Cost-Push Shocks at the Zero Lower Bound*

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Abstract

I introduce positive cost-push shocks into a New Keynesian model with a zero lower bound (ZLB) on nominal interest rates and analyze its effect on equilibrium properties. I consider two types of ZLB: negative demand shocks (fundamental-driven ZLB model) and sunspot shocks (expectation-driven ZLB model). I find that ZLB binds in a state where both demand/sunspot shock and cost-push shock exist when the impact of the demand/sunspot shock is greater than the impact of the cost-push shock under the optimal discretionary policy. Moreover, a unique reasonable equilibrium exists in the fundamental-driven ZLB model. However, there may be multiple reasonable equilibria in the expectation-driven ZLB model.

Keywords: Cost-Push Shocks, Fundamental-Driven Liquidity Traps, Expectation-Driven Liquidity Traps, Monetary Policy, Discretion
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1 Introduction

In the recent few years, advanced economies have been hit by a rapid surge in inflation. In many advanced countries, inflation has exceeded central bank targets, but the response varies from country to country. For example, FRB, ECB, and BOE raised interest rates, while the BOJ continued to negative interest rate policy. Considering these conditions, how to fight against inflation has become an important question in monetary policy. However, there is no clear answer yet, and indeed the inflation rate has not been successfully suppressed. One factor complicating the issue is the ZLB. Until a few years ago, many developed countries were plagued by the ZLB, especially Japan, which had been mired in deflation and low-interest rates for nearly two decades. Therefore, in considering monetary policy today, we need to consider not only cost-push shocks but also shocks that push nominal interest rates down to the ZLB. However, few theoretical studies of monetary policy have simultaneously considered ZLB and cost-push shocks.

Given this situation, I investigate how the equilibrium properties change when cost-push shocks are introduced in an economy where ZLB binds nominal interest rates. More precisely, my questions are (i) whether ZLB binds or not, (ii) under what conditions equilibrium exists, and (iii) how the allocations and prices look in equilibrium. To answer these questions, I analytically derive the conditions for the existence of Markov perfect equilibrium using a standard New Keynesian model with the ZLB and cost-push shocks. I assume that the central bank sets the nominal interest rate under the optimal discretionary policy.

I consider two types of shocks causing ZLB. The first is negative demand shocks (fundamental-driven ZLB model), which capture persistent negative natural interest rates (Hansen,1939, Summers, 2013). The second is sunspot shocks (expectation-driven ZLB model), which represent decreases in agents' confidence (Benhabib et al. 2001, 2002). I analyze two types of equilibria. In one equilibrium, the cost-push shock is not enough to end a ZLB caused by the demand or sunspot shock, and the ZLB binds whenever demand or confidence is low (Type-I equilibrium). In the other, it is optimal to set the interest rate at a positive level when the positive cost-push shock hits, even if demand or confidence is low (Type-II equilibrium). Consequently, there are four equilibria of interest overall, depending on whether the economy is affected by low demand or low confidence at the time of the cost-push shock.

While there are not many empirical studies on the expectation-driven ZLB, Aruoba et al. (2018) assess the factors of liquidity traps in the United States and Japan using an estimated New Keynesian model. The analysis reveals that Japan in the late 1990s fell into a liquidity trap due to a decline in agent's confidence, while the United States fell into a liquidity trap due to a deterioration in economic fundamentals. Cuba-Borda and Singh (2019), using an estimated DSGE model incorporating government bonds, demonstrate that a DSGE model with expectation-driven ZLB fits Japanese data better than a DSGE model with fundamental-driven ZLB. [1]

The main results of this paper are twofold. First, I find that ZLB binds in a state

[1]They attribute the better fit of the expectation-driven ZLB to the negative correlation between the growth rate of output and inflation. Under the fundamental-driven ZLB, shocks that lead to an increase in inflation also lower real interest rates and increase consumption, resulting in a positive correlation between output and inflation.

where both demand/sunspot shock and cost-push shock exist when the impact of the demand/sunspot shock is greater than the impact of the cost-push shock. This result is intuitive. On one hand, when only negative demand/sunspot shocks are present, it is optimal for the central bank to lower nominal interest rates to mitigate the recession. On the other hand, when only cost-push shocks are present, it is optimal for the central bank to raise nominal interest rates to fight against inflation. Since the optimal actions for the central bank differ for each shock, the ZLB binds or does not in equilibrium depending on which shock, negative demand/sunspot shocks, or positive cost-push shocks, are larger.

Second, I find that in the fundamental-driven ZLB model, either Type-I or Type-II equilibrium exists, while in the expectation-driven ZLB model, Type-I and Type-II equilibrium may exist simultaneously. The intuitive interpretation of this result is as follows. Even though a positive cost-push shock exists, self-fulfilling ZLB binds nominal interest rates if the decline in the agent's confidence is sufficiently persistent. In addition to the equilibrium with self-fulfilling ZLB, there is also an ordinary equilibrium, and thus multiple equilibria exist.

The motivation behind the analysis in this paper is the post-COVID inflation, which has gradually become a subject of research.[2] Walsh (2022) analyzes the impact of delayed responses to rapid inflation. Eggertsson and Kohn (2023) evaluate the policies implemented by the FRB. Harding et al (2023) utilize a macroeconomic model incorporating a non-linear Phillips curve, where the slope becomes steep during high inflationary pressure and flat during weak pressure. Comin et al (2023), using an open-economy New Keynesian model, reveal that approximately half of the increase in inflation in the United States during 2021-2022 can be interpreted as a supply-side constraint.

This paper builds on previous studies that have analyzed the conditions for the existence of Markov perfect equilibrium in a New Keynesian model (Armenter, 2018, Boneva et al, 2016, Eggertsson, 2011, Nakata, 2018). Ascari and Mavroeidis (2022) discuss the existence conditions for the MSV solution (McCallum, 1983, 2004) in a more general framework. The novelty of this paper, compared to previous research, lies in simultaneously considering the ZLB and cost-push shocks.

Finally, there exists a wealth of prior research on optimal monetary and fiscal policies under the ZLB driven by fundamental factors. Regarding optimal monetary policy, notable contributions include works by Eggertsson and Woodford (2003), Adam and Billi (2006, 2007), and Nakov (2008), among others. Optimal fiscal policy is studied by Eggertsson and Woodford (2006), Eggertsson (2006), Schmidt (2013, 2017), Nakata (2016, 2017), Bilbiie et al. (2018). All these papers, however, don't address cost-push shocks coincident with the ZLB constraint: my contribution is to derive the existence condition of Markov perfect equilibrium in models with the ZLB constraint and cost-push shocks.

The structure of this paper is as follows. In Section 2, I describe the model. Section 3 provides analytical results of the conditions for the equilibrium existence and the properties of allocations. In Section 4, I interpret the results obtained in Section 3 using numerical examples. Finally, Section 5 concludes the paper.

[2] In the main text, I introduce papers that have studied post-COVID inflation. Also, there are papers studying the effects of cost-push shocks at the ZLB. Wieland (2019) provides empirical evidence that negative supply shocks are contractionary at the ZLB. Miyamoto et al (2024) reveal that negative oil supply news shocks are less contractionary at the ZLB compared to normal periods.

2 Model

I use a standard infinite-horizon New Keynesian model formulated in discrete time. The economy is inhabited by identical households who consume and work, goods-producing firms that act under monopolistic competition and are subject to Calvo (1983) type of price rigidities, and a central bank. From here, I assume that the one-period nominal interest rate is the only policy instrument. More detailed descriptions of the model can be found in Clarida, Galí and Gertler (1999), Woodford (2003), and Galí (2015). I work with a semi-loglinear version of the model that can be solved in closed form and allows me to derive analytical results.

2.1 Private sector and welfare

Aggregate private sector behavior is represented by a Phillips curve and a consumption Euler equation.

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1} + \varepsilon_t \tag{1}$$

$$y_t = E_t y_{t+1} - \sigma \left(i_t - E_t \pi_{t+1} - r_t^n \right) \tag{2}$$

The equations above are obtained by log-linearizing the equilibrium conditions of the private sector around a zero-inflation steady state. Where π_t is the inflation rate between periods $t-1$ and t, y_t is the output gap, i_t is the nominal interest rate level between periods t and $t+1$, ε_t is the cost-push shocks, r_t^n is the exogenous real natural interest rate, and E_t is the rational expectations operator. The structure of shocks is explained in detail later in this section. The parameters are defined as follows: $\beta \in (0,1)$ is the household's subjective discount rate, $\sigma > 0$ is the elasticity of intertemporal substitution, and κ represents the slope of the Phillips curve. κ is generated as a structural parameter of multiple models, $\kappa = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+\eta\theta)}(\sigma^{-1} + \eta)$, where $\alpha \in (0,1)$ is the proportion of firms unable to optimize prices in a given period, $\eta > 0$ is the inverse of the elasticity of labor supply, and $\theta > 1$ is the price elasticity of demand for differentiated products.

Households welfare at time t is given by the expected discounted sum of current and future utility flows. A second-order approximation to household preferences can be written as

$$V_t = -\frac{1}{2} E_t \sum_{j=0}^{\infty} \beta^j \left[\pi_{t+j}^2 + \lambda y_{t+j}^2 \right] \tag{3}$$

where $\lambda = \frac{\kappa}{\theta}$.

2.2 Central bank

The central bank sets the nominal interest rate under the optimal discretionary policy. In this paper, I assume that the central bank's objective function is equal to the social objective function (3). The central bank, given the value function and policy function for period $t+1$ at each period t, chooses the inflation rate, the output gap, and the nominal interest rate to maximize the objective function under the constraints of the private sector ((1)-(2)) and the zero lower bound constraint ($i_t \geq 0$).

The first-order necessary conditions for this problem are

$$[\kappa\pi_t + \lambda y_t]\, i_t = 0 \tag{4}$$

where $\kappa\pi_t + \lambda y_t = 0$ whenever $i_t > 0$ and $\kappa\pi_t + \lambda y_t < 0$ when $i = 0$. In other words, in each period the central bank aims to stabilize a weighted sum of the current period's inflation rate and the output gap.

2.3 Cost-push shock

This paper assumes that cost-push shocks follow a two-state Markov process.[3] In the high state (hereafter H state), the cost-push shock is $\varepsilon_t = \varepsilon > 0$, while in the normal state (hereafter N state), it is $\varepsilon_t = 0$. The transition probabilities of cost-push shocks are as follows.

$$\text{Prob}(\varepsilon_{t+1} = \varepsilon | \varepsilon_t = \varepsilon) = p_H \tag{5}$$

$$\text{Prob}(\varepsilon_{t+1} = \varepsilon | \varepsilon_t = 0) = 0 \tag{6}$$

The term p_H represents the probability that the economy is in a H state in period t and transitions to a N state in the next period. This probability can be interpreted as the persistence of the shock. Similarly, regarding equation (6), where the shock frequency is 0, it implies that the N state acts as an absorbing state.

2.4 Demand shock

In the fundamental-driven ZLB model, demand shocks and cost-push shocks are introduced. In this model, it is assumed that r_t^n follows a two-state Markov process. In the low state (hereafter L state), $r_t^n = r_L^n < 0$ while in the normal state (hereafter N state), $r_t^n = r^n = 1/\beta - 1$. The transition probabilities for demand shocks are as follows.

$$\text{Prob}(r_{t+1}^n = r_L^n | r_t^n = r_L^n) = p_L \tag{7}$$

$$\text{Prob}(r_{t+1}^n = r_L^n | r_t^n = r^n) = 0 \tag{8}$$

Similar to the case of cost-push shocks, the frequency is 0.

2.5 Sunspot shock

In the expectation-driven ZLB model, sunspot shocks and cost-push shocks are introduced. As mentioned in Mertens and Ravn (2014) and Nie and Roulleau-Pasdeloup (2023), even in the absence of fundamental shocks, agents confidence can change due to sunspot shocks, leading to the possibility of facing a ZLB. In the expectation-driven ZLB model, for all t, $r_t^n = r^n$. In this economy, agent's expectations are influenced by sunspot shocks. Sunspot shocks follow a two-state Markov process, namely the deflation state (hereafter, D state) and the normal state (hereafter, N state), where $(\xi_t \in (\xi_D, \xi_N))$. The transition probabilities are as follows.

$$\text{Prob}(\xi_{t+1} = \xi_D | \xi_t = \xi_D) = p_D \tag{9}$$

[3] This shock structure is a common assumption in theoretical literature (see Christiano et al., 2011, Eggertsson, 2011, Eggertsson and Woodford, 2003, Bilbiie, 2019).

$$\text{Prob}(\xi_{t+1} = \xi_D | \xi_t = \xi_N) = 0 \tag{10}$$

Similar to other shocks, the N state is assumed to be an absorbing state.

2.6 Markov-Perfect equilibrium (fundamental-driven ZLB model)

In kl state ($k \in \{L, N\}$, $l \in \{H, N\}$), the value of the variable x is denoted as x_{kl}. There are 16 potential Markov perfect equilibria depending on whether the ZLB binds in each state. However, analyzing all equilibria is difficult, this paper focuses on the analysis of the following two realistic equilibria[4]: (1) Equilibrium where ZLB binds in the LH and LN states (F-Type-I equilibrium). (2) Equilibrium where ZLB only binds in the LN state (F-Type-II equilibrium).

