Multivariable calculus #1

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June 29, 2020

1 Introduction

In this post, we talk about local linearization, which is the analog to finding the tangent line to a single-variable function. Basically, we try to find the equation of the tangent **plane** to a multivariable function and use some vector notation to extend it to more dimensions. We will also discuss quadratic approximations next time, which is a bit harder.

2 Finding tangent planes

Before we try to figure out the tangent plane to a specific function, let's figure out the equation of a plane in 3D. The easiest way to represent a plane is

$$P(x, y) = ax + by + c$$

From this, we have

$$\frac{\partial P}{\partial x} = a \, , \; \frac{\partial P}{\partial y} = b$$

To guarantee that the plane passes through a specific point (x_0, y_0, z_0), let's change our equation a little bit

$$L(x, y) = a(x - x_0) + b(y - y_0) + c$$

So, if we wanted to find the tangent plane to $f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$, we could easily express it as

$$L_f(x, y) = a(x - x_0) + b(y - y_0) + f(x_0, y_0)$$

We can also see check that $\frac{\partial L_f}{\partial x} = a$ and $\frac{\partial L_f}{\partial y} = b$.Finally, we obtain the formula for a tangent plane:

$$L_f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

3 Generalization with vectors

The formula

$$L_f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

is also called the local linearization of f near (x_0, y_0). It satisfies 2 important properties. It has the same value and the same partial derivatives at (x_0, y_0) as f. We can see that it contains a constant term, the partial derivatives of f at (x_0, y_0) multiplied with a variable term minus the constant term respectively.
Now let $\mathbf{x_0} = (x_0, y_0)^T$ and $\mathbf{x} = (x, y)^T$. Then we can rewrite it as

$$L_f(x,y) = f(\mathbf{x_0}) + \nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0})$$

Where $\nabla f(\mathbf{x_0}) = (f_x(x_0,y_0), f_y(x_0,y_0))^T$ is the gradient of f at (x_0, y_0).

4 Example

Let's say we wanted to approximate

$$a = \sqrt{6.99 + \sqrt{2.01 + \sqrt{3.99}}}$$

We can let $f(x, y, z) = \sqrt{x + \sqrt{y + \sqrt{z}}}$ and find the local linearization of f near $(x_0) = (7, 2, 4)$. First, we evaluate the constant term

$$f(\mathbf{x_0}) = f(7, 2, 4) = \sqrt{7 + \sqrt{2 + \sqrt{4}}} = 3$$

Then we evaluate the partial derivatives and plug in the values

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)^T$$

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x + \sqrt{y + \sqrt{z}}}}, \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x + \sqrt{y + \sqrt{z}}}} \cdot \frac{1}{2\sqrt{y + \sqrt{z}}}$$

$$\frac{\partial f}{\partial z} = \frac{1}{2\sqrt{x + \sqrt{y + \sqrt{z}}}} \cdot \frac{1}{2\sqrt{y + \sqrt{z}}} \cdot \frac{1}{2\sqrt{z}}$$

$$\nabla f(\mathbf{x_0}) = \left(\frac{1}{6}, \frac{1}{24}, \frac{1}{96}\right)^T$$

Plugging everything in the formula,

$$L_f(x, y, z) = f(\mathbf{x_0}) + \nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0})$$

$$= f(\mathbf{x_0}) + f_x(\mathbf{x_0})(x - x_0) + f_y(\mathbf{x_0})(y - y_0) + f_z(\mathbf{x_0})(z - z_0)$$

$$= 3 + \frac{1}{6}(x - 7) + \frac{1}{24}(y - 2) + \frac{1}{96}(z - 4)$$

To approximate a, substitute $(x, y, z) = (6.99, 2.01, 3.99)$

$$L_f(x, y, z)) = 3 + \frac{1}{6}(6.99 - 7) + \frac{1}{24}(2.01 - 2) + \frac{1}{96}(3.99 - 4) \approx 2.9986458$$

Using a calculator, $a \approx 2.9986453$, and our approximation is very accurate.