The Di-Gamma Function

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$$

$$\psi(s) = -\gamma + H_{s-1}$$

Swipe For Derivations $\Rightarrow$

The Di-Gamma function is defined as the derivative of $\ln(\Gamma(s))$

$$\psi(s) = \frac{d}{dx}\ln(\Gamma(s)) = \frac{\Gamma'(s)}{\Gamma(s)}$$

To obtain an expression for it, let's use the Weirstrass Representation of $\Gamma(s)$

$$\Gamma(s) = \frac{e^{-\gamma s}}{s}\prod_{k=1}^{\infty} e^{s/k}\left(1+\frac{s}{k}\right)^{-1}$$

$$\ln(\Gamma(s)) = -\gamma s - \ln(s) + \sum_{k=1}^{\infty}\frac{s}{k} - \ln\left(1+\frac{s}{k}\right)$$

We can differentiate with respect to s on both sides now

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma - \frac{1}{s} + \sum_{k=1}^{\infty}\frac{1}{k} - \frac{\frac{1}{k}}{1+\frac{s}{k}} = -\gamma - \frac{1}{s} + \sum_{k=1}^{\infty}\frac{1}{k} - \frac{1}{k+s}$$

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma + \sum_{k=1}^{\infty}\frac{1}{k} - \frac{1}{k+s-1}$$

Notice that the $\frac{1}{s}$ is in the summation now (at $k=1$)

$$\psi(s) = -\gamma + \sum_{k=0}^{\infty}\frac{1}{k+1} - \frac{1}{k+s}$$

$$\psi(s) = -\gamma + H_{s-1}$$

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$$

$$\psi(s) = -\gamma + H_{s-1}$$

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