Solution to Mathemaddict's Problem

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1 Question

$$I = \int_1^\infty \frac{2x\{x\} - \{x\}^2}{x^2\lfloor x \rfloor^2}\, dx = \frac{\pi^A}{B} - \phi^C$$

Then,

$$L = \lim_{\theta \to C} \frac{\sin(A\theta)\sec(\theta)}{B\theta} = ?$$

Note: $\{x\} = x - \lfloor x \rfloor$

2 Solution

Let's expand the integral using the definition of the fractional part

$$I = \int_1^\infty \frac{2x^2 - 2x\lfloor x \rfloor - x^2 + 2x\lfloor x \rfloor - \lfloor x \rfloor^2}{x^2\lfloor x \rfloor^2}\, d$$

Simplifying and by the definition of the floor function

$$I = \sum_{k=1}^{\infty} \int_k^{k+1} \frac{1}{k^2} - \frac{1}{x^2}\, dx$$

Therefore,

$$I=\sum_{k=1}^{\infty}\left(\frac{x}{k^2}\bigg|_k^{k+1}+\frac{1}{x}\bigg|_k^{k+1}\right)$$
$$=\sum_{k=1}^{\infty}\left(\frac{1}{k^2}+\frac{1}{k+1}-\frac{1}{k}\right)$$

Evaluating the series we get

$$I=\frac{\pi^2}{6}-1$$

Clearly,

$$A=2,\ B=6,\ C=1$$

Substituting in the limit, we get

$$\lim_{\theta\to 0}\frac{\sin(2\theta)\sec(\theta)}{6\theta}=\frac{1}{3}$$

Our final answer is

$$L=\frac{1}{3}$$