Calculus Q.2

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1 Question

Evaluate the Integral:

$$\int_0^\infty \frac{\left(x^2-1\right)\ln x}{1+x^6}\,dx$$

2 Solution

Taking Substitution,

$$x = \tan^{1/3}\theta$$
$$dx = \frac{1}{3}\tan^{-2/3}\theta\sec^2\theta\,d\theta$$

The Integral Becomes,

$$\Rightarrow I = \frac{1}{3}\int_0^{\pi/2} \frac{\left(\tan^{2/3}\theta - 1\right)\ln(\tan\theta)}{1+\tan^2\theta}\left(\frac{\tan^{-2/3}\theta\sec^2\theta\,d\theta}{3}\right)$$
$$= \frac{1}{9}\int_0^{\pi/2}\left(1-\tan^{-2/3}\theta\right)\ln\tan\theta\,d\theta$$

We know the Formula,

$$\int_0^{\pi/2} \tan^n(x)\log(\tan(x))dx = \pi^2\sin^3\left(\frac{\pi n}{2}\right)\csc^2(\pi n)$$

[For $-1 < n < 1$]

Using this, and Splitting the Integral,

$$\Rightarrow I = \frac{1}{9}\left[\int_0^{\pi/2}\ln(\tan\theta)\,d\theta - \int_0^{\pi/2}\left(\tan^{-2/3}\theta\right)\ln(\tan\theta)\,d\theta\right]$$
$$= \frac{1}{9}\left[0 - \pi^2\sin^3\left(\frac{\pi}{3}\right)\csc^2\left(\frac{\pi}{3}\right)\right]$$

Thus, the final answer is:

$$\boxed{I = \frac{\pi^2}{6\sqrt{3}}}$$