I3 – TD4

(Hypotheses Testing)

1. Let $X_1, X_2, \dots, X_{20}$ be a random sample from a distribution with probability mass function

$$f(x,p) = \begin{cases} p^x(1-p)^{1-x} & , if\ x = 0,1 \\ 0 & , Otherwis \end{cases}$$

Where $0 < p \le \frac{1}{2}$ is a parameter. The hypothesis $H_o : p = \frac{1}{2}$ to be tested against $H_a : p < \frac{1}{2}$ if H_o is rejected when $\sum_{i=1}^{20} X_i \le 6$, then what is the probability of type I error ?

Answer ~ 1

Find probability of type I error

We have $X_1, X_2, \dots, X_n \overset{iid}{\sim}$ Ber(p) where $0 < p < \frac{1}{2}$

And Hypotheses $H_o : p = \frac{1}{2}$ $\quad versus \quad H_a : p > \frac{1}{2}$

$$CR = \left\{ (x_1, \dots, x_{20}) : \sum_{i=1}^{20} x_i \le 6 \ \ that\ \ H_o\ is\ rejected \right\}$$

- R, C, RR or CR is notationally of **rejection region** or **critical region**.

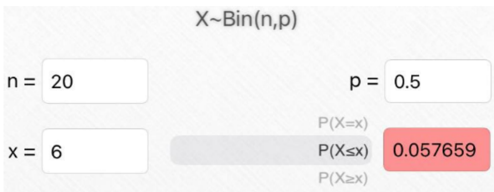
The probability of Type I Error denoted by α where

$$\alpha = \mathrm{P(\,Type\ I\ Error\,)} = \mathrm{P(Reject\ H_o | H_o\ is\ true\,)}$$

$$= P\left(\sum_{i=1}^{20} X_i \leq 6\right)$$

Since : $X \sim Ber\left(\frac{1}{2}\right) \Rightarrow \sum_{i=1}^{20} X_i \sim Bin\left(20, \frac{1}{2}\right)$

$$= \sum_{i=1}^{6} \binom{20}{k} p^k (1-p)^{20-k} = 0.0576$$

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> pbinom(6,20,1/2,lower.tail = T)
[1] 0.05765915
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$\boxed{\text{Therefore, } \alpha = 0.0576}$

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2. Let p represent the proportion of defectives in a manufacturing process. To test H_o : p $\leq \frac{1}{4}$ versus H_a : p $> \frac{1}{4}$, a random sample of size 5 is taken from the process. If the number of defectives is 4 or more, the null hypothesis is rejected. What is the probability of rejecting Ho if p $= \frac{1}{5}$

Answer ~ 2

Find the probability of rejecting H_o if p $= \frac{1}{5}$

Let X be a number of defectives

$$X \sim Bin\,(n, p) = Bin\left(5, \frac{1}{5}\right)$$

we have : H_o : p $\leq \frac{1}{4}$ versus H_a : p $> \frac{1}{4}$

Since it critical region : $CR = \{ x = defectives \geq 4 \}$

Then, the probability of type I Error is given by α where

$$\alpha = \text{P(Type I Error)} = \text{P(Reject H}_\text{o}|\text{H}_\text{o}\text{ is true)}$$

$$= P(\, X \geq 4\,)$$

$$= 1 - \sum_{i=1}^{3}\binom{5}{k}p^k(1-p)^{5-k}$$

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> 1-pbinom(3,5,1/5,lower.tail = T)
[1] 0.00672
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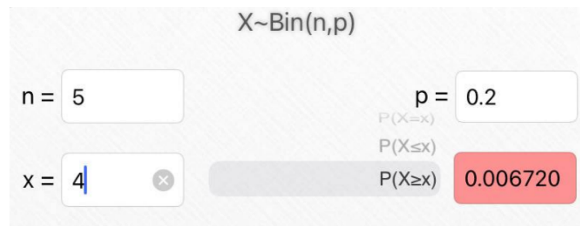

> Therefore, the probability of type I error is $\alpha = 0.0067$

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3. A random sample of size 4 is taken from a normal distribution with unknown mean μ and variance $\sigma^2 > 0$. To test H_o : μ = 0 against Ha : μ < 0 the following test is used: "Reject H_o if and only if $X_1 + X_2 + X_3 + X_4 < -20$."
Find the value of σ so that the significance level of this test will be closed to 0.14.

Answer ~ 3

Find the value of σ so that the significance level of this test will be closed to 0.14

We have : $X_1, \dots, X_4 \overset{\text{iid}}{\sim} N(\,\mu, \sigma\,)$

The hypotheses H_o : μ = 0 against Ha : μ < 0

Then, the probability of type I Error is given by α where

$$\alpha = \text{P(Type I Error)} = \text{P(Reject } H_o | H_o \text{ is true)} = 0.14$$

$$= P\ (\ X_1 + X_2 + X_3 + X_4 < -20\ |\ \mu = 0\ ) = 0.14$$

Since, it iid (independent identical distribution)

$$\Longrightarrow \alpha = P\ (\ 4\bar{X} < -20 |\ \mu = 0\ ) = 0.14$$

$$= P\ (\ \bar{X} < \ -\frac{20}{4} | \mu = 0\ ) = 0.14$$

$$= P\ (\ \bar{X} < -5\ ) = 0.14$$

Where $\bar{X} \sim N\ \left(0\ , \frac{\sigma^2}{n}\right) \sim \left(0, \frac{\sigma^2}{4}\right)$

$$P\ (\bar{X} < a\ ) = \phi\ \left(\frac{a-\mu}{\sigma}\right)$$

$$= \phi\ \left(-\frac{5}{\sigma}\right) = 0.14$$

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> qnorm(0.14)
[1] -1.080319
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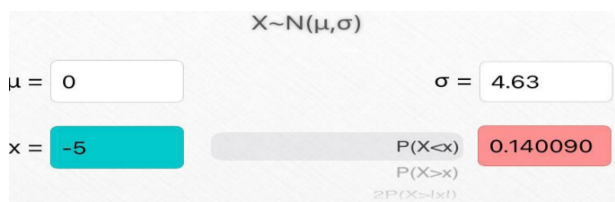
$$\Longrightarrow \quad -\frac{5}{\sigma} \ = \ -1.08$$

$$\Longrightarrow \quad \sigma \quad = 4.629$$

$$\boxed{\text{Therefore, } \sigma = 4.63}$$



4. A normal population has a standard deviation of 16. The rejection region for testing $H_o : \mu = 5$ versus the alternative $H_a : \mu = k$ is $\overline{X} > k - 2$. What would be the value of the constant k and the sample size n which would allow the probability of Type I error to be 0.0228 and the probability of Type II error to be 0.1587 ?

Answer ~ 4

Find the value of the constant k and the sample size n

We have : $X \sim N(\mu, 16^2)$

The hypotheses $H_o : \mu = 5 \quad vs \quad H_a : \mu = k : \bar{X} > k - 2$

$$CR = \{(x_1, \dots, x_n) : \bar{x} > k - 2\} \; where \; \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

When $H_0 : \mu = 5$ is true, then $\bar{X} \sim N\left(\frac{5, 16^2}{n}\right)$

$$\begin{cases} \alpha = P(\,Reject\ H_o\ |H_o\ is\ true\,) \\ \beta = P(\,Accept\ H_o|H_a\ is\ true\,) \end{cases}$$

$$\Leftrightarrow \begin{cases} 0.0228 = P\,(\bar{X} > k - 2\,), \bar{X} \sim N\left(5, \frac{16^2}{n}\right) \\ 0.1587 = P\,(\,\bar{X} \le k - 2\,), \bar{X} \sim N\left(k, \frac{16^2}{n}\right) \end{cases}$$

$$\Leftrightarrow \begin{cases} 0.0228 = 1 - \phi\left(\frac{k-2-5}{\frac{16}{\sqrt{n}}}\right) \\ 0.1587 = \phi\left(\frac{k-2-5}{\frac{16}{\sqrt{n}}}\right) \end{cases}$$

$$\Leftrightarrow \begin{cases} \phi\left(\frac{k-2-5}{\frac{16}{\sqrt{n}}}\right) = 1 - 0.0228 \\ \phi\left(-\frac{\sqrt{n}}{8}\right) = 0.1587 \end{cases}$$

