Orthogonal Polynomials and Applications

Naoki Fujita

Based on Chapter 8 from *T.J. Sullivan, 2015*

January 12, 2025

Outline

1. 8.1 Basic Definitions and Properties
2. 8.2 Recurrence Relations
3. 8.3 Differential Equations
4. 8.4 Roots of Orthogonal Polynomials
5. 8.5 Polynomial Interpolation
6. 8.6 Polynomial Approximation
7. 8.7 Multivariate Orthogonal Polynomials

Definition of Orthogonal Polynomials

Orthogonal Polynomials:

- Polynomials $q_n(x)$ are orthogonal with respect to a measure $\mu(x)$:

$$\int_{\mathbb{R}} q_m(x) q_n(x) \mu(x) dx = \gamma_n \delta_{mn},$$

  where γ_n is a normalization constant.
- $\{q_n\}$ are set of orthogonal basis for $L^2_\mu(\mathbb{R})$
- Orthogonal polynomials serve as efficient basis functions for approximations.
- if γ_n is 1, it is orthonormal

Examples of Orthogonal Polynomials

Classical Examples:

- **Legendre Polynomials:** Orthogonal on $[-1, 1]$ with uniform weight.
- **Hermite Polynomials:** Orthogonal on $\mathbb{R}$ with Gaussian weight.
- **Chebyshev Polynomials:** Orthogonal with weight $(1 - x^2)^{-1/2}$.

Properties:

- Real roots within the domain.
- Generated via Gram-Schmidt process or recurrence relations.
- eigenfunctions for differential operators

Applications:

- Numerical integration (Gaussian quadrature).
- Polynomial approximations.

Gram-Schmidt Process for Orthogonalization

Procedure:

- Start with monomials: $1, x, x^2, \ldots$.
- Use Gram-Schmidt orthogonalization:

$$q_n(x) = x^n - \sum_{k=0}^{n-1} \frac{\langle x^n, q_k \rangle}{\langle q_k, q_k \rangle} q_k(x).$$

- Result: Orthogonal polynomials $q_n(x)$ up to degree n.

Numerical Stability:

- Direct Gram-Schmidt can be unstable.
- Recurrence relations preferred in practice.

Key Properties of Orthogonal Polynomials

Key Theorems:

- **Hankel Determinants:** Positive definite if orthogonal polynomials exist.
- **Completeness:** An infinite family of orthogonal polynomials does not grantee to form a complete orthogonal basis for $L_\mu(\mathbb{R})$

Outline

1. 8.1 Basic Definitions and Properties
2. 8.2 Recurrence Relations
3. 8.3 Differential Equations
4. 8.4 Roots of Orthogonal Polynomials
5. 8.5 Polynomial Interpolation
6. 8.6 Polynomial Approximation
7. 8.7 Multivariate Orthogonal Polynomials

Three-Term Recurrence Relation

every system of orthogonal polynomials satisfies a three-term recurrence relation of the form:

$$q_{n+1}(x) = (A_n x + B_n) q_n(x) - C_n q_{n-1}(x),$$

where A_n, B_n, C_n are coefficients determined by the measure μ and $q_0(x) = 1, q_{-1}(x) = 0$

If $\{q_n\}$ are monic ($deg(p) = n$and $c_n = 1$)

$$q_{n+1}(x) = (x + \alpha_n) q_n(x) - \beta_n q_{n-1}(x),$$

Examples:

- **Legendre Polynomials:**

$$Le_{n+1}(x) = \frac{2n+1}{n+1} x Le_n(x) - \frac{n}{n+1} Le_{n-1}(x).$$

- **Hermite Polynomials:**

$$He_{n+1}(x) = x He_n(x) - n He_{n-1}(x).$$

- **Chebyshev Polynomials:**

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x).$$

Favard's Theorem

Statement:

- A sequence satisfies a three-term recurrence relation with some conditions on the coefficients represents orthogonal polynomials with respect to some measure.

Implications:

- Establishes existence of measure μ for given recurrence coefficients.

Outline

1. 8.1 Basic Definitions and Properties
2. 8.2 Recurrence Relations
3. 8.3 Differential Equations
4. 8.4 Roots of Orthogonal Polynomials
5. 8.5 Polynomial Interpolation
6. 8.6 Polynomial Approximation
7. 8.7 Multivariate Orthogonal Polynomials

Differential Equations for Orthogonal Polynomials

In addition to their orthogonality and recurrence properties, the classical orthogonal polynomials are eigenfunctions for second-order differential operators

Form of Differential Operator:

$$\mathcal{L}[q_n](x) = \lambda_n q_n(x),$$

where $\mathcal{L} = Q(x)\frac{d^2}{dx^2} + L(x)\frac{d^2}{dx}$ with $Q(x)$ quadratic and $L(x)$ linear.
the eigenvalue is $\lambda_n = n(\frac{n-1}{2}Q^{''} + L^{'})$

Examples of Differential Equations

Classical Cases:

- **Leguerre Polynomials:**

$$xq_n''(x) - (1 + \alpha - x)q_n'(x) = -nq_n(x).$$

- **Hermite Polynomials:**

$$q_n''(x) - xq_n'(x) = -nq_n(x).$$

Properties of Differential Equations

Key Properties:

- Differential operators for orthogonal polynomials are derived from the recurrence relations
- The converse (the existence of orthogonal polynomials when you have this type of differential equation) is also true with some condition on Q and L

Applications of Differential Equations

Applications:

- Spectral methods for solving PDEs.
- Stability analysis in numerical solutions.

