1 Summary

Solve the Poisson equation in circular coordinate

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial p\left(r,\theta\right)}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 p\left(r,\theta\right)}{\partial \theta^2}=q\left(r,\theta\right), \\ r\in\left[r_-,r_+\right],\theta\in\left[\theta_-,\theta_+\right], \\ \left.\dfrac{\partial p}{\partial r}\right|_{r=r_-}=\left.\dfrac{\partial p}{\partial r}\right|_{r=r_+}=0,\theta_-=\theta_+, \end{cases} \tag{1}$$

where p is the answer, while q the input. Those variables are defined on the center of staggered cell (pressure position). Fourier transform is used in the azimuthal direction, while tri-diagonal matrix solver is adopted in the radial direction.

2 Detail

By using the second-order central difference scheme, the above equation is discretized as

$$\frac{1}{r_{i,j}}\frac{\left(r\frac{\partial p}{\partial r}\right)_{i+\frac{1}{2},j}-\left(r\frac{\partial p}{\partial r}\right)_{i-\frac{1}{2},j}}{\Delta r}+\frac{1}{r_{i,j}^2}\frac{p_{i,j+1}-2p_{i,j}+p_{i,j-1}}{\Delta\theta^2}=q_{i,j}, \tag{2}$$

where the subscripts i and j are used to denote the indices in r and θ directions, respectively. The half-index points are further re-arranged as

$$\begin{cases} \left(r\dfrac{\partial p}{\partial r}\right)_{i+\frac{1}{2},j}=r_{i+\frac{1}{2},j}\dfrac{p_{i+1,j}-p_{i,j}}{\Delta r}, \\ \left(r\dfrac{\partial p}{\partial r}\right)_{i-\frac{1}{2},j}=r_{i-\frac{1}{2},j}\dfrac{p_{i,j}-p_{i-1,j}}{\Delta r}, \end{cases} \tag{3}$$

thus we solve

$$\begin{aligned} &\frac{1}{r_{i,j}}\frac{r_{i+\frac{1}{2},j}\left(p_{i+1,j}-p_{i,j}\right)-r_{i-\frac{1}{2},j}\left(p_{i,j}-p_{i-1,j}\right)}{\Delta r^2} \\ +&\frac{1}{r_{i,j}^2}\frac{p_{i,j+1}-2p_{i,j}+p_{i,j-1}}{\Delta\theta^2}=q_{i,j}. \end{aligned} \tag{4}$$

3 Test code

In test code, the following equation is solved as a benchmark [1]

$$p=\left(\frac{1}{12}r^4+C_1r^2+C_2\frac{1}{r^2}\right)\cos\left(2\theta\right), \tag{5a}$$

$$q=r^2\cos\left(2\theta\right), \tag{5b}$$

where the coefficients are

$$C_1 = \frac{1}{6}\frac{r_+^3 r_-^3}{r_+^4 - r_-^4}\left(-\frac{r_+^3}{r_-^3} + \frac{r_-^3}{r_+^3}\right), \tag{6a}$$

$$C_2 = \frac{1}{6}\frac{r_+^3 r_-^3}{r_+^4 - r_-^4}\left(-r_+^3 r_- + r_+ r_-^3\right). \tag{6b}$$

References

[1] V. P. Pikulin and S. I. Pohozaev, *Equations in mathematical physics: a practical course.* Springer Science & Business Media, 2012.