Paper

The category of relational $\mathbf{T}$-algebras isomorphic to the category of topological spaces

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Contents

1 Introduction

A network structure, a relation between two objects, is frequently used in applied mathematics, such as telecommunications field, transportation network, computatinal algorithms' data structure. Relational theory, the basis of a network structure, has been reserached 200 years ago [8], and in recent years, has been developed such as the relational calculus library of the theorem prover Coq[10]. In this thesis, I adjusted the theory of relational algebras over the category of sets Set, and , especially, I prove fundamental properties using relational calculus, and studied the relational **T**-algebra generated by a monad **T**.

I first briefly describe the histry of relational theory. In 19th century, the relational theory stareted with creation of Boolean algebra, which is a algebraic formulization for logical operators or set operators, and also has already been studied Morgan[11] or Peirce[13]. Afterwards, their results was arranged and strictly axiomized by Tarski[17]. Furthermore, the categorical reserach on the relational theory had been begun by MacLane[7], Puppe[14], Kawahara[6] e.t.c., and was formulized by Oliver and Serrato in 1980 as Dedekind category, the cathegory of relations[12], and was formulized by Fryed and Scedrov in 1990 as Allegory, the cathegory of relations[3]. Thanks to their reserach, relational algebras is thought among the distinct sets not just among the same sets. Finally, in 2015, Furusawa axiomized Dedekind category equipped with the point axiom[4].

The relational calculus has applied to computer science so far[8]. Hoare[5] started applying for the programming theory, and which succeeds to the extension for a topos, representationability of algebraic system. In addition, the relational calculus formulization has already been applied to the calculus model like automata or graph transformation, and the data model like relational database or formal concept analysis.

When you study progrmming theory, you should consider it from abstract different views, not only the specific programming languages, the calculus systems in use, or the accustomed procedure of coding. Scott introduced the denotational semantics for programming languages and the calculus theory over lattices in order to reserach the qualitative properties for calculus process[15], and then he created continuous lattices to describe calculus process[16]. The continuous lattices are defined by the abstraction for the calculus process and the axiomization for required properties to the ordered set, and then they have mathematical remarkable properties enough to have been reserached by many mathematicians. In the meantime, Day proved that Clos is isomorphic to Set from the category thoery, where Clos is the category of continuous lattices and functions preserving the supremum on the directed subset and the infimum[2]. Furthermore, Barr proved that Top is isomorphic to Rel(**U**) by setting the relational **T**-algebra over Set[1], where Top is the category of topological spaces and Rel(**U**) is the category of relational algebras over ultra filter monads. In this thesis, I reported the rewriting for the above formal proofs by Day and Barr by using the relational theory.

Finally, I describe the composition of this thesis.

In the section 2, I prepare many terms and properties for the relational theory, the theory of multivalued correspondances. Functions have single-valued property, where one independent variable (input) must correspond to one dependent variable(output), however never do relations. Because of this property, relations can be considered to the extended idea of functions, and you may call them multivalued functions or nondeterministic functions. Furthermore, I set various ideas by using the category theory, where you reserach mathematical structures by using the features of functions, not by using the set theory including ideas like points or variables. For example, I, a set with single element, is defined to be the existence of single function from any set X to I. Besides, I algebraically formalized relational operators, and then prepare many required terms or properties such as monads, **T** algebras, the rationality or the axiom of choice so as to formalize proofs in the section 4.

In the section 3, I explain first the definition or the properties of filters and second the ultra filter monads[2]. I use this idea to prove the isomorphism between Rel(**U**) and Top to compose interactive functors between them.

In the section 4, I explained the theory of the relational **T**-algebras extended from **T**-algebras. Barr proved had defined relational $\mathbf{T}$-algebras over Set for the ultra fliter monad $\mathbf{U}$, and had proved that Rel($\mathbf{U}$) is isomorphic to Top [1]. Furthermore, Day had proved that Clos is isomorphic to the reflexive subcategory of the category of $\mathbf{T}$-algebras over filter monads [2]. Moreover, Mizoguchi extended it to the theory over the category of sets with monoid actions [9]. In this thesis, their results were briefly rewritten by using relational calculus, which is checkable and formal, including notations and terms defined in the section 2 and in the section 3.

2 Preliminaries

In this section, we define several terms used in section 3 and 4 before explaining their properties.

Let A, B, C, D be sets and I a unit set.

Definition 2.1. *A subset α of $A \times B$ is called a relation from A to B, and is denoted as $\alpha : A \rightharpoonup B$.*

Definition 2.2. *Let n be a positive integer and $\{X_k\}_{1 \leq k \leq n}$ a family of sets, then X^n is a set defined by $X^n = X_1 \times X_2 \times \cdots \times X_n$ if $X_k = X$ holds for any $k = 1, 2, \cdots, n-1$.*

Definition 2.3. *For relations $\alpha : A \rightharpoonup B$, $\alpha' : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$,*

(1) *a composite relation $\beta \cdot \alpha$ of α and β is defined by*

$$\beta \cdot \alpha = \{(a, c) \in A \times C;\ ^{\exists}b \in B,\ \langle\langle(a, b) \in \alpha\rangle \wedge \langle(b, c) \in \beta\rangle\rangle\},$$

where $\beta \cdot \alpha$ is also denoted by $\beta\alpha$.

(2) *the inverse relasion $\alpha^\sharp$ of α is defined by*

$$\alpha^\sharp = \{(b, a) \in B \times A; (a, b) \in \alpha\}.$$

(3) *$\alpha \sqsubseteq \alpha'$ is denoted if we have*

$$(a, b) \in \alpha \rightarrow (a, b) \in \alpha'$$

holds for a pair $(a, b) \in A \times B$.

(4) *a union $\alpha \sqcup \beta$ of α and β is defined by*

$$\alpha \sqcup \beta = \{(a, b) \in A \times B; (a, b) \in \alpha \text{ or } (a, b) \in \beta\}.$$

(5) *an intersection or a meet $\alpha \sqcap \beta$ of α and β is defined by*

$$\alpha \sqcap \beta = \{(a, b) \in A \times B; (a, b) \in \alpha \text{ and } (a, b) \in \beta\}.$$

(6) *α is called total if $\alpha\alpha^\sharp \sqsubseteq 1_B$ holds.*

(7) *α is called univalent if $\alpha^\sharp\alpha \sqsupseteq 1_A$ holds.*

(8) *α is called a function from A to B if α is univalent and total, and then it is denoted by $\alpha : A \to B$.*

Definition 2.4. *The relation $1_A : A \rightharpoonup A$ defined by $1_A := \{(a, a); a \in A\}$ is called the identity function over A.*

Definition 2.5. *For a function $f : A \to B$,*

(1) *f is surjective if $ff^\sharp \sqsupseteq 1_B$ holds, and then it is denoted by $f : A \twoheadrightarrow B$.*

(2) *f is injective if $f^\sharp f \sqsubseteq 1_A$ holds, and then it is denoted by $f : A \rightarrowtail B$.*

Proposition 2.6. *For a point $a \in A$, the relation $\hat{a} : I \rightharpoonup A$ defined by $\hat{a} := \{(*, a)\}$ is a function.*

Proposition 2.7. *For relations $\alpha, \alpha' : A \rightharpoonup B, \beta, \beta' : B \rightharpoonup C$, and $\gamma, \gamma' : C \rightharpoonup D$, the following propositions hold :*

(1) [associativity] $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$

(2) [identity] $\langle \alpha \cdot 1_A = \alpha \rangle \wedge \langle 1_B \cdot \beta = \beta \rangle$

(3) [monotonicity] $\langle\langle\alpha \sqsubseteq \alpha'\rangle \wedge \langle\beta \sqsubseteq \beta'\rangle\rangle \to \langle\beta \cdot \alpha \sqsubseteq \beta' \cdot \alpha'\rangle$
(4) [monotonicity] $\langle\langle\alpha \sqsubseteq \alpha'\rangle \wedge \langle\beta \sqsubseteq \beta'\rangle\rangle \to \langle\beta \cdot \alpha \sqsubseteq \beta' \cdot \alpha'\rangle$.

Proposition 2.8. *For relations $\alpha, \alpha' : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$, the following propositions hold:*

(1) $(\beta \cdot \alpha)^\sharp = \alpha^\sharp \cdot \beta^\sharp$
(2) [monotonicity] $\langle\alpha \sqsubseteq \alpha'\rangle \to \langle\alpha^\sharp \sqsubseteq (\alpha')^\sharp\rangle$

Proposition 2.9. (1) *Let S be a subset of a set A, then a relation $i_{S,A} : S \rightharpoonup A$ defined by $i_{S,A} := \{(s, a) \in S \times A; s = a\}$ is an injective function, and is called the inclusion function from S to A, and is denoted by $i : S \hookrightarrow A$.*
(2) *Let $f : A \to B$ and $f' : A \to B$ be functions, then $f = f'\rangle$ holds if $f \sqsubseteq f'$ holds.*
(3) *Let $f : A \to B$ and $g : B \to C$ be functions, then the composite function $g \cdot f$ is a function.*

Proof. (1)

$$\begin{array}{rcll}
(a, a') \in i_{S,A} \cdot (i_{S,A})^\sharp & \leftrightarrow & (a, s) \in (i_{S,A})^\sharp \wedge (s, a') \in i_{S,A} & (\because \text{proposition2.7(2)}) \\
& \sqsupseteq & (s, a) \in i_{S,A} \wedge (s, a') \in i_{S,A} & (\because \text{the univalence of } f') \\
& \sqsupseteq & s = as = a' & (\because \text{equation(1)}) \\
& = & a = a' & (\because \text{proposition2.7(1)}) \\
& \sqsupseteq & (a, a') \in 1_A & (\because \text{the totality of } f)
\end{array}$$

$$\begin{array}{rcll}
(s, s') \in (i_{S,A})^\sharp \cdot i_{S,A} & \leftrightarrow & (a, s') \in (i_{S,A})^\sharp \wedge (s, a) \in i_{S,A} & (\because \text{proposition2.7(2)}) \\
& \sqsupseteq & (s', a) \in i_{S,A} \wedge (s, a) \in i_{S,A} & (\because \text{the univalence of } f') \\
& \sqsupseteq & s' = as = a & (\because \text{equation(1)}) \\
& = & s = s' & (\because \text{proposition2.7(1)}) \\
& \sqsupseteq & (s, s') \in 1_S & (\because \text{the totality of } f)
\end{array}$$

(2) Since $f \sqsubseteq f'$ holds by assumption, and it follows from proposition2.8(2).

$$f^\sharp \sqsubseteq f'^\sharp \tag{1}$$

holds, then we attain the following:

$$\begin{array}{rcll}
f & = & 1_B \cdot f & (\because \text{proposition2.7(2)}) \\
& \sqsupseteq & (f' \cdot f'^\sharp) \cdot f & (\because \text{the univalence of } f') \\
& \sqsupseteq & (f' \cdot f^\sharp) \cdot f & (\because \text{equation(1)}) \\
& = & f' \cdot (f^\sharp \cdot f) & (\because \text{proposition2.7(1)}) \\
& \sqsupseteq & f' \cdot 1_A & (\because \text{the totality of } f) \\
& = & f' & (\because \text{proposition2.7(2)}).
\end{array}$$

(3) $g \cdot f$ is univalent since

$$\begin{array}{rcll}
(g \cdot f)^\sharp \cdot (g \cdot f) & = & f^\sharp \cdot g^\sharp \cdot g \cdot f & (\because \text{proposition2.7(2)}) \\
& \sqsupseteq & f^\sharp \cdot 1_B \cdot f & (\because \text{proposition2.7(1)}) \\
& = & f^\sharp \cdot f & (\because \text{the totality of } f) \\
& \sqsupseteq & 1_A & (\because \text{proposition2.7(2)}).
\end{array}$$

holds, and $g \cdot f$ is total since

$$\begin{array}{rcll}
1_C & \sqsupseteq & g \cdot g^\sharp & (\because \text{proposition2.7(2)}) \\
& \sqsupseteq & g \cdot 1_B \cdot g^\sharp & (\because \text{proposition2.7(1)}) \\
& \sqsupseteq & g \cdot f \cdot f^\sharp \cdot g^\sharp & (\because \text{the totality of } f) \\
& = & (g \cdot f) \cdot (g \cdot f)^\sharp & (\because \text{proposition2.7(2)}).
\end{array}$$

hold.

