ElMag Zusammenfassung

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1 Vector Analysis Recap

1.1 Partial derivatives

Definition 1 (Partial derivative)**.** A vector valued function $f : \mathbb{R}^m \to \mathbb{R}$, with $\mathbf{v} \in \mathbb{R}^m$, has a partial derivative with respect to v_i defined as

$$\partial_{v_i} f(\mathbf{v}) = \frac{\partial f}{\partial v_i} = \lim_{h \to 0} \frac{f(\mathbf{v} + h\hat{\mathbf{e}}_i) - f(\mathbf{v})}{h}$$

Theorem 1 (Integration of partial derivatives)**.** Let $f : \mathbb{R}^m \to \mathbb{R}$ be a partially differentiable function of many x_i. When x_i is *indipendent* with respect to all other x_j $(0 < j \leq m, j \neq i)$ then

$$\int \partial_{x_i} f \, dx_i = f + C,$$

where C is a function of $x_1, \dots, x_m$ but not of x_i.

To illustrate the previous theorem, in a simpler case with $f(x, y)$, we get

$$\int \partial_x f(x, y) \, dx = f(x, y) + C(y).$$

Beware that this is valid only if x and y are indipendent. If there is a relation $x(y)$ or $y(x)$ the above does not hold.

1.2 Vector derivatives

Definition 2 (Gradient vector)**.** The *gradient* of a function $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^m$ is a column vector containing the partial derivatives in each direction.

$$\boldsymbol{\nabla} f(\mathbf{x}) = \sum_{i=1}^{m} \partial_{x_i} f(\mathbf{x}) \hat{\mathbf{e}}_i = \begin{pmatrix} \partial_{x_1} f(\mathbf{x}) \\ \vdots \\ \partial_{x_m} f(\mathbf{x}) \end{pmatrix}$$

Theorem 2 (Gradient in curvilinear coordinates)**.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field. In cylindrical coordinates (r, ϕ, z)

$$\boldsymbol{\nabla} f = \hat{\mathbf{r}} \, \partial_r f + \hat{\boldsymbol{\phi}} \, \frac{1}{r} \partial_\phi f + \hat{\mathbf{z}} \, \partial_z f,$$

and in spherical coordinates (r, θ, ϕ)

$$\boldsymbol{\nabla} f = \hat{\mathbf{r}} \, \partial_r f + \hat{\boldsymbol{\theta}} \, \frac{1}{r} \partial_\theta f + \hat{\boldsymbol{\phi}} \, \frac{1}{r \sin\theta} \partial_\phi f.$$

Definition 3 (Divergence)**.** Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$ be a vector field. The divergence of $\mathbf{F} = (F_{x_1}, \dots, F_{x_m})^t$ is

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \sum_{i=1}^{m} \partial_{x_i} F_{x_i},$$

as suggested by the (ab)use of the dot product notation.

Theorem 3 (Divergence in curvilinear coordinates)**.** Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a field. In cylindrical coordinates (r, ϕ, z)

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r} \partial_r (r F_r) + \frac{1}{r} \partial_\phi F_\phi + \partial_z F_z,$$

and in spherical coordinates (r, θ, ϕ)

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r^2} \partial_r (r^2 F_r) &+ \frac{1}{r \sin\theta} \partial_\theta (\sin\theta F_\theta) \\ &+ \frac{1}{r \sin\theta} \partial_\phi F_\phi \end{aligned}$$

Theorem 4 (Divergence theorem, Gauss's theorem)**.** Because the flux on the boundary ∂V of a volume V contains information of the field inside of V, it is possible relate the two with

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F} \, dv = \oint_{\partial V} \mathbf{F} \cdot d\mathbf{s}.$$

Definition 4 (Curl)**.** Let $\mathbf{F}$ be a vector field. In 2 dimensions

$$\boldsymbol{\nabla} \times \mathbf{F} = (\partial_x F_y - \partial_y F_x) \, \hat{\mathbf{z}}.$$

And in 3D

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{pmatrix} \partial_y F_z - \partial_z F_y \\ \partial_z F_x - \partial_x F_z \\ \partial_x F_y - \partial_y F_x \end{pmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix}.$$

Definition 5 (Curl in curvilinear coordinates)**.** Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a field. In cylindrical coordinates (r, ϕ, z)

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{F} = &\left(\frac{1}{r} \partial_\phi F_z - \partial_z F_\phi \right) \hat{\mathbf{r}} \\ &+ (\partial_z F_r - \partial_r F_z) \, \hat{\boldsymbol{\phi}} \\ &+ \frac{1}{r} \Big[\partial_r (r F_\phi) - \partial_\phi F_r \Big] \hat{\mathbf{z}}, \end{aligned}$$

and in spherical coordinates (r, θ, ϕ)

