y} d\omega \\ &= \frac{1}{2\pi\sqrt{N}} \sum_{s=1}^{N} \int \hat{G}(\omega, \boldsymbol{x}_j, \boldsymbol{y}_s) \hat{n}_s(\omega) \exp(-i\omega t) d\omega \end{aligned}$$

The empirical cross correlation of the signals recorded at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ for the recording time T is

$$C_{T,N}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{T - |\tau|} \int_0^{T-|\tau|} u(t, \boldsymbol{x}_1) u(t + \tau, \boldsymbol{x}_2) dt. \tag{6}$$

Its expectation with respect to the distribution of the emitted signals is

$$C_N(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi N} \sum_{s=1}^{N} \int d\omega \hat{F}(\omega) \overline{\hat{G}(\omega, \boldsymbol{x}_1, \boldsymbol{y}_s)} \hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y}_s) e^{-i\omega\tau}. \tag{7}$$

The expectation of the empirical cross correlation with respect to the distribution of the emitted signals and the source positions is

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi} \int d\omega \int d\boldsymbol{y} K(\boldsymbol{y}) \hat{F}(\omega) \overline{\hat{G}(\omega, \boldsymbol{x}_1, \boldsymbol{y})} \hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y}) e^{-i\omega\tau}. \tag{8}$$

If $T \gg 1$, then the theory predicts that

$$C_{T,N}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \overset{T \to \infty}{\longrightarrow} C_N(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \tag{9}$$

If $N \gg 1$, then the theory predicts that

$$C_N(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \overset{N \to \infty}{\longrightarrow} C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \tag{10}$$

If the illumination were isotropic (which is not the case here), then we would have:

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \approx C_{\rm asy}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2), \tag{11}$$

$$\text{with } \frac{\partial}{\partial \tau} C_{\rm asy}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = -\big[F * G(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) - F * G(-\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)\big], \tag{12}$$

up to a multiplicative constant.

4) Questions.

1. We first consider the case where $T \to +\infty$.

a. Compute and plot $\tau \to C_N(\tau, \boldsymbol{x}_5, \boldsymbol{x}_1)$ for $\tau \in [-200, 200]$.

b. Compute and plot the KM image:

$$\mathcal{I}_N(\boldsymbol{y}^S) = \sum_{j,l=1}^{5} C_N\big(|\boldsymbol{x}_j - \boldsymbol{y}^S| + |\boldsymbol{y}^S - \boldsymbol{x}_l|, \boldsymbol{x}_j, \boldsymbol{x}_l\big) \tag{13}$$

for $\boldsymbol{y}^S$ in the plane (x, z) in a square window with size 20×20 and centered on the reflector location $\boldsymbol{z}_{\rm r}$.

c. Study the resolution properties of the image.

2. We consider the case where $T < +\infty$.

a. Compute and plot $\tau \to C_{N,T}(\tau, \boldsymbol{x}_5, \boldsymbol{x}_1)$ for $\tau \in [-150, 150]$ for different values of T ($T \geq 500$).

In order to increase the stability, we can consider M (non-overlaping) time windows of size T, which give independent realizations of the recorded signals $(u^{(m)}(t, \boldsymbol{x}_j))_{j=1,\ldots,5,\, t\in[0,T]}$, $m = 1, \ldots, M$. We can then compute M empirical cross correlations $C^{(m)}_{T,N}$, $m = 1, \ldots, M$:

$$C^{(m)}_{T,N}(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l) = \frac{1}{T - |\tau|} \int_0^{T-|\tau|} u^{(m)}(t, \boldsymbol{x}_j) u^{(m)}(t + \tau, \boldsymbol{x}_l) dt$$

and compute the average $C_{T,N,M}$ of the M empirical cross correlations:

$$C_{T,N,M}(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l) = \frac{1}{M} \sum_{m=1}^{M} C^{m)}_{T,N}(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$$

b. Compute and plot the CC-KM image:

$$\mathcal{I}_{T,N,M}(\boldsymbol{y}^S) = \sum_{j,l=1}^{5} C_{T,N,M}\big(|\boldsymbol{x}_j - \boldsymbol{y}^S| + |\boldsymbol{y}^S - \boldsymbol{x}_l|, \boldsymbol{x}_j, \boldsymbol{x}_l\big) \tag{14}$$

c. Study the stability properties of the image (with respect to T and M).