Mathinity Factorial Problem

Shreenabh Agrawal

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1 Question

$$Let\ S_n = 1! + 2! + \ldots + n!$$

Find all values of n such that S_n is a perfect square and prove that these are the only ones.

2 Solution

Let us start by calculating some values of this summation:

$$\begin{aligned} S_1 &= 1! \\ &= 1 \\ S_2 &= 1! + 2! \\ &= 1 + 2 \\ &= 3 \\ S_3 &= 1! + 2! + 3! \\ &= 1 + 2 + 6 \\ &= 9 \\ S_4 &= 1! + 2! + 3! + 4! \\ &= 1 + 2 + 6 + 24 \\ &= 33 \end{aligned}$$

Here, we can easily find the trivial solutions

$$n = 1 \,\&\, n = 3$$

Next, from S_5 onwards, there will always be multiples of 10 that would be added to S_4 (because 5! and consequently all larger factorials have 10 as a factor). Hence, we can write as (for $n > 4$)

$$\begin{aligned} S_n &\equiv (S_4 + 10k) \bmod 10 \\ &\equiv S_4 \bmod 10 \\ &\equiv 3 \bmod 10 \end{aligned}$$

Which as we know can never be perfect squares. Thus our only answers are:

$$\boxed{n = 1,\ 3}$$