$\phi(-a) = 1 - \phi(a)$

$$\Leftrightarrow \begin{cases} \phi\left(\frac{\sqrt{n}(k-7)}{16}\right) = 1 - 0.0228 = 0.9772 \\ \phi\left(\frac{\sqrt{n}}{8}\right) = 1 - 0.1587 = 0.8413 \end{cases}$$

```
> qnorm(0.9772)
[1] 1.999077
> qnorm(0.8413)
[1] 0.9998151
```

$$\Leftrightarrow \begin{cases} \left(\frac{\sqrt{n}(k-7)}{16}\right) = 1.99 \approx 2 \\ \left(\frac{\sqrt{n}}{8}\right) = 0.99 \approx 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} k = \frac{2(16)}{8} + 7 = 11 \\ \sqrt{n} = 8 \Longrightarrow n = 64 \end{cases}$$

Therefore, $k = 11$ & $n = 64$

5. Let $X_1, X_2, \dots, X_{25}$ be a random sample of size 25 drawn from a normal distribution with unknown mean μ and variance σ^2 = 100. It is desired to test the null hypothesis H_o : μ = 4 against the alternative H_a : μ = 6. What is the power at μ = 6 of the test

with rejection rule: reject μ = 4 if $\sum_{i=1}^{25} X_i \geq 125$?

Answer~5

What is the power at μ = 6 of the test ?

with rejection rule: reject μ = 4 if $\sum_{i=1}^{25} X_i \geq 125$

The sampling distribution of $\bar{X}$ is normal with mean $\mu = 6$ & $variance : \frac{\sigma^2}{n} = \frac{100}{25} = 4$

$\bar{X} \sim (6, 4)$

The hypotheses $H_o : \mu = 4 \; vs \; H_a : \mu = 6$

$$CR = \left\{ (x_1, \dots, x_{25}) : \sum_{i=1}^{25} X_i \geq 125 \right\}$$

We have : $\pi\,(6) = P(Reject\ H_0 | \mu = 6\,)\ is\ a\ true\ power\ at\ \mu = 6$

$$= P\,(\textstyle\sum_{i=1}^{25} X_i \geq 125 | \mu = 6\,)$$

$$= P\,(\bar{X} \geq 5\,)$$

$$= 1 - \phi\left(\frac{5-6}{2}\right)$$

$P(X \geq b) = 1 - \Phi\left(\frac{b-\mu}{\sigma}\right)$

$$= 1 - \phi(-0.5)$$

-0.5	.30854

$$= 1 - 0.30854$$

Therefore, the true power of the test at $\mu = 6$ is 0.69146

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> 1-pnorm(-0.5)
[1] 0.6914625
```

Method2

$$X_1, X_2, \dots, X_{25} \overset{iid}{\sim} N(\mu, \sigma^2), \sigma^2 = 100 \text{ , then}$$

$$T_o = \sum_{i=1}^{25} X_i \sim N(25\mu, 25\sigma^2) \sim N(25 \times 6, 25 \times 100) \sim N(150, 2500)$$

So, $\pi(6) = P(\,Reject\ H_o | \mu = 6\,)$

$$= P(T_o \geq 125)$$

$$= 1 - P(T_o < 125)$$

$$= 1 - \phi\left(\frac{125 - 150}{50}\right)$$

$$= 1 - \phi(-0.5)$$

$$= 1 - 0.30854$$

-0.5	.30854

```
> 1-pnorm(-0.5)
[1] 0.6914625
```

Therefore, the true power of the test at $\mu = 6$ is 0.69146

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6. A urn contains 7 balls, θ of which are red. A random sample of size 2 is drawn without replacement to test $H_o: \theta \leq 1$ against $H_a: \theta > 1$. If the null hypothesis is rejected if one or more red balls are drawn, find the power of the test when θ = 2.

Answer~6

- $R_{red\ ball} = \theta$
- $\bar{R} : 7 - \theta$

Population

→ 2 balls is drawn without replacement

n = 2

Sample

Let $x =$ the number of red balls in the sample. since, drawn without replacement

$$\Longrightarrow X \sim Hypergeometric\,(n, M, N) \quad \text{where} \quad \begin{cases} n = 2 \text{ (sample)} \\ M = \theta \text{ (successes)} \\ N = 7 \text{ (total)} \end{cases}$$

Find the power of test when $\theta = 2$

$$\pi(2) = P(\,Reject\ H_o | \theta = 2\,)$$

$$= P(X \geq 1)\ ,\ X \sim Hyp\,(n, M, N) \quad \text{where} \quad \begin{cases} n = 2 \\ M = 2 \\ N = 7 \end{cases}$$

$$= 1 - P(X = 0)$$

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> 1-dhyper(0,2,5,2)
[1] 0.5238095
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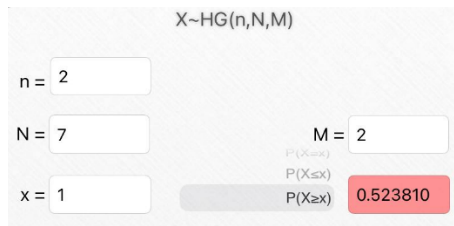

Therefore, $\pi(2) = 0.5238$

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7. Let $X_1, X_2, \dots, X_8$ be a random sample of size 8 from a Poisson distribution with Parameter λ. Reject the null hypothesis H_o : λ = 0.5 if the observed sum $\sum_{i=1}^{8} x_i \geq 8$ First, compute the significance level α of the test. Second, find the power function π(λ) of the test as a sum of Poisson probabilities when H_a is true.

Solution~7

Complete the significance level α of the test

We have : $X_1, \dots, X_8 \overset{iid}{\sim} Poi(\lambda)$, then

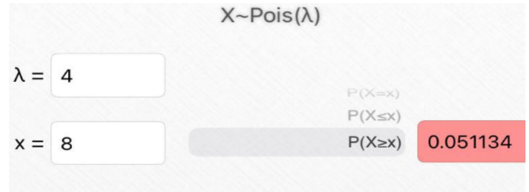
$$T_o = \sum_{i=i}^{n} X_i \sim Poi(n\lambda)$$

$$= \sum_{i=1}^{8} X_i \sim Poi(8 \times 0.5) \sim Poi(4)$$

- $\alpha \; = P\;(Reject\;H_o|H_o\;is\;true\;)$

  $= P\;(\;T_o \geq 8\;) \quad , T_o \sim Poi(4)$

  $= 1 - P(T_o \leq 7)$



Therefore, $\alpha = 0.051$

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> ppois(q=7,lambda=4,lower.tail = F)
[1] 0.05113362
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Find the power function $\pi(\lambda)$ of the test as a sum of Poisson probability when H_a is true

$k(\lambda) = \pi(\lambda) = P\;(Reject\;H_o|H_a\;\;is\;true\;)$

$= P(\;T_o \geq 8\;), \qquad T_o \sim Poi(8\lambda)\;,\;\lambda \neq \frac{1}{2}\;,$ because H_a true $\Longrightarrow \lambda_{H_a} \neq \lambda_{H_o}$

$= 1 - P(T_o \leq 7)$

$= 1 - \sum_{t=0}^{7} \frac{e^{-8\lambda}(8\lambda)^t}{t!}$

$$\frac{e^{-\lambda}\lambda^x}{x!}$$

Therefore, $\pi(\lambda) = 1 - \sum_{t=0}^{7} \frac{e^{-8\lambda}(8\lambda)^t}{t!}$

8. Let $X_1, X_2, \dots, x_n$ be a random sample from $N(0, \sigma^2)$
   (a) Show that RR = $\{ (x_1, x_2, \dots, x_n) : \sum_{i=1}^{n} x_i^2 \geq c \}$ is a best rejection region for testing $H_o : \sigma^2 = 4$ against $H_a : \sigma^2 = 16$

   (b) If n = 15, find the value of c so that α = 0.05. [Hint: Recall that $\sum_{i=1}^{n} \frac{X_i^2}{\sigma^2}$ is for $\chi^2(n)$]

   (c) If n = 15 and c is the value found in part (b), find the approximate value of $\beta = P\left(\sum_{i=1}^{n} X_i^2 < c \,|\sigma^2 = 16 \right)$

Solution~8

(a) Show that RR = $\{ (x_1, x_2, \dots, x_n) : \sum_{i=1}^{n} x_i^2 \geq c \}$ is a best rejection region for testing $H_o : \sigma^2 = 4$ against $H_a : \sigma^2 = 16$

Recall : if $X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2)$, *then*

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1) \quad \text{and} \quad \chi^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2(n)$$

We have : $X_1, X_2, \dots, X_n \sim N(0, \sigma^2)$, *then*

$$\chi^2 = \sum_{i=1}^{n} \frac{X_i^2}{\sigma^2} \sim \chi^2(n)$$

Given that : $H_0 : \sigma^2 = 4 \quad vs \quad H_a : \sigma^2 = 16$

➢ By using Neyman – pearson lemma (NPL)

- $L(\sigma^2) = \prod_{i=1}^{n} f(x_i, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i - 0)^2}$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} . e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2}$$

- $\frac{L(\sigma_0^2)}{L(\sigma_a^2)} = \frac{L(4)}{L(16)} = \frac{(8\pi)^{-\frac{n}{2}} . e^{-\frac{1}{8}\sum_{i=1}^{n} x_i^2}}{(32\pi)^{-\frac{n}{2}} . e^{-\frac{1}{32}\sum_{i=1}^{n} x_i^2}}$

$$= (4\pi)^{\frac{n}{2}} . e^{\left(\frac{1}{32}-\frac{1}{8}\right)\sum_{i=1}^{n} x_i^2}$$

$$= (4\pi)^{\frac{n}{2}} . e^{-\frac{3}{32}\sum_{i=1}^{n} x_i^2}$$

- For $K > 0$ and $\forall(x_1, x_2, \dots, x_n) \in RR$, *by NPL* , we have

$$\frac{L(\sigma_o^2)}{L(\sigma_a^2)} \le k \iff (4\pi)^{\frac{n}{2}} . e^{-\frac{3}{32}\sum_{i=1}^{n} x_i^2} \le k$$

$$\iff -\frac{3}{32}\sum_{i=1}^{n} x_i^2 \le \ln\left(\frac{k}{(4\pi)^{\frac{n}{2}}}\right)$$

$$\iff \sum_{i=1}^{n} x_i^2 \ge -\frac{32}{3}\ln\left(\frac{k}{(4\pi)^{\frac{n}{2}}}\right) = C$$

So, $RR = \{ (x_1, x_2, \dots x_n) : \sum_{i=1}^{n} x_i^2 \ge c \}$ where C is constant defined by α

$$\alpha = P(Reject\ H_0 | H_0\ is\ true\ )$$

$$= P\ (\sum_{i=1}^{n} x_i^2 \ge c\ ),\ where\ \chi^2 = \sum_{i=1}^{n} \frac{X_i^2}{4} \sim \chi(n)$$

$$= P\left(\chi^2 \ge \frac{C}{4}\right)$$

$$\Longrightarrow C = 4\chi^2_{\alpha,n}$$

(b) If n = 15, find the value of c so that α = 0.05. [Hint: Recall that $\sum_{i=1}^{n} \frac{X_i^2}{\sigma^2}$ is for $\chi^2(n)$]

From (a) ,

we have $C = 4\chi^2_{0.05,15}$

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> 4*qchisq(0.05,15,lower.tail = F)
[1] 99.98316
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Therefore, $C = 99.983$

(c) If n = 15 and c is the value found in part (b), find the approximate value of
$\beta = P\left(\sum_{i=1}^{n} X_i^2 < c \mid \sigma^2 = 16\right)$

We have : $\beta = P(Accept\ H_o | H_a\ is\ true\ )$

$$= P\left(\sum_{i=1}^{n} X_i^2 < C \mid \sigma^2 = 16\right)$$

$$= P\left(\chi^2 < \frac{C}{16}\right), \qquad \chi^2 \sim \chi^2(n)$$

$$= P\left(\chi^2 < \frac{99.983}{16}\right), \qquad \chi^2 \sim \chi^2(15)$$

