Policy Gradient Algorithms

Ashwin Rao

ICME, Stanford University

Overview

1. Motivation and Intuition
2. Definitions and Notation
3. Policy Gradient Theorem and Proof
4. Policy Gradient Algorithms
5. Compatible Function Approximation Theorem and Proof
6. Natural Policy Gradient

Why do we care about Policy Gradient (PG)?

- Let us review how we got here
- We started with Markov Decision Processes and Bellman Equations
- Next we studied several variants of DP and RL algorithms
- We noted that the idea of *Generalized Policy Iteration* (GPI) is key
- Policy Improvement step: $\pi(s, a)$ derived from $\text{argmax}_a \, Q(s, a)$
- How do we do argmax when action space is large or continuous?
- Idea: Do Policy Improvement step with a Gradient Ascent instead

“Policy Improvement with a Gradient Ascent??”

- We want to find the Policy that fetches the “Best Expected Returns”
- Gradient Ascent on “Expected Returns” w.r.t params of Policy func
- So we need a func approx for (stochastic) Policy Func: $\pi(s, a; \boldsymbol{\theta})$
- In addition to the usual func approx for Action Value Func: $Q(s, a; \boldsymbol{w})$
- $\pi(s, a; \boldsymbol{\theta})$ called *Actor* and $Q(s, a; \boldsymbol{w})$ called *Critic*
- Critic parameters $\boldsymbol{w}$ are optimized w.r.t $Q(s, a; \boldsymbol{w})$ loss function min
- Actor parameters $\boldsymbol{\theta}$ are optimized w.r.t Expected Returns max
- We need to formally define “Expected Returns”
- But we already see that this idea is appealing for continuous actions
- GPI with Policy Improvement done as **Policy Gradient (Ascent)**

Value Function-based and Policy-based RL

- Value Function-based
  - Learn Value Function (with a function approximation)
  - Policy is implicit - readily derived from Value Function (eg: ϵ-greedy)
- Policy-based
  - Learn Policy (with a function approximation)
  - No need to learn a Value Function
- Actor-Critic
  - Learn Policy (Actor)
  - Learn Value Function (Critic)

Advantages and Disadvantages of Policy Gradient approach

Advantages:

- Finds the best *Stochastic* Policy (Optimal Deterministic Policy, produced by other RL algorithms, can be unsuitable for POMDPs)
- Naturally *explores* due to Stochastic Policy representation
- Effective in high-dimensional or continuous action spaces
- Small changes in $\boldsymbol{\theta} \Rightarrow$ small changes in π, and in state distribution
- This avoids the convergence issues seen in argmax-based algorithms

Disadvantages:

- Typically converge to a local optimum rather than a global optimum
- Policy Evaluation is typically inefficient and has high variance
- Policy Improvement happens in small steps $\Rightarrow$ slow convergence

Notation

- Assume episodic with $0 \leq \gamma \leq 1$ or non-episodic with $0 \leq \gamma < 1$
- Usual notation of discrete-time, countable-spaces, stationary MDPs
- We lighten $\mathcal{P}(s, a, s')$ notation to $\mathcal{P}_{s,s'}^a$ and $\mathcal{R}(s, a)$ notation to $\mathcal{R}_s^a$
- Initial State Probability Distribution denoted as $p_0 : \mathcal{N} \rightarrow [0, 1]$
- Policy Function Approximation $\pi(s, a; \boldsymbol{\theta}) = \mathbb{P}[A_t = a | S_t = s; \boldsymbol{\theta}]$

PG coverage is quite similar for non-discounted non-episodic, by considering average-reward objective (we won't cover it)

“Expected Returns” Objective

Now we formalize the “Expected Returns” Objective $J(\boldsymbol{\theta})$

$$J(\boldsymbol{\theta}) = \mathbb{E}_\pi[\sum_{t=0}^{\infty} \gamma^t \cdot R_{t+1}]$$

