DCC

EXERCISES

IN

CHOICE

AND

CHANCE

WHITWORTH

DEIGHTON BELL & Co.

# DCC EXERCISES

IN

*CHOICE AND CHANCE*

*By the same Author.*

# CHOICE AND CHANCE,

AN ELEMENTARY TREATISE ON

PERMUTATIONS, COMBINATIONS, AND PROBABILITY,

WITH 640 EXERCISES.

*FOURTH EDITION, ENLARGED.*

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# DCC EXERCISES

INCLUDING HINTS FOR THE SOLUTION
OF ALL THE QUESTIONS IN

*CHOICE AND CHANCE*

BY

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LATE FELLOW OF ST JOHN'S COLLEGE CAMBRIDGE

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# PREFACE.

THE intelligent student does not desire what is commonly called a "Key" to a collection of problems. He welcomes, at the most, some hints as to the method by which a problem is to be solved, or some indication of the principal steps in the process.

I make no apology therefore for the abridged form of many of the solutions in this volume. Merely arithmetical or algebraical reductions are left to the student, except when it is desired to exhibit some particular expedient. In many cases I have described a geometrical construction and left the student to draw the figure. I have used language convenient for its brevity which to the purist may appear slovenly. For instance I have not scrupled to say "The Choice is ten" when I mean that there are ten courses amongst which one may choose.

I have tried to illustrate as many principles and methods as possible; and therefore when a group of questions might have been solved by processes identical in principle, I have preferred to treat different questions of the group by different methods.

My thanks are due to Mr C. H. Brooks, whose name was familiar 40 years ago to the mathematical readers of the once famous *Lady's and Gentleman's Diary*, for the careful revision of the proof sheets, by which I trust a high degree of accuracy has been secured.

W. A. WHITWORTH

ALL SAINTS' MARGARET STREET
*September* 11, 1897

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# INTRODUCTION I.

## ON THE SUMMATION OF CERTAIN SERIES.

1. A LARGE number of questions in Combinations and Probability require for their solution the summation of series. Conversely, the summation of many series can be effected at sight by simple considerations of choice or chance. In the following articles we give some examples.

We assume the notation of *Choice and Chance* where the no. of permutations of $n$ things $r$ at a time is denoted by $P_r^n$; the no. of combinations by $C_r^n$, $C_{n-r}^n$ or $C_{n-r,\,r}$ indifferently; and the no. of combinations, when repetitions are allowed, by $R_r^n$. It is also useful to write $Y_n$ for the no. of combinations of $2n$ things $n$ at a time, or the no. of ways of dividing $2n$ things into two equal parcels. We have

$$Y_n = C_n^{2n} = C_{n,\,n} = R_n^{n+1} = 2R_n^n.$$

$$P_r^n = C_r^n \underline{|r}. \quad R_r^n = C_r^{n+r-1} = C_{n-1,\,r}.$$

Also $$R_n^{m+1} = R_m^{n+1}; \text{ and if } m > n,\ C_{m-n+1}^m = R_{m-n+1}^n.$$

[The student who has read the Integral Calculus will note that in virtue of Wallis' formula

$$Y_n < \frac{2^{2n}}{\sqrt{\pi n}} \text{ and } Y_n > \frac{2^{2n}}{\sqrt{\pi\left(n+\frac{1}{2}\right)}}.$$

This enables us readily to obtain by the aid of logarithms useful approximations to factorial expressions of a high order. For instance the no. of ways in which 52 cards can be distributed to four whist players is

$$\underline{|52} \div (\underline{|13})^4 = Y_{26}\,(Y_{13})^2.$$

By Wallis' formula we easily calculate that this number lies between

549497... to 29 figures and 524129... to 29 figures.

The geometric mean of these is 536663... to 29 figures which differs from the number given in Qn. 399 by less than ·04 per cent.]

2. The combinations of $n$ things, 0, 1, 2, 3, ... or $n$ together may be formed by dealing with each of the $n$ things in either of 2 ways, i.e. either selecting or rejecting each. Hence we write down

$$1 + C_1^n + C_2^n + C_3^n + \ldots + C_n^n = 2^n.$$

If one of the things (say $\alpha$) is always to be used there are $2^{n-1}$ combinations. But in the group $C_r^n$ there are $rC_r^n$ letters and therefore $\alpha$ occurs $rC_r^n \div n$ times. Hence

$$C_1^n + 2C_2^n + 3C_3^n + \ldots + nC_n^n = n2^{n-1}.$$

3. Again, let us have $m+1$ letters of which one is $\alpha$. The no. of combinations $n$ together when repetitions are allowed is $R_n^{m+1}$. Analyse these according as $\alpha$ occurs $n$, $n-1$, $n-2$, ... or 1 or 0 times and we write

$$1 + R_1^m + R_2^m + R_3^m + \ldots + R_n^m = R_n^{m+1},$$

or

$$1 + C_{1,\,m+1} + C_{2,\,m-1} + \ldots + C_{n,\,m-1} = C_{m,\,n}.$$

4. Out of an alphabet of $x$ consonants and $y$ vowels an $n$ lettered word can be formed in $(x+y)^n$ ways. If we analyse these according as they contain 0, 1, 2, 3, ... or $n$ vowels, we write down

$$(x+y)^n = x^n + C_1^n x^{n-1}y + C_2^n x^{n-2}y^2 + \ldots + y^n,$$

which proves the Binomial theorem when the exponent is a positive integer.

Out of the same alphabet a selection of $n$ letters without repetitions can be made in $C_n^{x+y}$ ways. Considering the selections when 0, 1, 2, 3 ... or $n$ vowels are used we find

$$C_n^x + C_{n-1}^x C_1^y + C_{n-2}^x C_2^y + C_{n-3}^x C_3^y + \dots + C_n^y = C_n^{x+y}.$$

So if repetitions be allowed

$$R_n^x + R_{n-1}^x R_1^y + R_{n-2}^x R_2^y + R_{n-3}^x R_3^y + \dots + R_n^y = R_n^{x+y}.$$

5. Out of the $n+2$ numbers 1, 2, 3, ... $(n+2)$ a selection of 3 numbers can be made in $C_3^{n+2}$ ways. The lowest no. chosen may be $n, n-1, n-2, \dots 3, 2, 1$. Counting these separately we get

$$C_2^2 + C_2^3 + C_2^4 + \dots + C_2^{n+1} = C_3^{n+2},$$

or $$1 \,.\, 2 + 2 \,.\, 3 + 3 \,.\, 4 + \dots + n(n+1) = \tfrac{1}{3}n(n+1)(n+2),$$

which is a particular case of the series in Art. 3. Or we may count the ways separately according as the middle number is 2, 3, 4, ... or $(n+1)$. Then

$$1 \,.\, n + 2(n-1) + 3(n-2) + \dots + n \,.\, 1 = \tfrac{1}{6}n(n+1)(n+2).$$

6. Another series

$$1 + C_1^m C_1^n + C_2^m C_2^n + \dots + C_n^m C_n^n = C_n^{m+n}$$

($m$ not being less than $n$) simply expresses the fact that if an equal no. of men and women are to be selected out of $m$ men and $n$ women, all possible results can be effected by selecting $n$ persons out of the whole number and combining the selected men with the non-selected women.

This series is of frequent occurrence in the particular case when $m = n$,

$$1 + (C_1^n)^2 + (C_2^n)^2 + \dots + (C_n^n)^2 = Y_n.$$

7. Again, let $\frac{1}{2}n(n+1)$ balls be placed together so as to form an equilateral triangle, each edge containing $n$ balls. Let one of the balls be marked at random, and let it be asked what is the

expectation of the number of balls in the line parallel to the base, which contains the marked ball. If instead of the line parallel to the base we take the three lines through the marked ball parallel to the three sides we get the constant sum $2n+1$. Hence the expectation in each direction is $\frac{1}{3}(2n+1)$. But if we consider the chances of the marked ball being in the 1st, 2nd, 3rd, ... or $n$th line, counting from the vertex, we find the expectation

$$(1^2+2^2+3^2+\ldots+n^2) \div \tfrac{1}{2}n(n+1).$$

Hence $$1^2+2^2+3^2+\ldots+n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

8. Again, suppose a bag contains $m$ coins, of which $n$ are sovereigns and the rest worthless imitations. If I am to draw as long as I continue to draw sovereigns my expectation is evidently £$n \div (m-n+1)$. For the $m-n$ worthless coins divide the sovereigns into $m-n+1$ lots of equal expectation, of which lots I receive one. But my expectation may be expressed as a series in either of two ways. (i) Considering the sum of the expectations of the several drawings we write down

$$\frac{n}{m}+\frac{n(n-1)}{m(m-1)}+\frac{n(n-1)(n-2)}{m(m-1)(m-2)}+\ldots \text{ to } n \text{ terms} = \frac{n}{m-n+1}.$$

(ii) Considering the sum of the several expectations of drawing £1, £2, £3, &c., and writing $m+1$ for $m$ we obtain

$$\frac{n}{m}+2\frac{n(n-1)}{m(m-1)}+3\frac{n(n-1)(n-2)}{m(m-1)(m-2)}+\ldots \text{ to } n \text{ terms}$$
$$= \frac{n(m+1)}{(m-n+1)(m-n+2)}.$$

9. Again, suppose that I and $m$ other persons are to seat ourselves at random on $m+n+1$ chairs. Since all orders are equally likely the chance that exactly $p$ persons sit above me is $1 \div (m+1)$. But the chance of my occupying the $(x+p+1)$th chair is $1 \div (m+n+1)$, and the no. of ways in which $p$ chairs above me and $m-p$ below me can be occupied is $C_{p,x}C_{m-p,n-x}$,

while the total no. of ways in which $m$ chairs can be selected is $C_{m,n}$. Hence giving $x$ all values from 0 to $n$ we have

$$\Sigma \{C_{p,x} C_{m-p,n-x}\} = C_{m+1,n},$$

or

$$\Sigma \{C_{p,x} C_{q,n-x}\} = C_{p+q+1,n},$$

a very useful summation, reducing when $p = 0$ to that of Art. 3.

10. The Binomial Theorem, for a negative integral exponent, may be established as follows.

Suppose we have to throw a die of $p$ faces till we have thrown $n$ aces. Since one ace is to be expected in $p$ throws we must expect $np$ throws, $= n \div (1-q)$; if $p = 1 \div (1-q)$.

But the chance of succeeding at the $(n+x)$th throw is

$$C_{n-1,x} q^x (1-q)^n,$$

$$\therefore\ n \div (1-q) = \Sigma \{(n+x) C_{n-1,x} q^x (1-q)^n\},$$

or

$$(1-q)^{-n-1} = \Sigma \{C_{n,x} q^x\}.$$

Writing $n$ for $n+1$, we have

$$(1-q)^{-n} = 1 + R_1^n q + R_2^n q^2 + R_3^n q^3 + \dots.$$

11. The foregoing are but a few examples of a method which is very largely applicable. In the tables which follow we record the result of summation in a number of other cases. In Table I. the $x$ of the general term takes the values $0, 1, 2, \dots n$, and the summation is to $n+1$ terms, unless the general term vanishes when $x = 0$, when the sum is to $n$ terms. But as $n$ is not a parameter of the general term, $n$ can have any desired value. That is, the series can be summed to any assigned number of terms. In Table II. the general term is a function of $n$, and only the *complete* sum can generally be obtained; that is, the sum of all the terms between the limits at which the terms vanish or become impossible. There are, however, a few cases in which the first half or some other definite portion of the series can be summed.

## TABLE I.

| | General term | Limits of $x$ | Sum |
|---|---|---|---|
| 1 | $R_x^m$ | 0 to $n$ | $R_n^{m+1}$ |
| 2 | $R_m^x$ | 0 to $n$ | $R_{m+1}^n$ |
| 3 | $C_{m, x}$ | 0 to $n$ | $C_{m+1, n}$ |
| 4 | $C_{m, a+x}$ | 0 to $n$ | $C_{m+1, a+n} - C_{m+1, a-1}$ |
| 5 | $C_{m, a-x}$ | 0 to $n$ | $C_{m+1, a} - C_{m+1, a-n-1}$ |
| 6 | $P_m^{m+x}$ | 0 to $n$ | $P_{m+1}^{m+n+1} \div (m+1)$ |
| 7 | $xR_x^m$ | 1 to $n$ | $mR_{n-1}^{m+2}$ |
| 8 | $xR_m^x$ | 1 to $n$ | $(mn+n+1)\, R_{m+1}^n \div (m+2)$ |
| 9 | $xC_{m, x}$ | 1 to $n$ | $(m+1)\, C_{m+2, n-1}$ |
| 10 | $xC_{m, a+x}$ | 1 to $n$ | $nC_{m+1, a+n} - C_{m+2, a+n-1} + C_{m+2, a-1}$ |
| 11 | $xC_{m, a-x}$ | 1 to $n$ | $C_{m+2, a-1} - C_{m+2, a-n-1} - nC_{m+1, a-n-1}$ |
| 12 | $xP_m^{m+x}$ | 1 to $n$ | $P_{m+2}^{m+n+1} \div (m+2)$ |
| 13 | $C_r^{m+x}$ $m > r$ | 0 to $n$ | $C_{r+1}^{m+n+1} - C_{r+1}^m$ |
| 14 | $P_r^{m+x}$ $m > r$ | 0 to $n$ | $(P_{r+1}^{m+n+1} - P_{r+1}^m) \div (r+1)$ |
| 15 | $\dfrac{1}{P_r^{m+x}}$ $m > r$ | 0 to $n$ | $\left(\dfrac{1}{P_{r-1}^{m-1}} - \dfrac{1}{P_{r-1}^{m+n}}\right) \div (r-1)$ |
| 16 | $x$ | 1 to $n$ | $\frac{1}{2}n\,(n+1) = s$, suppose |
| 17 | $x^3$ | 1 to $n$ | $s^2$ |
| 18 | $x^5$ | 1 to $n$ | $s^2\,(4s-1) \div 3$ |

## TABLE I. (*continued*)

| | General term | Limits of $x$ | Sum |
|---|---|---|---|
| 19 | $x^7$ | 1 to $n$ | $s^2(6s^2-4s+1)\div 3$ |
| 20 | $x^2$ | 1 to $n$ | $\frac{1}{6}n(n+1)(2n+1)=S$, suppose |
| 21 | $x^4$ | 1 to $n$ | $S(6s-1)\div 5$ |
| 22 | $x^6$ | 1 to $n$ | $S(12s^2-6s+1)\div 7$ |
| 23 | $x^8$ | 1 to $n$ | $S(40s^3-40s^2+18s-3)\div 15$ |
| 24 | $x(x+1)$ | 1 to $n$ | $\frac{1}{3}n(n+1)(n+2)$ |
| 25 | $x(x-1)$ | 2 to $n$ | $\frac{1}{3}(n-1)n(n+1)$ |
| 26 | $x(x+1)(x+2)$ | 1 to $n$ | $\frac{1}{4}n(n+1)(n+2)(n+3)$ |
| 27 | $x(x-1)(x-2)$ | 3 to $n$ | $\frac{1}{4}(n-2)(n-1)n(n+1)$ |
| 28 | $a^x$ | 0 to $n$ | $(a^{n+1}-1)\div(a-1)$ |
| 29 | $xa^{x-1}$ | 1 to $n$ | $\dfrac{1-a^n(1+n-an)}{(1-a)^2}$ |
| 30 | $\dfrac{\underline{\lvert a+x}}{\underline{\lvert b+x}}$ | 0 to $n$ | $\dfrac{\dfrac{\underline{\lvert a+n+1}}{\underline{\lvert b+n}}-\dfrac{\underline{\lvert a}}{\underline{\lvert b-1}}}{a-b+1}$ |
| 31 | $\dfrac{\underline{\lvert a-x}}{\underline{\lvert b-x}}$ | 0 to $n$ | $\dfrac{\dfrac{\underline{\lvert a+1}}{\underline{\lvert b}}-\dfrac{\underline{\lvert a-n}}{\underline{\lvert b-n-1}}}{a-b+1}$ |
| 32 | $\dfrac{Y_{x-1}}{x4^x}$ | 1 to $n$ | $\dfrac{1}{2}\left(1-\dfrac{Y_n}{4^n}\right)$ |

# TABLE II.

| | General term | Limits of $x$ | Sum |
|---|---|---|---|
| 1 | $x(n-x)$ | 1 to $n-1$ | $n(n^2-1) \div 6$ |
| 2 | $x^2(n-x)$ | 1 to $n-1$ | $n^2(n^2-1) \div 12$ |
| 3 | $x^2(n-x)^2$ | 1 to $n-1$ | $n(n^2-1)(n^2+1) \div 30$ |
| 4 | $x^3(n-x)$ | 1 to $n-1$ | $n(n^2-1)(3n^2-2) \div 60$ |
| 5 | $x^3(n-x)^2$ | 1 to $n-1$ | $n^2(n^2-1)(n^2+1) \div 60$ |
| 6 | $x^4(n-x)$ | 1 to $n-1$ | $n^2(n^2-1)(2n^2-3) \div 60$ |
| 7 | $x^3(n-x)^3$ | 1 to $n-1$ | $n(n^2-1)(3n^4+3n^2+10) \div 420$ |
| 8 | $C_x^n$ | 0 to $n$ | $2^n$ |
| 9 | $C_{2x}^{2n}$ | 0 to $n$ | $2^{2n-1}$ |
| 10 | $C_x^{2n}$ | 0 to $2n$ | $2^{2n}$ |
| 11 | $C_x^{2n}$ | 0 to $n$ | $\frac{1}{2}(2^{2n}+Y_n)$ |
| 12 | $C_x^{2n+1}$ | 0 to $2n+1$ | $2^{2n+1}$ |
| 13 | $C_x^{2n+1}$ | 0 to $n$ | $2^{2n}$ |
| 14 | $xC_x^{2n}$ | 1 to $2n$ | $n2^{2n}$ |
| 15 | $xC_x^{2n}$ | 1 to $n$ | $n2^{2n-1}$ |
| 16 | $xC_x^{2n+1}$ | 1 to $2n+1$ | $(2n+1)\,2^{2n}$ |

## TABLE II. (*continued*).

| | General term | Limits of $x$ | Sum |
|---|---|---|---|
| 17 | $xC_x^{2n+1}$ | 1 to $n$ | $\frac{1}{2}(2n+1)(2^{2n}-Y_n)$ |
| 18 | $x^2C_x^n$ | 1 to $n$ | $n(n+1)2^{n-2}$ |
| 19 | $x^3C_x^n$ | 1 to $n$ | $n^2(n+3)2^{n-3}$ |
| 20 | $C_x^pC_q^x \quad p>q$ | $q$ to $p$ | $C_q^p 2^{p-q}$ |
| 21 | $C_x^pC_{n-x}^q$ | | $C_n^{p+q}$ |
| 22 | $R_x^pR_{n-x}^q$ | | $R_n^{p+q}$ |
| 23 | $C_{p,\,x}C_{q,\,n-x}$ | | $C_{p+q+1,\,n}$ |
| 24 | $C_x^pC_{m+x}^{m+q}$ | | $C_q^{m+p+q}$ |
| 25 | $C_x^pC_x^q \quad p>q$ | 0 to $q$ | $C_{p,\,q}$ |
| 26 | $xC_x^pC_x^q \quad p>q$ | 1 to $q$ | $C_{p,\,q}\,pq\div(p+q)$ |
| 27 | $Y_xY_{n-x}$ | 0 to $n$ | $4^n$ |
| 28 | $\dfrac{Y_xY_{n-x}}{x+1}$ | 0 to $n$ | $\frac{1}{2}Y_{n+1}$ |
| 29 | $\dfrac{Y_xC_{m-x,\,n-x}}{x+1}$ | 0 to $n$ | $C_{m+1,\,n} \qquad m>n$ |
| 30 | $4^xR_x^mY_{n-x}$ | 0 to $n$ | $C_{m,\,n}Y_{m+n}\div Y_m$ |
| 31 | $C_x^n\div C_x^m \quad m>n$ | 0 to $n$ | $(m+1)\div(m-n+1)$ |
| 32 | $C_x^n\div C_x^m \quad m>n$ | 1 to $n$ | $n\div(m-n+1)$ |

In the cases in which we have not entered the limits of $x$ in the table the summation is complete, but the limits depend upon the comparative magnitude of the parameters of the general term. For instance, in no. 21 the superior limit of $x$ is $p$ or $n$ (whichever is the less), and the inferior limit is $n-q$ if $n>q$; otherwise it is 0. But in all cases the sum between these limits (that is, the complete sum) is as given in the table.

12. As $C_1^n$, $C_2^n$, $C_3^n$ ... are the coefficients in the expansion of $(1+x)^n$, $n$ being a positive integer, so $R_1^n$, $R_2^n$, $R_3^n$ ... are the coefficients in the expansion of $(1-x)^{-n}$. We also find $Y_1$, $Y_2$, $Y_3$ ... useful in certain expansions. Thus

$$(1-x)^{-n} = 1 + R_1^n x + R_2^n x^2 + R_3^n x^3 + \dots$$

$$(1-4x)^{-\frac{1}{2}} = 1 + Y_1 x + Y_2 x^2 + Y_3 x^3 + \dots$$

$$Y_n (1-4x)^{-n-\frac{1}{2}} = Y_n + Y_{n+1} C_{n,1} x + Y_{n+2} C_{n,2} x^2 + \dots$$

$$\frac{(1-4x)^{n-\frac{1}{2}}}{Y_n} = \frac{1}{Y_n} - \frac{C_1^n x}{Y_{n-1}} + \frac{C_2^n x^2}{Y_{n-2}} - \frac{C_3^n x^3}{Y_{n-3}} + \dots$$

$$\pm x^n \left\{1 + \frac{Y_1 x}{C_{n,1}} + \frac{Y_2 x^2}{C_{n,2}} + \dots\right\},$$

$$\frac{1-\sqrt{1-4x}}{2x} = 1 + \frac{Y_1 x}{2} + \frac{Y_2 x}{3} + \frac{Y_3 x}{4} + \dots.$$

This last is a particular case of the more general theorem :

$$\frac{1}{p}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^p = \frac{1}{p} + \frac{R_1^{p+1} x}{p+1} + \frac{R_2^{p+2} x^2}{p+2} + \frac{R_3^{p+3} x^3}{p+3} + \dots.$$

# INTRODUCTION II.

## A GRESHAM LECTURE*.

WHEN an eminent professor has discarded the reasonings of Laplace and has maintained that the whole theory of what is called Inverse Probability is a delusion and a snare, there is a call to us to reconsider, and if need be to restate, the fundamental principles of the Science of Probability, re-examining with the utmost rigour the inferences that have been drawn therein.

Consider any undetermined occurrence which may or may not eventuate in a particular way, which particular way we will call the favourable event. A very concrete question may be asked concerning the event, to which question there must in the reason of things be a definite answer whether we can discover the answer or not. The question is, What sum ought you fairly to give or take now, while the event is undetermined, in exchange for the assurance that you shall receive a stated sum (say £1,000) if the favourable event occur? The chance of receiving £1,000 is worth something. It is not as good as the certainty of receiving £1,000, and therefore it is worth less than £1,000. But the

* Sir Thomas Gresham's Reader in Geometry was unable, owing to ill-health, to give his usual course of four lectures in the Easter Term, 1893. In his place Dr Venn, Professor Weldon, Mr Whitworth and Sir Robert Ball were asked to give one lecture each on *Applications of the Laws of Chance*. The present lecture was delivered in Gresham College on Thursday, April 20, 1893.

prospect or expectation or chance, however slight, is a commodity which may be bought and sold. It must have its price somewhere between zero and £1,000. Later on I will give you the test by which a "fair" price is to be defined. Meanwhile I only insist that as zero is certainly an unfairly low price and £1,000 as certainly an unfairly high price, there must exist between these limits a price which may be called the fair price, whether we are able to determine it or not. This fair price is said to be the value of the expectation, or for brevity it is often said that the expectation is so much. And the ratio of the expectation to the full sum to be received is what is called the chance of the favourable event. For instance, if we say that the chance is $\frac{1}{5}$, it is equivalent to saying that £200 is the fair price of the contingent £1,000. If we say that the chance is $\frac{7}{10}$, we are asserting that £700 is the fair sum to pay for the prospect of receiving £1,000.

The doubtful event may be of almost any kind. It may be that you are to receive £1,000 if a coin on being tossed turns up *head*, or if a pair of dice on being thrown turn up *sixes*, or if a certain ship at sea safely arrives in port, or if a certain horse wins a certain race, or if a certain man of stated age outlives another man also of stated age. The variety of event is endless. But in any of these cases, if we can arrive at the fair price at which you may transfer to another person your interest or expectation of receiving £1,000 (always excluding any consideration of discount for deferred payment), the ratio of this fair price to the full £1,000 expresses what is termed the chance or probability of the event.

The fair price can sometimes be calculated mathematically from *à priori* considerations: sometimes it can be deduced from statistics, that is, from the recorded results of observation and experiment. Sometimes it can only be estimated generally, the estimate being founded on a limited knowledge or experience. If your expectation depends on the drawing of a ticket in a raffle, the fair price can be calculated from abstract considerations: if it depend upon your outliving another person, the fair price can be inferred from recorded statistics: if it depend upon a benefactor not revoking his will, the fair price depends upon the character of your benefactor, his habit of changing his mind, and other circumstances upon the knowledge of which you base your estimate. But if in any of these cases you determine that £300 is the sum which you ought fairly to accept for your prospect, this is equivalent to saying that your chance, whether calculated or estimated, is $\frac{3}{10}$. But whether it can be calculated or not, or estimated with any degree of accuracy, there must exist the price which will answer to our definition of being fair between the parties to the bargain, and the ratio of this price to the full £1,000 is *upon the given data* the chance of the event.

"Upon the given data." For all chance is a function of limited knowledge. If we were omniscient we should say with certainty such an event will happen—or will not happen—as the case may be. If we knew all the causes at work in the psychical as well as the material world and could work out their effects, we should be able to say for certain whether our ship would come safely to port, whether the penny tossed by a particular person under the particular forces of his frame and will would fall head or tail, whether the testator would revoke his legacy, whether we should outlive a competitor.