```
> pchisq(99.983/16,15,lower.tail = T)
[1] 0.0247441
```

Therefore, $\beta = 0.02474$

9. Let X have a Pareto distribution with parameter θ > 0; that is, the pdf of X is

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} x^{\left(-\frac{1}{\theta}-1\right)} & x > 1 \\ 0 & \text{Otherwise} \end{cases}$$

Let $X_1, X_2, \dots X_n$ be a random sample from this distribution.

(a) Let $Y_n = \frac{2}{\theta}\sum_{i=1}^{n} lnX_i$. Show that Y_n has chi – squared distribution with Degree of freedom $2n$ ($that\ is, Y_n \sim \chi^2(2n)$)

(b) Using Neyman-Pearson lemma, show that the best critical region for testing $H_0 : \theta = \theta_o$ against $H_a : \theta = \theta_a$, $\theta_a > \theta_0 > 0$, at level of test α, is

$$RR = \{ (x_1, \dots, x_n) : \sum_{i=1}^{n} \ln x_i \geq c \}$$

where c satisfies $P\left(Y_n \geq \frac{2n}{\theta_o}\right) = \alpha$.

(c) Is the above critical region RR is uniformly most powerful for testing $H_0 : \theta = \theta_0$ against $H_a : \theta > \theta_0$ at level of test α? Justify your answer.

(d) If n = 12, α = 0.10, $H_0 : \theta = 3$ and $H_a : \theta = 5$. Determine the critical region RR.

Solution~9

(a) Let $Y_n = \frac{2}{\theta}\sum_{i=1}^{n} lnX_i$. Show that Y_n has chi – squared distribution with Degree of freedom $2n$ ($that\ is, Y_n \sim \chi^2(2n)$)

We have :

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} x^{\left(-\frac{1}{\theta}-1\right)} & x > 1 \\ 0 & \text{Otherwise} \end{cases}$$

Find mgf of Y_n

We have :

$$M_{Y_n}(t) = E(e^{tY_n})$$

$$= E\left(e^{\frac{2t}{\theta}\sum_{i=1}^{n} lnX_i}\right)$$

$$= E\left(e^{\frac{2t}{\theta} lnX_1}\right) \times E\left(e^{\frac{2t}{\theta} lnX_2}\right) \times \ldots.\times E\left(e^{\frac{2t}{\theta} lnX_n}\right)$$

$$= \left[E\left(e^{\frac{2t}{\theta} lnX}\right)\right]^n$$

$$= \left[M_{lnX}\left(\frac{2t}{\theta}\right)\right]^n$$

We have

$$M_{lnX}(t) = E\left(e^{tlnX}\right)$$

$$= E(X^t)$$

$$= \int_1^{\infty} x^t f(x,\theta)\, dx$$

$$= \int_1^{\infty} \frac{x^t}{\theta} x^{-\frac{1}{\theta}-1}\, dx$$

$$= \int_1^{\infty} \frac{1}{\theta} x^{-\frac{1}{\theta}-1+t}\, dx$$

$$= \frac{1}{1-t\theta}$$

So, $\quad M_{Y_n}(t) = \left[\frac{1}{1-t\theta}\right]^n = (1 - t\theta)^{-n}$

Therefore, $Y_n \sim \chi^2(2n)$

(b) Using Neyman-Pearson lemma, show that the best critical region for testing $H_0 : \theta = \theta_o$ against $H_a : \theta = \theta_a$, $\theta_a > \theta_0 > 0$, at level of test α, is

$$RR = \{ (x_1, \dots, x_n) : \sum_{i=1}^{n} \ln x_i \geq c \}$$

where c satisfies P $\left(Y_n \geq \frac{2n}{\theta_o}\right)$ = α.

By Neymar – pearson lemma, we have

$$RR = \left\{(x_1, \dots., x_n) : \frac{L(\theta_o)}{L(\theta_a)} \leq k \right\}$$

We have

$$L(\theta) = \frac{1}{\theta^n} \prod_{i=1}^{n} x_i^{-1-\frac{1}{\theta}}$$

Then,

$$\frac{L(\theta_o)}{L(\theta_a)} = \left(\frac{\theta_a}{\theta_o}\right)^n \prod_{i=1}^{n} x^{\frac{1}{\theta_a} - \frac{1}{\theta_a}} \leq k$$

$$= \quad \prod_{i=1}^{n} x^{\frac{1}{\theta_a} - \frac{1}{\theta_a}} \quad \leq \quad k \left(\frac{\theta_a}{\theta_o}\right)^n$$

$$= \ln \prod_{i=1}^{n} x^{\frac{1}{\theta_a} - \frac{1}{\theta_a}} \quad \leq \quad lnk \left(\frac{\theta_a}{\theta_o}\right)^n$$

$$= \frac{1}{\theta_a} - \frac{1}{\theta_o} \sum_{i=1}^{n} lnx_i \quad \leq \quad lnk \left(\frac{\theta_a}{\theta_o}\right)^n$$

$$= \quad \sum_{i=1}^{n} lnx_i \quad \geq \quad \frac{\theta_0 \theta_a}{\theta_0 - \theta_a} lnk \left(\frac{\theta_a}{\theta_o}\right)^n$$

Let $c \; = \; \frac{\theta_0 \theta_a}{\theta_0 - \theta_a} lnk \left(\frac{\theta_a}{\theta_o}\right)^n$

Therefore, $RR = \{ (x_1, \dots, x_n) : \sum_{i=1}^{n} \ln x_i \geq c \}$ is the best critical region

$P(\,Reject\ H_o|H_o) = \alpha$

$$P\left(\sum_{i=1}^{n} lnX_i \geq c\ |\theta_0)\right) = \alpha$$

$P\left(\frac{2}{\theta_0}\sum_{i=1}^{n} lnX_i \geq \frac{2c}{\theta_0}\right) = P\left(Y_n \geq \frac{2c}{\theta_0}\right)$

Since, $Y_n \sim \chi^2(2n)$

Then, $\frac{2c}{\theta_0} = \chi^2_{2n,\alpha}$

Therefore, $c = \frac{\theta_0}{2}\chi^2_{2n,\alpha}$

(c) Is the above critical region RR is uniformly most powerful for testing $H_0 : \theta = \theta_0$ against Ha : θ > θ0 at level of test α? Justify your answer.

Since, RR is not depend on θ_a , $\forall\ \theta_a > \theta_0 > 0\ , then$

This RR is the UMP rejection region for $\begin{cases} H_0 : \theta = \theta_o \\ H_a : \theta > \theta_0 \end{cases}$

(d) If $n = 12,\ \ \alpha = 0.10,\ \ H_0 : \theta = 3\ \ and\ \ H_a : \theta = 5$

Determine the critical region RR.

We have $RR = \left\{ (x_1, \dots, x_n) :\ \sum_{i=1}^{n} lnx_i \geq c = \frac{\theta_0}{2}\chi^2_{2n,\alpha} \right\}$

Then, $RR = \left\{ (x_1, \dots, x_n) :\ \sum_{i=1}^{n} lnx_i \geq c = \frac{3}{2}\chi^2_{24\,,0.1} \right\}$

Therefore, $RR = \{\, (x_1, \dots, x_n) :\ \sum_{i=1}^{n} lnx_i \geq 49.794\, \}$

