When looking at this problem it seems natural to draw a parallel to the single variable differential equation approach to

$$\frac{dx}{dt} = Ax,\ x(t_0) = x_0$$

who's solution is given by

$$x(t) = e^{(t-t_0)A}x_0$$

However, this same operation generalizes for differential equations of vectors (as long as the dimensions of all the matrices allows for the multiplication of course). So we also have,

$$\frac{d\vec{x}}{dt} = A\vec{x},\ \vec{x}(t_0) = \vec{x}_0$$

who's solution is given by

$$\vec{x}(t) = e^{(t-t_0)A}\vec{x}_0$$

In this case we have $t_0 = 0$ so we just need to calculate

$$\vec{x}(t) = e^{At}\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

But how do we compute e to a matrix?? Well, we can use the power series definition of e^x to create a sum of powers of the matrix that will evaluate to it. So for a square matrix we can see

$$e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!}A^n = I + tA + \frac{t^2}{2!}A^2 + ...$$

So it seems clear we will need a formula for any power of A. We will need to diagonalize A. This is because for any diagonal matrix (say 3×3 in this case) we have

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}^n = \begin{bmatrix} a_1^n & 0 & 0 \\ 0 & a_2^n & 0 \\ 0 & 0 & a_3^n \end{bmatrix}$$

So if we write A as a product

$$A = PDP^{-1}$$

for a diagonal matrix D, then

$$A^n = PD^nP^{-1}$$

Lets get to finding eigenvalues and eigenvectors. First, we set

$$\det(A - \lambda I_3) = \begin{vmatrix} 6-\lambda & 3 & -2 \\ -4 & -1-\lambda & 2 \\ 13 & 9 & -3-\lambda \end{vmatrix} = 0$$

After suitable algebra we find solutions

$$\lambda \in \{1, 2, -1\}$$

Then, to find corresponding eigenvectors we solve the system

$$\begin{bmatrix} 6-\lambda & 3 & -2 \\ -4 & -1-\lambda & 2 \\ 13 & 9 & -3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

for each value of lambda and find the corresponding eigenvectors

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

So to create a product PDP^{-1} to represent A, we write D as a diagonal matrix whose entries are the eigenvalues of A, P as a matrix who's column vectors are the corresponding eigenvectors to A, and P^{-1} as the inverse to this matrix. So,

$$\begin{aligned} A &= PDP^{-1} \\ e^{At} &= Pe^{Dt}P^{-1} \\ &= \begin{bmatrix} 1 & -1 & 1/2 \\ -1 & 2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 5 & 3 & -1 \\ 1 & 1 & 0 \\ -6 & -4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5e^t - 2e^{2t} - 3e^{-t} & 3e^t - e^{2t} - 2e^{-t} & -e^t + e^{-t} \\ -5e^t + 2e^{2t} + 3e^{-t} & -3e^t + 2e^{2t} + 2e^{-t} & e^t - e^{-t} \\ 5e^t + e^{2t} - 6e^{-t} & 3e^t + e^{2t} - 4e^{-t} & -e^t + 2e^{-t} \end{bmatrix} \end{aligned}$$

Finally, since we have

$$\vec{x}(t) = e^{At} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

we can perform the multiplication to find the final solution to the differential equation,

$$\vec{x}(t) = \begin{bmatrix} -11e^t + e^{2t} + 8e^{-t} \\ 11e^t - 2e^{2t} - 8e^{-t} \\ -11e^t - e^{2t} + 16e^{-t} \end{bmatrix}$$