A Guided Tour of Chapter 9: Reinforcement Learning for Prediction

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RL does not have access to a probability model

- DP/ADP assume access to probability model (knowledge of $\mathcal{P}_R$)
- Often in real-world, we do not have access to these probabilities
- Which means we'd need to *interact* with the *actual environment*
- Actual Environment serves up individual experiences, not probabilities
- Even if MDP model is available, model updates can be challenging
- Often real-world models end up being too large or too complex
- Sometimes estimating a *sampling model* is much more feasible
- So RL interacts with either *actual* or *simulated* environment
- Either way, we receive *individual experiences* of next state and reward
- RL learns Value Functions from a stream of individual experiences
- How does RL solve Prediction and Control with such limited access?

The RL Approach

- Like humans/animals, RL doesn't aim to estimate probability model
- Rather, RL is a "trial-and-error" approach to linking actions to returns
- This is hard because actions have overlapping reward sequences
- Also, sometimes actions result in *delayed rewards*
- The key is incrementally updating Q-Value Function from experiences
- Appropriate Approximation of Q-Value Function is also key to success
- RL algorithms are founded on the *Bellman Equations*
- Moreover, RL Control is based on *Generalized Policy Iteration*
- This lecture/chapter focuses on RL for Prediction

RL for Prediction

- Prediction: Problem of estimating MDP Value Function for a policy π
- Equivalently, problem of estimating π-implied MRP's Value Function
- Assume interface serves an *atomic experience* of (next state, reward)
- Interacting with this interface repeatedly provides a *trace experience*

$$S_0, R_1, S_1, R_2, S_2, \ldots$$

- *Value Function* $V : \mathcal{N} \rightarrow \mathbb{R}$ of an MRP is defined as:

$$V(s) = \mathbb{E}[G_t | S_t = s] \text{ for all } s \in \mathcal{N}, \text{ for all } t = 0, 1, 2, \ldots$$

where the *Return* G_t for each $t = 0, 1, 2, \ldots$ is defined as:

$$G_t = \sum_{i=t+1}^{\infty} \gamma^{i-t-1} \cdot R_i = R_{t+1} + \gamma \cdot R_{t+2} + \gamma^2 \cdot R_{t+3} + \ldots = R_{t+1} + \gamma \cdot G_{t+1}$$

Code interface for RL Prediction

An *atomic experience* is represented as a TransitionStep [S]

```
@dataclass(frozen=True)
class TransitionStep(Generic[S]):
    state: S
    next_state: S
    reward: float
```

Input to RL prediction can be either of:

- Atomic Experiences as Iterable [TransitionStep [S]], or
- Trace Experiences as Iterable [Iterable [TransitionStep [S]]]

Note that Iterable can be either a Sequence or an Iterator (i.e., *stream*)

Monte-Carlo (MC) Prediction

- Supervised learning with states and returns from trace experiences
- Incremental estimation with update method of FunctionApprox
- x-values are states S_t, y-values are returns G_t
- Note that updates can be done only at the end of a trace experience
- Returns calculated with a backward walk: $G_t = R_{t+1} + \gamma \cdot G_{t+1}$

$$\mathcal{L}_{(S_t,G_t)}(\boldsymbol{w}) = \frac{1}{2} \cdot (V(S_t;\boldsymbol{w}) - G_t)^2$$

$$\nabla_{\boldsymbol{w}} \mathcal{L}_{(S_t,G_t)}(\boldsymbol{w}) = (V(S_t;\boldsymbol{w}) - G_t) \cdot \nabla_{\boldsymbol{w}} V(S_t;\boldsymbol{w})$$

$$\Delta \boldsymbol{w} = \alpha \cdot (G_t - V(S_t;\boldsymbol{w})) \cdot \nabla_{\boldsymbol{w}} V(S_t;\boldsymbol{w})$$

Structure of the parameters update formula

$$\Delta \boldsymbol{w} = \alpha \cdot (G_t - V(S_t; \boldsymbol{w})) \cdot \nabla_{\boldsymbol{w}} V(S_t; \boldsymbol{w})$$

The update $\Delta \boldsymbol{w}$ to parameters $\boldsymbol{w}$ should be seen as product of:

- *Learning Rate* α
- *Return Residual* of the observed return G_t relative to the estimated conditional expected return $V(S_t; \boldsymbol{w})$
- *Estimate Gradient* of the conditional expected return $V(S_t; \boldsymbol{w})$ with respect to the parameters $\boldsymbol{w}$

This structure (as product of above 3 entities) will be a repeated pattern.