```
> 3/2*qchisq(0.10,24,lower.tail = F)
[1] 49.79437
```

11. Let X_1, X_2, X_3 denote a random sample of size 3 from a population X with probability

mass function

$$f(x,\theta) = \begin{cases} \frac{e^{-\theta}\theta^{x}}{x!} & , if\ x = 0,1,2,3,\ldots \\ 0 & , Otherwise \end{cases}$$

where θ > 0 is a parameter. What is the likelihood ratio critical region for testing

$H_o : \theta = 0.1 \;\; versus \;\; H_a : \theta \neq 0.1$

Solution~11

- $L(\theta) = \prod_{i=1}^{3} f(x_i\,,\theta) = \prod_{i=1}^{3} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-3}\theta^{\sum^3 x_i}}{\prod_{i=1}^{3} x_i!}$

- $\frac{L(\hat{\theta}_o)}{L(\hat{\theta})} = \frac{L(0.1)}{L(\bar{x})}$

$$= \frac{e^{-3(0.1)}(0.1)^{\sum^3 x_i}}{e^{-3\bar{x}}.\,\bar{x}^{\sum^3 x_i}}$$

$$= e^{3(\bar{x}-0.1)}\bar{x}^{-\sum^3 x_i}(0.1)^{\sum^n x_i}$$

$$= e^{3\bar{x}}\, e^{-0.3}(\bar{x})^{-3\bar{x}}(0.1)^{\sum^n x_i}$$

$$= e^{3\bar{x}}\left(\frac{0.1}{\bar{x}}\right)^{3\bar{x}} e^{-0.3}$$

- For $k \in (0,1), and\ \forall (x_1, x_2, x_3) \in RR$, $we\ have$

$$\frac{L(\hat{\theta}_o)}{L(\hat{\theta})} \leq k \Leftrightarrow e^{3\bar{x}}\left(\frac{0.1}{\bar{x}}\right)^{3\bar{x}} \leq e^{0.3}k = C$$

Therefore, $RR = \left\{ (x_1, x_2, x_3) : e^{3\bar{x}}\left(\frac{0.1}{\bar{x}}\right)^{3\bar{x}} \leq C \right\}$

---

13. The melting point of each of 16 samples of a certain brand of hydrogenated vegetable oil was determined, resulting in $\bar{x} = 94.32$. Assume that the distribution of the melting point is normal with σ = 1.20.

(a) Test H_0 : μ = 95 versus H_a : μ ≠ 95 using a two-tailed level 0.01 test.
(b) If a level 0.01 test is used, what is $\beta(94)$, the probability of a type II error when μ = 94?
(c) What value of n is necessary to ensure that $\beta(94) = 0.1$ when $\sigma = 0.1$?

Solution~13

We have : $\begin{cases} n = 16 \\ \bar{x} = 94.32 \\ \sigma = 1.2 \end{cases}$

(a)) Test H_0 : μ = 95 versus H_a : μ ≠ 95 using a two-tailed level 0.01 test.

Given : $\begin{cases} H_0 : \mu = 95 \\ H_a : \mu \neq 95 \\ \alpha = 0.01 \end{cases}$

- Test statistic value

since, the parameter $is\ \sigma$, so we use Z to test

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$= \frac{94.32-95}{\frac{1.2}{\sqrt{16}}} = -2.266$$

```
> (94.32-95)/(1.2/sqrt(16))
[1] -2.266667
```

- $RR = \left\{ Z : |Z| \geq Z_{\frac{\alpha}{2}} = \phi^{-1}\left(1-\frac{\alpha}{2}\right) = 2.575 \right\}$

```
> qnorm(1-0.01/2)
[1] 2.575829
```

Since, $|Z| = 2.266 < 2.575 \Longrightarrow Z \notin RR$, So H_0 is not rejected

Therefore, H_0 is not rejected

Method2 : We use P – value method

- Test statistic is $Z = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$

  when H_0 is true, $Z = \frac{\bar{X}-\mu_o}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$

- The value of the test statistic is $Z = -2.266$

- $P - value = 2\,[\,1 - \phi(|Z|)]$

$$= 2\,[\,1 - \phi(2.266)\,]$$

```
> qnorm(0.98809)
[1] 2.26002
```

$$= 2\,[\,1 - 0.98809\,]$$

$$= 0.02382$$

- Since, $\alpha = 0.01 < 0.0238 \Longrightarrow P - value > \alpha$

Therefore, H_0 is not rejected

14. The desired percentage of SiO2 in a certain type of aluminous cement is 5.5. To test whether the true average percentage is 5.5 for a particular production facility, 16 independently obtained samples are analyzed. Suppose that the percentage of SiO2 in a sample is normally distributed with σ = 0.3 and that .x = 5.25.

(a) Does this indicate conclusively that the true average percentage differs from 5.5?

(b) If the true average percentage is μ = 5.6 and a level α = 0.01 test based on n = 16 is used, what is the probability of detecting this departure from H_o?

(c) What value of n is required to satisfy α = 0.01 and β(5.6) = 0.01 ?

Solution~14

(a) Find the true average differ from 5.5

We have: $\begin{cases} \bar{x} = 5.25 \\ n = 16 \\ \sigma = 0.3 \\ \mu_0 = 5.5 \end{cases}$

Test $H_0 : \mu = 5.5$ vs $H_a : \mu \neq 5.5$

- Test statistic

$$Z = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}, \text{ under } H_0 \Rightarrow Z = \frac{\bar{X}-\mu_0}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

- Test statistic value

$$Z = \frac{5.25+5.5}{\frac{0.3}{\sqrt{16}}} = -3.33$$

