A Guided Tour of Chapter 10: Reinforcement Learning for Control

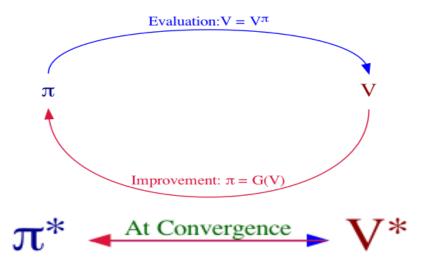
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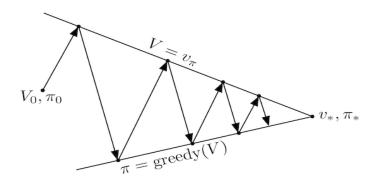
RL does not have access to a probability model

- ullet DP/ADP assume access to probability model (knowledge of \mathcal{P}_R)
- Often in real-world, we do not have access to these probabilities
- Which means we'd need to interact with the actual environment
- Actual Environment serves up individual experiences, not probabilities
- Even if MDP model is available, model updates can be challenging
- Often real-world models end up being too large or too complex
- Sometimes estimating a sampling model is much more feasible
- So RL interacts with either actual or simulated environment
- Either way, we receive individual experiences of next state and reward
- We saw how RL Prediction learns from individual experiences
- Now we extend those ideas to RL Control: Learning Optimal VF
- We start with Tabular RL Control

Let us recall the Policy Iteration algorithm



The idea of Generalized Policy Iteration (GPI)



- Any Policy Evaluation method, Any Policy Improvement method
- ullet Policy Evaluation estimates $V^{(\pi)}$, eg: Iterative Policy Evaluation
- Policy Improvement produces $\pi' \geq \pi$, eg: Greedy Policy Improvement
- Policy Evaluation and Policy Improvement alternate until convergence

Natural Idea: GPI with Tabular Monte-Carlo Evaluation

- Let us explore GPI with Tabular Monte-Carlo evaluation
- So we will do Policy Evaluation with Tabular MC evaluation
- And we will do the usual Greedy Policy Improvement
- But Greedy Policy Improvement requires a model of MDP

$$\pi'(s) \leftarrow \argmax_{a \in \mathcal{A}} \{ \mathcal{R}(s, a) + \gamma \sum_{s' \in \mathcal{N}} \mathcal{P}(s, a, s') \cdot V^{\pi}(s') \}$$

However, it works if we were working with Action-Value Function

$$\pi'(s) \leftarrow \arg\max_{a \in \mathcal{A}} Q^{\pi}(s, a)$$

- ullet This means: Policy Evaluation for Action-Value Function $Q^\pi(s,a)$
- Following a policy π , update Q-value for each (S_t, A_t) each episode:

$$Count(S_t, A_t) \leftarrow Count(S_t, A_t) + 1$$

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{1}{Count(S_t, A_t)} \cdot (G_t - Q(S_t, A_t))$$

ϵ-Greedy Policy Improvement

- A full Policy Evaluation with MC takes too long
- So we typically improve policy after each episode
- This can lead to some actions not being tried enough
- Which can lead to premature (greedy) domination of an action
- Which can lead to other actions getting "locked-out"
- Same as Explore v/s Exploit dilemma of Multi-Armed Bandit problem
- Simple solution: Perform an ϵ -Greedy Policy Improvement
- ullet All $|\mathcal{A}|$ actions are tried with non-zero probability (for each state)
- ullet Pick the greedy action with probability $1-\epsilon$
- ullet With probability ϵ , randomly choose one of the $|\mathcal{A}|$ actions

Stochastic Policy
$$\pi(s,a) = \begin{cases} \frac{\epsilon}{|\mathcal{A}|} + 1 - \epsilon & \text{if } a = \arg\max_{b \in \mathcal{A}} Q(s,b) \\ \frac{\epsilon}{|\mathcal{A}|} & \text{otherwise} \end{cases}$$

ϵ -Greedy improves the policy

Theorem

For any ϵ -greedy policy π , the ϵ -greedy policy π' with respect to Q^{π} is an improvement, i.e., $\boldsymbol{V}^{\pi'}(s) \geq \boldsymbol{V}^{\pi}(s)$ for all $s \in \mathcal{N}$.

