Stochastic Control for Optimal Trade Order Execution

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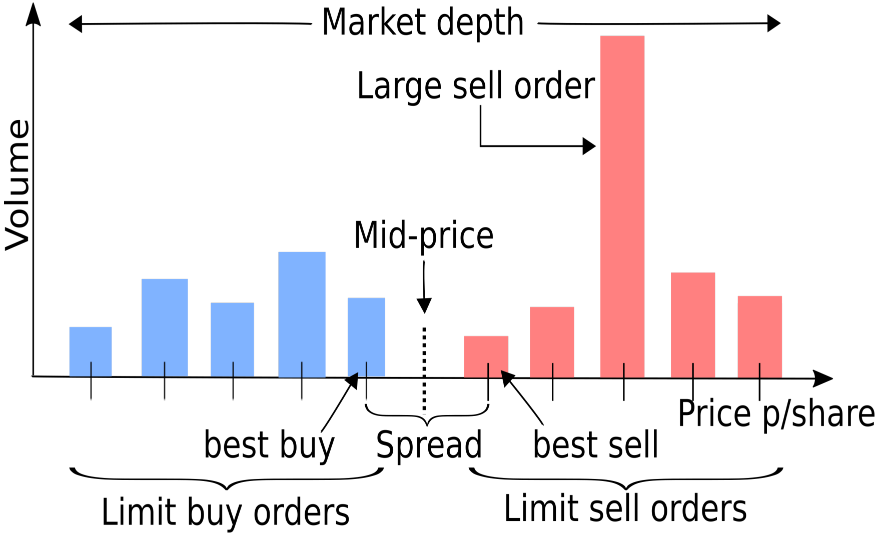
Overview

Trading Order Book (TOB)



Basics of Trading Order Book (TOB)

- Buyers/Sellers express their intent to trade by submitting bids/asks
- These are Limit Orders (LO) with a price P and size N
- Buy LO (P, N) states willingness to buy N shares at a price $\leq P$
- Sell LO (P, N) states willingness to sell N shares at a price $\geq P$
- Trading Order Book aggregates order sizes for each unique price
- So we can represent with two sorted lists of (Price, Size) pairs

$$\text{Bids: } [(P_i^{(b)}, N_i^{(b)}) \mid 1 \leq i \leq m], P_i^{(b)} > P_j^{(b)} \text{ for } i < j$$

$$\text{Asks: } [(P_i^{(a)}, N_i^{(a)}) \mid 1 \leq i \leq n], P_i^{(a)} < P_j^{(a)} \text{ for } i < j$$

- We call $P_1^{(b)}$ as simply *Bid*, $P_1^{(a)}$ as *Ask*, $\frac{P_1^{(a)} + P_1^{(b)}}{2}$ as *Mid*
- We call $P_1^{(a)} - P_1^{(b)}$ as *Spread*, $P_n^{(a)} - P_m^{(b)}$ as *Market Depth*
- A Market Order (MO) states intent to buy/sell N shares at the *best possible price(s)* available on the TOB at the time of MO submission

Trading Order Book (TOB) Activity

- A new Sell LO (P, N) potentially removes best bid prices on the TOB

$$\text{Removal: } [(P_i^{(b)}, \min(N_i^{(b)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(b)}))) \mid (i : P_i^{(b)} \geq P)]$$

- After this removal, it adds the following to the asks side of the TOB

$$(P, \max(0, N - \sum_{i:P_i^{(b)} \geq P} N_i^{(b)}))$$

- A new Buy MO operates analogously (on the other side of the TOB)
- A Sell Market Order N will remove the best bid prices on the TOB

$$\text{Removal: } [(P_i^{(b)}, \min(N_i^{(b)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(b)}))) \mid 1 \leq i \leq m]$$

- A Buy Market Order N will remove the best ask prices on the TOB

$$\text{Removal: } [(P_i^{(a)}, \min(N_i^{(a)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(a)}))) \mid 1 \leq i \leq n]$$

