`ElMag` Zusammenfassung

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1 Vector Analysis Recap

1.1 Partial derivatives

Definition 1 (Partial derivative)**.** A vector valued function $f : \mathbb{R}^m \to \mathbb{R}$, with $\mathbf{v} \in \mathbb{R}^m$, has a partial derivative with respect to v_i defined as

$$\partial_{v_i} f(\mathbf{v}) = \frac{\partial f}{\partial v_i} = \lim_{h \to 0} \frac{f(\mathbf{v} + h\hat{\mathbf{e}}_i) - f(\mathbf{v})}{h}$$

Theorem 1 (Integration of partial derivatives)**.** Let $f : \mathbb{R}^m \to \mathbb{R}$ be a partially differentiable function of many x_i. When x_i is *indipendent* with respect to all other x_j $(0 < j \leq m, j \neq i)$ then

$$\int \partial_{x_i} f \, dx_i = f + C,$$

where C is a function of $x_1, \dots, x_m$ but not of x_i.

To illustrate the previous theorem, in a simpler case with $f(x, y)$, we get

$$\int \partial_x f(x, y) \, dx = f(x, y) + C(y).$$

Beware that this is valid only if x and y are indipendent. If there is a relation $x(y)$ or $y(x)$ the above does not hold.

1.2 Vector derivatives

Definition 2 (Gradient vector)**.** The *gradient* of a function $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^m$ is a column vector containing the partial derivatives in each direction.

$$\boldsymbol{\nabla} f(\mathbf{x}) = \sum_{i=1}^{m} \partial_{x_i} f(\mathbf{x}) \hat{\mathbf{e}}_i = \begin{pmatrix} \partial_{x_1} f(\mathbf{x}) \\ \vdots \\ \partial_{x_m} f(\mathbf{x}) \end{pmatrix}$$

Theorem 2 (Gradient in curvilinear coordinates)**.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field. In cylindrical coordinates (r, ϕ, z)

$$\boldsymbol{\nabla} f = \hat{\mathbf{r}} \, \partial_r f + \hat{\boldsymbol{\phi}} \, \frac{1}{r} \partial_\phi f + \hat{\mathbf{z}} \, \partial_z f,$$

and in spherical coordinates (r, θ, ϕ)

$$\boldsymbol{\nabla} f = \hat{\mathbf{r}} \, \partial_r f + \hat{\boldsymbol{\theta}} \, \frac{1}{r} \partial_\theta f + \hat{\boldsymbol{\phi}} \, \frac{1}{r \sin\theta} \partial_\phi f.$$

Definition 3 (Divergence)**.** Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$ be a vector field. The divergence of $\mathbf{F} = (F_{x_1}, \dots, F_{x_m})^t$ is

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \sum_{i=1}^{m} \partial_{x_i} F_{x_i},$$

as suggested by the (ab)use of the dot product notation.

Theorem 3 (Divergence in curvilinear coordinates)**.** Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a field. In cylindrical coordinates (r, ϕ, z)

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r} \partial_r (r F_r) + \frac{1}{r} \partial_\phi F_\phi + \partial_z F_z,$$

and in spherical coordinates (r, θ, ϕ)

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r^2} \partial_r (r^2 F_r) &+ \frac{1}{r \sin\theta} \partial_\theta (\sin\theta F_\theta) \\ &+ \frac{1}{r \sin\theta} \partial_\phi F_\phi \end{aligned}$$

Theorem 4 (Divergence theorem, Gauss's theorem)**.** Because the flux on the boundary ∂V of a volume V contains information of the field inside of V, it is possible relate the two with

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F} \, dv = \oint_{\partial V} \mathbf{F} \cdot d\mathbf{s}.$$

Definition 4 (Curl)**.** Let $\mathbf{F}$ be a vector field. In 2 dimensions

$$\boldsymbol{\nabla} \times \mathbf{F} = (\partial_x F_y - \partial_y F_x) \, \hat{\mathbf{z}}.$$