Tabular MC Prediction

- Finite state space, let's say non-terminal states $\mathcal{N} = \{s_1, s_2, \ldots, s_m\}$
- Denote $V_n(s_i)$ as estimate of VF after the n-th occurrence of s_i
- Denote $Y_i^{(1)}, Y_i^{(2)}, \ldots, Y_i^{(n)}$ as returns for first n occurrences of s_i
- Denote count_to_weight_func attribute of Tabular as $f(\cdot)$
- Then the Tabular update at the n-th occurrence of s_i is:

$$\begin{aligned} V_n(s_i) &= (1 - f(n)) \cdot V_{n-1}(s_i) + f(n) \cdot Y_i^{(n)} \\ &= V_{n-1}(s_i) + f(n) \cdot (Y_i^{(n)} - V_{n-1}(s_i)) \end{aligned}$$

- So update to VF for s_i is *Latest Weight* times *Return Residual*
- For default setting of count_to_weight_func as $f(n) = \frac{1}{n}$:

$$V_n(s_i) = \frac{n-1}{n} \cdot V_{n-1}(s_i) + \frac{1}{n} \cdot Y_i^{(n)} = V_{n-1}(s_i) + \frac{1}{n} \cdot (Y_i^{(n)} - V_{n-1}(s_i))$$

Tabular MC Prediction

- Expanding the incremental updates across values of n, we get:

$$
\begin{aligned}
V_n(s_i) = & f(n) \cdot Y_i^{(n)} + (1 - f(n)) \cdot f(n-1) \cdot Y_i^{(n-1)} + \ldots \\
& \ldots + (1 - f(n)) \cdot (1 - f(n-1)) \cdots (1 - f(2)) \cdot f(1) \cdot Y_i^{(1)}
\end{aligned}
$$

- For default setting of count_to_weight_func as $f(n) = \frac{1}{n}$:

$$
\begin{aligned}
V_n(s_i) = & \frac{1}{n} \cdot Y_i^{(n)} + \frac{n-1}{n} \cdot \frac{1}{n-1} \cdot Y_i^{(n-1)} + \ldots \\
& \ldots + \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdots \frac{1}{2} \cdot \frac{1}{1} \cdot Y_i^{(1)} = \frac{\sum_{k=1}^{n} Y_i^{(k)}}{n}
\end{aligned}
$$

- Tabular MC is simply incremental calculation of averages of returns
- Exactly the calculation in the update method of Tabular class
- View Tabular MC as an application of Law of Large Numbers

Tabular MC as a special case of Linear Func Approximation

- Features functions are indicator functions for states
- Linear-approx parameters are Value Function estimates for states
- count_to_weight_func plays the role of learning rate
- So tabular Value Function update can be written as:

$$w_i^{(n)} = w_i^{(n-1)} + \alpha_n \cdot (Y_i^{(n)} - w_i^{(n-1)})$$

- $Y_i^{(n)} - w_i^{(n-1)}$ represents the gradient of the loss function
- For non-stationary problems, algorithm needs to "forget" distant past
- With constant learning rate α, time-decaying weights:

$$\begin{aligned} V_n(s_i) &= \alpha \cdot Y_i^{(n)} + (1-\alpha) \cdot \alpha \cdot Y_i^{(n-1)} + \ldots + (1-\alpha)^{n-1} \cdot \alpha \cdot Y_i^{(1)} \\ &= \sum_{j=1}^{n} \alpha \cdot (1-\alpha)^{n-j} \cdot Y_i^{(j)} \end{aligned}$$

- Weights sum to 1 asymptotically: $\lim_{n \to \infty} \sum_{j=1}^{n} \alpha \cdot (1-\alpha)^{n-j} = 1$

Each-Visit MC and First-Visit MC

- The MC algorithm we covered is known as *Each-Visit Monte-Carlo*
- Because we include each occurrence of a state in a trace experience
- Alternatively, we can do *First-Visit Monte-Carlo*
- Only the first occurrence of a state in a trace experience is considered
- Keep track of whether a state has been visited in a trace experience
- MC Prediction algorithms are easy to understand and implement
- MC produces unbiased estimates but can be slow to converge
- Key disadvantage: MC requires complete trace experiences

Temporal-Difference (TD) Prediction

- To understand TD, we start with Tabular TD Prediction
- Key: Exploit recursive structure of VF in MRP Bellman Equation
- Replace G_t with $R_{t+1} + \gamma \cdot V(S_{t+1})$ using atomic experience data
- So we are *bootstrapping* the VF ("estimate from estimate")
- The tabular MC Prediction update (for constant α) is modified from:

$$V(S_t) \leftarrow V(S_t) + \alpha \cdot (G_t - V(S_t))$$

  to:

$$V(S_t) \leftarrow V(S_t) + \alpha \cdot (R_{t+1} + \gamma \cdot V(S_{t+1}) - V(S_t))$$