```
> Z=(5.25-5.5)/(0.3/sqrt(16))
> Z
[1] -3.333333
> qnorm(1-0.01/2)
[1] 2.575829
```

- Critical region

$$RR = \left\{Z: |Z| \geq Z_{\frac{\alpha}{2}} = \phi^{-1}\left(1 - \frac{\alpha}{2}\right) = 2.575\right\}$$

Since: $|Z| = 3.3333 \geq 2.575 \Rightarrow Z \in RR$

So , H_0 is rejected at $\alpha = 0.01$

Therefore, There is enough evidence to support the claim that $\mu \neq 5.5$

(b) If the true average percentage is $\mu = 5.6$ and a level $\alpha = 0.01$ test based on $n = 16$ is used, what is the probability of detecting this departure from H_0 ?

We have: $\begin{cases} \mu' = 5.6 \\ \mu_0 = 5.5 \\ \alpha = 0.01 \end{cases}$

$$\pi(\mu') = 1 - \beta(\mu')$$

$$Since,\ H_a : \mu \neq \mu_o \Longrightarrow \beta(\mu') = \phi\left(Z_{\frac{\alpha}{2}} + \frac{\mu_0 - \mu'}{\frac{\sigma}{\sqrt{n}}}\right) + \phi\left(-Z_{\frac{\alpha}{2}} + \frac{\mu_0 - \mu'}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$\Longrightarrow \pi(\mu') = 1 - \phi\left(Z_{\frac{\alpha}{2}} + \frac{\mu_0 - \mu'}{\frac{\sigma}{\sqrt{n}}}\right) + \phi\left(-Z_{\frac{\alpha}{2}} + \frac{\mu_0 - \mu'}{\frac{\sigma}{\sqrt{n}}}\right) = 0.1063$$

(By រូបមន្តដាក់ខាងក្រោម All bruhh បើមិនយល់ទៀតដឹងយាយមិចទេ)

```
R 4.1.1 · ~/
> 1-pnorm(2.58+(5.6-5.5)/(0.3/sqrt(16)))+pnorm(-2.58+(5.6-5.5)/(0.3/sqrt(16)))
[1] 0.1063054
```

Therefore, The probability of detecting this departure from H_o is about 0.1063

β and Sample Size Determination

Alternative Hypothesis	$\beta(\mu')$ for a Level α Test
$H_a : \mu > \mu_0$	$\Phi\left(z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$
$H_a : \mu < \mu_0$	$1 - \Phi\left(-z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$
$H_a : \mu \neq \mu_0$	$\Phi\left(z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right) - \Phi\left(-z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$

where $\Phi(z)$=the standard normal cdf.

(c) Find value of n

we have : $\begin{cases} \alpha = 0.01 \\ \beta(5.6) = 0.01 \Longrightarrow Z_\beta = 2.33 \end{cases}$

$$n = \left[\frac{\sigma\left(Z_{\frac{\alpha}{2}} + Z_\beta\right)}{\mu_o - \mu'}\right]^2 = \left[\frac{0.3\ (2.575 + 2.33)}{5.5 - 5.6}\right]^2$$

β and Sample Size Determination

The sample size n for which a level α test also has at the $\beta(\mu') = \beta$ at the alternative value μ' is

$$n = \begin{cases} \left[\frac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu'}\right]^2 & \text{for a one-tailed (lower or upper) test} \\ \left[\frac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu'}\right]^2 & \text{for a two-tailed test (an proximation solution)} \end{cases}$$

15. The article "Uncertainty Estimation in Railway Track Life- Cycle Cost" (J. of Rail and Rapid Transit, 2009) presented the following data on time to repair (min) a rail break in the high rail on a curved track of a certain railway line.

159 120 480 149 270 547 340 43 228 202 240 218

A normal probability plot of the data shows a reasonably linear pattern, so it is plausible that the population distribution of repair time is at least approximately normal. The sample mean and standard deviation are 249.7 and 145.1, respectively.

(a) Is there compelling evidence for concluding that true average repair time exceeds 200 min? Carry out a test of hypotheses using a significance level of 0.05.

(b) Using σ = 150, what is the type II error probability of the test used in (a) when true average repair time is actually 300 min? That is, what is β(300)?

Solution~15

(a) Test $H_o: \mu = 200 \ vs \ H_a: \mu > 200, \alpha = 0.05$

- Test statistic

$$T = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}, under\ H_o \longrightarrow T = \frac{\bar{X} - \mu_o}{\frac{S}{\sqrt{n}}} \sim t_{\alpha,n-1}$$

- Test statistic value

$$t = \frac{\bar{X}-\mu_o}{\frac{S}{\sqrt{n}}} = \frac{249.7-200}{\frac{145.1}{\sqrt{12}}} = 1.1865$$

```
> (249.7-200)/(145.1/sqrt(12))
[1] 1.186532
```

- $RR = \left\{t: t \geq t_{\alpha,n-1} = t_{0.05,11} = 1.796\right\}$

```
> qt(0.05,11,lower.tail = F)
[1] 1.795885
```

Since: $t = 1.186 < 1.796 \implies t \notin RR$

H_o is not rejected at $\alpha = 0.05$. There is not enough evidence to support the claim that true average repair time exceed 200 min

Recall :

Alternative Hypothesis $\qquad \Phi\left(z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$

$H_a : \mu > \mu_0$

(b) Find β when $\mu' = 300$

We have : $\begin{cases} \mu = 200 \\ \mu' = 300 \\ \sigma = 150 \\ n = 12 \\ \alpha = 0.05 \Longrightarrow \phi^{-1}(1-\alpha) = 1.645 \end{cases}$

```
> qnorm(1-0.05)
[1] 1.644854
```

Since, $H_a : \mu > 200$ where, $\mu_o = 200$

By Alternative Hypothesis $H_a : \mu > \mu_o$

$$\Longrightarrow \beta(\mu') = \phi\left(Z_\alpha + \frac{\mu_o - \mu'}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$= \phi\left(1.645 + \frac{200-300}{\frac{150}{\sqrt{12}}}\right)$$