And in 3D

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{pmatrix} \partial_y F_z - \partial_z F_y \\ \partial_z F_x - \partial_x F_z \\ \partial_x F_y - \partial_y F_x \end{pmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix}.$$

Definition 5 (Curl in curvilinear coordinates)**.** Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a field. In cylindrical coordinates (r, ϕ, z)

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{F} = &\left(\frac{1}{r} \partial_\phi F_z - \partial_z F_\phi \right) \hat{\mathbf{r}} \\ &+ (\partial_z F_r - \partial_r F_z) \, \hat{\boldsymbol{\phi}} \\ &+ \frac{1}{r} \Big[\partial_r (r F_\phi) - \partial_\phi F_r \Big] \hat{\mathbf{z}}, \end{aligned}$$

and in spherical coordinates (r, θ, ϕ)

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{F} = &\frac{1}{r \sin\theta} \Big[\partial_\theta (\sin\theta F_\phi) - \partial_\phi F_\theta \Big] \hat{\mathbf{r}} \\ &+ \frac{1}{r} \left[\frac{1}{\sin\theta} \partial_\phi F_r - \partial_r (r F_\phi) \right] \hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r} \Big[\partial_r (r F_\theta) - \partial_\theta F_r \Big] \hat{\boldsymbol{\phi}}. \end{aligned}$$

Theorem 5 (Stokes' theorem)**.**

$$\int_S \boldsymbol{\nabla} \times \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

1.3 Second vector derivatives

Definition 6 (Laplacian operator)**.** A second vector derivative is so important that it has a special name. For a scalar function $f : \mathbb{R}^m \to \mathbb{R}$ the divergence of the gradient

$$\nabla^2 f = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = \sum_{i=1}^{m} \partial_{x_i}^2 f_{x_i}$$

is called the *Laplacian operator*.

Theorem 6 (Laplacian in curvilinear coordinates)**.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field. In cylindrical coordinates (r, ϕ, z)

$$\nabla^2 f = \frac{1}{r} \partial_r (r \partial_r f) + \frac{1}{r^2} \partial_\phi^2 f + \partial_z^2 f$$

and in spherical coordinates (r, θ, ϕ)

$$\nabla^2 f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 f.$$

Definition 7 (Vector Laplacian)**.** The Laplacian operator can be extended on a vector field $\mathbf{F}$ to the *Laplacian vector* by applying the Laplacian to each component:

$$\boldsymbol{\nabla}^2\,\mathbf{F} = (\nabla^2\,F_x)\hat{\mathbf{x}} + (\nabla^2\,F_y)\hat{\mathbf{y}} + (\nabla^2\,F_z)\hat{\mathbf{z}}.$$

The vector Laplacian can also be defined as

$$\boldsymbol{\nabla}^2\,\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}).$$

Theorem 7 (Product rules and second derivatives)**.** Let f, g be sufficiently differentiable scalar functions $D \subseteq \mathbb{R}^m \to \mathbb{R}$ and $\mathbf{A}, \mathbf{B}$ be sufficiently differentiable vector fields in $\mathbb{R}^m$ (with $m = 2$ or 3 for equations with the curl).

- Rules with the gradient

$$\begin{aligned}
\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) &= \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} + \boldsymbol{\nabla}^2\,\mathbf{A}\\
\boldsymbol{\nabla}(f\cdot g) &= (\boldsymbol{\nabla} f)\cdot g + f\cdot\boldsymbol{\nabla} g\\
\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}\\
&\quad + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})
\end{aligned}$$

- Rules with the divergence

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f) &= \nabla^2\,f\\
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) &= 0\\
\boldsymbol{\nabla}\cdot(f\cdot\mathbf{A}) &= (\boldsymbol{\nabla} f)\cdot\mathbf{A} + f\cdot(\boldsymbol{\nabla}\cdot\mathbf{A})\\
\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) &= (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})
\end{aligned}$$