Value Function $V^\pi(s)$ and Action Value function $Q^\pi(s,a)$ defined as:

$$V^\pi(s) = \mathbb{E}_\pi[\sum_{k=t}^{\infty} \gamma^{k-t} \cdot R_{k+1} | S_t = s] \text{ for all } t = 0, 1, 2, \ldots$$

$$Q^\pi(s,a) = \mathbb{E}_\pi[\sum_{k=t}^{\infty} \gamma^{k-t} \cdot R_{k+1} | S_t = s, A_t = a] \text{ for all } t = 0, 1, 2, \ldots$$

$$\text{Advantage Function } A^\pi(s,a) = Q^\pi(s,a) - V^\pi(s)$$

Also, $p(s \rightarrow s', t, \pi)$ will be a key function for us - it denotes the probability of going from state s to s' in t steps by following policy π

Discounted-Aggregate State-Visitation Measure

$$J(\boldsymbol{\theta}) = \mathbb{E}_\pi[\sum_{t=0}^{\infty} \gamma^t \cdot R_{t+1}] = \sum_{t=0}^{\infty} \gamma^t \cdot \mathbb{E}_\pi[R_{t+1}]$$

$$= \sum_{t=0}^{\infty} \gamma^t \cdot \sum_{s \in \mathcal{N}} (\sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot p(S_0 \rightarrow s, t, \pi)) \cdot \sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta}) \cdot \mathcal{R}_s^a$$

$$= \sum_{s \in \mathcal{N}} (\sum_{S_0 \in \mathcal{N}} \sum_{t=0}^{\infty} \gamma^t \cdot p_0(S_0) \cdot p(S_0 \rightarrow s, t, \pi)) \cdot \sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta}) \cdot \mathcal{R}_s^a$$

Definition

$$J(\boldsymbol{\theta}) = \sum_{s \in \mathcal{N}} \rho^\pi(s) \cdot \sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta}) \cdot \mathcal{R}_s^a$$

where $\rho^\pi(s) = \sum_{S_0 \in \mathcal{N}} \sum_{t=0}^{\infty} \gamma^t \cdot p_0(S_0) \cdot p(S_0 \rightarrow s, t, \pi)$ is the key function (for PG) we'll refer to as *Discounted-Aggregate State-Visitation Measure*.

Policy Gradient Theorem (PGT)

Theorem

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(s, a; \boldsymbol{\theta}) \cdot Q^{\pi}(s, a)$$

- Note: $\rho^{\pi}(s)$ depends on $\boldsymbol{\theta}$, but there's no $\nabla_{\boldsymbol{\theta}} \rho^{\pi}(s)$ term in $\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$
- Note: $\nabla_{\boldsymbol{\theta}} \pi(s, a; \boldsymbol{\theta}) = \pi(s, a; \boldsymbol{\theta}) \cdot \nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta})$
- So we can simply generate sampling traces, and at each time step, calculate $(\nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta})) \cdot Q^{\pi}(s, a)$ (probabilities implicit in paths)
- Note: $\nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta})$ is Score function (Gradient of log-likelihood)
- We will estimate $Q^{\pi}(s, a)$ with a function approximation $Q(s, a; \boldsymbol{w})$
- We will later show how to avoid the estimate bias of $Q(s, a; \boldsymbol{w})$
- This numerical estimate of $\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$ enables **Policy Gradient Ascent**
- Let us look at the score function of some canonical $\pi(s, a; \boldsymbol{\theta})$

Canonical $\pi(s, a; \boldsymbol{\theta})$ for finite action spaces

- For finite action spaces, we often use Softmax Policy
- $\boldsymbol{\theta}$ is an m-vector $(\theta_1, \ldots, \theta_m)$
- Features vector $\boldsymbol{\phi}(s, a) = (\phi_1(s, a), \ldots, \phi_m(s, a))$ for all $s \in \mathcal{N}, a \in \mathcal{A}$
- Weight actions using linear combinations of features: $\boldsymbol{\phi}(s, a)^T \cdot \boldsymbol{\theta}$
- Action probabilities proportional to exponentiated weights:

$$\pi(s, a; \boldsymbol{\theta}) = \frac{e^{\boldsymbol{\phi}(s,a)^T \cdot \boldsymbol{\theta}}}{\sum_{b \in \mathcal{A}} e^{\boldsymbol{\phi}(s,b)^T \cdot \boldsymbol{\theta}}} \text{ for all } s \in \mathcal{N}, a \in \mathcal{A}$$