The **F-Type-I equilibrium** is defined as a vector $\{y_{NH}, \pi_{NH}, i_{NH}, y_{NN}, \pi_{NN}, i_{NN}, y_{LH}, \pi_{LH}, i_{LH}, y_{LN}, \pi_{LN}, i_{LN}\}$ that solves following system of linear equations

$$\begin{aligned}
y_{NH} &= [p_H y_{NH} + (1-p_H) y_{NN}] + \sigma [p_H \pi_{NH} + (1-p_H)\pi_{NN} - i_{NH} + r^n] \\
\pi_{NH} &= \kappa y_{NH} + \beta [p_H \pi_{NH} + (1-p_H)\pi_{NN}] + \varepsilon \\
0 &= \lambda y_{NH} + \kappa \pi_{NH} \\
y_{NN} &= y_{NN} + \sigma (\pi_{NN} - i_{NN} + r^n) \\
\pi_{NN} &= \kappa y_{NN} + \beta \pi_{NN} \\
0 &= \lambda y_{NN} + \kappa \pi_{NN} \\
y_{LH} &= [p_H p_L y_{LH} + p_H(1-p_L) y_{NH} + (1-p_H) p_L y_{LN} + (1-p_H)(1-p_L) y_{NN}] \\
&+ \sigma [p_H p_L \pi_{LH} + p_H(1-p_L)\pi_{NH} + (1-p_H) p_L \pi_{LN} + (1-p_H)(1-p_L)\pi_{NN} - i_{LH} + r^n_L] \\
\pi_{LH} &= \kappa y_{LH} + \beta [[p_H p_L \pi_{LH} + p_H(1-p_L)\pi_{NH} + (1-p_H) p_L \pi_{LN} + (1-p_H)(1-p_L)\pi_{NN}] + \varepsilon \\
i_{LH} &= 0 \\
y_{LN} &= [p_L y_{LN} + (1-p_L) y_{NN}] + \sigma [p_L \pi_{LN} + (1-p_L)\pi_{NN} - i_{LN} + r^n_L] \\
\pi_{LN} &= \kappa y_{LN} + \beta [p_L \pi_{LN} + (1-p_L)\pi_{NN}] \\
i_{LN} &= 0
\end{aligned}$$

and satisfies the following inequality constraints

$$\begin{aligned}
i_{NH} &> 0 \\
i_{NN} &> 0 \\
0 &> \lambda y_{LH} + \kappa \pi_{LH} \\
0 &> \lambda y_{LN} + \kappa \pi_{LN}
\end{aligned}$$

The **F-Type-II equilibrium** is defined as a vector $\{y_{NH}, \pi_{NH}, i_{NH}, y_{NN}, \pi_{NN}, i_{NN}, y_{LH}, \pi_{LH}, i_{LH}, y_{LN}, \pi_{LN}, i_{LN}\}$ that solves following system of linear equations

$$\begin{aligned}
y_{NH} &= [p_H y_{NH} + (1-p_H) y_{NN}] + \sigma [p_H \pi_{NH} + (1-p_H)\pi_{NN} - i_{NH} + r^n] \\
\pi_{NH} &= \kappa y_{NH} + \beta [p_H \pi_{NH} + (1-p_H)\pi_{NN}] + \varepsilon
\end{aligned}$$

[4] I discuss the existence of other equilibria in the appendix.

$$
\begin{aligned}
0 &= \lambda y_{NH} + \kappa \pi_{NH} \\
y_{NN} &= y_{NN} + \sigma \left(\pi_{NN} - i_{NN} + r^n \right) \\
\pi_{NN} &= \kappa y_{NN} + \beta \pi_{NN} \\
0 &= \lambda y_{NN} + \kappa \pi_{NN} \\
y_{LH} &= \left[p_H p_L y_{LH} + p_H (1-p_L) y_{NH} + (1-p_H) p_L y_{LN} + (1-p_H)(1-p_L) y_{NN} \right] \\
&+ \sigma \left[p_H p_L \pi_{LH} + p_H (1-p_L) \pi_{NH} + (1-p_H) p_L \pi_{LN} + (1-p_H)(1-p_L) \pi_{NN} - i_{LH} + r^n_L \right] \\
\pi_{LH} &= \kappa y_{LH} + \beta \left[\left[p_H p_L \pi_{LH} + p_H (1-p_L) \pi_{NH} + (1-p_H) p_L \pi_{LN} + (1-p_H)(1-p_L) \pi_{NN} \right] + \varepsilon \right. \\
0 &= \lambda y_{LH} + \kappa \pi_{LH} \\
y_{LN} &= \left[p_L y_{LN} + (1-p_L) y_{NN} \right] + \sigma \left[p_L \pi_{LN} + (1-p_L) \pi_{NN} - i_{LN} + r^n_L \right] \\
\pi_{LN} &= \kappa y_{LN} + \beta \left[p_L \pi_{LN} + (1-p_L) \pi_{NN} \right] \\
i_{LN} &= 0
\end{aligned}
$$

and satisfies the following inequality constraints

$$
\begin{aligned}
i_{NH} &> 0 \\
i_{NN} &> 0 \\
i_{LH} &> 0 \\
0 &> \lambda y_{LN} + \kappa \pi_{LN}
\end{aligned}
$$

2.7 Markov-Perfect equilibrium (expectation-driven ZLB model)

In the kl state ($k \in \{D, N\}$, $l \in \{H, N\}$), the value of the variable x is denoted as x_{kl}. Similar to the fundamental-driven ZLB model, I focus on the following two equilibria: (1) Equilibrium where ZLB binds in the DH and DN states (E-Type-I equilibrium). (2)Equilibrium where ZLB only binds in the DN state (E-Type-II equilibrium).

The **E-Type-I equilibrium** is defined as a vector $\{y_{NH}, \pi_{NH}, i_{NH}, y_{NN}, \pi_{NN}, i_{NN}, y_{DH}, \pi_{DH}, i_{DH}, y_{DN}, \pi_{DN}, i_{DN}\}$ that solves following system of linear equations

$$
\begin{aligned}
y_{NH} &= \left[p_H y_{NH} + (1-p_H) y_{NN} \right] + \sigma \left[p_H \pi_{NH} + (1-p_H) \pi_{NN} - i_{NH} + r^n \right] \\
\pi_{NH} &= \kappa y_{NH} + \beta \left[p_H \pi_{NH} + (1-p_H) \pi_{NN} \right] + \varepsilon \\
0 &= \lambda y_{NH} + \kappa \pi_{NH} \\
y_{NN} &= y_{NN} + \sigma \left(\pi_{NN} - i_{NN} + r^n \right) \\
\pi_{NN} &= \kappa y_{NN} + \beta \pi_{NN} \\
0 &= \lambda y_{NN} + \kappa \pi_{NN} \\
y_{DH} &= \left[p_H p_D y_{DH} + p_H (1-p_D) y_{NH} + (1-p_H) p_D y_{DN} + (1-p_H)(1-p_D) y_{NN} \right] \\
&+ \sigma \left[p_H p_D \pi_{DH} + p_H (1-p_D) \pi_{NH} + (1-p_H) p_D \pi_{DN} + (1-p_H)(1-p_D) \pi_{NN} - i_{DH} + r^n \right] \\
\pi_{DH} &= \kappa y_{DH} + \beta \left[\left[p_H p_D \pi_{DH} + p_H (1-p_D) \pi_{NH} + (1-p_H) p_D \pi_{DN} + (1-p_H)(1-p_D) \pi_{NN} \right] + \varepsilon \right. \\
i_{DH} &= 0 \\
y_{DN} &= \left[p_D y_{DN} + (1-p_D) y_{NN} \right] + \sigma \left[p_D \pi_{DN} + (1-p_D) \pi_{NN} - i_{DN} + r^n \right] \\
\pi_{DN} &= \kappa y_{DN} + \beta \left[p_D \pi_{DN} + (1-p_D) \pi_{NN} \right]
\end{aligned}
$$

$$
\begin{aligned}
i_{DN} &= 0
\end{aligned}
$$

and satisfies the following inequality constraints

$$
\begin{aligned}
i_{NH} &> 0 \\
i_{NN} &> 0 \\
0 &> \lambda y_{DH} + \kappa \pi_{DH} \\
0 &> \lambda y_{DN} + \kappa \pi_{DN}
\end{aligned}
$$

The **E-Type-II equilibrium** is defined as a vector $\{y_{NH}, \pi_{NH}, i_{NH}, y_{NN}, \pi_{NN}, i_{NN}, y_{DH}, \pi_{DH}, i_{DH}, y_{DN}, \pi_{DN}, i_{DN}\}$ that solves following system of linear equations

$$
\begin{aligned}
y_{NH} &= [p_H y_{NH} + (1-p_H) y_{NN}] + \sigma\,[p_H \pi_{NH} + (1-p_H)\pi_{NN} - i_{NH} + r^n] \\
\pi_{NH} &= \kappa y_{NH} + \beta\,[p_H \pi_{NH} + (1-p_H)\pi_{NN}] + \varepsilon \\
0 &= \lambda y_{NH} + \kappa \pi_{NH} \\
y_{NN} &= y_{NN} + \sigma\,(\pi_{NN} - i_{NN} + r^n) \\
\pi_{NN} &= \kappa y_{NN} + \beta \pi_{NN} \\
0 &= \lambda y_{NN} + \kappa \pi_{NN} \\
y_{DH} &= [p_H p_D y_{DH} + p_H(1-p_D) y_{NH} + (1-p_H) p_D y_{DN} + (1-p_H)(1-p_D) y_{NN}] \\
&+ \sigma\,[p_H p_D \pi_{DH} + p_H(1-p_D)\pi_{NH} + (1-p_H) p_D \pi_{DN} + (1-p_H)(1-p_D)\pi_{NN} - i_{DH} + r^n] \\
\pi_{DH} &= \kappa y_{DH} + \beta\,[[p_H p_D \pi_{DH} + p_H(1-p_D)\pi_{NH} + (1-p_H) p_D \pi_{DN} + (1-p_H)(1-p_D)\pi_{NN}] + \varepsilon \\
0 &= \lambda y_{DH} + \kappa \pi_{DH} \\
y_{DN} &= [p_D y_{DN} + (1-p_D) y_{NN}] + \sigma\,[p_D \pi_{DN} + (1-p_D)\pi_{NN} - i_{DN} + r^n] \\
\pi_{DN} &= \kappa y_{DN} + \beta\,[p_D \pi_{DN} + (1-p_D)\pi_{NN}] \\
i_{DN} &= 0
\end{aligned}
$$

and satisfies the following inequality constraints

$$
\begin{aligned}
i_{NH} &> 0 \\
i_{NN} &> 0 \\
i_{DH} &> 0 \\
0 &> \lambda y_{DN} + \kappa \pi_{DN}
\end{aligned}
$$

3 Basic properties of Markov-Perfect equilibria

In this section, I describe the analytical results concerning the existence conditions of each equilibrium and allocations in each equilibrium.

3.1 Fundamental-driven ZLB model

In propositions of this section, I assume that $\frac{1}{\frac{(1-p_H p_L)(1-\beta p_H p_L)}{\sigma\kappa} - p_H p_L} \neq 0$.

3.1.1 Equilibrium existence

The following proposition establishes the necessary and sufficient conditions for the existence of the F-Type-I equilibrium.

Proposition 1 The F-Type-I equilibrium exists if and only if

$$\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa} - p_L > 0 \tag{11}$$

and

$$\frac{1}{\frac{(1-p_H p_L)(1-\beta p_H p_L)}{\sigma\kappa} - p_H p_L}\left(F^{F,I}\cdot\varepsilon + G^{F,I}\cdot r_L^n\right) < 0 \tag{12}$$

where $F^{F,I}$ and $G^{F,I}$ are composed of the set of parameter values excluding ε and r_L^n.

Proof: See Appendix.

(11) pertains to the LN state. This implies that the persistence of demand shocks should be sufficiently low.[5] (12) relates to the LH state. When (11) is satisfied,

$$\frac{(1-p_H p_L)(1-\beta p_H p_L)}{\sigma\kappa} - p_H p_L > 0$$

holds. Hence, when considering (12), it is sufficient to focus on the following inequality:

$$F^{F,I}\cdot\varepsilon + G^{F,I}\cdot r_L^n < 0$$

In the appendix, it is elaborated that when (11) is fulfilled, $F^{F,I} > 0$ and $G^{F,I} > 0$. Therefore, the first term of the LHS represents the weighted size of cost-push shocks, while the second term reflects the weighted size of demand shocks. The interpretation is that equilibrium conditions are met when the impact of cost-push shocks is greater than that of demand shocks. A more detailed explanation will be provided in later chapters along with numerical examples. Additionally, for NH and NN states, equilibrium conditions are always satisfied regardless of parameter values.