Chance implies a limited knowledge, and in every question of chance there are certain data known and certain conditions unknown.

In defining what I mean by the "fair" price to be given for a contingent gain I shall bring myself into harmony with Dr Venn, who insists that probability has always to do with a series of events.

Whatever price be paid for a particular contingent prize, in the single case one of the parties must lose and one must gain. If I pay you £300 as the price of your interest in a contingent £1,000, I must either lose my £300 for nothing, or make a profit of £700. And if I lose you gain, or what I gain you lose. The single transaction must thus turn out to the advantage of one party and the disadvantage of the other. But think of the transaction as one of a series of equal and similar transactions. If 1,000 times or 1,000,000 times I pay £300 for a contingent £1,000, but if there be reason to expect that in the long run the event by which I lose £300 will occur seven times for every three occurrences of the event by which I gain £700, it is plain that in the long run I cannot expect either to gain or lose.

And this is the test of the fair price to be paid for a contingent prize. The transaction is a fair one, if on the supposition that it were repeated indefinitely there would be no presumption that either party would in the long run have an advantage.

The expression "in the long run" implies that though the transaction be repeated millions or billions of times there is still no presumption in favour of either party. The first may become richer and the second poorer, or the second richer and the first poorer, but no one can say beforehand that either of these events is more likely than the other. We do not assert that in 1,000,000 transactions the gains and losses would balance: we do not assert that they would nearly balance: the discrepancy may turn out to be very great indeed. But the fair price is arrived at if the terms are such that under the given limitations of knowledge we have no reason to expect an advantage to the one party rather than to the other. Chance has to do altogether with what we have reason to expect. It therefore depends upon our knowledge or upon our ignorance. It is a function of our knowledge, but that necessarily a limited and imperfect knowledge. This is a point which both Dr Venn and Prof. Chrystal appear to me to miss.

In many cases the fair price, and therefore the chance, can be determined by *à priori* considerations alone, though Dr Venn asserts that it must always be the result of experiment. For example, I toss this coin and I promise you a shilling if it turns up head. Plainly the value of your expectation is sixpence, for if we continue the operation indefinitely (you staking 6*d.* every time that I toss the coin and receiving 1*s.* if it turns up head), we have no reason to expect that the head will turn up oftener or less often than the tail. On every head you win 6*d.*, on every tail you lose 6*d.* In the long run there is no presumption of advantage or disadvantage. The price is fair. But this result is not based on the result of experiment. It does not depend upon the fact that when we have tossed coins heads and tails have approximately been equal. It is based rather on our not knowing anything about such experimental results. It is enough that we know no reason in favour either of head or tail turning up.

But in other cases the fair price can only be arrived at by the results of experiment or by statistics. What price, for instance, ought to be paid for £1,000 contingent on a certain man aged 20 living till he is 60? Recorded observations give the result that out of 1,000 persons who have reached the age of 20, 392 have died before the age of 60, and 608 have survived. If therefore you pay £608 for the expectation and repeat the transaction an indefinite number of times you may expect to win 608 times for every 392 that you lose. But when you win you net £392, and when you lose you lose £608. The results balance. You have no ground to expect a net advantage or a net disadvantage, and so the test of the fair price is satisfied.

This is the very example that Prof. Chrystal uses to establish his view that there is such a thing as the absolute probability of an event independent of the conditions of the persons concerned in it. He writes:

> When an actuary says that the probability that a man of 20 will live to be 60 is 59/97 ...... no one knows better than the actuary that this statement is a fact established (under certain circumstances and with certain limitations) by experience, and that it has nothing whatever to do with the mental attitude of any one*.

We agree that it has nothing to do with the mental attitude of anyone. But it has much to do with the limitations of know-

* *Transactions of the Actuarial Society of Edinburgh*, Vol. II. No. 13.

ledge in the parties concerned. After all, the statement cannot come to more than this, that according to the best tables of mortality to which the actuary has access the chance is 59/97. At the same time another actuary may have access to tables based on a larger observation from which he deduces the chance (say) 61/100. The chance of the event is different to the two observers because they have different degrees of knowledge.

Dr Venn indeed says:

> We shall naturally assume the observers to be in this as in other sciences all of them equally well informed. If they are not it is their own fault.

But as Probability is the science that has to do with the measure of limited knowledge, it is an arbitrary and absurd restriction to impose upon it, to assert that we must always assume all conceivable observations to have been made. If such an assumption is to be accepted, where shall we stop short of the omniscience which should observe everything and be able to deliver the future event with certainty from the matrix of cause in which it lies hid? Every question of probability is of the limited form. Certain observations having been made, certain conditions known, other conditions unknown, what is the chance that such and such an event will occur?

I allow, of course, that in determining the fair price to be paid for any venture we must assume the buyer and the seller to have the same knowledge; otherwise the price that one should fairly pay would be different from the price that the other should fairly receive. The morals of commerce demand that the conditions of the commodity should be patent to both parties, otherwise one is trading upon the ignorance of the other. But of this more anon.

It is because they have not grasped the idea of Probability as a science concerned with limited knowledge that some writers feel a repugnance to speaking of chance in regard to an event already decided. But the chance *to us* that a coin will turn up head is the same thing whether the coin be still spinning in the air or whether it have already fallen upon the table and been covered without our seeing its face.

Or, again, a friend sets out for a voyage on a vessel which is carrying 50 passengers, of whom 30, including our friend, are Englishmen and the rest are foreigners. A telegram is published that a passenger fell overboard and was lost. Apart from any knowledge I may have of our friend's ability to take care of himself the chance that the lost passenger is our friend is 1/50. The event is actually decided. To the captain of the vessel there is no probability whatever; he knows for certain which passenger is lost, but in my limited knowledge the odds are 49 to 1 against its being our friend.

But a second telegram is received which describes the lost passenger as an Englishman. The accession of new knowledge at once raises the chance to 1/30, or reduces the odds to 29 to 1 against the loss of our friend.

But the accession of knowledge which modifies the probability of an event is often of an indirect character. We receive perhaps no direct information about the event, but we observe something else to happen which is more likely to have happened as the consequence of the event in question than to have happened in some other way. If so, the observation certainly increases to us the chance that the event in question has happened.

For instance, here are two purses externally alike. I have put into one of them 5 sovereigns and a valueless token, and into the other 5 such tokens and one sovereign. Each purse therefore contains six coins indistinguishable, let us say, to the touch, and the purses themselves are externally indistinguishable. I ask a child to select one of the purses. What will you give for it? The two purses together contain £6. Taken at random the expectation of the two must be equal. The fair price of each is therefore £3.

But now let us examine the selected purse, the other being put altogether aside. The selected purse may contain £5, it may contain only £1. I shake it up and draw a coin. It is a sovereign. I put the coin back and draw again. It is a sovereign. I repeat the operation, and again it is a sovereign. Are you not beginning to suspect that this purse is the one that contains £5? We are not certain. It may be that this is the purse containing only one sovereign and that we have by chance drawn the same coin three times, an event of which the probability is only 1/216; or else this is the purse containing 5 sovereigns and a single token, and I have by chance drawn a sovereign each time, an event of which the probability is 125/216. There is a strong presumption that this is the richer purse. According to Laplace the probability is now 125 : 1 in favour of its being so, and the fair price to give for it is no longer £3 but £4. 19*s.* 4*d.* Some feel a difficulty in accepting Laplace's result, but Prof. Chrystal is the only writer of repute who denies that such observations as we have made by drawing out and replacing the coins have any legitimate effect whatever upon our conclusion as to which is the richer purse. On his theory it remains still an even chance whether this purse or that one be the richer. He is driven to this conclusion by his obstinate hypothesis that the chance of an event is in some way inherent in the event and is independent of the observer. The chance that the child selected the richer bag was originally $\frac{1}{2}$ and therefore he says it must always remain $\frac{1}{2}$. No experiment of ours can alter it.

And yet I suppose he would admit that another experiment might altogether alter the chance. If we were to draw two coins at once out of the purse and find them both sovereigns, this new observation would have the effect of raising the chance from $\frac{1}{2}$ to 1; but only to the person who had cognisance of the result of the experiment.

That I may not by any possibility misrepresent the Professor I will take the example of his own choice.

A bag contains 5 balls which are known to be either all black or all white, and both these are equally probable. A white ball is dropped into the bag and then a ball is drawn out at random and found to be white. What is now the chance that the original balls were white? The proper answer to this question is that the chance is precisely what it was before (viz.: $\frac{1}{2}$).

I maintain, on the contrary, that there is now a strong presumption that the bag contains white balls. In fact the odds are 6 to 1 in favour of this presumption.

For there were originally 12 possible events. The balls might be black, and any one of the 6 balls (5 black and one white) might be drawn. Or the balls might be white, and any one of the 6 white balls might be drawn. If $B$ and $W$ represent the cases of the original balls being black or white, and the small letters represent the different balls that may be drawn, the 12 possible events may be represented thus,

$$\begin{array}{cccccc} Bb_1 & Bb_2 & Bb_3 & Bb_4 & Bb_5 & Bw, \\ Ww_1 & Ww_2 & Ww_3 & Ww_4 & Ww_5 & Ww, \end{array}$$

and all these twelve are equally likely. But observation shews that a white ball is drawn. The first five events are therefore discarded and the remaining seven remain all equally likely. But one of these is of the form $Bw$ and six are of the form $Ww$. Hence the odds are 6 to 1 that what has occurred is $Ww$ and not $Bw$, or the odds are 6 to 1 in favour of $W$ as against $B$.

This is what Prof. Chrystal calls "applying the *à priori* rule in the usual mechanical way." It is, however, an argument as pure and sound as any in the Science of Probability, unsatisfactory only to those who are infatuated with the theory that the chance of an event is something inherent in the event itself, independent of the conditions of the observer. I assert that the mathematical probability of the same event may be different to different observers possessed of different data, and different to the same observer as he becomes possessed of greater information. If there be such a thing as the absolute probability of an event, it is not that with which we are concerned or that with which we profess to deal in treatises on Chance.

I can give no meaning to the question what is the absolute probability that a given marksman will win a shooting match. But if I have certain information as to his past achievements and the achievements of the other competitors I can calculate the presumption to be drawn from those particular data. I can say therefore what is the chance to me when I am in possession of those data, though the chance may become something different when further data are furnished.

The mental attitude of the observer does not affect a chance, but there are many chances which cannot be mathematically calculated, but can only be estimated by an exercise of judgment. I have already instanced the chance that a testator who has made a will in your favour will revoke it. The chance depends upon his character and habits. Does he often change his mind? Is he obstinate in pursuing a course to which he has once set himself? If you, with your knowledge of his character, are able to estimate that the odds are say 2 to 1 against his revoking his will, the chance thus estimated can be dealt with just as a calculated chance and you will say that £200 is the value of your expectation of a legacy of £300. But vain attempts are often made to determine numerically chances which can only be estimated by an exercise of judgment. For instance it is asked,

> If I take up a packet of 20 tickets numbered from 1 to 20 and find them arranged in order, what are the odds that the arrangement is due to design and not to accident?

And it is sometimes said that as the chance that accident should produce this particular arrangement is only $1 \div \underline{|20}$, the odds are $\underline{|20}$ to 1 in favour of the arrangement having come from design. But this argument is quite unsound. At least it involves the assumption that if the tickets are arranged by design the arrangement in order of the numbers is the only arrangement that could be designed. As a matter of fact any assigned arrangement of the tickets, however promiscuous it may appear, is equally unlikely to be produced by chance; and the same argument might be used to shew that whatever the arrangement of the tickets there were enormous odds against the arrangement having come by chance. The problem proposed really involves the question, if the cards are to be arranged by design what are the respective chances of the different arrangements being devised? This is a question concerning taste and will, and it is impossible to make it the subject of mathematical calculation. By confusion of thought here, many common arguments for design are vitiated, as well as some of the arguments in vogue amongst astronomers upon the distribution of stars.

But in practice, when we get beyond the region of cards and dice, probabilities are generally of a mixed kind, that is, they involve elements which can only be estimated by an exercise of judgment as well as others which lend themselves to arithmetical calculation. The probability of a man of 20 living to be 60 has been stated as 59/97, but if you have reason to think the particular man to be stronger or less strong than the average, if you know him to belong to a long-lived or a short-lived family, the price at which you would buy a remainder contingent on his reaching the age of 60 will be modified, and the modifying element will be one which can only be estimated, not calculated. Doubtless an Insurance Company is obliged to take bad lives with good, subject to the limit imposed by a medical examination, and its premiums are loaded not only to meet working expenses and to provide a reasonable profit, but also to cover a margin of risk.

As I have dissented from some of Prof. Chrystal's views, let me here express my full agreement with him when he denounces what he calls the Rule of Succession. He states the law in these terms:

> If the probability of an event is entirely unknown and it has been observed to happen $n$ times in succession the chance that it happens the next time is $(n+1)\div(n+2)$.

This so-called law of succession, as it is commonly applied, is extremely fallacious. It involves an hypothesis which can very seldom be justified. It assumes rightly enough that there is an unknown cause of the observed event: but it assumes unwarrantably that if $x$ be the chance (at any trial) of the cause producing the event, *$x$ is à priori equally likely to have any value from* 0 *to* 1.

It is easy to reduce it *ad absurdum.* Suppose for instance that in a collection of letters some are capitals, some are small letters, and among the small letters some are Greek. If we draw a letter at random 5 times and find it always small-Greek, the rule would give the same chance that the sixth letter drawn should be small-Greek as that it should be small.

Prof. Chrystal quotes the following from Mr Crofton:

> If a person knowing nothing of the water he is fishing draws up a fish each time in four casts of his line the chance that he succeeds a fifth time is $\frac{5}{6}$.

If this is true, it is (says Prof. Chrystal) "a physical fact depending on the nature of waters, fish, anglers, lines and seasons": and if so it is absurd to suppose that the result can be deduced by pure mathematics, which take no account of "water, fish, anglers, lines and seasons."

Nevertheless, we must say this for Mr Crofton, that if his question means that the "waters, fish, lines and seasons" are supposed to be adequate, and success depends only on the angler's skill, then there is a reasonable hypothesis on which his result is true.

The so-called Rule of succession ought to be stated as the Rule of presumptive skill*. We measure a person's skill in

* The distinction will be clearly seen if we contrast the two following experiences. (i) We throw a coin five times and it always turns up head. This is a mere succession of events and it affords no evidence or presumption for or against the next throw being head. The chance of head at the sixth throw remains as it was at the beginning $=\frac{1}{2}$. (ii) We watch an unknown archer shooting at a target. He hits the bull's-eye five times in succession. This continued success is so much evidence in favour of his skill. Our expectation of his hitting the bull's-eye at the sixth trial is therefore much greater than it was at first. If we deemed it at first an even chance whether he would hit the bull's-eye, we must now, on the hypothesis of the rule, estimate the odds as 6 to 1 in favour of his hitting at any assigned trial.

If Laplace applied the rule as is alleged to calculate the probability that the sun should rise to-morrow the application was altogether invalid unless a particular experiment by the chance that he succeeds at a given trial. If he is as likely as not to succeed we say that his skill is $\frac{1}{2}$. A man's skill thus measured may be expressed by any fraction between 0 and 1.

The Rule of presumptive skill is based on the assumption (which we admit is quite arbitrary*) that if nothing whatever is known of a man's skill, it is equally likely to have any value whatever between 0 and 1. As often as you see the man make the experiment and succeed, the presumptive value of his skill is increased. As often as you see him fail, the presumptive value of his skill is decreased. The rule then comes in. It asserts that if you have seen him make the experiment $n$ times, and succeed $p$ times out of the $n$, then the presumptive value of his skill is $(p+1)\div(n+2)$. Of course if he succeeds every time, the value becomes, as Prof. Chrystal gives it, $(n+1)\div(n+2)$.

The rule is applicable not to a succession of events in general, but to a succession of trials of skill only. If in Mr. Crofton's question success is simply a matter of the angler's skill, and if the angler is unknown to us, then, according to the rule, when he has succeeded 4 times we estimate his skill at $\frac{5}{6}$, and therefore we reckon at $\frac{5}{6}$ his chance of success at the next trial.

The Rule is easily established on the principles of Inverse Probability, and on this account it is obnoxious to those who wish to reject these principles. The rule, however, satisfies some stringent tests quite independent of Inverse Probability †.

it were assumed that Dame Nature were trying experimentally every morning to make the sun rise and on each occasion might or might not succeed.

* See note at the end of the volume.

† Let $S_n^p$ denote the presumptive skill of the operator when he has succeeded $p$ times in $n$ trials. Then $S_n^p$ is the chance of success at any assigned trial. But if we deal with failures instead of successes $S_n^{n-p}$ must be the chance of failing at a given trial. Therefore

$$S_n^p + S_n^{n-p} = 1 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(i)}.$$

Then there is the chance $S_n^p$ that the next trial will raise the presumptive

But though Mr Crofton's question be set aside on the ground that its solution requires information which we do not possess regarding the nature of "waters, fish, anglers, lines and seasons" we are not to dismiss every question simply on the ground that we have not all the data affecting the probability of the event enquired about. For this absence of some conceivable data constitutes the very ignorance on which the question of probability is based. Take for instance this question :—

It has been observed that in fishing at a particular point on an average three fishes per hour are drawn up. What is the chance that an angler fishing for 20 minutes will catch a fish?

There are many conceivable conditions not given us in regard to this question. Are more fish drawn up in the morning or in the evening? in fine weather or in wet? At what hour and in what weather did this particular angler go to fish? Was his tackle better or worse than the average? and so on. We do not know. But the question is, what is the chance to us upon the knowledge that we possess. Details which might be obtained by further observation are outside the question. So in questions of expectation of life the Insurance Company will grant its policy on the statistics of average life without attempting to weigh the effect of every incident in the past life of the candidate for insurance. Concerning the angler, we only know that on an average 30 fishes are drawn up in 10 hours, or 3,000 in 1,000 hours. We have no reason to suppose that one time is more

skill to $S_{n+1}^{p+1}$ and the chance $S_n^{n-p}$ that it will depress it to $S_{n+1}^p$. Therefore the chance of success in the next trial but one is $S_n^p S_{n+1}^{p+1} + S_n^{n-p} S_{n+1}^p$. But this must be equal to $S_n^p$. That is

$$S_n^p S_{n+1}^{p+1} + S_n^{n-p} S_{n+1}^p = S_n^p;$$

or in virtue of (i) we have

$$S_n^{n-p} S_{n+1}^p = S_n^p S_{n+1}^{n-p} \quad \text{(ii)}.$$

Any true expression for $S_n^p$ must satisfy the tests (i) and (ii). But the expression given by the rule plainly satisfies them both: which as far as it goes is a confirmation of the truth of the rule.

favourable than another, or if it be we have no ground to assume that the particular 20 minutes in question is favourable or unfavourable. We only know that in the long run a success occurs every 20 minutes. We can only treat these successes as occurring at random intervals, and on this hypothesis we arrive at a result not by empiric rule, but by strict and careful reasoning. We find that the chance of not getting a fish in the assigned 20 minutes is ·367879 or say 7/19 (the reciprocal of $e$ the base of Napierian logarithms) and the chance that the angler gets at least one fish is 12/19.

This is a direct application of a general theorem which I published in 1886*, which met with rather rough treatment at the hands of a reviewer in *The Academy*†. My theorem is this:

> If an event happen at random on an average once in time $t$ the chance of its not happening in an assigned period $T$ is $e^{-\frac{T}{t}}$.

For example, if an earthquake be observed on an average $n$ times in a year, the chance that in a given year there should be no earthquake is $e^{-n}$. Again, if one passenger be killed on a railway for every ten millions of miles that a passenger is carried, the chance that I shall make a journey of 1,000 miles in safety is $e^{-\frac{1}{10000}} = {\cdot}9999$ nearly.

The reviewer objects that there is no meaning in these questions unless we know the law of distribution of earthquakes or of railway accidents respectively. Representing the succession of time by a right line, and using a set of points $\alpha, \beta, \gamma, \ldots \omega$ to represent the instants at which earthquakes occur, he says:

> We ought to know something about the distribution of $\alpha$, $\beta$, $\gamma$.... It may be that they are at very equal intervals or it might be that $\alpha$, $\beta$, $\gamma$ are huddled together and then after a long blank interval occur $\delta$, $\epsilon$.

Of course if we had this additional knowledge our result might be modified. And if only our knowledge of the distri-

* *Choice and Chance*, Prop. LI. (4th Edition).

† *The Academy*, Oct. 16, 1886.

bution were carried far enough no question of chance would remain: we should assign the intervals with certainty. Or if we accepted a popular theory that railway accidents occur in groups of three with long intervals between the groups our result would be modified. But the question is, what is the chance when we know nothing of a law of distribution, but only that the event occurs at the stated *average* intervals. If there be no meaning in such a question it might as well be alleged that there is no meaning in the deduction from tabulated observations that 59/97 is the chance of a person aged 20 living to be 60, unless we have taken into account the law of distribution of epidemics. In all these cases alike we argue from the observed average frequency of an event without entering into the causes that produce the event, or the laws that govern its repetition.

I spoke at the beginning of the fair price to be paid for any contingent expectation. If you desire an example of an unfair price you have only to take up any newspaper and observe the betting on any race. The terms are always unfair, and they must be so if the bookmaker is to get a living. I do not mean that they are fraudulent; but they are always to the advantage of the bookmaker and against the backers of the horses. Just as in *rouge et noir* or any other game on a public gambling table there must be odds in favour of the banker to cover the expenses of his establishment, so there must be odds in favour of the bookmaker if the trade of the bookmaker is to exist.

I take last Saturday's *Times* and I find that for the Esher Stakes the betting was

| | | | | | |
|---|---|---|---|---|---|
| 9 to 4 | against | $A$ | 10 to 1 | against | $F$ |
| 9 to 2 | ,, | $B$ | 100 to 7 | ,, | $G$ |
| 25 to 3 | ,, | $C$ | 100 to 6 | ,, | $H$ |
| 25 to 3 | ,, | $D$ | 20 to 1 | ,, | $J$ |
| 9 to 1 | ,, | $E$ | 25 to 1 | ,, | $K$ |

If you calculate from these odds the chance of each horse winning you get

| | | | |
|---|---|---|---|
| *A* | ·3077... | *F* | ·0909... |
| *B* | ·1818... | *G* | ·0654... |
| *C* | ·1072 | *H* | ·0566... |
| *D* | ·1071 | *J* | ·0476... |
| *E* | ·1000 | *K* | ·0385... |

The sum of these chances is 1·1028.... But if they were the true chances of the several horses winning their sum would be unity. If the prices were fair the sum would be unity. The public therefore who back the horses are paying 10 per cent. too much, this advantage accruing to the bookmakers who bet against all the horses.

In fact if a bookmaker can make bets at the published rates to any extent he chooses he can secure a net profit of any extent. For instance if he bet against all the horses so that the sum at stake on every horse is the same, he will win a fixed amount whichever horse wins. (By the sum at stake on a horse I mean the addition of the money staked on him and staked against him.) In the case cited of the Esher Stakes the bookmaker might bet

| | | | | |
|---|---|---|---|---|
| £6923 | to | £3077 | against | *A* |
| 8182 | ,, | 1818 | ,, | *B* |
| 8928 | ,, | 1072 | ,, | *C* |
| 8928 | ,, | 1072 | ,, | *D* |
| 9000 | ,, | 1000 | ,, | *E* |
| 9091 | ,, | 909 | ,, | *F* |
| 9346 | ,, | 654 | ,, | *G* |
| 9434 | ,, | 566 | ,, | *H* |
| 9524 | ,, | 476 | ,, | *J* |
| 9615 | ,, | 385 | ,, | *K* |

and whichever horse comes in first he has a certain net profit of £1028.

The bookmaker's position is thus secure and the public in general must be the losers. Of course an individual among the public who is a good judge of horseflesh may back his judgment so successfully as to do more than compensate for the odds against him. But the public in general must lose if the bookmakers are to get their living by their trade, and every list of betting prices in the newspaper, when analysed, exhibits considerable odds to the bookmaker's advantage. It is common enough to find the advantage to be 6, 10, 12 or 15 per cent.: but sometimes it exceeds 20 per cent.

The most ardent gambler would scarcely care to play at pitch and toss on the condition that he should receive 10*d.* for every head and pay 11*d.* or 1*s.* for every tail, though this is precisely what he does when without technical judgment he backs race-horses. If the public knew a little of the principles of Probability they would be much less addicted to gambling.

But if the bookmaker is to make his book without exhibiting too obtrusively the advantage which he secures there must be more than two competitors in the race. The guileless boy has sense enough to know that if the odds be 7 to 6 against Cambridge they cannot at the same time be 6 to 5 against Oxford. This is undoubtedly the reason why there is less professional betting on Inter-University matches than on an ordinary horse-race.

But even when the price is mathematically fair there is a mathematical disadvantage to the individual in gambling. You may think of his position as symbolised by the oscillation of the compass needle about the pole. The needle swings to the east: let that represent a gain. It swings to the west: let that represent a loss. The needle oscillates, but its mean direction is the pole of neutrality. So if a man gamble on terms mathematically fair his gains may be expected in the long run to balance his losses and his mean position becomes the neutral one.

Yes, but we have not represented all the conditions. There is a point westward of the pole, which if the needle once touches, it sticks there, and no more oscillation is possible. That is to say the man may win and lose, win tens and lose tens, win hundreds and lose hundreds, win thousands and lose thousands and still his destiny may oscillate. But there is on the losing side one point at which, if he ever reaches it, he remains stationary. It is the point of ruin. If he once lose all that he has or all that he can afford to stake there is an end to his gambling. But the point at which the needle sticks is only on the losing side. It is an element of disadvantage with nothing to compensate it on the side of advantage.

