Messy Integral (@Vibingmath)

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1 Question

$$\int_1^e \frac{x - x\ln x + 1}{x\,(x+1)^2 + x\ln^2 x}\,dx$$

2 Solution

Let us take substitution $\ln x = t$ so that $dx = e^t dt$, thus our integral becomes

$$\int_0^1 \frac{(e^t \;-\; te^t \;+\; 1)}{\cancel{e^t}\left[(e^t+1)^2 \;+\; t^2\right]}\,\cancel{e^t}\,dt$$

Now dividing both numerator and denominator by t^2 we get,

$$\int_0^1 \frac{\frac{e^t}{t^2} + \frac{1}{t^2} - \frac{e^t}{t}}{\left(\frac{e^t}{t} + \frac{1}{t}\right)^2 + 1}\,dt$$

Now taking a final substitution

$$\frac{e^t}{t} + \frac{1}{t} = u$$

so that

$$\left(\frac{e^t}{t} - \frac{e^t}{t^2} - \frac{1}{t^2}\right)dt \;=\; du$$

The integral simplifies as:

$$\int_{e+1}^{\infty} \frac{1}{1+u^2}\,du$$

Plugging in the limits, the answer is

$$\boxed{= \tan^{-1}\left(\frac{1}{1+e}\right)}$$