CSPC ASSIGNMENT-5

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ASSIGNMENT - 5
LINEAR ALGEBRA

Q.1.

a) The set of all polynomials (of degree at most n) over R

$$V = \{a_n x^n + \dots + a_1 x + a_0\} \quad a_i \in R \quad 0 \leq i \leq n$$

$\oplus$ be addition of polynomials

$\odot$ be multiplication of vector by a scalar.

1. Checking 'abelian group property' of $(V, \oplus)$

- $P_1 + P_2 = P \in V \quad (\forall P_1, P_2 \in V)$

→ Closure

- $P_1 + (P_2 + P_3) = (P_1 + P_2) + P_3 \quad (\forall P_1, P_2, P_3 \in V)$

→ associative

- $P_1 + 0 = P_1 = 0 + P_1$

  $0 \in V$

→ There exists identity $(\forall P_1 \in V)$

- $P_1 + (-P_1) = 0$

  if $P_1 \in V$ then $-P_1 \in V$

→ There exists inverse $(\forall P_1 \in V)$

- $P_1 + P_2 = P_2 + P_1 \quad (\forall P_1, P_2 \in V)$

→ commutative

→ it's an abelian group.

2. $s \cdot p = sp \in V \; (\forall p \in V) \; \& \; (\forall s \in R)$

→ satisfies

3. a. $1 \odot v = v$

$1(P) = P \quad (\forall P \in V)$

→ satisfies

b. $(c_1c_2) \odot v = c_1 \odot (c_2 \odot v)$

$(c_1c_2)(p) = c_1(c_2p)$

$c_1c_2p = c_1c_2p \quad (\forall p \in V)$

c. $c \odot (\alpha + \beta) = c \odot \alpha \oplus c \odot \beta$

$c(p_1 + p_2) = cp_1 + cp_2 \quad (\forall p \in V)$

d. $(c_1+c_2) \odot \alpha = (c_1 \odot \alpha) \oplus (c_2 \odot \alpha)$

$(c_1+c_2)\, p = (c_1p + c_2p) \; \forall (p \in V)$

since it satisfies all the conditions, v is vector space over R.

ii. Symmetric 2 X 2 matrices.

$V = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad a, b, c \in R$

$\oplus$ - addition of matrices

$\odot$ - matrix is multiplied by scalar.

1. Checking abelian group property of $(V, \oplus)$

$m + n = p \in V \quad (\forall m, n \in V)$

→ closure

$(m+n)+p = m + (n+p) \quad (\forall m, n, p \in V)$

→ associative.

$m + 0 = m = 0 + m \qquad \left\{0 \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$

$0 \in V$

→ exists identity '0' $(\forall m \in V)$

$m + (-m) = 0$

if $m \in V$ then $-m \in V$

→ exists inverse

$m+n = n+m \qquad \forall\, m, n \in V$

$\rightarrow$ closure

$\rightarrow$ $(V, \oplus)$ makes an abelian group.

2. $s \odot v \in V$

$s.m \in V \quad (\forall\, m \in V \ \& \ \forall\, s \in R)$

3. a. $1 \odot v = v$

$1.m = m$

b. $(c_1 c_2) \odot v = c_1 \odot (c_2 \odot v)$

$c_1 c_2 . m = c_1 \odot c_2 m$

$c_1 c_2 m = c_1 c_2 m \qquad \forall\, m \in V$

c. $c \odot (\alpha \oplus \beta) = (c \odot \alpha) \oplus (c \odot \beta)$

$c(m+n) = cm + cn \quad \forall\ (m, n \in V)$

d. $(c_1 + c_2) \odot \alpha = (c_1 \odot \alpha) \oplus (c_2 \odot \alpha)$

$(c_1 + c_2) m = c_1 m + c_2 m \quad \forall\ (m \in V)$

since it satisfies all the conditions, V is vector space over R.

c. set of all pairs (x, y) over R such that

$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$ and $c(x, y) = (c^2 x, c^2 y)$

$V = \{(x, y)\} \quad x, y \in R$

To be a vector space, it should satisfy the condition

$(c_1 + c_2) \odot \alpha = (c_1 \odot \alpha) \oplus (c_2 \odot \alpha)$

$((c_1 + c_2)^2 x, (c_1 + c_2)^2 y) = (c_1^2 x, c_1^2 y) + (c_2^2 x + c_2^2 y)$

$((c_1 + c_2)^2 x, (c_1 + c_2)^2 y) \neq ((c_1^2 + c_2^2) x, (c_1^2 + c_2^2) y)$

$\rightarrow$ V not a vector space over R.

d. The set of polynomials of degree exactly 3 over R.