□

Lemma 2.10. *Let A and C be subsets of a set X, then we have*

$$(i_{A,X})^\sharp(C) = C \cap A.$$

Definition 2.11. (1) Set *is the category of sets and functions.*
(2) Rel *is the category of sets and relations.*
(3) *Let O_1, O_2 be objects in a category, then the notation $O_1 \cong O_2$ means the isomorphism between them.*

Proposition 2.12 (Rationality of relations)**.** *Let A, B be sets, then for a relation $\alpha : A \rightharpoondown B$, there exsists a set R_α and relations $f_\alpha : R_\alpha \to A$, $g_\alpha : R_\alpha \to B$ such that $\alpha = g_\alpha \cdot (f_\alpha)^\sharp$ and $f_\alpha^\sharp f_\alpha \sqcap g_\alpha^\sharp g_\alpha = 1_{R_\alpha}$ are satisfied.*

Proof. Let $R_\alpha := \alpha$ for a relation $\alpha : A \rightharpoondown B$, and $f_\alpha : R_\alpha \rightharpoondown A$ and $g_\alpha : R_\alpha \rightharpoondown B$ relations defined by

$$((x,y),x) \in f_\alpha$$

and

$$((x,y),y) \in g_\alpha,$$

then we only have to show that f_α, g_α are functions; i.e.

$$f_\alpha \cdot (f_\alpha)^\sharp \sqsubseteq 1_A$$

and

$$1_B \sqsubseteq (f_\alpha)^\sharp \cdot f_\alpha.$$

Since

$$\begin{array}{lll}
 & (x,y) \in f_\alpha \cdot (f_\alpha)^\sharp & \\
\leftrightarrow & {}^\exists(x',y') \in R_\alpha, \langle (x,(x',y')) \in (f_\alpha)^\sharp \rangle \wedge \langle ((x',y'),y) \in f_\alpha \rangle & \\
\leftrightarrow & {}^\exists(x',y') \in R_\alpha, \langle ((x',y'),x) \in f_\alpha \rangle \wedge \langle ((x',y'),y) \in f_\alpha \rangle & (\because \text{definition 2.3(2)}) \\
\leftrightarrow & {}^\exists(x',y') \in R_\alpha, \langle x \in A \rangle \wedge \langle y \in B \rangle \wedge \langle x' = x \rangle \wedge \langle x' = y \rangle & \\
\rightarrow & \langle x = y \rangle \wedge \langle x \in A \rangle \wedge \langle y \in B \rangle & \\
\leftrightarrow & (x,y) \in 1_A & (\because \text{proposition 2.9(1)})
\end{array}$$

holds, then f_α is univalent, and so is g_α in the same way. Furthermore, since

$$\begin{array}{lll}
 & ((x,y),(x',y')) \in 1_{R_\alpha} & \\
\leftrightarrow & \langle (x,y) \in R_\alpha \rangle \wedge \langle (x',y') \in R_\alpha \rangle \wedge \langle (x,y) = (x',y') \rangle & \\
\rightarrow & \langle ((x,y),x) \in f_\alpha \rangle \wedge \langle ((x',y'),x') \in f_\alpha \rangle \wedge \langle (x,y) = (x',y') \rangle & \\
\leftrightarrow & ((x,y),(x',y')) \in (f_\alpha)^\sharp \cdot f_\alpha & (\because \text{definition 2.3(2)}),
\end{array}$$

holds, and then f_α is total, in addition, so is g_α in the same way. As shown above, f_α and g_α are functions.

Secondly, it follows that $\alpha = g_\alpha \cdot (f_\alpha)^\sharp$ since

$$\begin{array}{lll}
 & (x,y) \in g_\alpha \cdot (f_\alpha)^\sharp & \\
\leftrightarrow & \exists(x',y') \in \alpha, \langle (x,(x',y')) \in (f_\alpha)^\sharp \rangle \wedge \langle ((x',y'),y) \in g_\alpha \rangle & \\
\leftrightarrow & \exists(x',y') \in \alpha, \langle ((x',y'),x) \in f_\alpha \rangle \wedge \langle ((x',y'),y) \in g_\alpha \rangle & (\because \text{definition 2.3(2)}) \\
\leftrightarrow & \exists(x',y') \in \alpha, \langle x' = x \rangle \wedge \langle y' = y \rangle \wedge \langle x \in A \rangle \wedge \langle y \in B \rangle & \\
\leftrightarrow & (x,y) \in \alpha. &
\end{array}$$

Finally, we have $(f_\alpha)^\sharp \cdot f_\alpha \sqcap (g_\alpha)^\sharp \cdot g_\alpha = 1_{R_\alpha}$ since

$$\begin{array}{lll}
& ((x,y),(x',y')) \in (f_\alpha)^\sharp \cdot f_\alpha \sqcap (g_\alpha)^\sharp \cdot g_\alpha & \\
\leftrightarrow & \langle ((x,y),(x',y')) \in (f_\alpha)^\sharp \cdot f_\alpha \rangle \wedge \langle ((x,y),(x',y')) \in (g_\alpha)^\sharp \cdot g_\alpha \rangle & \\
\leftrightarrow & [{}^\exists x'' \in A, \langle ((x,y),x') \in f_\alpha \rangle \wedge \langle (x'',(x',y')) \in (f_\alpha)^\sharp \rangle] & \\
& \wedge [{}^\exists y'' \in B, \langle ((x,y),y'') \in g_\alpha \rangle \wedge \langle (y'',(x',y')) \in (g_\alpha)^\sharp \rangle] & \\
\leftrightarrow & [{}^\exists x'' \in A, \langle ((x,y),x') \in f_\alpha \rangle \wedge \langle ((x',y'),x'') \in f_\alpha \rangle] & \\
& \wedge [{}^\exists y'' \in B, \langle ((x,y),y'') \in g_\alpha \rangle \wedge \langle ((x',y'),y'') \in g_\alpha \rangle] & \\
\leftrightarrow & [{}^\exists x'' \in A, \langle x = x'' \rangle \wedge \langle x' = x'' \rangle \wedge \langle (x,y) \in R_\alpha \rangle \wedge \langle (x',y') \in R_\alpha \rangle] & \\
& \wedge [{}^\exists y'' \in B, \langle y = y'' \rangle \wedge \langle y' = y'' \rangle \wedge \langle (x,y) \in R_\alpha \rangle \wedge \langle (x',y') \in R_\alpha \rangle] & \\
\leftrightarrow & [\langle x = x' \rangle \wedge \langle (x,y) \in R_\alpha \rangle \wedge \langle (x',y') \in R_\alpha \rangle] & \\
& \wedge [\langle y = y' \rangle \wedge \langle (x,y) \in R_\alpha \rangle \wedge \langle (x',y') \in R_\alpha \rangle] & \\
\leftrightarrow & \langle x = x' \rangle \wedge \langle y = y' \rangle \wedge \langle (x,y) \in R_\alpha \rangle \wedge \langle (x',y') \in R_\alpha \rangle & (\because \text{definitions of } f_\alpha, g_\alpha) \\
\leftrightarrow & ((x,y),(x',y')) \in 1_{R_\alpha} & (\because \text{proposition 2.9(1)})
\end{array}$$

is saisfied.

As shown above, this proposition is completely verified. □

Proposition 2.13 (axiom of choice)**.** *Let $\alpha : A \rightharpoonup B$ be a total function, then there exists a function $c : A \to B$ such that $c \sqsubseteq \alpha$ holds.*

If we admit proposition 2.13, we can prove proposition 2.14.

Proposition 2.14. (1) *A function $f : X \to Y$ is surjective if and only if there exists a function $g : Y \to X$ for f such that $f \cdot g = 1_Y$ holds.*

(2) *The function $T(f) : T(X) \to T(Y)$ is a surjection for both a surjection $f : X \to Y$ and a functor $T : \mathrm{Set} \to \mathrm{Set}$.*

Proof. (1)($\to$) We only have to get a function $g : Y \to X$ such that

$$f \cdot g = 1_Y \tag{2}$$

holds. If $f : X \to Y$ is a surjection, it follows that $f^\sharp$ is total, so by applying proposition2.13 for $f^\sharp$, we get a function $c : Y \to X$ such that

$$c \sqsubseteq f^\sharp \tag{3}$$

holds. Hence, given that

$$g := c, \tag{4}$$

then we have

$$\begin{array}{rcll}
f \cdot g & = & f \cdot c & (\because \text{equation(4)}) \\
& \sqsubseteq & f \cdot f^\sharp & (\because \text{equation(3)}) \\
& = & 1_Y & (\because \text{the surjection of } f),
\end{array}$$

so we attain $f \cdot g \sqsubseteq 1_Y$. Because both sides are functions, then we get equation(2) by using proposition2.9(2).

($\leftarrow$) Let $f : X \to Y$ be a function, then $f^\sharp$ is total; i.e. $f \cdot f^\sharp \sqsubseteq 1_Y$, so we only have to verify

$$f \cdot f^\sharp \sqsupseteq 1_Y. \tag{5}$$

Assuming that there exists a function $g : Y \to X$ for f such that

$$f \cdot g = 1_Y \tag{6}$$

holds, then we obtain equation(5) because

$$\begin{array}{rcll} f \cdot f^\sharp & = & f \cdot 1_X \cdot f^\sharp & (\because \text{proposition2.7(2)}) \\ & \sqsupseteq & f \cdot g \cdot g^\sharp \cdot f^\sharp & (\because \text{the univalence of } g) \\ & = & f \cdot g \cdot (f \cdot g)^\sharp & (\because \text{proposition2.8(1)}) \\ & = & 1_Y \cdot (1_Y)^\sharp & (\because \text{equation(6)}) \\ & = & 1_Y & (\because \text{proposition2.7(2)}) \end{array}$$

is satisfied.

(2) Let $f : X \to Y$ be a surjection and $T : \mathrm{Set} \to \mathrm{Set}$ a functor. Note that proposition2.14(1), we only have to get a function $k : T(Y) \to T(X)$ such that

$$T(f) \cdot k = 1_{T(Y)} \tag{7}$$

is satisfied. By applying proposition2.14(1) for f, we get a function $g : Y \to X$ satisfied with

$$f \cdot g = 1_Y. \tag{8}$$

Therefore, given

$$k := T(g), \tag{9}$$

then we get equation(7) since

$$\begin{array}{rcll} T(f) \cdot k & = & T(f) \cdot T(g) & (\because \text{equation(9)}) \\ & = & T(f \cdot g) & (\because T \text{ is a functor}) \\ & = & T(1_Y) & (\because \text{equation(8)}) \\ & = & 1_{T(Y)} & (\because T \text{ is a functor}) \end{array}$$

is satisfied.

□

Lemma 2.15. *Let $\mathscr{C}_1, \mathscr{C}_2$ be categories, F_1, F_2, F_3 functors from $\mathscr{C}_1$ to $\mathscr{C}_2$, $\tau_1 : F_1 \to F_2$ and $\tau_2 : F_2 \to F_3$ arrows, then if $\tau_1 \in \mathrm{Nat}[F_1, F_2]$ and $\tau_2 \in \mathrm{Nat}[F_2, F_3]$ hold, an arrow $\tau_2 \bullet \tau_1 : \mathscr{C}_1 \to \mathscr{C}_3$ defined by*

$$(\tau_2 \bullet \tau_1)_{c_1} := (\tau_2)_{c_1} \cdot (\tau_1)_{c_1}$$

for an object $c_1 \in \mathrm{Ob}(\mathscr{C}_1)$ is a natural transformation.