- $R_{t+1} + \gamma \cdot V(S_{t+1})$ known as *TD target*
- $\delta_t = R_{t+1} + \gamma \cdot V(S_{t+1}) - V(S_t)$ known as *TD Error*
- TD Error is the crucial quantity - it represents "sample Bellman Error"
- VF is adjusted so as to bridge TD error (on an expected basis)

TD updates after each atomic experience

- Unlike MC, we can use TD when we have incomplete traces
- Often in real-world situations, experiments gets curtailed/disrupted
- Also, we can use TD in non-episodic (known as *continuing*) traces
- TD updates VF after each atomic experience ("continuous learning")
- So TD can be run on *any* stream of atomic experiences
- This means we can chop up the input stream and serve in any order

TD Prediction with Function Approximation

- Each atomic experience leads to a parameters update
- To understand how parameters update work, consider:

$$\mathcal{L}_{(S_t,S_{t+1},R_{t+1})}(\boldsymbol{w}) = \frac{1}{2} \cdot (V(S_t;\boldsymbol{w}) - (R_{t+1} + \gamma \cdot V(S_{t+1};\boldsymbol{w})))^2$$

- Above formula replaces G_t (of MC) with $R_{t+1} + \gamma \cdot V(S_{t+1}, \boldsymbol{w})$
- Unlike MC, in TD, we don't take the gradient of this loss function
- "Cheat" in gradient calc by ignoring dependency of $V(S_{t+1};\boldsymbol{w})$ on $\boldsymbol{w}$
- This "gradient with cheating" calculation is known as *semi-gradient*
- So we pretend the only dependency on $\boldsymbol{w}$ is through $V(S_t;\boldsymbol{w})$

$$\Delta\boldsymbol{w} = \alpha \cdot (R_{t+1} + \gamma \cdot V(S_{t+1};\boldsymbol{w}) - V(S_t;\boldsymbol{w})) \cdot \nabla_{\boldsymbol{w}} V(S_t;\boldsymbol{w})$$

Structure of the parameters update formula

$$\Delta \boldsymbol{w} = \alpha \cdot (R_{t+1} + \gamma \cdot V(S_{t+1}; \boldsymbol{w}) - V(S_t; \boldsymbol{w})) \cdot \nabla_{\boldsymbol{w}} V(S_t; \boldsymbol{w})$$

The update $\Delta \boldsymbol{w}$ to parameters $\boldsymbol{w}$ should be seen as product of:

- *Learning Rate* α
- *TD Error* $\delta_t = R_{t+1} + \gamma \cdot V(S_{t+1}; \boldsymbol{w}) - V(S_t; \boldsymbol{w})$
- *Estimate Gradient* of the conditional expected return $V(S_t; \boldsymbol{w})$ with respect to the parameters $\boldsymbol{w}$

So parameters update formula has same product-structure as MC

TD's many benefits

- "TD is the most significant and innovative idea in RL" - Rich Sutton
- Blends the advantages of DP and MC
- Like DP, TD learns by bootstrapping (drawing from Bellman Eqn)
- Like MC, TD learns from experiences without access to probabilities
- So TD overcomes curse of dimensionality and curse of modeling
- TD also has the advantage of not requiring entire trace experiences
- Most significantly, TD is akin to human (continuous) learning

Bias, Variance and Convergence of TD versus MC

- MC uses G_t is an unbiased estimate of the Value Function
- This helps MC with convergence even with function approximation
- TD uses $R_{t+1} + \gamma \cdot V(S_{t+1}; \boldsymbol{w})$ as a biased estimate of the VF
- Tabular TD prediction converges to true VF in the mean for const α
- And converges to true VF under Robbins-Monro learning rate schedule

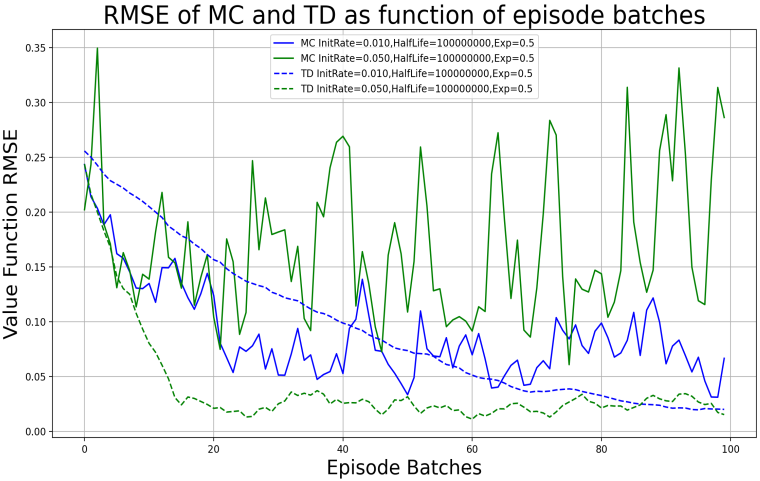
$$\sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} \alpha_n^2 < \infty$$