- Rules with the curl

$$\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) &= \mathbf{0}\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}^2\,\mathbf{A}\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}^2\,\mathbf{A}) &= \boldsymbol{\nabla}^2(\boldsymbol{\nabla}\times\mathbf{A})\\
\boldsymbol{\nabla}\times(f\cdot\mathbf{A}) &= (\boldsymbol{\nabla} f)\times\mathbf{A} + f\cdot\boldsymbol{\nabla}\times\mathbf{A}\\
\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) &= (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}\\
&\quad + \mathbf{A}\cdot(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}\cdot(\boldsymbol{\nabla}\cdot\mathbf{A})
\end{aligned}$$

2 Electrodynamics Recap

2.1 Maxwell's equations

Maxwell's equations in matter in their integral form are

$$\oint_{\partial S}\mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}, \tag{1a}$$

$$\oint_{\partial S}\mathbf{H}\cdot d\mathbf{l} = \int_S(\mathbf{J} + \partial_t\mathbf{D})\cdot d\mathbf{s}, \tag{1b}$$

$$\oint_{\partial V}\mathbf{D}\cdot d\mathbf{s} = \int_V \rho\,dv, \tag{1c}$$

$$\oint_{\partial V}\mathbf{B}\cdot d\mathbf{s} = 0. \tag{1d}$$

Where $\mathbf{J}$ and ρ are the *free current density* and *free charge density* respectively.

2.2 Linear materials and boundary conditions

Inside of so called isotropic linear materials fluxes and current densities are proportional and parallel to the fields, i.e.

$$\mathbf{D} = \varepsilon\mathbf{E}, \qquad \mathbf{J} = \sigma\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}.$$

Where two materials meet the following boundary conditions must be satisfied. For the perpendicular component:

$$\hat{\mathbf{n}}\cdot\mathbf{D}_1 = \hat{\mathbf{n}}\cdot\mathbf{D}_2 + \rho_s \tag{2a}$$

$$\hat{\mathbf{n}}\cdot\mathbf{J}_1 = \hat{\mathbf{n}}\cdot\mathbf{J}_2 - \partial_t\rho_s \tag{2b}$$

$$\hat{\mathbf{n}}\cdot\mathbf{B}_1 = \hat{\mathbf{n}}\cdot\mathbf{B}_2 - \partial_t\rho_s \tag{2c}$$

and for the tangential component:

$$\hat{\mathbf{n}}\times\mathbf{E}_1 = \hat{\mathbf{n}}\times\mathbf{E}_2 \tag{3a}$$

$$\hat{\mathbf{n}}\times\mathbf{H}_1 = \hat{\mathbf{n}}\times\mathbf{H}_2 + \mathbf{J}_s \tag{3b}$$

$$\hat{\mathbf{n}}\times\mathbf{M}_1 = \hat{\mathbf{n}}\times\mathbf{M}_2 + \mathbf{J}_{s,m} \tag{3c}$$

2.3 Potentials

Because $\mathbf{E}$ is often conservative ($\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$), and $\boldsymbol{\nabla}\cdot\mathbf{B}$ is always zero, it is often useful to use *potentials* to describe these quantities instead. The electric scalar potential and magnetic vector potentials are in their integral form:

$$\varphi = \int_{\mathsf{A}}^{\mathsf{B}}\mathbf{E}\cdot d\mathbf{l}, \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}dv}{R}.$$

With differential operators:

$$\mathbf{E} = -\boldsymbol{\nabla}\varphi, \qquad \mu_0\mathbf{J} = -\,\boldsymbol{\nabla}^2\,\mathbf{A}.$$

By taking the divergence on both sides of the equation with the electric field we get $\rho/\varepsilon = -\,\nabla^2\,\varphi$, which also contains the Laplacian operator. We will study equations with of form in §4. Potentials, are continuous quantities from any direction, so we have the boundary conditions

$$\varphi_1 = \varphi_2, \qquad \mathbf{A}_1 = \mathbf{A}_2.$$

However their derivatives inherit the discontinuity

$$\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\varphi_1 = \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\varphi_2, \qquad \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{A}_1 = \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\mathbf{A}_2.$$

