$$I = \int_0^\infty \frac{\ln x}{(1+x^2)(1+x^3)}\, dx$$

This may be a long one, but let's get into it. We break up the bounds of integration.

$$I = \int_0^1 \frac{\ln x}{(1+x^2)(1+x^3)}\, dx + \int_1^\infty \frac{\ln x}{(1+x^2)(1+x^3)}\, dx$$

Now we will introduce a substitution $x \to 1/x$ into the second integral,

$$\begin{aligned}
&= \int_0^1 \frac{\ln x}{(1+x^2)(1+x^3)}\, dx + \int_1^0 \frac{\ln(1/x)}{(1+(1/x)^2)(1+(1/x)^3)} \frac{-dx}{x^2} \\
&= \int_0^1 \frac{\ln x}{(1+x^2)(1+x^3)}\, dx - \int_0^1 \frac{x^3 \ln x}{(1+x^2)(1+x^3)}\, dx \\
&= \int_0^1 \frac{\ln x(1-x^3)}{(1+x^2)(1+x^3)}\, dx
\end{aligned}$$

We can now perform partial fractional decompostion on the rational multiple of $\ln x$ to show that

$$I = \underbrace{\int_0^1 \frac{x \ln x}{1+x^2}\, dx}_{I_1} - \frac{2}{3} \underbrace{\int_0^1 \frac{\ln x(2x-1)}{x^2-x+1}\, dx}_{I_2} + \frac{1}{3} \underbrace{\int_0^1 \frac{\ln x}{1+x}\, dx}_{I_3}$$

So,

$$\boxed{I = I_1 - \frac{2}{3} I_2 + \frac{1}{3} I_3} \tag{1}$$

Let's begin by solving I_1,

$$I_1 = \int_0^1 \frac{x \ln x}{1+x^2}\, dx$$

We introduce the substitution $u = x^2$,

$$\begin{aligned} I_1 &= \frac{1}{2}\int_0^1 \frac{\ln\sqrt{u}}{1+u}\, du \\ &\frac{1}{4}\int_0^1 \frac{\ln u}{1+u} = \end{aligned}$$

We can now integrate by parts,

D	I
$+\ln u$	$\frac{1}{1+u}$
$-\frac{1}{u}$	$\ln(1+u)$

to get

$$I_1 = \underbrace{\frac{1}{4}\ln u \ln(1+u)\Big|_0^1}_{=0} - \frac{1}{4}\int_0^1 \frac{\ln(1+u)}{u}\, du$$

We now let $u = -x$, and have

$$\begin{aligned} &= \frac{1}{4}\int_0^{-1} \frac{\ln(1-x)}{-x}\, dx \\ &= \frac{1}{4}\int_{-1}^{0} \frac{\ln(1-x)}{x}\, dx \\ &= \frac{1}{4}\mathrm{Li}_2(-1) = \frac{1}{4}\left(-\frac{\pi^2}{12}\right) = -\frac{\pi^2}{48} \end{aligned}$$

So,

$$\boxed{I_1 = -\frac{\pi^2}{48}} \tag{2}$$

Now we must calculate I_2, which will be slightly more difficult.

$$I_2 = \int_0^1 \frac{\ln x(2x-1)}{x^2-x+1}\,dx$$

We now once again integrate by parts

D	I
$+\ln x$	$\frac{2x-1}{x^2-x+1}$
$-\frac{1}{x}$	$\ln(x^2-x+1)$

$$\begin{aligned} I_2 &= \underbrace{\ln x \ln(x^2-x+1)\Big|_0^1}_{=0} - \int_0^1 \frac{\ln(x^2-x+1)}{x}\,dx \\ &= -\int_0^1 \frac{\ln(x^2-x+1)}{x}\,dx \end{aligned}$$

Lets now factorize the polynomial argument of the logarithm and see what we can do. We notice that

$$x^2-x+1=0 \implies x = \frac{1}{2} \pm \frac{\sqrt{3}}{2} = e^{\pm i\frac{\pi}{3}}$$