$$V = \{a_1x^3 + a_2x^2 + a_3x + a_4\} \quad a_1 \neq 0$$

$a_2, a_3, a_4 \in R$

$x^3 \in V$

$-x^3 \in V$

$x^3 + (-x^3) = 0 \notin V$

→ not closure

→ doesn't form abelian group

→ V is not vector space.

e. The set of integers over field of R

$V = \{z\} \quad z \in \mathbb{Z}$

Checking $s \odot v = V$ property

$s \in R$

$v = z \in \mathbb{Z}$

Let $s = \sqrt{2}$ and $z = 1$

$\sqrt{2} \cdot 1 = \sqrt{2} \notin V$ ($\sqrt{2}$ is not an integer)

→ V is not vector space over R.

2) $S = \{(1,1), (0,1)\}$ Field : GF(2)

$F = \{0, 1\}$

Set of linear combinations of S over field GF (2) is $= span(S) = \left\{ \begin{matrix} (0,0), (1,2), \\ (0,1), (1,1) \end{matrix} \right\}$

3) Vectors in span (2,3) over R...

The vectors in span of (2,3) can be represented as. $V = \{(2x+3y)\}$ $x, y \in R$.

since There are infinite combinations of x, y in R.

There will be infinite number of vectors in span of (2,3) over R

4) a) $S = \{(1,2,0), (0,3,1), (-1,0,1)\}$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -1 & 0 & 1 \end{bmatrix} \quad \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$|A| \neq 0$

$\therefore$ S has linearly independent vectors

b) $S = \{(-1,2,1), (3,0,-1), (-5,4,3)\}$

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 0 & -1 \\ -5 & 4 & 3 \end{bmatrix} \quad \begin{vmatrix} -1 & 2 & 1 \\ 3 & 0 & -1 \\ -5 & 4 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 1 \\ 0 & 6 & 2 \\ 0 & -6 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 0 \end{vmatrix}$$

$|A| = 0$

$\therefore$ S has ~~has~~ linearly dependent vectors

6) a) $S = \{1, x+1, x^2+x+1, x^3+x^2+x+1\}$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$|A| \neq 0$

$\therefore$ S has linearly independent vectors and S spans P^3. $\therefore$ S is basis for P^3 with dimension 4

b) $S = \{(1,2,3), (2,3,4), (3,4,5)\}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{vmatrix}$$

$|A| = 0$

S has dependent vectors.

$\therefore$ S is not basis for R^3

c) $S = \{(1,-1,0,0), (0,1,2,0), (0,0,-2,1), (-1,0,0,1)\}$

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ -1 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} \qquad |A| \neq 0$$

S has independent vectors & S spans R^4

$\therefore$ S is a basis for R^4 with dimension 4

d) $S = \{1+x, x+x^2, x^2+x^3, x^3+1\}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} \qquad |A| = 0$$

S has dependent vectors.

$\therefore$ S is not basis for P^3.

5) $S = \{(1,2,0), (0,3,1), (-1,0,1)\}$

$(x\ y\ z) = a(1\ 2\ 0) + b(0\ 3\ 1) + c(-1\ 0\ 1)$ $a, b, c \in R$

Since S has independent vectors, S spans R^3

$(x, y, z) \in R^3$

$x = a - c \quad y = 2a + 3b \quad z = b + c$

$$\cancel{\left[\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 0 & 3 & 2 & y-2x \\ 0 & 0 & 1 & 3z-y+2 \end{array}\right]} \quad \left|\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 2 & 3 & 0 & y \\ 0 & 1 & 1 & z \end{array}\right| = \left|\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 0 & 3 & 2 & y-2x \\ 0 & 1 & 1 & z \end{array}\right|$$

$$= \left|\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 0 & 3 & 2 & y-2x \\ 0 & 0 & 1 & 3z-y+2x \end{array}\right| \qquad \begin{aligned} c &= 3z - y + 2x \\ b &= -2z + y - 2x \\ a &= 3x - y + 3z \end{aligned}$$

$\therefore (x, y, z)$ can be expressed as:

$(x\ y\ z) = a(1\ 2\ 0) + b(0\ 3\ 1) + c(-1\ 0\ 1)$ where

$a = 3x - y + 3z \quad b = -2x + y - 2z \quad c = 2x - y + 3z$

CSPC
ASSIGNMENT-6

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Linear Recurrence – Worksheet

1) Solve the following recurrence problem.