```
> pnorm(1.645 + (200-300)/(150/sqrt(12)))
[1] 0.2532168
```

Therefore, $\beta(300) = 0.2532$

16. Given the accompanying sample data on expense ratio (%) for large-cap growth Mutual funds:

0.52	1.06	1.26	2.17	1.55.	0.99	1.10	1.07	1.81	2.05
0.91	0.79	1.39	0.62	1.52	1.02	1.10	1.78.	1.01	1.15

A normal probability plot shows a reasonably linear pattern.

(a) Is there compelling evidence for concluding that the population mean expense ratio exceeds 1%? Carry out a test of the relevant hypotheses using a significance level of 0.01.

(b) Referring back to (a), describe in context type I and II errors and say which error you might have made in reaching your conclusion. The source from which the data was obtained reported that $\mu = 1.33$ for the population of all 762 such funds. So did you actually commit an error in reaching your conclusion?

(c) Supposing that $\sigma = 0.5$, determine and interpret the power of the test in (a) for the actual value of μ stated in(b).

Solution~16

17. A random sample of 50 measurements resulted in a sample mean of 62 with a Sample standard deviation 8. It is claimed that the true population mean is at least 64.

(a) Is there sufficient evidence to refute the claim at the 2% level of significance?

(b) What is the p-value?

(c) What is the smallest value of α for which the claim will be rejected?

Solution~17

18. A random sample of 78 observations produced the following sums:

$$\sum_{i=1}^{78} x_i = 22.8 \, , \sum_{i=1}^{78} (x_i - \bar{x})^2 = 2.05$$

(a) Test the null hypothesis that μ = 0.45 against the alternative hypothesis that μ < 0.45 using α = 0.01. Also find the p-value.

(b) Test the null hypothesis that μ = 0.45 against the alternative hypothesis that μ ≠ 0.45 using α = 0.01. Also find the p-value.

(c) What assumptions did you make for solving (a) and (b)?

Solution~18

19. A common characterization of obese individuals is that their body mass index is at least 30 [BMI = $\frac{weight}{(height)^2}$, where height is in meters and weight is in kilograms]. The article " The Impact of Obesity on Illness Absence and Productivity in an Industrial Population of Petrochemical Workers" (Annals of Epidemiology, 2008: 8–14) reported that in a sample of female workers, 262 had BMIs of less than 25, 159 had BMIs that were at least 25 but less than 30, and 120 had BMIs exceeding 30. Is there compelling evidence for concluding that more than 20% of the individuals in the sampled population are obese ?

Solution~19

20. A manufacturer of nickel-hydrogen batteries randomly selects 100 nickel plates for Test cells, cycles them a specified number of times, and determines that 14 of the Plates have blistered.

(a) Does this provide compelling evidence for concluding that more than 10% of all plates blister under such circumstances? State and test the appropriate hypotheses
using a significance level of 0.05. In reaching your conclusion, what type of error might you have committed?

(b) If it is really the case that 15% of all plates blister under these circumstances and a sample size of 100 is used,how likely is it that the null hypothesis of part (a) will not be rejected by the level 0.05 test? Answer this question for a sample size of 200.

(c) How many plates would have to be tested to have $\beta(0.15) = 0.10$for the test of part (a)?

Solution~20

21. An experiment to compare the tension bond strength of polymer latex modified Mortar (Portland cement mortar to which polymer latex emulsions have been added During mixing) to that of unmodified mortar resulted in $\bar{x} = 18.12kgf/cm^3$ for the Modified mortar (m = 40) and $\bar{y}$ = 16.87kgf/cm2 for the unmodified mortar (n = 32). Let μ_1 and μ_2 be the true average tension bond strengths for the modified and Unmodified mortars, respectively. Assume that the bond strength distributions are both normal.

(a) Assuming that σ_1= 1.6 and σ_2 = 1.4, test H_o : $\mu_1 - \mu_2$ = 0 versus Ha : $\mu_1 - \mu_2 \neq 0$ at level 0.01

(b) Compute the probability of a type II error for the test of part (a) when $\mu_1 - \mu_2$ = 1

(c) Suppose the investigator decided to use a level 0.05 test and wished β = 0.10 when $\mu_1 - \mu_2$ = 1. If m = 40, what value of n is necessary?

(d) How would the analysis and conclusion of part (a) change if σ_1 and $\sigma_2$were unknown but s_1 = 1.6 and s_2 = 1.4 ?

Solution~21

(a) Assuming that σ_1= 1.6 and σ_2 = 1.4, test H_o : $\mu_1 - \mu_2$ = 0 versus Ha : $\mu_1 - \mu_2 \neq 0$ at level 0.01

We have : $\begin{cases} \bar{x} = 18.12 \\ \bar{y} = 16.87 \\ \sigma_1^2 = (1.6)^2 = 2.56 \\ \sigma_2^2 = (1.4)^2 = 1.96 \\ m = 40 \\ n = 32 \\ \alpha = 0.01 \end{cases}$

Since, we known both value of σ_1^2 & σ_2^2, and both population distribution are normal

So, Null hypothesis : H_o : $\mu_1 - \mu_2 = \Delta_o$

Alternative Hypothesis : $H_a : \mu_1 - \mu_2 \neq 0 \Longrightarrow R = \left\{ z : z \leq -z_{\frac{\alpha}{2}} \;\; or \;\; z \geq z_{\frac{\alpha}{2}} \right\}$

- Test statistic value :

$$Z = \frac{\bar{x}-\bar{y}-\Delta_o}{\sqrt{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}} = \frac{18.12-16.87}{\sqrt{\frac{2.56}{40}+\frac{1.96}{32}}} = 3.532$$

By using two – tailed level for 0.01 test rejection region is

$H_a : \mu_1 - \mu_2 \neq \Delta_0$ $\qquad R = \{z : z \leq -z_{\alpha/2} \text{ or } z \geq z_{\alpha/2}\}$

For $\alpha = 0.01 \Longrightarrow z_{\frac{\alpha}{2}} = \phi^{-1}\left(1-\frac{\alpha}{2}\right) = 2.576$

```
> qnorm(1-0.005)
[1] 2.575829
```

$\Longrightarrow CR = \{ z : z \leq -2.576 \ or \ z \geq 2.576 \}$

Since, $|z| = 3.532 > 2.576 \Longrightarrow Z \in RR$

Therefore, H_o is rejected at $\alpha = 0.01$

(b) Compute the probability of a type II error for the test of part (a) when $\mu_1 - \mu_2 = 1$

$$\beta(\Delta') = \phi\left(Z_{\frac{\alpha}{2}} - \frac{\Delta' - \Delta_o}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}\right) - \phi\left(-Z_{\frac{\alpha}{2}} - \frac{\Delta' - \Delta_o}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}\right)$$

$\beta(1) = \phi(-0.25) - \phi(-5.4)$

$= 0.40129 - 0$

```
> pnorm(-0.25)
[1] 0.4012937
> pnorm(-5.4)
[1] 3.332045e-08
```

Therefore, $\beta(1) = 0.40129$

(c) Suppose the investigator decided to use a level 0.05 test and wished $\beta = 0.10$ when $\mu_1 - \mu_2 = 1$. If m = 40, what value of n is necessary?

we have $$\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n} = \frac{(\Delta' - \Delta_0)^2}{(z_\alpha + z_\beta)^2}$$

For a two-tailed test, α is replaced by $\alpha/2$.

$$\Leftrightarrow \frac{2.56}{40} + \frac{1.96}{n} = \frac{1}{(1.96 + 1.28)^2}$$

$$\begin{cases} z_\alpha = z_{\frac{\alpha}{2}} = \phi^{-1}\left(1 - \frac{\alpha}{2}\right) = 1.96 \\ z_\beta = 0.10 = \phi^{-1}(1 - \alpha) = 1.281 \end{cases}$$

```
> qnorm(1-0.05/2)
[1] 1.959964
> qnorm(1-0.10)
[1] 1.281552
```

$$\Longrightarrow n = \frac{-1.96}{\frac{2.56}{40} - \frac{1}{10.497}} = 62.689$$

```
> (-1.96)/((2.56/40)-(1/10.497))
[1] 62.68928
```

Therefore, $n = 63$

(d) How would the analysis and conclusion of part (a) change if σ_1 and σ_2 were unknown but $s_1 = 1.6$ and $s_2 = 1.4$?

Large-Sample Tests

The assumptions of normal population distributions and known values of σ_1 and σ_2 are unnecessary when both sample sizes are large($m \geq 30$ and $n \geq 30$). In this case, the Central Limit Theorem guarantees that $\bar{X} - \bar{Y}$ has approximately a normal distribution regardless of the underlying population distributions. Furthermore, using S_1^2 and S_2^2 in place of σ_1^2 and σ_2^2 gives a variable whose distribution is approximately standard normal:

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}}$$

Since, $\sigma_1 = s_1$ & $\sigma_2 = s_2 \Longrightarrow$ *the result remain the same*

Therefore, The result remain the same

22. Let X and Y equal the forces required to pull stud No. 3 and stud No. 4 out of a window that has been manufactured for an automobile. Assume that the distributions of X and Y are $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively

(a) If m = n = 10 observations are selected randomly, define a test statistic and a critical region for testing $H_o : \mu_1 - \mu_2 = 0$ against a two-sided alternative hypothesis. Let α = 0.05. Assume that the variances are equal.

(b) Given m = 10 observations of X, namely,

Solution~22

23. A new diet and exercise program has been advertised as remarkable way to reduce blood glucose levels in diabetic patients. Ten randomly selected diabetic patients are put on the program, and the results after 1 month are given by the following table:

Before	268	225	252	192	307	228	246	298	231	185
After	106	186	223	110	203	101	211	176	194	203

Do the data provide sufficient evidence to support the claim that the new program reduces blood glucose level in diabetic patients? Use α = 0.05

Solution~23

- Null hypothesis : $H_o : \mu_D = 0 \ versus \ H_a : \mu_D > 0$

  Since, D = Before – After

Before	268	225	252	192	307	228	246	298	231	185
After	106	186	223	110	203	101	211	176	194	203
μ_D	162	39	29	82	104	127	35	122	37	-18

We have : $\begin{cases} n = 10 \\ s_D = 56.1554 \\ \bar{d} = 71.9 \end{cases}$

