# Nachrichtentechnik 2

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## 1  Probability

**Definition 1** (Random experiment). An experiment is called *random* if its outcome cannot be predicted.

**Definition 2** (Sample space and events). The set of all possible outcomes of a random experiment is called the *sample space $S$*. An $\lambda \in S$ is called a *sample point*. A set of sample points $A \subseteq S$ is called an *event*. The set with no sample points $\emptyset$ is the *impossible event*.

**Definition 3** (Complement). The complement of an event $A$ is $\overline{A} = \{\lambda \in S : \lambda \notin A\}$. That is, the event when $A$ does not happen.

**Definition 4** (Union and intersection). Given two events $A, B \subseteq S$ their *union* and *intersection* are respectively:

- $A \cup B = \{\lambda \in S : (\lambda \in A) \vee (\lambda \in B)\}$
- $A \cap B = \{\lambda \in S : (\lambda \in A) \wedge (\lambda \in B)\}$

If $A \cap B = \emptyset$ then $A$ and $B$ are said to be *disjoint*. Both $\cup$ and $\cap$ are associative, commutative and distributive on each other:

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**Theorem 1** (DeMorgan's Law). Let $A, B \subseteq S$

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{A \cap B} = \overline{A} \cup \overline{B} \qquad (1)$$

**Definition 5** (Probability). We assign a real number between 0 and 1 called *probability* to each event in $S$. The following 3 axioms are enough to define it.

1. $P(A) \geq 0$,
2. $P(S) = 1$,
3. $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$.

**Lemma 1** (Useful laws of probability). Although the axiomatic definition is complete, it is not very useful on its own. Here are some other expressions that can be derived from the axioms:

1. $P(A) \leq 1$
2. $P(\overline{A}) = 1 - P(A)$
3. $P(\emptyset) = 0$
4. $P(A) \leq P(B)$ if $A \subseteq B$
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

**Remark 1.** In the case where $S$ has a finite number of events, the probability can be thought of as a function $P : \mathcal{P}(S) \to [0, 1]$ from the power set of $S$ to the interval between 0 and 1.

## 2  Random Variables

**Definition 6** (Random Variable). For a random experiment with sample space $S$, we assign to each sample point $\lambda$

## 3  Random Processes

Previously we defined a random variable as a function $X : S \to E \subset \mathbb{R}$ that assigns a *number* to each event. We will now extend this concept.

**Definition 7** (Random Process). For a random experiment with sample space $S$, we assign to every sample point $\lambda \in S$ a *function* of time. The *random process $X(t, \lambda)$* is effectively $X : \mathbb{R} \times S \to E \subseteq \mathbb{R}$.

**Remark 2.** Notice that for a fixed $\lambda_i$, $X(t, \lambda_i) = x_i(t)$ is indeed a function of time. Whereas for a fixed time $t_j$ is $X(t_j, \lambda) = X_j$, a random variable. Only when both a $\lambda_i$ and $t_j$ are given $X(t_j, \lambda_i)$ is a number.

**Definition 8** (Probability distribution and density function). A random process $X(t, \lambda)$ at a particular time $t$ *is* a random variable with distribution

$$F_X(t, x) = P(X \leq x) \qquad (2)$$
$$= P(\{\lambda \in S : X(t, \lambda) \leq x\}) \qquad (3)$$

and a corresponding density function

$$f_X(t,x) = \frac{\partial F_X(t,x)}{\partial x} \qquad (4)$$

**Remark 3.**