• Applying ${m B}^{\pi'}$ repeatedly (starting with ${m V}^{\pi}$) converges to ${m V}^{\pi'}$:

$$\lim_{i o \infty} ({m{B}}^{\pi'})^i({m{V}}^\pi) = {m{V}}^{\pi'}$$

• So the proof is complete if we prove that:

$$(oldsymbol{\mathcal{B}}^{\pi'})^{i+1}(oldsymbol{\mathcal{V}}^{\pi}) \geq (oldsymbol{\mathcal{B}}^{\pi'})^{i}(oldsymbol{\mathcal{V}}^{\pi})$$
 for all $i=0,1,2,\ldots$

• Increasing tower of Value Functions $[({m B}^{\pi'})^i({m V}^\pi)|i=0,1,2,\ldots]$ with repeated applications of ${m B}^{\pi'}$

Proof of ϵ -Greedy improving the policy

To prove base case (proof by induction), note: $m{B}^{\pi'}(m{V}^{\pi})(s) = Q^{\pi}(s,\pi'(s))$

$$\begin{split} Q^{\pi}(s, \pi'(s)) &= \sum_{a \in \mathcal{A}} \pi'(s, a) \cdot Q^{\pi}(s, a) \\ &= \frac{\epsilon}{|\mathcal{A}|} \cdot \sum_{a \in \mathcal{A}} Q^{\pi}(s, a) + (1 - \epsilon) \cdot \max_{a \in \mathcal{A}} Q^{\pi}(s, a) \\ &\geq \frac{\epsilon}{|\mathcal{A}|} \cdot \sum_{a \in \mathcal{A}} Q^{\pi}(s, a) + (1 - \epsilon) \cdot \sum_{a \in \mathcal{A}} \frac{\pi(s, a) - \frac{\epsilon}{|\mathcal{A}|}}{1 - \epsilon} \cdot Q^{\pi}(s, a) \\ &= \sum_{a \in \mathcal{A}} \pi(s, a) \cdot Q^{\pi}(s, a) = \boldsymbol{V}^{\pi}(s) \end{split}$$

Induction step is proved by monotonicity of \mathbf{B}^{π} operator (for any π):

Monotonicity Property of ${m B}^\pi: {m X} \geq {m Y} \Rightarrow {m B}^\pi({m X}) \geq {m B}^\pi({m Y})$

So
$$({m B}^{\pi'})^{i+1}({m V}^\pi) \geq ({m B}^{\pi'})^i({m V}^\pi) \Rightarrow ({m B}^{\pi'})^{i+2}({m V}^\pi) \geq ({m B}^{\pi'})^{i+1}({m V}^\pi)$$

GLIE

Definition

Greedy in the Limit with Infinite Exploration (GLIE):

All state-action pairs are explored infinitely many times

$$\lim_{k\to\infty} N_k(s,a) = \infty$$

• The policy converges to a greedy policy

$$\lim_{k \to \infty} \pi_k(s, a) = \mathbb{I}_{a = \operatorname{arg} \max_{b \in \mathcal{A}} Q(s, b)}$$

 ϵ -greedy can be made GLIE if ϵ is reduced as: $\epsilon_k = \frac{1}{k}$

GLIE Tabular Monte-Carlo Control

- Sample k-th episode using π : $\{S_0, A_0, R_1, S_1, A_1, \dots, R_T, S_T\} \sim \pi$
- For each state S_t and action A_t in the episode:

$$Count(S_t, A_t) \leftarrow Count(S_t, A_t) + 1$$

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{1}{Count(S_t, A_t)} \cdot (G_t - Q(S_t, A_t))$$

Improve policy at end of episode based on updated Q-Value function:

$$\begin{aligned} \epsilon &\leftarrow \frac{1}{k} \\ \pi &\leftarrow \epsilon\text{-greedy}(Q) \end{aligned}$$

Theorem

GLIE Tabular Monte-Carlo Control converges to the Optimal Action-Value function: $Q(s, a) \rightarrow Q^*(s, a)$.