The following proposition establishes the necessary and sufficient conditions for the existence of the F-Type-II equilibrium.

Proposition 2 The F-Type-II equilibrium exists if and only if

$$\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa} - p_L > 0$$

and

$$F^{F,II}\cdot\varepsilon + G^{F,II}\cdot r_L^n > 0 \tag{13}$$

where $F^{F,II}$ and $G^{F,II}$ are composed of the set of parameter values excluding ε and r_L^n.

Proof: See Appendix.

[5] This result is consistent with Nakata and Schmidt (2019) when the frequency of demand shocks is set to zero.

First inequality is the same as (11). In the appendix, it is elaborated that when (11) is fulfilled, $F^{F,II} > 0$ and $G^{F,II} > 0$. Therefore, the first term of the LHS represents the weighted size of cost-push shocks, while the second term reflects the weighted size of demand shocks. The interpretation is that equilibrium conditions are met when the impact of cost-push shocks is greater than that of demand shocks. Additionally, for NH and NN states, equilibrium conditions are always satisfied regardless of parameter values.

3.1.2 Allocations and prices

The allocations and prices in each state can be solved in closed form. The signs of inflation and output gap in each state are as follows.

Proposition 3 Suppose that $\sigma - \theta < 0$ holds. In the F-Type-I equilibrium, $\pi_{NH} > 0, y_{NH} < 0, \pi_{NN} = y_{NN} = 0, \pi_{LN} < 0, y_{LN} < 0$, $y_{LH} < 0$ and the sign of π_{LH} depends on structural parameters.

Proof: See Appendix.

In the NH state, there is a positive cost-push shock outside the ZLB. As the N state is absorbing, there is no possibility of being bounded by the ZLB in the future. Therefore, the inflation rate is positive, and the output gap is negative. In the NN state, no shocks are present, and the N state is absorbing, both the inflation rate and the output gap stabilize completely. In the LN state, ZLB binds, and there is no cost-push shock. Thus, both the inflation rate and the output gap become negative. The interpretation of the LH state is complex as it involves the presence of two shocks. The difference from a model with only negative demand shock lies in facing the ZLB under mild positive inflation.

Proposition 4 Suppose that $\sigma - \theta < 0$ holds. In the F-Type-II equilibrium, $\pi_{NH} > 0, y_{NH} < 0, \pi_{NN} = y_{NN} = 0, \pi_{LN} < 0, y_{LN} < 0$, $\pi_{LH} > 0$ and $y_{LH} < 0$.

Proof: See Appendix.

For NN, NH, and LN states, the allocations, and prices are similar to those of F-Type-I equilibrium by definition. In the LH state, although the natural interest rate is negative, the impact of cost-push shocks is relatively more significant. Hence, the LH state is free from the ZLB, with positive inflation and a negative output gap.

3.2 Expectation-driven ZLB model

In propositions of this section, I assume that $\frac{1}{\frac{(1-p_H p_D)(1-\beta p_H p_D)}{\sigma\kappa} - p_H p_D} \neq 0$.

3.2.1 Equilibrium existence

The following proposition establishes the necessary and sufficient conditions for the existence of the E-Type-I equilibrium.

Proposition 5 The E-Type-I equilibrium exists if and only if

$$\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D < 0 \tag{14}$$

and

$$\frac{1}{\frac{(1-p_H p_D)(1-\beta p_H p_D)}{\sigma\kappa} - p_H p_D}\left(F^{E,I}\cdot\varepsilon + G^{E,I}\right) < 0 \tag{15}$$

where $F^{E,I}$ and $G^{E,I}$ are composed of the set of parameter values excluding ε.

Proof: See Appendix.

(14) represents the condition for the DN state, implying that the D state must be sufficiently persistent. As noted by Nakata and Schmidt (2022), this result is the opposite in the case of fundamental-driven ZLB. (15) is a condition for the DH state. When (14) is fulfilled, $F^{E,I} > 0$ and $G^{E,I} < 0$. Therefore, the first term of the formula in brackets represents the impact of cost-push shocks, while the second term reflects the declining agent's confidence. The difference with the F-Type-I equilibrium is that even though (14) is satisfied, the sign of

$$\frac{1}{\frac{(1-p_H p_D)(1-\beta p_H p_D)}{\sigma\kappa} - p_H p_D}$$

can be either positive or negative. More precisely, under rational parameter settings, it takes a negative value only when both p_H and p_D are sufficiently high, otherwise, it takes a positive value. Additionally, for the NH and NN states, equilibrium conditions are always satisfied regardless of parameter values.

The following proposition establishes the necessary and sufficient conditions for the existence of the E-Type-II equilibrium.

Proposition 6 The E-Type-II equilibrium exists if and only if

$$\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D < 0$$

and

$$F^{E,II}\cdot\varepsilon + G^{E,II} > 0 \tag{16}$$

where $F^{E,II}$ and $G^{E,II}$ are composed of the set of parameter values excluding ε.

Proof: See Appendix.

First inequality represents the condition for the DN state and is the same as the Type-I equilibrium. (16) is a condition for the DH state. When (14) is fufilled, $F^{E,II} > 0$ and $G^{E,II} < 0$. A more detailed explanation of this inequality will be provided later, similar to the E-Type-I case. Additionally, for the NH, and NN states, equilibrium conditions are always satisfied regardless of parameter values.

3.2.2 Allocations and prices

The allocations and prices in each state can be solved in closed form. The signs of inflation and output gap in each state are as follows.

Proposition 7 In the E-Type-I equilibrium, $\pi_{NH} > 0, y_{NH} < 0, \pi_{NN} = y_{NN} = 0, \pi_{DN} < 0, y_{DN} < 0$.

Proof: See Appendix.

The NN and NH states are the same as those of the fundamental-driven model. In the DN state, there is no cost-push shock. In the case of a sunspot shock in the D state, agents' confidence is low, leading to a decrease in consumption and production due to the anticipation of low income. The central bank lowers interest rates to mitigate these effects, but if people are excessively pessimistic, the ZLB binds nominal interest rates. Therefore, both the output gap and inflation rate become negative.

Here, we don't address the sign of the allocation in the DH state. It is apparent that both y_{DH} and π_{DH} cannot be positive, and there are cases where both y_{DH} and π_{DH} are negative. However, proving analytically, that one of y_{DH} and π_{DH} is positive while the other is negative is challenging. Therefore, I interpret the allocation in the DH state using numerical examples in next section.

Proposition 8 Suppose that $\sigma - \theta < 0$ holds. In the E-Type-II equilibrium, $\pi_{NH} > 0, y_{NH} < 0, \pi_{NN} = y_{NN} = 0, \pi_{DN} < 0, y_{DN} < 0$. The signs of π_{DH} and y_{DH} depend on parameters.

Proof: See Appendix.

For NN, NH, and DN states, the allocations and prices are similar to those of E-Type-I equilibrium. In the DH state, although the agents' confidence decreases, the impact of cost-push shocks is relatively more significant. Moreover, π_{DN} is generally positive, but it becomes slightly negative (almost 0) when p_D approaches 1 infinitely and p_H is sufficiently small. This aspect is distinct from the E-Type-II equilibrium in the fundamental-driven ZLB model.

4 Numerical example

I illustrate the properties in the previous section with specific parameter values. The structural parameters are calibrated using the parameter values from Nakata and Schmidt (2022), as listed in Table 1. The size of the cost-push shock is adjusted to make the decline in the output gap a realistic value, set at 0.1%

Table 1: Parameterization

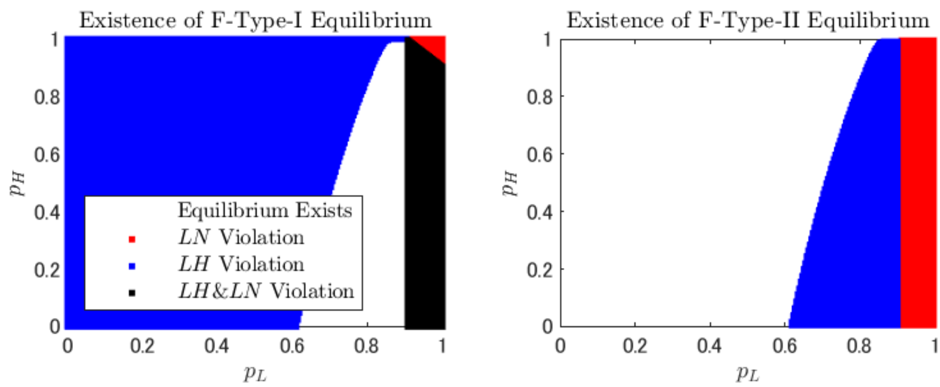
Parameter	Value	Economic interpretation
β	0.9975	Subjective discount factor
σ	0.5	Intertemporal elasticity of substitution in consumption
η	0.47	Inverse labor supply elasticity
θ	10	Price elasticity of demand
α	0.8106	Share of firms per period keeping prices unchanged
r^n	$\frac{1}{\beta} - 1$	N state natural real rate
r^n_L	-0.005	L state natural real rate
ε	0.001	Size of cost-push shock

4.1 Fundamental-driven ZLB model

4.1.1 Equilibrium existence

Figure 1 shows the existence of two possible equilibria for different combinations of p_L (persistence of demand shock) and p_H (persistence of cost-push shock). In each panel, white areas show the combinations of persistence of two shocks under which the equilibrium of a particular type exists. Colored areas indicate that either one or both of the relevant inequality constraints are violated and thus the equilibrium does not exist. Different colors indicate different reasons why the equilibrium does not exist. I say the LN violation occurs if the inequality related to the LN state is violated, LH violation if the inequality related to the LH state is violated. Red, blue, and black dots respectively indicate LN, LH, and $LN\&LH$ violations.

Figure 1: Existence of Two Markov-Perfect Equilibria



As I mentioned, the red and black regions illustrate equilibrium conditions for the LN state, corresponding to (11). In this paper, I assume the frequency of each shock to be 0. Therefore, the equilibrium condition for the LN state is independent of cost-push shocks. Consequently, the red and black regions can be interpreted as the areas where equilibrium does not exist in a model without cost-push shocks. Therefore, we need only focus on the blue and white regions to see how introducing cost-push shocks changes the region in which equilibrium exists. There are three findings. Firstly, when p_L is fixed, F-Type-I exists when p_H is small and F-Type-II exists when p_H exceeds a certain threshold. The intuitive explanation is as follows: since p_H can be interpreted as the severity of the cost-push shock, when p_H is low (mild cost-push shock), the nominal interest rate is bounded by the ZLB in the LH state (F-Type-I), but when p_H is high (severe cost-push shock) the nominal interest rate deviates from zero in the LH state.

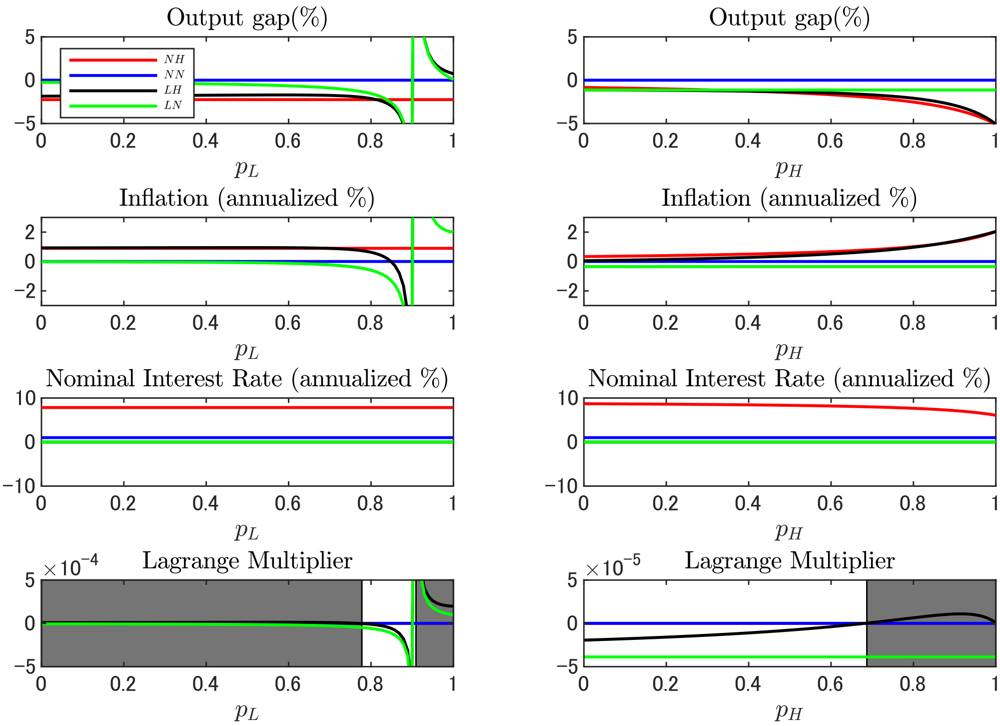
Secondly, the threshold value of p_H at which the equilibrium shifts from F-Type-I to F-Type-II decreases as p_L decreases and reaches 0 when p_L is around 0.6. Intuitively, deviation from the ZLB in the LH state when p_L is small, i.e., when there is a mild demand shock, even if the effect of the cost-push shock is not severe.