Definition 2.16. *The arrow $\tau_2 \bullet \tau_1$ on definition 2.15 is called a (vertical) composition of τ_1 and τ_2.*

Lemma 2.17. *Let $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3 \in \mathrm{Cat}$ be categories.*

(1) *Let $F_1, F_2 \in \mathrm{Func}[\mathscr{C}_1, \mathscr{C}_2]$ $F'_1 \in \mathrm{Func}[\mathscr{C}_2, \mathscr{C}_3]$ be functors, and $\tau \in \mathrm{Nat}[F_1, F_2]$ a natural transformation, then an arrow $F'_1 \circ \tau : F'_1 \circ F_1 \to F'_1 \circ F_2$ defined by*

$$(F'_1 \circ \tau)_{c_1} = F'_1(\tau_{c_1})$$

for an object $c_1 \in \mathrm{Ob}(\mathscr{C}_1)$ is a natural transformation. In case of $F'_1 := \mathrm{id}_{\mathscr{C}_2}$, we have $\mathrm{id}_{\mathscr{C}_2} \circ \tau = \tau$.

$$\begin{tikzcd} \mathscr{C}_1 \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathscr{C}_2 \arrow[r, "F'_1"] & \mathscr{C}_3 \end{tikzcd}$$

(2) *Let $F_1 \in \mathrm{Func}[\mathscr{C}_1, \mathscr{C}_2]$, $F'_1, F'_2 \in \mathrm{Func}[\mathscr{C}_2, \mathscr{C}_3]$ be functors, and $\tau' \in \mathrm{Nat}[F'_1, F'_2]$ a natural transformation, then an arrow $F'_1 \circ \tau : F'_1 \circ F_1 \to F'_1 \circ F_2$ defined by*

$$(\tau' \circ F_1)_{c_1} = \tau'_{F_1(c_1)}$$

for an object $c_1 \in \mathscr{C}_1$ is a natural transformation. In case of $F_1 := \mathrm{id}_{\mathscr{C}_1}$, we have $\tau' \circ \mathrm{id}_{\mathscr{C}_1} = \tau'$.

$$\begin{tikzcd} \mathscr{C}_1 \arrow[r, "F_1"] & \mathscr{C}_2 \arrow[r, bend left, "F_1'"{name=U}] \arrow[r, bend right, "F_2'"'{name=D}] & \mathscr{C}_3 \arrow[from=U, to=D, phantom, "\Downarrow\tau'"] \end{tikzcd}$$

Definition 2.18 (Monads over Set)**.** *For a functor $T : \mathrm{Set} \to \mathrm{Set}$, natural transformations $\eta : 1_{\mathrm{Set}} \to T, \mu : T^2 \to T$, the pair $\mathbf{T} = (T, \eta, \mu)$ is called a monad over* Set *if the following conditions are satisfied :*

(a1) $\mu \bullet (T \circ \mu) = \mu \bullet (\mu \circ T)$ $\left(\begin{tikzcd} T^3 \arrow[r, "\mu\circ T"] \arrow[d, "T\circ\mu"'] \arrow[dr, phantom, "\circlearrowleft"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}\right)$

(a2) $\mu \bullet (\eta \circ T) = 1_T = \mu \bullet (T \circ \eta)$ $\left(\begin{tikzcd} T \arrow[r, "\eta\circ T"] \arrow[dr, bend right, "1_T"'] \arrow[dr, phantom, "\circlearrowleft"] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\circ\eta"'] \arrow[dl, bend left, "1_T"] \arrow[dl, phantom, "\circlearrowleft"] \\ & T & \end{tikzcd}\right)$.

Definition 2.19 ($\mathbf{T}$-algebras over Set)**.** *Let $\mathbf{T} = (T, \eta, \mu)$ be an arbitary monad over*Set*. For a set X and $h : TX \to X$, the pair (X, h) is called a $\mathbf{T}$-algebra if the following propositions*

(a1) $h \cdot T(h) = h \cdot (\mu \circ T)_X$ $\left(\begin{tikzcd} T^2(X) \arrow[r, "(\mu\circ T)_X"] \arrow[d, "T(h)"'] \arrow[dr, phantom, "\circlearrowleft"] & T(X) \arrow[d, "h"] \\ T(X) \arrow[r, "h"'] & X \end{tikzcd}\right)$

(a2) $1_X = h \cdot \eta_X$ $\left(\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, bend right, "1_X"'] \arrow[dr, phantom, "\circlearrowleft"] & T(X) \arrow[d, "h"] \\ & X \end{tikzcd}\right)$

are satisfied. For $\mathbf{T}$-algebras $(X, h), (Y, k)$, $f : (X, h) \to (Y, k)$ is said to be a $\mathbf{T}$-morphism if

$(a3)$ $f \cdot h = k \cdot T(f)$ $\left(\begin{tikzcd} T(X) \arrow[r, "T(f)"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & T(Y) \arrow[d, "k"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}\right)$.

Under the previous circumstance, let $\mathrm{Set}^{\mathbf{T}}$ *be the category of $\mathbf{T}$-algebras and $\mathbf{T}$-morphisms.*

3 Filters

This section provides definitions and properties for filters [2].

Definition 3.1. *For a set X, $\mathcal{S} \subseteq 2^X$ is called a filter over X if the following condituins are satisified for $n \in \mathbb{Z}_{>0}$, a set $A_\lambda \in \mathcal{S}$ $(\lambda = 1, 2, \cdots, n)$ and $B \subseteq X$:*

(a) $\cap_{k=1}^n A_k \in \mathcal{S}$
(b) $A_1 \subseteq B \;\to\; B \in \mathcal{S}$.

Proposition 3.2. *For a set X and a subfamily $\mathcal{S} \sqsubseteq 2^X$, the following propositions are equivalent*:

(1) $\mathcal{S}$ *is a filter over* X
(2) $X \in \mathcal{S}$.

Proof. $(\rightarrow)$ By definition(3), let $\mathcal{S}$ be a filter over X, and then $X \in \mathcal{S}$ holds.

$(\leftarrow)$ Assuming that $X \in \mathcal{S}$ is satisfied, for $n \in \mathbb{Z}_{>0}$ and for all $A_\lambda \in \mathcal{S}(\lambda = 1, 2, \cdots, n)$, since we have $\cap_{\lambda=1}^{n} A_\lambda \subseteq X$, $\cap_{\lambda=1}^{n} A_\lambda \in \mathcal{S}$ holds. Additionally, by the adjacent assumption, for all $Q_1 \in \mathcal{S}$ and for all $Q_2 \in \mathcal{S}$, $Q_2 \in X$ is satisisfied such that $Q_1 \subseteq Q_2$ holds. Therefore, $\mathcal{S}$ is a filter over X.

□

Proposition 3.3. *For a filter $\mathcal{F}$ over a set X, the following proposition holds*:

$$\emptyset \in \mathcal{F} \quad \leftrightarrow \quad \mathcal{F} = 2^X.$$

Proof. $(\rightarrow)$ If you have $\emptyset \in \mathcal{F}$, $C \supseteq \emptyset$ holds for all $C \subseteq X$, then by the definition of filters, you get $C \in \mathcal{F}$. Therefore, note that $\mathcal{F}$ is a filter, because $\mathcal{F} \subseteq 2^X$ is satisified, the proof is complete.

$(\leftarrow)$ It is clear that $\emptyset \in 2^X$ holds, so if $\mathcal{F} = 2^X$ is satisfied, then you obtain $\emptyset \in \mathcal{F}$. Hence, the proof is complete.

□

Definition 3.4. *For a set X, a filter $\mathcal{U}$ not including $\emptyset$ is an ultra filter over a set X if it is satisfied with the following condition for a filter $\mathcal{F}$ not including $\emptyset$ over X*:

$$\langle \mathcal{U} \subseteq \mathcal{F} \rangle \rightarrow \langle \mathcal{U} = \mathcal{F} \rangle.$$

Proposition 3.5. *For a filter $\mathcal{F}$ over a set X not including $\emptyset$, there exists the ultrafilter $\mathcal{U}$ over X such that $\mathcal{F} \subseteq \mathcal{U}$ holds.*

Proposition 3.6. *For an ultrafilter $\mathcal{U}$ over a set X and subsets $A_1, A_2 \subseteq X$, the following propositions hold*:

(1) $A_1 \cup A_2 \in \mathcal{U} \quad \leftrightarrow \quad A_1 \in \mathcal{U}$ or $A_2 \in \mathcal{U}$
(2) $A_1 \cup (X \setminus A_1) \in \mathcal{U} \quad \leftrightarrow \quad A_1 \in \mathcal{U}$ or $(X \setminus A_1) \in \mathcal{U}$.

Proof. (1)$(\leftarrow)$ $X \in \mathcal{U}$ is correct by the definition of filters, and note that $A_1 \cup A_2 \subseteq X$ holds, then $A_1 \cup A_2 \in \mathcal{U}$ is satisfied.

$(\rightarrow)$ If $A_1 \in \mathcal{U}$ is satisified, by using the definition of filters, $A_1 \cup A_2 \in \mathcal{U}$ holds for $A_1 \subseteq A_1 \cup A_2$. In addition, if $A_2 \in \mathcal{U}$ holds, by the definition of filters, $A_1 \cup A_2 \in \mathcal{U}$ is satisfied for$A_2 \subseteq A_1 \cup A_2$. As shown above, the proof completes.

(2) You only have to be applied to the result of (1).

□

Definition 3.7. *Let X be a set and $\mathcal{A} := \{A_\lambda; \lambda \in \Lambda\}$ a family of subsets of X, and then $\mathcal{A}$ is said to have the finite intersection property* (FIP) *if $\cap_{k \in K} A_k \neq \emptyset$ holds for a finite set $K \subseteq \Lambda$.*

Proposition 3.8. *Let X be a set and $\mathcal{S}$ be a family of subsets of X, then if $\mathcal{S}$ has* FIP*, there exists the ultrafilter over X including $\mathcal{S}$.*

Proposition 3.9. *Let $\mathcal{S}$ be a family of subsets of X satisfied with* FIP*, then $\mathcal{S}$ is an ultrafilter over X if $A \in \mathcal{S}$ or $X \setminus A \in \mathcal{S}$ hold for all $A \subseteq X$.*

Proposition 3.10. *Let U : Set $\rightarrow$ Set be a functor, $\eta^U : 1_{\mathrm{Set}} \rightarrow U$ and $\mu^U : U^2 \rightarrow U$ natural transformations, then a set $U(I \times X)$ is defined as the collection of all ultra filters over a set X. Given a set Y and a function $\Psi : I \times X \rightarrow I \times Y$, then functions $U(\Psi) : U(I \times X) \rightarrow U(I \times Y)$, $(\eta^U)_X : I \times X \rightarrow U(I \times X)$, $(\mu^U)_X : U^2(I \times X) \rightarrow U(I \times X)$ are defined by*

$$\begin{aligned} U(\Psi)(\mathcal{U}) &:= \{I \times S \sqsubseteq I \times Y \mid \Psi^\sharp(I \times S) \in \mathcal{U}\}, \\ (\eta^U)_X((*, s)) &:= \{I \times S \sqsubseteq I \times X \mid s \in S\}, \\ (\mu^U)_X(\mathscr{U}) &:= \{I \times S \sqsubseteq I \times X \mid (\pi^U)_X(I \times S) \in \mathscr{U}\}, \end{aligned}$$

where a function $(\pi^U)_X : I \times X \to U(I \times X)$ is defined for $S \subseteq X$ by

$$(\pi^U)_X(I \times S) := \{\mathcal{U} \in U(I \times X) |\, I \times S \in \mathcal{U}\}.$$

Under this circumstance, $\mathbf{U} = (U, \eta^U, \mu^U)$ is a monad over Set *and called an ultrafilter monad.*

4 Relational algebras

This section presents the derivation of the isomorphism between the category of topological spaces and the category of relational algebras.

Let $I := \{*\}$ be a unit set.

Lemma 4.1.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "v"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & D \end{tikzcd} \tag{10}$$

Let X, A, B, D be sets and u, v, f, g in diagram(10) *functions, then the following* (1) *and* (2) *are equivallent*:

(1) $fu = gv$
(2) $vu^\sharp \sqsubseteq g^\sharp f$.