- However, Robbins-Monro schedule is not so useful in practice
- TD Prediction with func-approx does not always converge to true VF
- Most convergence proofs are for Tabular, some for linear func-approx
- TD Target $R_{t+1} + \gamma \cdot V(S_{t+1}; \boldsymbol{w})$ has much lower variance that G_t
- G_t depends on many random rewards whose variances accumulate
- TD Target depends on only the next reward, so lower variance

Speed of Convergence of TD versus MC

- We typically compare algorithms based on:
    - Speed of convergence
    - Efficiency in use of limited set of experiences data
- There are no formal proofs for MC v/s TD on above criterion
- MC and TD have significant differences in their:
    - Usage of data
    - Nature of updates
    - Frequency of updates
- So unclear exactly how to compare them apples to apples
- Typically, MC and TD are compared with constant α
- Practically/empirically, TD does better than MC with constant α
- Also, MC is not very sensitive to initial Value Function, but TD is

Convergence of MC versus TD with constant α



RMSE of MC versus TD as function of episodes

- Symmetric random walk with barrier $B = 10$ and no discounting
- Graph depicts RMSE after every 7th episode (700 episodes in all)
- Blue curves for constant $\alpha = 0.01$, green for constant $\alpha = 0.05$
- Notice how MC has significantly more variance
- RMSE progression is quite slow on blue curves (small learning rate)
- MC progresses quite fast initially but then barely progresses
- TD gets to fairly small RMSE quicker than corresponding MC
- This performance of TD versus MC is typical for constant α

Fixed-Data Experience Replay on TD versus MC

- So far, we've understood *how* TD learns versus *how* MC learns
- Now we want to understand *what* TD learns versus *what* MC learns
- To illustrate, we consider a finite set of trace experiences
- The agent can tap into this finite set of traces experiences endlessly
- But everything is ultimately sourced from this finite data set
- So we'd end up tapping into these experiences repeatedly
- We call this technique *Experience Replay*

```
data: Sequence[Sequence[Tuple[str, float]]] = [
    [('A', 2.), ('A', 6.), ('B', 1.), ('B', 2.)],
    [('A', 3.), ('B', 2.), ('A', 4.), ('B', 2.), ('B',
    [('B', 3.), ('B', 6.), ('A', 1.), ('B', 1.)],
    [('A', 0.), ('B', 2.), ('A', 4.), ('B', 4.), ('B',
    [('B', 8.), ('B', 2.)]
]
```

MC and TD learn different Value Functions

- It is quite obvious what MC Prediction algorithm would learn
- MC Prediction is simply supervised learning with (state, return) pairs
- But here those pairs ultimately come from the given finite pairs
- So, MC estimates Value Function as average returns in the finite data
- Running MC Prediction algo matches explicit average returns calc
- But running TD Prediction algo gives significantly different answer
- So what is TD Prediction algorithm learning?
- TD drives towards VF of MRP *implied* by the finite experiences
- Specifically, learns MLE for $\mathcal{P}_R$ from the given finite data

$$\mathcal{P}_R(s, r, s') = \frac{\sum_{i=1}^N \mathbb{I}_{S_i=s, R_{i+1}=r, S_{i+1}=s'}}{\sum_{i=1}^N \mathbb{I}_{S_i=s}}$$

- TD is advantageous in Markov environments, MC in non-Markov

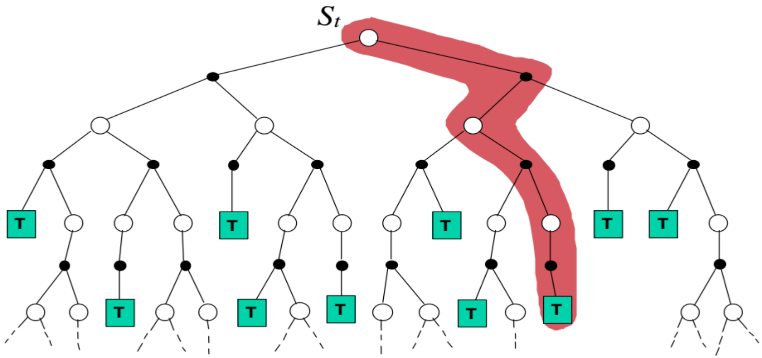
Bootstrapping and Experiencing

- We summarize MC, TD and DP in terms of whether they:
    - Bootstrap: Update to VF utilizes a current or prior estimate of the VF
    - Experience: Interaction with actual or simulated environment
- TD and DP *do bootstrap* (updates use current/prior estimate of VF)
- MC *does not bootstrap* (updates use trace experience returns)
- MC and TD *do experience* (actual/simulated environment interaction)
- DP *does not experience* (updates use transition probabilities)
- Bootstrapping means backups are *shallow* (MC backups are *deep*)
- Experiencing means backups are *narrow* (DP backups are *wide*)