Finally, if p_L is small enough (outside of the red and black regions), then either F-Type-I or F-Type-II always exists. In other words, the region where equilibrium exists in a model without cost-push shock is divided into regions where either F-Type-I or F-Type-II exists, depending on whether the impact of the demand shock or the cost-push shock is larger.

4.1.2 Allocations and prices

Figure 2 plots how the solution to the system of linear equations of F-Type-I equilibrium defined in the previous section depends on p_L and p_H. The left panels show how the consumption, inflation, nominal interest rate, and the Lagrange multiplier that solves the linear system vary with p_L, holding p_H constant at 0.75. The right panels show how they vary with p_H holding p_L constant at 0.75. The shaded area in the Lagrange multiplier panels shows the region where it is positive (i.e., the equilibrium condition is violated).

Figure 2: Allocations in F-Type-I equilibrium

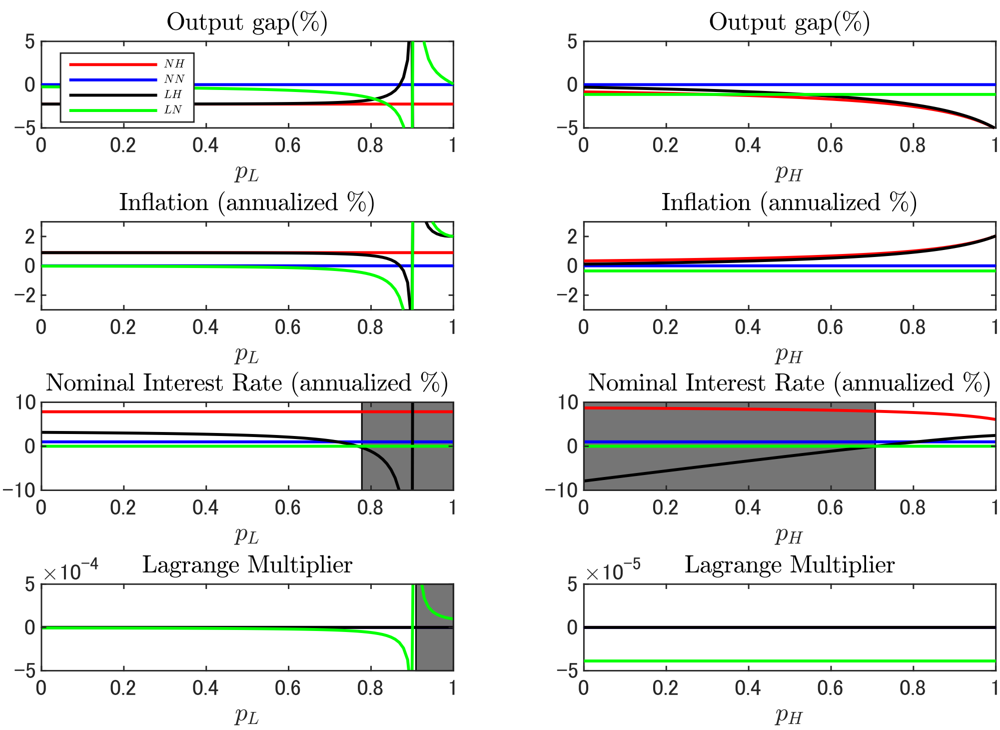


Left panels show how allocations and the Lagrange multiplier vary with p_L holding $p_H = 0.75$ while right panels show how they vary with p_H holding $p_L = 0.75$. In all panels, red blue black, and green lines are for NH, NN, LH, and LN states respectively. The shaded area in the Lagrange multiplier panels shows the region where it is positive.

As mentioned in Section 3, the existence of the F-Type-I equilibrium depends on the equilibrium conditions for the LH and LN states. According to the left panel of Figure 2, when p_L is not sufficiently high, the inflation rate in the LH state is positive. This is because the impact of a positive cost-push shock is considered to be greater than the impact of a negative demand shock. Therefore, for the central bank, it is optimal to set a positive nominal interest rate in the LH state, and as a result, the F-Type-I equilibrium does not exist. Furthermore, when p_L is very high, it is possible to self-fulfill both a positive output gap and a positive inflation rate. In this case, it is optimal for the central bank to set a positive nominal interest rate in the LN state, and therefore, the Type-I equilibrium does not exist.

Additionally, examining the right panel of Figure 2 reveals that the F-Type-I equilibrium does not exist when p_H is sufficiently high. The high persistence of cost-push shocks implies a prolonged force in inflation, allowing for a severe positive inflation rate even in the presence of negative demand shocks.

Figure 3: Allocations in F-Type-II equilibrium



Left panels show how allocations and the Lagrange multiplier vary with p_L holding $p_H = 0.75$ while right panels show how they vary with p_H holding $p_L = 0.75$. In all panels, red blue black, and green lines are for NH, NN, LH, and LN states respectively. Shaded areas in the panels for the nominal interest rate show the parameter region where the nominal interest rate is below zero, and the shaded area in the Lagrange multiplier panels shows the region where it is positive.

The existence of the F-Type-II equilibrium also depends on the equilibrium conditions for the LH and LN states. According to the left panel of Figure 3, it can be observed that equilibrium does not exist when p_L is sufficiently high. As p_L increases, the output gap in the LH state transitions from negative to positive, and the inflation rate goes positive to negative. This shift occurs because the central bank implements optimal discretionary policy, disregarding the ZLB. However, since the nominal interest rate required to achieve such allocation is negative, equilibrium does not exist.

Furthermore, looking at the right panel of Figure 3, it is evident that the F-Type-II equilibrium does not exist when p_H is sufficiently low. When p_H is low, except for the NN state, the output gap is negative in the three states, and the inflation rate is mildly positive in the NH and LH states, and negative in the LN state. Then, the nominal interest rates in the LH state are negative, which violates the ZLB.

4.1.3 Aggregate demand and aggregate supply analysis

To better understand how equilibrium and allocation change with the introduction of cost-push shocks, I illustrate the aggregate demand (AD) and aggregate supply (AS) curves.

The AD curve is the set of pairs of inflation rates and output gaps consistent with the Euler equation where the nominal interest rate is set in line with the target criterion, and the AS curve is the set of pairs of inflation rates and output gaps consistent with the Phillips curve. From the definition of shock structure, it is sufficient to consider the curves of the LH state as those of a model with cost-push shocks and the curves of the LN state as those of a model without cost-push shocks. The two curves in the LH state are given by

AD curve:

$$y_{LH} = \min\left[\frac{\sigma}{1-p_H p_L}\left\{p_H(1-p_L)\left(1-\frac{\kappa}{\sigma\lambda}\right)\pi_{NH} + (1-p_H)p_L\left(\frac{1-\beta}{\sigma\kappa}+1\right)\pi_{LN} + r_L^n\right\} + \frac{\sigma p_H p_L}{1-p_H p_L}\pi_{LH}, -\frac{\kappa}{\lambda}\pi_{LH}\right] \tag{17}$$

AS curve:

$$y_{LH} = \frac{1-\beta p_H p_L}{\kappa}\pi_{LH} - \frac{\beta p_H(1-p_L)}{\kappa}\pi_{NH} - \frac{\beta(1-p_H)p_L}{\kappa}\pi_{LN} - \frac{\varepsilon}{\kappa} \tag{18}$$

The two curves in the LN state are given by

AD curve:

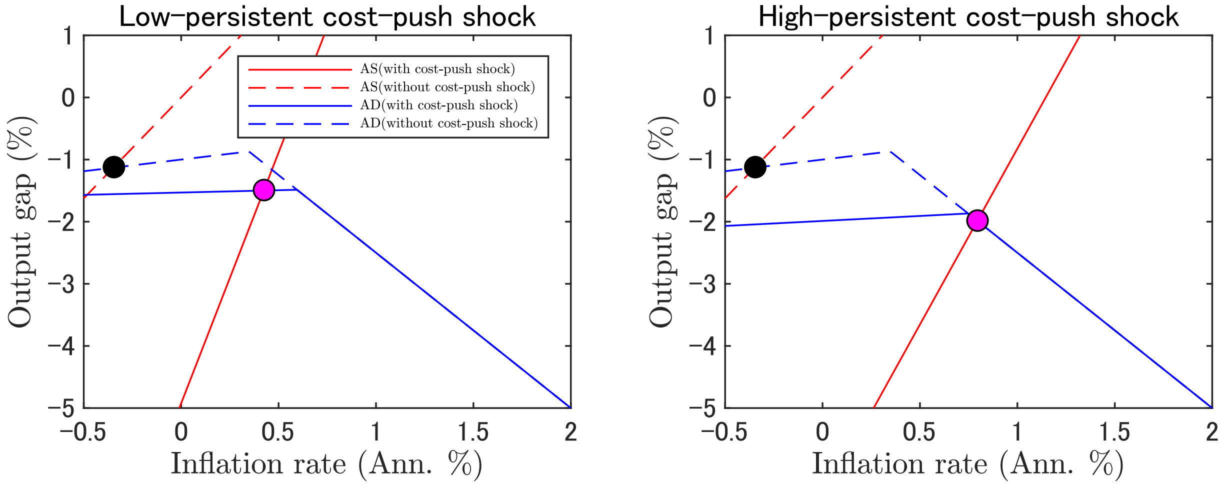
$$y_{LN} = \min\left(\frac{\sigma}{1-p_L}r_L^n + \frac{\sigma p_L}{1-p_L}\pi_{LN}, -\frac{\kappa}{\lambda}\pi_{LN}\right) \tag{19}$$

AS curve:

$$y_{LN} = \frac{1-\beta p_L}{\kappa}\pi_{LN} \tag{20}$$

Figure 4 plots AD-AS curves for the fundamental-driven ZLB model. I set $p_L = 0.75$, implying an average duration of the ZLB episodes is one year. I set $p_H = 0.5$ in the left panel and $p_H = 0.75$ in the right panel, meaning the average duration of cost-push episodes is 1/2 year, and one year respectively. In both panels, red solid, red dashed, blue solid, and blue dashed lines are for the AS curve with cost-push shock, the AS curve without cost-push shock, the AD curve with cost-push shock, and the AD curve without cost-push shock respectively. In both panels, the black dot, and magenta dot represent an equilibrium without cost-push shock and with cost-push shock.

Figure 4: AD-AS with and without cost-push shock



I set $p_H = 0.5$ in the left panel and $p_H = 0.75$ in the right panel, meaning the average duration of cost-push episodes is 1/2 year, and one year respectively. In both panels, red solid, red dashed, blue solid, and blue dashed lines are for the AS curve with cost-push shock, the AS curve without cost-push shock, the AD curve with cost-push shock, and the AD curve without cost-push shock respectively. In both panels, the black and magenta dots represent an equilibrium without cost-push shock and with cost-push shock.

The AD curve has a kink because of the ZLB. To the left of the kink, the ZLB binds, and to the right of the kink, the ZLB is slack. The AD curve is upward-sloping to the left of the kink. Aggregate demand increases inflation when the ZLB binds because an increase in inflation lowers the real interest rate. Also, the AD curve is downward-sloping to the right of the kink. Aggregate demand decreases in inflation when the ZLB is slack because the central bank raises the policy rate more than one-for-one with inflation. The AS curve is monotonically upward-sloping. Then, an increase in demand leads to an increase in inflation.

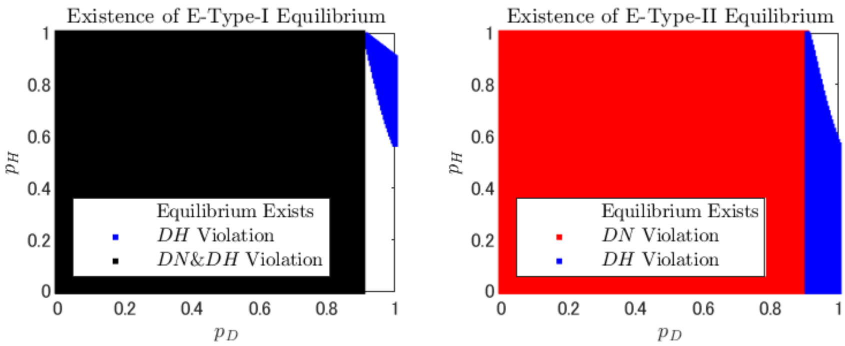
In both panels, introducing cost-push shocks causes the equilibrium to shift southeastward. The inflation rate increases from negative to positive, but the output gap decreases significantly. The left figure represents the case where the persistence of cost-push shocks is low, and even in the presence of cost-push shocks, the LH state remains constrained by the ZLB (F-Type-I). The right figure shows the case where the persistence of cost-push shocks is high, and because of the presence of cost-push shocks, the LH state is free from the ZLB (F-Type-II).