Proof. ($\to$) Assuming (1), then we have (2)

$$\begin{array}{rcll} vu^\sharp & \sqsubseteq & g^\sharp gvu^\sharp & (\because \text{the totality of } g) \\ & = & g^\sharp fuu^\sharp & (\because (1)) \\ & \sqsubseteq & g^\sharp f & (\because \text{the univalence of } u). \end{array}$$

($\leftarrow$) Assuming (2), then we have

$$\begin{array}{rcll} gv & \sqsubseteq & gvu^\sharp u & (\because \text{the totality of } u) \\ & \sqsubseteq & gg^\sharp fu & (\because (2)) \\ & \sqsubseteq & fu & (\because \text{the univalence of } g), \end{array}$$

so $gv \sqsubseteq fu$ holds. Since both sides are fuctions, then we attain (1) by using proposition2.9(2). □

Definition 4.2.

$$\begin{tikzcd} Z \arrow[r, "u"] \arrow[d, "v"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & U \end{tikzcd} \tag{11}$$

Let U, X, Y, Z be sets and u, v, f, g in the commuted diagram(11) *functions, then diagram*(11) *is weak-pullback if there exists a function $\Psi : W \to Z$ such that both $u\Psi = k$ and $v\Psi = h$ are satisified for a set W and functions $h : W \to X$ and $k : W \to Y$.*

$$\begin{tikzcd} W \arrow[dr, "\Psi"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\ & Z \arrow[r, "u"] \arrow[d, "v"'] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & U \end{tikzcd} \tag{12}$$

Proposition 4.3. *The following propositions are equivallent on definition*4.2:

(1) *Diagram*(11) *is weak-pullback*
(2) *Diagram*(11) *is satisified with* $vu^\sharp = g^\sharp f$.

Proof. ($\rightarrow$) In order to prove (2), we only have to verify $vu^\sharp = g^\sharp f$; i.e.

$$(x, y) \in g^\sharp f \rightarrow (x, y) \in vu^\sharp \tag{13}$$

$$(x, y) \in g^\sharp f \leftarrow (x, y) \in vu^\sharp. \tag{14}$$

[eq.(13)] We just choice $z \in Z$ such that both

$$u(z) = x \tag{15}$$

and

$$v(z) = y \tag{16}$$

hold for any $(x, y) \in g^\sharp f$. Assuming (1), then there exists a function $\Psi : I \rightarrow Z$ such that

$$u\Psi = \hat{x} \tag{17}$$

and

$$v\Psi = \hat{y} \tag{18}$$

hold for functions $\hat{x} : I \rightarrow X$ and $\hat{y} : I \rightarrow Y$, so given

$$z := \Psi(*), \tag{19}$$

then this satisfies equations (15) and (16) because the following hold:

$$\begin{array}{rcll} u(z) & = & u(\Psi(*)) & (\because \text{equation}(19)) \\ & = & u\Psi(*) & \\ & = & \hat{x}(*) & (\because \text{equation}(17)) \\ & = & x & (\because \text{proposition2.6}), \end{array}$$

$$\begin{array}{rcll} v(z) & = & v(\Psi(*)) & (\because \text{equation}(19)) \\ & = & v\Psi(*) & \\ & = & \hat{y}(*) & (\because \text{equation}(18)) \\ & = & y & (\because \text{proposition2.6}). \end{array}$$

$$\begin{tikzcd} I \arrow[dr, "\Psi"] \arrow[drr, bend left, "\hat{x}"] \arrow[ddr, bend right, "\hat{y}"'] & & \\ & Z \arrow[r, "u"] \arrow[d, "v"'] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & U \end{tikzcd}$$

[eq.(14)] We have only to verify

$$f(x) = g(y) \tag{20}$$

for any $(x, y) \in vu^\sharp$. Assuming (1), then we have

$$fu = gv. \tag{21}$$

Due to the definition of (x, y), there exists $z \in Z$ such that both

$$v(z) = x \tag{22}$$

and

$$u(z) = y \tag{23}$$

hold, so we get equation(20) because the following holds:

$$\begin{array}{rcll} f(x) & = & f(u(z)) & (\because \text{equation}(22)) \\ & = & fu(z) & \\ & = & gv(z) & (\because \text{equation}(21)) \\ & = & g(v(z)) & \\ & = & g(y) & (\because \text{equation}(23)). \end{array}$$

($\leftarrow$) In order to prove (1), if

$$fu = gv \tag{24}$$

holds, then we only have to get a function $\Psi : W \to Z$ such that

$$u\Psi = h \tag{25}$$

and

$$v\Psi = k \tag{26}$$

hold for a set W and any functions $h : W \to X$ and $k : W \to Y$ satisfied with

$$fk = gh. \tag{27}$$

Assuming (2), then we have

$$vu^\sharp = g^\sharp f. \tag{28}$$

Note that we have

$$hk^\sharp \sqsubseteq g^\sharp f \tag{29}$$

by applying lemma 4.1 for equation(27), then we have

$$hk^\sharp \sqsubseteq vu^\sharp \tag{30}$$

because

$$\begin{array}{rcll} hk^\sharp & \sqsubseteq & g^\sharp f & (\because \text{equation}(29)) \\ & = & vu^\sharp & (\because \text{equation}(28)) \end{array}$$

holds. Therefore, $(k(w), h(w)) \in vu^\sharp$ holds for all $w \in W$, so Ψ can be defined as follows:

$${}^{\forall}w \in W, {}^{\exists}z \in Z, \langle h(w) = u(z) \rangle \wedge \langle k(w) = v(z) \rangle. \tag{31}$$

Here, we already get equations (25) and (26) because the double following statements hold:

$$u\Psi(w) = u(\Psi(w)) = u(z) = h(w),$$

$$v\Psi(w) = v(\Psi(w)) = v(z) = k(w).$$

$$\begin{tikzcd} W \arrow[dr, "\Psi"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\ & Z \arrow[r, "u"] \arrow[d, "v"'] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & U \end{tikzcd}$$

$\square$

Definition 4.4 (Pseudo functors)**.** *Let $\alpha, \gamma : A \rightharpoonup B,\ \beta : B \rightharpoonup C$ be relations, then a functor $T : \mathrm{Rel} \to \mathrm{Rel}$ is called a pseudo functor if the following conditions are satisfied*:

(a) $(T(\alpha))^\sharp = T(\alpha^\sharp)$
(b) [monotonicity] $\langle \alpha \sqsubseteq \gamma \rangle \to \langle T(\alpha) \sqsubseteq T(\gamma) \rangle$
(c) $T(\beta \cdot \alpha) \sqsubseteq T(\beta) \cdot T(\alpha)$.

Proposition 4.5. *Let $\alpha : A \rightharpoonup B$ be a relation, then there exists a set R_α and $f_\alpha : R_\alpha \to A$ and $g_\alpha : R_\alpha \to B$ such that*

$$f_\alpha^\sharp f_\alpha \sqcap g_\alpha^\sharp g_\alpha = 1_{R_\alpha}$$

$$\alpha = g_\alpha \cdot (f_\alpha)^\sharp \tag{32}$$

hold. Under this condition, a functor $\bar{T} : \mathrm{Rel} \to \mathrm{Rel}$ defined by

$$\bar{T}(\alpha) := T(g_\alpha) \cdot (T(f_\alpha))^\sharp$$

for a functor $T : \mathrm{Set} \to \mathrm{Set}$ is a pseudo functor.

Proof. (P0) Here we want to verify that $\bar{T}$ is uniquely determined no matter how α is decomposed; i.e.

$$^\forall W, {}^\exists u : W \to A, {}^\exists v : W \to B, \langle \alpha = v \cdot u^\sharp \rangle \to \langle \langle \bar{T}(\alpha) = T(v) \cdot (T(u))^\sharp \rangle \rangle,$$

but we get this equation by using the definition of $\bar{T}$.

(P1) We already have the proof of

$$(\bar{T}(\alpha))^\sharp = \bar{T}(\alpha^\sharp) \tag{33}$$

because the following deformation

$$\begin{array}{rcll} (\bar{T}(\alpha))^\sharp & = & (T(g_\alpha) \cdot (T(f_\alpha))^\sharp)^\sharp & (\because \text{the definition of } \bar{T}) \\ & = & T(f_\alpha) \cdot (T(g_\alpha))^\sharp & \\ & = & \bar{T}(\alpha^\sharp) & (\because \text{the definition of } \bar{T}) \end{array}$$

is satisfied.

(P2) Here we want to prove

$$^\forall \gamma : A \to B, \langle \alpha \sqsubseteq \gamma \rangle \to \langle \bar{T}(\alpha) \sqsubseteq \bar{T}(\gamma) \rangle. \tag{34}$$

By applying proposition 2.12 for γ, it follows that there exists R_γ, and there exists $l_\gamma : R_\gamma \to A$ and $m_\gamma : R_\gamma \to B$ such that the following propositions are satisfied:

$$l_\gamma^\sharp l_\gamma \sqcap m_\gamma^\sharp m_\gamma = 1_{R_\gamma},$$

$$\gamma = m_\gamma \cdot (l_\gamma)^\sharp. \tag{35}$$

Therefore, we already proved equation(34) since the following deformation holds:

$$\begin{array}{rcll} \alpha \sqsubseteq \gamma & \leftrightarrow & g_\alpha \cdot (f_\alpha)^\sharp \sqsubseteq m_\gamma \cdot (l_\gamma)^\sharp & (\because \text{equation}(32), (35)) \\ & \leftrightarrow & f_\alpha \cdot (l_\gamma)^\sharp = g_\alpha \cdot (m_\gamma)^\sharp & (\because \text{lemma4.1}) \\ & \to & T((f_\alpha \cdot (l_\gamma)^\sharp) = T(g_\alpha \cdot (m_\gamma)^\sharp) & \\ & \leftrightarrow & T(f_\alpha) \cdot T((l_\gamma)^\sharp) = T(g_\alpha) \cdot T((m_\gamma)^\sharp) & \\ & \leftrightarrow & T(f_\alpha) \cdot (T(l_\gamma))^\sharp = T(g_\alpha) \cdot (T(m_\gamma))^\sharp & (\because \text{equation}(33)) \\ & \leftrightarrow & T(g_\alpha) \cdot (T(f_\alpha))^\sharp \sqsubseteq T(m_\gamma) \cdot (T(l_\gamma))^\sharp & (\because \text{lemma4.1}) \\ & \leftrightarrow & \bar{T}(\alpha) \sqsubseteq \bar{T}(\gamma) & (\because \text{proposition2.12}). \end{array}$$

(P3) For sets B, C and for a relation $\beta : B \rightharpoonup C$, we want to justify

$$\bar{T}(\beta \cdot \alpha) \sqsubseteq \bar{T}(\beta) \cdot \bar{T}(\alpha). \tag{36}$$