MC versus TD Control

- TD learning has several advantages over MC learning:
 - Lower variance
 - Online
 - Can work with incomplete traces or continuing traces
 - Generic interface of Iterable of atomic experiences allows for serving up atomic experiences in any order (eg: atomic experience replays)
- So use TD instead of MC in our Control loop
 - Apply TD to Q(S, A) (instead of V(S))
 - Use ϵ -greedy Policy Improvement
 - Update Q(S, A) after each atomic experience
 - \bullet $\epsilon\text{-greedy}$ policy automatically updated after each atomic experience

Tabular SARSA Algorithm

- Tabular SARSA is our first TD Control algorithm
- Like Tabular MC Control, Policy Improvement is ϵ -greedy
- But here Policy Evaluation is with a TD target, as below:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \cdot (R_{t+1} + \gamma \cdot Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t))$$

- Note that Q(S, A) is updated after each atomic experience
- ullet ϵ -greedy policy automatically updated after each atomic experience
- Action A_t is chosen from State S_t based on ϵ -greedy policy
- Action A_{t+1} is chosen from State S_{t+1} based on ϵ -greedy policy
- Note: Instead of ϵ -greedy, we could employ a more sophisticated exploratory policy derived from Q-value function (ϵ -greedy is just our default simple exploratory policy derived from Q-value function)

SARSA Visualization



Convergence of Tabular SARSA

Theorem

Tabular SARSA converges to the Optimal Action-Value function, $Q(s, a) \rightarrow Q^*(s, a)$, under the following conditions:

- GLIE sequence of policies $\pi_t(s, a)$
- ullet Robbins-Monro sequence of step-sizes $lpha_t$

$$\sum_{t=1}^{\infty} \alpha_t = \infty$$

$$\sum_{t=1}^{\infty} \alpha_t^2 < \infty$$

Tabular *n*-step SARSA

Tabular SARSA bootstraps the Q-Value Function with update:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t))$$

• So it's natural to extend this to bootstrapping with 2 steps ahead:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma R_{t+2} + \gamma^2 Q(S_{t+2}, A_{t+2}) - Q(S_t, A_t))$$

• Generalize to bootstrapping with $n \ge 1$ time steps ahead:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(G_{t,n} - Q(S_t, A_t))$$

• $G_{t,n}$ (call it *n*-step bootstrapped return) is defined as:

$$G_{t,n} = \sum_{i=t+1}^{t+n} \gamma^{i-t-1} \cdot R_i + \gamma^n \cdot Q(S_{t+n}, A_{t+n})$$

= $R_{t+1} + \gamma \cdot R_{t+2} + \ldots + \gamma^{n-1} \cdot R_{t+n} + \gamma^n \cdot Q(S_{t+n}, A_{t+n})$

Tabular λ -Return SARSA

• Instead of $G_{t,n}$, a valid target is a weighted-average target:

$$\sum_{n=1}^{N} u_n \cdot G_{t,n} + u \cdot G_t \text{ where } u + \sum_{n=1}^{N} u_n = 1$$

- ullet Any of the u_n or u can be 0, as long as they all sum up to 1
- The λ -Return target is a special case of weights u_n and u

$$u_n = (1 - \lambda) \cdot \lambda^{n-1}$$
 for all $n = 1, \dots, T - t - 1$ $u_n = 0$ for all $n \ge T - t$ and $u = \lambda^{T - t - 1}$

• We denote the λ -Return target as $G_t^{(\lambda)}$, defined as:

$$G_t^{(\lambda)} = (1 - \lambda) \cdot \sum_{n=1}^{T-t-1} \lambda^{n-1} \cdot G_{t,n} + \lambda^{T-t-1} \cdot G_t$$
$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \cdot (G_t^{(\lambda)} - Q(S_t, A_t))$$

Tabular SARSA (λ)

