CSPC
ASSIGNMENT-5

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LINEAR ALGEBRA

Q.1.

a) The set of all polynomials (of degree at most n) over R

$$V = \{a_n x^n + \dots + a_1 x + a_0\} \quad a_i \in R \quad 0 \leq i \leq n$$

$\oplus$ be addition of polynomials

$\odot$ be multiplication of vector by a scalar.

1. Checking 'abelian group property' of $(V, \oplus)$

- $P_1 + P_2 = P \in V \quad (\forall P_1, P_2 \in V)$

→ Closure

- $P_1 + (P_2 + P_3) = (P_1 + P_2) + P_3 \quad (\forall P_1, P_2, P_3 \in V)$

→ associative

- $P_1 + 0 = P_1 = 0 + P_1$

  $0 \in V$

  → There exists identity $(\forall P_1 \in V)$

- $P_1 + (-P_1) = 0$

  if $P_1 \in V$ then $-P_1 \in V$

  → There exists inverse $(\forall P_1 \in V)$

- $P_1 + P_2 = P_2 + P_1 \quad (\forall P_1, P_2 \in V)$

  → commutative

→ it's an abelian group.

2. $s \cdot p = sp \in V \; (\forall p \in V) \;\&\; (\forall s \in R)$

→ satisfies

3. a. $1 \odot v = v$

   $1(P) = P \quad (\forall P \in V)$

   → satisfies

b. $(c_1 c_2) \odot v = c_1 \odot (c_2 \odot v)$

$(c_1 c_2)(p) = c_1 (c_2 p)$

$c_1 c_2 p = c_1 c_2 p \quad (\forall p \in V)$

c. $c \odot (\alpha + \beta) = c \odot \alpha \oplus c \odot \beta$

$c(p_1 + p_2) = cp_1 + cp_2 \quad (\forall p \in V)$

d. $(c_1 + c_2) \odot \alpha = (c_1 \odot \alpha) \oplus (c_2 \odot \alpha)$

$(c_1 + c_2) p = (c_1 p + c_2 p) \ \forall (p \in V)$

since it satisfies all the conditions, V is vector space over R.

ii. Symmetric 2×2 matrices.

$$V = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad a, b, c \in R$$

$\oplus$ - addition of matrices

$\odot$ - matrix is multiplied by scalar.

1. Checking 'abelian group property' of $(V, \oplus)$

$m + n = p \in V \quad (\forall m, n \in V)$

→ closure

$(m+n)+p = m + (n+p) \quad (\forall m, n, p \in V)$

→ associative.

$m + 0 = m = 0 + m \qquad \left\{ 0 \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

$0 \in V$

→ exists identity '0' $(\forall m \in V)$

$m + (-m) = 0$

if $m \in V$ then $-m \in V$

→ exists inverse

$m+n = n+m \quad \forall\, m, n \in V$

$\rightarrow$ closure

$\rightarrow$ $(V, \oplus)$ makes an abelian group.

2. $s \odot v \in V$

$s.m \in V \quad (\forall\, m \in V \;\&\; \forall\, s \in R)$

3. a. $1 \odot v = v$

$1.m = m$

b. $(c_1 c_2) \odot v = c_1 \odot (c_2 \odot v)$

$c_1 c_2 . m = c_1 \odot c_2 m$

$c_1 c_2 m = c_1 c_2 m \quad \forall\, m \in V$

c. $c \odot (\alpha \oplus \beta) = (c \odot \alpha) \oplus (c \odot \beta)$

$c(m+n) = cm + cn \quad \forall\, (m, n \in V)$

d. $(c_1 + c_2) \odot \alpha = (c_1 \odot \alpha) \oplus (c_2 \odot \alpha)$

$(c_1 + c_2) m = c_1 m + c_2 m \quad \forall\, (m \in V)$

since it satisfies all the conditions, V is vector space over R.

c. set of all pairs (x, y) over R such that

$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$ and $c(x, y) = (c^2 x, c^2 y)$

$V = \{(x, y)\} \quad x, y \in R$

To be a vector space, it should satisfy the condition

$(c_1 + c_2) \odot \alpha = (c_1 \odot \alpha) \oplus (c_2 \odot \alpha)$

$((c_1+c_2)^2 x, (c_1+c_2)^2 y) = (c_1^2 x, c_1^2 y) + (c_2^2 x + c_2^2 y)$

$((c_1+c_2)^2 x, (c_1+c_2)^2 y) \neq ((c_1^2 + c_2^2)x, (c_1^2 + c_2^2) y)$

$\rightarrow$ V not a vector space over R.