Outline

1. 8.1 Basic Definitions and Properties
2. 8.2 Recurrence Relations
3. 8.3 Differential Equations
4. 8.4 Roots of Orthogonal Polynomials
5. 8.5 Polynomial Interpolation
6. 8.6 Polynomial Approximation
7. 8.7 Multivariate Orthogonal Polynomials

Properties of Roots

The points x at which an orthogonal polynomial $q_n(x) = 0$ are its roots

Roots of Orthogonal Polynomials:

- Real and distinct.
- The roots $\boldsymbol{z}^n$of q_n and $\boldsymbol{z}^{n+1}$of q_{n+1} alternate i.i.e:
  $z_1^{n+1} < z_1^n < z_2^{n+1} < \cdots < z_n^{n+1} < z_n^n < z_{n+1}^{n+1}$

Applications:

- Gaussian quadrature nodes.
- Basis for interpolation methods.

Jacobi Matrix:

- Roots are eigenvalues of the tridiagonal Jacobi matrix:

$$J = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & 0 & \cdots \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & \cdots \\ 0 & \sqrt{\beta_2} & \alpha_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

where α_n and β_n are coefficients of recurrent relations.

Zeros and Quadrature

Gaussian Quadrature:

- Integration nodes: Roots of orthogonal polynomials.
- Weights computed from Christoffel-Darboux formula.

Key Property:

$$\int_a^b f(x)\mu(x)dx \approx \sum_{i=1}^{n} w_i f(x_i).$$

Outline

1. 8.1 Basic Definitions and Properties
2. 8.2 Recurrence Relations
3. 8.3 Differential Equations
4. 8.4 Roots of Orthogonal Polynomials
5. 8.5 Polynomial Interpolation
6. 8.6 Polynomial Approximation
7. 8.7 Multivariate Orthogonal Polynomials

Lagrange Interpolation

Definition:

- Polynomial passing through $(x_0, y_0), \ldots, (x_n, y_n)$:

$$L(x) = \sum_{i=0}^{n} y_i \ell_i(x),$$

where $\ell_i(x)$ are Lagrange basis polynomials.

Runge's Phenomenon

Observation:

- Uniformly spaced nodes lead to oscillations near boundaries.
- Example: Interpolation of $f(x) = \frac{1}{1+25x^2}$.

Solution:

- Use Chebyshev nodes: Dense near endpoints.

Chebyshev Interpolation

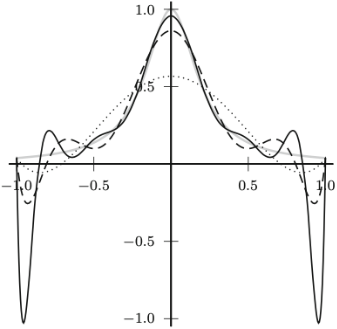
Chebyshev Nodes:

$$x_k = \cos\left(\frac{2k+1}{2n}\pi\right), \quad k = 0, \ldots, n-1.$$

Advantages:

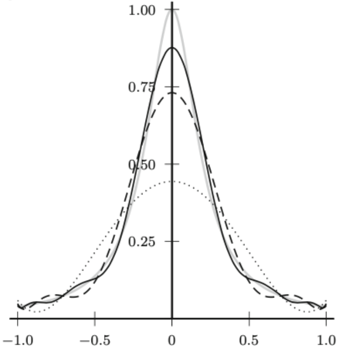
- Minimize interpolation error.
- Avoid Runge's phenomenon.

a



Uniformly spaced nodes.

b



Chebyshev nodes.

Outline

1. 8.1 Basic Definitions and Properties
2. 8.2 Recurrence Relations
3. 8.3 Differential Equations
4. 8.4 Roots of Orthogonal Polynomials
5. 8.5 Polynomial Interpolation
6. 8.6 Polynomial Approximation
7. 8.7 Multivariate Orthogonal Polynomials

Weierstrass Approximation Theorem

Statement:

- Any continuous function on $[a, b]$ can be approximated by polynomials:

$$\sup_{x \in [a,b]} |f(x) - p_n(x)| \to 0 \quad \text{as } n \to \infty.$$

Orthogonal Polynomial Approximation

Best Approximation:

- Orthogonal projection onto $q_0, \ldots, q_n$ is the best approximation in the mean square sense:

$$\Pi_n f = \sum_{k=0}^{n} c_k q_k(x).$$

- Coefficients determined by:

$$c_k = \frac{\langle f, q_k \rangle}{\langle q_k, q_k \rangle}.$$

Spectral convergence of Legendre expansions

There is a constant $C_k \geq 0$ that may depend upon k but is independent of d and f such that, for all $f \in H^k([-1,1], dx)$,

$$\|f - \Pi_d f\|_{L^2(\mu)} \leq C_k d^{-k} \|f\|_{H^k(\mu)}.$$

Bounds:

- Depend on derivatives of f.
- Asymptotically higher smoothness $\implies$ faster convergence.

