Solution to Calculus with Calvin's Problem

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1 Question

$$I = \int_0^1 \frac{\ln(x+1)\ln x}{x}\,dx$$

2 Solution

Knowing,

$$\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$$

We have

$$I = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\int_0^1 x^{n-1}\ln x\,dx$$

Applying IBP, we get

$$I = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{x^n\ln x}{n}\Big|_0^1 - \frac{1}{n}\int_0^1 x^{n-1}\,dx\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

Let's try to find a closed expression for this sum. We know:

$$\zeta\left(3\right)=\sum_{n=1}^{\infty}\frac{1}{n^3}$$

As you can see, our sum is equal to the even inverse cubes minus the odd ones. So, let's find an expression for the odd ones

$$\sum_{n=1}^{\infty}\frac{1}{\left(2n+1\right)^3}=\sum_{n=1}^{\infty}\left(\frac{1}{n^3}-\frac{1}{\left(2n\right)^3}\right)$$
$$=\frac{7\zeta\left(3\right)}{8}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n^3}=\sum_{n=1}^{\infty}\left(\frac{1}{\left(2n\right)^3}-\frac{1}{\left(2n+1\right)^3}\right)$$

Thus,

$$I=\frac{-3\zeta\left(3\right)}{4}$$