Price Impact and Trading Order Book Dynamics

- We focus on how a Market order (MO) alters the TOB
- A large-sized MO often results in a big *Spread* which could soon be replenished by new LOs, potentially from either side
- So a large-sized MO moves the Bid/Ask/Mid (*Price Impact* of MO)
- Subsequent Replenishment activity is part of *TOB Dynamics*
- Models for TOB Dynamics can be quite complex
- We will cover a few simple Models in this lecture
- Models based on how a Sell MO will move the TOB *Bid Price*
- Models of Buy MO moving the TOB *Ask Price* are analogous

Optimal Trade Order Execution Problem

- The task is to sell a large number N of shares
- We are allowed to trade in T discrete time steps
- We are only allowed to submit Market Orders
- We consider both *Temporary* and *Permanent* Price Impact
- For simplicity, we consider a model of just the *Bid Price* Dynamics
- Goal is to maximize Expected Total Utility of Sales Proceeds
- By breaking N into appropriate chunks (timed appropriately)
- If we sell too fast, we are likely to get poor prices
- If we sell too slow, we risk running out of time
- Selling slowly also leads to more uncertain proceeds (lower Utility)
- This is a Dynamic Optimization problem
- We can model this problem as a Markov Decision Process (MDP)

Problem Notation

- Time steps indexed by $t = 1, \ldots, T$
- P_t denotes Bid Price at start of time step t
- N_t denotes number of shares sold in time step t
- $R_t = N - \sum_{i=1}^{t-1} N_i$ = shares remaining to be sold at start of time step t
- Note that $R_1 = N, N_T = R_T$
- Price Dynamics given by:

$$P_{t+1} = f_t(P_t, N_t, \epsilon_t)$$

  where $f_t(\cdot)$ is an arbitrary function incorporating:
  - Permanent Price Impact of selling N_t shares
  - Impact-independent market-movement of Bid Price over time step t
  - ϵ_t denotes source of randomness in Bid Price market-movement
- Sales Proceeds in time step t defined as:

$$N_t \cdot Q_t = N_t \cdot (P_t - g_t(P_t, N_t))$$

  where $g_t(\cdot)$ is an arbitrary func representing Temporary Price Impact
- Utility of Sales Proceeds function denoted as $U(\cdot)$

Markov Decision Process (MDP) Formulation

- Order of MDP activity in each time step $1 \leq t \leq T$:
  - Observe *State* $:= (t, P_t, R_t)$
  - Perform *Action* $:= N_t$
  - Receive *Reward* $:= U(N_t \cdot Q_t) = U(N_t \cdot (P_t - g_t(P_t, N_t)))$
  - Experience Price Dynamics $P_{t+1} = f_t(P_t, N_t, \epsilon_t)$
- Goal is to find a Policy $\pi^*(t, P_t, R_t) = N_t$ that maximizes:

$$\mathbb{E}[\sum_{t=1}^{T} \gamma^t \cdot U(N_t \cdot Q_t)] \text{ where } \gamma \text{ is MDP discount factor}$$

A Simple Linear Impact Model with No Risk-Aversion

- We consider a simple model with Linear Price Impact
- N, N_t, P_t are all continuous-valued ($\in \mathbb{R}$)
- Price Dynamics: $P_{t+1} = P_t - \alpha N_t + \epsilon_t$ where $\alpha \in \mathbb{R}_{\geq 0}$
- ϵ_t is i.i.d. with $\mathbb{E}[\epsilon_t | N_t, P_t] = 0$
- So, Permanent Price Impact is αN_t
- Temporary Price Impact given by βN_t, so $Q_t = P_t - \beta N_t$ ($\beta \in \mathbb{R}_{\geq 0}$)
- Utility function $U(\cdot)$ is the identity function, i.e., no Risk-Aversion
- MDP Discount factor $\gamma = 1$
- This is an unrealistic model, but solving this gives plenty of intuition
- Approach: Define Optimal Value Function & invoke Bellman Equation

Optimal Value Function and Bellman Equation

- Denote Value Function for policy π as:

$$V^\pi(t, P_t, R_t) = \mathbb{E}_\pi[\sum_{i=t}^{T} N_i(P_i - \beta N_i)|(t, P_t, R_t)]$$

