INF367A Exercise 10

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1 Introduction

This exercise is about using Gibbs sampling to estimate the posterior even with missing data.

2 Gibbs sampler

2.1 Deriving conjugate prior for μ

We assume here that we have the missing values, so we simply need to derive the conjugate prior of the likelihood of the "assumed complete data".

$$\begin{aligned}
\log P(\mu|x,z) &= \sum_{i=1}^{n}\left[-\frac{1}{2}(x_i-\mu)^T\Sigma^{-1}(x_i-\mu)\right] \\
&= -\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^T\Sigma^{-1}(x_i-\mu) \\
&= -\frac{1}{2}\sum_{i=1}^{n} x_i^T\Sigma^{-1}x_i - x_i^T\Sigma^{-1}\mu - \mu^T\Sigma^{-1}x_i + \mu^T\Sigma^{-1}\mu \\
&\propto -\frac{1}{2}\sum_{i=1}^{n} -2x_i^T\Sigma^{-1}\mu + \mu^T\Sigma^{-1}\mu \\
&= \sum_{i=1}^{n} x_i^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu \\
&= N\bar{X}^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T N\Sigma^{-1}\mu \\
&= -\frac{1}{2}\mu^T \underbrace{\Sigma^{-1}}_{=A}\mu + \underbrace{\bar{X}^T\Sigma^{-1}}_{=b^T}\mu
\end{aligned}$$

Completing the square:

$$\begin{aligned}
S &= A^{-1} = \Sigma \\
m &= Sb = \Sigma\Sigma^{-1}\bar{X} = \bar{X} \\
&\underline{\underline{\Rightarrow \mu \sim \mathcal{N}(\bar{X},\Sigma)}}
\end{aligned}$$

For future self: think before deriving something obvious!!!

2.2 Deriving conjugate prior for Z

Conveniently, marginalizing away features from multivariate gaussians is simply done by dropping said features from the multivariate function [1]. In this case we will remain with a univariate Gaussian. Let $\sigma^2 = \Sigma_{11}$

$$
\begin{aligned}
P(Z|X,\mu) &= \prod_{j=1}^{k} N(z_{j1}|\mu_1,\sigma) \\
&\Rightarrow \underline{\underline{P(z_{j1}|\mu_1,\mu) = N(\mu_1,\Sigma_{11})}}
\end{aligned}
$$

2.3 Implementation

The implementation of the Gibbs sampler is then very simple.

1. Initialize values for Z and μ
2. For iterations t in T do:
   - 2.1. Sample $\mu_t \sim \mathcal{N}(\bar{X},\Sigma)$ (X here is data with sampled Z for missing data)
   - 2.2. Sample Z_t by sampling for each $z_{i1} \sim N(\mu_1,\Sigma_{11})$

References

[1] Ruye Wang, "Marginal distribution of multivariate gaussian," 2020, [accessed 22-March-2020]. [Online]. Available: http://fourier.eng.hmc.edu/e161/lectures/gaussianprocess/node7.html