3 Boundary value problems

3.1 Steady-state flow analysis

The equation for the steady-state analysis is

$$\nabla^2\,\varphi = 0 \quad \text{for} \quad \mathbf{r}\in\Omega, \tag{4}$$

with its boundary conditions: $\varphi = 0$ for $\mathbf{r}\in\Gamma_e, \varphi = U$ for $\mathbf{r}\in\Gamma_b, \nabla_{\hat{\mathbf{n}}}\varphi = 0$ for $\mathbf{r}\in\Gamma_s$.

3.2 Magnetostatic analysis

The equation for the magnetostatic analysis is

$$\boldsymbol{\nabla}^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad \text{for} \quad \mathbf{r} \in \Omega. \tag{5}$$

3.3 Magnetoquasistatic analysis

The equation for the magnetoquasistatic analysis is

$$\boldsymbol{\nabla}^2 \mathbf{A} - \mu_0 \sigma \partial_t \mathbf{A} = -\mu_0 \mathbf{J}_q \quad \text{for} \quad \mathbf{r} \in \Omega, \tag{6}$$

with its boundary conditions $\hat{\mathbf{n}} \times \mathbf{A} = \mathbf{0}$ for $\mathbf{r} \in \Gamma_i$.

3.4 Electrodynamic analysis

The equations for the electrodynamic analysis are

$$\boldsymbol{\nabla}^2 \mathbf{E} - \mu\sigma\partial_t \mathbf{E} - \mu\varepsilon\partial_t^2 \mathbf{E} = \mathbf{0}, \tag{7a}$$

$$\boldsymbol{\nabla}^2 \mathbf{H} - \mu\sigma\partial_t \mathbf{H} - \mu\varepsilon\partial_t^2 \mathbf{H} = \mathbf{0}. \tag{7b}$$

4 Laplace and Poisson's equations

The so called *Poisson's equation* has the form

$$\nabla^2 \varphi = -\frac{\rho}{\epsilon}.$$

When the right side of the equation is zero, it is also known as *Laplace's equation*.

4.1 Easy solutions of Laplace and Poisson's equations

Geometry with zenithal and azimuthal symmetries (Übung 2) Suppose we have a geometry where, using spherical coordinates, there is a symmetry such that the solution does not depend on ϕ or θ. Then Laplace's equation reduces down to

$$\nabla^2 \varphi = \frac{1}{r^2}\partial_r(r^2 \partial_r \varphi) = 0,$$

which has solutions of the form

$$\varphi(r) = \frac{C_1}{r} + C_2.$$

Geometry with azimuthal and translational symmetry (Übung 3) Suppose that when using cylindrical coordinates, the solution does not depend on ϕ or z. Then Laplace's equation becomes

$$\nabla^2 A_z = \frac{1}{r}\partial_r(r\partial_r A_z) = 0.$$

5 Solved problems

Concentric rings

> Two concentric rings of conductor ($\sigma \to \infty$) have a tall rectangular cross section (ignore boundary effects). The distance from the center to the outer edge of the first ring is a, and the distance to the inner edge of the second is $b > a$. In other words, the air gap between the two rings has width $b - a$. Both rings are coated with a dielectric ($\varepsilon_{r1}, \varepsilon_{r3} > 1$) of thickness d. The inner ring is grounded, while the second has a voltage $u > 0$. Compute the electric field $\mathbf{E}$ between the rings.