Because of proposition2.12, there exists a set R_β and relations $p_\beta : R_\beta \to B$ and $q_\beta : R_\beta \to C$ such that

$$p_\alpha^\sharp p_\beta \sqcap q_\beta^\sharp q_\beta = 1_{R_\beta}$$

$$\beta = q_\beta \cdot (p_\beta)^\sharp \tag{37}$$

are satisfied. In addition, by applying proposition2.12 for $p_\beta^\sharp \cdot g_\alpha$, there exists a set Q and relations $r : Q \to R_\alpha, s : Q \to R_\beta$ such that

$$r^\sharp r \sqcap s^\sharp s = 1_Q$$

$$(p_\beta)^\sharp \cdot g_\alpha = s \cdot r^\sharp \tag{38}$$

are satisfied. Note that we have

$$\bar{T}((p_\beta)^\sharp \cdot g_\alpha) \sqsubseteq (T(p_\beta))^\sharp \cdot T(g_\alpha) \tag{39}$$

because the following deformation

$$\begin{array}{rcll}
\text{equation(38)} & \leftrightarrow & r^\sharp \cdot (g_\alpha)^\sharp \sqsubseteq s^\sharp \cdot (p_\beta)^\sharp & (\because \text{lemma4.1}) \\
& \leftrightarrow & g_\alpha \cdot r \sqsubseteq p_\beta \cdot s & \\
& \to & g_\alpha \cdot r = p_\beta \cdot s & (\because \text{proposition(2.9)}) \\
& \to & T(p_\beta \cdot s) = T(g_\alpha \cdot r) & \\
& \leftrightarrow & T(p_\beta) \cdot T(s) = T(g_\alpha) \cdot T(r) & \\
& \leftrightarrow & T(s) \cdot (T(r))^\sharp \sqsubseteq (T(p_\beta))^\sharp \cdot T(g_\alpha) & (\because \text{lemma4.1}) \\
& \leftrightarrow & T(s) \cdot T(r^\sharp) \sqsubseteq (T(p_\beta))^\sharp \cdot T(g_\alpha) & (\because \text{equation(33)}) \\
& \leftrightarrow & T(s \cdot r^\sharp) \sqsubseteq (T(p_\beta))^\sharp \cdot T(g_\alpha) & \\
& \leftrightarrow & \bar{T}((p_\beta)^\sharp \cdot g_\alpha) \sqsubseteq (T(p_\beta))^\sharp \cdot T(g_\alpha) & (\because \text{the definition of } \bar{T})
\end{array}$$

is satisfied, then we acquire equation(36) because the following deformation

$$\begin{array}{rcll}
\bar{T}(\beta \cdot \alpha) & = & \bar{T}(q_\beta \cdot (p_\beta)^\sharp \cdot g_\alpha \cdot (f_\alpha)^\sharp) & (\because \text{equation(32)}, (37)) \\
& = & \bar{T}(q_\beta \cdot s \cdot r^\sharp \cdot (f_\alpha)^\sharp) & (\because \text{equation(38)}) \\
& = & \bar{T}(q_\beta \cdot s \cdot (f_\alpha \cdot r)^\sharp) & \\
& = & T(q_\beta \cdot s) \cdot (T(f_\alpha \cdot r))^\sharp & (\because \text{the definition of } \bar{T}) \\
& = & T(q_\beta) \cdot T(s) \cdot (T(f_\alpha) \cdot T(r))^\sharp & \\
& = & T(q_\beta) \cdot T(s) \cdot (T(r))^\sharp \cdot (T(f_\alpha))^\sharp & \\
& = & T(q_\beta) \cdot \bar{T}(s \cdot r^\sharp) \cdot (T(f_\alpha))^\sharp & (\because \text{the definition of } \bar{T}) \\
& = & T(q_\beta) \cdot \bar{T}((p_\beta)^\sharp \cdot g_\alpha) \cdot (T(f_\alpha))^\sharp & (\because \text{equation(38)}) \\
& \sqsubseteq & \bar{T}(q_\beta) \cdot (\bar{T}(p_\beta))^\sharp \cdot \bar{T}(g_\alpha) \cdot (T(f_\alpha))^\sharp & (\because \text{equation(39)}) \\
& = & \bar{T}(\beta) \cdot \bar{T}(\alpha) & (\because \text{the definition of } \bar{T})
\end{array}$$

is satisfied.

$$\begin{tikzcd}
A & & B & & C \\
& R_\alpha \arrow[ul, "f_\alpha"'] \arrow[ur, "g_\alpha"] & & R_\beta \arrow[ul, "p_\beta"'] \arrow[ur, "q_\beta"] & \\
& & Q \arrow[ul, "r"'] \arrow[ur, "s"] & &
\end{tikzcd}$$

The above subproofs (P0), (P1), (P2), (P3) present a complete proof of this proposition.

□

Definition 4.6 (Relational **T**-algebras). *Let* $\mathbf{T} = (T, \eta, \mu)$ *be a monad over* Set. *For a set* X *and a relation* $x : T(I \times X) \rightharpoondown I \times X$*, the pair* $(I \times X, x)$ *is said to be a relational* **T***-algebra if the following equations are satisfied:*

(a1) $x \cdot T(x) \sqsubseteq x \cdot \mu_X$
$$\left(\begin{tikzcd} T^2(I \times X) \arrow[r, "\mu_X"] \arrow[d, "T(x)"'] \arrow[dr, phantom, "\sqsubseteq"] & T(I \times X) \arrow[d, "x"] \\ T(I \times X) \arrow[r, "x"'] & I \times X \end{tikzcd}\right)$$

(a2) $1_X \sqsubseteq x \cdot \eta_X$
$$\left(\begin{tikzcd} I \times X \arrow[r, "\eta_X"] \arrow[dr, bend right, "1_{I \times X}"'] \arrow[dr, phantom, "\sqsubseteq"] & T(I \times X) \arrow[d, "x"] \\ & I \times X \end{tikzcd}\right).$$

Besides, for relational $\mathbf{T}$*-algebras* $(I \times X, x), (I \times X', x')$, $f : (I \times X, x) \to (I \times X', x')$ *is a relational* $\mathbf{T}$*-morphism if*

(a3) $f \cdot x \sqsubseteq x' \cdot T(f)$
$$\left(\begin{tikzcd} T(I \times X) \arrow[r, "T(f)"] \arrow[d, "x"'] \arrow[dr, phantom, "\sqsubseteq"] & T(I \times X') \arrow[d, "x'"] \\ I \times X \arrow[r, "f"'] & I \times X' \end{tikzcd}\right).$$

$\mathrm{Rel}(\mathbf{T})$ *means the category of relational* $\mathbf{T}$*-algebras and relational* $\mathbf{T}$*-morphisms.*

Definition 4.7. *Let* X *be a set and* $\alpha, \alpha' : I \rightharpoondown X$ *relations, then a function* $\Gamma : 2^{I \times X} \to 2^{I \times X}$ *is called a closure over* X *if the follwing conditions* (1), (2), (3) *are satisfied.*

(1) $\alpha \sqsubseteq \Gamma(\alpha)$
(2) [monotonicity] $\langle \alpha \sqsubseteq \alpha' \rangle \to \langle \Gamma(\alpha) \sqsubseteq \Gamma(\alpha') \rangle$
(3) [idempotence] $\Gamma^2(\alpha) = \Gamma(\alpha)$

Moreover, Γ *is called a topological closure if it is satisfied with the following condition* (4) *as well.*

(4) [distributivity] $\Gamma(\alpha \sqcup \alpha') = \Gamma(\alpha) \sqcup \Gamma(\alpha')$.

A closure space is defined as the pair of a set and a closure of this set, and a topolosical space is defined as the pair of a set and a topolosical closure of this set. For closure spaces $(I \times X, \Gamma), (I \times X', \Gamma')$, *a function* $f : (I \times X, \Gamma) \to (I \times X', \Gamma')$ *is called continuous if*

$$f(\Gamma(\alpha)) \sqsubseteq \Gamma'(f(\alpha))$$

holds. Clos *means the category of closure spaces and continuous functions between them, and* Top *means the category of topological spaces and continuous functions between them.*

Definition 4.8. *For a monad* $\mathbf{T} = (T, \eta, \mu)$ *over* Set, *a functor* $C : \mathrm{Rel}(\mathbf{T}) \to \mathrm{Clos}$ *is defined as follows*:

(1) $C(I \times X, x) := (I \times X, \Gamma_x) \qquad ((I \times X, x) \in \mathrm{Rel}(\mathbf{T}))$
(2) $C(f) := f \qquad (f \in \mathrm{Hom}_{\mathrm{Rel}(\mathbf{T})}((I \times X, x), (I \times X', x'))$,

where a function $\Gamma_x : 2^{I \times X} \to 2^{I \times X}$ *is defined by using* $\bar{T}$ *used at proposition*4.5 *as*

$$\Gamma_x(\xi) := x \cdot \bar{T}(\xi) \cdot \eta_I \qquad (\xi : (I \times I) \rightharpoondown (I \times X)).$$

$$I \times I \xrightarrow{\eta_I} T(I \times I) \xrightarrow{\bar{T}(\xi)} T(I \times X) \xrightarrow{x} I \times X$$

Lemma 4.9. *Let* $\bar{T}$ *be the pseudo functor used in proposition* 4.5, *then for sets* Y_1, Y_2 *and for a relation*

$\alpha : Y_1 \rightharpoondown Y_2$, *the following equation holds*:

$$\mu_{Y_2} \cdot \bar{T}^2(\alpha) \sqsubseteq \bar{T}(\alpha) \cdot \mu_{Y_1} \qquad \left(\begin{tikzcd} \bar{T}^2(Y_1) \arrow[r, "\mu_{Y_1}"] \arrow[d, "\bar{T}^2(\alpha)"'] \arrow[dr, phantom, "\sqsubseteq"] & \bar{T}(Y_1) \arrow[d, "\bar{T}(\alpha)"] \\ \bar{T}^2(Y_2) \arrow[r, "\mu_{Y_2}"'] & \bar{T}(Y_2) \end{tikzcd} \right).$$

Proof. Equation(32) in proposition4.5 proffers

$$\bar{T}(\alpha) = T(g_\alpha) \cdot (T(f_\alpha))^\sharp, \tag{40}$$

which yields

$$\bar{T}^2(\alpha) = T^2(g_\alpha) \cdot (T^2(f_\alpha))^\sharp. \tag{41}$$

Thereby, we attain

$$(T^2(f_\alpha))^\sharp \sqsubseteq (\mu_{R_\alpha})^\sharp \cdot (T(f_\alpha))^\sharp \cdot \mu_{Y_1} \tag{42}$$

because

$$\begin{array}{rcll} \text{equation(40)} & \leftrightarrow & \mu_{R_\alpha} \cdot T(f_\alpha) = \mu_{Y_1} \cdot T^2(f_\alpha) & (\because \mu \text{ is a transformation}) \\ & \leftrightarrow & \mu_{R_\alpha} \cdot (T^2(f_\alpha))^\sharp \sqsubseteq (T(f_\alpha))^\sharp \cdot \mu_{Y_1} & (\because \text{lemma4.1}) \\ & \rightarrow & (\mu_{R_\alpha})^\sharp \cdot \mu_{R_\alpha} \cdot (T^2(f_\alpha))^\sharp \sqsubseteq (\mu_{R_\alpha})^\sharp \cdot (T(f_\alpha))^\sharp \cdot \mu_{Y_1} & \\ & \leftrightarrow & \text{equation(42)} & (\because \text{the totality of } \mu_{R_\alpha}) \end{array}$$

is satisfied. In addition, we have

$$T^2(g_\alpha) \cdot (\mu_{R_\alpha})^\sharp \sqsubseteq (\mu_{Y_2})^\sharp \cdot T(g_\alpha) \tag{43}$$

since

$$\begin{array}{rcll} \text{equation(41)} & \leftrightarrow & \mu_{R_\alpha} \cdot T(g_\alpha) = \mu_{Y_2} \cdot T^2(g_\alpha) & (\because \mu \text{ is a transformation}) \\ & \leftrightarrow & \mu_{R_\alpha} \cdot (T^2(g_\alpha))^\sharp \sqsubseteq (T(g_\alpha))^\sharp \cdot \mu_{Y_2} & (\because \text{lemma4.1}) \\ & \leftrightarrow & \text{equation(43)} & (\because \text{proposition2.8(2)}) \end{array}$$

is satisfied.

$$\begin{tikzcd} T^2(Y_1) \arrow[d, "\mu_{Y_1}"'] \arrow[rr, bend left, "\bar{T}^2(\alpha)"] \arrow[dr, phantom, "\circlearrowleft"] & T^2(R_\alpha) \arrow[l, "T^2(f_\alpha)"'] \arrow[r, "T^2(g_\alpha)"] \arrow[d, "\mu_{R_\alpha}"] \arrow[dr, phantom, "\circlearrowright"] & T^2(Y_2) \arrow[d, "\mu_{Y_2}"] \\ T(Y_1) \arrow[rr, bend right, "\bar{T}(\alpha)"'] & T(R_\alpha) \arrow[l, "T(f_\alpha)"] \arrow[r, "T(g_\alpha)"'] & T(Y_2) \end{tikzcd}$$

