Shreenabh Agrawal's Daily Challenge 7

Jose Bedoya

June 28, 2020

Prove that if

$$a = 2 + 2\sqrt{28n^2 + 1}$$

is an integer, then it is a perfect square.

Solution

Notice that for a to be an integer

$$28n^2 + 1$$

must be a perfect square. Therefore,

$$28n^2 + 1 = q^2, \text{ for some } q \in \mathbf{N}$$

It's easy to see that q is an odd number. Hence,

$$7n^2 = \left(\frac{q+1}{2}\right)\left(\frac{q-1}{2}\right)$$

As these are two consecutive numbers, they are prime relatives (no common divisors). This is, they cannot form a perfect square if not by separate. In addition, as 7 is prime, it must divide just one of them. Therefore,

$$\frac{q+1}{2} = 7r^2, \quad \frac{q-1}{2} = s^2$$

or

$$\frac{q+1}{2} = s^2, \quad \frac{q-1}{2} = 7r^2$$

for some $s, r \in \mathbf{N}$.

Notice the first case is not possible as $s^2 \equiv -1 \pmod 7$ In other words, there are no perfect squares congruent to $-1 \pmod 7$. This is trivial to see, just make a list of numbers from 1 to 6, square each of them and you will see no final number has remainder of 6 (same as saying -1) when divided by 7.

Hence,

$$a = 2 + 2\sqrt{28n^2 + 1} = 2(q + 1)$$
$$\Rightarrow a = 4s^2 = (2s)^2$$

which is always a perfect square.