- The score function is:

$$\nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta}) = \boldsymbol{\phi}(s, a) - \sum_{b \in \mathcal{A}} \pi(s, b; \boldsymbol{\theta}) \cdot \boldsymbol{\phi}(s, b) = \boldsymbol{\phi}(s, a) - \mathbb{E}_{\pi}[\boldsymbol{\phi}(s, \cdot)]$$

Canonical $\pi(s, a; \boldsymbol{\theta})$ for continuous action spaces

- For continuous action spaces, we often use Gaussian Policy
- $\boldsymbol{\theta}$ is an m-vector $(\theta_1, \ldots, \theta_m)$
- State features vector $\boldsymbol{\phi}(s) = (\phi_1(s), \ldots, \phi_m(s))$ for all $s \in \mathcal{N}$
- Gaussian Mean is a linear combination of state features $\boldsymbol{\phi}(s)^T \cdot \boldsymbol{\theta}$
- Variance may be fixed σ^2, or can also be parameterized
- Policy is Gaussian, $a \sim \mathcal{N}(\boldsymbol{\phi}(s)^T \cdot \boldsymbol{\theta}, \sigma^2)$ for all $s \in \mathcal{N}$
- The score function is:

$$\nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta}) = \frac{(a - \boldsymbol{\phi}(s)^T \cdot \boldsymbol{\theta}) \cdot \boldsymbol{\phi}(s)}{\sigma^2}$$

Proof of Policy Gradient Theorem

We begin the proof by noting that:

$$J(\boldsymbol{\theta}) = \sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot V^{\pi}(S_0) = \sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot Q^{\pi}(S_0, A_0)$$

Calculate $\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$ by parts $\pi(S_0, A_0; \boldsymbol{\theta})$ and $Q^{\pi}(S_0, A_0)$

$$\begin{aligned} \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = & \sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot Q^{\pi}(S_0, A_0) \\ & + \sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot \nabla_{\boldsymbol{\theta}} Q^{\pi}(S_0, A_0) \end{aligned}$$

Proof of Policy Gradient Theorem

Now expand $Q^\pi(S_0, A_0)$ as:

$$\mathcal{R}_{S_0}^{A_0} + \sum_{S_1 \in \mathcal{N}} \gamma \cdot \mathcal{P}_{S_0,S_1}^{A_0} \cdot V^\pi(S_1) \text{ (Bellman Policy Equation)}$$

$$= \sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot Q^\pi(S_0, A_0) +$$
$$\sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot \nabla_{\boldsymbol{\theta}} (\mathcal{R}_{S_0}^{A_0} + \sum_{S_1 \in \mathcal{N}} \gamma \cdot \mathcal{P}_{S_0,S_1}^{A_0} \cdot V^\pi(S_1))$$

Note: $\nabla_\theta \mathcal{R}_{S_0}^{A_0} = 0$, so remove that term

$$= \sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot Q^\pi(S_0, A_0) +$$
$$\sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot \nabla_{\boldsymbol{\theta}} (\sum_{S_1 \in \mathcal{N}} \gamma \cdot \mathcal{P}_{S_0,S_1}^{A_0} \cdot V^\pi(S_1))$$

Proof of Policy Gradient Theorem

Now bring the $\nabla_{\boldsymbol{\theta}}$ inside the $\sum_{S_1 \in \mathcal{N}}$ to apply only on $V^\pi(S_1)$

$$= \sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot Q^\pi(S_0, A_0) +$$
$$\sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot \sum_{S_1 \in \mathcal{N}} \gamma \cdot \mathcal{P}_{S_0, S_1}^{A_0} \cdot \nabla_{\boldsymbol{\theta}} V^\pi(S_1)$$

Now bring $\sum_{S_0 \in \mathcal{N}}$ and $\sum_{A_0 \in \mathcal{A}}$ inside the $\sum_{S_1 \in \mathcal{N}}$

$$= \sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot Q^\pi(S_0, A_0) +$$
$$\sum_{S_1 \in \mathcal{N}} \sum_{S_0 \in \mathcal{N}} \gamma \cdot p_0(S_0) \cdot (\sum_{A_0 \in \mathcal{A}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot \mathcal{P}_{S_0, S_1}^{A_0}) \cdot \nabla_{\boldsymbol{\theta}} V^\pi(S_1)$$