To solve this problem we let C be a cylinder centered on the axis of the rings, with radius r and height h. Because the outer ring has a higher potential ($u > 0$) we can assume that $\hat{\mathbf{E}} = -\hat{\mathbf{r}}$. The we use Gauss's law:

$$Q = \oint_C \mathbf{D} \cdot d\mathbf{s} = \oint_C \varepsilon(r)\mathbf{E} \cdot d\mathbf{s} = -2\pi r h \varepsilon(r) E,$$

where

$$\varepsilon(r) = \begin{cases} \varepsilon_{r1}\varepsilon_0 & a \leq r < a + d \\ \varepsilon_0 & a + d \leq r < b - d \\ \varepsilon_{r3}\varepsilon_0 & b - d \leq r < b \end{cases}$$

thus

$$\mathbf{E} = \frac{-\hat{\mathbf{z}}Q}{2\pi h r \varepsilon(r)}.$$

The unknown charge Q can be expressed in terms of the voltage u since

$$u = \int_b^a \mathbf{E} \cdot d\mathbf{r} = \frac{Q}{2\pi h}\int_a^b \frac{dr}{r\varepsilon(r)}.$$

Therefore

$$Q = 2\pi h u \left(\int_a^b \frac{dr}{r\varepsilon(r)}\right)^{-1},$$

which when substituted into $\mathbf{E}$ gives the solution

$$\mathbf{E} = \frac{-u\hat{\mathbf{z}}}{r\varepsilon(r)}\left(\int_a^b \frac{dr}{r\varepsilon(r)}\right)^{-1}.$$

We shall now solve again for $\mathbf{E} = -\boldsymbol{\nabla}\varphi$, but this time using a boundary value problem in terms of φ. By taking the divergence on both sides we get

$$\boldsymbol{\nabla} \cdot \mathbf{E} = -\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\varphi \implies \frac{\rho}{\varepsilon} = -\nabla^2 \varphi,$$

i.e. Poisson's equation. Since between a and b the charge density is zero and thanks to cylindrical symmetry the problem reduces down to:

$$\begin{aligned} 0 = -\nabla^2 \varphi &= -\frac{1}{r}\partial_r(r\partial_r\varphi) \\ \implies 0 &= \frac{d}{dr}\left(r\frac{d\varphi}{dr}\right). \end{aligned} \tag{8}$$

Before solving (8) we need to define the boundary conditions. First, the voltage is fixed at the electrodes to zero and u. Secondly, physically the electric potential must be continuous. Third, the derivative must also be continuous because the flux $\mathbf{D} = -\varepsilon\boldsymbol{\nabla}\varphi$ is continuous. Thus between a and b there are 3 regions: φ_1 in the first dielectic, φ_2 in the air between and φ_3 in the second dielectric. Algebraically:

$$\varphi_1(a) = 0, \tag{9a}$$
$$\varphi_3(b) = u, \tag{9b}$$
$$\varphi_1(a+d) = \varphi_2(a+d), \tag{9c}$$
$$\varphi_2(b-d) = \varphi_3(b-d), \tag{9d}$$
$$\varepsilon_{r1}\partial_r\varphi_1(a+d) = \partial_r\varphi_2(a+d), \tag{9e}$$
$$\partial_r\varphi_2(b-d) = \varepsilon_{r3}\partial_r\varphi_3(b-d). \tag{9f}$$

The ODE (8) has solutions of the form:

$$\varphi_i = K_i + C_i \ln r. \tag{10}$$

The unknowns can be found by using the boundary conditions.

Grounded pole

> A pole (mast) is anchored to the ground using a metal half sphere ($\sigma \to \infty$) of radius a, that is surrounded by the ground which has a conductivity $\sigma(r)$. What is the resistance of the ground?

