Joukowsky Airfoils

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Chapter 1

Elementary flow functions

Given an element of volume having dimensions of dx, dy, and dz (see Fig. 1.1) the mass entering the left face has a density ρ and a velocity u. The area of the flow is $dydz$. Assume now that the mass leaving the right face has a slightly different velocity and density. The mass leaving the right face is, therefore, $[\rho u + (\partial \rho u/\partial x)dx]dydz$. We may write similar terms for the mass entering and leaving the volume in the y direction: $\rho udxdz$ and $[\rho v+(\theta \rho v/dy)dy]dxdz$. In the z direction the same procedure leads to $\rho wdxdy$ and $[\rho w + (\partial \rho w/\partial z)dz]dxdy$. The change in the mass inside the volume is given by

$$\frac{\partial \rho}{\partial t} dxdydz \tag{1.1}$$

Equating the change in the mass inside the volume to the sum of the inflows less the sum of the outflows results in the following equation:

$$\begin{aligned}\frac{\partial \rho}{\partial t} dx\,dy\,dz &= \rho u\,dy\,dz + \rho v\,dx\,dz + \rho w\,dx\,dy \\ &\quad - \left(\rho u + \frac{\partial \rho u}{\partial x} dx\right) dy\,dz \\ &\quad - \left(\rho v + \frac{\partial \rho v}{\partial y} dy\right) dx\,dz - \left(\rho w + \frac{\partial \rho w}{\partial z} dz\right) dx\,dy \\ &= - \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) dx\,dy\,dz\end{aligned} \tag{1.2}$$

However, $dxdydz$ is simply the differential volume. Dividing both sides by the differential volume we have the law of mass conservation or the continuity equation as applied to a fluid medium

$$\frac{\partial \rho}{\partial t} + \left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right) = 0 \tag{1.3}$$

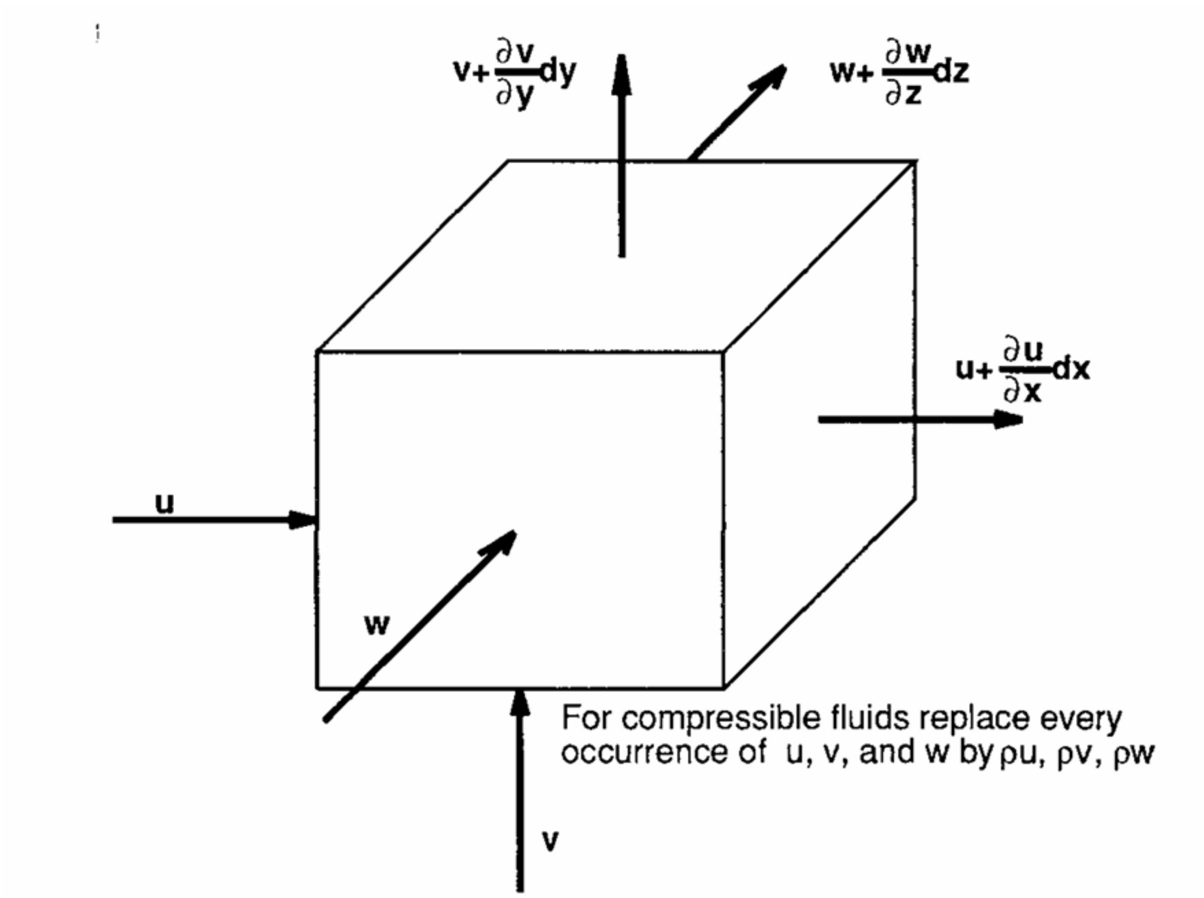


Figure 1.1: Element of volume

For steady, incompressible flow in two dimensions, that is with ρ constant, the continuity equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{1.4}$$