Policy Gradient Theorem

Note that $\sum_{A_0 \in \mathcal{A}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot \mathcal{P}_{S_0,S_1}^{A_0} = p(S_0 \rightarrow S_1, 1, \pi)$

$$= \sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot Q^\pi(S_0, A_0) +$$
$$\sum_{S_1 \in \mathcal{N}} \sum_{S_0 \in \mathcal{N}} \gamma \cdot p_0(S_0) \cdot p(S_0 \rightarrow S_1, 1, \pi) \cdot \nabla_{\boldsymbol{\theta}} V^\pi(S_1)$$

Now expand $V^\pi(S_1)$ to $\sum_{A_1 \in \mathcal{A}} \pi(S_1, A_1; \boldsymbol{\theta}) \cdot Q^\pi(S_1, A_1)$

$$= \sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot Q^\pi(S_0, A_0) +$$
$$\sum_{S_1 \in \mathcal{N}} \sum_{S_0 \in \mathcal{N}} \gamma \cdot p_0(S_0) \cdot p(S_0 \rightarrow S_1, 1, \pi) \cdot \nabla_{\boldsymbol{\theta}} (\sum_{A_1 \in \mathcal{A}} \pi(S_1, A_1; \boldsymbol{\theta}) \cdot Q^\pi(S_1, A_1))$$

Proof of Policy Gradient Theorem

We are now back to when we started calculating gradient of $\sum_a \pi \cdot Q^\pi$. Follow the same process of splitting $\pi \cdot Q^\pi$, then Bellman-expanding Q^π (to calculate its gradient), and iterate.

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \sum_{S_0 \in \mathcal{N}} p_0(S_0) \cdot \sum_{A_0 \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(S_0, A_0; \boldsymbol{\theta}) \cdot Q^\pi(S_0, A_0) +$$

$$\sum_{S_1 \in \mathcal{N}} \sum_{S_0 \in \mathcal{N}} \gamma \cdot p_0(S_0) \cdot p(S_0 \rightarrow S_1, 1, \pi) \cdot (\sum_{A_1 \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(S_1, A_1; \boldsymbol{\theta}) \cdot Q^\pi(S_1, A_1) + \ldots)$$

This iterative process leads us to:

$$= \sum_{t=0}^{\infty} \sum_{S_t \in \mathcal{N}} \sum_{S_0 \in \mathcal{N}} \gamma^t \cdot p_0(S_0) \cdot p(S_0 \rightarrow S_t, t, \pi) \cdot \sum_{A_t \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(S_t, A_t; \boldsymbol{\theta}) \cdot Q^\pi(S_t, A_t)$$

Proof of Policy Gradient Theorem

$$\text{Bring } \sum_{t=0}^{\infty} \text{ inside } \sum_{S_t \in \mathcal{N}} \sum_{S_0 \in \mathcal{N}} \text{ and note that}$$

$$\sum_{A_t \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(S_t, A_t; \boldsymbol{\theta}) \cdot Q^{\pi}(S_t, A_t) \text{ is independent of } t$$

$$= \sum_{s \in \mathcal{N}} \sum_{S_0 \in \mathcal{N}} \sum_{t=0}^{\infty} \gamma^t \cdot p_0(S_0) \cdot p(S_0 \rightarrow s, t, \pi) \cdot \sum_{a \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(s, a; \boldsymbol{\theta}) \cdot Q^{\pi}(s, a)$$

$$\text{Reminder that } \sum_{S_0 \in \mathcal{N}} \sum_{t=0}^{\infty} \gamma^t \cdot p_0(S_0) \cdot p(S_0 \rightarrow s, t, \pi) \overset{\text{def}}{=} \rho^{\pi}(s). \text{ So,}$$

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(s, a; \boldsymbol{\theta}) \cdot Q^{\pi}(s, a)$$

$$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$$

Monte-Carlo Policy Gradient (REINFORCE Algorithm)