We are looking for $R = U/I$, so we try to find an expression $U(I)/I$. To solve this problem we consider a current I on the pole towards the ground, which creates a current density $\mathbf{J}$ on a half sphere S of radius r, that is related to I by:

$$I = \int_S \mathbf{J}\cdot d\mathbf{s} = \int_S \sigma(r)\mathbf{E}\cdot d\mathbf{s}$$
$$= \sigma(r)E\int_S r^2\sin(\theta)d\theta d\phi.$$

Because $\mathbf{E}$ and $\sigma(r)$ are constant on the surface (do not depend on θ or ϕ), we can solve for the electric field to get

$$\mathbf{E}(r) = \frac{I\hat{\mathbf{r}}}{2\pi r^2\sigma(r)}.$$

With the electric field as a function of I, we can now find the voltage as a function of the current:

$$U(I) = \int_a^\infty \mathbf{E}(r)\cdot d\mathbf{r} = \frac{I}{2\pi}\int_a^\infty \frac{dr}{r^2\sigma(r)},$$

and finally divide by I to get

$$R = \frac{1}{2\pi}\int_a^\infty \frac{dr}{r^2\sigma(r)}. \tag{11}$$

Reformulating this problem as a boundary value problem, we look for the potential distribution φ to find $R = U/I(U)$. This problem is formulated with the Laplace equation

$$\nabla^2\varphi = \frac{1}{r^2}\partial_r(r^2\partial_r\varphi) = 0,$$

and the boundary conditions:

$$\varphi(a) = U, \tag{12a}$$
$$\varphi(r\to\infty) = 0, \tag{12b}$$

and the requirements for the derivative $-\sigma(r)\boldsymbol{\nabla}\varphi = \mathbf{J}$ to be continuous. If we let

$$\sigma(r) = \begin{cases}\sigma_1, & a \le r < b\\ \sigma_2, & b \le r\end{cases}$$

then we get the boundary condition:

$$\sigma_1\partial_r\varphi_1 = \sigma_2\partial_r\varphi_2, \tag{12c}$$

where φ_1 is in the first conductor, where the potential at its end equals U, and φ_2 is in the second conductor.

Magnetic field around a conductor

> Though a conductor of radius a flows a homogeneous currrent density $J_0\hat{\mathbf{z}}$. Compute the flux density $\mathbf{B}$.

We let C be an Amperian contour, that is a disc of radius r around the axis of the conductor, and then simply use Ampere's law

$$\oint_{\partial C}\mathbf{H}\cdot d\mathbf{l} = \int_C \mathbf{J}(r)\cdot d\mathbf{s}.$$

Because of symmetry, the left side simplifies down to $2\pi r H$, thus

$$\mathbf{H} = \frac{\hat{\boldsymbol{\phi}}}{2\pi r}\int_C \mathbf{J}(r)\cdot d\mathbf{s} = \begin{cases} rJ_0\hat{\boldsymbol{\phi}}, & r\le a\\ a^2J_0\hat{\boldsymbol{\phi}}/(2r), & r>a\end{cases}$$

and finally $\mathbf{B} = \mu\mathbf{H}$. So the flux density is linear in the material, and proportional to $1/r$ outside.

If we now wish to formulate this as a boundary value problem, we apply Stoke's theorem to Ampere's law to get

$$\int_S \boldsymbol{\nabla}\times\mathbf{H}\cdot d\mathbf{s} = \int_S \mathbf{J}\cdot d\mathbf{s},$$

then by removing the integrals and multiplying both sides by μ we get

$$\mu\mathbf{J} = \boldsymbol{\nabla}\times\mu\mathbf{H} = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A}$$
$$= \big(\underbrace{\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})}_{=0} - \boldsymbol{\nabla}^2\mathbf{A}\big) = -\boldsymbol{\nabla}^2\mathbf{A}.$$

We know that in $\hat{\mathbf{A}} = \hat{\mathbf{J}} = \hat{\mathbf{z}}$, so in cylindrical coordinates the vector Laplacian simplifies down to a single dimension

$$\mu J(r) = -\frac{1}{r}\partial_r(r\partial_r A_z), \tag{13}$$

an ODE in r. The boundary conditions for this problem are: that at the border between the conductor the A_z is continuous, and the derivative $\partial_r A_z$ is also continuous (because $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ is continuous), lastly when $r=0$ we want A_z to not be infinite, so we set it to zero. Algebraically, if 1 is inside the conductor and 2 outside:

$$A_{z1}(0) = 0, \tag{14a}$$
$$A_{z1}(a) = A_{z2}(a), \tag{14b}$$
$$\partial_r A_{z1}(a) = \partial_r A_{z2}(a), \tag{14c}$$