(i) $a_n = a_{n-1} + n$

$a_n - a_{n-1} = n$ — (1)

Homogeneous part:

$$a_n - a_{n-1} = 0$$

Let $a_n = A\alpha^n$

$A\alpha^n - A\alpha^{n-1} = 0$

$A\alpha^{n-1}(\alpha - 1) = 0$

$\Rightarrow$ $\boxed{\alpha = 1}$ is characteristic equation.

$$a_n(h) = A(1)^n = A.$$

$a_n(p)$: $f(n) = n.1^n$

Assume $a_n(p) = n(P_0 + P_1 n)(1)^n$

Substitute in eq (1),

$n(P_0 + P_1 n) - (n-1)(P_0 + P_1(n-1)) = n$

equating coefficients of 'n'

$P_0 - P_0 + P_1 + P_1 = 1$

$2P_1 = 1 \Rightarrow P_1 = 1/2$

Now by equating constants

$P_0 - P_1 = 0$

$P_0 = P_1 = 1/2$

$a_n(h) = A$

$a_n(P) = \frac{n}{2}(1+n)$

$a_n = a_n(h) + a_n(P)$

$a_n = A + \frac{n(n+1)}{2}$

ii) $f(n) = f(n/2) + n$

$a_n = a_{n/2} + n$

$a_n(h)$:

$a_n - a_{n/2} = 0.$

Let $a_n = A\alpha^n$.

$A\alpha^n - A\alpha^{n/2} = 0$

$A\alpha^{n/2}(\alpha^{n/2} - 1) = 0.$

$\alpha^{n/2} - 1 = 0$ → characteristic equation

$\alpha^{n/2} = 1$

$n = 0$ $\alpha = 1$

Not possible for all 'n'.

$\boxed{a_n = A}$

$a_n(p)$:

$f(n) = n$

$= n(1)^n$

$= n(P_0 + P_1 n)(1)^n.$

$a_n(p) = n(P_0 + P_1 n)$

2) $a_n - 3a_{n-1} - 4a_{n-2} = 4^n$.

Let $a_n = A\alpha^n$

$a_n(h)$: $A\alpha^n - 3A\alpha^{n-1} - 4A\alpha^{n-2} = 0$

$A\alpha^{n-2}[\alpha^2 - 3\alpha - 4] = 0$.

$(\alpha+1)(\alpha-4) = 0$.

$\alpha = -1, 4$

$a_n(h) = A_1(-1)^n + A_2(4)^n$

$a_n(p)$:

Assume $a_n = nP_0 \cdot 4^n$

Substitute,

$nP_0 4^n - 3(n-1)P_0 4^{n-1} - 4(n-2)P_0 4^{(n-2)} = 4^n$.

$P_0 \cancel{4^{n-2}}[n(16) - 3(n-1)4 - 4(n-2)] = \cancel{4^n}\, 16$.

$P_0[\cancel{16n} - \cancel{12n} + 12 - \cancel{4n} + 8] = 16$.

$P_0 = \frac{16}{20} = \frac{4}{5}$.

equating coefficients of n,

$P_0 - \frac{3}{4}P_0 - \frac{P_0}{4} = 0$.

$0 = 0$.

$a_n = a_n(h) + a_n(p)$

$a_n = A_1(-1)^n + A_2(4)^n + \frac{n \cdot 4^{n+1}}{5}$

Substituting,

$n(P_0 + P_1 n) - \frac{n}{2}\left(P_0 + \frac{P_1 n}{2}\right) = n$

Equating n-terms,

$P_0 - P_0/2 = 1$

$\frac{P_0}{2} = 1 \Rightarrow \boxed{P_0 = 2}$

equating n^2 terms on both sides

$P_1 - \frac{P_1}{4} = 0 \Rightarrow \boxed{P_1 = 0}$

$a_n(P) = n(2 + 0n)$

$= \boxed{2n}$

$a_n = a_n(h) + a_n(P)$

$a_n = A + 2n$

iii) $f(n) = 2f(n/2) + n$

Homogeneous:

$a_n(h)$:

$a_n - 2a_{n/2} = 0$

Let $a_n = A\alpha^n$

$A\alpha^n - 2A\alpha^{n/2} = 0.$

$A\alpha^{n/2}[\alpha^{n/2} - 2] = 0.$

characteristic equation.

$\alpha^{n/2} = 2$

$\hookrightarrow \alpha = 2^{2/n}.$

(3) $a_n - 2a_{n-1} + a_{n-2} = 7$

Let $a_n = A\alpha^n$

$a_n(h)$:

$$A\alpha^n - 2A\alpha^{n-1} + A\alpha^{n-2} = 0$$

$$A\alpha^{n-2}(\alpha^2 - 2\alpha + 1) = 0$$

$(\alpha - 1)^2 = 0 \rightarrow$ characteristic equation.