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{F} = &\frac{1}{r \sin\theta} \Big[\partial_\theta (\sin\theta F_\phi) - \partial_\phi F_\theta \Big] \hat{\mathbf{r}} \\ &+ \frac{1}{r} \left[\frac{1}{\sin\theta} \partial_\phi F_r - \partial_r (r F_\phi) \right] \hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r} \Big[\partial_r (r F_\theta) - \partial_\theta F_r \Big] \hat{\boldsymbol{\phi}}. \end{aligned}$$

Theorem 5 (Stokes' theorem)**.**

$$\int_S \boldsymbol{\nabla} \times \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

1.3 Second vector derivatives

Definition 6 (Laplacian operator)**.** A second vector derivative is so important that it has a special name. For a scalar function $f : \mathbb{R}^m \to \mathbb{R}$ the divergence of the gradient

$$\nabla^2 f = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = \sum_{i=1}^{m} \partial_{x_i}^2 f_{x_i}$$

is called the *Laplacian operator*.

Theorem 6 (Laplacian in curvilinear coordinates)**.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field. In cylindrical coordinates (r, ϕ, z)

$$\nabla^2 f = \frac{1}{r} \partial_r (r \partial_r f) + \frac{1}{r^2} \partial_\phi^2 f + \partial_z^2 f$$

and in spherical coordinates (r, θ, ϕ)

$$\nabla^2 f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 f.$$

Definition 7 (Vector Laplacian)**.** The Laplacian operator can be extended on a vector field $\mathbf{F}$ to the *Laplacian vector* by applying the Laplacian to each component:

$$\boldsymbol{\nabla}^2\,\mathbf{F} = (\nabla^2 F_x)\hat{\mathbf{x}} + (\nabla^2 F_y)\hat{\mathbf{y}} + (\nabla^2 F_z)\hat{\mathbf{z}}.$$

The vector Laplacian can also be defined as

$$\boldsymbol{\nabla}^2\,\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}).$$

Theorem 7 (Product rules and second derivatives)**.** Let f, g be sufficiently differentiable scalar functions $D \subseteq \mathbb{R}^m \to \mathbb{R}$ and $\mathbf{A}, \mathbf{B}$ be sufficiently differentiable vector fields in $\mathbb{R}^m$ (with $m = 2$ or 3 for equations with the curl).

- Rules with the gradient

$$\begin{aligned}
\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) &= \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} + \boldsymbol{\nabla}^2\,\mathbf{A}\\
\boldsymbol{\nabla}(f\cdot g) &= (\boldsymbol{\nabla}f)\cdot g + f\cdot\boldsymbol{\nabla}g\\
\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}\\
&\quad + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})
\end{aligned}$$

- Rules with the divergence

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}f) &= \nabla^2 f\\
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) &= 0\\
\boldsymbol{\nabla}\cdot(f\cdot\mathbf{A}) &= (\boldsymbol{\nabla}f)\cdot\mathbf{A} + f\cdot(\boldsymbol{\nabla}\cdot\mathbf{A})\\
\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) &= (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})
\end{aligned}$$

- Rules with the curl

$$\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f) &= \mathbf{0}\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}^2\,\mathbf{A}\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}^2\,\mathbf{A}) &= \boldsymbol{\nabla}^2(\boldsymbol{\nabla}\times\mathbf{A})\\
\boldsymbol{\nabla}\times(f\cdot\mathbf{A}) &= (\boldsymbol{\nabla}f)\times\mathbf{A} + f\cdot\boldsymbol{\nabla}\times\mathbf{A}\\
\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) &= (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}\\
&\quad + \mathbf{A}\cdot(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}\cdot(\boldsymbol{\nabla}\cdot\mathbf{A})
\end{aligned}$$

2 Electrodynamics Recap

2.1 Maxwell's equations

Maxwell's equations in matter in their integral form are

$$\oint_{\partial S}\mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S\mathbf{B}\cdot d\mathbf{s}, \tag{1a}$$

$$\oint_{\partial S}\mathbf{H}\cdot d\mathbf{l} = \int_S(\mathbf{J} + \partial_t\mathbf{D})\cdot d\mathbf{s}, \tag{1b}$$

$$\oint_{\partial V}\mathbf{D}\cdot d\mathbf{s} = \int_V\rho\,dv, \tag{1c}$$

$$\oint_{\partial V}\mathbf{B}\cdot d\mathbf{s} = 0. \tag{1d}$$

Where $\mathbf{J}$ and ρ are the *free current density* and *free charge density* respectively.

