

Solution to Mathemaddict's Problem

Jose Bedoya

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1 Question

$$I = \int_1^\infty \frac{2x\{x\} - \{x\}^2}{x^2\lfloor x \rfloor^2} dx = \frac{\pi^A}{B} - \phi^C$$

Then,

$$L = \lim_{\theta \rightarrow C} \frac{\sin(A\theta) \sec(\theta)}{B\theta} = ?$$

Note: $\{x\} = x - \lfloor x \rfloor$

2 Solution

Let's expand the integral using the definition of the fractional part

$$I = \int_1^\infty \frac{2x^2 - 2x\lfloor x \rfloor - x^2 + 2x\lfloor x \rfloor - \lfloor x \rfloor^2}{x^2\lfloor x \rfloor^2} dx$$

Simplifying and by the definition of the floor function

$$I = \sum_{k=1}^{\infty} \int_k^{k+1} \frac{1}{k^2} - \frac{1}{x^2} dx$$

Therefore,

$$\begin{aligned}
I &= \sum_{k=1}^{\infty} \left(\left. \frac{x}{k^2} \right|_k^{k+1} + \left. \frac{1}{x} \right|_k^{k+1} \right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k^2} + \frac{1}{k+1} - \frac{1}{k} \right)
\end{aligned}$$

Evaluating the series we get

$$I = \frac{\pi^2}{6} - 1$$

Clearly,

$$A = 2, B = 6, C = 0$$

Substituting in the limit, we get

$$\lim_{\theta \rightarrow 0} \frac{\sin(2\theta) \sec(\theta)}{6\theta} = \frac{1}{3}$$

Our final answer is

$$L = \frac{1}{3}$$