- Update $\boldsymbol{\theta}$ by stochastic gradient ascent using PGT
- Using $G_t = \sum_{k=t}^{T} \gamma^{k-t} \cdot R_{k+1}$ as an unbiased sample of $Q^\pi(S_t, A_t)$

$$\Delta\boldsymbol{\theta} = \alpha \cdot \gamma^t \cdot \nabla_{\boldsymbol{\theta}} \log \pi(S_t, A_t; \boldsymbol{\theta}) \cdot G_t$$

Algorithm 4.1: REINFORCE($\cdot$)

```
Initialize $\boldsymbol{\theta}$ arbitrarily
for each episode $\{S_0, A_0, R_1, S_1, \ldots, S_{T-1}, A_{T-1}, R_T, S_T\} \sim \pi(\cdot, \cdot; \boldsymbol{\theta})$
do  for $t \leftarrow 0$ to $T$
        do  $G \leftarrow \sum_{k=t}^{T} \gamma^{k-t} \cdot R_{k+1}$
            $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha \cdot \gamma^t \cdot \nabla_{\boldsymbol{\theta}} \log \pi(S_t, A_t; \boldsymbol{\theta}) \cdot G$
```

Reducing Variance using a Critic

- Monte Carlo Policy Gradient has high variance
- We use a Critic $Q(s, a; \boldsymbol{w})$ to estimate $Q^\pi(s, a)$
- Actor-Critic algorithms maintain two sets of parameters:
    - Critic updates parameters $\boldsymbol{w}$ to approximate Q-function for policy π
    - Critic could use any of the algorithms we learnt earlier:
        - Monte Carlo policy evaluation
        - Temporal-Difference Learning
        - $TD(\lambda)$ based on Eligibility Traces
        - Could even use LSTD (if critic function approximation is linear)
    - Actor updates policy parameters $\boldsymbol{\theta}$ in direction suggested by Critic
    - This is Approximate Policy Gradient due to *Bias* of Critic

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) \approx \sum_{s \in \mathcal{N}} \rho^\pi(s) \cdot \sum_{a \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(s, a; \boldsymbol{\theta}) \cdot Q(s, a; \boldsymbol{w})$$

So what does the algorithm look like?

- Generate a sufficient set of sampling traces $S_0, A_0, R_1, S_1, A_1, R_2, S_2 \ldots$
- S_0 is sampled from the distribution $p_0(\cdot)$
- A_t is sampled from $\pi(S_t, \cdot; \boldsymbol{\theta})$
- Receive atomic experience (R_{t+1}, S_{t+1}) from the environment
- At each time step t, update $\boldsymbol{w}$ proportional to gradient of appropriate (MC or TD-based) loss function of $Q(s, a; \boldsymbol{w})$
- Sum $\gamma^t \cdot (\nabla_{\boldsymbol{\theta}} \log \pi(S_t, A_t; \boldsymbol{\theta})) \cdot Q(S_t, A_t; \boldsymbol{w})$ over t and over paths
- Update $\boldsymbol{\theta}$ using this (biased) estimate of $\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$
- Iterate with a new set of sampling traces ...

Reducing Variance with a Baseline

- We can reduce variance by subtracting a baseline function $B(s)$ from $Q(s, a; \boldsymbol{w})$ in the Policy Gradient estimate
- This means at each time step, we replace $\gamma^t \cdot \nabla_{\boldsymbol{\theta}} \log \pi(S_t, A_t; \boldsymbol{\theta}) \cdot Q(S_t, A_t; \boldsymbol{w})$ with $\gamma^t \cdot \nabla_{\boldsymbol{\theta}} \log \pi(S_t, A_t; \boldsymbol{\theta}) \cdot (Q(S_t, A_t; \boldsymbol{w}) - B(S_t))$
- Note that Baseline function $B(s)$ is only a function of s (and not a)
- This ensures that subtracting Baseline $B(s)$ does not add bias

$$
\begin{aligned}
&\sum_{s \in \mathcal{N}} \rho^\pi(s) \sum_{a \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(s, a; \boldsymbol{\theta}) \cdot B(s) \\
=&\sum_{s \in \mathcal{N}} \rho^\pi(s) \cdot B(s) \cdot \nabla_{\boldsymbol{\theta}} (\sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta})) \\
=&\sum_{s \in \mathcal{N}} \rho^\pi(s) \cdot B(s) \cdot \nabla_{\boldsymbol{\theta}} 1 \\
=&0
\end{aligned}
$$