4.2 Expectation-driven ZLB model

4.2.1 Equilibrium existence

Figure 5 shows the existence of two possible equilibria for different combinations of p_D (persistence of sunspot shock) and p_H (persistence of cost-push shock). In each panel, white areas show the combinations of persistence of two shocks under which the equilibrium of a particular type exists. Colored areas indicate that either one or both of the relevant inequality constraints are violated and thus the equilibrium does not exist. We say the DN violation occurs if the inequality related to the DN state is violated, and the DH violation if the inequality related to the DH state is violated. Red, blue, and black dots respectively indicate DN, DH, and $DN\&DH$ violations.

Figure 5: Existence of Two Markov-Perfect Equilibria



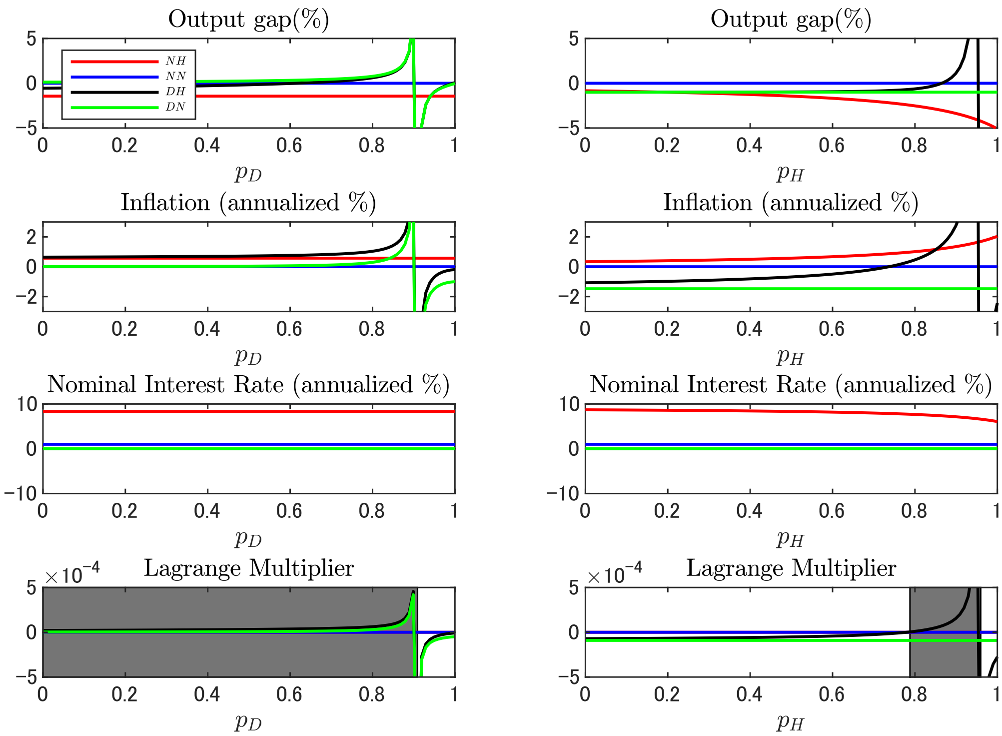
Similar to the fundamental-driven ZLB model, the equilibrium condition for the DN state is independent of cost-push shocks. Thus, the red and black regions can be interpreted as the areas where equilibrium does not exist in a model without cost-push shocks. This implies that a self-fulfilling ZLB requires sufficiently high persistence of sunspot shocks. Next, I confirm how the introduction of cost-push shocks affects the areas of equilibrium existence. Similar to the fundamental-driven ZLB model, I focus on the blue region.

There are three main findings. Firstly, when p_D is fixed, E-Type-I exists when p_H is low or extremely high and E-Type-II exists when p_H exceeds a certain threshold. The difference from the fundamental-driven ZLB model is the existence of the E-Type-I equilibrium when the persistence of cost-push shocks is extremely high. Secondly, the threshold value of p_H at which equilibrium shifts from Type-I to Type-II increases as p_D decreases. Finally, if p_D and p_H are extremely high, Type-I and Type-II coexist. Interpreting the second result, and the existence of multiple equilibria, as revealed by the first and third results, solely from this figure is challenging. Therefore, I provide details in the following subsections.

4.2.2 Allocations and prices

Figure 6 plots how the solution to the system of linear equations of E-Type-I equilibrium defined in the previous section depends on p_D and p_H. The left panels show how the consumption, inflation, nominal interest rate, and the Lagrange multiplier that solves the linear system vary with p_D, holding p_H constant at 0.5. The right panels show how they vary with p_H holding p_D constant at 0.95. The shaded area in the Lagrange multiplier panels shows the region where it is positive (i.e., the equilibrium condition is violated).

Figure 6: Allocations in E-Type-I equilibrium

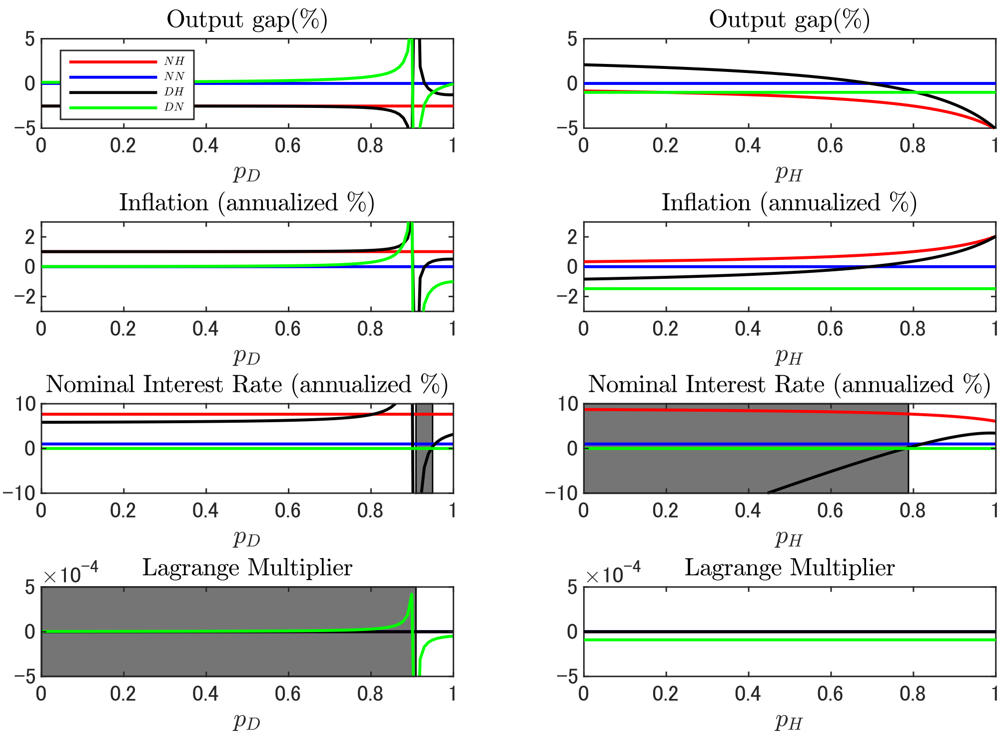


Left panels show how allocations and the Lagrange multiplier vary with p_D holding $p_H = 0.5$ while right panels show how they vary with p_H holding $p_D = 0.95$. In all panels, red blue black, and green lines are for NH, NN, DH, and DN states respectively. The shaded area in the Lagrange multiplier panels shows the region where it is positive.

As mentioned in Section 3, the existence of E-Type-I equilibrium depends on equilibrium conditions related to the DH and DN states. According to the left panel in Figure 6, when p_D is not excessively high, the output gap in the DH state is slightly negative, and the inflation rate is positive. This is because the impact of a positive cost-push shock is considered to be greater than the declining confidence. Therefore, it is optimal for the central bank to set a positive nominal interest rate in the DH state, and as a result, E-Type-I equilibrium does not exist. In addition, the second finding mentioned in the previous subsection can be explained by the left panels. From the figures, it is clear that when p_D exceeds a certain value, the signs of the output gap and inflation rate change from positive to negative. Additionally, near this threshold, there is a significant decline in both the output gap and the inflation rate. Therefore, the closer to the threshold, the cost-push shocks must be more persistent to deviate from the ZLB in the LH state.

Furthermore, examining the right panel in Figure 6 reveals that E-Type-I equilibrium does not exist when p_H is within a certain range. As p_H increases, the output gap and inflation rates increase, making it optimal to raise nominal interest rates. Then, after exceeding a certain threshold, the output gap and inflation rate are negative, leading to the reemergence of E-Type-I equilibrium.

Figure 7: Allocations in E-Type-II equilibrium



Left panels show how allocations and the Lagrange multiplier vary with p_D holding $p_H = 0.8$ while right panels show how they vary with p_H holding $p_D = 0.95$. In all panels, red blue black, and green lines are for NH, NN, DH, and DN states respectively. Shaded areas in the panels for the nominal interest rate show the parameter region where the nominal interest rate is below zero, and the shaded area in the Lagrange multiplier panels shows the region where it is positive.

The existence of E-Type-II equilibrium also depends on equilibrium conditions related to the DH and DN states. According to the left panel in Figure 7, equilibrium does not exist when p_D is not sufficiently high. This is because a self-fulfilling deflation does not occur in the DN state. Moreover, to understand the presence of shadows in the left panel of nominal interest rates, it is important to confirm the reasons behind it. The shadow section corresponds to the part where the impact of sunspot shocks is most significant, leading to highly unstable output gaps and inflation rates. These effects dominate the fluctuations caused by cost-push shocks, resulting in an optimal nominal interest rate that becomes negative. Therefore, the E-Type-II equilibrium does not exist.

Finally, I explain the reason for the presence of shadows in the right panel of nominal interest rates. The output gap is non-positive in states other than DH (and the output gap in DH state becomes negative with the increase in p_H), and the inflation rate is non-positive in states other than NH. As a result, the optimal interest rate becomes negative. Therefore, the E-Type-II equilibrium does not exist.

4.2.3 Aggregate demand and aggregate supply analysis

To understand how equilibrium properties change with the introduction of cost-push shocks, I show the AD and AS curves. Similar to the fundamental-driven ZLB model, it is sufficient to consider the curves of the DH state as those of a model with cost-push shocks and the curves of the DN state as those of a model without cost-push shocks. The two curves in the DH state are given by

AD curve:

$$y_{DH} = \min\left[\frac{\sigma}{1-p_Hp_D}\left\{p_H(1-p_D)\left(1-\frac{\kappa}{\sigma\lambda}\right)\pi_{NH} + (1-p_H)p_D\left(\frac{1-\beta}{\sigma\kappa}+1\right)\pi_{DN} + r^n\right\} + \frac{\sigma p_H p_D}{1-p_Hp_D}\pi_{DH}, -\frac{\kappa}{\lambda}\pi_{DH}\right] \tag{21}$$

AS curve:

$$y_{DH} = \frac{1-\beta p_H p_D}{\kappa}\pi_{DH} - \frac{\beta p_H(1-p_D)}{\kappa}\pi_{NH} - \frac{\beta(1-p_H)p_D}{\kappa}\pi_{DN} - \frac{\varepsilon}{\kappa} \tag{22}$$

The two curves in the DN state are given by

AD curve:

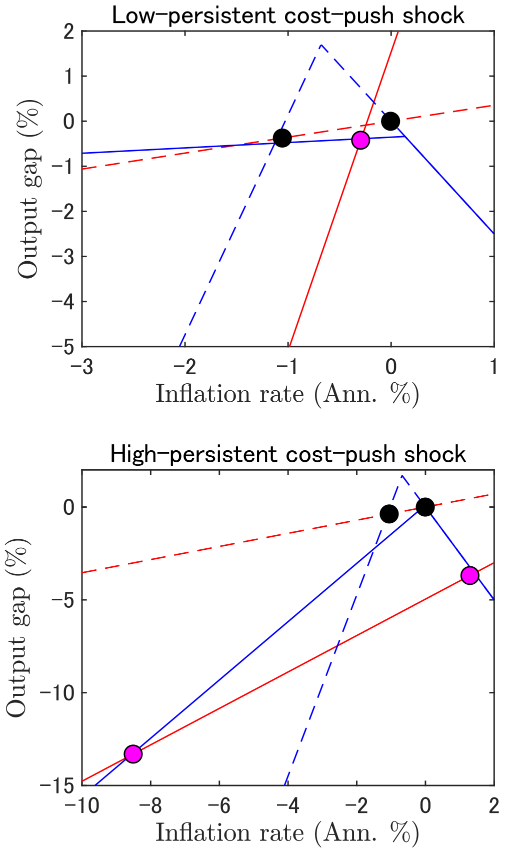
$$y_{DH} = \min\left(\frac{\sigma}{1-p_D}r^n + \frac{\sigma p_D}{1-p_D}\pi_{DN}, -\frac{\kappa}{\lambda}\pi_{DN}\right) \tag{23}$$

AS curve:

$$y_{DN} = \frac{1-\beta p_D}{\kappa}\pi_{DN} \tag{24}$$

Figure 8 plots AD-AS curves for the expectation-driven ZLB model. I set $p_D = 0.975$, implying an average duration of the ZLB episodes is ten years. I also set $p_H = 0.5$ in the top left panel, $p_H = 0.75$ in the top right panel, and $p_H = 0.95$ in the bottom left panel, meaning the average duration of cost-push episodes is 1/2 year, one year, and five years respectively. In all panels, red solid, red dashed, blue solid, and blue dashed lines are for the AS curve with cost-push shock, the AS curve without cost-push shock, the AD curve with cost-push shock, and the AD curve without cost-push shock respectively. In all panels, the black dot, and magenta dot represents an equilibrium without cost-push shock and with cost-push shock.