1.1 potential functions

It can be shown that the statement that the line integral between two points in a plane is independent of the path of integration between the two points is equivalent to the statement that the line integral around any closed path is zero. This equivalence leads to the following theorem: In a simply connected region in which $u(x, v)$, $v(x, y)$, and their first partial derivatives are continuous, the necessary and sufficient condition that the integral

$$\int udx + vdy \tag{1.5}$$

around a closed path should be zero and that the integral along a path connecting two points should be independent of the path is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1.6}$$

Further, if Eq. (1.6) is satisfied for any two functions u and v, there exists a function

ϕ for which

$$\frac{\partial \phi}{\partial x} = u; \quad \frac{\partial \phi}{\partial y} = v \tag{1.7}$$

Therefore,

$$udx + vdy = d\phi \tag{1.8}$$

is an exact differential.
Suppose we have a flow with a component of velocity in the x direction desig nated as u and a component of velocity in the y direction designated as D. Further, let u and v be continuous differentiate functions of position. Then if Eq. (2.3) is satisfied we conclude that a velocity potential ϕ exists. We have demonstrated previously that when two-dimensional, inviscid, incompressible flows satisfy the requirement for conservation of mass, the velocity components are related through the equation (1.6)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.9}$$

Using the relationship between ϕ and u and v we may write Eq. (1.9) as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1.10}$$

This is Laplace's equation, probably the most studied partial differential equation in the mathematical literature.

1.2 Laplace's Partial Differential Equation and the Stream Function

aerodynamics is the branch of mathematical physics concerned with the prop erties of functions that satisfy Laplace's partial differential equation. We will re strict our consideration at this level to the two-dimensional form. In rectangular coordinates we can restate Laplace's equation, written here in terms of the stream function, as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{1.11}$$

In polar coordinates Laplace's equation is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} = 0 \tag{1.12}$$

Because this equation is a linear partial differential equation, a sum of solutions is also a solution. For example, if

$$\psi = Ur \sin\theta \tag{1.13}$$

are solutions, then

$$\psi = \frac{K}{2\pi} \ln r \tag{1.14}$$

is also a solution.
We should note that certain stream functions, like certain potential functions, satisfy the Laplace equation. They differ mathematically from potential functions by how their

boundary conditions must be specified. The physical basis of stream functions is discussed in the following discussion. Assume we seek to find functions that model some physical fluid flow. (By model we mean that the graph of the function looks like a picture of the physical flow.) It is readily seen that if the product $r \sin\theta$ is a constant regardless of the value of r or θ, then $y = Ur \sin\theta$ represents a straight line. By choosing $\langle r \sin\theta \rangle = (r \sin\theta)_{-1} + C$

$$\psi = Ur \sin\theta \tag{1.15}$$

represents a series of straight lines each separated by a distance D from the line below it and by a distance C from the line above it. If these lines are parallel to the direction of the flowing stream and the fluid anywhere between any pair of lines is of the same constant density and it flows at the same constant velocity, then the fluid flow is termed a uniform stream. Now, a curve that is always parallel to the local direction of a flowing stream is called a streamline. A function x and y or r and θ that describes a series of streamlines is called a stream function. Because there is no flow across streamlines (else the streamline would not be parallel to the local flow direction) the fluid volume per unit depth per unit time flowing between a pair of streamlines remains constant. Obviously, then, there is a relationship between fluid flow as modeled by stream functions and the continuity equation. We shall seek to determine this relationship and some aspects of its characteristics.

Suppose we consider the function

$$\psi = Ur \sin\theta \tag{1.16}$$

We can readily see that this function satisfies the definition for a stream function we have given. In this case, the stream function represents a uniform stream.

We are going to restrict our attention to fluid flows for which the density remains essentially constant. Water and air at low Mach numbers can be treated as constant density fluids for purposes of our analysis. The mass flowing past a given streamline section is given by

$$\rho v h = \dot{m} \tag{1.17}$$

where ρ is the density, v is the average velocity, h is the distance between the streamlines, d is the depth of the fluid, and is assumed to remain constant, and $\dot{m}$ is the mass flow rate.

In order for a steady flow with no additions or removals, the mass flowing between two streamlines remains constant, as we have already noted. When the streamlines converge, the average velocity must increase; when the streamlines diverge, the average velocity must decrease; when the streamlines are parallel, the velocity remains constant. The value of the stream function is then a measure of the mass flow (or the volumetric flow) between the streamline and the reference streamline. It has the units of length $\times$ length per unit time.

When one goes from one streamline to an adjacent one the change in the volumetric flow rate is a measure of the velocity of fluid parallel to the stream.

If we take as our coordinates the streamwise direction and x as being parallel to the y-axis, we write the expressions for the stream function, ψ, then

$$\frac{\partial\psi}{\partial y} = u \quad v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \tag{1.18a}$$

and

$$\frac{\partial \psi}{\partial x} = -v \quad v_\theta = -\frac{\partial \psi}{\partial r} \tag{1.18b}$$

The negative sign points in the direction that ψ increases as one moves in the stream-wise direction, while the streamlines are converging and the velocity component along the y-axis points in the negative direction.