Using State Value function as Baseline

- A good baseline $B(s)$ is state value function $V(s; \boldsymbol{v})$
- Rewrite Policy Gradient algorithm using advantage function estimate

$$A(s, a; \boldsymbol{w}, \boldsymbol{v}) = Q(s, a; \boldsymbol{w}) - V(s; \boldsymbol{v})$$

- Now the estimate of $\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$ is given by:

$$\sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(s, a; \boldsymbol{\theta}) \cdot A(s, a; \boldsymbol{w}, \boldsymbol{v})$$

- At each time step, we update both sets of parameters $\boldsymbol{w}$ and $\boldsymbol{v}$

TD Error as estimate of Advantage Function

- Consider TD error δ^π for the *true* Value Function $V^\pi(s)$

$$\delta^\pi = r + \gamma \cdot V^\pi(s') - V^\pi(s)$$

- δ^π is an unbiased estimate of Advantage function $A^\pi(s,a)$

$$\mathbb{E}_\pi[\delta^\pi | s, a] = \mathbb{E}_\pi[r + \gamma \cdot V^\pi(s') | s, a] - V^\pi(s) = Q^\pi(s,a) - V^\pi(s) = A^\pi(s,a)$$

- So we can write Policy Gradient in terms of $\mathbb{E}_\pi[\delta^\pi | s, a]$

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \sum_{s \in \mathcal{N}} \rho^\pi(s) \cdot \sum_{a \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(s, a; \boldsymbol{\theta}) \cdot \mathbb{E}_\pi[\delta^\pi | s, a]$$

- In practice, we can use func approx for TD error (and sample):

$$\delta(s, r, s'; \boldsymbol{v}) = r + \gamma \cdot V(s'; \boldsymbol{v}) - V(s; \boldsymbol{v})$$

- This approach requires only one set of critic parameters $\boldsymbol{v}$

TD Error can be used by both Actor and Critic

Algorithm 4.2: ACTOR-CRITIC-TD-ERROR($\cdot$)

```
Initialize Policy params $\boldsymbol{\theta}$ and State VF params $\boldsymbol{v}$ arbitrarily
for each episode
    do
        Initialize $s$ (first state of episode)
        $P \leftarrow 1$
        while $s$ is not terminal
            do
                $a \sim \pi(s, \cdot; \boldsymbol{\theta})$
                Take action $a$, receive $r, s'$ from the environment
                $\delta \leftarrow r + \gamma \cdot V(s'; \boldsymbol{v}) - V(s; \boldsymbol{v})$
                $\boldsymbol{v} \leftarrow \boldsymbol{v} + \alpha_{\boldsymbol{v}} \cdot \delta \cdot \nabla_{\boldsymbol{v}} V(s; \boldsymbol{v})$
                $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha_{\boldsymbol{\theta}} \cdot P \cdot \delta \cdot \nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta})$
                $P \leftarrow \gamma \cdot P$
                $s \leftarrow s'$
```

Using Eligibility Traces for both Actor and Critic

Algorithm 4.3: ACTOR-CRITIC-ELIGIBILITY-TRACES($\cdot$)

```
Initialize Policy params $\boldsymbol{\theta}$ and State VF params $\boldsymbol{v}$ arbitrarily
for each episode
do {
    Initialize $s$ (first state of episode)
    $\boldsymbol{z_\theta}, \boldsymbol{z_v} \leftarrow 0$ (eligibility traces for $\boldsymbol{\theta}$ and $\boldsymbol{v}$)
    $P \leftarrow 1$
    while $s$ is not terminal
    do {
        $a \sim \pi(s, \cdot; \theta)$
        Take action $a$, observe $r, s'$
        $\delta \leftarrow r + \gamma \cdot V(s'; \boldsymbol{v}) - V(s; \boldsymbol{v})$
        $\boldsymbol{z_v} \leftarrow \gamma \cdot \lambda_{\boldsymbol{v}} \cdot \boldsymbol{z_v} + \nabla_{\boldsymbol{v}} V(s; \boldsymbol{v})$
        $\boldsymbol{z_\theta} \leftarrow \gamma \cdot \lambda_{\boldsymbol{\theta}} \cdot \boldsymbol{z_\theta} + P \cdot \nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta})$
        $\boldsymbol{v} \leftarrow \boldsymbol{v} + \alpha_{\boldsymbol{v}} \cdot \delta \cdot \boldsymbol{z_v}$
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha_{\boldsymbol{\theta}} \cdot \delta \cdot \boldsymbol{z_\theta}$
        $P \leftarrow \gamma \cdot P, s \leftarrow s'$
```

