SBM Dynamics

August 2019

1 Simplified Biped Model

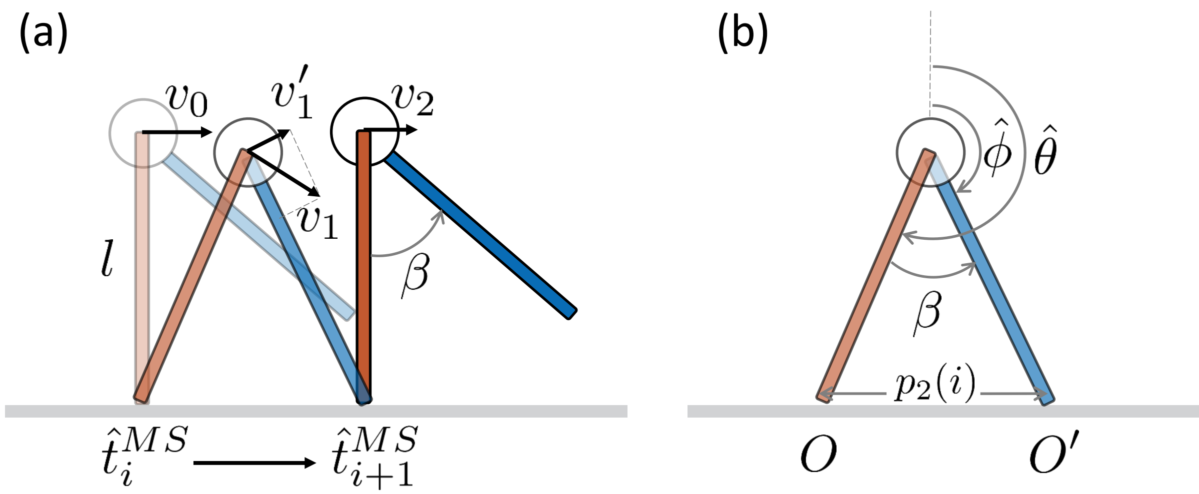

Figure 1: (a) SBM walking from the i-th mid-stance to the $(i+1)$-st mid-stance. (b) SBM at the touch-down moment.

Consider a *Simplified Biped Model* (SBM) adopted from [1], illustrated in Fig. 1. The model consists of a point-mass (also called *hip*, shown as black circle) and two mass-less legs each with constant length l. The stance leg is colored in red, and the swing leg is colored in blue. The stance leg angle with respect to the upright direction is denoted as $\hat{\theta}$, and the swing leg angle with respect to the upright direction is denoted as $\hat{\phi}$. In the scope of this work we consider the walking motion of SBM starting from the mid-stance position $\hat{\theta} = \pi$ with positive hip velocity $v_0 > 0$. During single stance phase, the stance leg rotates around the pivot point O, and the swing leg swings forward instantaneously to form an angle β relative to the stance leg. Notice $\beta = \hat{\theta} - \hat{\phi}$, and the value of β stays constant during the stance phase. Given any control parameter $p_2(i)$ that represents the step length during the i-th step, such β can be obtained using the Law of Cosines (Fig. 1(b)):

$$\beta = \arccos\left(\frac{2l^2 - p_2(i)^2}{2l^2}\right). \tag{1}$$

As the single stance phase continues, the touch-down event, described by the guard condition $\hat{\theta}+\hat{\phi} = 2\pi$, will eventually be triggered, and an instantaneous stance phase takes place as shown in Fig. 1(b). Subsequent to the double stance phase, an impact with the ground happens with a coefficient of restitution of 0. That is, the axial component of v_1 resets to zero after the impact, but the lateral component v_1' remains unchanged. The stance leg is then pivoted at a new point O' and the system keeps evolving forward.