But the disadvantage of the individual gambler is shewn in another way. If two men bet on fair terms, and repeat the same bet indefinitely (as for instance if they play at pitch and toss on equal terms) it is only a question of time which of them will be "cleared out." And the chance of either being cleared out can be shewn to be inversely proportional to his means. But practically an individual gambles not with an individual but with the whole race of gamblers one after another. He may clear out a dozen men, but there is always a 13th ready to continue the fray. He is therefore with his own limited resources gambling with an adversary whose means are unlimited. Dr Venn indeed objects that the adversaries with whom the gambler plays "are not one body with a common purse like the bank in a gambling establishment." But this makes no difference. The gambler is practically playing with the gambling public, and the gambling public is never exhausted or cleared out. The man may happily die while his good fortune is at its height, but if he live long enough and play long enough he must sooner or later be cleared out. His limited means may be exhausted, but the means of the public never. It must be assumed that the amount staked is something comparable with the man's resources or else we must demand unlimited time for the working out of the fatal result. But even so the tendency of gambling is shewn.

But it may be objected that if a man carefully proportion his stakes to his resources he can never be actually ruined. Yet in this case also the disadvantage of gambling is most manifest. Suppose as the simplest case that the cautious gambler play an even game (such as pitch and toss) always staking the same fraction $f$ of his resources. Every time that he wins he multiplies his means by $1+f$; every time that he loses he multiplies his means by $1-f$. But a gain and a loss do not balance one another. One gain and one loss will multiply his means by $(1+f)(1-f)=1-f^2$ which is less than unity, and 1000 gains and 1000 losses will multiply his resources by $(1-f^2)^{1000}$, a still smaller fraction.

Of late years gambling has been denounced in unmeasured terms from platform and from pulpit by speakers who have never taken the trouble to define the thing which they denounce or to discover wherein its evil consists. They insist however that the evil is not to be found merely in the abuse or excess of a practice which within moderate limits might be legitimate. The thing itself they are sure is wrong, whatever be the scale upon which it is practised.

But if the assertions thus hastily made be followed out to their legitimate conclusions, we find not only that it is wrong to play whist for penny points, or to join in a raffle, or to bet a pair of gloves with a lady on an event in which she is interested, but that it is equally wrong to compete for a prize in athletic sports, it is equally wrong to buy a piece of land on the expectation that its value will improve, it is equally wrong for a trader to purchase a commodity largely because he thinks a large demand will arise for it. In a word, if the principles of these moralists are to be rigorously applied, money must never be risked in the expectation of receiving a contingent advantage: trade and commerce must come to an end.

If 12 boys pay entrance fees of 6*d.* each for a race in which the winner is to receive a prize worth six shillings, each boy is precisely in the same position as if he accepted a bet of 12 to 1 in sixpences against his winning the race. If such a competition be right, such a bet cannot be wrong. One form of the speculation may be discouraged rather than the other, on grounds of ethical expediency, but inherently the two operations are equivalent and their morality identical.

If I give £120 for a piece of land whose agricultural value is £100 because I think that a public improvement will be carried out which will enable me to sell the land for £150, the transaction is precisely the same as if I were to make a bet of £30 to £20 that the improvement would be effected. If the speculative purchase is right the wager cannot be wrong.

But this example suggests a point of real importance. No reasonable person would condemn my speculative purchase of the land if I were only exercising the same judgment which anyone else might equally exercise as to the prospect of the betterment. But if I have private information which is out of the reach of the seller that this public improvement will be effected, the morality of the transaction is altogether changed. If I speculate upon my judgment the transaction is legitimate: if I speculate upon private information it is otherwise.

Now in every wager a man is backing either (1) his mere luck, or (2) his skill whether in judgment or bodily or intellectual prowess, or (3) his knowledge or private information. I assert that the first is generally foolish, the second within limits is legitimate, the third is contrary to good morals.

As an illustration of the instance in which a man is backing his mere luck, we may take the case of two rational men sitting down to play a game of mere chance for stakes. There is no skill in the game, no credit or satisfaction in winning, no end whatever to be gained except the possible gathering up of the stakes. If the stakes are small and inconsiderable we can hardly imagine a more frivolous waste of time: it is almost inconceivable that rational men should employ themselves to so little purpose. If on the other hand the stakes are of considerable value, all that we have said of the disadvantage of gambling will have its force. The players gratify the gambler's passion and that is all. As a means of gaining money, the process is indefensible. To appeal to the casting of the dice to determine whether I shall have my neighbour's wealth or he shall have mine seems to me to verge on profanity.

Yet there are sometimes circumstances which redeem from triviality on the one hand or folly on the other, the appeal to mere chance. If persons have a joint interest in some chattel or some work of art, which is in its nature indivisible, they may well draw lots for it whose it shall be. Somewhat similar is the case in which a costly object has been presented to be sold for the benefit of a charity and no one is found ready to purchase it outright. There seems to be justification in such a case for the common raffle, in which for instance 80 persons pay their half-crowns for the fair chance of obtaining an object worth £10. Probably the desire to benefit the charity is a stronger influence with most of the contributors than the hope of winning the prize. Charity is also the ruling motive in many raffles among workmen. One of their comrades is dying, and his family are in sore need. The men get up a raffle for the tools which the dying man will never use again, and their charitable motive is apparent in the fact that they are not careful to limit the number of tickets that the price of each may be mathematically fair. Say the tools are worth £3 : they will probably sell 100 tickets at a shilling, and so realize £5 for the suffering family.

I take this as an evidence of charitable purpose; and yet it is certain that many unlimited lotteries simply impose upon the popular ignorance of chance. The uneducated man perceives that if he is lucky enough to get for a shilling an object worth £3 he will have reason to congratulate himself. But he does not understand that it makes any difference to him whether 60 or 100 tickets be sold. A little more instruction in the principles of chance would be of immense service in saving the uneducated classes from the fraudulent designs of crafty schemers who make it a matter of business to get up unlimited raffles. At present it is far too common to see displayed in some poor window the announcement that a valuable object is to be raffled for. The value of the object is stated, and the price of the tickets, but there is not a word about the number of tickets. To the ignorant this is matter of no concern, but of course the promoter of the raffle will sell as many tickets as he can.

Besides the cases in which a man backs his luck purely there are many cases in which luck and skill are mixed, and the latter element may be of sufficient weight, especially if the stakes be small, to save the transaction from the condemnation passed on the backing of luck alone. To play whist or piquet for trifling stakes is perhaps an instance in point.

But as a general rule it is foolish to stake anything on mere luck. To back one's skill, one's industry, one's judgment, one's prowess is a very different thing. See the difference in a familiar example. I may promise sixpence to whichever of my boys finds the greatest variety of wild flowers, or to whichever pulls up most weeds in the garden, or to whichever climbs a tree to the highest branch. Few will think that I am demoralising the boys when the reward depends upon their quickness of intelligence, their industry or their agility. But I should certainly think that I were demoralising them if I offered the sixpence for the highest throw with dice.

When competitors enter for a race they are backing their running powers: when they enter for a shooting match they are backing their skill: when a man buys land which he thinks will improve in value he is backing his judgment: when he competes for a prize for some puzzle he is backing his ingenuity. All these are legitimate transactions, for it is skill of some sort that is backed.

But if one of those who enters for a cross-country race has private knowledge of some of the obstacles he is backing his private information, and the transaction becomes as immoral as when one over-reaches in a bargain. It is equally immoral if you play with dice, when you know but your adversary does not know that they are loaded; or if you spin a bent coin when you know but your adversary does not know that it has a special tendency to turn up head. As a rule it is unjustifiable to back your knowledge or even to back jointly your knowledge and your skill.

I must apologise for seeming so far to infringe upon the department of the moralist while I speak only as a mathematician. But it is as a mathematician that I protest against the loose language in which the moralist is apt to indulge when he approaches the subject of chance. It is in the name of Science that I demand something like a scientific treatment of the morality of those ventures which differ little in character, whether they take the form of a wager, or a commercial speculation, or the competition for a prize; but which do differ very much in character, whether those who make the ventures are pledging their luck or their skill or their private information.

To pledge my luck is generally foolish, to stake my skill may be legitimate, to back my private knowledge must be deemed immoral.

# EXERCISES IN CHOICE AND CHANCE.

1. If there are five routes from London to Cambridge, and three routes from Cambridge to Lincoln, how many ways are there of going from London to Lincoln viâ Cambridge?

2. Two hostile companies of 100 men each agreed to settle their dispute by single combat. In how many ways could the two champions be chosen?

3. Having four seals and five sorts of sealing wax, in how many ways can we seal a letter?

4. In how many ways can a consonant and a vowel be chosen out of the letters of the word *a l m o s t*?

5. In how many ways can a consonant and a vowel be chosen out of the letters of the word *o r a n g e*?

6. A die of six faces and a teetotum of eight faces are thrown. In how many ways can they fall?

7. There are five routes to the top of a mountain, in how many ways can a person go up and down?

8. Out of 20 knives and 24 forks, in how many ways can a man choose a knife and fork? And then, in how many ways can another man take another knife and fork?

9. Out of a list of 12 regular and 5 irregular verbs in how many ways can we choose an example of each?

10. Out of 12 masculine words, 9 feminine and 10 neuter, in how many ways can we choose an example of each?

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**1 to 9, 12.** Direct applications of *Choice*, Rule I.

**10, 13, 18.** Direct applications of *Choice*, Rule II.

**8.** By Rule I, $20 \times 24 = 480$. Then there remain 19 knives and 23 forks, and the second choice is $19 \times 23 = 437$.

11. Having six pairs of gloves, in how many ways can you take a right-hand and a left-hand glove without taking a pair?

12. Out of 5 bibles and 12 prayer books, in how many ways can I select a bible and prayer book?

13. Out of three bibles, seven prayer books and seven hymn books, in how many ways can I select a bible, prayer book and hymn book?

14. A bookseller shews me six bibles, three prayer books and four hymn books, also five volumes each containing a bible and a prayer book bound together, and seven volumes each containing a prayer book and hymn book bound together. In how many ways can I select a bible, prayer book and hymn book?

15. If in addition the bookseller produced three volumes, each containing a bible and a hymn book, what would my choice now be?

16. A basket contains 12 apples and 10 oranges, John is to choose an apple *or* an orange, and then Tom is to choose an apple *and* an orange. Shew that if John chooses an apple, Tom will have more choice than if John chooses an orange.

17. How many changes can be rung upon eight bells? And in how many of these will an assigned bell be rung last?

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**11.** When a right-hand glove is chosen; there remain only five eligible left-hand gloves. Hence Choice $= 6 \times 5 = 30$.

**14.** We may represent the books thus:

$$B^6,\ P^3,\ H^4,\ (BP)^5,\ (PH)^7.$$

If we select $(BP)$ and $H$, the choice is $5 \times 4 = 20$. If we select $(PH)$ and $B$, it is $7 \times 6 = 42$. If we select $B$, $P$, $H$, the choice is $6 \times 3 \times 4 = 72$. Total choice $= 20 + 42 + 72 = 134$.

**15.** The addition of $(BH)^3$ gives us the further choice of $(BH)$ and $P$. Choice $= 3 \times 3 = 9$. Total choice $= 143$.

**16.** We have $A^{12}$, $O^{10}$. If John choose $A$, Tom's choice is $11 \times 10 = 110$. If John choose $O$, Tom's choice is $12 \times 9 = 108$. He has $\therefore$ "more choice" in the former case.

**17.** By Rule III, $\underline{|8} = 40320$. If an assigned bell is to be rung last the only choice is in arranging the other seven bells. Choice $= \underline{|7} = 5040$.

18. There are three teetotums, having respectively 6, 8, 10 sides. In how many ways can they fall? And in how many of these will at least two aces be turned up?

19. Having bunting of five different colours, in how many ways can I select three colours for a tricolour flag?

20. In how many ways can I make the tricolour, the three selected colours being arranged in horizontal order?

21. A publisher proposes to issue a set of dictionaries to translate from any one language to any other. If he confines his system to five languages, how many dictionaries must he publish?

22. If he extend his system to ten languages, how many *more* dictionaries must he issue?

23. A man lives within reach of three boys' schools and four girls' schools. In how many ways can he send his three sons and two daughters to school?

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**18.** Choice $= 6 \times 8 \times 10 = 480$. The first can fall not-ace and the others ace in $5 \times 1 = 5$ ways. The second not-ace and the others ace in $7 \times 1 = 7$ ways. The third not-ace and the others ace in $9 \times 1 = 9$ ways. All three aces in 1 way. Total

$$= 5 + 7 + 9 + 1 = 22.$$

**19.** By Rule VIII, Choice $= 5 . 4 . 3 \div 1 . 2 . 3 = 10$.

**20.** The selection having been made in 10 ways, the colours can be arranged (by Rule III) in $1 . 2 . 3 = 6$ ways. Total choice $= 10 \times 6 = 60$.

**21.** The language *from* which the translation is to be made can be selected in 5 ways. Then the language *into* which the translation is to be made can be selected in 4 ways (as it must be different from the former). $\therefore$ Choice $= 5 \times 4 = 20$. Or 20 dictionaries must be published.

**22.** As before, the choice $= 10 \times 9 = 90$. $\therefore$ A second set of 5 languages will require 70 more dictionaries.

**23.** $B_1$, $B_2$, $B_3$ can each be sent to school in 3 ways and $G_1$, $G_2$ each in 4 ways. $\therefore$ Choice $= 3 . 3 . 3 . 4 . 4 = 432$.

24. If there be 300 Christian names, in how many ways can a child be named without giving it more than three Christian names?

25. In how many ways can four persons sit at a round table, so that all shall not have the same neighbours in any two arrangements?

26. In how many ways can seven persons sit as in the last question? And in how many of these will two assigned persons be neighbours? And in how many will an assigned person have the same two neighbours?

27. Five ladies and three gentlemen are going to play at croquet; in how many ways can they divide themselves into sides of four each, so that the gentlemen may not be all on one side?

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**24.** One name in 300 ways. Two names in $300 \times 299 = 89700$. Three names in $300 \times 299 \times 298 = 26730600$. Total 26820600 ways.

**25.** If I am one of the four I must have a different person opposite to me each time. There are 3 other persons. $\therefore$ Choice $= 3$.

**26.** Wherever I am placed the other six can sit relatively to me in $\underline{|6} = 720$ ways. But any arrangement from right to left and its reverse from left to right give each person the same two neighbours. Therefore these two arrangements will count as one and the choice is reduced to $720 \div 2 = 360$.

In these 360 arrangements I have 720 neighbours. $\therefore$ Any assigned person must be my neighbour, $720 \div 6 = 120$ times.

If two assigned persons be my neighbours they can be placed right and left of me in two ways: the other four can then be arranged in $\underline{|4} = 24$ ways, and as before the total number must be halved. Hence Choice $= 24$.

**27.** The gentlemen must be divided into 1 and 2, the ladies into 3 and 2. By Rule VIII, one gentleman and three ladies can be chosen in $\frac{3}{1} \times \frac{5 \,.\, 4 \,.\, 3}{1 \,.\, 2 \,.\, 3} = 30$ ways. Those who are not so chosen will form the opposite side. $\therefore$ Sides can be made in 30 ways.

28. I have six letters to be delivered, and three boys offer their services to deliver them. In how many ways have I the choice of sending the letters?

29. $A$ has seven different books, $B$ has nine different books; in how many ways can one of $A$'s books be exchanged for one of $B$'s?

30. In the case of the last question, in how many ways can two books be exchanged for two?

31. Five men, $A$, $B$, $C$, $D$, $E$, are going to speak at a meeting; in how many ways can they take their turns without $B$ speaking before $A$?

32. In how many ways, so that $A$ speaks *immediately* before $B$?

33. A company of soldiers consists of three officers, four sergeants and sixty privates. In how many ways can a detachment be made consisting of an officer, two sergeants and twenty privates? In how many of these ways will the captain and the senior sergeant appear?

34. Out of a party of twelve ladies and fifteen gentlemen, in how many ways can four gentlemen and four ladies be selected for a dance?

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**28.** Each letter can be sent in 3 ways: all in $3^6 = 729$ ways.

**29.** $A$'s choice $= 7$, $B$'s choice $= 9$. So there are $7 \times 9 = 63$ ways.

**30.** By Rule VIII, $A$'s choice $= 7 \,.\, 6 \div 1 \,.\, 2$, $B$'s choice $= 9 \,.\, 8 \div 1 \,.\, 2$. So there are $21 \times 36 = 756$ ways.

**31.** The total no. of ways in which they might speak is $\underline{|5} = 120$. In half of these $A$ will speak before $B$ and in half $B$ will speak before $A$. $\therefore$ Choice $= 120 \div 2 = 60$.

**32.** In this case we may think of $(AB)$ as one counter to be permuted with $C$, $D$, $E$. Choice $= \underline{|4} = 24$.

**33.** By Rule VIII, the choice is $\dfrac{3}{1} \times \dfrac{4 \,.\, 3}{1 \,.\, 2} \times \dfrac{\underline{|60}}{\underline{|20}\,\underline{|40}} = \dfrac{18\,\underline{|60}}{\underline{|20}\,\underline{|40}}$.

In the second case the only choice is the selection of one serjeant out of 3 and 20 privates out of 60. Choice $= 3\,\underline{|60} \div \underline{|20}\,\underline{|40}$.

**34.** By Rule VIII, the choice is

$$\frac{12 \,.\, 11 \,.\, 10 \,.\, 9}{1 \,.\, 2 \,.\, 3 \,.\, 4} \times \frac{15 \,.\, 14 \,.\, 13 \,.\, 12}{1 \,.\, 2 \,.\, 3 \,.\, 4} = 675675.$$

35. A man belongs to a club of thirty members, and every day he invites five members to dine with him, making a different party every day. For how many days can he do this?

36. Shew that the letters of *anticipation* can be arranged in three times as many ways as the letters of *commencement*.

37. How many five-lettered words can be made out of 26 letters, repetitions being allowed, but no consecutive repetitions (*i.e.* no letter must follow itself in the same word)?

38. A boat's crew consists of eight men, of whom two can only row on the stroke side of the boat, and three only on the bow side. In how many ways can the crew be arranged?

39. Having three copies of one book, two copies of a second book, and one copy of a third book, in how many ways can I give them to a class of twelve boys, (1) so that no boy receives more than one book; and (2) so that no boy receives more than one copy of any book?

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**35.** The number of ways of selecting five out of 29

$$=29 \,.\, 28 \,.\, 27 \,.\, 26 \,.\, 25 \div 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 = 118755.$$

**36.** The words may be represented thus:

$$i^3 a^2 n^2 t^2 c p o \text{ and } m^3 e^3 c^2 n^2 t o.$$

The former can be arranged by Rule VII in $\underline{|12} \div \underline{|3}\, \underline{|2}\, \underline{|2}\, \underline{|2}$ ways and the latter in $\underline{|12} \div \underline{|3}\, \underline{|3}\, \underline{|2}\, \underline{|2}$ ways. The former number is 3 times the latter.

**37.** The first place can be filled in 26 ways: each of the other four places in 25 ways. Choice $= 26 \,.\, 25^4 = 10156250$.

**38.** One of the three free men must be chosen to complete the bow side. Choice $= 3$. Then each side can be arranged in $\underline{|4}$ ways. Total choice $= 3\underline{|4} \,.\, \underline{|4} = 1728$.

**39.** (1) I must select 3 boys out of 12 for the first books; 2 boys out of 9 for the second; one out of 7 for the third. Choice $= \dfrac{12 \,.\, 11 \,.\, 10}{1 \,.\, 2 \,.\, 3} \times \dfrac{9 \,.\, 8}{1 \,.\, 2} \times \dfrac{7}{1} = 55440$.

40. In how many ways can a set of twelve black and twelve white draught-men be placed on the black squares of a draught-board?

41. In how many ways can the letters of the word *possessions* be arranged?

42. In how many ways can the letters of *cockatoo* be arranged?

43. Shew that the letters of *cocoon* can be arranged in twice as many ways as the letters of *cocoa*.

44. In how many ways can the letters of *pallmall* be arranged without letting all the $l$'s come together?

45. In how many ways can the letters of *oiseau* be arranged so as to have the vowels in their natural order?

46. In how many ways can the letters of *cocoa* be arranged so that $a$ may have the middle place?

47. In how many ways can the letters of *quartus* be arranged so that $q$ may be followed by $u$?

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(2) I must select 3 boys out of 12 for the first books; 2 out of 12 for the second; one out of 12 for the third. Choice

$$=\frac{12\,.\,11\,.\,10}{1\,.\,2\,.\,3}\times\frac{12\,.\,11}{1\,.\,2}\,.\,\frac{12}{1}=174240.$$

**40.** Choose 12 squares out of 32 for the black men; then 12 out of the remaining 20 for the white men. Choice $=\underline{|32}\div\underline{|12}\,\underline{|12}\,\underline{|8}$.

**41** to **43.** Direct applications of Rule VII.

**44.** The letters are $l^4 a^2 p\, m$. Total no. of arrangements $=\underline{|8}\div\underline{|4}\,\underline{|2}=840$. If the four $l$'s are to come together we may regard them as making one counter to permute with $a^2 p\, m$. No. of arrangements $=\underline{|5}\div\underline{|2}=60$. $\therefore$ The no. in which the $l$'s do not come together is $840-60=780$.

**45.** The vowels being in their natural order, our only choice is in inserting the $s$. Choice $=6$.

**46.** We can arrange *coco* in $\underline{|4}\div\underline{|2}\,\underline{|2}=6$ ways. The $a$ can then be inserted in only one way. Choice $=6$.

**47.** The $(qu)$ may be regarded as one block, the second $u$

48. In how many ways can the letters of *u b i q u i t o u s* be arranged so that $q$ may be followed by $u$?

49. In how many ways may the letters of *q u i s q u i s* be arranged so that each $q$ may be followed by $u$?

50. In how many ways can the letters of *i n d i v i s i b i l i t y* be arranged without letting two $i$'s come together?

51. In how many ways can the letters of *f a c e t i o u s* be arranged without two vowels coming together?

52. In how many ways can the letters of *f a c e t i o u s* be rearranged without changing the order of the vowels?

53. In how many ways can the letters of *a b s t e m i o u s l y* be rearranged without changing the order of the vowels?

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being left free. We have then 6 counters to permute. Choice $= \underline{|6} = 720$.

**48.** We have $(qu)\, u^2 i^2 b\, t\, o\, s$. No. of arrangements

$$= \underline{|9} \div \underline{|2}\, \underline{|2} = 90720.$$

**49.** We have $(qu)^2 i^2 s^2$. Choice $= \underline{|6} \div \underline{|2}\, \underline{|2}\, \underline{|2} = 90$.

**50.** The 8 other letters can be arranged in $\underline{|8}$ ways. The six $i$'s have then to be inserted in six of the nine places between the other letters including the two ends. This can be done in $9 \,.\, 8 \,.\, 7 \div 1 \,.\, 2 \,.\, 3 = 84$ ways. $\therefore$ The total no. of arrangements is $84 \,.\, \underline{|8} = 3386880$.

**51.** The odd places can be filled with the vowels in $\underline{|5}$ ways and the even places with the consonants in $\underline{|4}$ ways. Hence Choice $= \underline{|5} \times \underline{|4} = 2880$.

**52.** Total no. of arrangements of the letters $= \underline{|9}$. Among these the vowels will appear equally in all possible orders, i.e. in $\underline{|5}$ orders. Hence the no. of arrangements with the vowels in assigned order is $\underline{|9} \div \underline{|5} = 3024$. One of these is the original order. $\therefore$ There are 3023 rearrangements.

**53, 54.** Follow the method of Qn 52.

54. In how many ways can the letters of *parallelism* be rearranged without changing the order of the vowels?

55. In how many ways can the letters of *almost* be rearranged, keeping the vowels at their present distance apart?

56. In how many ways can the letters of *logarithms* be rearranged, so that the second, fourth and sixth places may be occupied by consonants?

57. In how many ways can two consonants and a vowel be chosen out of the word *logarithms*, and in how many of these will the letter *s* occur?

58. In how many ways can the letters of *syzygy* be arranged without letting the three *y*'s come together?

59. In how many ways without letting two *y*'s come together?

60. In how many ways can we arrange the letters of the words *choice and chance* without letting two *c*'s come together?

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**55.** The consonants can be arranged in $\underline{|4} = 24$ ways. The vowels can be arranged in 2 orders. Then they must occupy either the 1st or 4th places or 2nd and 5th or 3rd and 6th, giving a choice of 3 positions. Hence we have $24 \times 2 \times 3 = 144$ arrangements.

**56.** The 2nd, 4th and 6th places can be filled in $7 . 6 . 5$ ways. The remaining places can be filled by the other seven letters in $\underline{|7}$ ways. Hence Choice $= 210 \times 5040 = 1058400$.

**57.** There are 7 consonants and 3 vowels. Choice $= 21 \times 3 = 63$. If $s$ is to be one of the consonants, Choice $= 6 \times 3 = 18$.

**58.** Total arrangements $= \underline{|6} \div \underline{|3} = 120$. The three $y$'s together in $\underline{|4} = 24$. The eligible ways are therefore $120 - 24 = 96$.

**59.** The consonants can have $\underline{|3} = 6$ orders. Then out of the 4 spaces between them three places can be chosen for the $y$'s in 4 ways. $\therefore$ Choice $= 6 \times 4 = 24$.

**60.** We have the letters $c^4 h^2 e^2 a^2 n^2 oid$. We can arrange the 11 not-$c$'s in $\underline{|11} \div 16$ ways. We have then 12 places (10 between the letters and the 2 ends) in which to place 4 $c$'s. This can be

61. In how many ways can ten books be made into five parcels of two books each?

62. In how many ways can nine books be made up in four parcels of two books each and one book over?

63. In how many ways can they be made into three parcels of three books each?

64. Ten men and their wives are to be formed into five parties, each consisting of two men and two women. In how many ways can this be done?

65. In how many of these ways will an assigned man find himself in the same party with his wife?

66. In how many will two assigned men and their wives be together in one party?

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done in $12 \,.\, 11 \,.\, 10 \,.\, 9 \div 24$ ways. $\therefore$ The whole no. of arrangements is $\lfloor 11 \times 12 \,.\, 11 \,.\, 10 \,.\, 9 \div 16 \,.\, 24 = 1234926000$.

