Di-Gamma Reflection Formula

$$\psi(1-s)-\psi(s)=\pi\cot(\pi s)$$

And Functional Equation

$$\psi(s+1)=\psi(s)+\frac{1}{s}$$

Swipe For TWO Proofs $\Rightarrow$

1 First Proof (Reflection Formula)

Start off with Euler's Reflection Formula (derived on 17th June)

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

Taking the natural log on both sides gives

$$\ln(\Gamma(s)) + \ln(\Gamma(1-s)) = \ln(\pi) - \ln(\sin(\pi s))$$

Differentiating both sides gives

$$\frac{\Gamma'(s)}{\Gamma(s)} - \frac{\Gamma'(1-s)}{\Gamma(1-s)} = -\pi\cot(\pi s)$$

Don't forget to bring out the negative signs due to the chain rule! Since the terms on the left are the definition of the Di-Gamma function, we can say

$$\psi(s) - \psi(1-s) = -\pi\cot(\pi s)$$

Which we can rewrite as

$$\psi(1-s) - \psi(s) = \pi\cot(\pi s)$$

This is the classic proof, on the next slide I'll show another proof that you can get from playing around with the Mittag-Lefler expansion of the cotangent.

2 Second Proof(Reflection Formula)

First recall the Mittag-Lefler expansion of the cotangent. I showed and provided a derivation of this on my account on a post on 13th June (3rd slide specifically)

$$\pi\cot(\pi z) = \sum_{n=-\infty}^{\infty} \frac{z}{z^2-n^2} = \frac{1}{z} + 2\sum_{n=1}^{\infty} \frac{z}{z^2-n^2}$$

Let's recall the definition of $\psi(z)$ and $\psi(1-z)$

$$\psi(z) = -\gamma + \sum_{k=0}^{\infty} \frac{1}{k+1} - \frac{1}{k+z} \tag{1}$$

$$\psi(1-z) = -\gamma + \sum_{k=0}^{\infty} \frac{1}{k+1} - \frac{1}{k+1-z} \tag{2}$$

Doing $(2)-(1)$ yields

$$\psi(1-z) - \psi(z) = \sum_{k=0}^{\infty} \frac{-1}{k+1-z} + \frac{1}{k+z}$$

$$\sum_{k=0}^{\infty} \frac{1}{k+z} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{k+z} \quad \textbf{and} \quad \sum_{k=0}^{\infty} \frac{-1}{k+1-z} = \sum_{k=1}^{\infty} \frac{-1}{k-z}$$

$$\psi(1-z) - \psi(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{k+z} - \frac{1}{k-z} = \frac{1}{z} + 2\sum_{k=1}^{\infty} \frac{z}{k^2-z^2} = \pi\cot(\pi z)$$

3 Functional Equation

This one's pretty short and sweet

$$\psi(z+1) = \psi(z) + \frac{1}{z}$$

Starting with the definition of $\psi(z+1)$ and $\psi(z)$, we have

$$\psi(z+1) = -\gamma + \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+z} \tag{3}$$

$$\psi(z) = -\gamma + \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+z-1} \tag{4}$$

Doing (3) $-$ (4) yields us

$$\psi(z+1) - \psi(z) = \sum_{k=1}^{\infty} \frac{1}{k+z-1} - \frac{1}{k+z}$$

This is a telescoping sum so if you write out a few terms you get

$$\psi(z+1) - \psi(z) = \left(\frac{1}{z} - \cancel{\frac{1}{z+1}} + \cancel{\frac{1}{z+1}} - \cancel{\frac{1}{z+2}} + \cancel{\frac{1}{z+2}} - \cancel{\frac{1}{z+3}} \cdots \right) = \frac{1}{z}$$

$$\psi(z+1) = \psi(z) + \frac{1}{z}$$

$$\psi(1-s)-\psi(s)=\pi\cot(\pi s)$$

$$\psi(s+1)=\psi(s)+\frac{1}{s}$$

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