Therefore, we attained a proof of this lemma since the following deformation

$$\begin{array}{rcll} \mu_{Y_2} \cdot \bar{T}^2(\alpha) & = & \mu_{Y_2} \cdot T^2(g_\alpha) \cdot (T^2(f_\alpha))^\sharp & (\because \text{equation(41)}) \\ & \sqsubseteq & \mu_{Y_2} \cdot T^2(g_\alpha) \cdot (\mu_{R_\alpha})^\sharp \cdot (T(f_\alpha))^\sharp \cdot \mu_{Y_1} & (\because \text{equation(42)}) \\ & \sqsubseteq & \mu_{Y_2} \cdot (\mu_{Y_2})^\sharp \cdot T(g_\alpha) \cdot (T(f_\alpha))^\sharp \cdot \mu_{Y_1} & (\because \text{equation(43)}) \\ & \sqsubseteq & T(g_\alpha) \cdot (T(f_\alpha))^\sharp \cdot \mu_{Y_1} & (\because \text{the univalence of } \mu_{Y_2}) \\ & = & \bar{T}(\alpha) \cdot \mu_{Y_1} & (\because \text{equation(40)}) \end{array}$$

is satisfied. □

Proposition 4.10. *On definition* 4.8,

(1) *$C(I \times X, x)$ is a closure space.*
(2) *$C(f)$ is a continuous function.*

Proof. (1) By definition, we only have to prove the following 3 equations for relations $\alpha, \alpha' : I \rightharpoondown X$:

$$\alpha \sqsubseteq \Gamma_x(\alpha) \tag{44}$$

$$\langle \alpha \sqsubseteq \alpha' \rangle \to \langle \Gamma_x(\alpha) \sqsubseteq \Gamma_x(\alpha') \rangle \tag{45}$$

$$\Gamma_x^2(\alpha) = \Gamma_x(\alpha). \tag{46}$$

$< 1\mathrm{a} >$ [eq.(44)] Because η is the transformation, then we have

$$\eta_I \cdot f_\alpha = T(f_\alpha) \cdot \eta_{R_\alpha} \tag{47}$$

and

$$\eta_X \cdot g_\alpha = T(g_\alpha) \cdot \eta_{R_\alpha}. \tag{48}$$

$$\begin{tikzcd}
I \arrow[d, "\eta_I"'] \arrow[rr, bend left, "\alpha"] & R_\alpha \arrow[l, "f_\alpha"'] \arrow[r, "g_\alpha"] \arrow[d, "\eta_{R_\alpha}"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\eta_X"] \\
T(I) \arrow[rr, bend right, "\bar{T}(\alpha)"'] & T(R_\alpha) \arrow[l, "T(f_\alpha)"] \arrow[r, "T(g_\alpha)"'] & T(X)
\end{tikzcd}$$

Thus, we have

$$\alpha \cdot (\eta_I)^\sharp \sqsubseteq (\eta_X)^\sharp \cdot \bar{T}(\alpha) \tag{49}$$

since the following deformation holds:

$$\begin{array}{rcll}
\eta_X \cdot \alpha & \sqsubseteq & \eta_X \cdot \alpha \cdot (\eta_I)^\sharp \cdot \eta_I & (\because \text{the totality of } \eta_I) \\
& = & \eta_X \cdot g_\alpha \cdot (f_\alpha)^\sharp \cdot (\eta_I)^\sharp \cdot \eta_I & (\because \text{equation(32) in proposition4.5}) \\
& = & \eta_X \cdot g_\alpha \cdot (\eta_I \cdot f_\alpha)^\sharp \cdot \eta_I & (\because \text{proposition2.8(2)}) \\
& = & T(g_\alpha) \cdot \eta_{R_\alpha} \cdot (\eta_I \cdot f_\alpha)^\sharp \cdot \eta_I & (\because \text{equation(48)}) \\
& = & T(g_\alpha) \cdot \eta_{R_\alpha} \cdot (T(f_\alpha) \cdot \eta_{R_\alpha})^\sharp \cdot \eta_I & (\because \text{equation(47)}) \\
& = & T(g_\alpha) \cdot \eta_{R_\alpha} \cdot (\eta_{R_\alpha})^\sharp \cdot (T(f_\alpha))^\sharp \cdot \eta_I & (\because \text{proposition2.8(2)}) \\
& \sqsubseteq & T(g_\alpha) \cdot (T(f_\alpha))^\sharp \cdot \eta_I & (\because \text{the univalence of } \eta_{R_\alpha}) \\
& = & \bar{T}(\alpha) \cdot \eta_I. & (\because \text{the definition of } \bar{T})
\end{array}$$

Therefore, note that we have

$$(\eta_X)^\sharp \sqsubseteq x \tag{50}$$

since the deformation

$$\begin{array}{rll}
& 1_X \sqsubseteq x \cdot \eta_X & (\because (X, x) \text{ is the relational } \mathbf{T}\text{-algebra}) \\
\to & 1_X \cdot (\eta_X)^\sharp \sqsubseteq x \cdot \eta_X \cdot (\eta_X)^\sharp & (\because \text{proposition2.7(3)}) \\
\to & (\eta_X)^\sharp \sqsubseteq x & (\because \text{the univalence of } \eta_X)
\end{array}$$

is satisfied, then we already acquire equation(44) because the following deformation holds:

$$\begin{array}{rcll}
\alpha & \sqsubseteq & \alpha \cdot (\eta_I)^\sharp \cdot \eta_I & (\because \text{the totality of } \eta) \\
& \sqsubseteq & (\eta_X)^\sharp \cdot \bar{T}(\alpha) \cdot \eta_I & (\because \text{equation(49)}) \\
& \sqsubseteq & x \cdot \bar{T}(\alpha) \cdot \eta_I & (\because \text{equation(50)}) \\
& = & \Gamma_x(\alpha). & (\because \text{the definition of } \Gamma_x)
\end{array}$$

$< 1\text{b} >$ [eq.(45)]

$$\begin{array}{rlll} \alpha \subseteq \alpha' & \rightarrow & \bar{T}(\alpha) \sqsubseteq \bar{T}(\alpha') & (\because \bar{T} \text{ is the pseudo-functor}) \\ & \rightarrow & x \cdot \bar{T}(\alpha) \cdot \eta_I \sqsubseteq x \cdot \bar{T}(\alpha') \cdot \eta_I & (\because \text{proposition2.7(3)}) \\ & \leftrightarrow & \Gamma_x(\alpha) \sqsubseteq \Gamma_x(\alpha'). & (\because \text{the definition of } \Gamma_x) \end{array}$$

$< 1\text{c} >$ [eq.(46)] $\Gamma_x(\check{\alpha}) \sqsubseteq \Gamma_x^2(\check{\alpha})$ holds by applying propposition4.10(1) and proposition2.7(3), in addition, we have $\Gamma_x(\check{\alpha}) \sqsupseteq \Gamma_x^2(\check{\alpha})$ since the following deformation

$$\begin{array}{rcll} \Gamma_x^2(\alpha) & = & \Gamma_x(\Gamma_x(\alpha)) & \\ & = & x \cdot \bar{T}(\Gamma_x(\alpha)) \cdot \eta_I & (\because \text{the definition of } \Gamma_x) \\ & = & x \cdot \bar{T}(x \cdot \bar{T}(\alpha) \cdot \eta_I) \cdot \eta_I & (\because \text{the definition of } \Gamma_x) \\ & \sqsubseteq & x \cdot \bar{T}(x) \cdot \bar{T}^2(\alpha) \cdot \bar{T}\eta_I \cdot \eta_I & (\because \bar{T} \text{ is the pseudo-functor}) \\ & \sqsubseteq & x \cdot \mu_X \cdot \bar{T}^2(\alpha) \cdot \bar{T}(\eta_I) \cdot \eta_I & (\because (X,x) \text{ is the relational } \mathbf{T}\text{-algebra}) \\ & \sqsubseteq & x \cdot \bar{T}(\alpha) \cdot \mu_I \cdot \bar{T}(\eta_I) \cdot \eta_I & (\because \text{lemma4.9}) \\ & = & x \cdot \bar{T}(\alpha) \cdot \mu_I \cdot T(\eta_I) \cdot \eta_I & (\because \text{the definition of } \bar{T}) \\ & = & x \cdot \bar{T}(\alpha) \cdot \eta_I & (\because \mathbf{T} \text{ is the monad}) \\ & = & \Gamma_x(\alpha) & (\because \text{the definition of } \Gamma_x) \end{array}$$

is satisfied.

The above subproofs $< 1\text{a} >, < 1\text{b} >, < 1\text{c} >$ ensure that $C(X,x)$ is a closure space.

(2) All we have to do is to prove the following equation to verify proposition4.10(2):

$$^{\forall}\alpha : I \rightharpoondown X, \langle f \cdot \Gamma_x(\alpha) \sqsubseteq \Gamma_{x'}(f \cdot \alpha) \rangle. \tag{51}$$

Note that we have

$$\bar{T}(f) \cdot \bar{T}(\alpha) = \bar{T}(f \cdot \alpha) \tag{52}$$

since the following deformation

$$\begin{array}{rcll} \bar{T}(f) \cdot \bar{T}(\alpha) & = & \bar{T}(f) \cdot T(g_\alpha) \cdot (T(f_\alpha))^\sharp & (\because \text{the definition of } \bar{T}) \\ & = & T(f) \cdot T(g_\alpha) \cdot (T(f_\alpha))^\sharp & (\because \text{the definition of } \bar{T}) \\ & = & T(f \cdot g_\alpha) \cdot (T(f_\alpha))^\sharp & (\because T \text{ is the functor}) \\ & = & \bar{T}(f \cdot g_\alpha \cdot (f_\alpha)^\sharp) & (\because \text{the definition of } \bar{T}) \\ & = & \bar{T}(f \cdot \alpha) & (\because \text{equation(32) in proposition4.5}) \end{array}$$

is satisfied, then we already have equation(51) because the following deformation

$$\begin{array}{rcll} f \cdot \Gamma_x(\alpha) & = & f \cdot x \cdot \bar{T}(\alpha) \cdot \eta_I & (\because \text{the definition of } \Gamma_x) \\ & \sqsubseteq & x' \cdot \bar{T}(f) \cdot \bar{T}(\alpha) \cdot \eta_I & (\because f \text{ is the relational } \mathbf{T}\text{-morphism}) \\ & = & x' \cdot \bar{T}(f \cdot \alpha) \cdot \eta_I & (\because \text{equation(52)}) \\ & = & \Gamma_{x'}(f \cdot \alpha) & (\because \text{the definition of } \Gamma_{x'}) \end{array}$$

is satisfied.

□

Let $\mathbf{U} = (U, \eta^U, \mu^U)$ be the ultra filter monad over Set defined at definition3.10.