**61** to **63**. Direct applications of Rule VI. But the equal parcels being indifferent we have to divide in the three cases by $\lfloor 5$, $\lfloor 4$, and $\lfloor 3$ respectively. Thus we get the results

$$\frac{\lfloor 10}{\lfloor 5\,(\lfloor 2)^5} = 945; \quad \frac{\lfloor 9}{\lfloor 4\,(\lfloor 2)^4} = 945; \quad \frac{\lfloor 9}{\lfloor 3\,(\lfloor 3)^3} = 280.$$

**64**. By Rule VI, the men can be formed into parties in $\dfrac{\lfloor 10}{\lfloor 5\,(\lfloor 2)^5}$ ways and the women in the same no. of ways. Then the parties of women can be combined with the parties of men in $\lfloor 5$ orders. Hence Choice $= (\lfloor 10)^2 \div (\lfloor 2)^{10}\, \lfloor 5 = 107163000$.

**65**. He will find himself with all the wives equally in turn. $\therefore$ In $\frac{1}{5}$th of the whole no. he will find himself with his wife, i.e. in 21432600 arrangements.

**66**. If one party is made up of two assigned men and their wives, the choice is only in making the other 8 men and their wives into 4 parties. The no. of ways is $(\lfloor 8)^2 \div (\lfloor 2)^8\, \lfloor 4 = 264600$.

67. In how many ways can a purchaser select half a dozen handkerchiefs at a shop where seven sorts are kept?

68. A choir contains 10 members. In how many ways can a different six be selected every day for three days?

69. A man has six friends and he invites three of them to dinner every day for twenty days. In how many ways can he do this without having the same party twice?

70. Find the total number of selections that can be made out of the letters *ned needs nineteen nets.*

71. Find the total number of selections that can be made out of the letters *daddy did a deadly deed.*

72. How many selections of three letters can be made out of the letters in the last question?

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**67.** By Rule X, the no. of selections of 6 things out of 7 when repetitions are allowed is the same as the no. of selections of 6 things out of 12 without repetitions $= \underline{|12} \div \underline{|6}\,\underline{|6} = 924$.

**68.** There are 210 different selections of 6 men out of 10. Out of these 210 selections choice has to be made for three days. The no. of ways is $210 \,.\, 209 \,.\, 208 = 9129120$.

**69.** The no. of possible selections is 20. He must use all these and his only choice is in the order in which he takes them. Choice $= \underline{|20}$.

**70.** We have the letters $e^7 n^6 d^2 s^2 t^2 i$. We can deal with the $e$'s in 8 ways; i.e. we can take 0 or 1 or 2 or 3 or 4 or 5 or 6 or 7. So with the $n$'s in 7 ways. And so on. Hence we can deal with all in $8 \,.\, 7 \,.\, 3 \,.\, 3 \,.\, 3 \,.\, 2 = 3024$. This however includes the case when all the letters are rejected. Excluding this case we have 3023 selections.

**71.** Here we have $d^9 a^3 e^3 y^2 l\, i$. As in the last question the no. of selections is $10 \,.\, 4 \,.\, 4 \,.\, 3 \,.\, 2 \,.\, 2 - 1 = 1919$.

**72.** The selection may have ($\alpha$) 3 letters all alike, or ($\beta$) 2 letters alike or ($\gamma$) all different.

($\alpha$) must be either $d^3$, $a^3$ or $e^3$; i.e. it can be made in 3 ways.

73. How many arrangements of three letters can be made out of the same?

74. Shew that there are 8 combinations 3-together of the letters *veneer*, and 16 combinations 4-together of the letters *veneered*.

75. How many are the combinations 3-together of the letters *wedded*?

76. How many are the combinations 4-together of the letters *redeemed*?

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($\beta$) The two letters alike may be chosen in 4 ways and a single letter can be conjoined with them in 5 ways; i.e. there are 20 ways.

($\gamma$) is the selection of 3 letters out of 6, which can be made in 20 ways. Total: $3+20+20=43$.

**73.** The 3 selections ($\alpha$) give only 3 arrangements. The 20 selections ($\beta$) give $20 \times 3 = 60$ arrangements. The 20 selections ($\gamma$) give $20 \times 6 = 120$ arrangements. Total: $3 + 60 + 120 = 183$.

**74.** (i) We have $e^3 v n r$. We get 1 selection, 3 alike; $1 \times 3 = 3$ selections, 2 alike and 1 different; 4 selections all three different. Total: $1+3+4=8$.

(ii) We have $e^4 vnrd$. We get

1 selection all 4 alike;
4 selections 3 alike;
6 selections 2 alike;
5 selections all different.

Total: $1+4+6+5=16$.

**75.** We have $d^3 e^2 w$. We get 1 selection, all alike; $2 \times 2 = 4$ selections, 2 alike; 1 selection, all different. Total:

$$1+4+1=6.$$

**76.** We have $e^4 d^2 r m$. We get 1 selection, all alike; 3 selections, 3 alike; $2 \times 3 = 6$ selections, 2 alike; 1 selection, two and two alike; 1 selection, all different. Total:

$$1+3+6+1+1=12.$$

77. The number of combinations 5-together of the letters *ever esteemed* is 63.

78. In how many ways can three persons divide among themselves the letters *ever esteemed?*

79. In how many ways can they divide them so that each may take four?

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**77.** We have $e^6 vrstmd$. 1 selection, all alike; 6 selections, 4 alike; 15 selections, 3 alike; 20 selections, 2 alike; 21 selections, all different. Total: $1+6+15+20+21=63$.

**78.** The $e$'s can be divided as follows:

$$2+2+2=1 \text{ way};$$

$$\left.\begin{array}{l}6+0+0\\4+1+1\\0+3+3\end{array}\right\} \text{ 3 ways each; } =9 \text{ ways};$$

$$\left.\begin{array}{l}5+1+0\\4+2+0\\3+2+1\end{array}\right\} \text{ 6 ways each; } =18 \text{ ways}.$$

Total: 28 ways of distributing the $e$'s. And each of the other letters can be allocated in 3 ways. Hence the total no. of distributions is $28 \,.\, 3^6 = 20412$.

This however includes blank lots. There are $7 \,.\, 2^6 - 2 = 446$ ways of dividing the letters between 2 persons. The first person will $\therefore$ have a blank lot in 446 ways. So for the second and so for the third. Hence we have $3 \,.\, 446 = 1338$ ineligible ways owing to the existence of a single blank lot. There are also 3 ineligible ways in which all the letters will fall to one person. Hence the no. of distributions without blank lots is

$$20412 - 1338 - 3 = 19071.$$

**79.** The six letters $vrstmd$ can be distributed to 3 persons in $3^6 = 729$ ways. The first person will however have all 6 letters in 1 way, and he will have 5 letters in 12 ways, making 13 ineligible ways. And so for the second person and the third.

80. *A* has the letters *esteem*, *B* has *feeble*, and *C* has *veneer*. In how many ways can they exchange so that each shall have six?

81. In how many ways can the letters of *falsity* be arranged, keeping the consonants in their natural order, and the vowels also in their natural order?

82. In how many ways can the letters of *affection* be arranged, keeping the vowels in their natural order and not letting the two *f*'s come together?

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Hence there are 39 ineligible ways, leaving 690 ways of distributing the six letters so that no person has more than 4. The *e*'s must then be distributed in the one way necessary to bring each lot up to four letters. Hence 690 is the total no. of ways required.

**80.** Here we have $e^9 stmfblvnr$. The 9 not-*e*'s can be distributed to three persons in $3^9 = 19683$ ways. From these we must exclude the ways in which any person would have more than 6. The first person will have 9 in 1 way; 8 in $9 \times 2 = 18$ ways; 7 in $36 \times 4 = 144$ ways; making 163 ineligible ways. And so for the second person and the third. Hence there are 489 ineligible ways, leaving 19194 ways of distributing the nine letters so that no person has more than 6. The *e*'s must now be distributed so as to bring every share up to 6 letters. Hence 19194 is the total no. of ways in which each person can hold 6. One of these is the original arrangement. There are $\therefore$ 19193 ways of making an exchange.

**81.** The seven places can be assigned, 4 to consonants and 3 to vowels in $7 \,.\, 6 \,.\, 5 \div 1 \,.\, 2 \,.\, 3 = 35$ ways. There is then only one way of placing the consonants in their places and the vowels in theirs. $\therefore$ Choice $= 35$.

**82.** Total no. of arrangements with the vowels in natural order $= \underline{|9} \div \underline{|2}\,\underline{|4} = 7560$. No. in which the *ff* come together $= \underline{|8} \div \underline{|4} = 1680$. No. in which the *ff* do not come together

$$= 7560 - 1680 = 5880.$$

83. In how many ways can the letters of *kaffeekanne* be arranged so that the consonants and vowels come alternately?

84. In how many ways can the letters of *delete* be arranged, keeping the consonants in their natural order?

85. In how many ways can the letters of *delete* be arranged, keeping the consonants in their natural order and not letting two *e*'s come together?

86. In how many ways can the letters of *delirious* be arranged, keeping the consonants in their natural order and the vowels in their natural order?

87. What would be the answer of the last question if the two *i*'s must not come together?

88. In how many ways can the letters of *fulfil* be arranged without letting two letters which are alike come together?

89. In how many ways can the letters of *murmur* be arranged without letting two letters which are alike come together?

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**83.** We have the consonants $k^2 f^2 n^2$, and the vowels $e^3 a^2$. The consonants can be arranged in the odd places in $|\underline{6} \div 8 = 90$ ways, and the vowels in the even places in $|\underline{5} \div |\underline{3}\, |\underline{2} = 10$ ways. Total $90 \times 10 = 900$.

**84.** $|\underline{6} \div |\underline{3}\, |\underline{3} = 20$.

**85.** Compare Qn. 59. The consonants can be placed in one way and the *e*'s interposed in 4 ways. Choice $= 4$.

**86.** Out of 9 places we have to assign 4 to the consonants and 5 to the vowels. Choice $= |\underline{9} \div |\underline{5}\, |\underline{4} = 126$.

**87.** The $i\,i$ come together in $|\underline{8} \div |\underline{4}\, |\underline{4} = 70$ ways. There remain 56 eligible ways.

**88.** Total no. of arrangements $= |\underline{6} \div |\underline{2}\, |\underline{2} = 180$. But $ff$ come together in $|\underline{5} \div |\underline{2} = 60$ ways; $l\,l$ come together in 60 ways; $ff$ come together and $l\,l$ together in $|\underline{4} = 24$ ways. These 24 are counted in each of the two 60's. Hence the ineligible ways are $2 \,.\, 60 - 24 = 96$, and the eligible ways are $180 - 96 = 84$.

**89.** If we begin with $m\,u$ the two $r$'s must occupy either ($\alpha$) the 4th and 6th places, or ($\beta$) the 3rd and 5th, or ($\gamma$) the 3rd and 6th. In the first case the remaining $m\,u$ can only be in-

90. How many selections of four letters can be made out of the letters of *m u r m u r*; and how many arrangements of four letters?

91. How many arrangements of four letters can be made out of the letters of *f u l f i l*?

92. How many arrangements of five letters can be made out of the letters of *p a l l m a l l*?

93. How many arrangements of four letters can be made out of the letters of *k a f f e e k a n n e* without letting the three *e*'s come together?

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serted in one way; in either of the other cases two ways; total, 5 ways beginning with *mu*. And so for each of the six ways of writing the first two letters. Hence Choice $= 6 \times 5 = 30$.

**90.** Two alike and two different in 3 ways: two alike and two alike in 3 ways; total, 6 selections. The former can be arranged in 12 ways each: the latter in 6 ways each. Total: $3 \times 12 + 3 \times 6 = 54$ arrangements.

**91.** We have $f^2 l^2 u i$. 4 different: 1 selection $= 24$ arrangements; 2 alike and 2 diff.: 6 selections $= 72$ arrangements; 2 alike and 2 alike: 1 selection $= 6$ arrangements. Total:

$$24 + 72 + 6 = 102 \text{ arrangements.}$$

**92.** We have $l^4 a^2 p m$. We may make

| | | | |
|---|---|---|---|
| 4 alike and 1 diff.: | 3 selections | = | 15 arrangements |
| 3 alike and 2 alike: | 1 selection | = | 10 ,, |
| 3 alike and 2 diff.: | 3 selections | = | 60 ,, |
| 2 alike, 2 alike, 1 diff.: | 2 selections | = | 60 ,, |
| 2 alike and 3 diff.: | 2 selections | = | 120 ,, |
| Total: | $15 + 10 + 60 + 60 + 120$ | = | 265 arrangements. |

**93.** We have $e^3 k^2 a^2 f^2 n^2$. We may make

3 alike and 1 diff.: 4 selections = 8 arrangements

the odd letter coming between the *e*'s,

| | | | |
|---|---|---|---|
| 2 alike and 2 alike: | 10 selections | = | 60 ,, |
| 2 alike and 2 diff.: | 30 selections | = | 360 ,, |
| 4 different: | 5 selections | = | 120 ,, |

Total: $8 + 60 + 360 + 120 = 548$ arrangements.

94. How many arrangements of six letters can be made out of the letters of *nineteen tennis nets*?

95. How many arrangements of six letters can be made out of the letters of *little pipe* without letting two letters which are alike come together?

96. How many arrangements of five letters can be made out of the letters of *murmurer* without letting the three *r*'s come together?

97. In how many ways can the letters of *quisquis* be arranged without letting two letters which are alike come together?

98. In how many ways can the letters of *feminine* be arranged without letting two letters which are alike come together?

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**94, 95, 96** are solved by the same method as the three preceding. *Results:* 12289, 4020, 416.

**97.** Total no. of arrangements $= \lfloor 8 \div 2^4$:

$qq$, $uu$, $ii$, $ss$ occur in couplets in $\lfloor 4 \ldots\ldots\ldots\ldots (\alpha)$,

$qq$, $uu$, $ii$, in $\frac{1}{2}\lfloor 5$; $\therefore qq$, $uu$, $ii$ without $ss$ in $\frac{1}{2}\lfloor 5 - \lfloor 4 \ldots (\beta)$,

$qq$, $uu$, in $\frac{1}{4}\lfloor 6$; $\therefore qq$, $uu$, and no other couplet in

$$\tfrac{1}{4}\lfloor 6 - 2\{\tfrac{1}{2}\lfloor 5 - \lfloor 4\} - \lfloor 4 = \tfrac{1}{4}\lfloor 6 - 2 \,.\, \tfrac{1}{2}\lfloor 5 + \lfloor 4 \ldots\ldots\ldots\ldots (\gamma),$$

$qq$ in $\frac{1}{8}\lfloor 7$; $\therefore$ $qq$ and no other couplet in

$$\tfrac{1}{8}\lfloor 7 - 3\{\tfrac{1}{4}\lfloor 6 - 2 \,.\, \tfrac{1}{2}\lfloor 5 + \lfloor 4\} - 3\{\tfrac{1}{2}\lfloor 5 - \lfloor 4\} - \lfloor 4$$
$$= \tfrac{1}{8}\lfloor 7 - 3 \,.\, \tfrac{1}{4}\lfloor 6 + 3 \,.\, \tfrac{1}{2}\lfloor 5 - \lfloor 4 \ldots (\delta).$$

The whole no. of ineligible cases will be $\alpha + 4\beta + 6\gamma + 4\delta$

$$= 4 \,.\, \tfrac{1}{8}\lfloor 7 - 6 \,.\, \tfrac{1}{4}\lfloor 6 + 4 \,.\, \tfrac{1}{2}\lfloor 5 - \lfloor 4,$$

and the no. of eligible cases will be

$$\frac{\lfloor 8}{2^4} - 4\frac{\lfloor 7}{2^3} + 6\frac{\lfloor 6}{2^2} - 4\frac{\lfloor 5}{2} + \lfloor 4 = 864.$$

**98.** Following the method of the last question, we obtain

$$\frac{\lfloor 8}{2^3} - 3\frac{\lfloor 7}{2^2} + 3\frac{\lfloor 6}{2} - \lfloor 5 = 2220.$$

99. In how many ways can the letters of *m u h a m m a d a n* be arranged without letting three letters which are alike come together?

100. In how many ways can the same letters be arranged without letting two letters which are alike come together?

101. Out of 20 consecutive numbers, in how many ways can two be selected whose sum shall be odd?

102. Out of 30 consecutive integers, in how many ways can three be selected whose sum shall be even?

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**99.** Total no. of arrangements $= \underline{|10} \div \underline{|3}.\underline{|3}$. *mmm* will be found together in $\underline{|8} \div \underline{|3}$. So will *aaa*. Both triplets will be found together in $\underline{|6}$ ways. Hence the eligible no. is

$$\underline{|10} \div 36 - 2 . \underline{|8} \div 6 + \underline{|6} = 88080.$$

**100.** First consider the arrangements without the *aaa*. We have $m^3uhdn$. These can be arranged in $\underline{|7} \div \underline{|3} = 840$ ways. Amongst them there are $\underline{|5} = 120$ ways ($\alpha$) in which the *mmm* come together and $\underline{|6} - \underline{|5} = 600$ ways ($\beta$) in which *mm* but not *mmm* come together; leaving 120 ways ($\gamma$) in which no *m*'s come together. Now between the seven letters we have 8 places including the ends into which the *a*'s may be introduced. In ($\alpha$) there are two places between the *m*'s into which *a*'s *must* be put: the remaining *a* must be put into one of the remaining 6 places. Hence ($\alpha$) gives $120 \times 6 = 720$ arrangements.

In ($\beta$) we must put an *a* between the two *m*'s. The other two *a*'s must have 2 out of the remaining 7 places. Hence ($\beta$) gives $600 \times 21 = 12600$ arrangements.

In ($\gamma$) we may put the three *a*'s into any three of the 8 places. Hence ($\gamma$) gives $120 \times 56 = 6720$ arrangements.

$$\text{Total} = 720 + 12600 + 6720 = 20040.$$

**101.** One no. must be even and one odd. Each can be selected in 10 ways. Total choice $= 100$.

103. Out of an unlimited number of pence, half-pence and farthings, in how many ways can twenty coins be selected?

104. A person holds five coins and asks you to guess what they are. Knowing that he can only have sovereigns, half-sovereigns, crowns, half-crowns, florins, shillings, sixpences, fourpences, or threepences, how many wrong guesses is it possible to make?

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**102.** Three even nos. can be selected in

$$15 \,.\, 14 \,.\, 13 \div 1 \,.\, 2 \,.\, 3 = 455 \text{ ways.}$$

Two odd and one even in $15 \times 15 \,.\, 14 \div 1 \,.\, 2 = 1575$ ways. Total: 2030 ways.

**103.** If we take no pennies, we must have

$$\begin{array}{lcccccc} \text{halfpennies} & 20, & 19, & 18 \ldots & \text{or } 2, & 1, & 0, \\ \text{and farthings} & 0, & 1, & 2 \ldots & 18, & 19, & 20, \end{array}$$

i.e. 21 ways. Similarly if we take 1 penny we complete the choice in 20 ways; with 2 pennies in 19 ways; and so on. Hence Choice $= 21 + 20 + 19 + \ldots$ to 21 terms $= 231$.

Second Solution. Arrange 20 spaces for the coins, to be filled up with pennies from one end, then with halfpennies, and the rest with farthings. We have to insert two points of partition amongst the spaces, either in the 19 intervals between them or at the two ends. There are $\therefore$ 21 available places, and as blank lots are admissible the two partitions may be both in the same space. Hence the choice is the selection of 2 out of 21, repetitions allowed. By Rule X, Choice $= 21 \,.\, 22 \div 2 = 231$.

**104.** Following the steps of the second solution of Qn. 103, we have to put 8 points of partition into the intervals or ends of a row of 5 spaces. It is the selection of 8 things out of 6, repetitions allowed. Choice

$$= 6 \,.\, 7 \,.\, 8 \,.\, 9 \,.\, 10 \,.\, 11 \,.\, 12 \,.\, 13 \div \underline{|8} = 1287.$$

One of the 1287 possible guesses must be the right one. Hence there are 1286 wrong ones.

105. In the ordinary scale of notation how many numbers are there consisting of five digits? In how many of these is every digit an odd number? In how many is every digit an even number? In how many is there no digit lower than 6? In how many is there no digit higher than 3? How many contain all the digits 1, 2, 3, 4, 5? and how many contain all the digits 0, 2, 4, 6, 8?

106. How many different sums can be thrown with two dice, the faces of each die being numbered 0, 1, 3, 7, 15, 31?

107. How many different sums can be thrown with three dice, the faces of each die being numbered 1, 4, 13, 40, 121, 364?

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**105.** The first digit must not be a cypher. It can $\therefore$ be chosen in 9 ways and each of the other digits in 10 ways. Hence choice $= 9 \times 10 \times 10 \times 10 \times 10 = 90000$.

(ii) Each digit can be chosen in 5 ways. Choice

$$= 5^5 = 3125.$$

(iii) The first digit in 4 ways; the others in 5 each. Choice

$$= 4 \,.\, 5^4 = 2500.$$

(iv) Each digit can be chosen in 4 ways. Choice

$$= 4^5 = 1024.$$

(v) The first digit can be chosen in 3 ways, the others in 4 each. Choice $= 3 \,.\, 4^4 = 768$.

(vi) These 5 digits can be placed in $\underline{|5} = 120$ orders.

(vii) Out of $\underline{|5}$ orders, $\underline{|4}$ are ineligible beginning with the cypher. Choice $= 120 - 24 = 96$.

**106.** Any two faces give a different sum. Doublets can be thrown in 6 ways, and not-doublets in $6 \,.\, 5 \div 1 \,.\, 2 = 15$ ways. Total $= 21$.

**107.** The choice is the number of selections of 3 things out of 6, repetitions allowed. By Rule X, Choice

$$= 6 \,.\, 7 \,.\, 8 \div 1 \,.\, 2 \,.\, 3 = 56.$$

108. At a post-office they keep ten sorts of postage-stamps. In how many ways can a person buy twelve stamps? In how many ways can he buy eight stamps? In how many ways can he buy eight different stamps?

109. In how many ways can a pack of 52 cards be dealt to 13 players, four to each, (1) so that every one may have a card of each suit, (2) so that one may have a card of each suit, and no one else have cards of more than one suit?

110. In how many ways can a pack of cards be dealt to four players, subject to the condition that each player shall have three cards of each of three suits and four cards of the remaining suit?

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**108.** The selection of 12 out of 10, repetitions allowed, is 293930. The selection of 8 out of 10, repetitions allowed, is 24310. The selection of 8 out of 10 without repetitions is 45.

**109.** (1) Each suit can be dealt in $\underline{|13}$ ways. All in $(\underline{|13})^4$ ways.

(2) The one person can be selected in 13 ways: his cards in $13^4$ ways. Then there remain 12 cards in each suit. The 12 can be made into three indifferent parcels of four in $\underline{|12} \div (\underline{|4})^3 \underline{|3}$ ways, and all the 48 cards into 12 parcels in the fourth power of this number. Thus we get 12 parcels each containing four cards of one suit. These can be assigned to the 12 players in $\underline{|12}$ ways. Hence Choice $= 13^5 (\underline{|12})^5 \div (\underline{|4})^{12} (\underline{|3})^4 = \underline{|13}^5 \div (\underline{|4})^{12} (\underline{|3})^4 = (\underline{|13})^5 \div 2^{40} . 3^{16}$.

**110.** The assignment to each player of the suit in which he is to have the extra card can be made in $\underline{|4}$ ways. When this is settled each suit can be distributed in $\underline{|13} \div \underline{|4}\,\underline{|3}\,\underline{|3}\,\underline{|3}$ ways, and all the cards in the fourth power of this number. Hence Choice

$$= (\underline{|13})^4 \div (\underline{|4})^3 (\underline{|3})^{12} = (\underline{|13})^4 \div 3^{15} . 2^{21}.$$

111. In how many ways can 18 different things be divided among five persons, (1) so that four of them have four each, and the fifth has two things; (2) so that three of them have four each and the other two have three each?

112. If there be 14 sorts of things, and two things of each sort, find the total number of selections that can be made.

113. If there be twenty sorts of things and nine things of each sort, shew that 99999999999999999999 different selections can be made.

114. The game of bagatelle is played with 8 balls all alike and 1 different. The object is to get as many as possible of these balls into 9 different holes, no hole being capable of more than one ball. How many different dispositions of balls in the holes are possible?

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**111.** Direct application of Rule VI, but note the caution after Question 46 in *Choice and Chance.* The results are

$$5 \,.\, \underline{|18} \div (\underline{|4})^4 \underline{|2} = 5 \,.\, \underline{|17} \div 2^{12} \,.\, 3^2,$$

$$10 \,.\, \underline{|18} \div (\underline{|4})^3 (\underline{|3})^2 = 5 \underline{|17} \div 2^9 \,.\, 3^3.$$

**112.** Of the first sort we can take 0 or 1 or 2; and so of the others. Choice $= 3^{14}$ including the cases in which we take all or take none. Excluding the latter case, $3^{14} - 1 = 4782968$.

**113.** As in Qn. 112, Choice $= 10^{20} - 1$.

**114.** If the black ball is in a hole, this can occur in 9 ways; the other 8 holes can be either empty or full in $2^8$ ways; this makes $9 \,.\, 2^8$ ways. If the black ball be not in a hole the nine holes can be either empty or full (excluding the case of *all* being full) in $2^9 - 1$ ways. Hence total $= 9 \,.\, 2^8 + 2^9 - 1 = 2815$.

**115.** Proceeding as in Qn. 114. If the two black balls are in holes we have $36 \,.\, 2^7$ ways. If one of them is in a hole we have $9\,(2^8 - 1)$ ways. If neither is in a hole we have $2^9 - 1 - 9$ ways. Total: $36 \,.\, 2^7 + 9 \,.\, 2^8 - 9 + 2^9 - 10 = 7405$.