Overcoming Bias

- We've learnt a few ways of how to reduce variance
- But we haven't discussed how to overcome bias
- All of the following substitutes for $Q^\pi(s,a)$ in PG have bias:
  - $Q(s,a;\boldsymbol{w})$
  - $A(s,a;\boldsymbol{w},\boldsymbol{v})$
  - $\delta(s,s',r;\boldsymbol{v})$
- Turns out there is indeed a way to overcome bias
- It is called the *Compatible Function Approximation Theorem*

Compatible Function Approximation Theorem

Theorem

If the following two conditions are satisfied:

1. *Critic gradient is* compatible *with the Actor score function*

$$\nabla_{\boldsymbol{w}} Q(s, a; \boldsymbol{w}) = \nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta})$$

2. *Critic parameters* $\boldsymbol{w}$ *minimize the following mean-squared error:*

$$\epsilon = \sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta}) \cdot (Q^{\pi}(s, a) - Q(s, a; \boldsymbol{w}))^2$$

Then the Policy Gradient using critic $Q(s, a; \boldsymbol{w})$ *is exact:*

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(s, a; \boldsymbol{\theta}) \cdot Q(s, a; \boldsymbol{w})$$

Proof of Compatible Function Approximation Theorem

For $\boldsymbol{w}$ that minimizes

$$\epsilon = \sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta}) \cdot (Q^{\pi}(s, a) - Q(s, a; \boldsymbol{w}))^2,$$

$$\sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta}) \cdot (Q^{\pi}(s, a) - Q(s, a; \boldsymbol{w})) \cdot \nabla_{\boldsymbol{w}} Q(s, a; \boldsymbol{w}) = 0$$

But since $\nabla_{\boldsymbol{w}} Q(s, a; \boldsymbol{w}) = \nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta})$, we have:

$$\sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta}) \cdot (Q^{\pi}(s, a) - Q(s, a; \boldsymbol{w})) \cdot \nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta}) = 0$$

$$\begin{aligned}\text{Therefore, } & \sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta}) \cdot Q^{\pi}(s, a) \cdot \nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta}) \\ &= \sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta}) \cdot Q(s, a; \boldsymbol{w}) \cdot \nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta})\end{aligned}$$

Proof of Compatible Function Approximation Theorem

$$\text{But } \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta}) \cdot Q^{\pi}(s, a) \cdot \nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta})$$

$$\begin{aligned} \text{So, } \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) &= \sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta}) \cdot Q(s, a; \boldsymbol{w}) \cdot \nabla_{\boldsymbol{\theta}} \log \pi(s, a; \boldsymbol{\theta}) \\ &= \sum_{s \in \mathcal{N}} \rho^{\pi}(s) \cdot \sum_{a \in \mathcal{A}} \nabla_{\boldsymbol{\theta}} \pi(s, a; \boldsymbol{\theta}) \cdot Q(s, a; \boldsymbol{w}) \end{aligned}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

This means with conditions (1) and (2) of Compatible Function Approximation Theorem, we can use the critic func approx $Q(s, a; \boldsymbol{w})$ and still have the exact Policy Gradient.

