Solution to Integral Fun's Problem

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1 Question

$$I(\alpha, n) = \int_0^\infty \frac{\cos \alpha t}{t^n} \, dt$$

with $\mid n \mid < 1$

2 Solution

Knowing,

$$\int_0^\infty f(t)g(t) \, dt = \int_0^\infty \mathcal{L}\{f(t)\}(s)\mathcal{L}^{-1}\{g(t)\}(s) \, ds$$

And by choosing

$$f(t) = \cos(\alpha t)$$

$$g(t) = \frac{1}{t^n}$$

we have

$$\mathcal{L}\{f(t)\}(s) = \frac{s}{s^2 + \alpha^2}$$

$$\mathcal{L}^{-1}\{g(t)\}(s) = \frac{s^{n-1}}{\Gamma(n)}$$

Substituting on our integral

$$I(\alpha, n) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{s^n}{s^2 + \alpha^2} \, ds$$

Let $s = \alpha \tan(\theta)$

$$\begin{aligned} I(\alpha, n) &= \frac{\alpha^{n-1}}{\Gamma(n)} \int_0^{\frac{\pi}{2}} \tan^n(\theta) \, d\theta \\ &= \frac{\alpha^{n-1}}{\Gamma(n)} \int_0^{\frac{\pi}{2}} \sin^n(\theta) \cos^{-n}(\theta) \, d\theta \end{aligned}$$

Recall the definition of the Beta Function

$$\begin{aligned} B(x, y) &= 2 \int_0^{\frac{\pi}{2}} \sin^{2x-1}(\theta) \cos^{2y-1}(\theta) \, d\theta \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \end{aligned}$$

where $\Gamma(n) = (n-1)!$.

Notice, in our integral:

$$\begin{aligned} x &= \frac{n+1}{2} \\ y &= \frac{1-n}{2} \end{aligned}$$

Therefore,

$$I(\alpha, n) = \frac{\alpha^{n-1}}{2\Gamma(n)} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{1-n}{2}\right)$$

Here we can simplify the expression using the reflection property of the Gamma Function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

for z not integer.

If we let $z = \frac{n+1}{2}$

$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi(n+1)}{2}\right)}$$
$$= \pi \sec\left(\frac{\pi n}{2}\right)$$

we get

$$I(\alpha, n) = \frac{\pi}{4}\frac{\alpha^{n-1}}{\Gamma(n)} \sec\left(\frac{\pi n}{2}\right)$$