Problem #15

@integral_dx

June 29, 2020

Problem:
Factorise $a^3(b^2-c^2)+b^3(a^2-c^2)+c^3(a^2-b^2)$.
Solution:
Let $f(a,b,c)=a^3(b^2-c^2)+b^3(a^2-c^2)+c^3(a^2-b^2)$.
Consider $f(a,-b,c)$.

$$f(a,-b,c)=a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$$

$f(a,-b,c)$ is a homogeneous polynomial of degree 5.
Let $g(a,b,c)=f(a,-b,c)$.
Consider $g(b,b,c)$.

$$g(b,b,c)=b^3(b^2-c^2)+b^3(c^2-b^2)+c^3(b^2-b^2)=0$$

By factor theorem, $(a-b)$ is a factor of $g(a,b,c)$.
Because $g(a,b,c)$ is homogeneous, $(b-c)$ and $(c-a)$ are also factors.
Now, $g(a,b,c)=(a-b)(b-c)(c-a)Q(a,b,c)$.
If the $Q(a,b,c)$ is the product of two linear factors, then there must be another linear factor because $g(a,b,c)$ is homogeneous. Therefore, $Q(a,b,c)$ must be a linear combination of all homogeneous polynomials of degree 2.
Let $Q(a,b,c)=\lambda_1(a^2+b^2+c^2)+\lambda_2(ab+bc+ca)$.
Therefore, $g(a,b,c)=(a-b)(b-c)(c-a)(\lambda_1(a^2+b^2+c^2)+\lambda_2(ab+bc+ca))$.
Substituting $(a,b,c)=(1,2,3)$ and $(a,b,c)=(1,0,-1)$ respectively, we have

$$\begin{cases} -22=2(14\lambda_1+11\lambda_2) \\ -2=-2(2\lambda_1-\lambda_2) \end{cases}$$

Solving gives $(\lambda_1,\lambda_2)=(0,-1)$.
Therefore, $g(a,b,c)=-(a-b)(b-c)(c-a)(ab+bc+ca)$
Because $g(a,b,c)=f(a,-b,c)$,

$$f(a,b,c)=-(a+b)(-b-c)(c-a)(-ab-bc+ca)$$

Simplifying gives

$$\boxed{f(a,b,c)=(a+b)(b+c)(c-a)(ac-cb-ba)}$$