So we can factorize the polynomial

$$x^2-x+1=0=(x-e^{i\frac{\pi}{3}})(x-e^{-i\frac{\pi}{3}})$$

So our integral becomes

$$\begin{aligned}
I_2 &= -\int_0^1 \frac{\ln((x-e^{i\frac{\pi}{3}})(x-e^{-i\frac{\pi}{3}}))}{x}\,dx \\
&= -\int_0^1 \frac{\ln(x-e^{i\frac{\pi}{3}})}{x}\,dx - \int_0^1 \frac{\ln(x-e^{-i\frac{\pi}{3}})}{x}\,dx
\end{aligned}$$

Let's simplify with some algebra and log properties.

$$\begin{aligned}
&= -\int_0^1 \frac{\ln(-e^{i\frac{\pi}{3}}(-e^{-i\frac{\pi}{3}}x+1))}{x}\,dx - \int_0^1 \frac{\ln(-e^{-i\frac{\pi}{3}}(-e^{i\frac{\pi}{3}}x+1))}{x}\,dx \\
&= -\int_0^1 \frac{\ln(-e^{-i\frac{\pi}{3}}x+1)}{x}\,dx - \int_0^1 \frac{\ln(-e^{i\frac{\pi}{3}}x+1)}{x}\,dx - \cancel{\left[\ln(-e^{-i\frac{\pi}{3}}) + \ln(-e^{i\frac{\pi}{3}})\right]}\int_0^1 \frac{dx}{x} \\
&= \int_1^0 \frac{\ln(1-e^{-i\frac{\pi}{3}}x)}{x}\,dx \int_1^0 \frac{\ln(1-e^{i\frac{\pi}{3}}x)}{x}\,dx
\end{aligned}$$

Where the last term cancels because the coefficient of the integral is 0, as can be shown with logarithm properties. So now let's the substitution $e^{-i\frac{\pi}{3}}x \to x$ in the first integral and $e^{i\frac{\pi}{3}}x \to x$ in the second to get

$$\begin{aligned}
I_2 &= \int_{e^{-i\frac{\pi}{3}}}^0 \frac{\ln(1-x)}{x}\,dx + \int_{e^{i\frac{\pi}{3}}}^0 \frac{\ln(1-x)}{x}\,dx \\
&= \mathrm{Li}_2(e^{i\frac{\pi}{3}}) + \mathrm{Li}_2\left(\frac{1}{e^{i\frac{\pi}{3}}}\right)
\end{aligned}$$

We can now apply the dilogarithm identity

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(-z)$$

So

$$I_2 = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(-e^{i\frac{\pi}{3}}) = -\frac{\pi^2}{6} - \frac{1}{2}\left(-\frac{2i\pi}{3}\right)^2 = \frac{\pi^2}{18}$$

And we have

$$\boxed{I_2 = \frac{\pi^2}{18}} \tag{3}$$

Finally, we evaluate I_3 which is the simplest of the 3.

$$I_3 = \int_0^1 \frac{\ln x}{1+x}\,dx$$

We will once again integrate by parts

D	I
$+\ln u$	$\frac{1}{1+u}$
$-\frac{1}{u}$	$\ln(1+u)$

$$\begin{aligned} I_3 &= \underbrace{\ln u \ln(1+u)\Big|_0^1}_{=0} - \int_0^1 \frac{\ln(1+u)}{u}\,du \\ &= \int_1^0 \frac{\ln(1+u)}{u}\,du \end{aligned}$$

Now letting $u = -x$,

$$= \int_{-1}^{0} \frac{\ln(1-x)}{x}\,dx = \mathrm{Li}_2(-1) = -\frac{\pi^2}{12}$$

And therefore

$$\boxed{I_3 = -\frac{\pi^2}{12}} \tag{4}$$

We can now put everything together from equations (1), (2), (3), and (4) to show

$$I = -\frac{\pi^2}{48} - \frac{2}{3}\left(\frac{\pi^2}{18}\right) + \frac{1}{3}\left(-\frac{\pi^2}{12}\right)$$

and we have the final result,

$$\boxed{\int_0^\infty \frac{\ln x}{(1+x^2)(1+x^3)}\,dx = -\frac{37\pi^2}{432}}$$