Euler's Reflection Formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

Swipe For Proof $\Rightarrow$

Recall the Gauss Representation of $\Gamma(s)$. If you're unfamiliar with this representation you can check out the post published on 15th June on my account **@creative_math_**. The Gauss representation states

$$\Gamma(s) = \lim_{n\to\infty}\left(\frac{n^s}{s}\prod_{k=1}^{n}\frac{k}{s+k}\right)$$

Let's substitute $s \to -s$

$$\Gamma(-s) = \lim_{n\to\infty}\left(\frac{n^{-s}}{-s}\prod_{k=1}^{n}\frac{k}{k-s}\right)$$

Let's multiply these expressions together to give

$$\Gamma(s)\Gamma(-s) = \lim_{n\to\infty}\left(\frac{n^s}{s}\prod_{k=1}^{n}\frac{k}{s+k}\right) \times \lim_{n\to\infty}\left(\frac{n^{-s}}{-s}\prod_{k=1}^{n}\frac{k}{k-s}\right)$$

$$\Gamma(s)\Gamma(-s) = \lim_{n\to\infty}\left(\frac{n^s}{s} \times \frac{n^{-s}}{-s}\prod_{k=1}^{n}\frac{k^2}{k^2-s^2}\right)$$

We can simplify this further to give

$$\Gamma(s)\Gamma(-s) = \lim_{n\to\infty}\left(\frac{1}{-s^2}\prod_{k=1}^{\infty}\frac{1}{1-\frac{s^2}{k^2}}\right)$$

$$\Gamma(s)\Gamma(-s) = \frac{1}{-s^2}\left(\prod_{k=1}^{\infty}1-\frac{s^2}{k^2}\right)^{-1}$$

This product here is the product for $\sin(\pi s)$. Recall that

$$\sin(\pi s) = \pi s \prod_{k=1}^{\infty} \left(1 - \frac{s^2}{k^2}\right)$$

This was derived on my account on 13th June using Fourier Series

$$\Gamma(s)\Gamma(-s) = \frac{1}{-s^2} \left(\prod_{k=1}^{\infty} 1 - \frac{s^2}{k^2}\right)^{-1} = \frac{1}{-s^2} \times \frac{\pi s}{\sin(\pi s)}$$

$$\Gamma(s)\Gamma(-s) = \frac{1}{-s} \times \frac{\pi}{\sin(\pi s)}$$

$$\Gamma(s) \times -s\Gamma(-s) = \frac{\pi}{\sin(\pi s)}$$

We know that $n\Gamma(n) = \Gamma(n+1)$. This gives us our desired result

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

This is known as Euler's Reflection Formula, We can use it to solve questions like the one given here

$$\sum_{x=1}^{1729} \Gamma\left(\frac{1+2x}{2}\right) \Gamma\left(\frac{1-2x}{2}\right)$$

Put your answers in the comments let's see who can do this ;)

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$