We denote a hybrid execution of the SBM as a pair $(\hat{\mathcal{I}}, \hat{a})$ where $\hat{\mathcal{I}} = \{\hat{I}_i\}_{i=0}^N$ is a hybrid time set with $\hat{I}_i := [\hat{\tau}_i^+, \hat{\tau}_{i+1}^-]$ and $\hat{a} = \{\hat{a}_i(\cdot)\}_{i=0}^N$ is a finite sequence of solutions to the SBM's equations of motion. In the scope of this work we require $\hat{\theta} \in [\pi/2, 3\pi/2]$, and only consider the motion of SBM in the duration of $\hat{I}_i \cup \hat{I}_{i+1}$.

2 Dynamics

In this section we derive the functions $f_{\hat{y}_1}$, $f_{\hat{y}_2}$, $f_{\hat{y}_3}$, and $f_{\hat{y}_4}$ in our manuscript by writing down $\hat{y}_1(i+1)$, $\hat{y}_2(i)$, $\hat{y}_3(i)$, $\hat{y}_4(i)$ explicitly.

Ideally we expect SBM to walk from mid-stance to mid-stance as shown in Fig. 1 (a). Assume SBM arrives at the i-th mid-stance at t_i^{MS} with positive hip velocity v_0 and positive stance leg angular velocity $\dot{\hat{\theta}}(t_i^{MS})$. As SBM moves forward, denote v_1 as the hip velocity when touch-down happens, and v_1' as the projection of v_1 to the direction that is perpendicular to the swing leg. Eventually, SBM should reach the $(i+1)$-th mid-stance at t_{i+1}^{MS} with hip velocity v_2. Notice that v_1, v_1' and v_2 all remain to be computed. From [1, (3)] we know

$$\ddot{\hat{\theta}}(t) = \frac{g\sin(\hat{\theta}(t))}{l} \tag{2}$$

and thus $\dot{\hat{\theta}}(t) > 0$ for all $t \in [\hat{t}_i^{MS}, \tau_{i+1}^-]$.

We want to point out an important observation at the touch-down moment as shown in Fig. 1(b). Given any $p_2(i)$, the angle between stance and swing leg, defined as $\beta \coloneqq \hat{\theta} - \hat{\phi}$, can be obtained using the Law of Cosines:

$$\beta = \arccos\left(\frac{2l^2 - p_2(i)^2}{2l^2}\right), \tag{3}$$

which is independent of the states $\hat{\theta}$ and $\hat{\phi}$. Moreover, for all $p_2(i) \in [0.15, 0.7]$ considered in this work, $0 < \beta \leq \pi/3$.

By conservation of energy, we have:

$$\begin{align} 0.5(l \cdot \dot{\hat{\theta}}(t_i^{MS}))^2 + g(l - l\cos(\beta/2)) &= 0.5({\color{red}v_1})^2, \tag{4}\\ {\color{red}v_1'} &= {\color{red}v_1} \cdot \cos\beta, \tag{5}\\ 0.5({\color{red}v_1'})^2 - g(l - l\cos(\beta/2)) &= 0.5({\color{red}v_2})^2, \tag{6} \end{align}$$

where the unknowns are marked in red. Notice from (3) that $0 < \beta \leq \pi/3$ for all $p_2(i) \in [0.15, 0.7]$ considered in this work, therefore $\cos\beta > 0$. The solution to the system of equations (4)-(6) may or may not exists, depending on whether SBM eventually reaches the $(i+1)$-st mid-stance. These two cases are discussed separately as follows.

1. If $0.5(v_1')^2 - g(l - l\cos(\beta/2)) \geq 0$, the reader can solve for a positive v_2 from (4)-(6), and thus

$$\hat{y}_1(i+1) = v_2/l. \tag{7}$$

where v_2 is a function of $\hat{y}_1(i) = \dot{\hat{\theta}}(t_i^{MS})$. Furthermore, we have

$$\hat{y}_2(i) = \pi - \beta/2. \tag{8}$$

Notice that $\hat{\theta}(t) > \pi$, $\dot{\hat{\theta}}(t) > 0$ for all $t \in [\hat{t}_i^{MS}, \hat{\tau}_{i+1}^-]$, and $\hat{\theta}(t) < \pi$, $\dot{\hat{\theta}}(t) > 0$ for all $t \in [\hat{\tau}_{i+1}^-, \hat{t}_{i+1}^{MS}]$. We then have

$$\begin{align} \hat{y}_3(i) = \hat{\theta}(\hat{\tau}_{i+1}^-) = \hat{y}_2(i) = \pi - \beta/2, \tag{9}\\ \hat{y}_4(i) = \hat{\theta}(\hat{\tau}_{i+1}^-) = \pi + \beta/2. \tag{10} \end{align}$$