2.2 Linear materials and boundary conditions

Inside of so called isotropic linear materials fluxes and current densities are proportional and parallel to the fields, i.e.

$$\mathbf{D} = \varepsilon\mathbf{E}, \qquad \mathbf{J} = \sigma\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}.$$

Where two materials meet the following boundary conditions must be satisfied. For the perpendicular component:

$$\hat{\mathbf{n}}\cdot\mathbf{D}_1 = \hat{\mathbf{n}}\cdot\mathbf{D}_2 + \rho_s \tag{2a}$$

$$\hat{\mathbf{n}}\cdot\mathbf{J}_1 = \hat{\mathbf{n}}\cdot\mathbf{J}_2 - \partial_t\rho_s \tag{2b}$$

$$\hat{\mathbf{n}}\cdot\mathbf{B}_1 = \hat{\mathbf{n}}\cdot\mathbf{B}_2 - \partial_t\rho_s \tag{2c}$$

and for the tangential component:

$$\hat{\mathbf{n}}\times\mathbf{E}_1 = \hat{\mathbf{n}}\times\mathbf{E}_2 \tag{3a}$$

$$\hat{\mathbf{n}}\times\mathbf{H}_1 = \hat{\mathbf{n}}\times\mathbf{H}_2 + \mathbf{J}_s \tag{3b}$$

$$\hat{\mathbf{n}}\times\mathbf{M}_1 = \hat{\mathbf{n}}\times\mathbf{M}_2 + \mathbf{J}_{s,m} \tag{3c}$$

2.3 Potentials

Because $\mathbf{E}$ is often conservative ($\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$), and $\boldsymbol{\nabla}\cdot\mathbf{B}$ is always zero, it is often useful to use *potentials* to describe these quantities instead. The electric scalar potential and magnetic vector potentials are in their integral form:

$$\varphi = \int_{\mathsf{A}}^{\mathsf{B}}\mathbf{E}\cdot d\mathbf{l}, \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}dv}{R}.$$

With differential operators:

$$\mathbf{E} = -\boldsymbol{\nabla}\varphi, \qquad \mu_0\mathbf{J} = -\boldsymbol{\nabla}^2\,\mathbf{A}.$$

By taking the divergence on both sides of the equation with the electric field we get $\rho/\varepsilon = -\nabla^2\varphi$, which also contains the Laplacian operator. We will study equations with of form in §4. Potentials, are continuous quantities from any direction, so we have the boundary conditions

$$\varphi_1 = \varphi_2, \qquad \mathbf{A}_1 = \mathbf{A}_2.$$

However their derivatives inherit the discontinuity

$$\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\varphi_1 = \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\varphi_2, \qquad \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{A}_1 = \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{A}_2.$$

2.4 Classic problems

Grounded pole

> A pole (mast) is anchored to the ground using a metal half sphere ($\sigma \to \infty$) of radius a, that is surrounded by the ground which has a conductivity $\sigma(r)$. What is the resistance of the ground?

We are looking for $R = U/I$, so we try to find an expression $U(I)/I$. To solve this problem we consider a current I on the pole towards the ground, which creates a current density $\mathbf{J}$ on a half sphere S of radius r, that is related to I by:

$$\begin{aligned} I &= \int_S \mathbf{J} \cdot d\mathbf{s} = \int_S \sigma(r)\mathbf{E} \cdot d\mathbf{s} \\ &= \sigma(r) E \int_S r^2 \sin(\theta) d\theta d\phi. \end{aligned}$$

Because $\mathbf{E}$ and $\sigma(r)$ are constant on the surface (do not depend on θ or ϕ), we can solve for the electric field to get

$$\mathbf{E}(r) = \frac{I\hat{\mathbf{r}}}{2\pi r^2 \sigma(r)}.$$

With the electric field as a function of I, we can now find the voltage as a function of the current:

$$U(I) = \int_a^\infty \mathbf{E}(r) \cdot d\mathbf{r} = \frac{I}{2\pi} \int_a^\infty \frac{dr}{r^2\sigma(r)},$$

and finally divide by I to get

$$R = \frac{1}{2\pi} \int_a^\infty \frac{dr}{r^2 \sigma(r)}. \tag{4}$$

Magnetic field around a conductor

> Though a conductor of radius a flows a homogeneous currrent density $J_0\hat{\mathbf{z}}$. Compute the flux density $\mathbf{B}$.