Figure 8: AD and AS in the DH state



I set $p_H = 0.5$ in the top left panel, $p_H = 0.75$ in the top right panel, and $p_H = 0.95$ in the bottom left panel. In all panels, red solid, red dashed, blue solid, and blue dashed lines are for the AS curve with cost-push shock, the AS curve without cost-push shock, the AD curve with cost-push shock, and the AD curve without cost-push shock respectively. In all panels, the black dot, and magenta dot represent an equilibrium without cost-push shock and with cost-push shock.

There are various similarities between the AD and AS curves of the fundamental-driven ZLB Model and the expectation-driven ZLB Model. The AD curve has a kink because of the ZLB. To the left of the kink, the ZLB binds, and to the right of the kink, the ZLB is slack. The AD curve is upward-sloping to the left of the kink and downward-sloping to the right of the kink. The AS curve is monotonically upward-sloping. Comparing the two panels in the upper row, when the size of the cost-push shock increases, nominal interest rates in the DH state rise from zero to positive.

The difference from the fundamental-driven ZLB model is that when p_H is extremely high, the slope of the AD curve is steeper than that of the AS curve. As a result, E-Type-I and E-Type-II equilibrium coexist. In the E-Type-I equilibrium, the output gap and inflation rate experience a significant decline. In Figure 8, p_H is set to 0.95, indicating an expectation that the cost-push shock will persist for five years. Analytically, E-Type-

I equilibrium exists. However, A situation where both p_H and p_D are extremely high, indicating a sustained positive cost-push shock and sufficient decline in agents' confidence, would be unusual.

5 Conclusion

In this paper, I have analyzed the conditions for the existence of a Markov perfect equilibrium in an economy with ZLB and cost-push shocks, using a standard New Keynesian model. I have used two models: one incorporating negative demand shocks and positive cost-push shocks (fundamental-driven ZLB model), and another including sunspot shocks and positive cost-push shocks (expectation-driven ZLB model). I have focused on two types of equilibria: Type-I; equilibrium where the ZLB binds in the two states facing demand/sunspot shocks, regardless of the presence of cost-push shocks, and Type-II; equilibrium where the ZLB binds in a state where only demand/sunspot shocks exist.

I have found that Type-I exists when the impact of demand/sunspot shocks is greater than that of cost-push shocks, and vice versa, Type-II exists. Furthermore, Type-I or Type-II equilibrium exists in the fundamental-driven ZLB model. In contrast, in the expectation-driven ZLB model, Type-I and Type-II equilibrium may exist simultaneously when the persistence of all shocks is sufficiently high.

I have assumed that the N state is absorbing to simplify the analysis. However, in reality, once escaping from the ZLB, the constraint may bind again. Therefore, it is considered desirable to conduct analysis using a model where there is a positive probability of transitioning from a state without shocks to a state with shocks. Additionally, I have assumed that the central bank conducts optimal discretionary policy. Considering the policy operations of real central banks, there is a need for analysis under optimal commitment policies and various monetary policy rules.

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Appendix

Proof of Proposition 1

Proposition A.1

There exists a vector $\{y_{NH}, \pi_{NH}, i_{NH}, y_{NN}, \pi_{NN}, i_{NN}, y_{LH}, \pi_{LH}, i_{LH}, y_{LN}, \pi_{LN}, i_{LN}\}$ that solves the system of linear equation defined in Section 2.6.

Proof

The equations for the NN state are as follows.

$$\begin{aligned} y_{NN} &= y_{NN} + \sigma(\pi_{NN} - i_{NN} + r^n) && (25) \\ \pi_{NN} &= \kappa y_{NN} + \beta \pi_{NN} && (26) \\ 0 &= \lambda y_{NN} + \kappa \pi_{NN} && (27) \end{aligned}$$

From (26) and (27), $y_{NN} = \pi_{NN} = 0$ holds. Substituting this result into (25), we obtain $i_{NN} = r^n$.

The equation for the NH state are as follows.

$$\begin{aligned} y_{NH} &= [p_H y_{NH} + (1 - p_H) y_{NN}] + \sigma [p_H \pi_{NH} + (1 - p_H)\pi_{NN} - i_{NH} + r^n] \\ \pi_{NH} &= \kappa y_{NH} + \beta [p_H \pi_{NH} + (1 - p_H)\pi_{NN}] + \varepsilon \\ 0 &= \lambda y_{NH} + \kappa \pi_{NH} \end{aligned}$$

Substituting the result of the NN state into the above equations,

$$\begin{aligned} y_{NH} &= p_H y_{NH} + \sigma [p_H \pi_{NH} - i_{NH} + r^n] && (28) \\ \pi_{NH} &= \kappa y_{NH} + \beta p_H \pi_{NH} + \varepsilon && (29) \\ 0 &= \lambda y_{NH} + \kappa \pi_{NH} && (30) \end{aligned}$$

hold. From (29) and (30), we obtain

$$\pi_{NH} = \frac{\lambda}{\kappa^2 + \lambda(1 - \beta p_H)} \varepsilon \qquad (31)$$

Substituting (31) into (30), we get y_{NH}. Then, substituting obtained π_{NH} and y_{NH} into (28), i_{NH} is determined. Hence

$$y_{NH} = -\frac{\kappa}{\kappa^2 + \lambda(1 - \beta p_H)} \varepsilon \qquad (32)$$

$$i_{NH} = \left(\frac{(1 - p_H)\kappa}{\sigma} + p_H \lambda \right) \frac{1}{\kappa^2 + \lambda(1 - \beta p_H)} \varepsilon + r^n \qquad (33)$$

The equations for the LN state are as follows.

$$\begin{aligned} y_{LN} &= [p_L y_{LN} + (1 - p_L) y_{NN}] + \sigma [p_L \pi_{LN} + (1 - p_L)\pi_{LN} - i_{LN} + r^n_L] \\ \pi_{LN} &= \kappa y_{LN} + \beta [p_L \pi_{LN} + (1 - p_L)\pi_{NN}] \end{aligned}$$

$$i_{LN} = 0$$

Substituting the result of the NN state into the above equations,

$$\begin{aligned} y_{LN} &= p_L y_{LN} + \sigma \left[p_L \pi_{LN} - i_{LN} + r_L^n \right] && (34) \\ \pi_{LN} &= \kappa y_{LN} + \beta p_L \pi_{LN} && (35) \\ i_{LN} &= 0 && (36) \end{aligned}$$

From (35), $y_{LN} = \frac{1-\beta p_L}{\kappa} \pi_{LN}$ holds. Substituting this and $i_{LN} = 0$ into (34),

$$\pi_{LN} = \frac{r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa} - p_L} \tag{37}$$

holds. In this context, we assume that the denominator is nonzero. Using above result and (35),

$$y_{LN} = \frac{r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa} - p_L} \cdot \frac{1-\beta p_L}{\kappa} \tag{38}$$

The equations for the LH state are as follows.

$$\begin{aligned} y_{LH} &= \left[p_H p_L y_{LH} + p_H(1-p_L) y_{NH} + (1-p_H) p_L y_{LN} + (1-p_H)(1-p_L) y_{NN} \right] \\ &+ \sigma \left[p_H p_L \pi_{LH} + p_H(1-p_L)\pi_{NH} + (1-p_H) p_L \pi_{LN} + (1-p_H)(1-p_L)\pi_{NN} - i_{LH} + r_L^n \right] \\ \pi_{LH} &= \kappa y_{LH} + \beta \left[\left[p_H p_L \pi_{LH} + p_H(1-p_L)\pi_{NH} + (1-p_H) p_L \pi_{LN} + (1-p_H)(1-p_L)\pi_{NN} \right] + \varepsilon \right. \\ i_{LH} &= 0 \end{aligned}$$

Substituting the result of NN state and $i_{LH} = 0$ into above equations,

$$\begin{aligned} y_{LH} &= \left[p_H p_L y_{LH} + p_H(1-p_L) y_{NH} + (1-p_H) p_L y_{LN} \right] \\ &+ \sigma \left[p_H p_L \pi_{LH} + p_H(1-p_L)\pi_{NH} + (1-p_H) p_L \pi_{LN} + r_L^n \right] \\ \pi_{LH} &= \kappa y_{LH} + \beta \left[\left[p_H p_L \pi_{LH} + p_H(1-p_L)\pi_{NH} + (1-p_H) p_L \pi_{LN} \right] + \varepsilon \right. \end{aligned}$$

Using $y_{NH} = -\frac{\kappa}{\lambda}\pi_{NH}$, $y_{LN} = \frac{1-\beta p_L}{\kappa}\pi_{LN}$ and rearranging above two equations

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_{LH} \\ \pi_{LH} \end{bmatrix} = \begin{bmatrix} p_H(1-p_L)\left(\sigma - \frac{\kappa}{\lambda}\right)\pi_{NH} + (1-p_H)p_L\left(\frac{1-\beta p_L}{\kappa} + \sigma\right)\pi_{LN} + \sigma r_L^n \\ \beta p_H(1-p_L)\pi_{NH} + \beta(1-p_H)p_L\pi_{LN} + \varepsilon \end{bmatrix}. \tag{39}$$

where

$$\begin{aligned} A &= 1 - p_H p_L && (40) \\ B &= -\sigma p_H p_L && (41) \\ C &= -\kappa && (42) \\ D &= 1 - \beta p_H p_L && (43) \\ E &= AD - BC = (1-p_H p_L)(1-\beta p_H p_L) - \sigma\kappa p_H p_L && (44) \end{aligned}$$

Assumption 1 $E \neq 0$
Hence, we can invert the matrix on the left hand side of (39)

$$\begin{bmatrix} y_{LH} \\ \pi_{LH} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{bmatrix} p_H(1-p_L)\left(\sigma - \frac{\kappa}{\lambda}\right)\pi_{NH} + (1-p_H)p_L\left(\frac{1-\beta p_L}{\kappa} + \sigma\right)\pi_{LN} + \sigma r_L^n \\ \beta p_H(1-p_L)\pi_{NH} + \beta(1-p_H)p_L\pi_{LN} + \varepsilon \end{bmatrix}. \tag{45}$$

Thus,

$$\begin{aligned}
y_{LH} &= \frac{D}{E}\left\{p_H(1-p_L)\left(\sigma-\frac{\kappa}{\lambda}\right)\pi_{NH}+(1-p_H)p_L\left(\frac{1-\beta p_L}{\kappa}+\sigma\right)\pi_{LN}+\sigma r_L^n\right\}\\
&- \frac{B}{E}\left\{\beta p_H(1-p_L)\pi_{NH}+\beta(1-p_H)p_L\pi_{LN}+\varepsilon\right\}
\end{aligned}$$

and

$$\begin{aligned}
\pi_{LH} &= -\frac{C}{E}\left\{p_H(1-p_L)\left(\sigma-\frac{\kappa}{\lambda}\right)\pi_{NH}+(1-p_H)p_L\left(\frac{1-\beta p_L}{\kappa}+\sigma\right)\pi_{LN}+\sigma r_L^n\right\}\\
&+ \frac{A}{E}\left\{\beta p_H(1-p_L)\pi_{NH}+\beta(1-p_H)p_L\pi_{LN}+\varepsilon\right\}.
\end{aligned}$$

Substituting the result of π_{NH} and π_{LN}, we obtain

$$\begin{aligned}
y_{LH} = {}& \frac{\frac{(1-\beta p_H p_L)}{\kappa}\left(1-\frac{\kappa}{\sigma\lambda}\right)+\frac{\beta p_H p_L}{\kappa}}{\frac{(1-p_H p_L)(1-\beta p_H p_L)}{\sigma\kappa}-p_H p_L}\cdot\frac{p_H(1-p_L)\lambda}{\kappa^2+\lambda(1-\beta p_H)}\varepsilon\\
&+\frac{\frac{(1-\beta p_H p_L)}{\kappa}\left(\frac{1-\beta p_L}{\sigma\kappa}+1\right)+\frac{\beta p_H p_L}{\kappa}}{\frac{(1-p_H p_L)(1-\beta p_H p_L)}{\sigma\kappa}-p_H p_L}\cdot\frac{(1-p_H)p_L r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L}\\
&+\frac{\frac{(1-\beta p_H p_L)r_L^n+p_H p_L\varepsilon}{\kappa}}{\frac{(1-p_H p_L)(1-\beta p_H p_L)}{\sigma\kappa}-p_H p_L}
\end{aligned}\tag{46}$$

$$\begin{aligned}
\pi_{LH} = {}& \frac{\left(1-\frac{\kappa}{\sigma\lambda}\right)+\frac{\beta(1-p_H p_L)}{\sigma\kappa}}{\frac{(1-p_H p_L)(1-\beta p_H p_L)}{\sigma\kappa}-p_H p_L}\cdot\frac{p_H(1-p_L)\lambda}{\kappa^2+\lambda(1-\beta p_H)}\varepsilon\\
&+\frac{\left(\frac{1-\beta p_L}{\sigma\kappa}+1\right)+\frac{\beta(1-p_H p_L)}{\sigma\kappa}}{\frac{(1-p_H p_L)(1-\beta p_H p_L)}{\sigma\kappa}-p_H p_L}\cdot\frac{(1-p_H)p_L r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L}\\
&+\frac{r_L^n+\frac{(1-p_H p_L)}{\sigma\kappa}\varepsilon}{\frac{(1-p_H p_L)(1-\beta p_H p_L)}{\sigma\kappa}-p_H p_L}
\end{aligned}\tag{47}$$

Proposition A.2

Suppose equations defined in Section 2.6 are satisfied. Then $\lambda y_{LN}+\kappa\pi_{LN}<0$ if and only if $\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L>0$.