In two dimensions, we can change the flow field to express this condition by means of the continuity equation $\nabla \cdot \mathbf{u} = 0$. If we substitute for u and v in terms of the stream function ψ, we obtain

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \tag{1.19}$$

If the flow is irrotational (satisfies Laplace's equation), we are led to the introduction of μ and ν into

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{1.20}$$

which is regarded as a consequence of incompressibility, hence the flow this equation provides a complex derivative of the flow.

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{1.21}$$

which is Laplace's equation. Thus, we are led to the conclusion that the stream function is harmonic, and since the flow is irrotational, we need to express this relationship with respect to the flow direction.

Chapter 2

Airfoils & Joukowsky transform

2.1 Conformal Maps

A transformation can be said to map a region from one plane to a region in another plane. It is conformal if for every point in the first plane there is a unique point in the second plane and the angles between lines in one plane are preserved in the second. If we know for, example, the pressure at some point in the first plane the transformation tells us where this pressure exists in the second plane. Thus, if the transformation maps a circle in one plane to an airfoil shape in the second, we may take the solution for pressures in the first plane on the $\psi = 0$ line for a doublet and coincident vortex in a uniform stream and from this determine the pressures on the airfoil. Hence, if we construct an airfoil that has the same shape as that given by the transformation of a circular cylinder we immediately know a majority of its aerodynamic characteristics.

2.2 The Joukowsky Transform

The transformation devised by Joukowski can be written in the form:

$$z = \zeta + \frac{c^2}{\zeta} \tag{2.1}$$

where z, in the target plane, and ζ, in the source plane, are both complex numbers. We will set:

$$z = \xi + i\eta \tag{2.2}$$

and

$$\zeta = x + iy \tag{2.3}$$

then,

$$\xi + j\eta = x + jy + \frac{c^2}{x + jy} = x + jy + \frac{c^2(x - jy)}{x^2 + y^2} \tag{2.4}$$

From which it is seen that:

$$\zeta = x + \frac{xc^2}{x^2 + y^2} \tag{2.5}$$

$$\eta = y - \frac{yc^2}{x^2 + y^2} \tag{2.6}$$

These equations give us the coordinates of a point in the z plane in terms of the coordinates of the point in the £ plane, the plane in which the circle is located. We now demonstrate that if the center of the circle is at the origin, the circle transforms into a straight line of length 4c. First, let

$$x = r\cos\theta \tag{2.7}$$

$$y = r\sin\theta \tag{2.8}$$

Then, we have

$$\xi = r\cos\theta + \frac{rc^2\cos\theta}{r^2} \tag{2.9}$$

$$\eta = r\sin\theta - \frac{rc^2\sin\theta}{r^2} \tag{2.10}$$

We take the radius of the circle to be c with the result that

$$\zeta = \pm 2c \tag{2.11}$$

$$\eta = 0 \tag{2.12}$$

By moving the center of the circle away from the origin, we can affect the thickness and camber of the resulting airfoil. Camber is the curvature of mean line between upper and lower surface of the airfoil. We will define F as the displacement of the center along the y axis and ec as the displacement of the center along the x axis. Here, e is a number less than or equal to one. We will also define

$$\beta = \sin^{-1}(\frac{F}{c+ec}) \tag{2.13}$$

whereas α is the angle of attack and β is related to the camber of the airfoil, which results from the transformation.

Note that coordinates in the ζ plane still must be measured with respect to the origin despite the fact that the center of the circle may be displaced. If θ is measured with respect to the center of the circle, then the coordinates of points on the circle in the ζ plane are

$$y = F + a\sin\theta \tag{2.14}$$

$$x = ce + a\cos\theta \tag{2.15}$$

By plugging these into equation (2.1) yields us the coordinates of the transformed points:

$$\xi = ce + a\cos\theta + \frac{c^2(ce + a\cos\theta)}{(ce + a\cos\theta)^2 + (F + a\sin\theta)^2} \tag{2.16a}$$

$$\eta = F + a\sin\theta - \frac{c^2(F + a\sin\theta)}{(ce + a\cos\theta)^2 + (F + a\sin\theta)^2} \tag{2.16b}$$

Pressures at a particular value of 9 on the circle have that same value at the transformed point in the z plane.