We shall now solve (13). Because $J(r) = J_0$ for $r \leq a$ and zero elsewhere this is easy. First we do the outside

$$0 = -\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_{z2}}{dr}\right) \implies A_{z2} = K_2 + C_2 \ln r,$$

then similarly the inside gives

$$\mu J_0 = -\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_{z1}}{dr}\right) \implies A_{z1} = K_1 - \frac{\mu J_0 r^2}{4} + C_1 \ln r.$$

Finally with the boundary conditions we find the values for C_1, C_2, K_1, K_2:

- Because of (14a) $C_1 = K_1 = 0$;
- With (14b) and (14c) we get a system of equations

  $$-\frac{\mu J_0 a^2}{4} = K_2 + C_2 \ln a, \qquad \frac{\mu J_0 a}{2} = \frac{C_2}{a}.$$

  With the latter we find that $C_2 = \mu J_0 a^2/2$, and thus $K_2 = -\mu J_0 a^2(1 - 2\ln a)/2$

The final result is

$$\mathbf{A} = \begin{cases} -\mu J_0 r^2\, \hat{\mathbf{z}}/2, & r \leq a \\ -\mu J_0 a^2(1 + 2\ln(r/a))\, \hat{\mathbf{z}}/4, & r > a. \end{cases}$$

To get the flux density we use $\mathbf{B} = \nabla\times \mathbf{A}$:

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\partial_r A_z = \begin{cases} -\mu J_0 r\hat{\boldsymbol{\phi}}, & r \leq a \\ -\mu J_0 a^2\hat{\boldsymbol{\phi}}/(2r) & r > a. \end{cases}$$

Planar electromagnetic wave

> An electromagnetic wave $\tilde{\mathbf{E}} = \tilde{E}(x)\hat{\mathbf{y}}$ in air ($\sigma = 0, \varepsilon_r = 1$) hits a dielectric wall ($\sigma \approx 0, \varepsilon_r > 1$) on the zy plane at $x = 0$. Compute the reflected and transmitted waves $\tilde{\mathbf{E}}_r$, $\tilde{\mathbf{H}}_r$ $\tilde{\mathbf{E}}_t$ and $\tilde{\mathbf{H}}_t$.

This problem is solved using the wave equation (7a) in the frequency domain:

$$\nabla^2\, \tilde{\mathbf{E}} + \omega^2\mu\varepsilon\tilde{\mathbf{E}} = 0.$$

Because $\tilde{\mathbf{E}} = \tilde{E}(x)\hat{\mathbf{y}}$ the problem is reduced to an ODE:

$$\frac{d^2\tilde{E}}{dx^2} + \omega^2\mu\varepsilon\tilde{E} = 0. \tag{15}$$

With the ansatz $\tilde{E} = Ce^{\lambda x}$, we get the characteristic polynomial and its solutions:

$$(\lambda^2 + \omega^2\mu\varepsilon) = 0 \implies \lambda_{1,2} = \pm j\omega\sqrt{\mu\varepsilon}.$$

Thus the general solution is

$$\tilde{E} = C_t e^{-j\omega x\sqrt{\mu\varepsilon}} + C_r e^{j\omega x\sqrt{\mu\varepsilon}} = \tilde{E}_t + \tilde{E}_r.$$

