Golden Ratio Integral

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Question:

$$\int_1^{\phi} \frac{(x^2+1)}{(x^4-x^2+1)} \ln\left(1+x-\frac{1}{x}\right) dx$$

Answer: Dividing numerator and denominator of the fraction by x^2,

$$\int_1^{\phi} \frac{(1+\frac{1}{x^2})}{(x^2-1\ +\ \frac{1}{x^2})} \ln\left(x-\frac{1}{x}+1\right) dx$$

Now taking a substitution, $x-\frac{1}{x}$ as t, so that,

$$\left(1\ +\ \frac{1}{x^2}\right) dx = dt$$

Thus, the original integral (with new limits) can be rewritten as:

$$\int_0^1 \frac{\ln(1+t)}{1+t^2} dx$$

Taking the trigonometric substitution

$$\tan\theta\ =\ t$$

such that

$$\left(\sec^2\theta\ \right) d\theta\ =\ dt$$

The integral can be re-written with new limits as (let us denote it by (i))

$$\int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$$

By applying King's rule of sum of limits, the same integral can be re-written as

$$\int_0^{\pi/4} \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right) d\theta$$

Applying expansion for tangent function,

$$\int_0^{\pi/4} \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

This can be simplified as (let us denote it by (ii)):

$$\int_0^{\pi/4} (\ln 2\ -\ \ln(1+\tan\theta))\, d\theta$$

Now adding (i) and (ii), we get twice of the value of the required definite integral as:

$$\int_0^{\pi/4} \ln 2\ d\theta$$

Now substituting the limits and dividing by 2 (because we added two integrals in last step), we get our required answer as:

$$\frac{\pi}{8}\ln 2$$

The approximate value of which is: 0.272198261288