MC backup Diagram

Monte Carlo (Supervised Learning) (MC)

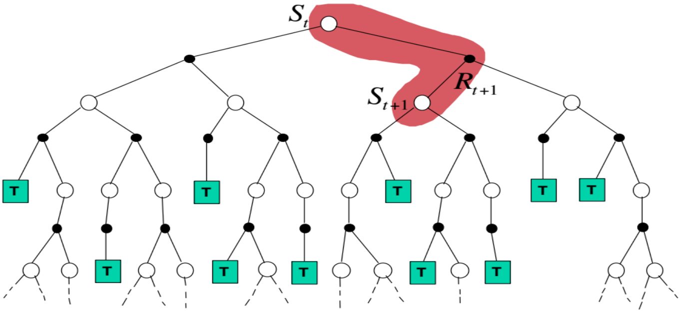
$$V(S_t) \leftarrow V(S_t) + \alpha\left[G_t - V(S_t)\right]$$



TD backup Diagram

Simplest TD Method

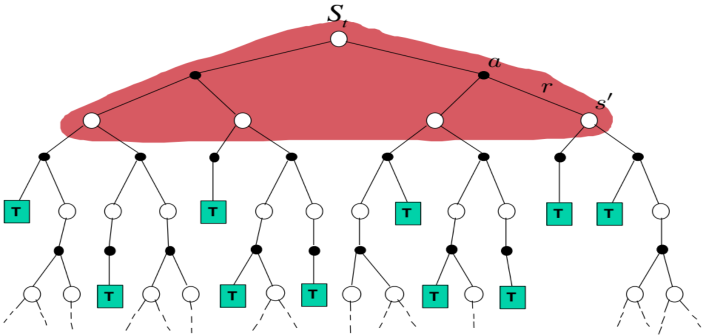
$$V(S_t) \leftarrow V(S_t) + \alpha\left[R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right]$$

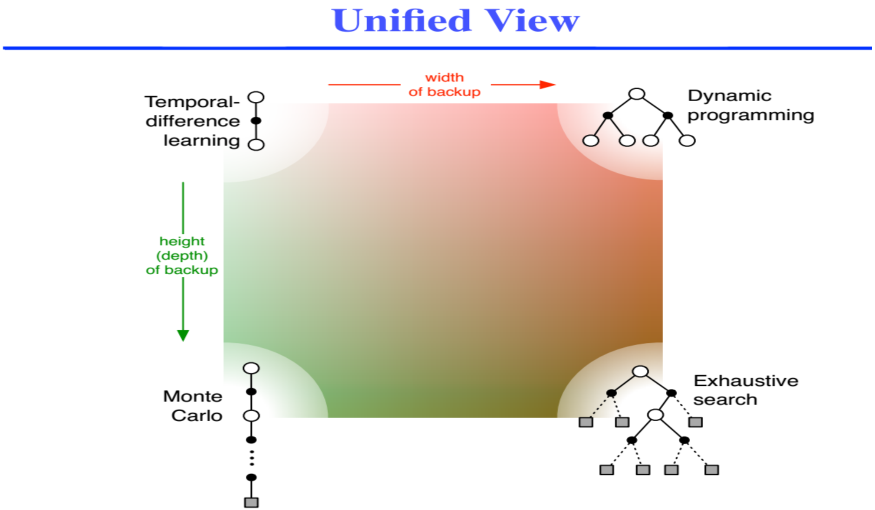


DP backup Diagram

cf. Dynamic Programming

$$V(S_t) \leftarrow E_\pi\left[R_{t+1} + \gamma V(S_{t+1})\right]$$




Unified View
width
of backup
Temporal-
difference
learning
Dynamic
programming
height
(depth)
of backup
Monte
Carlo
Exhaustive
search

Tabular n-step Bootstrapping

- Tabular TD Prediction bootstraps the Value Function with update:

$$V(S_t) \leftarrow V(S_t) + \alpha \cdot (R_{t+1} + \gamma \cdot V(S_{t+1}) - V(S_t))$$

- So it's natural to extend this to bootstrapping with 2 steps ahead:

$$V(S_t) \leftarrow V(S_t) + \alpha \cdot (R_{t+1} + \gamma \cdot R_{t+2} + \gamma^2 \cdot V(S_{t+2}) - V(S_t))$$