In the form just used the transform yields a cusp at the trailing edge. If a, the circle radius, in Eqs. (2.16) is replaced by

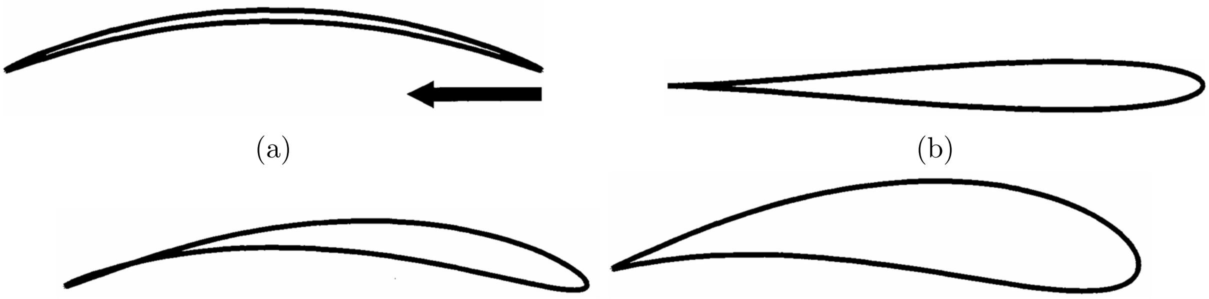
$$a = c(1 + f) \tag{2.17}$$

where the trailing-edge factor f is greater than or equal to 1.0 but usually not more that 2.0, the stagnation points will exist inside the circle leading to a finite angle trailing edge, and the point of maximum thickness on the airfoil exists a little farther aft than when $f = 1.0$.

To guarantee that the trailing edge is a stagnation point we make the circulation

$$K = 4\pi U_\infty a \sin(\alpha + \beta) \tag{2.18}$$

Notice that the amount of circulation does not affect the shape of the airfoil, which is determined entirely by the values of c, e, F, and f.
Figure 2.1 illustrates the characteristics of the Joukowski airfoils. The airfoil in Fig. 2.1a is termed a circular arc airfoil because it is made up of sections of circles. In this case the center of the circle is just moved upward on the y axis. The airfoil in Fig. 2.1b is one made by shifting the center of the circle to the right only. E in the program is 1/e in the preceding equations. The airfoil in Fig. 2.1c has both thickness and camber but still a cusp trailing edge. The airfoil in Fig. 2.1d has been generated with a greater thickness so that the finite trailing-edge angle is visible. The nominal circle radius for these airfoils is 1.0.



(a) (b)

(c) (d)

Figure 2.1: Joukowski airfoils: a) E = l.OD + 10, F = 0.2, full circulation, and cusp trailing edge; b) E = 14.0, F = 0.0, full circulation, and cusp trailing edge; c) E = 14.0, F = 0.2, full circulation, and cusp trailing edge; and d) E = 7.0, F = 0.2, full circulation, and trailing-edge factor = 1.25.

2.3 Stream functions for Joukowsky airfoils

With the use of this vortex strength, the stream function is

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta + \frac{4\pi U a}{2\pi}\sin(\alpha + \beta)\ln r \tag{2.19}$$

then,

$$v_\theta = U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{2U_a}{r}\sin(\alpha + \beta) \tag{2.20}$$

On the surface of the cylinder r = a and so here

$$v_\theta = 2U\sin\theta + 2U\sin(\alpha + \beta) \tag{2.21}$$

from which we see that $v_\theta = 0$ when $\theta = -(\alpha + \beta)$ and $\theta = \alpha + \beta - \pi$. These are the locations of the stagnation points.
The lift per unit span is

$$\rho U^2 4\pi a \sin(\alpha + \beta) \tag{2.22}$$

To normalize to the lift coefficient we divide by $\frac{1}{2}\rho U_\infty^2 \cdot \text{chord}$. The actual chord length is a little more than 4c, i.e.,

$$4c\left[1 + \frac{e^2}{1 + 2e}\right] \tag{2.23}$$

for a symmetrical Joukowski airfoil. Hence, for such an airfoil

$$C_l = 2\pi\left[\frac{1 + 3e + 2e^2}{1 + 2e + e^2}\right]\sin\alpha \tag{2.24}$$

For cambered Joukowski airfoils or airfoils with finite trailing-edge angles, it is preferable to write

$$C_\ell = 8\pi\frac{c(1 + fe)}{\text{chord}}\sin(\alpha + \beta) \tag{2.25}$$

where chord $= |\xi_{\text{max}}| + |\xi_{\text{min}}|$- The computer program uses this value. Notice the very important result that the lift curve slope is