- ullet λ can be tuned from SARSA ($\lambda=0$) to MC Control ($\lambda=1$)
- Note that for $\lambda > 0$, λ -Return SARSA is an Offline Algorithm
- SARSA(λ) is online "version" of λ -Return SARSA
- Similar to $TD(\lambda)$ for Prediction, $SARSA(\lambda)$ uses Eligibility Traces
- Eligibility Trace for a given trace experience at time t is a function

$$E_t: \mathcal{N} imes \mathcal{A}
ightarrow \mathbb{R}_{\geq 0}$$

$$E_0(s,a) = 0$$
, for all $s \in \mathcal{N}, a \in \mathcal{A}$

$$E_t(s,a) = \gamma \cdot \lambda \cdot E_{t-1}(s,a) + \mathbb{I}_{S_t=s,A_t=a}, \text{ for all } s \in \mathcal{N}, a \in \mathcal{A}, \text{ for all } t$$

• Tabular SARSA(λ) performs following update at each time step t in each trace experience (for each $s \in \mathcal{N}, a \in \mathcal{A}$):

$$Q(s, a) \leftarrow Q(s, a) + \alpha \cdot (R_{t+1} + \gamma \cdot Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)) \cdot E_t(s, a)$$

Off-Policy Learning

ullet Estimate VF for target policy π while following behavior policy μ

$$\{S_0, A_0, R_1, S_1, A_1, \dots, R_T, S_T\} \sim \mu$$

- Why is this important?
 - Learning from observing humans or other agents
 - Re-use experience generated from old policies $\pi_1, \pi_2, \dots, \pi_{t-1}$
 - Learn about optimal policy while following exploratory policy
 - Learn about multiple policies while following one policy

Importance Sampling for Off-Policy Learning

- Importance Sampling refers to methods to estimate properties of a distribution P, given access to samples from a different distribution Q
- We can calculate $\mathbb{E}_{X \sim P}[f(X)]$ given samples from Q as follows:

$$\mathbb{E}_{X \sim P}[f(X)] = \sum_{X \sim P} P(X) \cdot f(X)$$

$$= \sum_{X \sim Q} Q(X) \cdot \frac{P(X)}{Q(X)} \cdot f(X)$$

$$= \mathbb{E}_{X \sim Q} \left[\frac{P(X)}{Q(X)} \cdot f(X) \right]$$

Importance Sampling for Off-Policy Monte-Carlo

- ullet Use returns generated from μ to estimate Value Function for π
- \bullet Weight return G_t according to similarity between policies
- Multiply importance sampling corrections along whole episode

$$G_t^{\pi/\mu} = \frac{\pi(S_t, A_t)}{\mu(S_t, A_t)} \cdot \frac{\pi(S_{t+1}, A_{t+1})}{\mu(S_{t+1}, A_{t+1})} \cdots \frac{\pi(S_T, A_t)}{\mu(S_T, A_T)} \cdot G_t$$

Update value towards corrected return

$$V(S_t) \leftarrow V(S_t) + \alpha \cdot (G_t^{\pi/\mu} - V(S_t))$$

- Likewise for *Q*-Value Function for MC Control
- ullet Note: We cannot use this method if μ is zero when π is non-zero
- Importance sampling can dramatically increase variance

Importance Sampling for Off-Policy Temporal-Difference

- ullet Use TD targets generated from μ to evaluate Value Function for π
- Weight TD target $R + \gamma \cdot V(S')$ with importance sampling
- Here we only need a single importance sampling correction

$$V(S_t) \leftarrow V(S_t) + \alpha \cdot \left(\frac{\pi(S_t, A_t)}{\mu(S_t, A_t)} \cdot (R_{t+1} + \gamma \cdot V(S_{t+1})) - V(S_t)\right)$$

- Likewise for Q-Value Function for TD Control
- This has much lower variance than MC importance sampling
- Policies only need to be similar over a single step

