Solution to Linear algebra #1

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1 Problem

Let A be a square matrix. Then reflect its entries along the diagonal other than the diagonal and let the new matrix be B. Prove that $\det(A) = \det(B)$.

For example, if

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \implies B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

2 Solution

Let's use the example provided in the last page. First, let's take the transpose of A, which does not change its determinant.

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \implies A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

We can see that $\mathbf{r_1} \longleftrightarrow \mathbf{r_3}$ and $\mathbf{c_1} \longleftrightarrow \mathbf{c_3}$ turns A^T into B.

Similarly, we can deduce that for a $n \times n$ matrix A, the following EROs and ECOs can turn A^T into it's corresponding B:

$$\{\mathbf{r_1} \longleftrightarrow \mathbf{r_n}\ ,\ \mathbf{r_2} \longleftrightarrow \mathbf{r_{n-1}} \cdots \mathbf{r_{\lfloor n/2 \rfloor}} \longleftrightarrow \mathbf{r_{\lceil n/2 \rceil}}\}$$

$$\{\mathbf{c_1} \longleftrightarrow \mathbf{c_n}\ ,\ \mathbf{c_2} \longleftrightarrow \mathbf{c_{n-1}} \cdots \mathbf{c_{\lfloor n/2 \rfloor}} \longleftrightarrow \mathbf{c_{\lceil n/2 \rceil}}\}$$

It's trivial that both sets have the same number of operations. Let that number be m. Since these Type I operations multiply the determinant by -1, $\det(B)$ can be calculated like this.

$$\det(B) = \det(A^T)(-1)^m(-1)^m = \det(A)(-1)^{2m} = \det(A)$$

Therefore our claim is proved.