```
> d=c(162,39,29,82,104,127,35,122,37,-18)
> sd(d)
[1] 56.15544
> mean(d)
[1] 71.9
```

- Test statistic value

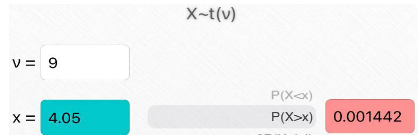
$$t = \frac{\bar{d}-\Delta_o}{\frac{s_D}{\sqrt{n}}} = \frac{79.1-0}{\frac{56.15544}{\sqrt{10}}} = 4.05$$

```
> (71.9-0)/((56.15/sqrt(10)))
[1] 4.049292
```

- P – value = $P\ (T \geq 4.05)$, $T \sim t(n-1=9)$

  $= 0.001442 < \alpha = 0.05$

  $\Longrightarrow H_o$ is rejected



Therefore, it seems that the new diet and exercise program reduce blood glucose level in diabetic patients

---

24. Consider the accompanying data on breaking load (kg/25 mm width) for various Fabrics in both an unabraded condition and an abraded condition ("The Effect of Wet Abrasive Wear on the Tensile Properties of Cotton and Polyester-Cotton Fabrics,"J. Testing and Evaluation, 1993: 84-93).
    Use the paired t test to test $H_o : \mu_D = 0$ versus $H_a : \mu_D > 0$ at significance level 0.01.

	Fabric							
	1	2	3	4	5	6	7	8
U	36.4	55	51.5	38.7	43.2	48.8	25.6	49.8
A	28.5	20	46	34.5	36.5	52.5	26.5	46.5

Solution~24

- Null hypothesis : $H_o : \mu_D = 0 \ versus \ H_a : \mu_D > 0$

  Analysis the given data :

	Fabric							
	1	2	3	4	5	6	7	8
U	36.4	55	51.5	38.7	43.2	48.8	25.6	49.8
A	28.5	20	46	34.5	36.5	52.5	26.5	46.5
μ_D	7.9	35	5.5	4.2	6.7	- 3.7	- 0.9	3.3

We have : $\begin{cases} n = 8 \\ \bar{d} = 7.25 \\ s_D = 11.86 \end{cases}$

```
> x=c(7.9,35,5.5,4.2,6.7,-3.7,- 0.9,3.3)
> sd(x)
[1] 11.86279
> mean(x)
[1] 7.25
```

- Test statistic value

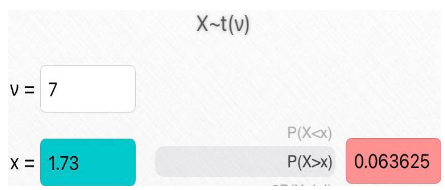
$$t = \frac{\bar{d} - \Delta_o}{\frac{s_D}{\sqrt{n}}} = \frac{7.25 - 0}{\frac{11.86}{\sqrt{8}}} = 1.73$$

```
> (7.25)/(11.86/sqrt(8))
[1] 1.729013
```

By using upper – tailed level 0.01 test the rejection region is given by :

- P – value = $P\ (\ T \geq 1.73\ )$, $T \sim t(n - 1 = 7\ )$

= 0.0636 > $\alpha = 0.01$



$\Longrightarrow H_o$ is not rejected

Therefore, H_o is not rejected

25. Recent incidents of food contamination have caused great concern among consumers. The article "How Safe Is That Chicken?" (Consumer Reports, Jan. 2010: 19-23) reported that 35 of 80 randomly selected Perdue brand broilers tested positively for either campylobacter or salmonella (or both), the leading bacterial causes of food-borne disease, whereas 66 of 80 Tyson brand broilers test positive.

(a) Does it appear that the true proportion of non-contaminated Perdue broilers differs from that for the Tyson brand? Carry out a test of hypotheses using a significance level 0.01 by obtaining a P-value.

(b) If the true proportions of non-contaminated chickens for the Perdue and Tyson brands are 0.50 and 0.25, respectively, how likely is it that the null hypothesis of equal proportions will be rejected when a 0.01 significance level is used and the sample sizes are both 80?

Solution~25

(a) Does it appear that the true proportion of non-contaminated Perdue broilers differs from that for the Tyson brand? Carry out a test of hypotheses using a significance level 0.01 by obtaining a P-value.

- Null hypothesis $H_o: p_1 - p_2 = 0$ and alternative hypothesis $H_a: p_1 - p_2 \neq 0$

We have:
$$\begin{cases} \text{35 Perdue brand broilers tested positively} \\ \text{45 Perdue broilers tested negatively} \end{cases}$$

$$\begin{cases} \text{66 Tyson brand broilers tested positively} \\ \text{14 Tyson brand broilers tested negatively} \end{cases}$$

Then: $\hat{p}_1 = \frac{80-35}{80} = 0.5625$

$\hat{p}_2 = \frac{80-66}{80} = 0.175$

- Test statistic value:

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{m}+\frac{1}{n}\right)}} \qquad \text{where : } \hat{p} = \frac{80\times0.5625+80\times0.175}{80+80} = 0.36875$$

$$\Longrightarrow z = \frac{0.5625 - 0.175}{\sqrt{0.36875 \times 0.63125 \times \frac{1}{40}}} = 5.0796$$

- P-value $= 2[1-\Phi(z)] = 2[1-\Phi(5.0796)]$
  $= 0.0000004 < \alpha = 0.01$

```
> pnorm(5.0796)
[1] 0.9999998
```

Therefore, H_o is totally rejected

(b) If the true proportions of non-contaminated chickens for the Perdue and Tyson brands are 0.50 and 0.25, respectively, how likely is it that the null hypothesis of equal proportions will be rejected when a 0.01 significance level is used and the sample sizes are both 80?

- Let γ be a probability of the event that the null hypothesis of equal proportions will be rejected.

Then, $\gamma(p_1, p_2) = 1 - \beta(\mathrm{p}_1, p_2)$

$$= 1 - \Phi\left[\frac{\left(z_{\frac{\alpha}{2}}\sqrt{\bar{p}.\bar{q}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}-(p_1-p_2)\right)}{\sigma}\right] + \Phi\left[\frac{\left(-z_{\frac{\alpha}{2}}\sqrt{\bar{p}.\bar{q}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}-(p_1-p_2)\right)}{\sigma}\right]$$

For $\sigma = 0.01$ then, $z_{\frac{\alpha}{2}} = \Phi^{-1}\left(1-\frac{\alpha}{2}\right) = 2.575$

And $\sigma = \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}} = 0.074$

$\bar{p} = \frac{p_1 + p_2}{2} = 0.375$ and $\bar{q} = 0.625$

Then $\gamma(p_1, p_2) = 1 - \Phi(-0.71) + \Phi(-6.05)$

$= 1 - 0.2388 + 0$

```
> pnorm(-0.71)
[1] 0.2388521
> pnorm(-6.05)
[1] 7.242292e-10
```

Therefore, $\gamma(p_1, p_2) = 0.7613$

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26. Olestra is a fat substitute approved by the FDA for use in snack foods. Because There have been anecdotal reports of gastrointestinal problems associated with olestra consumption, a randomized, double-blind, placebo-controlled experiment was carried out to compare olestra potato chips to regular potato chips with respect to GI symptoms ("Gastrointestinal Symptoms Following Consumption of Olestra or Regular Triglyceride Potato Chips," J. of the Amer. Med. Assoc., 1998: 150-152). Among 529 individuals in the TG control group, 17.6% experienced an adverse GI event, whereas among the 563 individuals in the olestra treatment group, 15.8% experienced such an event.

(a) Carry out a test of hypotheses at the 5% significance level to decide whether the incidence rate of GI problems for those who consume olestra chips according to the experimental regimen differs from the incidence rate for the TG control treatment.

(b) If the true percentages for the two treatments were 15% and 20%, respectively, what sample sizes (m = n) would be necessary to detect such a difference with probability 0.90?

Solution~26

(a) Carry out a test of hypotheses at the 5% significance level to decide whether the incidence rate of GI problems for those who consume 0lestra chips according to the experimental regimen differs from the incidence rate for the TG control treatment.

We have : $\begin{cases} \hat{p}_1 = 0.176 \\ n_1 = 529 \\ n_2 = 563 \\ \hat{p}_2 = 0.158 \\ \alpha = 0.05 \end{cases}$

Null hypothesis $H_0 : p_1 - p_2 = 0$,

Alternative hypothesis $H_a : p_1 - p_2 \neq 0$

- Test statistic value :

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{0.176 - 0.158}{\sqrt{(0.176+0.158)\times(1-0.176+0.158)\left(\frac{1}{529}+\frac{1}{563}\right)}}$$

$$= \frac{0.018}{0.0223} = 0.8$$

- By using two-tailed level for $\alpha = 0.05$ test, hence, the rejecting region is given by:

$$RR = \left\{Z : Z < -Z_{\frac{\alpha}{2}} = -Z_{0.025} = -\phi^{-1}(1 - 0.025) = -1.9599\right\}$$

