Solution to Shreenabh Agrawal's Daily Challenge 6

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$$\int_0^1 \ln x \ln(1-x)\, dx$$

Knowing,

$$\sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x}\,, \quad |x| < 1$$

Integrating both sides:

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x)$$

Therefore,

$$\int_0^1 \ln x \ln(1-x)\, dx = \int_0^1 \ln x \left(-\sum_{k=1}^{\infty} \frac{x^k}{k}\right) dx$$

Here we can change the order of integration and summation (think of it in an intuitive way)

$$= -\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 x^k \ln x\, dx$$

Applying IBP,

$$= -\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x^{k+1}\ln x}{k+1}\bigg|_0^1 - \frac{1}{k+1}\int_0^1 x^k\, dx \right)$$

$$= \sum_{k=1}^{\infty} \frac{1}{k(k+1)^2}$$

$$= \sum_{k=1}^{\infty} \frac{k+1-k}{k(k+1)^2} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} - \sum_{k=1}^{\infty} \frac{1}{(k+1)^2}$$

The first sum is telescopic, this means

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}\right)$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$$

As for the second sum, we know the Riemann Zeta Function evaluated at s is

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)^2} = \zeta(2) - 1$$

And,

$$\zeta(2) = \frac{\pi^2}{6}$$

Thus, our final answer is

$$\int_0^1 \ln x \ln(1-x)\, dx = 2 - \frac{\pi^2}{6}$$