2. If $0.5(v_1')^2 - g(l - l\cos(\beta/2)) < 0$, then $\dot{\hat{\theta}}(t)$ becomes 0 at some time $\hat{t}_{i+1}^0$ before the $(i+1)$-st mid-stance is reached, and SBM may fall backward as in Fig. 2.

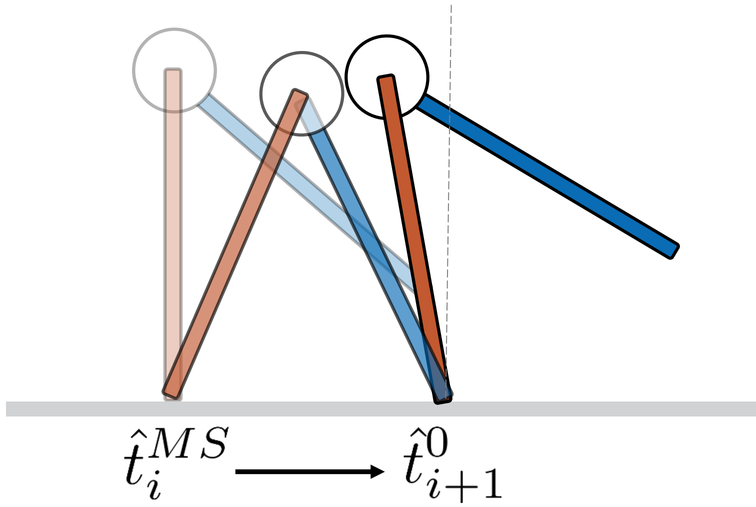

Figure 2: SBM fails to reach the $(i+1)$-th mid-stance.

By on conservation of energy, we have

$$0.5(v_1')^2 - g(l \cdot \cos(\hat{\theta}(\hat{t}_{i+1}^0)) - l\cos(\beta/2)) = 0, \tag{11}$$

where the unknown are again marked in red. Using (4), (5), and (11), one can compute that

$$\hat{y}_1(i+1) = -\sqrt{2gl\big(1-\cos(\hat{\theta}(\hat{t}_{i+1}^0))\big)}/l. \tag{12}$$

Again we have $\hat{y}_2(i) = \pi - \beta/2$. Since $\hat{\theta}(t) < \pi$ for all $t \geq \tau_{i+1}^+$, $t_i^{MS} = +\infty$, then we have $\hat{y}_3(i) = -\infty$ and $\hat{y}_4(i) = +\infty$.

Notice that β is actually a function of $p_2(i)$, one can then check that $\hat{y}_1(i+1)$ is essentially a function of $\hat{y}_1(i)$ and $p_2(i)$. Furthermore, since $\hat{y}_2(i) = \pi - \beta/2$, $\hat{y}_2(i)$ is then only a function of $p_2(i)$. Since the values of $y_3(i)$ and $y_4(i)$ depend on the positivity of $0.5(v_1')^2 - g(l - l\cos(\beta/2))$, then we can obtain the expressions for $y_3(i)$ and $y_4(i)$ both as functions of $\hat{y}_1(i)$ and $p_2(i)$.

References

[1] D. L. Wight, E. G. Kubica, and D. W. Wang, "Introduction of the foot placement estimator: A dynamic measure of balance for bipedal robotics," *Journal of computational and nonlinear dynamics*, vol. 3, no. 1, p. 011 009, 2008.