Calculus With Calvin QD. 27

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July 2, 2020

1 Question

Compute the Integral:

$$I = \int_0^1 \frac{\ln(1+x)\ln(x)}{x}\,dx$$

2 Solution

We know the Series Expansion:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$

$$\Rightarrow I = \int_0^1 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n-1}}{n}\ \ln(x)\,dx$$

$$\therefore I = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^1 x^{n-1}\ln(x)\,dx$$

We know the Formula,

$$I_{(m,n)} = \int_0^1 x^m \ln^n(x)\,dx = (-1)^n \frac{n!}{(m+1)^{n+1}}$$

[This can be derived using IBP and Recursion]

Comparing $I_{(m,n)}$ with I, we have,

$$m = n - 1$$

$$n = 1$$

$$\therefore I_{(n-1,1)} = \frac{(-1)}{n^2}$$

$$\begin{aligned}
\Rightarrow \quad I &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \times \frac{(-1)}{n^2} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}
\end{aligned}$$

Expanding this, we have:

$$\begin{aligned}
I &= -\frac{1}{1^3} + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - \dots \\
&= -\frac{1}{1^3} - \frac{1}{2^3} - \frac{1}{3^3} - \frac{1}{4^3} - \dots + 2\left[\frac{1}{2^3} + \frac{1}{4^3} + \dots\right] \\
&= -\zeta(3) + \left(\frac{1}{2^3 \cdot 1^3} + \frac{1}{2^3 \cdot 2^3} + \frac{1}{2^3 \cdot 3^3} + \dots\right) \\
&= -\zeta(3) + \frac{1}{4}\zeta(3)
\end{aligned}$$

Thus, the final answer is:

$$\boxed{I = -\frac{3\,\zeta(3)}{4}}$$