Tabular Q-Learning

- We now consider off-policy learning of action-values Q(s, a)
- No importance sampling is required
- The next action is chosen using behavior policy $A_{t+1} \sim \mu(S_t, \cdot)$
- But we consider an alternative successor action ${\it A}' \sim \pi({\it S}_t,\cdot)$
- Update $Q(S_t, A_t)$ towards value of alternative action

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \cdot (R_{t+1} + \gamma \cdot Q(S_{t+1}, A') - Q(S_t, A_t))$$

Off-Policy Control with Tabular Q-Learning

- We now allow both behavior and target policies to improve
- The target (deterministic) policy π_D is **greedy** w.r.t *Q*-Value Function

$$\pi_D(S_{t+1}) = \operatorname*{arg\,max}_{a' \in \mathcal{A}} Q(S_{t+1}, a')$$

- The behavior policy μ is also improving, eg: ϵ -greedy w.r.t. Q
- The Q-learning target then simplifies to:

$$R_{t+1} + \gamma \cdot Q(S_{t+1}, A')$$

$$= R_{t+1} + \gamma \cdot Q(S_{t+1}, \arg \max_{a' \in \mathcal{A}} Q(S_{t+1}, a'))$$

$$= R_{t+1} + \gamma \cdot \max_{a' \in \mathcal{A}} Q(S_{t+1}, a')$$

• Thus the update to Q-Value Function after each atomic experience is:

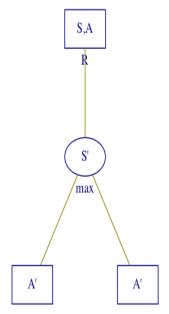
$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \cdot (R_{t+1} + \gamma \cdot \max_{a' \in A} Q(S_{t+1}, a') - Q(S_t, A_t))$$

Tabular Q-Learning Control Algorithm

Theorem

Tabular Q-Learning Control converges to the Optimal Action-Value Function, $Q(s,a) \rightarrow Q^*(s,a)$

Q-Learning Control Visualization



RL Control with Function Approximation

- Natural extension of Tabular Control to function approximations
- Instead of parameterized approximation $V(s; \boldsymbol{w})$, we have $Q(s, a; \boldsymbol{w})$
- GPI's Policy Evaluation is Approximate Q-Value Policy Evaluation
- Loss Function for Q(s, a; w) estimate relative to appropriate return
- For MC Control, the loss function for (S_t, A_t) in an episode is:

$$\mathcal{L}_{(S_t,A_t,G_t)}(\boldsymbol{w}) = \frac{1}{2} \cdot (Q(S_t,A_t;\boldsymbol{w}) - G_t)^2$$

• Therefore, the gradient of the loss function for this (S_t, A_t) is:

$$\nabla_{\boldsymbol{w}} \mathcal{L}_{(S_t, A_t, G_t)}(\boldsymbol{w}) = (Q(S_t, A_t; \boldsymbol{w}) - G_t) \cdot \nabla_{\boldsymbol{w}} Q(S_t, A_t; \boldsymbol{w})$$

• Therefore, parameters update for this (S_t, A_t) at episode-end is:

$$\Delta \mathbf{w} = \alpha \cdot (G_t - Q(S_t, A_t; \mathbf{w})) \cdot \nabla_{\mathbf{w}} Q(S_t, A_t; \mathbf{w})$$

ullet ϵ -greedy policy improvement is automatic from updated Q estimate

Bootstrapped Control with Function Approximation

• For SARSA, parameters update after each atomic experience:

$$\Delta \mathbf{w} = \alpha \cdot (R_{t+1} + \gamma \cdot Q(S_{t+1}, A_{t+1}; \mathbf{w}) - Q(S_t, A_t; \mathbf{w})) \cdot \nabla_{\mathbf{w}} Q(S_t, A_t; \mathbf{w})$$

• For Q-Learning, parameters update after each atomic experience:

$$\Delta \mathbf{w} = \alpha \cdot (R_{t+1} + \gamma \cdot \max_{\mathbf{a}' \in \mathcal{A}} Q(S_{t+1}, \mathbf{a}'; \mathbf{w}) - Q(S_t, A_t; \mathbf{w})) \cdot \nabla_{\mathbf{w}} Q(S_t, A_t; \mathbf{w})$$