How to enable Compatible Function Approximation

A simple way to enable Compatible Function Approximation $\frac{\partial Q(s,a;\boldsymbol{w})}{\partial w_i} = \frac{\partial \log \pi(s,a;\boldsymbol{\theta})}{\partial \theta_i}, \forall i$ is to set $Q(s,a;\boldsymbol{w})$ to be linear in its features.

$$Q(s,a;\boldsymbol{w}) = \sum_{i=1}^{m} \phi_i(s,a) \cdot w_i = \sum_{i=1}^{m} \frac{\partial \log \pi(s,a;\boldsymbol{\theta})}{\partial \theta_i} \cdot w_i$$

We note below that a compatible $Q(s,a;\boldsymbol{w})$ serves as an approximation of the advantage function.

$$\sum_{a \in \mathcal{A}} \pi(s,a;\boldsymbol{\theta}) \cdot Q(s,a;\boldsymbol{w}) = \sum_{a \in \mathcal{A}} \pi(s,a;\boldsymbol{\theta}) \cdot (\sum_{i=1}^{m} \frac{\partial \log \pi(s,a;\boldsymbol{\theta})}{\partial \theta_i} \cdot w_i)$$

$$= \sum_{a \in \mathcal{A}} (\sum_{i=1}^{m} \frac{\partial \pi(s,a;\boldsymbol{\theta})}{\partial \theta_i} \cdot w_i) = \sum_{i=1}^{m} (\sum_{a \in \mathcal{A}} \frac{\partial \pi(s,a;\boldsymbol{\theta})}{\partial \theta_i}) \cdot w_i$$

$$= \sum_{i=1}^{m} \frac{\partial}{\partial \theta_i} (\sum_{a \in \mathcal{A}} \pi(s,a;\boldsymbol{\theta})) \cdot w_i = \sum_{i=1}^{m} \frac{\partial 1}{\partial \theta_i} \cdot w_i = 0$$

Fisher Information Matrix

Denoting $[\frac{\partial \log \pi(s,a;\boldsymbol{\theta})}{\partial \theta_i}], i = 1, \ldots, m$ as the score column vector $\boldsymbol{SC}(s, a; \boldsymbol{\theta})$ and assuming compatible linear-approximation critic:

$$\begin{aligned}\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) &= \sum_{s \in \mathcal{N}} \rho^\pi(s) \cdot \sum_{a \in \mathcal{A}} \pi(s, a; \boldsymbol{\theta}) \cdot (\boldsymbol{SC}(s, a; \boldsymbol{\theta}) \cdot \boldsymbol{SC}(s, a; \boldsymbol{\theta})^T \cdot \boldsymbol{w}) \\ &= \mathbb{E}_{s \sim \rho^\pi, a \sim \pi}[\boldsymbol{SC}(s, a; \boldsymbol{\theta}) \cdot \boldsymbol{SC}(s, a; \boldsymbol{\theta})^T] \cdot \boldsymbol{w} \\ &= FIM_{\rho^\pi, \pi}(\boldsymbol{\theta}) \cdot \boldsymbol{w}\end{aligned}$$

where $FIM_{\rho_\pi, \pi}(\boldsymbol{\theta})$ is the Fisher Information Matrix w.r.t. $s \sim \rho^\pi, a \sim \pi$.

Natural Policy Gradient

- Recall the idea of Natural Gradient from Numerical Optimization
- Natural gradient $\nabla_{\boldsymbol{\theta}}^{nat} J(\boldsymbol{\theta})$ is the direction of optimal $\boldsymbol{\theta}$ movement
- In terms of the KL-divergence metric (versus plain Euclidean norm)
- Natural gradient yields better convergence (we won't cover proof)

$$\text{Formally defined as: } \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = FIM_{\rho_\pi,\pi}(\boldsymbol{\theta}) \cdot \nabla_{\boldsymbol{\theta}}^{nat} J(\boldsymbol{\theta})$$

$$\text{Therefore, } \nabla_{\boldsymbol{\theta}}^{nat} J(\boldsymbol{\theta}) = \boldsymbol{w}$$

This compact result is great for our algorithm:

- Update Critic params $\boldsymbol{w}$ with the critic loss gradient (at step t) as:

$$\gamma^t \cdot (R_{t+1} + \gamma \cdot \boldsymbol{SC}(S_{t+1}, A_{t+1}; \boldsymbol{\theta})^T \cdot \boldsymbol{w} - \boldsymbol{SC}(S_t, A_t; \boldsymbol{\theta})^T \cdot \boldsymbol{w}) \cdot \boldsymbol{SC}(S_t, A_t; \boldsymbol{\theta})$$

- Update Actor params $\boldsymbol{\theta}$ in the direction equal to value of $\boldsymbol{w}$