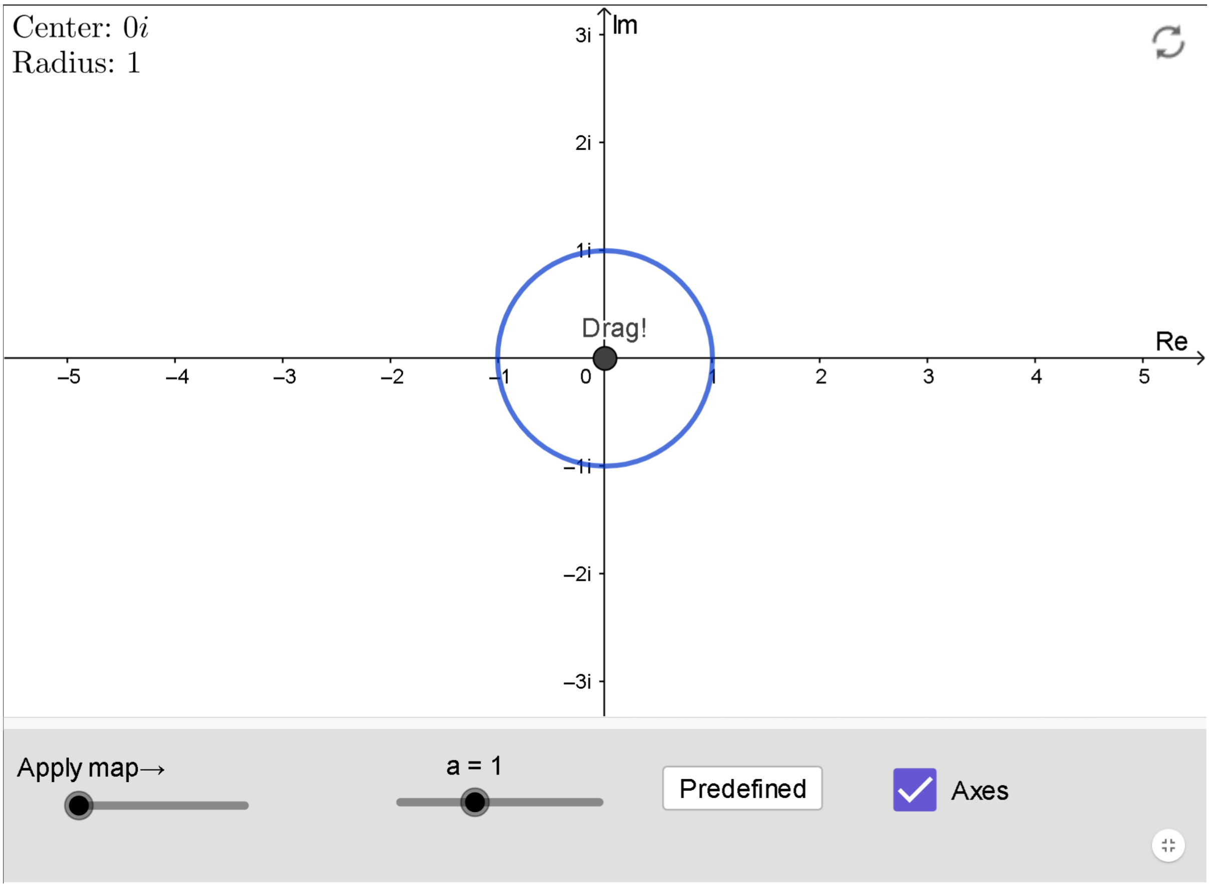
$$\frac{\partial C_\ell}{\partial \alpha} = 8\pi\frac{c(1 + fe)}{\text{chord}}\bigg|_{\alpha=0} \tag{2.26}$$

for all airfoils. This is another way of saying that the lift is approximately independent of the airfoil shape, as we noted previously. The lift curve slope does increase slightly as the airfoil thickness increases. This trend is seen for all airfoils, not just those of the Joukowski type.

Chapter 3

Program for visualizing a Joukowsky transform

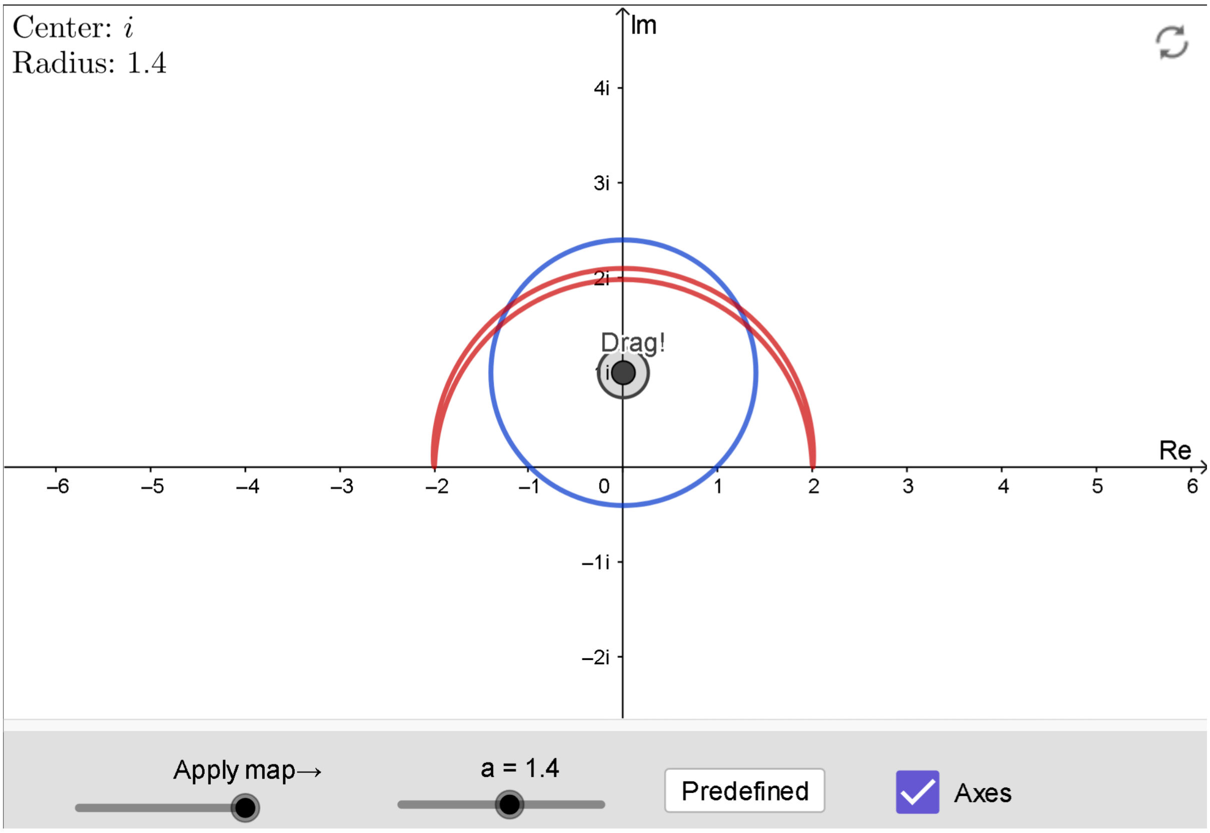
Below is the program that shows the circle before vs after applying the map:



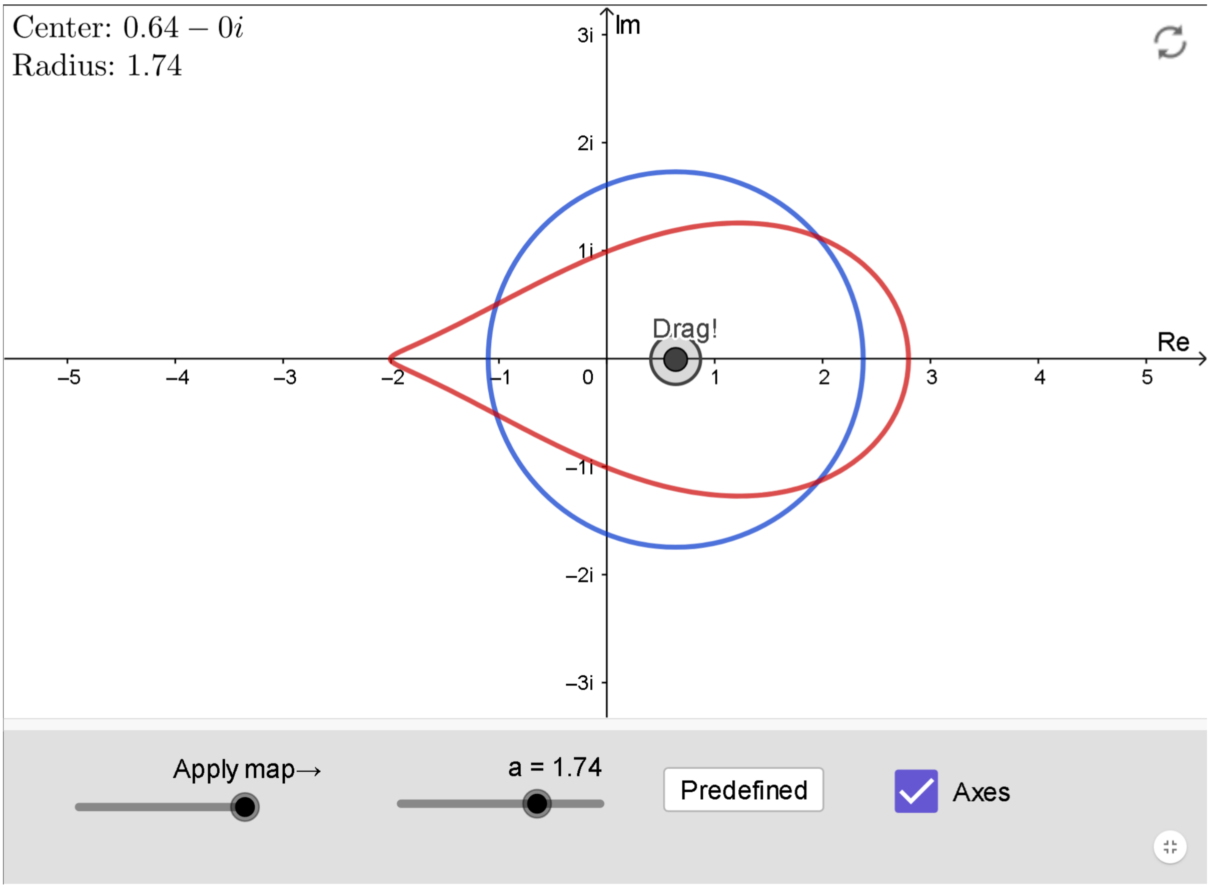
Where a is the radius of the circle and the "Apply map" slider is how we control the intensity of the Joukowsky transform, the slider controls a value α for