- Generalize to bootstrapping with $n \geq 1$ time steps ahead:

$$V(S_t) \leftarrow V(S_t) + \alpha \cdot (G_{t,n} - V(S_t))$$

- $G_{t,n}$ (known as n-step bootstrapped return) is defined as:

$$\begin{aligned} G_{t,n} &= \sum_{i=t+1}^{t+n} \gamma^{i-t-1} \cdot R_i + \gamma^n \cdot V(S_{t+n}) \\ &= R_{t+1} + \gamma \cdot R_{t+2} + \gamma^2 \cdot R_{t+3} + \ldots + \gamma^{n-1} \cdot R_{t+n} + \gamma^n \cdot V(S_{t+n}) \end{aligned}$$

n-step Bootstrapping with Function Approximation

- Generalizing this to the case of Function Approximation, we get:

$$\Delta \boldsymbol{w} = \alpha \cdot (G_{t,n} - V(S_t; \boldsymbol{w})) \cdot \nabla_{\boldsymbol{w}} V(S_t; \boldsymbol{w})$$

- This looks similar to formula for parameters update for MC and TD
- In terms of conceptualizing the change in parameters as product of:
    - *Learning Rate* α
    - *n-step Bootstrapped Error* $G_{t,n} - V(S_t; \boldsymbol{w})$
    - *Estimate Gradient* of the conditional expected return $V(S_t; \boldsymbol{w})$ with respect to the parameters $\boldsymbol{w}$
- n serves as a parameter taking us across the spectrum from TD to MC
- $n = 1$ is the case of TD while sufficiently large n is the case of MC

λ-Return Prediction Algorithm

- Instead of $G_{t,n}$, a valid target is a weighted-average target:

$$\sum_{n=1}^{N} u_n \cdot G_{t,n} + u \cdot G_t \text{ where } u + \sum_{n=1}^{N} u_n = 1$$

- Any of the u_n or u can be 0, as long as they all sum up to 1
- The λ-Return target is a special case of weights u_n and u

$$u_n = (1-\lambda) \cdot \lambda^{n-1} \text{ for all } n = 1, \ldots, T-t-1$$

$$u_n = 0 \text{ for all } n \geq T-t \text{ and } u = \lambda^{T-t-1}$$

- We denote the λ-Return target as $G_t^{(\lambda)}$, defined as:

$$G_t^{(\lambda)} = (1-\lambda) \cdot \sum_{n=1}^{T-t-1} \lambda^{n-1} \cdot G_{t,n} + \lambda^{T-t-1} \cdot G_t$$

$$\Delta \boldsymbol{w} = \alpha \cdot (G_t^{(\lambda)} - V(S_t; \boldsymbol{w})) \cdot \nabla_{\boldsymbol{w}} V(S_t; \boldsymbol{w})$$

Online versus Offline

- Note that for $\lambda = 0$, the λ-Return target reduces to the TD target
- Note that for $\lambda = 1$, the λ-Return target reduces to the MC target G_t
- λ parameter enables us to finely tune from TD ($\lambda = 0$) to MC ($\lambda = 1$)
- Note that for $\lambda > 0$, updates are made only at the end of an episode
- Algorithms updating at end of episodes known as *Offline Algorithms*
- Online algorithms (updates after each time step) are appealing:
  - Updated VF can be utilized immediately for next time step's update
  - This facilitates continuous/fast learning
- Can we have a similar λ-tunable online algorithm for Prediction?
- Yes - this is known as the TD(λ) Prediction algorithm

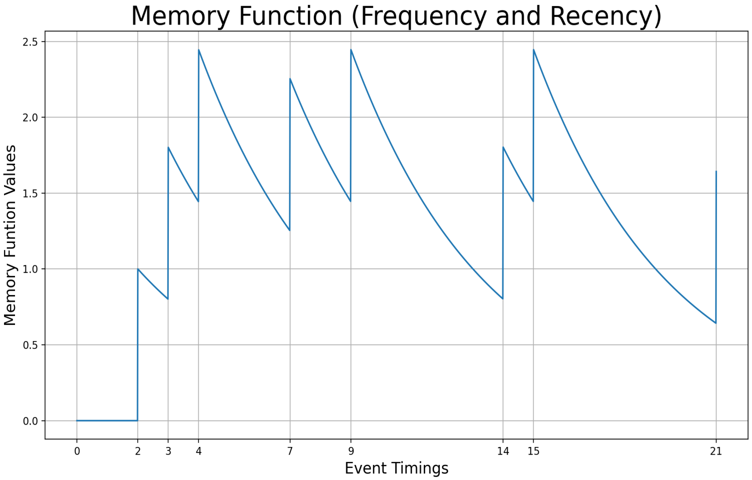
Memory Function

- TD(λ) algorithm is based on the concept of *Eligibility Traces*
- We introduce the concept by defining a *Memory Function* $M(t)$
- Assume an event occurs at times $t_1 < t_2 < \ldots < t_n \in \mathbb{R}_{\geq 0}$
- We want $M(t)$ to remember the $\#$ of times the event has occurred
- But we also want it to have an element of "forgetfulness"
- Recent event-occurrences remembered better than older occurrences
- We want $M(\cdot)$ to give us a time-decayed count of event-occurrences