• For offline λ -Return Control, parameters update at episode-end:

$$\Delta \mathbf{w} = \alpha \cdot (G_t^{(\lambda)} - Q(S_t, A_t; \mathbf{w})) \cdot \nabla_{\mathbf{w}} Q(S_t, A_t; \mathbf{w})$$

ullet For online SARSA(λ), update after each atomic experience:

$$\delta_t = R_{t+1} + \gamma \cdot Q(S_{t+1}, A_{t+1}; \mathbf{w}) - Q(S_t, A_t; \mathbf{w})$$

$$\mathbf{E}_t = \gamma \lambda \cdot \mathbf{E}_{t-1} + \nabla_{\mathbf{w}} Q(S_t, A_t; \mathbf{w})$$

$$\Delta \mathbf{w} = \alpha \cdot \delta_t \cdot \mathbf{E}_t$$

ullet ϵ -greedy policy improvement is automatic from updated Q estimate

Relationship between DP and TD

	Full Backup (DP)	Sample Backup (TD)
Bellman Expectation	$\begin{array}{c} (-V_{0}) \\ (a_{1}V_{0}) \\ (b_{2}V_{0}) \\ (b_{3}V_{0}) \\ (b_$	8
Equation for $V^{\pi}(s)$	Iterative Policy Evaluation	TD Learning
Bellman Expectation Equation for $Q^{\pi}(s,a)$		s A SARSA
Bellman Optimality Equation for $Q^*(s, a)$	$\begin{array}{c} \begin{array}{ccccccccccccccccccccccccccccccccc$	v V Q-Learning

Relationship between DP and TD

Full Backup (DP)	Sample Backup (TD)		
Iterative Policy Evaluation: $V(S)$ update	TD Learning: $V(S)$ update		
$\mathbb{E}[R + \gamma V(S') S]$	sample $R + \gamma V(S')$		
Q-Policy Evaluation: $Q(S, A)$ update	SARSA: $Q(S, A)$ update		
$\mathbb{E}[R + \gamma Q(S', A') S, A]$	sample $R + \gamma Q(S', A')$		
Q-Value Iteration: $Q(S, A)$ update	Q-Learning: $Q(S, A)$ update		
$\mathbb{E}[R + \gamma \max_{a'} Q(S', a') S, A]$	sample $R + \gamma \max_{a'} Q(S', a')$		

Convergence of Prediction Algorithms

On/Off Policy	Algorithm	Tabular	Linear	Non-Linear
On-Policy	MC	✓	✓	✓
	TD(0)	✓	✓	X
	$TD(\lambda)$	\checkmark	✓	X
Off-Policy	MC	✓	✓	1
	TD(0)	✓	X	X
	$TD(\lambda)$	✓	X	X

Gradient Temporal-Difference Learning

- TD does not follow the gradient of any objective function
- This is why TD can diverge:
 - when running off-policy, or
 - when using non-linear function approximation
- Gradient TD follows true gradient of projected Bellman Error

On/Off Policy	Algorithm	Tabular	Linear	Non-Linear
	MC	✓	✓	✓
On-Policy	TD	✓	✓	×
	Gradient TD	✓	✓	\checkmark
	MC	✓	✓	✓
Off-Policy	TD	✓	X	×
	Gradient TD	✓	✓	✓

Convergence of Control Algorithms

Algorithm	Tabular	Linear	Non-Linear
MC Control	✓	(✓)	Х
SARSA	✓	(✓)	X
Q-Learning	✓	X	×
Gradient Q-Learning	✓	✓	X

(\checkmark) means it chatters around near-optimal Value Function

Key Takeaways from this Chapter

- RL Control is based on the idea of Generalized Policy Iteration (GPI)
 - Policy Evaluation with Q-Value Function (instead of V)
 - Improved Policy needs to be exploratory, eg: ϵ -greedy
- On-Policy versus Off-Policy
- Deadly Triad := [Bootstrapping, Function Approximation, Off-Policy]