- Denote Optimal Value Function as $V^*(t, P_t, R_t) = max_\pi V^\pi(t, P_t, R_t)$
- Optimal Value Function satisfies the Bellman Equation ($\forall 1 \leq t < T$):

$$V^*(t, P_t, R_t) = \max_{N_t}(N_t(P_t - \beta N_t) + \mathbb{E}[V^*(t+1, P_{t+1}, R_{t+1})])$$

$$\text{Note: } V^*(T, P_T, R_T) = R_T(P_T - \beta R_T)$$

- From the above, we can infer $V^*(T-1, P_{T-1}, R_{T-1})$ as:

$$\max_{N_{T-1}}\{N_{T-1}(P_{T-1} - \beta N_{T-1}) + \mathbb{E}[R_T(P_T - \beta R_T)]\}$$

$$= \max_{N_{T-1}}\{N_{T-1}(P_{T-1} - \beta N_{T-1}) + \mathbb{E}[(R_{T-1} - N_{T-1})(P_T - \beta(R_{T-1} - N_{T-1}))\}$$

$$= \max_{N_{T-1}}\{N_{T-1}(P_{T-1} - \beta N_{T-1}) + (R_{T-1} - N_{T-1})(P_{T-1} - \alpha N_{T-1} - \beta(R_{T-1} - N_{T-1}))\}$$

Optimal Policy & Optimal Value Function for case $\alpha \geq 2\beta$

$$= \max_{N_{T-1}} \{R_{T-1} P_{T-1} - \beta R_{T-1}^2 + (\alpha - 2\beta)(N_{T-1}^2 - N_{T-1} R_{T-1})\}$$

- For the case $\alpha \geq 2\beta$, we have the trivial solution: $N_{T-1}^* = 0$ or R_{T-1}
- Substitute N_{T-1}^* in the expression for $V^*(T-1, P_{T-1}, R_{T-1})$:

$$V^*(T-1, P_{T-1}, R_{T-1}) = R_{T-1}(P_{T-1} - \beta R_{T-1})$$

- Continuing backwards in time in this manner gives:

$$N_t^* = 0 \text{ or } R_t$$

$$V^*(t, P_t, R_t) = R_t(P_t - \beta R_t)$$

- So the solution for the case $\alpha \geq 2\beta$ is to sell all N shares at any one of the time steps $t = 1, \ldots, T$ (and none in the other time steps) and the Optimal Expected Total Sale Proceeds $= N(P_1 - \beta N)$

Optimal Policy & Optimal Value Function for case $\alpha < 2\beta$

- For the case $\alpha < 2\beta$, differentiating w.r.t. N_{T-1} and setting to 0 gives:

$$(\alpha - 2\beta)(2N^*_{T-1} - R_{T-1}) = 0 \Rightarrow N^*_{T-1} = \frac{R_{T-1}}{2}$$

- Substitute N^*_{T-1} in the expression for $V^*(T-1, P_{T-1}, R_{T-1})$:

$$V^*(T-1, P_{T-1}, R_{T-1}) = R_{T-1}P_{T-1} - R^2_{T-1}(\frac{\alpha + 2\beta}{4})$$

- Continuing backwards in time in this manner gives:

$$N^*_t = \frac{R_t}{T - t + 1}$$

$$V^*(t, P_t, R_t) = R_t P_t - \frac{R^2_t}{2}(\frac{2\beta + (T-t)\alpha}{T - t + 1})$$

Interpreting the solution for the case $\alpha < 2\beta$

- Rolling forward in time, we see that $N_t^* = \frac{N}{T}$, i.e., uniformly split
- Hence, Optimal Policy is a constant (independent of *State*)
- Uniform split makes intuitive sense because Price Impact and Market Movement are both linear and additive, and don't interact
- Essentially equivalent to minimizing $\sum_{t=1}^{T} N_t^2$ with $\sum_{t=1}^{T} N_t = N$
- Optimal Expected Total Sale Proceeds = $NP_1 - \frac{N^2}{2}(\alpha + \frac{2\beta - \alpha}{T})$
- So, *Implementation Shortfall* from Price Impact is $\frac{N^2}{2}(\alpha + \frac{2\beta - \alpha}{T})$
- Note that Implementation Shortfall is non-zero even if one had infinite time available ($T \rightarrow \infty$) for the case of $\alpha > 0$
- If Price Impact were purely temporary ($\alpha = 0$, i.e., Price fully snapped back), Implementation Shortfall is zero with infinite time available