**116.** If the two special balls are different the no. of dispositions when either or both occupy a hole will be doubled. Hence Choice $= 72 \,.\, 2^7 + 18 \,.\, 2^8 - 18 + 2^9 - 10 = 14308$.

115. If there were 7 balls alike and 2 others alike, how many dispositions would be possible?

116. If there were 7 balls alike and 2 others different, how many dispositions would be possible?

117. In how many ways can 27 different books be distributed to $A$, $B$, $C$, so that $A$ and $C$ together may have twice as many as $B$?

118. Shew that out of 99 things the number of ways of selecting 70 is to the number of ways of selecting 30 as 3 to 7.

119. Four boys are in attendance at a telegraph office when 8 messages arrive. In how many ways can the messages be given to the boys without leaving any boy unemployed?

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**117.** $B$ can have 9 in $\underline{|27} \div \underline{|9}\,\underline{|18}$ ways. Then the other 18 can be divided between $A$ and $C$ in $2^{18}$ ways. Choice

$$= 2^{18} \,.\, \underline{|27} \div \underline{|9}\,\underline{|18}.$$

**118.** The no. of ways of selecting 70 is the same as the no. of ways of selecting 29 $= \dfrac{99 \,.\, 98 \,.\, 97 \,\ldots\, 73 \,.\, 72 \,.\, 71}{1 \,.\, 2 \,.\, 3 \,\ldots\, 27 \,.\, 28 \,.\, 29}$.

The no. of ways of selecting 30 is obtained from this by introducing the new factor 70 into the numerator and 30 into the denominator. The ratio of the former to the latter is $\therefore$ $30 : 70 = 3 : 7$.

**119.** The distribution may be made in the following ways:

$5+1+1+1$ (4 orders). Each $\underline{|8} \div \underline{|5} = 336$ ways,

$3+3+1+1$ (6 orders). Each $\underline{|8} \div \underline{|3}\,\underline{|3} = 1120$ ways,

$2+4+1+1$ (12 orders). Each $\underline{|8} \div \underline{|2}\,\underline{|4} = 840$ ways,

$3+2+2+1$ (12 orders). Each $\underline{|8} \div \underline{|3}\,\underline{|2}\,\underline{|2} = 1680$ ways,

$2+2+2+2$ (1 order). $\underline{|8} \div \underline{|2}\,\underline{|2}\,\underline{|2}\,\underline{|2} = 2520$ ways.

Choice $= 4 \,.\, 336 + 6 \,.\, 1120 + 12 \,.\, 840 + 12 \,.\, 1680 + 2520 = 40824$.

NOTE. When Chapter II has been read a simpler method can be applied. By help of Prop. XIV the choice is written $4^8 - 4 \,.\, 3^8 + 6 \,.\, 2^8 - 4 \,.\, 1^8 = 40824$.

120. Out of the first 100 integers, in how many ways can three be selected whose sum shall be divisible by 3?

121. The number of ways of selecting $x$ things out of $2x+2$ is to the number of ways of selecting $x$ things out of $2x-2$ as 99 to 7. Find $x$.

122. If $C_3^n + R_3^n = P_3^n$, find $n$. If $C_3^n + R_3^n = \frac{3}{4} P_3^n$, find $n$.

123. If $R_x^y = mR_y^x$ and $C_{x+1}^{x+y} = nC_{y+1}^{x+y}$, find $x$ and $y$.

124. Shew that $C_5^{n+1} : R_5^{n-1} = C_2^{n-2} : R_2^{n+2}$.

125. Shew that $(n-r)\, R_r^n\, C_r^n = nR_{2r}^{n-r}\, C_r^{2r}$.

126. Shew that $\{C_{r+1}^{n+1} - C_r^n\}\, C_{r-1}^{n-1} \div \{(C_r^n)^2 - C_{r+1}^{n+1}\, C_{r-1}^{n-1}\} = r$.

127. Shew that out of $m+n-1$ things the choice in selecting $m$ things is to the choice in selecting $n$ things as $n$ to $m$.

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**120.** There are 33 numbers divisible by 3; 34 numbers which leave remainder 1; and 33 which leave remainder 2. The sum will be divisible by 3, (1) if one number be taken out of each group, or (2) if all be taken out of any one group. Thus we have

$$33 \,.\, 34 \,.\, 33 + \frac{34 \,.\, 33 \,.\, 32}{1 \,.\, 2 \,.\, 3} + 2 \,.\, \frac{33 \,.\, 32 \,.\, 31}{1 \,.\, 2 \,.\, 3} = 53922 \text{ ways.}$$

**121.** We have $\dfrac{|\underline{2x+2}}{|\underline{x}\;|\underline{x+2}} : \dfrac{|\underline{2x-2}}{|\underline{x}\;|\underline{x-2}} = 99 : 7$.

$\therefore\ (2x+2)(2x+1)\, 2x\, (2x-1) : (x+2)(x+1)\, x\, (x-1) = 99 : 7$,

or $\qquad 4(2x+1)(2x-1) : (x+2)(x-1) = 99 : 7$,

whence $\qquad 13x^2 - 99x + 170 = 0$, and $x = 5$.

**122.** $$\frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3} + \frac{n(n+1)(n+2)}{1 \,.\, 2 \,.\, 3} = n(n-1)(n-2),$$

whence $n = 4$. So the second equation gives $n = 5$.

**123.** The equations reduce to $y = mx$ and $y^2 + y = n(x^2 + x)$.

**124.** Each ratio reduces to $(n-2)(n-3) : (n+2)(n+3)$.

**125.** Expressing each member in factors we immediately find that $(n-r)\, R_r^n\, C_r^n = nR_{2r}^{n-r}\, C_r^{2r}$.

128. There are $m$ parcels, of which the first contains $n$ things; the second $2n$ things; the third $3n$ things; and so on. Shew that the number of ways of taking $n$ things out of each parcel is $|\underline{mn} \div \{|\underline{n}\}^m$.

129. Shew that the number of ways of dividing $2n+2$ things into two equal parcels is the sum of the number of combinations of $2n$ things, $n$ and $n-1$ together.

130. Shew that when repetitions are allowed the number of selections of $n$ things out of $m+1$ is the same as the number of selections of $m$ things out of $n+1$.

131. Shew that $R_n^{2n} = 2R_{2n}^n$ and $R_n^{3n} = 3R_{3n}^n$.

132. When repetitions are allowed the choice of $m$ things out of $n$ is to the choice of $n$ things out of $m$ as $n$ to $m$.

133. A basket contains $2n+r$ apples and $2n-r$ pears. Shew that the choice of $n$ apples and $n$ pears is greatest when $r=0$, $n$ being constant.

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**126.** Dividing throughout by $(C_r^n)^2$, the first member of the equation becomes $\left\{\dfrac{n+1}{r+1} - 1\right\}\dfrac{r}{n} \div \left\{1 - \dfrac{n+1}{r+1} \cdot \dfrac{r}{n}\right\} = r.$

**127.** The factorial expressions are inversely as

$$|\underline{m}\,|\underline{n-1} : |\underline{n}\,|\underline{m-1},$$

that is directly as $n : m$.

**128.** Expressing $C_n^n$, $C_n^{2n}$ &c. in factorials we have

$$\frac{|\underline{2n}}{|\underline{n}\,|\underline{n}} \cdot \frac{|\underline{3n}}{|\underline{2n}\,|\underline{n}} \cdot \frac{|\underline{4n}}{|\underline{3n}\,|\underline{n}} \cdots \frac{|\underline{mn}}{|\underline{mn-n}\,|\underline{n}} = \frac{|\underline{mn}}{(|\underline{n})^m}.$$

**129.** $\frac{1}{2} \cdot C_{n+1}^{2n+2} = C_n^{2n} + C_{n-1}^{2n}.$

**130.** Each $= |\underline{m+n} \div |\underline{m}\,|\underline{n}.$

**131, 132.** $R_y^x : R_x^y = \dfrac{|\underline{x+y-1}}{|\underline{y}\,|\underline{x-1}} : \dfrac{|\underline{x+y-1}}{|\underline{x}\,|\underline{y-1}} = x : y.$

**133.** Let $Ch_x$ denote the choice when $r = x$; then we find

$$\frac{Ch_{x+1}}{Ch_x} = \frac{2n+x+1}{n+x+1} \cdot \frac{n-x}{2n-x};$$

134. Out of $3n$ consecutive integers, in how many ways can three be selected whose sum shall be divisible by 3?

135. If $pq+r$ different things are to be divided *as equally as possible* among $p$ persons, in how many ways can it be done? $(r<p)$.

136. Prove that the number of ways in which $p$ positive signs and $n$ negative signs may be placed in a row, so that no two negative signs shall be together, is equal to the number of combinations of $p+1$ things taken $n$ together. $(p>n)$.

137. The number of boys in the several classes of a school are in arithmetical progression, and a number of prizes equal to the common difference

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and therefore $Ch_{x+1}$ is less than $Ch_x$ if

$$2n^2+n-(n+1)\,x<2n^2+2n+(n-1)\,x,$$

i.e. for all values of $x$.

Therefore the choice is greatest when $x$ is least, that is, when $x=0$.

**134.** As in Qn. 120, the no. of ways is

$$n^3+3n\,(n-1)\,(n-2)\div 6=\tfrac{1}{2}n\,(3n^2-3n+2).$$

**135.** $r$ of the people must have $q+1$ things each and the others must have $q$ things. We can divide the persons into these two classes in $C^p_r$ ways. And then we may distribute the things in $\underline{|pq+r}\div(\underline{|q+1})^r\,(\underline{|q})^{p-r}$ ways. Hence the choice

$$=\underline{|p}\,\underline{|pq+r}\div(q+1)^r\,(\underline{|q})^p\,\underline{|r}\,\underline{|p-r}.$$

**136.** If we first write down the $p$ positive signs, with $p-1$ spaces between them, and two spaces at the ends, we have to choose for the $n$ negative signs $n$ spaces out of the $p+1$. Hence the proposition.

**137.** Let the no. of boys in the several classes be

$$a,\ a+b,\ a+2b,\ \dots\ a+(n-1)\,b.$$

of the progression is to be given to each class, no boy receiving more than one prize. Shew that if the prizes are all different the number of ways of giving them is the same as if all were to be given to the largest class.

138. The number of *different* throws that can be made with $n$ dice is

$$(1+n)\left(1+\frac{n}{2}\right)\left(1+\frac{n}{3}\right)\left(1+\frac{n}{4}\right)\left(1+\frac{n}{5}\right).$$

139. The number of ways of selecting four things out of $n$ different things is one-sixth of the number of ways of selecting four things out of $2n$ things which are two and two alike of $n$ sorts: find $n$.

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Then $b$ different prizes are to be given to each class. This can be done in

$$\frac{\lfloor a}{\lfloor a-b} \cdot \frac{\lfloor a+b}{\lfloor a} \cdot \frac{\lfloor a+2b}{\lfloor a+b} \ldots \&c. \, . \frac{\lfloor a+(n-1)b}{\lfloor a+(n-2)b} \text{ ways}$$
$$= \lfloor a+(n-1)b \div \lfloor a-b.$$

But the last expression is the no. of ways in which $nb$ prizes could be given to a class of $a+(n-1)b$ boys. Hence the proposition.

**138.** Every different throw with $n$ dice is a selection of $n$ numbers out of the six numbers on the faces of the dice, repetitions being allowed. Hence the no. of different throws is

$$R^6_n = C^{n+5}_n = C^{n+5}_5 = (n+1)(n+2)(n+3)(n+4)(n+5) \div \lfloor 5$$
$$= (1+n)\left(1+\frac{n}{2}\right)\left(1+\frac{n}{3}\right)\left(1+\frac{n}{4}\right)\left(1+\frac{n}{5}\right).$$

**139.** In selecting 4 things out of $2n$ things which are two and two alike of $n$ sorts, we may have the 4 things all different in $C^n_4$ ways; two alike and two different in $3C^n_3$ ways; and two alike and two alike in $C^n_2$ ways.

$$\therefore\ C^n_4 = \tfrac{1}{6}\{C^n_4 + 3C^n_3 + C^n_2\},$$

or
$$5C^n_4 = 3C^n_3 + C^n_2,$$

whence
$$5n^2 - 37n + 42 = 0, \text{ or } n = 6.$$

140. If bagatelle is played with $n$ balls alike and one different, and there are $n+1$ holes each capable of receiving one ball, the whole number of ways in which the balls can be disposed is $(n+3)\,2^n-1$.

141. In how many ways can a triangle be formed, having its angular points at three of the angular points of a given hexagon?

142. How many triangles can be formed having every side either 4 or 5 or 6 or 7 inches long?

143. How many different rectangular parallelepipeds can be constructed, the length of each edge being an integral number of inches not exceeding 10?

144. If four straight lines be drawn in a plane and produced indefinitely, how many points of intersection will there *generally* be?

145. If four straight lines be drawn in a plane, no two being parallel and no three concurrent, how many triangles will they form?

146. There are $n$ points in a plane, no three being in a straight line, except $p$ of them which are all in a straight line; how many triangles can be formed having some of these points for vertices?

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**140.** Proceeding as in Qn. 114. If the different ball be in a hole there are $(n+1)\,2^n$ dispositions possible. If it be not in a hole there are $2^{n+1}-1$. Total

$$=(n+1)\,2^n+2^{n+1}-1=(n+3)\,2^n-1.$$

**141.** Any three points will serve. $C_3^6=20$.

**142.** Any three of these lengths will serve: and they may be repeated. $R_3^4=20$.

**143.** We have to select three lengths out of the ten, repetitions being possible. $R_3^{10}=220$.

**144.** Any two will give a point of intersection. $C_2^4=6$.

**145.** Any three of the lines will be the sides of a triangle.

$$C_3^4=4.$$

**146.** Any three of the points will define a triangle unless they be *all* amongst the $p$ collinear points. Hence the no. is

$$C_3^n-C_3^p.$$

147. There are $p$ points in a straight line, and $q$ points on a parallel straight line; how many triangles can be formed having some of these points for vertices?

148. If there be $r$ more points lying on another parallel straight line, how many additional triangles can be formed, assuming that no three points lie on any transverse straight line?

149. Each side of a square is divided into $n$ parts. How many triangles can be formed having their vertices at points of section?

150. If $n$ straight lines be drawn in a plane, no two being parallel and no three concurrent, how many points of intersection will there be?

151. If $n$ straight lines be drawn in a plane, no two being parallel and no three concurrent, except $p$, which meet in one point, and $q$ which meet in another point, how many other points of intersection will there be?

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**147.** We must either take one point on the first line and two on the second, or two on the first and one on the second. Hence the no. of triangles

$$=\tfrac{1}{2}pq\,(q-1)+\tfrac{1}{2}qp\,(p-1)=\tfrac{1}{2}pq\,(p+q-2).$$

**148.** The additional triangles must have either one or two vertices on the new straight line. Hence the no. is

$$\tfrac{1}{2}r\,(p+q)\,(p+q-1)+\tfrac{1}{2}\,(p+q)\,r\,(r-1)$$
$$=\tfrac{1}{2}\,(p+q)\,r\,(p+q+r-2).$$

**149.** Any three of the $4\,(n-1)$ points will serve unless they be all on one side of the square.

$$C_3^{4n-4}-4C_3^{n-1}=2\,(n-1)^2\,(5n-8).$$

**150.** Any two will give a point of intersection.

$$C_2^n=\tfrac{1}{2}n\,(n-1).$$

**151.** Any two will give a new point of intersection unless they be *both* in the group $p$ or *both* in the group $q$. $C_2^n-C_2^p-C_2^q$ is the number exclusive of the two points of concurrence.

152. Into how many parts is an infinite plane divided by $n$ straight lines, of which no three are concurrent and no two parallel?

153. Into how many parts is infinite space divided by $n$ planes, of which no four meet in a point and no two are parallel?

154. A large floor is paved with square and triangular tiles, all the sides of the squares and triangles being equal. Every square is adjacent to four triangles, and every triangle is adjacent to two squares and a triangle. Shew that the number of angular points must be the same as the number of triangles, and that this must be double of the number of squares.

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**152.** Suppose there are $n$ lines, then introduce another. It will be cut by the old lines in $n$ points and will $\therefore$ be divided into $n+1$ segments. But each of these segments divides one of the former spaces into two, or each segment adds one to the no. of spaces. $\therefore$ The $(n+1)$th line increases the no. of spaces by $n+1$, whatever be $n$. But 1 line divides the plane into 2 spaces. $\therefore$ 2 lines will divide it into 4; 3 into 7; 4 into 11; and so on. And $n$ lines will divide it into

$$1+1+2+3+\ldots+n=1+\tfrac{1}{2}n(n+1)=\tfrac{1}{2}(n^2+n+2).$$

**153.** Proceeding as in the last question, a new $(n+1)$th plane would intersect the old planes in $n$ straight lines. These would make by the last result $\frac{1}{2}(n^2+n+2)$ enclosed plane-spaces each of which would divide one of the former volume-spaces into two. Hence the $(n+1)$th plane adds $\frac{1}{2}(n^2+n+2)$ to the no. of spaces. But 1 plane makes 2 spaces. $\therefore$ 2 planes will make 4 spaces; 3 planes will make 8; 4 planes will make 15; and so on. Hence $n$ planes will make

$$1+\tfrac{1}{2}\Sigma_0^{n-1}(n^2+n+2)=\tfrac{1}{6}(n+1)(n^2-n+6).$$

**154.** Every square is surrounded by 4 triangles, but as each triangle does duty for 2 squares we need only twice as many triangles as squares.

The no. of angles, being 3 for every triangle and 4 for every square, must amount to 10 times the no. of squares. But every

155. If, in a different pattern, the triangles be placed in blocks of three forming a trapezium, and every trapezium be adjacent to five squares, the same results will hold good as in the last question.

156. In how many ways can we form a triangle having each of its sides an integral number of inches, greater than $n$ and not greater than $2n$?

157. How many of these will be isosceles, and how many equilateral?

158. The number of triangles that can be formed having each of their sides an integral number of inches not exceeding $2n$ is $\frac{1}{6}n(n+1)(4n+5)$. And excluding all equilateral and isosceles triangles it is $\frac{1}{6}n(n-1)(4n-5)$.

159. If the number of inches is not to exceed $2n-1$ the number of triangles will be $\frac{1}{6}n(n+1)(4n-1)$. And excluding all equilateral and isosceles triangles it will be $\frac{1}{6}(n-1)(n-2)(4n-3)$.

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angular point is the concurrence of 5 adjacent angles. ∴ The no. of angular points is twice the no. of squares, or is equal to the no. of triangles.

**155.** The space will be covered by trapeziums, squares, and oblongs composed of two squares. Every trapezium is adjacent to two squares and two oblongs, but every square or oblong does duty for 4 trapeziums. ∴ For every two trapeziums there must be one square and one oblong, i.e. if we reduce the trapeziums to triangles and the oblongs to squares, for every six triangles there must be three squares or two triangles for one square.

The second paragraph in the solution of Qn. 154 applies identically here.

**156.** We have to select 3 out of the nos. $n+1$, $n+2$, ... $2n$, repetitions allowed. Choice

$$= n(n+1)(n+2) \div 6.$$

**157.** For an isosceles triangle we select the base in $n$ ways and the slant side in $n-1$. Choice $= n^2 - n$.

For an equilateral triangle one length can be selected in $n$ ways.

**158, 159.** Consider all possible triangles, whose longest side is $x$. Their perimeter must be

$$3x, \quad 3x-1, \quad 3x-2, \ \dots \ \&c. \ \dots \ \text{or } 2x+1.$$

160. If there be $n$ straight lines in one plane, no three of which meet in a point, the number of groups of $n$ of their points of intersection, in each of which no three points lie in one of the straight lines, is $\frac{1}{2}\,\underline{|n-1}$.

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With either perimeter $3x$ or $3x-1$ we can form only 1 triangle; with either perimeter $3x-2$ or $3x-3$ we can form 2 triangles; with either $3x-4$ or $3x-5$ we can form 3 triangles; and so on. Hence the no. of possible triangles is

$$1+1+2+2+3+3+\ldots \text{ to } x \text{ terms}$$
$$=\tfrac{1}{4}\{(x+1)^2-1\} \text{ if } x \text{ be even}$$
$$=\tfrac{1}{4}(x+1)^2 \text{ if } x \text{ be odd.}$$

Or if equilateral and isosceles triangles are excluded, then with either perimeter $3x-3$ or $3x-4$ we can form 1 triangle; with either $3x-5$ or $3x-6$ we can form 2 triangles; and so on. Hence the no. of possible triangles is

$$1+1+2+2+3+3+\ldots \text{ to } x-3 \text{ terms}$$
$$=\tfrac{1}{4}(x-2)^2 \text{ if } x \text{ be even}$$
$$=\tfrac{1}{4}\{(x-2)^2-1\} \text{ if } x \text{ be odd.}$$

For Qn. 158, we have to give $x$ all values from 1 to $2n$ and sum. Hence the no. of triangles is

$$\tfrac{1}{4}\Sigma_1^{2n}\{(x+1)^2\}-\tfrac{1}{4}n=\tfrac{1}{6}n(n+1)(4n+5).$$

Or if equilateral and isosceles triangles are excluded

$$\tfrac{1}{4}\Sigma_4^{2n}\{(x-2)^2\}-\tfrac{1}{4}(n-2)=\tfrac{1}{6}n(n-1)(4n-5).$$

For Qn. 159, we have to give $x$ all values from 1 to $2n-1$, and we have

$$\tfrac{1}{4}\Sigma_1^{2n-1}\{(x+1)^2\}-\tfrac{1}{4}(n-1)=\tfrac{1}{6}n(n+1)(4n-1),$$

and when equilateral and isosceles triangles are excluded

$$\tfrac{1}{4}\Sigma_4^{2n-1}\{(x-2)^2\}-\tfrac{1}{4}(n-2)=\tfrac{1}{6}(n-1)(n-2)(4n-3).$$

**160.** We must have 2 points on each of the $n$ lines, making $n$ points altogether. On one of the lines we can select points No. 1 and No. 2 in $\frac{1}{2}(n-1)(n-2)$ ways. From No. 2 we can

161. The number of ways of dividing $2n$ different things into two equal parts, is to the number of ways of similarly dividing $4n$ different things, as the continued product of the first $n$ odd numbers to the continued product of the $n$ odd numbers succeeding.

162. Shew that there are $2^{n-1}$ ways of selecting an odd number of things out of $n$ things.

163. If there be one card marked 1, two marked 2, three marked 3, and so on: the number of ways of selecting two cards on which the sum of the numbers shall be $n$ is $\frac{1}{12}n(n^2-1)$ or $\frac{1}{12}n(n^2-4)$ according as $n$ is odd or even.

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proceed along the transverse line and select a point No. 3 in $(n-3)$ ways, for we cannot use the points lying on either of the lines through No. 1. We can then proceed to No. 4 in $n-4$ ways, the points being excluded which lie on any of the three lines through Nos. 1 and 2. And so on. So we shall get No. "$(n-1)$" in 1 way, and finally we must take as No. "$n$" the intersection of our last line with the first line through No. 1. Thus we shall have $\frac{1}{2}\overline{n-1}\,.\,\overline{n-2}\ldots 3\,.\,2\,.\,1$ ways of making the group, or the no. of groups is $\frac{1}{2}\lfloor n-1$.

**161.**

$$\frac{\lfloor 2n}{\lfloor n\,\lfloor n} : \frac{\lfloor 4n}{\lfloor 2n\,\lfloor 2n} = \left(\frac{\lfloor 2n}{\lfloor n}\right)^2 : \frac{\lfloor 4n}{\lfloor 2n}$$

$$=(1\,.\,3\,.\,5\ldots\overline{2n-1})^2 \;:\; 1\,.\,3\,.\,5\ldots\overline{4n-1}$$

$$=1\,.\,3\,.\,5\ldots\overline{2n-1} \;:\; \overline{2n+1}\,.\,\overline{2n+3}\ldots\overline{4n-1}.$$

**162.** Omitting one of the things, the total no. of selections out of the rest is $2^{n-1}$ (Prop. XI). If in any selection we have an even no. of things we must add the omitted thing, but if we have an odd no. we must leave out the omitted thing. Hence choice $= 2^{n-1}$.

**163.** The no. of ways of selecting first a card marked $x$ and then a card marked $n-x$ is $x(n-x)$. If we give $x$ all values from 1 to $n-1$ the sum is $\frac{1}{6}n(n^2-1)$. If $n$ is odd, every way

164. There are $3n+1$ things of which $n$ are alike and the rest all different: shew that there are $2^{2n}$ ways of selecting $n$ things out of them.

165. In how many ways can three numbers in arithmetical progression be selected from the series $1, 2, 3 \ldots 2n$, and in how many ways from the series $1, 2, 3 \ldots (2n+1)$?

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has been counted twice, and the true choice is $\therefore \frac{1}{12}n(n^2-1)$. If $n$ is even, a further correction must be made. For we have counted the cards $\frac{n}{2}, \frac{n}{2}$ as if they could be drawn in

$$\frac{1}{2} \cdot \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{8} \text{ ways,}$$

whereas the choice is the selection of 2 out of $\frac{n}{2}$ cards $= \frac{1}{8}(n^2 - 2n)$.

We have $\therefore$ to deduct $\frac{n}{4}$ ways, and the choice becomes

$$\tfrac{1}{12}n(n^2-4).$$

**164.** Out of the $2n+1$ different things $2^{2n+1}$ selections can be made. Half of these will be less and half greater than $n+\frac{1}{2}$.

Take the $2^{2n}$ selections which do not exceed $n$, and add to each of them the requisite no. of the things all alike to bring them to selections of $n$ things. This completion of them will not alter their no., which thus remains $2^{2n}$.

