Solution to Calculus #2

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1 Problem

Evaluate the integral

$$\int_0^\infty \frac{(x^2-1)\ln x}{1+x^6}dx$$

2 Solution

In our last post, we had established this result for $\frac{m}{n} \notin \mathbb{Z}$

$$I(m,n) = \int_0^\infty \frac{x^{m-1}}{1+x^n} dx = \frac{\pi}{n \sin\left(\frac{m\pi}{m}\right)} = \frac{\pi}{n} \csc\left(\frac{m\pi}{n}\right) \tag{1}$$

Differentiating (1) with respect to m,

$$\frac{dI(m,n)}{dm} = \frac{d}{dm} \int_0^\infty \frac{x^{m-1}}{1+x^n} dx = \int_0^\infty \frac{\partial}{\partial m} \frac{x^{m-1}}{1+x^n} dx = \int_0^\infty \frac{x^{m-1} \ln x}{1+x^n} dx$$

We also have

$$\frac{dI(m,n)}{dm} = \frac{d}{dm}\left[\frac{\pi}{n} \csc\left(\frac{m\pi}{n}\right)\right] = -\frac{\pi^2}{n^2} \csc\left(\frac{m\pi}{n}\right) \cot\left(\frac{m\pi}{n}\right)$$

Rewriting our integral,

$$\begin{aligned}
\int_0^\infty \frac{(x^2-1)\ln x}{1+x^6} dx &= \int_0^\infty \frac{x^2 \ln x}{1+x^6} dx - \int_0^\infty \frac{\ln x}{1+x^6} dx \\
&= \frac{dI(3,6)}{dm} - \frac{dI(1,6)}{dm} \\
&= -\frac{\pi^2}{6^2} \csc\left(\frac{3\pi}{6}\right) \cot\left(\frac{3\pi}{6}\right) - \left(-\frac{\pi^2}{6^2} \csc\left(\frac{\pi}{6}\right) \cot\left(\frac{\pi}{6}\right)\right) \\
&= \frac{\pi^2}{6\sqrt{3}}
\end{aligned}$$

Using the Engineer's master theorem, let $\pi = 3$ and $\sqrt{3} = 1.5$.

$$= \frac{\pi^2}{6\sqrt{3}} = \frac{3^2}{6(1.5)} = 1$$

Therefore, we conclude that

$$\int_0^\infty \frac{(x^2-1)\ln x}{1+x^6} dx = 1$$