Solution to Number Theory # 1

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1 Problem

Let $S \subset \mathbb{N}$ such that $|S| = n$. Prove that $\exists A \subseteq S$ such that the sum of all elements in A are divisible by n.

2 Solution

Let $S = \{a_1, a_2, \cdots, a_n\}$. Then define the following sums:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$\vdots$$
$$S_n = a_1 + a_2 + \cdots + a_n$$

2.1 Case 1

If one of the above sums are divisible by n, then the claim is proved in this case.

2.2 Case 2

If none of the sums are divisible by n, then, by the Pigeonhole Principle, at least 2 of the sums must have the same remainder when divided by n. This implies that we can pick 2 sums S_b and S_c with $b > c$ such that

$$S_b - S_c = a_c + 1 + \cdots + a_b \equiv 0 (mod\, n)$$

Therefore the claim is proved in this case.