Lemma 4.11. *Let A be a subset of a set X and Γ_x the function used in definition4.8, then the following propositions are equivalent for an element $b \in A$ and a relation $\alpha : I \rightharpoondown A$:*

(1) $((*,*),(*,b)) \in \Gamma_x(\alpha)$
(2) *there exists $\mathcal{U} \in U(I \times X)$ such that both $\alpha \in \mathcal{U}$ and $(\mathcal{U},(*,b)) \in x$ hold.*

Proof. Note that

$$\alpha = i \cdot (\nabla_{A,I})^\sharp \tag{53}$$

holds, then we have

$$U(i_{I\times A,I\times X})(\mathcal{F}) \ni \alpha \tag{54}$$

for an ultrafilter $\mathcal{F} \in U(I\times A)$ because the following deformation

$$\begin{array}{rcll} U(i_{I\times A,I\times X})(\mathcal{F}) & = & \{\sigma \sqsubseteq I\times X; (i_{I\times A,I\times X})^\sharp(\sigma) \in \mathcal{F}\} & (\because \text{proposition3.10}) \\ & = & \{\sigma \sqsubseteq I\times X; \sigma \sqcap (I\times X) \in \mathcal{F}\} & (\because \text{lemma2.10}) \\ & = & \{\sigma \sqsubseteq I\times X; \sigma \in \mathcal{F}\} & \\ & \ni & \alpha & \end{array}$$

is satisfied. Therefore, note that we have

$$(\eta^U)_I((*,*)) = \{I\times I\} \tag{55}$$

by using proposition3.10, then we have a proof of proposition4.11 because the following replacement is satisfied:

$$\begin{array}{rcll} \text{lemma4.11(1)} & \leftrightarrow & ((*,*),(*,b)) \in x\cdot \bar{U}(\alpha)\cdot(\eta^U)_I & (\because \text{the definition of } \Gamma_x) \\ & \leftrightarrow & (\{I\times I\},(*,b)) \in x\cdot \bar{U}(\alpha) & (\because \text{equation(55)}) \\ & \leftrightarrow & (\{I\times I\},(*,b)) \in x\cdot U(i_{I\times A,I\times X})\cdot U((\nabla_{A,I})^\sharp) & (\because \text{equation(53)}) \\ & \leftrightarrow & {}^\exists\mathcal{G} \in U(I\times A), (\mathcal{G},(*,b)) \in x\cdot U(i_{I\times A,I\times X}) & \\ & \leftrightarrow & \text{lemma4.11(2)}. & (\because \text{equation(54)}) \end{array}$$

$$\begin{tikzcd} I\times I \arrow[r, "(\eta^U)_I"] \arrow[rrrr, bend left, "\alpha"] & U(I\times I) \arrow[rr, "\bar{U}(\alpha)"] \arrow[dr, "U((\nabla_{A,I})^\sharp)"'] & & U(I\times X) \arrow[r, "x"] & I\times X \\ & & U(I\times A) \arrow[ur, "U(i_{I\times A,I\times X})"'] & & \end{tikzcd}$$

□

Proposition 4.12. *Under the condition of lemma*4.11*, $C(X,x)$ is a topological space.*

Proof. We only have to verify

$$\Gamma_x(\alpha) \sqcup \Gamma_x(\beta) = \Gamma_x(\alpha \sqcup \beta) \tag{56}$$

for relations $\alpha, \beta : I \rightharpoondown X$; i.e.

$$\Gamma_x(\alpha) \sqcup \Gamma_x(\beta) \sqsubseteq \Gamma_x(\alpha \sqcup \beta) \tag{57}$$

$$\Gamma_x(\alpha) \sqcup \Gamma_x(\beta) \sqsupseteq \Gamma_x(\alpha \sqcup \beta). \tag{58}$$

($\sqsubseteq$) By applying definition4.7(2) for Γ, note that we have $\alpha \sqsubseteq \alpha \sqcup \beta$, then we obtain

$$\Gamma_x(\alpha) \sqsubseteq \Gamma_x(\alpha \sqcup \beta), \tag{59}$$

and in the same way, note that $\beta \sqsubseteq \alpha \sqcup \beta$, then we obtain

$$\Gamma_x(\beta) \sqsubseteq \Gamma_x(\alpha \sqcup \beta). \tag{60}$$

Hence, if $((*,*),(*,a)) \in \Gamma_x(\alpha)$ holds for any $((*,*),(*,a)) \in \Gamma_x(\alpha) \sqcup \Gamma_x(\beta)$, then we have $((*,*),(*,a)) \in \Gamma_x(\alpha \sqcup \beta)$ from equation(59); otherwise, i.e. $((*,*),(*,a)) \in \Gamma_x(\beta)$ holds, then we have $((*,*),(*,a)) \in \Gamma_x(\alpha \sqcup \beta)$ from equation(60). Therefore, we get $((*,*),(*,a)) \in \Gamma_x(\alpha \sqcup \beta)$, which induces equation(57).

($\sqsupseteq$) By applying lemma4.11 for any $((*,*),(*,b)) \in \Gamma_x(\alpha \sqcup \beta)$, there exists

$$\mathcal{U} \in U(I \times X) \tag{61}$$

such that both

$$\alpha \sqcup \beta \in \mathcal{U} \tag{62}$$

and $(\mathcal{U},(*,b)) \in x$ are satisfied. Therefore, by applying proposition3.6(1) for equation(62), then it follows $\alpha \in \mathcal{U}$ or $\beta \in \mathcal{U}$. If $\alpha \in \mathcal{U}$ holds, then we have $((*,*),(*,b)) \in \Gamma_x(\alpha)$ by using lemma4.11 ; otherwise, i.e. $\beta \in \mathcal{U}$ holds, then we have $((*,*),(*,b)) \in \Gamma_x(\beta)$ by using lemma4.11 in the same way. Therefore, we get $((*,*),(*,b)) \in \Gamma_x(\alpha) \sqcup \Gamma_x(\beta)$, which induces equation(58).

□

Definition 4.13. *Let $\mathcal{F}$ be a filter over a topological space $(I \times X, \Gamma)$, then a function* $\lim : U(I \times X) \to I \times X$ *defined by*

$$\lim \mathcal{F} := \sqcap_{\xi \in \mathcal{F}} \Gamma(\xi)$$

is called the limit of $\mathcal{F}$.

Definition 4.14. *A functor $J : \mathrm{Top} \to \mathrm{Rel}(\mathbf{U})$ is defined as follows*:

(1) $J(I \times X, \Gamma) := (I \times X, r_\Gamma) \qquad ((I \times X, \Gamma) \in \mathrm{Top})$
(2) $J(\Psi) := \Psi \qquad (\Psi \in \mathrm{Hom}_{\mathrm{Top}}((I \times X, \Gamma),(I \times X', \Gamma')))$,

where

$$(\mathcal{U},(*,a)) \in r_\Gamma \quad \leftrightarrow \quad (*,a) \in \lim \mathcal{U}$$

holds.

Lemma 4.15. *Under the condition on definition*4.14*, the following propositions are equivallent*:

(1) $((*,*),(*,b)) \in \Gamma(\alpha)$
(2) *there exists $\mathcal{U} \in U(I \times X)$ such that both $\alpha \in \mathcal{U}$ and $((*,*),(*,b)) \in \sqcap_{\xi \in \mathcal{U}} \Gamma(\xi)$ are satisfied*

for $\alpha : I \rightharpoondown X$ and $b \in X$.

Proof. ($\leftarrow$) Assuming that there exists $\mathcal{U} \in U(I \times X)$ such that

$$\alpha \in \mathcal{U} \tag{63}$$

and

$$((*,*),(*,b)) \in \sqcap_{\xi \in \mathcal{U}} \Gamma(\xi) \tag{64}$$

are satisfied, then we have $((*,*),(*,b)) \in \Gamma(\alpha)$ by applying equation(64) for equation(63) by definition.

($\rightarrow$) Assuming that

$$((*,*),(*,b)) \in \Gamma(\alpha) \tag{65}$$

holds and

$$V((*,x)) := \{\xi : I^2 \rightharpoondown I \times X; ((*,*),(*,x)) \in \xi, \Gamma(W \setminus \xi) = (W \setminus \xi), W := I^2 \times (I \times X)\}$$

for $x \in X$, then we have

$$\varepsilon \ni ((*,*),(*,b)) \tag{66}$$

for any

$$\varepsilon \in V((*,b)). \tag{67}$$

Hense, because of this equation and equation(65), we attain $\varepsilon \sqcap \Gamma(\alpha) \neq \emptyset$, so it follows that $V((*, b)) \sqcup \{\Gamma(\alpha)\}$ has FIP. Therefore, due to proposition3.8, there exists

$$\mathcal{U} \in U(I \times X) \tag{68}$$

such that

$$V((*, b)) \sqcup \{\Gamma(\alpha)\} \sqsubseteq \mathcal{U} \tag{69}$$

holds. If $((*, *), (*, b)) \notin \Gamma(\delta)$ holds for any $\delta \in \mathcal{U}$, then we are forced to hold

$$((*, *), (*, b)) \notin \delta \in \mathcal{U},$$

by using this equation and definition4.7(1), however we can get a contradiction

$$((*, *), (*, b)) \in \varepsilon \in \mathcal{U}$$

from equations(66), (67), (69), so we have $((*, *), (*, b)) \in \Gamma(\delta)$, which induces

$$((*, *), (*, b)) \in \sqcap_{\rho \in \mathcal{U}} \Gamma(\rho). \tag{70}$$

Furthermore, we get

$$\alpha \in \mathcal{U} \tag{71}$$

because the following deformation

$$\begin{array}{lll} \alpha & \sqsubseteq & \Gamma(\alpha) & (\because \text{definition4.7(1)}) \\ & \in & V((*, b)) \sqcup \{\Gamma(\alpha)\} & \\ & \sqsubseteq & \mathcal{U} & (\because \text{equation(69)}) \end{array}$$

holds. The above, equations (68), (70), (71) present a proof of $(\rightarrow)$ in this proposition.

□

Lemma 4.16. *On definition*4.14*,*

(1) $J(X, \Gamma)$ *is a relational* $\mathbf{U}$*-algebra.*
(2) $J(\Psi)$ *is a relational* $\mathbf{U}$*-morphism.*

Proof. (1) We only have to verify the following equations:

$$1_{I \times X} \sqsubseteq r_\Gamma \cdot (\eta^U)_X \qquad \left(\begin{tikzcd} I \times X \arrow[r, "(\eta^U)_X"] \arrow[dr, bend right, "1_{I\times X}"'] & U(I \times X) \arrow[d, "r_\Gamma"] \arrow[dl, phantom, "\Lsh"] \\ & I \times X \end{tikzcd} \right), \tag{72}$$

$$r_\Gamma \cdot U(r_\Gamma) \sqsubseteq r_\Gamma \cdot (\mu^U)_X \qquad \left(\begin{tikzcd} U^2(I \times X) \arrow[r, "(\mu^U)_X"] \arrow[d, "U(r_\Gamma)"'] \arrow[dr, phantom, "\Lsh"] & U(I \times X) \arrow[d, "r_\Gamma"] \\ U(I \times X) \arrow[r, "r_\Gamma"'] & I \times X \end{tikzcd} \right). \tag{73}$$

[eq.(72)] Here we only have to verify

$$((*, x), (*, x)) \in r_\Gamma \cdot (\eta^U)_X \tag{74}$$

for $x \in X$. By using definition4.7(1), we have

$$\alpha \sqsubseteq \Gamma(\alpha) \tag{75}$$

for a relation $\alpha : I \rightharpoondown X$. Therefore, we get equation(74) because the following replacement

$$\begin{array}{rcll}
((*,x),(*,x)) \in 1_{I\times X} & \to & \langle\langle((*,*),(*,x)) \in \alpha\rangle \to \langle((*,*),(*,x)) \in \Gamma(\alpha)\rangle\rangle & (\because \text{equation}(76)) \\
& \leftrightarrow & ((*,*),(*,x)) \in \sqcap_{\langle\xi \ni ((*,*),(*,x))\rangle \wedge \langle \xi : I \rightharpoondown X\rangle}\Gamma(\xi) & \\
& \leftrightarrow & ((*,*),(*,x)) \in \sqcap_{\xi \in (\eta^U)_X((*,x))}\Gamma(\xi) & (\because \text{the definition of } \eta^U) \\
& \leftrightarrow & (*,x) \in \lim((\eta^U)_X((*,x))) & (\because \text{the definition of } \lim) \\
& \leftrightarrow & ((\eta^U)_X((*,x)),(*,x)) \in r_\Gamma & (\because \text{the definition of } r_\Gamma) \\
& \leftrightarrow & ((*,x),(*,x)) \in r_\Gamma \cdot (\eta^U)_X &
\end{array}$$

is satisfied.