$$M(t) = \begin{cases} \mathbb{I}_{t=t_1} & \text{if } t \leq t_1, \\ M(t_i) \cdot \theta^{t-t_i} + \mathbb{I}_{t=t_{i+1}} & \text{if } t_i < t \leq t_{i+1} \text{ for any } 1 \leq i < n, \\ M(t_n) \cdot \theta^{t-t_n} & \text{otherwise (i.e., } t > t_n) \end{cases}$$

- There's an uptick of 1 each time the event occurs, but it decays by a factor of $\theta^{\Delta t}$ over any interval Δt where the event doesn't occur
- Thus, $M(\cdot)$ captures the notion of frequency as well as recency

Memory Function with $\theta = 0.8$



Eligibility Traces and Tabular TD(λ) Prediction

- Assume a finite state space with non-terminals $\mathcal{N} = \{s_1, s_2, \ldots, s_m\}$
- Eligibility Trace for each state $s \in S$ is defined as the Memory function $M(\cdot)$ with $\theta = \gamma \cdot \lambda$, and the event timings are the time steps at which the state s occurs in a trace experience
- Eligibility traces at time t is a function $E_t : \mathcal{N} \rightarrow \mathbb{R}_{\geq 0}$:

$$E_0(s) = 0, \text{ for all } s \in \mathcal{N}$$

$$E_t(s) = \gamma \cdot \lambda \cdot E_{t-1}(s) + \mathbb{I}_{S_t = s}, \text{ for all } s \in \mathcal{N}, \text{ for all } t = 1, 2, \ldots$$

- Tabular TD(λ) Prediction algorithm performs following update at each time step t in each trace experience:

$$V(s) \leftarrow V(s) + \alpha \cdot (R_{t+1} + \gamma \cdot V(S_{t+1}) - V(S_t)) \cdot E_t(s), \textit{ for all } s \in \mathcal{N}$$

"Equivalence" of TD(λ) and λ-Return

- TD(λ) is an online algorithm, similar to TD
- But unlike TD, we update the VF *for all* states at each time step
- VF update for each state is proportional to TD-Error δ_t (like TD)
- But here, δ_t is scaled by $E_t(s)$ for each state s at each t

$$V(s) \leftarrow V(s) + \alpha \cdot \delta_t \cdot E_t(s), \textit{ for all } s \in \mathcal{N}$$

- But how is TD(λ) Prediction linked to the λ-Return Prediction?
- It turns out that if we made all updates in an offline manner, then sum of updates for a fixed state $s \in \mathcal{N}$ over entire trace experience equals (offline) update for s in the λ-Return prediction algorithm

Theorem

$$\sum_{t=0}^{T-1} \alpha \cdot \delta_t \cdot E_t(s) = \sum_{t=0}^{T-1} \alpha \cdot (G_t^{(\lambda)} - V(S_t)) \cdot \mathbb{I}_{S_t=s}, \textit{ for all } s \in \mathcal{N}$$

Proof

$$
\begin{aligned}
G_t^{(\lambda)} =& (1-\lambda) \cdot \lambda^0 \cdot (R_{t+1} + \gamma \cdot V(S_{t+1})) \\
&+ (1-\lambda) \cdot \lambda^1 \cdot (R_{t+1} + \gamma \cdot R_{t+2} + \gamma^2 \cdot V(S_{t+2})) \\
&+ (1-\lambda) \cdot \lambda^2 \cdot (R_{t+1} + \gamma \cdot R_{t+2} + \gamma^2 \cdot R_{t+3} + \gamma^3 \cdot V(S_{t+2})) \\
&+ \ldots \\
=& (\gamma\lambda)^0 \cdot (R_{t+1} + \gamma \cdot V(S_{t+1}) - \gamma\lambda \cdot V(S_{t+1})) \\
&+ (\gamma\lambda)^1 \cdot (R_{t+2} + \gamma \cdot V(S_{t+2}) - \gamma\lambda \cdot V(S_{t+2})) \\
&+ (\gamma\lambda)^2 \cdot (R_{t+3} + \gamma \cdot V(S_{t+3}) - \gamma\lambda \cdot V(S_{t+3})) \\
&+ \ldots
\end{aligned}
$$