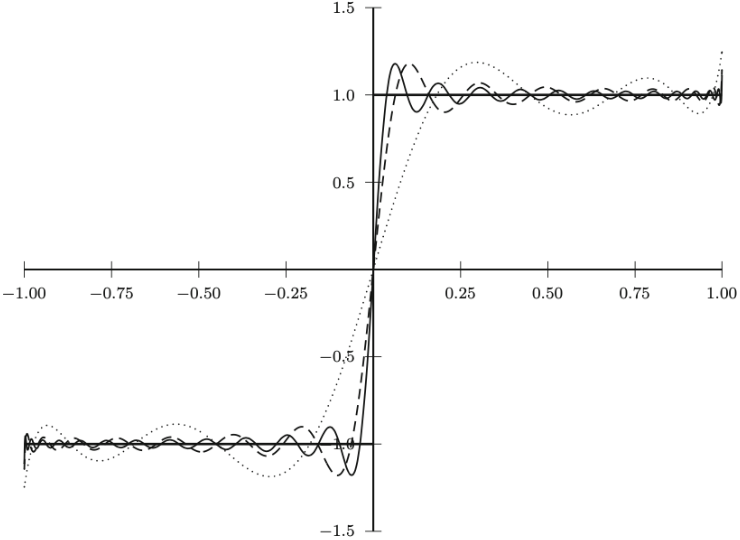


Fig. 8.3: Legendre expansions of the sign function on $[-1, 1]$ exhibit Gibbsian oscillations at 0 and at ± 1. The sign function is shown as the heavy black; also shown are the Legendre expansions (8.24) to degree $2N - 1$ for $N = 5$ (dotted), 15 (dashed), and 25 (solid).

Outline

1. 8.1 Basic Definitions and Properties
2. 8.2 Recurrence Relations
3. 8.3 Differential Equations
4. 8.4 Roots of Orthogonal Polynomials
5. 8.5 Polynomial Interpolation
6. 8.6 Polynomial Approximation
7. 8.7 Multivariate Orthogonal Polynomials

Definition:

- Multivariate orthogonal polynomials are extensions of univariate orthogonal polynomials to multiple dimensions.
- They satisfy:

$$\int_{\Omega} q_{\boldsymbol{\alpha}}(\boldsymbol{x}) q_{\boldsymbol{\beta}}(\boldsymbol{x}) \mu(\boldsymbol{x}) d\boldsymbol{x} = \gamma_{\boldsymbol{\alpha}} \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}},$$

  where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are multi-indices.

Multi-Index Notation

Multi-Index Representation:

- Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ be a multi-index.
- Monomials are represented as:

$$\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}.$$

- Degree of $\boldsymbol{\alpha}$:

$$|\boldsymbol{\alpha}| = \sum_{i=1}^{d} \alpha_i.$$

Weak and Strong Orthogonality

Orthogonality Concepts:

- **Weak Orthogonality:**

$$\int_{\Omega_i} q_{\boldsymbol{\alpha}}(\boldsymbol{x}) q_{\boldsymbol{\beta}}(\boldsymbol{x}) \mu(\boldsymbol{x}) d\boldsymbol{x} = 0, \quad \text{if } \boldsymbol{\alpha} \neq \boldsymbol{\beta}.$$

- **Strong Orthogonality:**

$$\int_{\Omega} q_{\boldsymbol{\alpha}}(\boldsymbol{x}) q_{\boldsymbol{\beta}}(\boldsymbol{x}) \mu(\boldsymbol{x}) d\boldsymbol{x} = \gamma_{\boldsymbol{\alpha}} \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}.$$

Using Product Measures:

- If $\mu(\boldsymbol{x}) = \prod_{i=1}^{d} \mu_i(x_i)$, then:

$$q_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \prod_{i=1}^{d} q_{\alpha_i}^{(i)}(x_i),$$

where $q_{\alpha_i}^{(i)}(x_i)$ are univariate orthogonal polynomials.

Applications of Multivariate Orthogonal Polynomials

Key Applications:

- Numerical integration in higher dimensions.
- Polynomial chaos expansions for random fields.
- Solving PDEs with stochastic inputs.

Challenges:

- Computational complexity grows exponentially with dimension ("curse of dimensionality").