11. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the exponential distribution whose pdf is $f(x;\theta) = (1/\theta)e^{-x/\theta}, 0 < x < \infty, 0 < \theta < \infty$.

(a) Show that $\bar{X}$ is an unbiased estimator of θ.

(b) Show that the variance of $\bar{X}$ is θ^2/n.

(c) What is a good estimate of θ if a random sample of size 5 yielded the sample values 3.5, 8.1, 0.9, 4.4, and 0.5?

Solution ~ 11

We have :

$$f(x_i, \theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} & , 0 < x_i < \infty \\ 0 & , \text{Otherwise} \end{cases}$$

(a) Show that $\bar{x}$ is an unbiased estimator of θ

$E(\bar{x}) = \mu = E(x) = \theta$ because X ~ Exp (θ)

$$\boxed{\text{Therefore, } \bar{x} \text{ is an unbiased estimator of } \theta}$$

(b) Show that the variance of $\overline{x}$ is $\frac{\theta^2}{n}$

$$V(\bar{x}) = \frac{\sigma^2}{n} = \frac{V(x)}{n} = \frac{\theta^2}{n}$$ because $X \sim \text{Exp}(\theta)$

Therefore, $\bar{x}$ of variance is $\frac{\theta^2}{n}$

(c) Find a good estimator of θ

$$\hat{\theta} = \bar{x} = \frac{1}{5}\sum_{i=1}^{5} x_i$$

$$= \frac{3{,}5+8{,}1+0{,}9+4{,}4+0{,}5}{5}$$

Therefore, A good estimator of θ is $\hat{\theta} = 3{,}48$

The Exponential Distribution

Definition 10

A continuous rv X is said to have an **exponential distribution** with parameter $\lambda (\lambda > 0)$ if the pdf of X is

$$f(x) = \begin{cases} \frac{1}{\lambda} e^{-\frac{x}{\lambda}}, & x > 0, \\ 0, & \text{otherwise} \end{cases}$$

In this case, we write $X \sim \text{Exp}(\lambda)$.

Theorem 9

If $X \sim \text{Exp}(\lambda)$, then

1. $F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\frac{x}{\lambda}}, & x \geq 0. \end{cases}$
2. $E(X) = \lambda, \qquad V(X) = \lambda^2, \qquad M(t) = \frac{1}{1-\lambda t} \quad , t < \frac{1}{\lambda}.$