```
> qnorm(1-0.025)
[1] 1.959964
```

Since, $Z = 0.8 > -1.9599$

The test statistic value is not included in rejection region,

Therefore, There is not enough evidence to conclude that Ha is true for the given α and more experiments may be needed.

(b) If the true percentages for the two treatments were 15% and 20%, respectively, what sample sizes (m = n) would be necessary to detect such a difference with probability 0.90?

For the case $m = n$, the level α test has type II error probability β at the alternative value p_1, p_2 with $p_1 - p_2 = d$ when

$$n = \frac{\left[Z_{\frac{\alpha}{2}}\sqrt{\frac{(p_1+p_2)(q_1-q_2)}{2}} + Z_\beta\sqrt{p_1q_1+p_2q_2}\right]^2}{d^2}$$

Where : $\begin{cases} p_2 = 0.20 \\ p_1 = 0.15 \\ d = p_1 - p_2 = -0.05 \end{cases}$

Since, the difference with probability is 0.9 so $\beta = 0.1 \Longrightarrow Z_\beta = 1.28$

```
> qnorm(1-0.1)
[1] 1.281552
```

$$n = \frac{1.96\sqrt{\frac{0.35(0.65)}{2}} + 1.28\sqrt{(0.15\times 0.85)+(0.2\times 0.8)}}{0.05^2} \approx 1211$$

Therefore, n = 1211

27. Toxaphene is an insecticide that has been identified as a pollutant in the Great Lakes ecosystem. To investigate the effect of toxaphene exposure on animals, groups of rats were given toxaphene in their diet. The article "Reproduction Study of Toxaphene in the Rat" (J. of Environ. Sci. Health, 1988: 101-126) reports weight gains (in grams) for rats given a low dose (4 ppm) and for control rats whose diet did not include the insecticide. The sample standard deviation for 23 female control rats was 32 g and for 20 female low-dose rats was 54 g. Does this data suggest that there is more variability in low-dose weight gains than in control weight gains? Assuming normality, carry out a test of hypotheses at significance level 0.05.

Solution~27

- Does this data suggest that there is more variability in low does weight gains than in control weight again ?
Assuming normality, carry out a test of hypothesis at significance level 0.05

We have : $\begin{cases} m = 23 \\ n = 20 \\ s_1 = 32 \\ s_2 = 54 \\ \alpha = 0.05 \end{cases}$

Null hypothesis : H_o : $\sigma_1^2 = \sigma_2^2$
Alternative hypothesis : H_a : $\sigma_1^2 < \sigma_2^2 \quad \Longrightarrow RR = \{f : f \leq F_{1-\alpha,m-1,n-1}\}$

- Test statistic value :

$$f = \frac{s_1^2}{s_2^2} = \frac{(32)^2}{(54)^2} = \frac{1024}{2916} = 0.3511$$

- By using upper tailed, so the rejection region is given by :

$$RR = \{ f \leq F_\alpha(v_1, v_2)\} \qquad \begin{cases} v_1 = m - 1 = 22 \\ v_2 = n - 1 = 19 \\ \alpha = 0.05 \end{cases}$$

$\Longrightarrow F_{0.05}(22,19) = 2.07$

Since, the test statistic value is included in rejection region

Therefore, This data suggest that there is more variability in low – dose weight gains than in control weight gains

28. In a study of copper deficiency in cattle, the copper values (mg Cu/100 mL blood) Were determined both for cattle grazing in an area known to have well-defined Molybdenum anomalies (metal values in excess of the normal range of regional variation) and for cattle grazing in a nonanomalous area ("An Investigation into Copper Deficiency in Cattle in the Southern Pennines," J. Agricultural Soc. Cambridge, 1972: 157–163), resulting in $s_1 = 21.5$($m = 48$) for the anomalous condition and $s_2 = 19.45$($n = 45$) for the nonanomalous condition. Test for the equality versus inequality of population variances at significance level 0.10 by using the P-value approach.

Solution~28

$Null\ hypothesis\ \ H_0: \sigma_1^2 = \sigma_2^2$

$Alternative\ hypothesis\ \ H_a: \sigma_1^2 \neq \sigma_2^2$

$Decision\ rule: Reject\ H_0\ if\ P\text{-}value \leq 0.10$

We have : $\begin{cases} m = 48 \\ n = 45 \\ s_1 = 21.5 \\ s_2 = 19.45 \\ \alpha = 0.10 \end{cases}$

- Test statistic value :

$$f = \frac{s_1^2}{s_2^2} = \frac{22.5^2}{19.45^2} \approx 1.22$$

- By using upper tailed, so the rejection region is given by :

$$RR = \{ f \leq F_\alpha(v_1, v_2)\} \qquad \begin{cases} v_1 = m - 1 = 47 \\ v_2 = n - 1 = 44 \\ \alpha = 0.10 \end{cases}$$

So, the value of test statistic F is 1.22 with $df = (47{,}44)$.

- $P\text{-}value = 2P(f_{df_1,df_2} \geq F)$

$$= 2P\left(f_{47,44} \geq 1.22\right) \approx 0.5079$$

Since P-value is greater than 0.10, fail to reject the null hypothesis that equality of population variances at a 10% significance level.

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29. To measure air pollution in a home, let X and Y equal the amount of suspended particulate matter (in mg/m3) measured during a 24-hour period in a home in which there is no smoker and a home in which there is a smoker,respectively.We shall test the null hypothesis $H_o : \frac{\sigma_1^2}{\sigma_2^2} = 1 \ Vs \ H_a : \frac{\sigma_1^2}{\sigma_2^2} > 1$

(a) If a random sample of size m = 9 yielded .x = 93 and sx = 12.9 while a random sample of size n = 11 yielded .y = 132 and sy = 7.1, define a critical region and give your conclusion if α = 0.05.

(b) Now test H0 : μ1 = μ2 against Ha : μ1 < μ2 if α = 0.05.

Solution~29

(a) If a random sample of size $m = 9$ yielded $\bar{x} = 93$ and $s_x = 12.9$ while a random sample of size $n = 11$ yielded $\bar{y} = 132$ and $s_y = 7.1$, define a critical region and give your conclusion if $\alpha = 0.05$

Null hypothesis $H_0: \sigma_1^2 = \sigma_2^2$ versus alternative hypothesis $H_a: \sigma_1^2 > \sigma_2^2$

- Test statistics value

$$f = \frac{s_x^2}{s_y^2} = \frac{12.9^2}{7.1^2} = 3.30$$

- The critical region is defined by using the lower-tailed test.

$$RR = \{f: f \geq F_\alpha(v_1, v_2)\} \qquad where: \begin{cases} v_1 = 8 \\ v_2 = 10 \\ \alpha = 0.05 \end{cases}$$

$\Longrightarrow F_{0.05}(8,10) = 3.07$

Therefore: $RR = \{f: f \geq 3.07\}$

Since the observed test statistic lies in the rejection region, then reject H_0

Therefore, H_o is reject

(b) Now test $H_0: \mu_1 = \mu_2$ against $H_a: \mu_1 < \mu_2$ if $\alpha = 0.05$.

Since the standard deviations are unequal, by using the Smith-Satterwaite procedure the test statistic value:

$$t_v = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_x^2}{m} + \frac{s_y^2}{n}}} = -8.12$$

Where v, the degree of freedom is defined by

$$v = \frac{\left[\frac{s_x^2}{m} + \frac{s_y^2}{n}\right]^2}{\frac{\left(\frac{S_x}{m}\right)^2}{m-1} + \frac{\left(\frac{S_y}{n}\right)^2}{n-1}} = 11.87 \approx 12$$

By using upper-tailed test the rejection region is given by:

$RR = \{t : t \leq -t_\alpha\}$ Where $t_\alpha = 1.796$

Then , $RR = \{t : t \leq -1.796\}$

Since the test statistic value is included in rejection region,

Therefore, H_0 is rejecte