Solution to Mathemaddict's Problem

Jose Bedoya

June 29, 2020

1 Question

$$I = \int_1^\infty \frac{x^2 - 1}{x^4 \ln x}\, dx$$

2 Solution

Consider the substitution $x \to \frac{1}{u}$

$$I = \int_0^1 \frac{u^2 - 1}{\ln u}\, du$$

Let

$$B(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}\, dx$$

Notice our desire integral is $B(2)$.

Differentiate both sides,

$$\frac{dB}{d\alpha} = \frac{d}{d\alpha} \int_0^1 \frac{x^\alpha - 1}{\ln x} \, dx$$

And, by the Leibniz rule for integrals

$$\frac{dB}{d\alpha} = \int_0^1 \frac{\partial}{\partial \alpha} \frac{x^\alpha - 1}{\ln x} \, dx$$

Therefore,

$$\begin{aligned} \frac{dB}{d\alpha} &= \int_0^1 x^\alpha \, dx \\ &= \frac{1}{\alpha + 1} \end{aligned}$$

Integrating both sides

$$\begin{aligned} B(\alpha) &= \int \frac{1}{\alpha + 1} \, d\alpha \\ &= \ln(\alpha + 1) + C \end{aligned}$$

If you let $\alpha = 0$, you will get $C = 0$.

Therefore,

$$I = \int_1^\infty \frac{x^2 - 1}{x^4 \ln x} \, dx = \ln 3$$