Proof

Using (37) and (38), we have

$$\lambda y_{LN}+\kappa\pi_{LN}=\left(\frac{(1-\beta p_L)\lambda}{\kappa}+\kappa\right)\frac{r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L}.\tag{48}$$

Notice that $\frac{(1-\beta p_L)\lambda}{\kappa}+\kappa>0$, and $r_L^n<0$. Thus, if $\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L>0$, then $\lambda y_{LN}+\kappa\pi_{LN}<0$. Similarly, if $\lambda y_{LN}+\kappa\pi_{LN}<0$, then $\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L>0$.

Proposition A.3

Suppose equations defined in Section 2.6 are satisfied. Then $\lambda y_{LH}+\kappa\pi_{LH}<0$ if and only if (12) is satisfied.

Proof

Using (46) and (47),

$$
\begin{aligned}
&\lambda y_{LH}+\kappa\pi_{LH}\\
&=\frac{1}{\frac{(1-p_Hp_L)(1-\beta p_Hp_L)}{\sigma\kappa}-p_Hp_L}\cdot\left[\sigma\left\{\lambda(1-\beta p_Hp_L)+\kappa^2\right\}\left\{p_H(1-p_L)\left(1-\frac{\kappa}{\sigma\lambda}\right)\frac{\lambda}{\kappa^2+\lambda(1-\beta p_H)}\varepsilon\right.\right.\\
&\left.+(1-p_H)p_L\left(\frac{1-\beta p_L}{\sigma\kappa}+1\right)\frac{r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L}+r_L^n\right\}\\
&\left.+\beta\left\{(\sigma\lambda-\kappa)p_Hp_L+\kappa\right\}\left\{p_H(1-p_L)\frac{\lambda}{\kappa^2+\lambda(1-\beta p_H)}\varepsilon+(1-p_H)p_L\frac{r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L}+\varepsilon\right\}\right]\\
&=\frac{1}{\frac{(1-p_Hp_L)(1-\beta p_Hp_L)}{\sigma\kappa}-p_Hp_L}\cdot\left[\frac{\left\{\kappa^2+\lambda(1-\beta p_Hp_L)\right\}\left\{\kappa(1-p_H)+\sigma\lambda p_H\right\}}{\kappa^2+\lambda(1-\beta p_H)}\varepsilon\right.\\
&+\frac{1}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L}\left[\frac{(1-\beta p_L)(2-p_L)}{\kappa}\left\{\kappa^2+\lambda(1-\beta p_Hp_L)\right\}+\sigma(\kappa^2+\lambda)(1-p_L)+\right.\\
&\left.\left.\sigma\beta\lambda p_Hp_L^2+\beta\kappa(1-p_Hp_L)\right]r_L^n\right]
\end{aligned}
\tag{49}
$$

I define

$$
F^{F,I}=\frac{\left\{\kappa^2+\lambda(1-\beta p_Hp_L)\right\}\left\{\kappa(1-p_H)+\sigma\lambda p_H\right\}}{\kappa^2+\lambda(1-\beta p_H)}>0
$$

and

$$
\begin{aligned}
G^{F,I}=&\frac{1}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L}\left[\frac{(1-\beta p_L)(2-p_L)}{\kappa}\left\{\kappa^2+\lambda(1-\beta p_Hp_L)\right\}+\sigma(\kappa^2+\lambda)(1-p_L)\right.\\
&\left.+\sigma\beta\lambda p_Hp_L^2+\beta\kappa(1-p_Hp_L)\right]
\end{aligned}
$$

when $\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L>0$, $G^{F,I}>0$.

Thus, $\lambda y_{LH}+\kappa\pi_{LH}<0$ if and only if (12) is satisfied.

Proof of Proposition 2

Proposition B.1

There exists a vector $\{y_{NH},\pi_{NH},i_{NH},y_{NN},\pi_{NN},i_{NN},y_{LH},\pi_{LH},i_{LH},y_{LN},\pi_{LN},i_{LN}\}$ that solves the system of linear equation defined in Section 2.6.

Proof

Based on the definition of equilibrium, NN, NH, and LN states are similar to f-type-I equilibrium. Therefore, I only have to consider the condition of the LH state. The equations for the LH state are as follows.

$$\begin{aligned}
y_{LH} &= [p_H p_L y_{LH} + p_H(1-p_L)y_{NH} + (1-p_H)p_L y_{LN} + (1-p_H)(1-p_L)y_{NN}] \\
&+ \sigma\left[p_H p_L \pi_{LH} + p_H(1-p_L)\pi_{NH} + (1-p_H)p_L\pi_{LN} + (1-p_H)(1-p_L)\pi_{NN} - i_{LH} + r_L^n\right] \\
\pi_{LH} &= \kappa y_{LH} + \beta\left[[p_H p_L \pi_{LH} + p_H(1-p_L)\pi_{NH} + (1-p_H)p_L\pi_{LN} + (1-p_H)(1-p_L)\pi_{NN}\right] + \varepsilon \\
0 &= \lambda y_{LH} + \kappa\pi_{LH}
\end{aligned}$$

Substituting the result of the NN state into the above equations,

$$\begin{aligned}
y_{LH} &= [p_H p_L y_{LH} + p_H(1-p_L)y_{NH} + (1-p_H)p_L y_{LN}] \\
&+ \sigma\left[p_H p_L \pi_{LH} + p_H(1-p_L)\pi_{NH} + (1-p_H)p_L\pi_{LN} - i_{LH} + r_L^n\right] \\
\pi_{LH} &= \kappa y_{LH} + \beta\left[[p_H p_L \pi_{LH} + p_H(1-p_L)\pi_{NH} + (1-p_H)p_L\pi_{LN}\right] + \varepsilon \\
0 &= \lambda y_{LH} + \kappa\pi_{LH}
\end{aligned}$$

Using the condition about zero lower bound, $y_{LH} = -\frac{\kappa}{\lambda}\pi_{LH}$. Substituting this into the second equation,

$$\pi_{LH} = \frac{1}{\kappa^2 + \lambda(1-\beta p_H p_L)}\left\{\beta\lambda p_H(1-p_L)\pi_{NH} + \beta\lambda(1-p_H)p_L\pi_{LN} + \lambda\varepsilon\right\}.$$

Substituting π_{NH} and π_{LN} into above equation, we obtain

$$\pi_{LH} = \frac{\lambda}{\kappa^2 + \lambda(1-\beta p_H p_L)}\left\{\frac{\kappa^2 + \lambda(1-\beta p_H p_L)}{\kappa^2 + \lambda(1-\beta p_H)}\varepsilon + \frac{\beta(1-p_H)p_L r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa} - p_L}\right\}.$$

Using this result and $y_{LH} = -\frac{\kappa}{\lambda}\pi_{LH}$, we obtain

$$y_{LH} = -\frac{\kappa}{\kappa^2 + \lambda(1-\beta p_H p_L)}\left\{\frac{\kappa^2 + \lambda(1-\beta p_H p_L)}{\kappa^2 + \lambda(1-\beta p_H)}\varepsilon + \frac{\beta(1-p_H)p_L r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa} - p_L}\right\}.$$

Substituting these results and result of NH and LN state into Euler equation,

$$\begin{aligned}
i_{LH} = &\left\{\left(p_H p_L + \frac{(1-p_H p_L)\kappa}{\sigma\lambda}\right)\frac{\beta\lambda p_H(1-p_L)}{\kappa^2 + \lambda(1-\beta p_H p_L)} + p_H(1-p_L)\left(1 - \frac{\kappa}{\sigma\lambda}\right)\right\}\frac{\lambda}{\kappa^2 + \lambda(1-\beta p_H)}\varepsilon \\
&+ \left\{\left(p_H p_L + \frac{(1-p_H p_L)\kappa}{\sigma\lambda}\right)\frac{\beta\lambda(1-p_H)p_L}{\kappa^2 + \lambda(1-\beta p_H p_L)} + (1-p_H)p_L\left(1 + \frac{1-\beta p_L}{\sigma\kappa}\right)\right\}\frac{r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa} - p_L} \\
&+ r_L^n + \left(p_H p_L + \frac{(1-p_H p_L)\kappa}{\sigma\lambda}\right)\frac{\lambda}{\kappa^2 + \lambda(1-\beta p_H p_L)}\varepsilon.
\end{aligned} \tag{50}$$

Proposition B.2

Suppose the equations defined in Section 2.6 are satisfied. Then, $i_{LH} > 0$ if and only if (13) is satisfied.

Proof

Rearranging (50),

$$\begin{aligned} i_{LH} = &\left\{\frac{(1-p_H)\kappa}{\sigma\lambda} + p_H\right\}\frac{\lambda}{\kappa^2+\lambda(1-\beta p_H)}\varepsilon \\ &+\left\{\left(p_H p_L + \frac{(1-p_H p_L)\kappa}{\sigma\lambda}\right)\frac{\beta\lambda(1-p_H)p_L}{\kappa^2+\lambda(1-\beta p_H p_L)}\right. \\ &\left.+\frac{(1-\beta p_L)(1-p_H p_L)}{\sigma\kappa} - p_H p_L\right\}\frac{r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L} \end{aligned}$$

We define

$$F^{F,II} = \left\{\frac{(1-p_H)\kappa}{\sigma\lambda} + p_H\right\}\frac{\lambda}{\kappa^2+\lambda(1-\beta p_H)} > 0$$

and

$$\begin{aligned} G^{F,II} = &\left\{\left(p_H p_L + \frac{(1-p_H p_L)\kappa}{\sigma\lambda}\right)\frac{\beta\lambda(1-p_H)p_L}{\kappa^2+\lambda(1-\beta p_H p_L)}\right. \\ &\left.+\frac{(1-\beta p_L)(1-p_H p_L)}{\sigma\kappa} - p_H p_L\right\}\frac{1}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L} \end{aligned}$$

when $\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa} - p_L > 0$, $G^{F,II} > 0$.

Thus, $i_{LH} > 0$ if and only if (13) is satisfied.

Proof of Proposition 3

From the proof of Proposition 1,

$$\pi_{NH} = \frac{\lambda}{\kappa^2+\lambda(1-\beta p_H)}\varepsilon > 0$$

$$y_{NH} = -\frac{\kappa}{\kappa^2+\lambda(1-\beta p_H)}\varepsilon < 0$$

$$\pi_{NN} = 0$$

$$y_{NN} = 0$$

$$\pi_{LN} = \frac{r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L} < 0$$

$$y_{LN} = \frac{r_L^n}{\frac{(1-p_L)(1-\beta p_L)}{\sigma\kappa}-p_L}\cdot\frac{1-\beta p_L}{\kappa} < 0$$

The Euler equation for LH state is

$$y_{LH} = [p_H p_L y_{LH} + p_H(1-p_L)y_{NH} + (1-p_H)p_L y_{LN}]$$

$$+ \quad \sigma \left[p_H p_L \pi_{LH} + p_H(1-p_L)\pi_{NH} + (1-p_H)p_L\pi_{LN} + r_L^n \right].$$

Rearranging this equation,

$$\begin{aligned} y_{LH} &= \frac{1}{1-p_H p_L}\left[p_H(1-p_L)\left(\sigma - \frac{\kappa}{\lambda}\right)\pi_{NH} + (1-p_H)p_L\left(\frac{1-\beta}{\kappa} + \sigma\right)\pi_{LN} + \sigma r_L^n \right] \\ &+ \frac{\sigma p_H p_L}{1-p_H p_L}\pi_{LH}. \end{aligned}$$

Let's focus on the first term of the RHS, $\sigma - \frac{\kappa}{\lambda}$. By definition, $\sigma - \frac{\kappa}{\lambda} = \sigma - \theta$. According to the definition provided in the text, the sign of this value is undetermined. However, considering that $\frac{1}{\theta - 1}$ represents the price markup rate, it is reasonable to consider $\sigma - \theta < 0$. Therefore, let's assume $\sigma - \theta < 0$.

