An Interesting Summation

Without Complex Analysis!

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi \coth(a\pi)}{a}$$

Swipe For EZ Derivation $\Rightarrow$

First we factor the expression in the denominator as

$$\frac{1}{n^2+a^2} = \frac{1}{(n+ai)(n-ai)} = \frac{1}{2ai}\left(\frac{1}{n-ai} - \frac{1}{n+ai}\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{2ai}\left(\sum_{n=1}^{\infty} \frac{1}{n-ai} - \frac{1}{n+ai}\right)$$

We can evaluate the inner summation by considering $\psi(1+ai)$ and $\psi(1-ai)$

$$\psi(1+ai) = -\gamma + \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+ai} \tag{1}$$

$$\psi(1-ai) = -\gamma + \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n-ai} \tag{2}$$

Doing (2) − (1) gives us

$$\psi(1+ai) - \psi(1-ai) = \sum_{n=1}^{\infty} \frac{1}{n-ai} - \frac{1}{n+ai}$$

Using the functional equation (derived last post)

$$\psi(1+ai) = \psi(ai) + \frac{1}{ai}$$

$$\sum_{n=1}^{\infty} \frac{1}{n-ai} - \frac{1}{n+ai} = \frac{1}{ai} + \psi(ai) - \psi(1-ai)$$

$$\sum_{n=1}^{\infty} \frac{1}{n-ai} - \frac{1}{n+ai} = \frac{1}{ai} + \psi(ai) - \psi(1-ai)$$

Using the reflection formula (derived last post), we would then have

$$\sum_{n=1}^{\infty} \frac{1}{n-ai} - \frac{1}{n+ai} = \frac{1}{ai} - \pi \cot(a\pi i)$$

Note the negative sign being there because the original reflection formula says

$$\psi(1-x) - \psi(x) = \pi \cot(\pi x)$$

Going back to the original sum, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{2ai}\left(\sum_{n=1}^{\infty} \frac{1}{n-ai} - \frac{1}{n+ai}\right) = \frac{1}{2ai}\left(\frac{1}{ai} - \pi \cot(a\pi i)\right)$$

Recalling $\coth(x) = i\cot(ix)$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{2ai}\left(\frac{1}{ai} - \frac{\pi \coth(a\pi)}{i}\right) = -\frac{1}{2a^2} + \frac{\pi \coth(a\pi)}{2a}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{a^2} + 2\left(-\frac{1}{2a^2} + \frac{\pi \coth(a\pi)}{2a}\right)$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi \coth(a\pi)}{a}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi \coth(a\pi)}{a}$$

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