Solution to Calculus #1

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1 Problem

Evaluate the integral

$$\int_0^\infty \frac{1}{\sqrt[4]{x}\,(1+x^3)}dx$$

2 Solution

Consider the generalized integral where $\frac{m}{n} \notin \mathbb{Z}$

$$I(m,n) = \int_0^\infty \frac{x^{m-1}}{1+x^n}dx$$

Therefore our desired integral is $I(\frac{3}{4}, 3)$. Let

$$u = \frac{1}{1+x^n} \implies x = \left(\frac{1-u}{u}\right)^{\frac{1}{n}} \implies dx = -\left(\frac{1-u}{u}\right)^{\frac{1}{n}-1} \frac{du}{nu^2}$$

Besides, $x = \infty \implies u = 0$ and $x = 0 \implies u = 1$. Therefore,

$$\begin{aligned} I(m,n) &= \int_1^0 \left(\left(\frac{1-u}{u}\right)^{\frac{1}{n}}\right)^{m-1} \left(-u\left(\frac{1-u}{u}\right)^{\frac{1}{n}-1}\right) \frac{du}{nu^2} \\ &= \frac{1}{n}\int_0^1 u\left(\frac{1-u}{u}\right)^{\frac{m}{n}-1} du = \frac{1}{n}\int_0^1 u^{1-\frac{m}{n}-1}(1-u)^{\frac{m}{n}-1}du \end{aligned} \tag{1}$$

Recall the definition of the Beta function and its special property

$$\mathrm{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}$$

Hence the last integral in (1) becomes

$$\frac{B\left(1-\frac{m}{n}, \frac{m}{n}\right)}{n} = \frac{1}{n}\frac{\Gamma(1-\frac{m}{n})\cdot\Gamma(\frac{m}{n})}{\Gamma(1-\frac{m}{n}+\frac{m}{n})} = \frac{1}{n}\,\Gamma\left(1-\frac{m}{n}\right)\cdot\Gamma\left(\frac{m}{n}\right) \tag{2}$$

Recall the Euler's reflection formula

$$\Gamma(z)\cdot\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \qquad z \notin \mathbb{Z}$$

Given that $\frac{m}{n} \notin \mathbb{Z}$, we use the Euler's reflection formula for (2)

$$I(m,n) = \frac{\pi}{n\sin(\frac{m\pi}{n})}$$

Therefore, our desired integral

$$I\left(\frac{3}{4}, 3\right) = \frac{\pi}{3\sin\left(\frac{\frac{3}{4}\pi}{3}\right)} = \frac{\sqrt{2}\pi}{3}$$