**165.** If $x$ be the common difference the extreme terms will include a range of $2x+1$ nos. They can $\therefore$ take $2n-2x$ positions in the series of $2n$ nos., and $2n+1-2x$ in the series of $2n+1$ nos. In the former case $x$ can range from 1 to $n-1$, in the latter from 1 to $n$. Hence the two results are

$$\Sigma_1^{n-1}(2n-2x) = n^2 - n \text{ and } \Sigma_1^n(2n+1-2x) = n^2.$$

166. By considering the selections of $n$ things out of $n$ things with repetitions, writing down separately the number of selections when 1, 2, 3... or $n$ *different* things are selected, shew that

$$\left(\frac{1}{\underline{|n-1}}\right)^2+\frac{1}{\underline{|1}\,\underline{|2}}\left(\frac{1}{\underline{|n-2}}\right)^2+\frac{1}{\underline{|2}\,\underline{|3}}\left(\frac{1}{\underline{|n-3}}\right)^2+\ldots\text{to } n \text{ terms}=\frac{\underline{|2n-1}}{(\underline{|n}\,\underline{|n-1})^2}.$$

167. Find the number of ways in which $3n$ different books may be distributed to three different persons so that their shares may be in Arithmetical Progression. (The cases in which the shares would be $0$, $n$, $2n$ or $n$, $n$, $n$ are to be included.)

168. If $M_r$ be the number of permutations of $m$ things taken $r$ together, and $N_r$ the number of permutations of $n$ things taken $r$ together, prove that the number of permutations of $m+n$ things $r$ together will be obtained by expanding $(M+N)^r$, and in the result replacing the indices by suffixes.

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**166.** The selections of $n$ things out of $n$ with repetitions are

$$\underline{|2n-1}\div\underline{|n}\,\underline{|n-1}.$$

If the selection is to include exactly $x$ different things out of the $n$, the $x$ things may be determined in $C^n_x$ ways. One of each may then be taken in 1 way, and the remaining $n-x$ may be selected out of the $x$ things with repetitions in $C^{n-1}_x$ ways. Hence the choice $=C^n_x C^{n-1}_x$. $\therefore\ \Sigma^n_1\{C^n_x C^{n-1}_x\}=\underline{|2n-1}\div\underline{|n}\,\underline{|n-1}$.

Giving $x$ the values $n$, $n-1$, $n-2$, &c. in order, and dividing throughout by $\underline{|n}\,\underline{|n-1}$ we obtain the required result.

**167.** The middle share can be chosen in $C^{3n}_n$ ways, and the person who is to have this share can be determined in 3 ways. The remaining $2n$ books can then be divided between the other two persons in $2^{2n}$ ways. Choice

$$=3\,.\,2^{2n}\,\underline{|3n}\div\underline{|2n}\,\underline{|n}.$$

**168.** Let there be $m$ capital letters and $n$ small letters. Classifying the selections of $r$ letters according as they contain 0, 1, 2, 3 ... or $x$ small letters, we get

$$C^{m+n}_r=C^m_r+C^m_{r-1}C^n_1+C^m_{r-2}C^n_2+\ldots+C^n_r.$$

169. In the expansion of $(a_1+a_2+\ldots+a_p)^n$ where $n$ is an integer not greater than $p$, there are $C^p_n$ terms, in none of which any one of the quantities $a_1$, $a_2 \ldots a_p$ occurs more than once as a factor; and the coefficient of each of these terms is $\underline{|n}$.

170. There are $2n$ letters, two and two alike of $n$ different sorts. Shew that the number of orders in which they may be arranged, so that no two letters which are alike may come together, is

$$\frac{1}{2^n}\left\{\underline{|2n} - \frac{n}{1} 2\, \underline{|2n-1} + \frac{n(n-1)}{1 \cdot 2} 2^2\, \underline{|2n-2} - \&c. \text{ to } n+1 \text{ terms}\right\}.$$

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Multiply by $\underline{|r}$;

$$P^{m+n}_r = P^m_r + r \cdot P^m_{r-1} C^n_1 + r(r-1) P^m_{r-2} C^n_2 + \ldots + \underline{|r}\, C^n_r$$

$$= P^m_r + \frac{r}{1} \cdot P^m_{r-1} P^n_1 + \frac{r(r-1)}{1 \cdot 2} P^m_{r-2} P^n_2 + \ldots + P^n_r,$$

that is

$$(M+N)_r = M_r + \frac{r}{1} M_{r-1} N_1 + \frac{r(r-1)}{1 \cdot 2} M_{r-2} N_2 + \ldots + N_r,$$

which proves the proposition.

**169.** If we write down

$$a_1 + a_2 + a_3 + \ldots + a_p$$

$n$ times and multiply the $n$ lines together every term in the continued product must be the product of a letter chosen out of each line. In the required term there are to be no repetitions, $\therefore$ the letters composing it can be chosen in $C^p_n$ ways, or there are $C^p_n$ such terms.

But for any such term the first letter can be chosen out of any of the $n$ lines, the second out of any of the remaining $n-1$ lines, and so on. Hence there are $\underline{|n}$ ways of obtaining the term, in other words it will have by addition a coefficient $\underline{|n}$.

**170.** When two letters alike come together, let them be called a couplet. The total no. of permutations is $\underline{|2n} \div 2^n$.

Any couplet such as $aa$ will occur in $\underline{|2n-1} \div 2^{n-1}$ permutations.

171. The number of ways in which $r$ different things may be distributed among $n+p$ persons so that certain $n$ of those persons may have one at least is

$$(n+p)^r - n(n+p-1)^r + \frac{n(n-1)}{1 \cdot 2}(n+p-2)^r - \&c.$$

172. Shew that, for $n$ different things, $1 -$ (number of partitions into two parts) $+ \lfloor 2$ (number of partitions into three parts) $- \ldots \pm \lfloor n-1$ (number of partitions into $n$ parts) $= 0$.

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Any two couplets in $\lfloor 2n-2 \div 2^{n-2}$ permutations, and so on.

$\therefore$ By Prop. XIV, the no. of arrangements free from couplets is

$$\frac{\lfloor 2n}{2^n} - \frac{n}{1} \cdot \frac{\lfloor 2n-1}{2^{n-1}} + \frac{n \cdot \overline{n-1}}{1 \cdot 2} \cdot \frac{\lfloor 2n-2}{2^{n-2}} + \ldots.$$

**171.** $r$ different things may be distributed among $n+p$ persons in $(n+p)^r$ ways. A given person will have a blank lot in $(n+p-1)^r$ ways; two given persons in $(n+p-2)^r$ ways, and so on.

By Prop. XIV, the ways in which $n$ given persons will not have blanks can be written down as in the question.

**172.** The partition into $r$ parts is, by Prop. XXIII, $\lfloor n$ times the coeff. of $x^n$ in $(e^x - 1)^r \div \lfloor r$. $\therefore$ The given series $= \lfloor n$ times the coeff. of $x^n$ in

$$(e^x - 1) - \frac{(e^x-1)^2}{2} + \frac{(e^x-1)^3}{3} - \&c. \text{ to } n \text{ terms,}$$

and the terms after the $n$th would contain the $(n+1)$th and higher powers of $e^x - 1$, that is, of $x + \frac{x^2}{2} + \&c.$; i.e. they would only contain powers of $x$ higher than the $n$th. Thus the continuance of the series to infinity will not affect the coeff. of $x^n$. But the series when continued to infinity represents

$$\log(1 + e^x - 1) = \log e^x = x.$$

$\therefore$ If $x > 1$ the coefficient of $x^n$ is zero, and $\therefore$ the sum of the given series is zero.

173. Shew that, if $m > n$,

$$\Sigma_{x=0}^{x=n} \{P^n_x \div P^m_x\} = (m+1) \div (m-n+1).$$

174. Shew that

$$\Sigma_{x=0}^{x=n} \{C^n_x C^n_r \div C^{2n}_{x+r}\} = (2n+1) \div (n+1).$$

175. Prove that, to $n$ terms,

$$R^m_1 + 2R^m_2 + 3R^m_3 + \ldots = (n+1)\, R^{m+1}_n - R^{m+2}_n = mR^{m+2}_{n-1}$$

and

$$R^m_1 + 2^2R^m_2 + 3^2R^m_3 + \ldots = (n+1)^2R^{m+1}_n - (2n+3)\, R^{m+2}_n + 2R^{m+3}_n$$
$$= m\,(m+1)\, R^{m+3}_{n-2} + mR^{m+2}_{n-1}.$$

Verify the results when $n=2$.

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**173.** $\Sigma\{P^n_x \div P^m_x\} = \Sigma P^{m-x}_{m-n}\, \underline{|n} \div \underline{|m}$

$$= \frac{\underline{|n}}{\underline{|m}}\{P^{m-n}_{m-n} + P^{m-n+1}_{m-n} + P^{m-n+2}_{m-n} + \ldots + P^m_{m-n}\},$$

and by *Choice and Chance*, Excursus 5,

$$= \frac{\underline{|n}}{\underline{|m}}\, P^{m+1}_{m-n+1} \div (m-n+1) = \frac{m+1}{m-n+1}.$$

**174.** By *Choice and Chance*, Excursus 10, we have

$$\Sigma^n_0\{C^n_x \div C^p_{q+x}\} = \frac{p+1}{q+1} \div C^{p+1-n}_{q+1};$$

write $2n$ for $p$, and $r$ for $q$, and the present theorem follows.

**175.** We have $xR^m_x = mR^{m+1}_{x-1}$. Give $x$ values from 1 to $n$ and add; then

$$R^m_1 + 2R^m_2 + 3R^m_3 + \ldots + nR^m_n = mR^{m+2}_{n-1}$$
$$= (n+1)\, R^{m+1}_n - R^{m+2}_n. \quad \text{Q. E. D. (1).}$$

Again $\quad x^2R^m_x = mxR^{m+1}_{x-1} = m\,(m+1)\, R^{m+2}_{x-2} + mR^{m+1}_{x-1}.$

Give $x$ values from 1 to $n$ and add; then

$$R^m_1 + 2^2R^m_2 + 3^2R^m_3 + \ldots + n^2R^m_n = m\,(m+1)\, R^{m+3}_{n-2} + mR^{m+2}_{n-1}$$
$$= (n+1)^2\, R^{m+1}_n - (2n+3)\, R^{m+2}_n + 2R^{m+3}_n. \quad \text{Q. E. D. (2).}$$

176. Prove that

$$\frac{m}{1}+\frac{m(m+1)}{1.2}+\frac{m(m+1)(m+2)}{1.2.3}+\dots \text{ to } n \text{ terms}$$
$$=\frac{n}{1}+\frac{n(n+1)}{1.2}+\frac{n(n+1)(n+2)}{1.2.3}+\dots \text{to } m \text{ terms.}$$

177. Shew that $\underline{|2n-1}\div(\underline{|n-1})^2$ is the sum to $n$ terms of the series

$$(C_1^n)^2+2(C_2^n)^2+3(C_3^n)^2+\dots.$$

178. Apply Prop. XIV. to shew that

$$\frac{\underline{|n+r-1}}{\underline{|r}}-\frac{n}{1}\frac{\underline{|n+r-3}}{\underline{|r-2}}+\frac{n(n-1)}{1.2}\frac{\underline{|n+r-5}}{\underline{|r-4}}-\text{\&c. (till it stops)}=\frac{\underline{|n}\,\underline{|n-1}}{\underline{|r}\,\underline{|n-r}}.$$

Or, putting $n=r$,

$$\frac{\underline{|2n-1}}{\underline{|n}}-\frac{n}{1}\frac{\underline{|2n-3}}{\underline{|n-2}}+\frac{n(n-1)}{1.2}\frac{\underline{|2n-5}}{\underline{|n-4}}-\text{\&c.}\dots\dots=\underline{|n-1}.$$

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**176.** The first series may be written (Excursus 4)

$$R_1^m+R_2^m+R_3^m+\dots+R_n^m=R_n^{m+1}.$$

The second $\quad R_1^n+R_2^n+R_3^n+\dots+R_m^n=R_m^{n+1}.$

But $\quad R_n^{m+1}=C_{m,\,n}=R_m^{n+1}, \quad \therefore$ &c. Q. E. D.

**177.** Suppose we have to choose $n-1$ letters out of $n$ capitals and $n-1$ small letters. The choice is

$$\underline{|2n-1}\div\underline{|n}\,\underline{|n-1}.$$

But if we analyse the selections according as we choose 0, 1, 2, ... or $n-1$ small letters the choice is

$$C_0^{n-1}C_{n-1}^n+C_1^{n-1}C_{n-2}^n+C_2^{n-1}C_{n-3}^n+\dots \text{ to } n \text{ terms.}$$

And the $x$th term is

$$C_{x-1}^{n-1}C_{n-x}^n=\frac{x}{n}C_x^nC_{n-x}^n=\frac{x}{n}(C_x^n)^2.$$

$$\therefore\ \underline{|2n-1}\div\underline{|n}\,\underline{|n-1}=\Sigma\left\{\frac{x}{n}(C_x^n)^2\right\},$$

or

$$\underline{|2n-1}\div(\underline{|n-1})^2=\Sigma\{x(C_x^n)^2\}$$
$$=(C_1^n)^2+2(C_2^n)^2+3(C_3^n)^2+\dots+n(C_n^n)^2.$$

**178.** Let us have $n$ things $\alpha, \beta, \gamma, \dots$ out of which we select $r$, repetitions being allowed. The no. of ways is $R_r^n$. Now $\alpha$ will

179. Shew that, to $n$ terms,

$$\frac{\underline{|a}}{\underline{|b}}+\frac{\underline{|a+1}}{\underline{|b+1}}+\frac{\underline{|a+2}}{\underline{|b+2}}+\ldots=\frac{\dfrac{\underline{|n+a}}{\underline{|n+b-1}}-\dfrac{\underline{|a}}{\underline{|b-1}}}{a-b+1}.$$

180. Shew that, to $n$ terms,

$$\frac{\underline{|a}}{\underline{|b}}+\frac{\underline{|a-1}}{\underline{|b-1}}+\frac{\underline{|a-2}}{\underline{|b-2}}+\ldots=\frac{\dfrac{\underline{|a+1}}{\underline{|b}}-\dfrac{\underline{|a-n+1}}{\underline{|b-n}}}{a-b+1}.$$

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occur more than once in $R^n_{r-2}$ selections $\alpha$ and $\beta$ more than once in $R^n_{r-4}$ selections. $\therefore$ The no. of ways in which no letter is repeated is, by Prop. XIV,

$$R^n_r-\frac{n}{1}R^n_{r-2}+\frac{n\,.\,\overline{n-1}}{1\,.\,2}R^n_{r-4}-\&c.$$

But without repetitions the no. of selections in $C^n_r$.

Equating these two expressions and multiplying throughout by $\underline{|n-1}$, we have

$$\frac{\underline{|n+r-1}}{\underline{|r}}-\frac{n}{1}\frac{\underline{|n+r-3}}{\underline{|r-2}}+\frac{n(n-1)}{1\,.\,2}\frac{\underline{|n+r-5}}{\underline{|r-2}}-\&c.=\frac{\underline{|n}\;\underline{|n-1}}{\underline{|r}\;\underline{|n-r}}.$$

Q. E. D.

**179, 180.** The following results are easily verified:

($\alpha$) $(r+1)P^n_r=P^{n+1}_{r+1}-P^n_{r+1}$,

($\beta$) $\dfrac{r-1}{P^n_r}=\dfrac{1}{P^{n-1}_{r-1}}-\dfrac{1}{P^n_{r-1}}$.

If we give $n$ all values from $x$ to $x+n-1$ inclusive and add, we get

($\gamma$) $P^x_r+P^{x+1}_r+P^{x+2}_r+\ldots+P^{x+n-1}_r=\dfrac{P^{x+n}_{r+1}-P^x_{r+1}}{r+1}$,

($\delta$) $\dfrac{1}{P^x_r}+\dfrac{1}{P^{x+1}_r}+\dfrac{1}{P^{x+2}_r}+\ldots+\dfrac{1}{P^{x+n-1}_r}=\dfrac{\dfrac{1}{P^{x-1}_{r-1}}-\dfrac{1}{P^{x+n-1}_{r-1}}}{r-1}$.

181. In how many ways can four black balls, four white balls, and four red balls be put into six pockets when one or more may be left empty?

182. In how many ways can 3 sovereigns and 10 shillings be put into 4 pockets? (One or more may be left empty.)

183. In how many ways can 12 sovereigns be distributed into five pockets, none being left empty?

184. In how many ways can 20 books be arranged in a bookcase containing five shelves, each shelf long enough to contain all the books?

185. In how many ways can a person wear five rings on the fingers (not the thumb) of one hand?

186. A debating society has to select one out of five subjects proposed. If thirty members vote, each for one subject, in how many ways can the votes fall?

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If $a>b$ put $x=a$, $r=a-b$ in $(\gamma)$; or if $a<b$ put $x=b$, $r=b-a$ in $(\delta)$, and we get the result of Qn. 179.

Again if $a>b$ put $x=a-n+1$, $r=a-b$ in $(\gamma)$; or if $a<b$ put $x=b-n+1$, $r=b-a$ in $(\delta)$, and we get the result of Qn. 180.

**181.** By Prop. XXVI, the balls of each colour can be put into the pockets in $C_5^9=126$ ways, and $\therefore$ all the balls in

$$126^3=2000376.$$

**182.** The sovereigns in $C_3^6=20$ ways; the shillings in

$$C_3^{13}=286 \text{ ways.}$$

All in $$286\times 20=5720 \text{ ways.}$$

**183.** By Prop. XXV, $C_4^{11}=330$.

**184.** By Prop. XV, $\underline{|24} \div \underline{|4} = \underline{|23}$.

**185.** By Prop. XV, $\underline{|8} \div \underline{|3} = 6720$.

**186.** It is meant to ask how many results of the poll are possible irrespective of the consideration of the persons giving the votes. The votes are $\therefore$ *indifferent*, the subjects *different*. Apply Prop. XXVI; $C_4^{34}=46376$.

187. A bookbinder has 12 different books to bind in red, green or black cloth. In how many different ways can he bind them, binding at least one in each colour?

188. In how many ways can 26 different letters be made into six words, each letter being used once and only once?

189. The *total* number of partitions of $n$ is equal to the number of $n$-partitions of $2n$.

190. Shew that the number of $(r+x)$-partitions of $2r+x$ is the same whatever be $x$.

191. In how many ways can an examiner assign a total of 30 marks to eight questions without giving less than two marks to any question?

192. A body of $n$ members has to elect one member as a representative of the body. If every member gives a vote, in how many ways can the votes be given? And how many different forms may the result of the poll assume,

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**187.** Apply Prop. XXII. The result is

$$3^{12} - 3 \,.\, 2^{12} + 3 = 519156.$$

**188.** The letters can be arranged in $|\underline{26}$ ways. Five points of partition can be introduced in $C^{25}_5$ ways. Total $C^{25}_5\ |\underline{26}$ ways, the words arranged in order. The order of the words being indifferent, choice $= C^{25}_5\ |\underline{26} \div |\underline{6}$.

**189.** To form the $n$-partitions of $2n$ we first place one of the $2n$ things in each parcel. We have then to distribute the remaining $n$ things in all possible ways. Our choice is $\therefore$ the total no. of partitions of $n$. Q. E. D.

**190.** By Prop. XXX Cor. $\Pi^{2r+x}_{r+x} = \Pi^{2r+x-1}_{r+x-1}$ provided

$$2r + x < 2\,(r + x)$$

which holds good as long as $x$ is positive. $\therefore$ For all positive values of $x$, $\Pi^{2r+x}_{r+x}$ is constant.

**191.** Having first given 2 marks to each question he has 14 marks left to distribute. By Prop. XXVI,

$$\text{Choice} = C^{21}_7 = 116280.$$

regarding only the number of votes given to each member and not the names of his supporters?

193. Find the number of ways in which $m$ indifferent black balls and $n$ indifferent white balls can be arranged in a row, so that there may be $2r-1$ contacts of black with white.

194. In how many ways can the same balls be arranged, so that there may be $2r$ contacts of black with white?

195. The number of ways in which two persons can divide $2n$ things of one sort, and $2n$ of another, and $2n$ of a third sort, so that each person may have $3n$ things is $3n^2+3n+1$.

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**192.** Each man may vote in $n$ ways. All in $n^n$ ways. Regarding the votes as indifferent we have, by Prop. XXVI,

$$\text{Choice} = C^{2n-1}_{n-1}.$$

**193.** Make $2r$ places in a line, each place being capable of holding any number of balls. Assign the odd numbered places to black and the even to white, or vice versa. Choice $= 2$. Distribute the black balls into their places without blank lots (Prop. XXV). Choice $= C^{m-1}_{r-1}$. So with the white: Choice $= C^{n-1}_{r-1}$. Total choice $= 2C^{m-1}_{r-1}\,C^{n-1}_{r-1}$.

**194.** Make $2r+1$ places. We can distribute the black into the odd numbered places and the white into the even in $C^{m-1}_{r}\,C^{n-1}_{r-1}$ ways. So we can distribute the black into the even and the white into the odd in $C^{m-1}_{r-1}\,C^{n-1}_{r}$ ways.

$$\text{Total choice} = C^{m-1}_{r}\,C^{n-1}_{r-1} + C^{m-1}_{r-1}\,C^{n-1}_{r} = C^{m-1}_{r-1}\,C^{n-1}_{r-1}\,(m+n+2r) \div r.$$

**195.** Let the things be $\alpha^{2n}$, $\beta^{2n}$, $\gamma^{2n}$ and call the two persons $A$ and $B$; $A$ can take $\alpha$'s and $\beta$'s in $(2n+1)^2$ ways. Amongst these ways there will be $1+2+3+\ldots+n = \frac{1}{2}n\,(n-1)$ in which he would take more than $3n$ altogether and the same number in which he would give $B$ more than $3n$. Excluding these cases there are $(2n+1)^2 - n\,(n+1) = 3n^2+3n+1$ ways in which $A$ and $B$ can divide the $\alpha$'s and $\beta$'s. The $\gamma$'s give no further choice as each man must take $\gamma$'s to make up his share to $3n$ letters.

196. If there be $2n$ more things of a fourth sort the number of ways in which the persons can take $4n$ things each is $\frac{1}{3}(2n+1)(8n^2+8n+3)$.

197. In the last two questions if the things were to be equally divided into two *indifferent* parcels the number of ways would be $\frac{1}{2}(3n^2+3n+2)$ and $\frac{1}{3}(n+1)(8n^2+4n+3)$ respectively. Explain why these results are not the halves of the previous results.

198. If there be $m$ sorts of things and $2n$ things of each sort, the number of ways in which two persons can divide them so that each may have $mn$ thing is

$$C_{m-1}^{mn+m-1} - C_1^m\, C_{m-1}^{mn+m-2n-2} + C_2^m\, C_{m-1}^{mn+m-4n-3} - \&c.$$

the $(x+1)$th term being $\pm C_x^m\, C_{m-1}^{mn+m-1-x(2n+1)}$, and the number of terms being the integer next less than $(mn+2n+1)\div(2n+1)$.

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Second Solution. If the number of the letters were indefinite $A$ could take $3n$ letters in $R_{3n}^3$ ways. He would have more that $2n$ $\alpha$'s in $R_{n-1}^3$ ways, and so for $\beta$'s and $\gamma$'s. Hence his eligible ways are $R_{3n}^3 - 3R_{n-1}^3 = 3n^2 + 3n + 1$.

**196.** Proceeding as in the second solution of Qn. 195 the no. of ways must be $R_{4n}^4 - 4R_{2n-1}^4 = \frac{1}{3}(2n+1)(8n^2+8n+3)$.

**197.** Every division into two parcels would give two divisions between $A$ and $B$, except when the two parcels are alike, each containing half the $\alpha$'s, half the $\beta$'s and so on. Hence the choice between two persons is 1 less than twice the choice in making indifferent parcels. The latter choice is $\therefore$ for 3 letters

$$\tfrac{1}{2}(3n^2+3n+2),$$

and for 4 letters

$$\tfrac{1}{6}(2n+1)(8n^2+8n+3) + \tfrac{1}{2} = \tfrac{1}{3}(n+1)(8n^2+4n+3).$$

**198.** As before, we have to deduct from $R_{mn}^m$ the number of selections in which $\alpha$ or $\beta$ or $\gamma$ or &c. would appear more than $2n$ times. Now $\alpha$ would be in excess in $R_{mn-(2n+1)}^m$ selections; $\alpha$ and $\beta$ in $R_{mn-2(2n+1)}^m$; and so on. $\therefore$ (Prop. XIV) the choice

$$= R_{mn}^m - C_1^m\, R_{mn-(2n+1)}^m + C_2^m\, R_{mn-2(2n+1)}^m - \&c.$$
$$= C_{m-1}^{mn+m-1} - C_1^m\, C^{mn+m-1-(2n+1)} + \&c.,$$

199. In how many ways can five black balls, five red balls, and five white balls be distributed into three different bags, five into each?

200. If there be 3 sorts of things, and $n$ things of each sort, the number of ways in which three persons can divide them so as to have $n$ things each is

$$C_2^{n+2}\,C_2^{n+2}-3\,C_4^{n+3}=\tfrac{1}{8}\,(n+1)\,(n+2)\,(n^2+3n+4).$$

201. In how many ways can six Englishmen, seven Russians, and ten Turks be arranged in a row, so that each Englishman may stand between a Russian and a Turk, and no Russian and Turk may stand together?

202. In how many ways can five Englishmen, seven Russians, and ten Turks be arranged in a row, so that each Englishman may stand between a Russian and a Turk, and no Russian and Turk may stand together?

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the number of terms being 1 more than the greatest no. of letters which can be simultaneously in excess $=1+$ greatest integer in $mn\div(2n+1)=$ greatest integer in $(mn+2n+1)\div(2n+1)$.

**199.** A particular case of the next question: $n=5$.

Choice $=231$.

**200.** Let the things be $\alpha^n\beta^n\gamma^n$, and call the persons $A$, $B$, $C$.

$A$, $B$ can take $n$ things each in $R_n^3\,.\,R_n^3$ ways, less the ways in which any letter occurs more than $n$ times.