$$z = \zeta + \alpha \frac{c^2}{\zeta}, \quad 0 \leq \alpha \leq 1$$

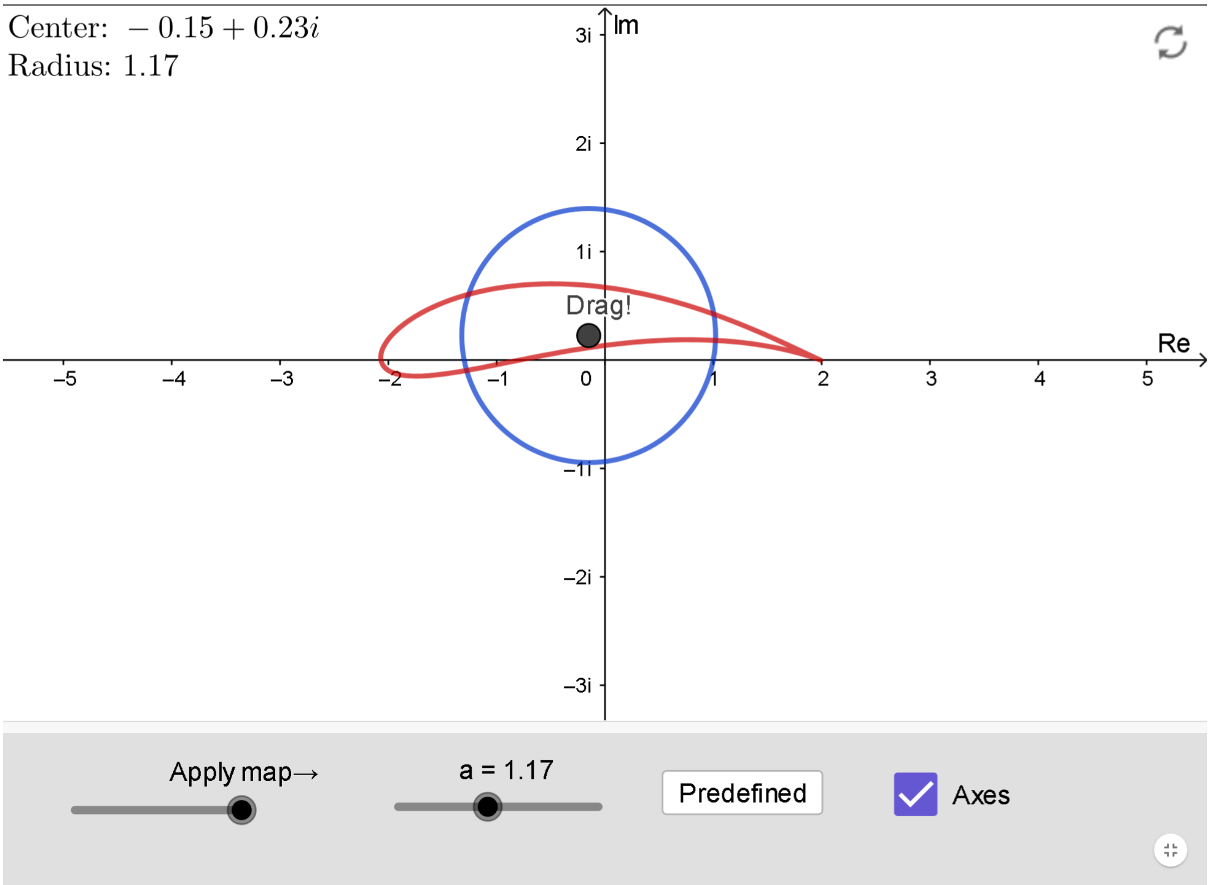
. Applying the map and moving the center of the circle upwards gives an airfoil like in Fig 2.1a.



Similarly, we obtain a symmetric airfoil like in fig 2.1b by moving the center of the circle to the right.



Now by simultaneously translating the circle's center in both upwards and the right(or left) yields airfoils similar in shape to the ones in figs 2.1c & 2.1d.



Chapter 4

Flow visualization over Joukowski airfoil

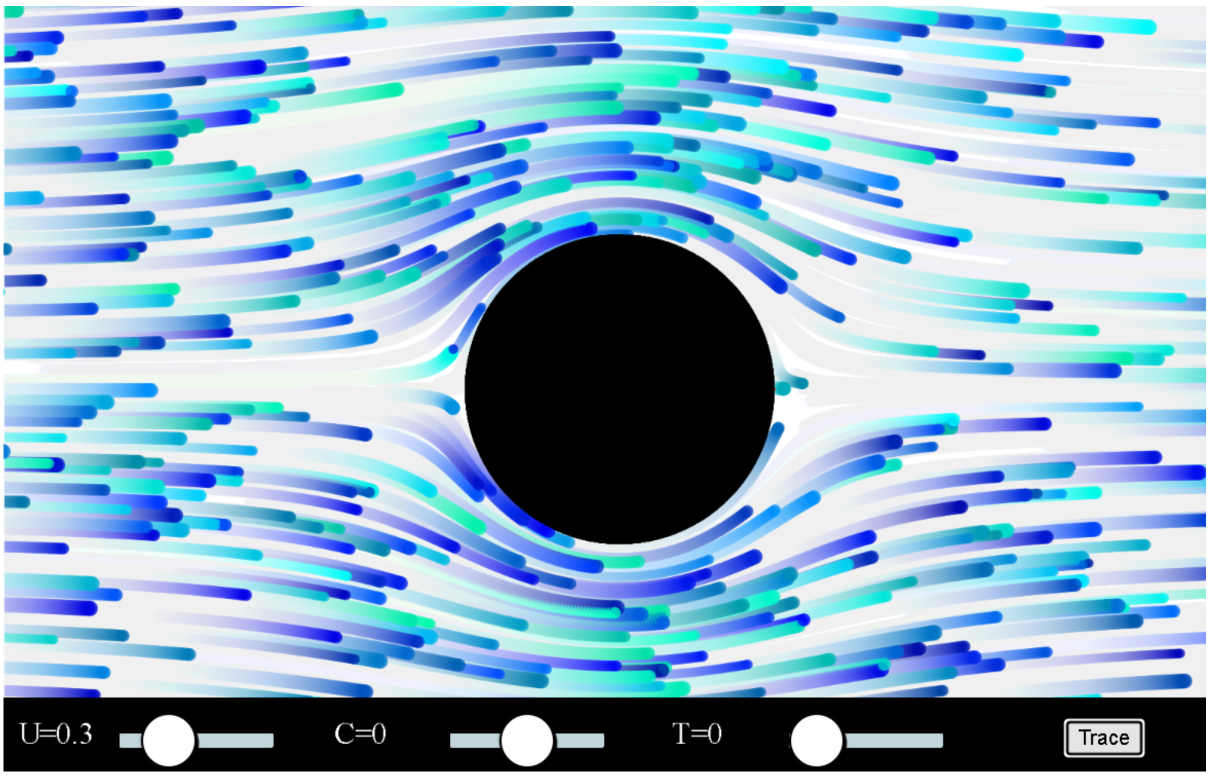
using an airfoil profile similar to the ones in figs 2.1c & 2.1d, we now consider the uniform flow around the unit circle with circulation K and speed $U > 0$ given by the complex potential