By inspecting this result we see that there are two waves: one traveling in the positive x direction (i.e. $\exp(-j\omega x\sqrt{\mu\varepsilon})$), and the other traveling in the negative x direction ($\exp(+j\omega x\sqrt{\mu\varepsilon})$). Further analyzing the exponent, we see that dimensionally we have distance over time times the units of $\sqrt{\mu\varepsilon}$. Because the exponent hast to be dimensionless, we can infer that the rest must be the reciprocal of a velocity:

$$\omega x\sqrt{\mu\varepsilon} = \frac{\omega x}{v}, \text{ where } v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c_0}{\sqrt{\mu_r\varepsilon_r}}.$$

Thus in the two materials (air and dielectric) we have the equations:

$$\tilde{E}_1 = C_{t1}e^{-j\omega x/c_0} + C_{r1}e^{j\omega x/c_0}, \tag{16a}$$

$$\tilde{E}_2 = C_{t2}e^{-j\omega x\sqrt{\varepsilon_r}/c_0}. \tag{16b}$$

Since in the dielectric we cannot possibly have a wave in the negative x direction, we can assume that $C_{r2} = 0$. The boundary conditions are given by the physical conditions $\hat{\mathbf{x}} \times \tilde{\mathbf{E}}_1 = \hat{\mathbf{x}} \times \tilde{\mathbf{E}}_2$ and $\hat{\mathbf{x}} \times \tilde{\mathbf{H}}_1 = \hat{\mathbf{x}} \times \tilde{\mathbf{H}}_2$, thus since

$$\tilde{\mathbf{H}} = \frac{-1}{j\omega\mu_0}\nabla\times\tilde{\mathbf{E}} = \frac{-\hat{\mathbf{z}}}{j\omega\mu_0}\frac{d\tilde{E}}{dx}$$

we get the boundary conditions

6 Finite element method

The finite element method (FEM) is a popular numerical method for solving partial differential equations, i.e. a boundary value problem. FEM works by discretizing space into a so called *mesh*, and computing the physical quantities (fields) only on it. This reduces a continuous problem (of inifinitely many infinitely small changes) into a linear system of equations that is solved using linear algebra.

6.1 Electrostatic FEM

For an electrostatic boundary value problem, we are trying to solve $\nabla^2\varphi = 0$ in a domain $\Omega \subset \mathbb{R}^n$, given a charge density distribution $\rho(\mathbf{r})$. From physics we know that in nature (solution) the energy of the system will be minimzed. So we need the energy, which is computed with

$$W_e = \int_\Omega \rho\,\varphi\, ds.$$

Because $\rho = \nabla\cdot\mathbf{D}$, we can rewrite it as

$$\begin{aligned} W_e &= \int_\Omega (\nabla\cdot\mathbf{D})\,\varphi\, ds = \int_\Omega \nabla\cdot(\mathbf{D}\varphi) - \mathbf{D}\cdot(\nabla\varphi)\, ds \\ &= \underbrace{\oint_{\partial\Omega} \mathbf{D}\varphi\cdot d\mathbf{s}}_{\varphi(\mathbf{r}\in\partial\Omega)=0} + \int_\Omega \mathbf{D}\cdot(-\nabla\varphi)\, ds \\ &= \int_\Omega \mathbf{D}\cdot\mathbf{E}\, ds = \int_\Omega \varepsilon E^2\, ds = \int_\Omega \varepsilon(\nabla\varphi)^2\, ds \end{aligned}$$

Now we can write the energy W_e both using charge density and potentials, as well as with fields. We can combine both into a single expression by taking half from each:

$$W_e = \frac{1}{2}\int_\Omega \rho\,\varphi\,ds + \frac{1}{2}\int_\Omega \varepsilon(\boldsymbol{\nabla}\varphi)^2\,ds. \tag{18}$$

This energy must be minimized, so $\boldsymbol{\nabla} W_e = \mathbf{0}$, but this cannot be practically solved, it is necessary to approximate φ. Of the domain Ω only a finite number of points M is taken, and the rest of omega is interpolated using a function $N(\mathbf{r})$

$$\varphi^e = \sum_k N_k \varphi^e \tag{19}$$