We let C be an Amperian contour, that is a disc of radius r around the axis of the conductor, and then simply use Ampere's law

$$\oint_{\partial C} \mathbf{H} \cdot d\mathbf{l} = \int_C \mathbf{J}(r) \cdot d\mathbf{s}.$$

Because of symmetry, the left side simplifies down to $2\pi r H$, thus

$$\mathbf{H} = \frac{\hat{\boldsymbol{\phi}}}{2\pi r} \int_C \mathbf{J}(r) \cdot d\mathbf{s} = \begin{cases} rJ_0\hat{\boldsymbol{\phi}}, & r \leq a \\ a^2 J_0 \hat{\boldsymbol{\phi}}/(2r), & r > a \end{cases}$$

and finally $\mathbf{B} = \mu\mathbf{H}$. So the flux density is linear in the material, and proportional to $1/r$ outside.

If we now wish to formulate this as a boundary value problem, we apply Stoke's theorem to Ampere's law to get

$$\int_S \boldsymbol{\nabla} \times \mathbf{H} \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s},$$

then by removing the integrals and multiplying both sides by μ we get

$$\begin{aligned} \mu\mathbf{J} &= \boldsymbol{\nabla} \times \mu\mathbf{H} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A} \\ &= \big(\underbrace{\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A})}_{=0} - \boldsymbol{\nabla}^2 \mathbf{A}\big) = -\boldsymbol{\nabla}^2 \mathbf{A}. \end{aligned}$$

We know that in $\hat{\mathbf{A}} = \hat{\mathbf{J}} = \hat{\mathbf{z}}$, so in cylindrical coordinates the vector Laplacian simplifies down to a single dimension

$$\mu J(r) = -\frac{1}{r}\partial_r(r\partial_r A_z), \tag{5}$$

an ODE in r.

The boundary conditions for this problem are: that at the border between the conductor the A_z is continuous, and the derivative $\partial_r A_z$ is also continuous (because $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ is continuous), lastly when $r = 0$ we want A_z to not be infinite, so we set it to zero. Algebraically, if 1 is inside the conductor and 2 outside:

$$A_{z1}(0) = 0, \tag{6a}$$
$$A_{z1}(a) = A_{z2}(a), \tag{6b}$$
$$\partial_r A_{z1}(a) = \partial_r A_{z2}(a), \tag{6c}$$

We shall now solve (5). Because $J(r) = J_0$ for $r \leq a$ and zero elsewhere this is easy. First we do the outside

$$0 = -\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_{z2}}{dr}\right) \implies A_{z2} = K_2 + C_2 \ln r,$$

then similarly the inside gives

$$\begin{aligned} \mu J_0 &= -\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_{z1}}{dr}\right) \\ &\implies A_{z1} = K_1 - \frac{\mu J_0 r^2}{4} + C_1 \ln r. \end{aligned}$$

Finally with the boundary conditions we find the values for C_1, C_2, K_1, K_2:

- Because of (6a) $C_1 = K_1 = 0$;
- With (6b) and (6c) we get a system of equations

  $$-\frac{\mu J_0 a^2}{4} = K_2 + C_2 \ln a, \qquad \frac{\mu J_0 a}{2} = \frac{C_2}{a}.$$

  With the latter we find that $C_2 = \mu J_0 a^2/2$, and thus $K_2 = -\mu J_0 a^2 (1 - 2\ln a)/2$

The final result is

$$\mathbf{A} = \begin{cases} -\mu J_0 r^2\, \hat{\mathbf{z}}/2, & r \leq a \\ -\mu J_0 a^2 (1 + 2\ln(r/a))\, \hat{\mathbf{z}}/4, & r > a. \end{cases}$$

To get the flux density we use $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$:

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \partial_r A_z = \begin{cases} -\mu J_0 r \hat{\boldsymbol{\phi}}, & r \leq a \\ -\mu J_0 a^2 \hat{\boldsymbol{\phi}}/(2r) & r > a. \end{cases}$$

3 Boundary value problems

3.1 Steady-state flow analysis

The equation for the steady-state analysis is

$$\nabla^2 \varphi = 0 \quad \text{for} \quad \mathbf{r} \in \Omega, \tag{7}$$

with its boundary conditions: $\varphi = 0$ for $\mathbf{r} \in \Gamma_e, \varphi = U$ for $\mathbf{r} \in \Gamma_b, \nabla_{\hat{\mathbf{n}}}\varphi = 0$ for $\mathbf{r} \in \Gamma_s$.