Proof

$$
\begin{aligned}
G_t^{(\lambda)} =& (\gamma\lambda)^0 \cdot (R_{t+1} + \gamma \cdot V(S_{t+1}) - \gamma\lambda \cdot V(S_{t+1})) \\
&+ (\gamma\lambda)^1 \cdot (R_{t+2} + \gamma \cdot V(S_{t+2}) - \gamma\lambda \cdot V(S_{t+2})) \\
&+ (\gamma\lambda)^2 \cdot (R_{t+3} + \gamma \cdot V(S_{t+3}) - \gamma\lambda \cdot V(S_{t+3})) \\
&+ \ldots
\end{aligned}
$$

$$
\begin{aligned}
G_t^{(\lambda)} - V(S_t) =& (\gamma\lambda)^0 \cdot (R_{t+1} + \gamma \cdot V(S_{t+1}) - V(S_t)) \\
&+ (\gamma\lambda)^1 \cdot (R_{t+2} + \gamma \cdot V(S_{t+2}) - V(S_{t+1})) \\
&+ (\gamma\lambda)^2 \cdot (R_{t+3} + \gamma \cdot V(S_{t+3}) - V(S_{t+2})) \\
&+ \ldots \\
=& \delta_t + \gamma\lambda \cdot \delta_{t+1} + (\gamma\lambda)^2 \cdot \delta_{t+2} + \ldots
\end{aligned}
$$

Proof

Now assume that a specific non-terminal state s appears at time steps $t_1, t_2, \ldots, t_n$. Then,

$$\begin{aligned}\sum_{t=0}^{T-1} \alpha \cdot (G_t^{(\lambda)} - V(S_t)) \cdot \mathbb{I}_{S_t=s} &= \sum_{i=1}^{n} \alpha \cdot (G_{t_i}^{(\lambda)} - V(S_{t_i})) \\ &= \sum_{i=1}^{n} \alpha \cdot (\delta_{t_i} + \gamma\lambda \cdot \delta_{t_i+1} + (\gamma\lambda)^2 \cdot \delta_{t_i+2} + \ldots) \\ &= \sum_{t=0}^{T-1} \alpha \cdot \delta_t \cdot E_t(s)\end{aligned}$$

$\square$

TD(0) and TD(1) with Offline Updates

- To be clear, TD(λ) Prediction is an online algorithm
- So *not the same* as *offline* λ-Return Prediction
- If we modified TD(λ) to be offline, they'd be equivalent
- Offline version of TD(λ) would not update VF at each step
- Accumulate changes in buffer, update VF offline with buffer contents
- If we set $\lambda = 0$, $E_t(s) = \mathbb{I}_{S_t=s}$ and so, the update reduces to:

$$V(S_t) \leftarrow V(S_t) + \alpha \cdot \delta_t$$

- This is exactly the TD update. So, TD is often refered to as TD(0)
- If we set $\lambda = 1$ with episodic traces, sum of all VF updates for a state over a trace experience is equal to it's VF update in Every-Visit MC
- Hence, Offline TD(1) is equivalent to Every-Visit MC

TD(λ) Prediction with Function Approximation

- Generalize TD(λ) to the case of function approximation
- Data-Type of eligibility traces same as func-approx parameters $\boldsymbol{w}$
- So here we denote eligibility traces at time t as simply $\boldsymbol{E}_t$
- Initialize $\boldsymbol{E}_0$ to 0 for each component in it's data type
- Then, for each time step $t > 0$, we define $\boldsymbol{E}_t$ recursively:

$$\boldsymbol{E}_t = \gamma\lambda \cdot \boldsymbol{E}_{t-1} + \nabla_{\boldsymbol{w}} V(S_t; \boldsymbol{w})$$

- VF approximation update at each time step t is as follows:

$$\Delta \boldsymbol{w} = \alpha \cdot (R_{t+1} + \gamma \cdot V(S_{t+1}; \boldsymbol{w}) - V(S_t; \boldsymbol{w})) \cdot \boldsymbol{E}_t$$

- Expressed more succinctly in terms of function-approx TD-Error δ_t:

$$\Delta \boldsymbol{w} = \alpha \cdot \delta_t \cdot \boldsymbol{E}_t$$

Key Takeaways from this Chapter

- Bias-Variance tradeoff of TD versus MC
- MC learns the mean of the observed returns while TD learns something "deeper" - it implicitly estimates an MRP from given data and produces the Value Function of the implicitly-estimated MRP
- Understanding TD versus MC versus DP from the perspectives of:
    - “Bootstrapping”
    - “Experiencing”
- “Equivalence” of λ-Return Prediction and TD(λ) Prediction
- TD is equivalent to TD(0) and MC is “equivalent” to TD(1)