d. The set of polynomials of degree exactly 3 over R.

$$V = \{ a_1x^3 + a_2x^2 + a_3x + a_4 \} \quad a_1 \neq 0$$
$$a_2, a_3, a_4 \in R$$

$x^3 \in V$

$-x^3 \in V$

$x^3 + (-x^3) = 0 \notin V$

→ not closure

→ doesn't form abelian group

→ V is not vector space.

e. The set of integers over field of R

$$V = \{ z \} \quad z \in \mathbb{Z}$$

Checking $s \odot v = V$ property

$s \in R$

$v = z \in \mathbb{Z}$

Let $s = \sqrt{2}$ and $z = 1$

$\sqrt{2} \cdot 1 = \sqrt{2} \notin V$ ($\sqrt{2}$ is not an integer)

→ V is not vector space over R.

2) $S = \{(1,1), (0,1)\}$ Field : $GF(2)$

$F = \{0, 1\}$

Set of linear combinations of S over field GF(2) is $\Big\} = span(S) = \begin{Bmatrix} (0,0), (1,2), \\ (0,1), (1,1) \end{Bmatrix}$

3) Vectors in span(2,3) over R...

The vectors in span of (2,3) can be represented as. $V = \{(2x+3y)\}$ $x, y \in R$.

since There are infinite combinations of x, y in R.

. There will be infinite number of vectors in span of (2,3) over R

4) a) $S = \{(1,2,0), (0,3,1), (-1,0,1)\}$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -1 & 0 & 1 \end{bmatrix} \quad \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$|A| \neq 0$

$\therefore S$ has linearly independent vectors

b) $S = \{(-1,2,1), (3,0,-1), (-5,4,3)\}$

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 0 & -1 \\ -5 & 4 & 3 \end{bmatrix} \quad \begin{vmatrix} -1 & 2 & 1 \\ 3 & 0 & -1 \\ -5 & 4 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 1 \\ 0 & 6 & 2 \\ 0 & -6 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 0 \end{vmatrix}$$

$|A| = 0$

$\therefore S$ has ~~has~~ linearly dependent vectors

6) a) $S = \{1, x+1, x^2+x+1, x^3+x^2+x+1\}$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$|A| \neq 0$

$\therefore S$ has linearly independent vectors and S spans P^3. $\therefore S$ is basis for P^3 with dimension 4

b) $S = \{(1,2,3), (2,3,4), (3,4,5)\}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{vmatrix}$$

$|A| = 0$

S has dependent vectors.

$\therefore S$ is not basis for R^3

c) $S = \{(1,-1,0,0), (0,1,2,0), (0,0,-2,1), (-1,0,0,1)\}$

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ -1 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} \qquad |A| \neq 0$$

S has independent vectors & S spans R^4

$\therefore$ S is a basis for R^4 with dimension 4

d) $S = \{1+x, x+x^2, x^2+x^3, x^3+1\}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} \qquad |A| = 0$$

S has dependent vectors.

$\therefore$ S is not basis for P^3.

5) $S = \{(1,2,0), (0,3,1), (-1,0,1)\}$

$(x\ y\ z) = a(1\ 2\ 0) + b(0\ 3\ 1) + c(-1\ 0\ 1) \quad a, b, c \in R$

Since S has independent vectors, S spans R^3

$(x, y, z) \in R^3$

$x = a - c \quad y = 2a + 3b \quad z = b + c$

$$\cancel{\left[\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 0 & 3 & 2 & y-2x \\ 0 & 0 & 1 & 3z-y+2 \end{array}\right]} \qquad \left|\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 2 & 3 & 0 & y \\ 0 & 1 & 1 & z \end{array}\right| = \left|\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 0 & 3 & 2 & y-2x \\ 0 & 1 & 1 & z \end{array}\right|$$

$$= \left|\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 0 & 3 & 2 & y-2x \\ 0 & 0 & 1 & 3z-y+2x \end{array}\right| \qquad \begin{aligned} c &= 3z - y + 2x \\ b &= -2z + y - 2x \\ a &= 3x - y + 3z \end{aligned}$$

$\therefore (x, y, z)$ can be expressed as :

$(x\ y\ z) = a(1\ 2\ 0) + b(0\ 3\ 1) + c(-1\ 0\ 1)$ where

$a = 3x - y + 3z \quad b = -2x + y - 2z \quad c = 2x - y + 3z$