[eq.(73)] Here we only have to verify

$$((\mu^U)_X(\mathscr{W}),(*,d)) \in r_\Gamma \tag{76}$$

for

$$(\mathscr{W},\mathcal{F}) \in U(r_\Gamma) \tag{77}$$

and

$$(\mathcal{F},(*,d)) \in r_\Gamma. \tag{78}$$

Note that

$$\xi \in \mathcal{F}$$

holds for a relation $\xi : I \rightharpoondown X$ from proposition3.2, and note that we have

$$^\forall\rho \in \mathcal{F}, ((*,*),(*,d)) \in \Gamma(\rho)$$

since

$$\begin{array}{rcll}
\text{equation}(78) & \leftrightarrow & (*,d) \in \lim \mathcal{F} & (\because \text{the definition of } r_\Gamma) \\
& \leftrightarrow & ((*,*),(*,d)) \in \sqcap_{\rho\in\mathcal{F}}\Gamma(\xi) & (\because \text{the definition of } \lim) \\
& \leftrightarrow & ^\forall\rho \in \mathcal{F}, ((*,*),(*,d)) \in \Gamma(\rho) &
\end{array}$$

is satisfied, then we have

$$((*,*),(*,d)) \in \Gamma(\xi). \tag{79}$$

Therefore, we get equation(76) since

$$\begin{array}{rcll}
\text{equation}(79) & \to & \langle(\pi^U)_X(\xi) \in \mathscr{W} \to ((*,*),(*,d)) \in \Gamma(\xi)\rangle & \\
& \leftrightarrow & \langle\xi \in (\mu^U)_X(\mathscr{W}) \to ((*,*),(*,d)) \in \Gamma(\xi)\rangle & (\because \text{the definition of } (\mu^U)_X) \\
& \leftrightarrow & ((*,*),(*,d)) \in \sqcap_{\xi\in(\mu^U)_X(\mathscr{W})}\Gamma(\xi) & \\
& \leftrightarrow & (*,d) \in \lim((\mu^U)_X(\mathscr{W})) & (\because \text{the definition of } \lim) \\
& \leftrightarrow & \text{equation}(76) & (\because \text{the definition of } r_\Gamma)
\end{array}$$

is satisfied.

(2) Given

$$J(I\times X',\Gamma') := (I\times X', r'_{\Gamma'}), \tag{80}$$

then we only have to verify

$$\Psi \cdot r_\Gamma \sqsubseteq r'_{\Gamma'} \cdot U(\Psi) \qquad \left(\begin{tikzcd} U(I\times X) \arrow[r, "U(\Psi)"] \arrow[d, "r_\Gamma"'] \arrow[dr, phantom, "\sqsubseteq"] & U(I\times X') \arrow[d, "r'_{\Gamma'}"] \\ I\times X \arrow[r, "\Psi"'] & I\times X' \end{tikzcd}\right);$$

i.e.

$$(U(\Psi)(\mathcal{F}), \Psi((*,x))) \in r'_{\Gamma'} \tag{81}$$

for a pear $(\mathcal{F},(*,x)) \in r_\Gamma$. Note that we have

$$\Psi(\Psi^\sharp(\rho)) \sqsubseteq \rho \tag{82}$$

because $\Psi(\Gamma(\rho)) \sqsubseteq \Gamma'(\Psi(\rho))$ holds for a relation $\rho : I \rightharpoondown X'$ by definition4.6(3), and note that

$$^\forall\rho \in \mathcal{F}, ((*,*), \Psi((*,x))) \in \Gamma'(\Psi(\rho)) \tag{83}$$

because

$$\begin{array}{rcl} (\mathcal{F},(*,x)) \in r_\Gamma & \leftrightarrow & (*,x) \in \lim \mathcal{F} \\ & \leftrightarrow & (*,x) \in \sqcap_{\rho\in\mathcal{F}}\Gamma(\rho) \\ & \leftrightarrow & ^\forall\rho \in \mathcal{F}, (*,x) \in \Gamma(\rho) \\ & \rightarrow & ^\forall\rho \in \mathcal{F}, \Psi((*,x)) \in \Psi(\Gamma(\rho)) \\ & \rightarrow & \text{equation(83)} \end{array}$$

is satisfied, then we attain equation(81) because

$$\begin{array}{lll} & \text{equation(83)} & \\ \rightarrow & ^\forall\rho \rightharpoondown I \times X', \Psi^\sharp(\rho) \in \mathcal{F} \rightarrow \Psi((*,x)) \in \Gamma'(\Psi(\Psi^\sharp(\rho))) & \\ \rightarrow & ^\forall\rho \rightharpoondown I \times X', \Psi^\sharp(\rho) \in \mathcal{F} \rightarrow \Psi((*,x)) \in \Gamma'(\rho) & \\ \leftrightarrow & ^\forall\rho \in U(\Psi)(\mathcal{F}), \Psi((*,x)) \in \Gamma'(\rho) & \\ \leftrightarrow & \Psi((*,x)) \in \sqcap_{\rho\in U(\Psi)(\mathcal{F})}\Gamma'(\rho) & (\because \text{equation(82)}) \\ \leftrightarrow & \Psi((*,x)) \in \lim(U(\Psi)(\mathcal{F})) & \\ \leftrightarrow & (U(\Psi)(\mathcal{F}), \Psi((*,x))) \in r'_{\Gamma'} & \end{array}$$

is satisfied.

□

Proposition 4.17. $C \cdot J = 1_{\mathrm{Top}}$

Proof. Given

$$(I \times X, r_\Gamma) := J(I \times X, \Gamma) \tag{84}$$

and

$$(I \times X, \Delta_{r_\Gamma}) := C(I \times X, r_\Gamma) \tag{85}$$

for a topological space$(I \times X, \Gamma)$, then we only have to verify $(I \times X, \Gamma) = (I \times X, \Delta_{r_\Gamma})$; i.e.

$$^\forall\alpha : I \rightharpoondown X, \Delta_{r_\Gamma}(\alpha) = \Gamma(\alpha).$$

We still get this equation because

$$\begin{array}{rcll} ((*,*),(*,b)) \in \Delta_{r_\Gamma}(\alpha) & \leftrightarrow & ^\exists\mathcal{U} \in U(I \times X), \langle\langle \alpha \in \mathcal{U}\rangle \wedge \langle(\mathcal{U},(*,b)) \in r_\Gamma\rangle\rangle & (\because \text{lemma4.11}) \\ & \leftrightarrow & ^\exists\mathcal{U} \in U(I \times X), \langle\langle \alpha \in \mathcal{U}\rangle \wedge \langle(*,b) \in \lim\mathcal{U}\rangle\rangle & (\because \text{definition4.14}) \\ & \leftrightarrow & ^\exists\mathcal{U} \in U(I \times X), \langle\langle \alpha \in \mathcal{U}\rangle \wedge \langle((*,*),(*,b)) \in \sqcap_{\rho\in\mathcal{U}}\Gamma(\rho)\rangle\rangle & (\because \text{definition4.13}) \\ & \leftrightarrow & ((*,*),(*,b)) \in \Gamma(\alpha) & (\because \text{lemma4.15}), \end{array}$$

is satisied. □

I prove lemma 4.18 necessary at the verification of proposition 4.19.

Lemma 4.18. *Let $(I \times X, x)$ be a relational $\mathbf{U}-$ algebra, $(*,a)$ an element of $I \times X$, and $\mathcal{U}$ an ultrafilter over $I \times X$, then the following $(1),(2)$ are equivalent*:

(1) $(\mathcal{U},(*,a)) \in x$.
(2) $^\forall\rho \in \mathcal{U},\ ^\exists\mathcal{A}_\rho \in U(I \times X),\ \langle\langle\rho \in \mathcal{A}_\rho\rangle \wedge \langle(\mathcal{A}_\rho,(*,a)) \in x\rangle\rangle$.

Proof. $(\rightarrow)$ Let $\mathcal{A}_\rho \in U(I \times X)$ be an ultrafilter defined by $\mathcal{A}_\rho := \mathcal{U}$ for a relation $\rho \in \mathcal{U}$. If (1) holds, then we get (2) because $\rho \in \mathcal{A}_\rho$ holds by definition and $(\mathcal{A}_\rho,(*,a)) \in x$ holds by assumption.

($\leftarrow$) If (2) and

$$\mathcal{U} = \{I \times A_\alpha \sqsubseteq I \times X; \alpha \in \Lambda\} \tag{86}$$

hold for an index set Λ, then there exists $\mathcal{A}_{\rho_\lambda} \in U(I \times X)$ for any $\lambda \in \Lambda$ such that

$$I \times A_\lambda \in \mathcal{A}_{\rho_\lambda} \tag{87}$$

and

$$(\mathcal{A}_{\rho_\lambda}, (*, a)) \in x \tag{88}$$

hold. Because of the definition of ρ and equation(87), we get $I \times A_\lambda \in \mathcal{A}_{\rho_\lambda}$, which induces $\mathcal{A}_{\rho_\lambda} = \mathcal{U}$. Therefore, by applying this equation for equation(88), we attain (1).

□

Proposition 4.19. $J \cdot C = 1_{\mathrm{Rel}(\mathbf{U})}$

Proof. Given

$$(I \times X, \Gamma_x) := C(I \times X, x) \tag{89}$$

and

$$(I \times X, r_{\Gamma_x}) := J(I \times X, \Gamma_x) \tag{90}$$

for a relational **U**-algebra $(I \times X, x)$, then we only have to verify

$$(I \times X, x) = (I \times X, r_{\Gamma_x}). \tag{91}$$

Because

$$\begin{array}{llll}
(\mathcal{U}, (*, a)) \in x & \leftrightarrow & {}^\forall\rho \in \mathcal{U},\ {}^\exists\mathcal{W} \in U(I \times X),\ \langle\langle \rho \in \mathcal{W}\rangle \wedge \langle(\mathcal{W}, (*, a)) \in x\rangle\rangle & (\because \text{lemma4.18}) \\
 & \leftrightarrow & {}^\forall\rho \in \mathcal{U},\ ((*, *), (*, a)) \in \Gamma_x(\rho) & (\because \text{lemma4.11}) \\
 & \leftrightarrow & ((*, *), (*, a)) \in \sqcap_{\rho \in \mathcal{U}} \Gamma_x(\rho) & \\
 & \leftrightarrow & (*, a) \in \lim \mathcal{U} & (\because \text{definition4.13}) \\
 & \leftrightarrow & (\mathcal{U}, (*, a)) \in r_{\Gamma_x} & (\because \text{definition4.14})
\end{array}$$

holds, then we have equation(91). □

The follwoing theorem 4.20can be pbtained by applying proposition 4.17 and proposition 4.19.

Thorem 4.20 ([1]corollary3.4)**.** Rel(**U**) $\cong$ Top

This paper does not present the proof of the convergence over a topological space, however it is known that proposition 4.21 and proposition 4.22 are satisified.

Proposition 4.21. *For a topological space $(I \times X, \Gamma)$, the following propositions are equivalent:*

(1) *an ultrafilter $\mathcal{U} \in U(I \times X)$ has the limit*
(2) *$(I \times X, \Gamma)$ is compact.*

Proposition 4.22. *For topological space $(I \times X, \Gamma)$, the following propositions are equivalent:*

(1) *an arbitary $\mathcal{U} \in U(I \times X)$ does not include more than 2 limits*
(2) *$(I \times X, \Gamma)$ is Hausdorff.*

Since proposition 4.21 and proposition 4.22, a topological space $(I \times X, \Gamma)$ is Hausdorff if and only if the limit for each $\mathcal{U} \in U(I \times X)$ is uniquely determined, so you get lemma4.23.

Lemma 4.23. *For the relational* **U***-algebra $(I \times X, r_\Gamma)$ corresponding to a topological space $(I \times X, \Gamma)$, the following propositions are equivalent:*

(1) *r_Γ is a function.*
(2) *$(I \times X, \Gamma)$ is compact Hausdorff.*

Let CH be the full subcategory which objects in Top is restricted to compact Hausdorff, then you have the following corollary from the results of theorem 4.20.

Corollary 4.24. $\mathrm{Set}^{\mathbf{U}} \cong \mathrm{CH}$

This corollary 4.24 corresponds to theorem 1 at section 9 in chapter 6 of [7].

5 Conclusion

Mac Lane proved that **T**-algebras for ultra filters are isomorphic to the compact Hausdorff spaces, and which he insisted is the algebraic object including the operator of convergence to the ultra filter [7]. Barr proved the isomorphism between genaral topological spaces and relational algebras for a monad **T**, by the extention from algebras for **T** to relational algebras for **T** [1]. This discovery says that we can rewrite the continuous mathematical theory of general topological space theory by using formal algebraical calculation in the use of relational theory. In this thesis, I reformallize the category of relational **T**-algebras to realize proofs of them, before the algebraic analysis for topological spaces and the composition of formal proofs by using relational calculus. The future works are the development of theorem provers by the implemention for those formal proofs into the tehorem provers such as Coq, and the consideration for the calculus theory on topological spaces.

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