Models in Bertsimas-Lo paper

- Bertsimas-Lo was the first paper on Optimal Trade Order Execution
- They assumed no risk-aversion, i.e. identity Utility function
- The first model in their paper is a special case of our simple Linear Impact model, with fully Permanent Impact (i.e., $\alpha = \beta$)
- Next, Betsimas-Lo extended the Linear Permanent Impact model
- To include dependence on Serially-Correlated Variable X_t

$$P_{t+1} = P_t - (\alpha N_t + \theta X_t) + \epsilon_t, X_{t+1} = \rho X_t + \eta_t, Q_t = P_t - (\alpha N_t + \theta X_t)$$

- ϵ_t and η_t are i.i.d. (and mutually independent) with mean zero
- X_t can be thought of as market factor affecting P_t linearly
- Bellman Equation on Optimal VF and same approach as before yields:

$$N_t^* = \frac{R_t}{T - t + 1} + h(t, \alpha, \theta, \rho) X_t$$

$$V^*(t, P_t, R_t, X_t) = R_t P_t - (\text{quadratic in } (R_t, X_t) + \text{ constant})$$

- Seral-correlation predictability ($\rho \neq 0$) alters uniform-split strategy

A more Realistic Model: LPT Price Impact

- Next, Bertsimas-Lo present a more realistic model called "LPT"
- *Linear-Percentage Temporary* Price Impact model features:
  - Geometric random walk: consistent with real data, & avoids prices ≤ 0
  - % Price Impact $\frac{g_t(P_t, N_t)}{P_t}$ doesn't depend on P_t (validated by real data)
  - Purely Temporary Price Impact

$$P_{t+1} = P_t e^{Z_t}, X_{t+1} = \rho X_t + \eta_t, Q_t = P_t(1 - \alpha N_t - \theta X_t)$$

- Z_t is a random variable with mean μ_Z and variance σ_Z^2
- With the same derivation as before, we get the solution:

$$N_t^* = c_t^{(1)} + c_t^{(2)} R_t + c_t^{(3)} X_t$$

$$V^*(t, P_t, R_t, X_t) = e^{\mu_Z + \frac{\sigma_Z^2}{2}} \cdot P_t \cdot (c_t^{(4)} + c_t^{(5)} R_t + c_t^{(6)} X_t + c_t^{(7)} R_t^2 + c_t^{(8)} X_t^2 + c_t^{(9)} R_t X_t)$$

Incorporating Risk-Aversion/Utility of Proceeds

- For analytical tractability, Bertsimas-Lo ignored Risk-Aversion
- But one is typically wary of *Risk of Uncertain Proceeds*
- We'd trade some (Expected) Proceeds for lower Variance of Proceeds
- Almgren-Chriss work in this Risk-Aversion framework
- They consider our simple linear model maximizing $E[Y] - \lambda Var[Y]$
- Where Y is the total (uncertain) proceeds $\sum_{i=1}^{T} N_i Q_i$
- λ controls the degree of risk-aversion and hence, the trajectory of N_t^*
- $\lambda = 0$ leads to uniform split strategy $N_t^* = \frac{N}{T}$
- The other extreme is to minimize $Var[Y]$ which yields $N_1^* = N$
- Almgren-Chriss derive *Efficient Frontier* and solutions for specific $U(\cdot)$
- Much like classical Portfolio Optimization problems

Real-world Optimal Trade Order Execution (& Extensions)

- Arbitrary Price Dynamics $f_t(\cdot)$ and Temporary Price Impact $g_t(\cdot)$
- Time-Heterogeneous/non-linear dynamics/impact require (Numerical) DP
- Frictions: Discrete Prices/Sizes, Constraints on Prices/Sizes, Fees
- Incorporating various markets factors in the State bloats State Space
- We could also represent the entire TOB within the State
- Practical route is to develop a simulator capturing all of the above
- Simulator is a *Market-Data-learnt Sampling Model* of TOB Dynamics
- In practice, we'd need to also capture *Cross-Asset Market Impact*
- Using this simulator and neural-networks func approx, we can do RL
- References: Nevmyvaka, Feng, Kearns; 2006 and Vyetrenko, Xu; 2019
- Exciting area for Future Research as well as Engineering Design