Geometry with complex numbers #1

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1   Introduction

In this series of posts, we discuss representations of geometrical objects with complex numbers and use them to prove some famous geometry theorems! We'll focus on lines and points in this post!
Enjoy :D

2 Lines with complex numbers

First, let's think about lines segments on the complex plane. Let $z_1, z_2 \in \mathbb{C}$. How can we represent the line segment from z_1 and z_2?

We know that the vector from z_1 to z_2 is $z_2 - z_1$, so we can represent the line segment by the equation

$$l = z_1 + t(z_2 - z_1) = z_1 + tv$$

Where t is a parameter so that $0 \leq t \leq 1$ and v is a non-zero vector in the direction of the line. Therefore, the line segment is the collection of all possible points l. Besides, if we let t be any real number, then we get the line passing through z_1 and z_2.

For example, let $z_1 = 3 - 2i$ and $z_2 = 4 + 5i$. Then the equation

$$L : 3 - 2i + t(1 + 7i)$$

With $0 \leq t \leq 1$ is the line segment from z_1 to z_2, and if $t \in \mathbb{R}$ then it is the line passing through z_1 and z_2.

3   Finding points of division

Let's say we have 2 points $z_1, z_2 \in \mathbb{C}$. Suppose we want to find a z_3 on the line segment $z_1 z_2$ such that it divides the length of the line segment by the ratio $a : b$, which is equivalent to

$$|z_1 - z_3| : |z_2 - z_3| = a : b$$

Consider the equation

$$L := z_1 + t(z_2 - z_1) \qquad 0 \le t \le 1$$

To find z_3,

$$t = \frac{a}{a+b} \implies z_3 = z_1 + \frac{a}{a+b}(z_2 - z_1) = \frac{bz_1 + az_2}{a+b}$$

For example, let $z_1 = 4+5i$, $z_2 = -4-3i$ and suppose we want to find z_3 such that it divides $z_1 z_2$ by $3 : 5$. Then

$$z_3 = \frac{5(4+5i) + 3(-4-3i)}{3+5} = \frac{8+16i}{8} = 1 + 2i$$

4   Parallel lines

Given 2 lines L_1 , L_2 and their equations

$$L_1 : z_1 + tv_1 \qquad t \in \mathbb{R}$$

$$L_1 : z_2 + sv_2 \qquad s \in \mathbb{R}$$

It is easy to see that if v_1 and v_2 are parallel, i.e.

$$\exists k \in \mathbb{R} : v_1 = kv_2$$

Stay tuned for the next post! We are going to use what we've learned here to easily prove a very useful theorem!