Assuming y_{LH} to be positive, given that the first term on the RHS is negative, it follows that π_{LH} must be positive. This contradicts $\lambda y_{LH} + \kappa \pi_{LH} < 0$, so in equilibrium, it must be $y_{LH} < 0$. According to the proof of Proposition 1, $\pi_{LH} < 0$ and $\lambda y_{LH} + \kappa \pi_{LH} < 0$ when ε and p_H are sufficiently small and p_L is sufficiently large. Also, when $p_H = 0.5, p_L = 0.75$ and remaining parameters are same as those of defined in Section 4, $\pi_{LH} > 0$ and $\lambda y_{LH} + \kappa \pi_{LH} < 0$. Thus the sign of π_{LH} depends on parameters.

Proof of Proposition 4

The sign of allocations in NN, NH, and LN states are shown in the proof of Proposition 3. Again, the Euler equation of LH state is

$$\begin{aligned} y_{LH} &= \left[p_H p_L y_{LH} + p_H(1-p_L)y_{NH} + (1-p_H)p_L y_{LN} \right] \\ &+ \sigma \left[p_H p_L \pi_{LH} + p_H(1-p_L)\pi_{NH} + (1-p_H)p_L\pi_{LN} - i_{LH} + r_L^n \right] \end{aligned}$$

Eliminating each state of y by using previous results and solving this equation for i_{LH},

$$\begin{aligned} \sigma i_{LH} &= (1-p_H p_L)\left(\frac{\kappa}{\lambda} + \sigma\right)\pi_{LH} + p_H(1-p_L)\left(-\frac{\kappa}{\lambda} + \sigma\right)\pi_{NH} \\ &+ (1-p_H)p_L\left(\frac{1-\beta p_L}{\kappa} + \sigma\right)\pi_{LN} + \sigma r_L^n. \end{aligned}$$

Since, second, third, and fourth term of the RHS are negative, the first term of the RHS must be positive when $i_{LH} > 0$. Notice that $(1 - p_H p_L)$ and $\frac{\kappa}{\lambda} + \sigma$ are positive. Thus, $\pi_{LH} > 0$ when $i_{LH} > 0$. Also, from $y_{LH} = -\frac{\kappa}{\lambda}\pi_{LH}$, $y_{LH} < 0$.

Proof of Proposition 5

From the definition of equilibria in the expectation-driven ZLB model in Section 2.7, it is understood that replacing r_L^n with r^n and p_L with p_D in the equilibria in the fundamental-driven ZLB model results in equilibria in the expectation-driven ZLB model. Therefore, proof can be significantly abbreviated, and it is sufficient to prove the propositions below.

Proposition C.1

Suppose the equations defined in Section 2.7 are satisfied. Then $\lambda y_{DN} + \kappa\pi_{DN} < 0$ if and only if $\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D < 0$.

Proof

$$\lambda y_{DN} + \kappa\pi_{DN} = \left(\frac{(1-\beta p_D)\lambda}{\kappa} + \kappa\right)\frac{r^n}{\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D}.$$

Notice that $\frac{(1-\beta p_D)\lambda}{\kappa} + \kappa > 0$, and $r^n > 0$. Thus, if $\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D < 0$, then $\lambda y_{DN} + \kappa\pi_{DN} < 0$. Similarly, if $\lambda y_{DN} + \kappa\pi_{DN} < 0$, then $\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D < 0$.

Proposition C.2

Suppose the equations defined in Section 2.7 are satisfied. Then $\lambda y_{DH} + \kappa\pi_{DH} < 0$ if and only if (15) is satisfied.

Proof

$$\begin{aligned}
&\lambda y_{DH} + \kappa\pi_{DH} \\
&= \frac{1}{\frac{(1-p_H p_D)(1-\beta p_H p_D)}{\sigma\kappa} - p_H p_D} \cdot \left[\frac{\{\kappa^2 + \lambda(1-\beta p_H p_D)\}\{\kappa(1-p_H) + \sigma\lambda p_H\}}{\kappa^2 + \lambda(1-\beta p_H)}\varepsilon\right. \\
&+ \frac{1}{\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D}\left[\frac{(1-\beta p_D)(2-p_D)}{\kappa}\left\{\kappa^2 + \lambda(1-\beta p_H p_D)\right\} + \sigma(\kappa^2+\lambda)(1-p_D) + \right. \\
&\left.\left.\sigma\beta\lambda p_H p_D^2 + \beta\kappa(1-p_H p_D)\right] r^n\right]
\end{aligned} \tag{51}$$

We define

$$F^{E,I} = \frac{\{\kappa^2 + \lambda(1-\beta p_H p_D)\}\{\kappa(1-p_H) + \sigma\lambda p_H\}}{\kappa^2 + \lambda(1-\beta p_H)} > 0$$

and

$$\begin{aligned}
G^{E,I} = &\frac{r^n}{\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D}\left[\frac{(1-\beta p_D)(2-p_D)}{\kappa}\left\{\kappa^2 + \lambda(1-\beta p_H p_D)\right\} + \sigma(\kappa^2+\lambda)(1-p_D)\right. \\
&\left.+\sigma\beta\lambda p_H p_D^2 + \beta\kappa(1-p_H p_D)\right]
\end{aligned}$$

when $\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D < 0$, $G^{E,I} < 0$.

Thus, $\lambda y_{DH} + \kappa\pi_{DH} < 0$ if and only if (15) is satisfied.

Proof of Proposition 6

See proof of Proposition 2 and proof of Proposition 5.

Proposition D.1

Suppose the equations defined in Section 2.7 are satisfied. Then $i_{DH} > 0$ if and only if (16) is satisfied.

Proof

Using previous results,

$$\begin{aligned}
i_{DH} = &\left\{ \frac{(1-p_H)\kappa}{\sigma\lambda} + p_H \right\} \frac{\lambda}{\kappa^2 + \lambda(1-\beta p_H)} \varepsilon \\
&+ \left\{ \left(p_H p_D + \frac{(1-p_H p_D)\kappa}{\sigma\lambda} \right) \frac{\beta\lambda(1-p_H)p_D}{\kappa^2 + \lambda(1-\beta p_H p_D)} \right. \\
&\left. + \frac{(1-\beta p_D)(1-p_H p_D)}{\sigma\kappa} - p_H p_D \right\} \frac{r^n}{\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D}
\end{aligned}$$

We define

$$F^{E,II} = \left\{ \frac{(1-p_H)\kappa}{\sigma\lambda} + p_H \right\} \frac{\lambda}{\kappa^2 + \lambda(1-\beta p_H)} = F^{F,II} > 0$$

and

$$\begin{aligned}
G^{E,II} = &\left\{ \left(p_H p_D + \frac{(1-p_H p_D)\kappa}{\sigma\lambda} \right) \frac{\beta\lambda(1-p_H)p_D}{\kappa^2 + \lambda(1-\beta p_H p_D)} \right. \\
&\left. + \frac{(1-\beta p_D)(1-p_H p_D)}{\sigma\kappa} - p_H p_D \right\} \frac{r^n}{\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D}
\end{aligned}$$

when $\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D > 0$, $G^{E,II} < 0$.

Thus, $i_{LH} > 0$ if and only if (16) is satisfied.

Proof of Proposition 7

The sign of allocations in the NN and NH states are shown in the proof of Proposition 3. From the proof of Proposition 5,

$$\pi_{DN} = \frac{r^n}{\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D} < 0$$

$$y_{DN} = \frac{r^n}{\frac{(1-p_D)(1-\beta p_D)}{\sigma\kappa} - p_D} \cdot \frac{1-\beta p_D}{\kappa} < 0$$

Proof of Proposition 8

The sign of allocations in the NN, NH, and DN states are shown in the proof of Proposition 7. Again, the Euler equation of DH state is

$$y_{DH} = [p_H p_D y_{DH} + p_H(1-p_D)y_{NH} + (1-p_H)p_D y_{DN}]$$

$$+ \quad \sigma\left[p_H p_D \pi_{DH} + p_H(1-p_D)\pi_{NH} + (1-p_H)p_D\pi_{DN} - i_{DH} + r^n\right]$$

Eliminating each state of y by using previous results and solving this equation for i_{LH},

$$\begin{aligned} \sigma i_{DH} &= (1-p_H p_D)\left(\frac{\kappa}{\lambda}+\sigma\right)\pi_{DH} + p_H(1-p_D)\left(-\frac{\kappa}{\lambda}+\sigma\right)\pi_{NH} \\ &+ (1-p_H)p_D\left(\frac{1-\beta p_D}{\kappa}+\sigma\right)\pi_{DN} + \sigma r^n. \end{aligned}$$

Since $(1-p_H p_D)\left(\frac{\kappa}{\lambda}+\sigma\right) > 0$, $\pi_{DH} > 0$ is consistent with $i_{DH} > 0$. Also, when $p_H = 0.59, p_D = 0.999$ and the remaining parameters are the same as those defined in Section 4, $\pi_{DH} < 0$ and $i_{DH} > 0$. Thus, the sign of π_{DH} depends on parameters. Since $y_{DH} = -\frac{\kappa}{\lambda}\pi_{DH}$, the sign of y_{DH} also depends on parameters.

Other Markov-Perfect equilibria

In this paper, I focus on two equilibria (Type-I, Type-II), but introducing two types of shocks following a two-state Markov process yields 16 potential equilibria. The purpose of this paper is to introduce cost-push shocks into a model that analyzes situations where the ZLB is bound due to demand/sunspot shocks and examine how this introduction changes the equilibrium properties. In the model with only demand/sunspot shocks, the ZLB is bound only when there are shocks, while in the absence of shocks, the ZLB is free. This is consistent with assumption $i_{NN} > 0$ and i_{LN} or $i_{DN} = 0$ in my model. Therefore, assuming $i_{NN} > 0$ and i_{LN} or $i_{DN} = 0$, remaining potential equilibria are as follows.

Type-III equilibrium:

$$i_{NN} > 0, i_{NH} = 0, i_{XN} = 0, i_{XH} = 0$$

Type-IV equilibrium:

$$i_{NN} > 0, i_{NH} = 0, i_{XN} = 0, i_{XH} > 0$$

$X = L$ in fundamental-driven ZLB model, and $X = D$ in expectation-driven ZLB model.

Figure 9: Existence of Two Markov-Perfect Equilibria: fundamental

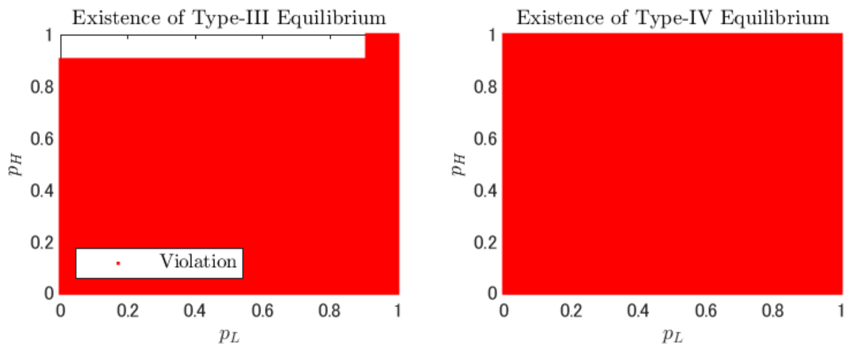

Figure 10: Existence of Two Markov-Perfect Equilibria: expectation

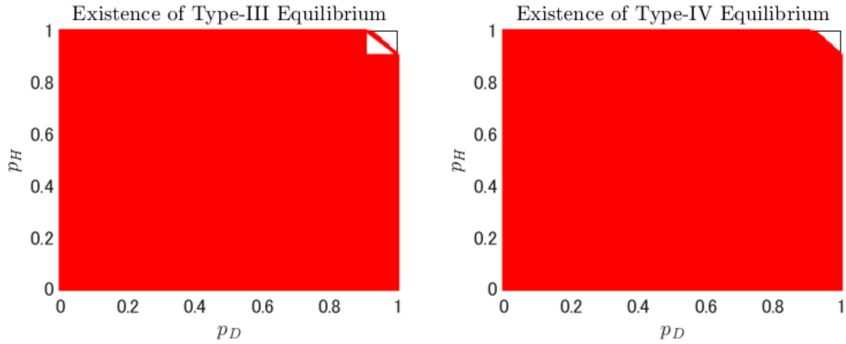

Figure 9 shows the existence of two possible equilibria in the fundamental-driven ZLB model for different combinations of p_L and p_H. In each panel, white areas show the combinations of persistence of two shocks under which the equilibrium of a particular type exists. Red areas indicate that either one or both of the relevant inequality constraints are violated and thus the equilibrium does not exist. Also, figure 10 shows the existence of two possible equilibria in the expectation-driven ZLB model for different combinations of p_D and p_H. From the figure, Type-III and Type-IV equilibria theoretically exist in some cases. However, it is unrealistic that a nominal interest rate is constrained by the ZLB when only positive cost-push shocks are present. Therefore, I don't address these two equilibria in this paper.