$$F(z) = Uz + \frac{U}{z} - \frac{iK}{2\pi} \log z. \tag{4.1}$$

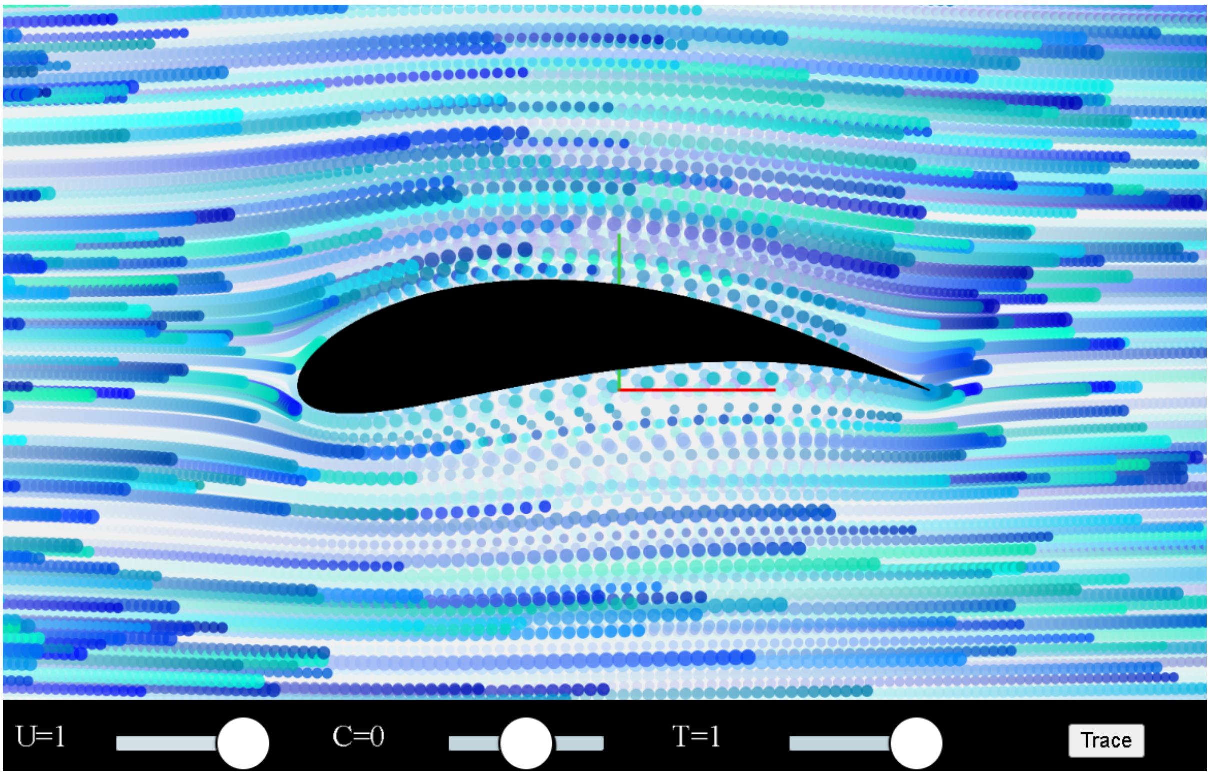
We can use the linear transformation

$$T(z) = -0.15 + i0.23\sqrt{13} - 0.23\sqrt{13 \cdot 2}z \tag{4.2}$$

to map this flow around $|z| = 1$ onto the flow around the circle c_1 with center $z_1 = -0.15 + i0.23\sqrt{13}$ and radius $r = -0.23\sqrt{13 \cdot 2}$.
Here is the preview before applying the transformation



and after the transformation



Done with circulation $C = 0, U_\infty = 1$, with $\alpha = 1$ for $z = \zeta + \alpha\frac{c^2}{\zeta}$.

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