Integral Fun Integral

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1 Question

$$I_{(\alpha,n)} = \int_0^\infty \frac{\cos(\alpha x)}{x^n}\,dx$$

2 Solution

We Know,

$$\int_0^\infty f(t)\,g(t)\,dt = \int_0^\infty \mathcal{L}\{f(t)\}\,\mathcal{L}^{-1}\{g(t)\}\,dt$$

Applying this,

$$\begin{aligned}
\therefore I_{(\alpha,n)} &= \int_0^\infty \mathcal{L}\{\cos(\alpha x)\}\,\mathcal{L}^{-1}\left\{\frac{1}{x^n}\right\}\,dx \\
&= \int_0^\infty \frac{s}{s^2+\alpha^2} \times \frac{s^{n-1}}{\Gamma(n)}\,ds \\
&= \frac{1}{\Gamma(n)} \int_0^\infty \frac{s^n}{s^2+\alpha^2}\,ds
\end{aligned}$$

Taking Substitution,

$$\begin{aligned}
s &= \alpha\ \tan\theta \\
ds &= \alpha\ \sec^2\theta\,d\theta
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad I_{(\alpha,n)} &= \frac{1}{\Gamma(n)} \int_0^{\pi/2} \frac{\alpha^n \tan^n\theta}{\alpha^2\sec^2\theta}\,\alpha\sec^2\theta\,d\theta \\
&= \frac{\alpha^{n-1}}{\Gamma(n)} \int_0^{\pi/2} \tan^n\theta\,d\theta
\end{aligned}$$

$$\boxed{I_{(\alpha,n)} = \frac{\pi\alpha^{n-1}}{2\Gamma(n)\cos\left(\frac{\pi n}{2}\right)}}$$