If $\alpha$ occur $n+x$ times $A$ may have

$$n,\ n-1,\ n-2\ \ldots\ldots\ \text{or}\ x,$$

and $B$

$$x,\ x+1,\ x+2\ \ldots\ldots\ \text{or}\ n.$$

The corresponding nos. of cases would be

$$R_0^2\,R_{n-x}^2,\ \ R_1^2\,R_{n-x-1}^2,\ \ R_2^2\,R_{n-x-2}^2\ \ldots\ R_{n-x}^2R_0^2\,.$$

But the sum of these is $R_{n-x}^4$. And $x$ varies from 1 to $n$. Hence the no. of selections in which $\alpha$ is in excess will be

$$R_0^4+R_1^4+R_2^4+\ \ldots\ +R_{n-1}^4=R_{n-1}^5\,.$$

More than one letter cannot be in excess at once. Hence the no. of eligible ways is

$$R_n^3\,R_n^3-3R_{n-1}^5=C_2^{n+2}\,C_2^{n+2}-3C_4^{n+3}.$$

**201, 202.** Take the case of $x$ Englishmen, $m$ Russians and $n$ Turks. The $m$ and $n$ have to be arranged in a line with $x$

203. Find the number of combinations that can be formed out of the letters of the following line (Soph. *Philoct.* 746):

απαππαπαι παπαππαπαππαπαππαπαι,

taking them (1) 5 together, and (2) 25 together.

204. What is the total number of combinations that can be made out of the same letters?

205. How many solutions can be given to the following problem? "Find two numbers whose greatest common measure shall be $G$ and their least common multiple $M = Ga^{\alpha} b^{\beta} c^{\gamma} d^{\delta}$; $a$, $b$, $c$, $d$ being prime numbers."

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contacts of Russians and Turks. The no. of ways is given by Qn. 194 when $x$ is even, Qn. 193 when $x$ is odd. The Englishmen have then to be interposed at each contact. And the three classes can be permuted among themselves in $\underline{|x}\,\underline{|m}\,\underline{|n}$ ways. Hence for Qn. 201,

$$\text{choice} = \{C_3^6 C_2^9 + C_2^6 C_3^9\} \underline{|6}\,\underline{|7}\,\underline{|10} = 1980 \,.\, \underline{|6}\,\underline{|7}\,\underline{|10};$$

and for Qn. 202,

$$\text{choice} = 2 \,.\, C_2^6 C_2^9 \underline{|5}\,\underline{|7}\,\underline{|10} = 1080 \,.\, \underline{|5}\,\underline{|7}\,\underline{|10}.$$

**203.** The letters are $\alpha^{12}\pi^{15}\iota^2$,

(1) If no $\iota$ is used: choice $= R_5^2$.

If one $\iota$ is used: $= R_4^2$.

If two $\iota$'s are used: $= R_3^2$.

$$\text{Choice} = R_5^2 + R_4^2 + R_3^2 = 6 + 5 + 4 = 15.$$

(2) To select 25 is the same thing as to select 4 out of the 29 letters. As in the former case

$$\text{choice} = R_4^2 + R_3^2 + R_2^2 = 5 + 4 + 3 = 12.$$

**204.** The $\alpha$'s may be chosen in 13 ways; the $\pi$'s in 16; the $\iota$'s in 3. Choice $= 13 \,.\, 16 \,.\, 3 = 624$, or excluding the case in which none are taken, 623.

**205.** Each number must contain $G$ as a factor. In addition, one number must contain $a^{\alpha}$ and the other must not

206. How many solutions can be given to the following problem? "Find two numbers of which $G$ shall be a common measure, and $M$ (as in the last question) a common multiple."

207. Out of 20 letters how many combinations of six letters can be made, no letter occurring more than twice in the same combination?

208. A certain symmetrical function of $x$, $y$, $z$ is such that $x$ is found in five of its terms, $xy$ in three terms, $xyz$ in two terms, and there is one term without $x$, or $y$, or $z$. How many terms are there altogether?

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contain the factor $a$ (unless it be contained in $G$). So one must contain $b^\beta$ and the other not $b$. So with $c^\gamma$ and $d^\delta$. Hence we can make the pair of numbers in $2^4$ ways. But as the order of the two numbers is indifferent, every different pair will occur twice. Hence the true result is $2^3 = 8$.

**206.** One such number can be formed by combining with $G$ the factor $a$,—0, 1, 2, 3 ... or $\alpha$ times; the factor $b$,—0, 1, 2, 3 ... or $\beta$ times; and so with $c$ and $d$. Hence the choice is

$$(\alpha + 1)(\beta + 1)(\gamma + 1)(\delta + 1) = n \text{ suppose.}$$

Two such numbers can be chosen in $\frac{1}{2}n(n-1)$ ways if they must be different, or in $\frac{1}{2}n(n+1)$ ways if both may be alike.

**207.** We have $C_6^{20}$ ways if no letter occurs twice, $C_1^{20} C_4^{19}$ if one letter occurs twice, $C_2^{20} C_2^{18}$ if two letters, and $C_3^{20}$ if three letters. Total $C_6^{20} + C_1^{20} C_4^{19} + C_2^{20} C_2^{18} + C_3^{20} = 146490$.

SECOND SOLUTION.

By Prop. XIV, the choice $= R_6^{20} - 20R_3^{20} + 190 = 146490$.

**208.** If $N$ be the no. of terms we have, by Prop. XIV,

$$1 = N - 3 \,.\, 5 + 3 \,.\, 3 \,.\, -2\,; \text{ which gives } N = 9.$$

E.g. the function might be such as

$$x^2y^2z^2 + xyz + ayz + bxz + cxy + a^2x + b^2y + c^2z + abc.$$

209. I have six friends, each of whom I have met at dinner 12 times. I have met every two of them 6 times, every three of them 4 times, and every four of them 3 times, every five twice, all six once, and I have dined out eight times without meeting any of them. How many times have I dined out altogether?

210. Two examiners, working simultaneously, examine a class of 12 boys, the one in classics the other in mathematics. The boys are examined individually for five minutes each in each subject. In how many ways can a suitable arrangement be made so that no boy may be wanted by both examiners at once?

211. Out of six pairs of gloves, in how many ways can six persons take each a right-handed and a left-handed glove without any person taking a pair?

212. Out of nine pairs of gloves, in how many ways can six persons take each a right-handed and a left-handed glove without any person taking a pair?

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**209.** If $x$ be the no. of times, we have, by Prop. XIV,

$$8 = x - 6 \,.\, 12 + 15 \,.\, 6 - 20 \,.\, 4 + 15 \,.\, 3 - 6 \,.\, 2 + 1,$$

which gives $x = 36$.

**210.** The boys may be arranged for classics in $\underline{|12}$ orders. Then (Prop. XXXI) they may be arranged for mathematics in $\underline{||12}$ ways.

$$\therefore \text{ Choice} = \underline{|12}\ \underline{||12} = 479001600 \times 176214841.$$

**211.** They may take each a right-handed glove in $\underline{|6}$ ways. The left-handed gloves can then be distributed to them (Prop. XXXI) in $\underline{||6}$ ways, so that no one may have a pair.

$$\therefore \text{ Choice} = \underline{|6}\ \underline{||6} = 720 \times 265 = 190800.$$

**212.** First they can take each a right-handed glove in

$$N = 9 \,.\, 8 \,.\, 7 \,.\, 6 \,.\, 5 \,.\, 4 = 60480 \text{ ways.}$$

Then they can take each a left-handed glove in $N$ ways, amongst which the first person will have a pair in $N_1$ ways, the first and

213. A square is divided into 16 equal squares by vertical and horizontal lines. In how many ways can 4 of these be painted white, 4 black, 4 red, and 4 blue, without repeating the same colour in the same vertical or horizontal row?

---

second will have pairs in $N_2$ ways, &c. where

$$\begin{aligned} N_1 &= 8.7.6.5.4 = 7560 \\ N_2 &= 7.6.5.4 = 945 \\ N_3 &= 6.5.4 = 135 \\ N_4 &= 5.4 = 20 \\ N_5 &= 4 \\ N_6 &= 1. \end{aligned}$$

By Prop. XIV Cor. 3, the no. of ways in which there will be no pairs is

$$N - 6N_1 + 15N_2 - 20N_3 + 15N_4 - 6N_5 + N_6 = 30637.$$

Hence the required result will be

$$60480 \times 30637 = 1852925760.$$

**213.** Let $\alpha\beta\gamma\delta$ represent the four colours. Let the top row be $\alpha\beta\gamma\delta$. One of the other rows must begin with $\beta$. It must be one of the following

$$\beta\,\alpha\,\delta\,\gamma \qquad \beta\,\gamma\,\delta\,\alpha \qquad \beta\,\delta\,\alpha\,\gamma.$$

If we now make rows beginning with $\gamma$ and $\delta$ and place them under the foregoing, we shall have the following arrangements:

$$\begin{array}{cccc} \alpha\beta\gamma\delta & \alpha\beta\gamma\delta & \alpha\beta\gamma\delta & \alpha\beta\gamma\delta \\ \beta\alpha\delta\gamma & \beta\alpha\delta\gamma & \beta\gamma\delta\alpha & \beta\delta\alpha\gamma \\ \gamma\delta\alpha\beta & \gamma\delta\beta\alpha & \gamma\delta\alpha\beta & \gamma\alpha\delta\beta \\ \delta\gamma\beta\alpha & \delta\gamma\alpha\beta & \delta\alpha\beta\gamma & \delta\gamma\beta\alpha \end{array}$$

In each of these squares the second, third and fourth lines may be interchanged in $\lfloor 3$ orders. Hence we have a total of $\lfloor 4$ squares with $\alpha\beta\gamma\delta$ at top. And the same with any other arrangement at top. $\therefore$ There are altogether

$$\lfloor 4 \times \lfloor 4 = 576 \text{ squares.}$$

214. The number of ways of deranging a row of $m+n$ terms so that $m$ are displaced and $n$ not displaced is $\underline{|m+n}\,\underline{||m} \div \underline{|m}\,\underline{|n}$.

215. Shew that, if $n$ be an integer, $\underline{|n^2}$ is divisible by $(\underline{|n})^{n+1}$; and if $m$ and $n$ are odd numbers $\underline{|mn}$ is divisible by

$$(\underline{|m})^{\frac{n+1}{2}}\,(\underline{|n})^{\frac{m+1}{2}}.$$

216. If there be an odd number of dominoes in a set that begins with *double-blank* and if the total number of pips on the whole set be even, the number of pips on the highest domino will be a multiple of 8.

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**214.** The $n$ terms to be not displaced can be chosen in $C_n^{m+n}$ ways. The others can then be deranged in $\underline{||m}$ ways.

$$\therefore \text{ Choice} = C_n^{m+n}\,\underline{||m} = \underline{|m+n}\,\underline{||m} \div \underline{|m}\,\underline{|n}.$$

**215.** Let there be $n$ sorts of letters $\alpha\,\beta\,\gamma\ldots$ and $n$ letters of each sort. Let them be arranged in all possible orders. The no. of arrangements is $\underline{|n^2} \div (\underline{|n})^n$.

These arrangements may be sorted out according to the order in which the letters $\alpha\,\beta\,\gamma\ldots$ *first* appear in any arrangement. But there are $\underline{|n}$ different orders and each must occur an equal no. of times, viz. $\underline{|n^2} \div (\underline{|n})^{n+1}$.

This must $\therefore$ be an integer; or $\underline{|n^2}$ is divisible by $(\underline{|n})^{n+1}$.

Q. E. D. (1).

Again let there be $m$ sorts of letters, $n$ of each sort. The same argument shews that $\underline{|mn}$ is divisible by $(\underline{|n})^m\,\underline{|m}$.

Similarly $\underline{|mn}$ is divisible by $(\underline{|m})^n\,\underline{|n}$.

$\therefore$ Multiplying, $(\underline{|mn})^2$ is divisible by $(\underline{|m})^{n+1}(\underline{|n})^{m+1}$; or if $m$ and $n$ are odd numbers $\underline{|mn}$ is divisible by

$$(\underline{|m})^{\frac{n+1}{2}} \cdot (\underline{|n})^{\frac{m+1}{2}}. \qquad \text{Q. E. D. (2).}$$

**216.** If a set of dominoes go from double blank to double $n$ the number of dominoes must be $R_2^{n+1} = \frac{1}{2}(n+1)(n+2)$.

217. If $f_n$ denote the number of derangements of $n$ terms in circular procession so that no term may follow the term which it followed originally,

$$f_n + f_{n+1} = \underline{\|n}.$$

218. Find the number of positive integral solutions of the equation $x+y+z+\ldots(p \text{ variables}) = m$, the variables being restricted to lie between $l$ and $n$, both inclusive.

219. If there be seven copies of one book, eight of another, and nine of another, in how many ways can two persons divide them, each taking 12 books?

---

As each domino contains two numbers, and there are $n+1$ different numbers each number must occur $n+2$ times. $\therefore$ The number of pips is

$$(n+2)\{0+1+2+\ldots+n\} = \tfrac{1}{2}n(n+1)(n+2).$$

In order that the no. of dominoes may be odd, $n$ must be of one of the forms $4r$ or $4r+1$; but in the latter case the total no. of pips would be odd. If $\therefore$ the total number of pips be even, $n$ must be of the form $4r$, and the no. of pips on the highest domino, being $2n$, must be divisible by 8.

**217.** By Prop. XXXIV

$$f_n = \underline{|n}\left\{\frac{1}{n} - \frac{1}{n-1} + \frac{1}{\underline{|2}\,(n-2)} - \frac{1}{\underline{|3}\,(n-3)} + \&c.\right\},$$

$$f_{n+1} = \underline{|n}\left\{1 - \frac{n+1}{n} + \frac{n+1}{\underline{|2}\,(n-1)} - \frac{n+1}{\underline{|3}\,(n-2)} + \&c.\right\}.$$

By addition

$$f_n + f_{n+1} = \underline{|n}\left\{1 - 1 + \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \&c.\right\} = \underline{\|n}.$$

**218.** By Prop. XXVIII the no. of ways in which the $m$ units can be distributed among the $p$ variables is the coefficient of $x^m$ in the expansion of $(x^l - x^{n+1})^p \div (1-x)^p$.

**219.** Call the books $\alpha^7\beta^8\gamma^9$. The first person has to take 12 books. If there were an unlimited no. of copies he might take these in $R^3_{12}$ ways. But amongst these ways $\alpha$ will occur 8 times

220. The squares on a chess-board are painted with 8 colours, 8 squares with each colour. The colours are to be arranged so that every horizontal line contains every colour, but the vertical lines are only subject to the condition that no two adjacent squares must be alike. How many arrangements are possible?

221. There are $n$ things alike and $n$ others all different; shew that there are $2^n$ ways of selecting $n$ things out of them, and that the number of orders in which all the $2n$ things can be arranged is $2^n \cdot 1 \cdot 3 \cdot 5 \ldots (2n-1)$.

222. If a set of dominoes be made from double blank up to double $n$, prove that the number of them whose pips are $n-r$ is the same as the number whose pips are $n+r$, and this number is $\frac{1}{4}(2n-2r+3\pm1)$; and the total number of dominoes is $\frac{1}{2}(n+1)(n+2)$.

---

or oftener in $R_4^3$ ways; $\beta$, 9 times or oftener in $R_3^3$ ways; and $\gamma$, 10 times or oftener in $R_2^3$. Hence

$$\text{Choice} = R_{12}^3 - R_4^3 - R_3^3 - R_2^3 = 91 - 15 - 10 - 6 = 60.$$

**220.** The colours in the top row can be arranged in $\lfloor 8$ ways; each of the other rows in $\|\underline{8}$ ways.

$$\text{Choice} = \lfloor 8 \,(\|\underline{8})^7 = 40320 \times (14833)^7.$$

**221.** Let the things be $\alpha^n\beta\gamma\delta \ldots$. We can take 0, 1, 2, 3 ... or $n$ of the $\beta\gamma\delta \ldots$ in $2^n$ ways; then the necessary no. of $\alpha$'s in one way. Choice $= 2^n$.

$$\text{Total arrangements} = \lfloor 2n \div \lfloor n = 2^n \cdot 1 \cdot 3 \cdot 5 \ldots (2n-1).$$

**222.** As in Qn. 216, the total no. of dominoes is

$$\tfrac{1}{2}(n+1)(n+2).$$

If the sum of the pips on a domino is to be $n-r$ the lower number on it can be 0, 1, 2, 3 ... &c. to $\frac{1}{2}(n-r)$, if $n-r$ be even; or $\frac{1}{2}(n-r-1)$, if $n-r$ be odd. $\therefore$ The no. of dominoes with $n-r$ pips is

$$1 + \tfrac{1}{2}\left(n - r - \tfrac{1}{2} \pm \tfrac{1}{2}\right) = \tfrac{1}{4}(2n - 2r + 3 \pm 1).$$

But corresponding to every domino marked $x$, $y$, if we arrange them from the other end, there is one marked $n-x$, $n-y$. $\therefore$ The no. with pips amounting to $n-r$ is the same as the no. with pips amounting to $n+r$.

223. If all the combinations of $n$ letters (1, 2, 3...or $n$ together) be written down, the number of times each letter will occur is $2^{n-1}$.

224. The total number of permutations [1, 2, 3...or $(m+n)$ together] which can be made out of $m$ things all alike and $n$ other things all alike is $C_{m+1, n+1}-2$.

225. If we write down all possible permutations [1, 2, 3...or $(m+n)$ together] of $m$ $\alpha$'s and $n$ $\beta$'s, we shall write $\alpha$ and $\beta$ respectively

$$1+\frac{mn+m-1}{n+2}C_{m+1, n+1} \text{ times; and } 1+\frac{mn+n-1}{m+2}C_{m+1, n+1} \text{ times.}$$

---

**223.** The total no. of letters used must be

$$C_1^n+2C_2^n+3C_3^n+\ldots+nC_n^n.$$

But $xC_x^n=nC_{x-1}^{n-1}$. $\therefore$ The sum becomes

$$n\{C_0^{n-1}+C_1^{n-1}+C_2^{n-1}+\ldots+C_{n-1}^{n-1}\}=n2^{n-1}.$$

But each of the $n$ letters must be used equally. $\therefore$ Each is used $2^{n-1}$ times.

**224.** Let $\alpha^m\beta^n$ represent the things. If we use $x$ $\alpha$'s we can permute with them 0, 1, 2 ... or $n$ $\beta$'s making

$$C_{x,0}+C_{x,1}+C_{x,2}+C_{x,3}+\ldots+C_{x,n}=C_{x+1,n} \text{ permutations.}$$

And $x$ varies from 0 to $m$. $\therefore$ The total no. of permutations is

$$C_{1,n}+C_{2,n}+C_{3,n}+\ldots+C_{m+1,n}=C_{m+1,n+1}-1.$$

This, however, includes the case of none being selected. The actual choice is $\therefore$ $C_{m+1,n+1}-2$.

**225.** We shewed that $x$ $\alpha$'s would occur in each of $C_{x+1,n}$ permutations. $\therefore$ The no. of times $\alpha$ occurs is $\Sigma\{xC_{x+1,n}\}$, $x$ varying from 0 to $m$.

But $$xC_{x+1,n}=(n+1)C_{x,n+1}-C_{x+1,n}.$$

Hence the total no. of $\alpha$'s is

$$(n+1)\Sigma C_{x,n+1}-\Sigma C_{x+1,n}=(n+1)C_{m,n+2}-(C_{m+1,n+1}-1)$$

$$=1+\frac{mn+m-1}{n+2}C_{m+1,n+1}.$$

And by interchanging $m$ and $n$ we have the no. of times that $\beta$ will occur.

226. Write down all the permutations 1, 2, 3, 4 or 5 together of the letters $m\,a\,m\,m\,a$, and verify the result of the last question.

227. The total number of permutations [1, 2, 3...or $(m+n+1)$ together] which can be made out of $m$ things alike, $n$ other things alike, and one different thing, always using the one different thing is,

$$1+\frac{mn+m+n}{m+n+4}C_{m+2,\,n+2}.$$

228. The total number of permutations [1, 2, 3 or...$(m+n+1)$ together] which can be made out of $m$ things alike, $n$ other things alike, and one different thing, is

$$\frac{(m+1)(n+1)}{m+n+3}C_{m+2,\,n+2}-1.$$

---

**226.** By Qn. 225, $a$ will occur 50 times and $m$ 71 times. If indices are used to denote the no. of arrangements of each selection, all the permutations including $a$ will be as follows:

$$a+(am)^2+(amm)^3+(ammm)^4,$$

and $$aa+(aam)^3+(aamm)^6+(aammm)^{10}.$$

The first line gives 10 $a$'s and the second gives 40. Total 50. And the no. of $m$'s may be similarly verified.

**227.** We have $\alpha^m\beta^n\gamma$. Now $x$ $\alpha$'s, $y$ $\beta$'s and one $\gamma$ can be permuted in

$$\underline{|x+y+1}\div\underline{|x}\,\underline{|y}=(x+1)\,C_{x+1,y}\text{ ways.}$$

Hence, if we use $x$ $\alpha$'s, the no. of permutations will be

$$(x+1)\{1+C_{x+1,1}+C_{x+1,2}+\ldots+C_{x+1,n}\}=(x+1)\,C_{x+2,n}.$$

The total no. is therefore

$$\begin{aligned}\Sigma\{(x+1)\,C_{x+2,n}\}&=\Sigma\{(n+1)\,C_{x+1,n+1}-C_{x+2,n}\}\\&=(n+1)(C_{m+1,n+2}-1)-(C_{m+2,n-1}-1-C_{n,1})\\&=1+(n+1)\,C_{m+1,n+2}-C_{m+2,n+1}\\&=1+\frac{mn+m+n}{m+n+4}C_{m+2,n+2}.\end{aligned}$$

**228.** We have to take the sum of the results in Qns. 224 and 227. We get

$$\frac{mn+m+n}{m+n+4}C_{m+2,n+2}+C_{m+1,n+1}-1=\frac{(m+1)(n+1)}{m+n+3}C_{m+2,n+2}-1.$$

229. Write down all the permutations, one or more together, of the letters *pepper*, and verify the result of the last question.

230. The total number of permutations of $n$ things [1, 2, 3...or $n$ together] is the integer nearest to $e\,\underline{|n}-1$.

231. The total number of permutations of $n$ things [1, 2, 3,...or $n$ together] is greater than $e\,\underline{|n}-1-1\div n$ and less than $e\,\underline{|n}-1-1\div(n+1)$.

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**229.** There will be found to be 6 combinations producing 1 permutation each; 3 producing 2 each; 4 producing 3 each; 2 producing 4 each; 2 producing 6 each; one producing 10; 2 producing 12 each; and one each producing 20, 30, 60. Total permutations:

$$6+6+12+8+12+10+24+20+30+60=188.$$

This agrees with the number obtained by putting $m=3$, $n=2$ in the result of Qn. 228.

**230, 231.** The no. of permutations is

$$n+n(n-1)+n(n-1)(n-2)+\&c.$$

to $n$ terms; or in reverse order

$$=\underline{|n}\left\{1+\frac{1}{1}+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+\ldots+\frac{1}{\underline{|n-1}}\right\}=e\,\underline{|n}-H,$$

where $$H=1+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\&c.$$

$H$ therefore lies between

$$1+\frac{1}{n+1}\ \text{ and }\ 1+\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\&c.,$$

that is between $1+\dfrac{1}{n+1}$ and $1+\dfrac{1}{n}$.

The no. of permutations therefore lies between $e\,\underline{|n}-1-\dfrac{1}{n}$ and $e\,\underline{|n}-1-\dfrac{1}{n+1}$ and is the integer nearest to $e\,\underline{|n}-1$.

232. If all the permutations of $n$ letters (1, 2, 3...or $n$ together) be written down, the number of times each letter will occur is the integer nearest to $e(n-1)\,\underline{|n-1}$.

233. The integer in the last question is always greater than $e(n-1)\,\underline{|n-1}$ and less than $e(n-1)\,\underline{|n-1} + 2\div(n^2-1)$.

234. In how many different orders can seven men and five women get into a carriage, one by one, so that there may never be more women than men in it?

---

**232, 233.** The total number of letters used in all the permutations is

$$\underline{|n}\left\{n+\frac{n-1}{1}+\frac{n-2}{1\,.\,2}+\ldots+\frac{1}{\underline{|n-1}}\right\}$$

$$=n\,\underline{|n}\left\{1+\frac{1}{1}+\frac{1}{1\,.\,2}+\ldots\text{ to } n \text{ terms}\right\}$$

$$-\underline{|n}\left\{1+\frac{1}{1}+\frac{1}{1\,.\,2}+\ldots\text{ to } (n-1) \text{ terms}\right\}$$

$$=n\{e\,\underline{|n}-H\}-n\{e\,\underline{|n-1}-H'\},$$

where $H'$ is the same function of $n-1$ that $H$ is of $n$.

Therefore each letter occurs $e\{\underline{|n}-\underline{|n-1}\}+H'-H$ times.

Now as $H$ lies between $1+\dfrac{1}{n}$ and $1+\dfrac{1}{n+1}$, $H'$ must lie between $1+\dfrac{1}{n-1}$ and $1+\dfrac{1}{n}$. Therefore $H'-H$ must lie between 0 and $\dfrac{1}{n-1}-\dfrac{1}{n+1}$, i.e. 0 and $\dfrac{2}{n^2-1}$. Hence the required number is the integer nearest to $e\{\underline{|n}-\underline{|n-1}\}$ or $e(n-1)\,\underline{|n-1}$, and the integer is greater than $e(n-1)\,\underline{|n-1}$ but less than

$$e(n-1)\,\underline{|n-1}+\frac{2}{n^2-1}.$$

Q. E. D.

**234.** Prop. XXXVIII. *Cor.* We have $J_{7,5}=\frac{3}{8}C_{7,5}=297$. This would be the result if the men were indifferent and the

235. In how many orders can $m$ positive units and $n$ negative units be arranged so that the sum to any number of terms may never be negative?

236. In how many orders can $m$ even powers of $x$ and $n$ odd powers of $x$ be arranged so that when $x=-1$ the sum to any number of terms may never be negative?