$\alpha = 1, 1$

$$a_n = (A_1 + A_2 n)(1)^n$$

$$= A_1 + A_2 n$$

$a_n(p)$:

$$f(n) = 7.(1)^n$$

Assume $a_n = n^2 P_0 . 1^n$

$$= n^2 P_0.$$

Substitute,

$$n^2 P_0 - 2(n-1)^2 P_0 + (n-2)^2 P_0 = 7.$$

equating constants,

$$0 - 2(1)P_0 + 4P_0 = 7.$$

$$P_0 = 7/2.$$

$$a_n = a_n(h) + a_n(p)$$

$$a_n = A_1 + A_2 n + \frac{7n^2}{2}.$$

(4) Solve:

$$a_{n+2} - 2a_{n+1} + a_n = 0$$

Homogeneous

Let $a_n = A\alpha^n$

$$A\alpha^n(\alpha^2 - 2\alpha + 1) = 0.$$

$$(\alpha - 1)^2 = 0.$$

$$\alpha = 1, 1$$

$$a_n = (A_1 + A_2 n)(1)^n.$$

$$= A_1 + A_2 n.$$

$f(0) = 2$

$$a_0 = 2 = A_1 + A_2(0) = A_1$$

$$\boxed{A_1 = 2}$$

$f(1) = 30$

$$30 = A_1 + A_2(1)$$

$$30 = 2 + A_2.$$

$$\boxed{A_2 = 28}$$

$$a_n = 2 + 28n.$$

(5) Solve:

$$a_{n+2} - 3a_{n+1} + 2a_n = 2^n.$$

Homogeneous part:

$a_n(h)$: $a_{n+2} - 3a_{n+1} + 2a_n = 0$

Let $a_n = A\alpha^n$.

$$A\alpha^n[\alpha^2 - 3\alpha + 2] = 0.$$

$$(\alpha - 1)(\alpha - 2) = 0.$$

$$\alpha = 1, 2$$

$$a_n = A_1(1)^n + A_2(2)^n.$$

$a_n(P)$:

$f(n) = 2^n$.

Assume $a_n = n P_0 2^n$.

$(n+2)P_0 2^{(n+2)} - 3(n+1)P_0 2^{(n+1)} + 2n P_0 2^n = 2^n$

Dividing by 2^n we will get:

$(n+2)4P_0 - 3(n+1).2P_0 + 2nP_0 = 1$.

equate n terms,

$4P_0 - 6P_0 + 2P_0 = 0$

$0 = 0$.

equating constants:

$8P_0 - 3(2)P_0 = 1$.

$2P_0 = 1$.

$P_0 = 1/2$.

$a_n(P) = \frac{n.2^n}{2}$.

$a_n = a_n(h) + a_n(P)$

$a_n = A_1(1)^n + A_2(2)^n + n.2^{n-1}$

6 $a_n - 5a_{n-1} + 6a_{n-2} = 2^n + n \hookrightarrow (1)$

$a_n(h)$:

Let $a_n = A\alpha^n$.

$A\alpha^n - 5A\alpha^{n-1} + 6A\alpha^{n-2} = 0$.

$A\alpha^{n-2}[\alpha^2 - 5\alpha + 6] = 0$.

$(\alpha-2)(\alpha-3) = 0$.

$\alpha = 2, 3$

$a_n = A_1(2)^n + A_2(3)^n$.

$a_n(P)$:

$f(n) = 2^n + n$

Assume $a_n = n^1 P_0 2^n + (P_1 + P_2 n)$

$a_n = n P_0 2^n + P_1 + P_2 n.$

Substituting in eq.①, we get

$(nP_0 2^n + P_1 + P_2 n) - 5((n-1)P_0 2^{n-1} + P_1 + P_2(n-1))$
$+ 6\left[(n-2)P_0 2^{n-2} + P_1 + P_2(n-2)\right] = 2^n + n$

Equating coefficient of n,

$P_2 - 5P_2 + 6P_2 = 1.$

$2P_2 = 1 \quad ; P_2 = 1/2.$

Equating constants,

$P_1 + 5P_2 + 6P_1 - 5P_1 - 12P_2 = 0$

$2P_1 - 7P_2 = 0.$

$2P_1 = 7/2. \Rightarrow P_1 = 7/4,$

Equating 2^n terms,

$\frac{5P_0}{2} - \frac{12P_0}{4} = 1.$

$\frac{-2P_0}{4} = 1$

$P_0 = -2.$

$a_n(P) = -2n \cdot 2^n + 7/4 + n/2.$

$a_n = a_n(h) + a_n(P)$

$a_n = A_1(2)^n + A_2(3)^n - n \cdot 2^{n+1} + 7/4 + n/2.$