3.2 Magnetostatic analysis

The equation for the magnetostatic analysis is

$$\boldsymbol{\nabla}^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad \text{for} \quad \mathbf{r} \in \Omega. \tag{8}$$

3.3 Magnetoquasistatic analysis

The equation for the magnetoquasistatic analysis is

$$\nabla^2 \mathbf{A} - \mu_0 \sigma \partial_t \mathbf{A} = -\mu_0 \mathbf{J}_q \quad \text{for} \quad \mathbf{r} \in \Omega, \tag{9}$$

with its boundary conditions $\hat{\mathbf{n}} \times \mathbf{A} = \mathbf{0}$ for $\mathbf{r} \in \Gamma_i$.

3.4 Electrodynamic analysis

The equationsfor the electrodynamic analysis are

$$\nabla^2 \mathbf{E} - \mu\sigma\partial_t \mathbf{E} - \mu\epsilon\partial_t^2 \mathbf{E} = \mathbf{0}, \tag{10a}$$

$$\nabla^2 \mathbf{H} - \mu\sigma\partial_t \mathbf{H} - \mu\epsilon\partial_t^2 \mathbf{H} = \mathbf{0}. \tag{10b}$$

4 Laplace and Poisson's equations

The so called *Poisson's equation* has the form

$$\nabla^2 \varphi = -\frac{\rho}{\epsilon}.$$

When the right side of the equation is zero, it is also known as *Laplace's equation*.

4.1 Easy solutions of Laplace and Poisson's equations

Geometry with zenithal and azimuthal symmetries (Übung 2) Suppose we have a geometry where, using spherical coordinates, there is a symmetry such that the solution does not depend on ϕ or θ. Then Laplace's equation reduces down to

$$\nabla^2 \varphi = \frac{1}{r^2}\partial_r(r^2\partial_r\varphi) = 0,$$

which has solutions of the form

$$\varphi(r) = \frac{C_1}{r} + C_2.$$

Geometry with azimuthal and translational symmetry (Übung 3) Suppose that when using cylindrical coordinates, the solution does not depend on ϕ or z. Then Laplace's equation becomes

$$\nabla^2 A_z = \frac{1}{r}\partial_r(r\partial_r A_z) = 0.$$

5 Finite element method

The finite element method (FEM) is a popular numerical method for solving partial differential equations, i.e. a boundary value problem. FEM works by discretizing space into a so called *mesh*, and computing the physical quantities (fields) only on it. This reduces a continuous problem (of inifinitely many infinitely small changes) into a linear system of equations that is solved using linear algebra.

5.1 Electrostatic FEM

For an electrostatic boundary value problem, we are trying to solve $\nabla^2 \varphi = 0$ in a domain $\Omega \subset \mathbb{R}^n$, given a charge density distribution $\rho(\mathbf{r})$. From physics we know that in nature (solution) the energy of the system will be minimzed. So we need the energy, which is computed with

$$W_e = \int_\Omega \rho\,\varphi\,ds.$$

Because $\rho = \nabla\cdot \mathbf{D}$, we can rewrite it as

$$\begin{aligned} W_e &= \int_\Omega (\nabla\cdot\mathbf{D})\,\varphi\,ds = \int_\Omega \nabla\cdot(\mathbf{D}\varphi) - \mathbf{D}\cdot(\nabla\varphi)\,ds \\ &= \underbrace{\oint_{\partial\Omega} \mathbf{D}\varphi\cdot d\mathbf{s}}_{\text{electrostatics } \therefore =0} + \int_\Omega \mathbf{D}\cdot(-\nabla\varphi)\,ds \\ &= \int_\Omega \mathbf{D}\cdot\mathbf{E}\,ds = \int_\Omega \varepsilon E^2\,ds = \int_\Omega \varepsilon(\nabla\varphi)^2\,ds \end{aligned}$$

Now we can write the energy W_e both using charge density and potentials, as well as with fields. We can combine both into a single expression by taking half from each:

$$W_e = \frac{1}{2}\int_\Omega \rho\,\varphi\,ds + \frac{1}{2}\int_\Omega \varepsilon(\nabla\varphi)^2\,ds. \tag{11}$$

This energy must be minimized, so $\nabla W_e = \mathbf{0}$, but this cannot be practically solved, it is necessary to approximate φ. Of the domain Ω only a finite number of points M is taken, and the rest of omega is interpolated using a function $N(\mathbf{r})$

$$\varphi^e = \sum_k N_k \varphi^e \tag{12}$$