Mathemaddict Arctan Integral

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1 Question

Evaluate:

$$\int_0^\infty \frac{\tan^{-1}\left(\frac{3}{2x}\right) - \tan^{-1}\left(\frac{1}{x}\right)}{x}\,dx$$

2 Solution

Taking Substitution,

$$\frac{1}{x} = t$$
$$\frac{-1}{x^2}\,dx = dt$$

Hence, our Integral becomes,

$$I = \int_0^\infty \frac{\tan^{-1}\left(\frac{3t}{2}\right) - \tan^{-1}(t)}{t}\,dt$$

Let's derive a general formula for:-

$$I_{(a,b)} = \int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x}\,dx$$

Applying Leibniz Rule w.r.t Parameter a,

$$\frac{\partial I_{(a,b)}}{\partial a} = \int_0^\infty \frac{1}{1+a^2x^2}\,dx = \frac{\pi}{2a}$$

Integrating it back,

$$\therefore \quad I_{(a,b)} = \frac{\pi}{2}\ln(a) + c$$

If we evaluate the special case for $a = b$, we can calculate the value of c as,

$$c = -\frac{\pi}{2}\ln b$$

Hence,

$$I_{(a,b)} = \frac{\pi}{2}\ln a - \frac{\pi}{2}\ln b$$
$$I_{(a,b)} = \frac{\pi}{2}\ln\left(\frac{a}{b}\right)$$

Now substituting:-

$$a = \frac{3}{2}, b = 1$$

Thus,

$$\boxed{I_{\left(\frac{3}{2},1\right)} = \frac{\pi}{2}\ln\frac{3}{2}}$$