2016 ISI Calculus Problem

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July 7, 2020

1 Question

Let f be a differentiable function such that $f(f((x)) = x$ for all $x \in [0,1]$ Suppose $f(0) = 1$. Determine the value of $\int\limits_0^1 (x - f(x))^{2016}\, dx$

2 Solution

$$f(f(x)) = x \implies f(x) = f^{-1}(x)$$

The graph of this is symmetric about $y = x$,

$$\therefore f(0) = 1 \implies f(1) = 0$$

Now, let us solve the integral (call it (i))

$$I = \int\limits_0^1 (x - f(x))^{2016}\, dx$$

Taking substitution,

$$\begin{aligned} f(x) &= t \\ x &= f^{-1}(t) = f(t) \\ dx &= f'(t)\, dt \end{aligned}$$

Plugging this in (call this (ii)),

$$\begin{aligned} \therefore \quad I &= \int\limits_1^0 (f(t) - t)^{2016} f'(t)\, dt \\ &= -\int\limits_0^1 (t - f(t))^{2016} f'(t)\, dt \end{aligned}$$

Now adding (i) and (ii),

$$2I = \int\limits_0^1 (x - f(x))^{2016} \left(1 - f'(x)\right)\, dx$$

Taking another substitution,

$$\begin{aligned} (x - f(x)) &= u \\ (1 - f'(x))\, dx &= du \end{aligned}$$

Finally,

$$\begin{aligned} \therefore \quad 2I &= \int\limits_{-1}^1 u^{2016}\, du \\ 2I &= 2 \cdot \left[\frac{u^{2017}}{2017}\right]_0^1 \end{aligned}$$

Thus, the value of ou original integral is:

$$\boxed{I = \frac{1}{2017}}$$