237. In how many orders can a man win $m$ games and lose $n$ games so as at no period to have lost more than he has won?

238. In how many different orders can a man possessed of $h$ pounds win $m$ wagers and lose $n$ wagers of 1 pound each without being ruined during the process?

239. If $p$, $q$, $r$ be integers, such that the two lowest are together greater than the highest, and $p+q+r=2s$, the number of ways in which two men can divide $p$ black balls, $q$ white and $r$ red, so that each may take $s$ balls, is $s^2+s+1-\frac{1}{2}(p^2+q^2+r^2)$.

---

women indifferent. We must multiply by $\underline{|7}$ and $\underline{|5}$ for the different orders in which the men can be arranged among themselves and the women amongst themselves. Result:

$$297 \times \underline{|7} \times \underline{|5} = 179625600.$$

**235, 236, 237.** Direct applications of Prop. XXXVIII.

$$\frac{m-n+1}{m+1} C_{m,n} = C_{m,n} - C_{m+1,n-1}.$$

**238.** By Prop. XXXIX. Result $= C_{m,n} - C_{m+h,n-h}$.

**239.** The first man has to take $s$ balls. His choice is $R^3_s$, less the number of cases in which he would have more than $p$ black, more than $q$ white, or more than $r$ red. Since $p+q>s$, no two of these can happen together. Hence choice $= R^3_s - R^3_{s-p-1} - R^3_{s-q-1} - R^3_{s-r-1}$, which reduces to

$$s^2+s+1-\tfrac{1}{2}(p^2+q^2+r^2).$$

Second Solution. Let the white balls be represented by units of length marked off from $O$ on a horizontal axis $OX$. And let the red balls be represented by units on a vertical axis $OY$. In $OX$ mark off $OS=s$, $OQ=q$, $OH=s-p$. In $OY$ mark

240. If the two lowest are together less than the highest $(q+r<p)$ the result in the last question must be increased by $\frac{1}{2}(p-s)(p-s-1)$.

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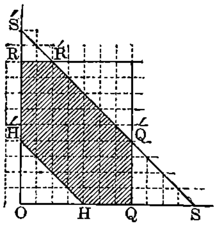
off $OS'=s$, $OR=r$, $OH'=s-p$. Then since $p+q>s$ and $p+r>s$, $Q$ will fall between $H$ and $S$, and $R$ between $H'$ and $S'$. Also since $q+r>s$ the horizontal line through $R$ and the vertical through $Q$ will meet beyond the line $SS'$. Let them cut $SS'$ in $R'$ and $Q'$. If now the space between the two axes be ruled into small squares of one square unit each, the intersections within the rectangle $QOR$ will represent all the different ways in which $A$ can take red and white balls. But the ways denoted by intersections beyond the line $SS'$ will be ineligible as $A$ would then have more than $s$ balls. Also those within the line $HH'$ will be ineligible as there would not be sufficient black balls to complete $A$'s share. The number of intersections in the area $HQQ'R'RH'$ will therefore be the number of eligible ways. This area is shaded in the diagram. This area $=s^2-\frac{1}{2}(p^2+q^2+r^2)$, but the number of intersections in it is equal to the number of units of area obtained by extending the shaded area by completing the half squares along $HH'$ and $Q'R'$ and adding whole squares along $R'RH'$.* The extension is




$$\tfrac{1}{2}(s-p)+\tfrac{1}{2}(q-s+r)+(r-s+p)+(s-r)+1=s+1.$$

Hence we have the result

$$s^2+s+1-\tfrac{1}{2}(p^2+q^2+r^2).$$

* For we may think of each square as a leaf growing out of a stalk at its lower right-hand corner. If then we are to have the no. of intersections equal to the no. of squares we must have the no. of stalks equal to the no. of leaves. Therefore the leaves corresponding to stalks within the shaded area must be completed: i.e. the half leaves along $HH'$ and $Q'R'$ must be made into whole ones, and whole leaves must be supplied to the stalks along $R'R$ and $RH'$.

241. If a letter be taken at random from the word *o r g a n i z e*, what is the chance that it is a vowel?

242. What is the chance that a letter taken at random from *r e s p l e n - d e n c e* should be $e$, and what is the chance that it should be $n$?

243. What is the chance that a letter taken at random from *p a r a - p h r a s e* should be a vowel, and if it be known to be a vowel what is the chance that it is $a$?

---

**240.** Proceeding as in Qn. 239, the intersections in the triangle $HOH'$ are no longer ineligible. Excluding those on the diagonal which have already been reckoned their number is $(s-p)(s-p+1)$, or as it must now be written $(p-s)(p-s-1)$. Hence the number of ways in which the balls can be divided is

$$s^2+s+1+(p-s)(p-s-1)-\tfrac{1}{2}(p^2+q^2+r^2),$$

which reduces on the substitution of $2s=p+q+r$ to the simple form $qr+q+r+1$ which as we should expect is independent of $p$.

Second Solution. We need only think of the division of the white and red balls, for as the black balls exceed the aggregate of white and red there are always enough of black to make up to each party the complement of $s$ balls.

The question is therefore the same as if it were asked in how many ways can two persons divide $q$ white balls and $r$ red balls between them, (the case of one person taking all being admissible). The white balls can be divided in $q+1$ ways and the red in $r+1$ ways. The division can therefore be made in $(q+1)(r+1)$ ways; as before.

**241.** A letter can be drawn in 8 ways of which 4 are favourable. Chance $=4\div 8=\frac{1}{2}$.

**242.** A letter can be drawn in 12 ways of which 4 are favourable to the first event and 2 to the second. Chances $\frac{1}{3}$ and $\frac{1}{6}$.

**243.** (i) Ten ways of which 4 are favourable. Chance $=\frac{2}{5}$. (ii) Four ways of which 3 are favourable. Chance $=\frac{3}{4}$.

244. If two letters are taken at random out of *e s t e e m e d*, shew that the odds against both being $e$ are the same as the odds in favour of one at least being $e$.

245. If two letters be taken at random out of *m u r m u r e r*, what is the chance that they should be both alike?

246. What is the chance that two letters taken at random from *o b s e q u i o u s* should both be vowels?

247. Three letters are taken at random from *a s s o c i a t i o n*, what is the chance that one of them is $c$?

248. What is the chance that at least one of them is $s$?

249. What is the chance that two of them are alike?

---

**244.** As there are an equal no. of $e$'s and not-$e$'s the odds against both being $e$'s must be the same as the odds against both being not-$e$'s. But the latter are the odds in favour of one at least being $e$.

**245.** Two letters can be drawn in $8 \cdot 7 \div 1 \cdot 2 = 28$ possible ways. But $mm$ in 1 way; $uu$ in 1 way; $rr$ in 3 ways. Total $1 + 1 + 3 = 5$ favourable ways. Chance $= \frac{5}{28}$.

**246.** Two letters in $10 \cdot 9 = 90$ ways. Two vowels in $6 \cdot 5 = 30$ ways. Chance $= \frac{1}{3}$.

Note that it is often indifferent as in this case, whether we think of the drawings as permutations or combinations.

**247.** Three letters can be chosen in $11 \cdot 10 \cdot 9 \div \underline{|3} = 165$ ways. Three letters not $c$ in $10 \cdot 9 \cdot 8 \div \underline{|3} = 120$ ways. $\therefore$ Chance that there is not $c$ is $\frac{8}{11}$ and chance that one is $c$ is $\frac{3}{11}$.

**248.** Three letters not $s$ can be chosen in $9 \cdot 8 \cdot 7 \div \underline{|3} = 84$ ways. $\therefore$ Chance that there is no $s$ is $\frac{28}{55}$ and chance that one is $s$ is $\frac{27}{55}$.

**249.** Two $a$'s and another letter can be chosen in 9 ways. Similarly for two $s$'s; and two $o$'s and two $i$'s. Total 36 ways with two letters alike. But the total choice is 165. Hence the required chance is $\frac{36}{165} = \frac{12}{55}$.

250. What is the chance that one and only one is *s*?

251. If the letters of *obsequious* be arranged at random, what is the chance that all the vowels come together?

252. If the letters of *oiseau* be arranged at random, what is the chance that the vowels occur in their natural order?

253. A letter is taken at random out of *assistant* and a letter out of *statistics*. What is the chance that they are the same letter?

254. A letter is taken at random out of *effete* and a letter out of *feet*; show that the odds are 5 to 3 against their being the same letter.

255. Two letters are taken at random out of *cocoa* and two out of *cocoon*. Find the chance that the four letters should be all different.

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**250.** An $s$ and two not-$s$'s can be chosen in $2 \times 36 = 72$ ways. Chance $= \frac{72}{165} = \frac{24}{55}$.

**251.** The fact that some of the letters are alike does not affect the question. We need only think of them as 6 vowels and 4 consonants. The vowels can be arranged together in $\lfloor 6$ ways. Think of them as a block to be permuted with the four consonants. This can be done in $\lfloor 5$ ways. Hence there are $\lfloor 6 \,.\, \lfloor 5$ favourable arrangements among $\lfloor 10$ possible arrangements. Chance $= \lfloor 6 \; \lfloor 5 \div \lfloor 10 = \frac{1}{42}$.

**252.** All orders of the vowels will be equally likely. And $\lfloor 5 = 120$ orders are possible. Chance $= \frac{1}{120}$.

**253.** $ss$ in $3 \,.\, 3 = 9$ ways; $tt$ in $2 \,.\, 3 = 6$ ways; $aa$ in $2 \,.\, 1 = 2$ ways; $ii$ in $1 \,.\, 2 = 2$ ways. Total $9 + 6 + 2 + 2 = 19$ favourable ways. Total $9 \,.\, 10 = 90$ possible ways. $\therefore$ Chance $= \frac{19}{90}$.

**254.** Out of 24 ways, 6 give $ee$; two give $ff$; one gives $tt$. Total 9. Odds $15 : 9 = 5 : 3$ against.

**255.** We must either have $cn$ from *cocoon* and $oa$ from *cocoa*; chance $= \frac{2}{15} \times \frac{1}{5} = \frac{2}{75}$. Or else $on$ from *cocoon* and $ca$ from *cocoa*; chance $= \frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$. Total chance $= \frac{1}{15}$.

256. If the letters of *replete* be arranged at random, find the chance that no two *e*'s come together.

257. Find the chance that all the *e*'s come together.

258. How are the last two results affected if the letters be arranged in a ring instead of a line?

259. What is the chance that three letters taken at random from *muhammadan* should be all different?

260. What is the chance that five taken at random should be all different?

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**256.** The letters *rplt* can be arranged in $\underline{|4} = 24$ ways. The spaces between them and the two ends make 5 places into which the *e*'s can be put. We select 3 of them in $C_3^5 = 10$ ways. Hence there are $24 \times 10 = 240$ arrangements in which the *e*'s will not come together. But there are $\underline{|7} \div \underline{|3} = 840$ ways of arranging the 7 letters. Chance $= 240 \div 840 = \frac{2}{7}$.

**257.** If all the *e*'s are to come together we can regard them as forming one counter to be permuted with the other four letters in $\underline{|5} = 120$ ways. Hence chance $= 120 \div 840 = \frac{1}{7}$.

**258.** The seven letters could make a ring in $\underline{|6} \div \underline{|3} = 120$ ways. The four consonants could make a ring in $\underline{|3} = 6$ ways, and the 3 *e*'s could be interposed in 3 out of the 4 spaces in 4 ways. So the chance of no two *e*'s being together would be

$$6 \times 4 \div 120 = \tfrac{1}{5}.$$

If all the *e*'s were in one block we should have $\underline{|4}$ ways of making the ring. Chance of all being together is

$$\therefore \underline{|4} \div 120 = \tfrac{1}{5}.$$

**259.** The letters are $m^3 a^3 u h d n$. $ma$ and another letter can be selected in $3 \times 3 \times 4 = 36$ ways. $m$ or $a$ and two of the other letters in $6 \times 6 = 36$ ways. Three of the single letters in 4 ways. Total 76 favourable ways out of 120 possible ways. Chance $= \frac{19}{30}$.

**260.** $ma$ and three others can be selected in $3 . 3 . 4 = 36$

261. If ten persons form a ring, what is the chance that two assigned persons will be together?

262. If ten persons stand in a line, what is the chance that two assigned persons will stand together?

263. A letter is chosen at random out of each of the words *tinsel* and *silent*. What is the chance that they are the same letter? And what does the chance become if it is known that they are either both consonants, or both vowels?

264. A letter is chosen at random out of each of the words *musical* and *amusing*: what is the chance that the same letter is chosen in each case?

265. A letter is taken at random out of each of the words *choice* and *chance*: what is the chance that they should be the same letter?

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ways. $m$ or $a$ with the other 4 in 6 ways. Total 42 favourable ways out of 252 possible ways. Chance $= \frac{1}{6}$.

**261.** Let $A$ and $B$ be the assigned persons. Out of his 9 comrades $A$ must have 2 next him. The chance that $B$ is next him is $\therefore \frac{2}{9}$.

**262.** $A$ and $B$ can be put together in 2 ways and permuted with the other 8 persons in $\lfloor 9$ ways. There are $\therefore 2 \lfloor 9$ favourable arrangements among $\lfloor 10$ possible arrangements. Chance $= \frac{1}{5}$.

**263.** Whatever letter be chosen out of the first there is a chance $\frac{1}{6}$ of getting the same letter from the second. $\frac{1}{6}$ is $\therefore$ the required chance. If it is known that both are consonants or both vowels the chance of getting a vowel from the first word is $\frac{1}{3}$ and then the chance of getting the same letter from the second is $\frac{1}{2}$. Also the chance of getting a consonant from the first is $\frac{2}{3}$ and the chance of getting the same letter from the second is $\frac{1}{4}$. Hence chance $= \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{3}$.

**264.** Chance of $mm = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$. Same for $uu$, $ss$, $ii$, $aa$. Total chance $= \frac{5}{49}$.

**265.** Chance of $cc = \frac{4}{36}$, $hh$ and $ee$ each $= \frac{1}{36}$. Total chance $= \frac{6}{36} = \frac{1}{6}$.

266. What is the chance that two letters taken at random out of the word *m y r r h* should be the same as two taken at random out of *m e r r y*?

267. 120 men are to be formed at random into a solid rectangle of 12 men by 10; all sides are equally likely to be in front. What is the chance that an assigned man is in the front?

268. If the 26 letters of the alphabet are written down in a *ring* so that no two vowels come together, what is the chance that $a$ is next to $b$?

269. If the 26 letters of the alphabet are written down in a *row* so that no two vowels come together, what is the chance that $a$ is next to $b$?

270. $A$, $B$, $C$ have equal claims for a prize. $A$ says to $B$, let us two draw lots, let the loser withdraw and the winner draw lots with $C$ for the prize. Is this fair?

271. $A$ and $B$ stand in a line with 10 other persons. If the arrangement is made at random what is the chance that there are exactly 3 persons between $A$ and $B$?

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**266.** Chance of both $m\,r$, or both $y\,r$, each $= \frac{1}{25}$. Chance of both $m\,y$, or both $r\,r$, each $= \frac{1}{100}$. Total chance $= \frac{1}{10}$.

**267.** Chance that a side of 12 is in front $= \frac{1}{2}$. Chance that the man is then in the front row $= \frac{1}{10}$. So chance that a side of 10 is in front $= \frac{1}{2}$. Chance that the man is then in the front row $= \frac{1}{12}$. Total chance $= \frac{1}{2} \cdot \frac{1}{10} + \frac{1}{2} \cdot \frac{1}{12} = \frac{11}{120}$.

**268.** $a$ has two consonants next it, out of the 20 consonants. Chance that one of them is $b = \frac{1}{10}$.

**269.** It matters not where the other 5 vowels are since they cannot come next $a$. All possible orders of $a$ and the 20 consonants are equally likely. If $a$ and $b$ be made into one counter and permuted with the other 19 consonants, we have $2\,\underline{|20}$ favourable orders among $\underline{|21}$ possible orders. Chance $= \frac{2}{21}$.

**270.** $A$ can only win the prize by winning two tosses; chance $= \frac{1}{4}$. Likewise $B$. $C$ only tosses once and his chance $= \frac{1}{2}$. Their expectations are $\therefore$ as $1 : 1 : 2$.

**271.** Select 3 out of the 10 other persons. They will make $10 \,.\, 9 \,.\, 8 = 720$ arrangements. Put $A$ and $B$ at the ends of this

272. What would the chance be if they stood in a ring instead of a line?

273. Five men, $A$, $B$, $C$, $D$, $E$, speak at a meeting, and it is known that $A$ speaks before $B$, what is the chance that $A$ speaks *immediately* before $B$?

274. Two numbers are chosen at random, find the chance that their sum is even.

275. A bag contains six black balls and one red. A person is to draw them out in succession, and is to receive a shilling for every ball he draws until he draws the red one. What is his expectation?

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row: (2 ways). There are thus 1440 ways of making a block of 5 persons with $A$ and $B$ at the ends. Permute this block with the 7 other persons, in $\lfloor\underline{8}$ ways. Thus there are $1440\,\lfloor\underline{8}$ favourable arrangements out of $\lfloor\underline{12}$ possible arrangements Chance

$$= 1440 \div 12 . 11 . 10 . 9 = \tfrac{4}{33}.$$

**272.** We could make the block as before in 1440 ways. The ring could then be made in $1440\,\lfloor\underline{7}$ favourable ways out of $\lfloor\underline{11}$ possible ways. Chance $= \frac{2}{11}$.

**273.** They can speak in $\lfloor\underline{5} = 120$ orders. In 60 of these $A$ speaks before $B$. But if he is to speak immediately before $B$ we permute the 4 things $(AB)\,C\,D\,E$ and get $\lfloor\underline{4}$ favourable ways. $\therefore$ Chance $= \frac{2}{5}$.

**274.** Chance that both are even $= \frac{1}{2} . \frac{1}{2} = \frac{1}{4}$. So chance that both are odd $= \frac{1}{4}$. $\therefore$ Chance that their sum is even

$$= \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

**275.** The red ball is equally likely to be drawn 1st, 2nd, 3rd, 4th, 5th, 6th or 7th. That is, he is equally likely to receive 0, 1, 2, 3, 4, 5 or 6 shillings. His expectation $= \frac{1}{7}$ of the sum

$$= 3 \text{ shillings}.$$

276. There are ten tickets, five of which are numbered 1, 2, 3, 4, 5, and the other five are blank. What is the probability of drawing a total of ten in three trials, one ticket being drawn out and replaced at each trial?

277. What is the probability in the preceding question if the tickets are not replaced?

278. A person has ten coins, which he throws down in succession. He is to receive one shilling if the first falls head, two shillings *more* if the second *also* falls head, four shillings *more* if the third *also* falls head, and so on, the amount doubling each time; but as soon as a coin falls tail, he ceases to receive anything. What is the value of his expectation?

279. $A$ and $B$ play at chess, and $A$ wins on an average two games out of three. Find the chance of $A$ winning exactly four games out of the first six, drawn games being disregarded.

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**276.** Ten can be drawn as follows:

| | | | | |
|---|---|---|---|---|
| $5+5+0$ | in 3 | orders and $\therefore$ | in $3\times 5=15$ | ways |
| $5+4+1$ | ,, 6 | ,, | 6 | ,, |
| $5+3+2$ | ,, 6 | ,, | 6 | ,, |
| $4+4+2$ | ,, 3 | ,, | 3 | ,, |
| $4+3+3$ | ,, 3 | ,, | 3 | ,, |

Total 33 favourable ways out of $10\,.\,10\,.\,10=1000$ possible ways. Chance $=\frac{33}{1000}$.

**277.** In this case regarding the blanks as different tickets any combination of 3 tickets is equally likely. But the only favourable combinations are 5, 4, 1 and 5, 3, 2. Out of a total of $C_3^{10}=120$ combinations. Chance $=\frac{1}{60}$.

**278.** His chance of receiving the first shilling is $\frac{1}{2}$. His chance of *also* receiving the 2 is $(\frac{1}{2})^2$. His chance of *also* receiving the $2^2$ is $(\frac{1}{2})^3$ and so on. Hence his whole expectation is

$$\tfrac{1}{2}+(\tfrac{1}{2})^2\,2+(\tfrac{1}{2})^3\,2^2+\&c. \text{ to 10 terms}=5 \text{ shillings}.$$

**279.** The chance of $A$ winning any assigned game $=\frac{2}{3}$, and of his losing it $=\frac{1}{3}$. So the chance of his winning four assigned games and losing two assigned games is $(\frac{2}{3})^4\,(\frac{1}{3})^2$.

280. Four flies come into a room in which there are four lumps of sugar, of different degrees of attractiveness, proportional to the numbers 8, 9, 10, 12; what is the chance that the flies will all select different lumps?

281. A teetotum of eight faces numbered from one to eight, and a common die are thrown: what is the chance that the same number is turned up on each?

282. Out of a set of dominoes, numbered from double one to double six, one is drawn at random. At the same time a pair of common dice are thrown. What is the chance that the numbers turned up on the dice will be the same as those on the domino? and what is the chance that they will have one number at least in common?

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But four games and two losses can be arranged in 15 different orders.

$$\therefore \text{ Chance} = 15\left(\tfrac{2}{3}\right)^4\left(\tfrac{1}{3}\right)^2 = \tfrac{80}{243}.$$

**280.** The flies can be arranged in $\underline{|4} = 24$ orders. Then the chance that the first fly chooses the first lump is $\frac{8}{39}$, the chance that the second chooses the second is $\frac{9}{39}$ and so on.

$$\therefore \text{ Chance} = 24 \,.\, 8 \,.\, 9 \,.\, 10 \,.\, 12 \div (39)^4 = \tfrac{256}{28561}.$$

**281.** Whatever way the die falls the chance of the teetotum giving the same number is $\frac{1}{8}$.

**282.** There are 21 dominoes in the set. Whatever throw be made with the dice there is one domino which will present the same nos. $\therefore$ Chance $= \frac{1}{21}$.

(2) If the throw with the dice is doublets (chance $= \frac{1}{6}$), there are 6 dominoes with one no. at least the same as on the dice. Chance of one of these $= \frac{6}{21}$. If the throw be not doublets (chance $= \frac{5}{6}$) there are 11 dominoes with at least one no. identical with one on the dice. Chance of one of these $= \frac{11}{21}$. Hence chance that at least one no. will be common to the dice and the domino

$$= \tfrac{1}{6} \cdot \tfrac{6}{21} + \tfrac{5}{6} \cdot \tfrac{11}{21} = \tfrac{61}{126}.$$

283. What are the odds against throwing seven twice at least in three throws with two dice?

284. Two persons play for a stake, each throwing two dice. They throw in turn, $A$ commencing. $A$ wins if he throws 6, $B$ if he throws 7: the game ceasing as soon as either event happens. Shew that $A$'s chance is to $B$'s as 30 to 31.

285. $A$, $B$, $C$ amongst them stake £9. 2*s.*, and throw in turn with a single die, until an ace is thrown, the thrower of the ace to take all the stakes. In what proportion ought they to contribute the stakes?

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**283.** Chance of 7 at any throw is $\frac{1}{6}$. Chance of 3 sevens

$$=\left(\tfrac{1}{6}\right)^3=\tfrac{1}{216}.$$

Chance of 2 sevens and one not-seven (noting that this can occur in three orders) $=3\left(\frac{1}{6}\right)^2\frac{5}{6}=\frac{15}{216}$. Total chance of "twice at least" $=\frac{16}{216}$. Odds 200 : 16 = 25 : 2 against it.

**284.** The chance of throwing 6 is $\frac{5}{36}$, of throwing 7 is $\frac{1}{6}$.

$A$ can win at the 1st, 3rd, 5th or &c. throw. But in order that he may win at any throw, all previous throws must have failed. The chance that two successive throws fail (one by $A$ and one by $B$) is $\frac{31}{36}\cdot\frac{5}{6}=\frac{155}{216}$. $A$'s chance is $\therefore$ the sum of the infinite series

$$\tfrac{5}{36}+\tfrac{155}{216}\cdot\tfrac{5}{36}+\left(\tfrac{155}{216}\right)^2\tfrac{5}{36}+\text{\&c. in G.P.}=\tfrac{30}{61}.$$

$B$'s chance must $\therefore$ be $\frac{31}{61}$. Ratio 30 : 31.

**285.** As in the last question $A$'s chance is

$$\tfrac{1}{6}+\left(\tfrac{5}{6}\right)^3\cdot\tfrac{1}{6}+\left(\tfrac{5}{6}\right)^6\cdot\tfrac{1}{6}+\ldots.$$

$B$'s chance is

$$\tfrac{5}{6}\cdot\tfrac{1}{6}+\left(\tfrac{5}{6}\right)^4\cdot\tfrac{1}{6}+\left(\tfrac{5}{6}\right)^7\tfrac{1}{6}+\ldots.$$

$C$'s chance is

$$\left(\tfrac{5}{6}\right)^2\tfrac{1}{6}+\left(\tfrac{5}{6}\right)^5\tfrac{1}{6}+\left(\tfrac{5}{6}\right)^8\tfrac{1}{6}+\ldots.$$

Their chances are $\therefore$ as

$$1:\tfrac{5}{6}:\tfrac{25}{36}=36:30:25.$$

$A$ must stake 36 florins, $B$ 30 florins, and $C$ 25 florins.

286. Two faces of a die are marked with even numbers and the other four faces with odd numbers. Shew that the odds are 41 to 40 in favour of the sum of four throws being an even number.

287. An archer hits his target on an average 3 times out of 4, find the chance that in the next four trials he will hit it three times exactly.

288. A man wins on an average 3 games out of 5, find the chance that out of the next five games he wins exactly 3.

289. On an average out of 10 games a player wins 3, loses 5, and the others are drawn. Find the chance that out of the next ten games he wins exactly 3, and loses exactly 5.

290. An experiment succeeds twice as often as it fails. Find the chance that in the next six trials there will be at least four successes.

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**286.** The sum will be even if there be 0 or 2 or 4 odd nos. Chance of an odd no. is always $\frac{2}{3}$.

The favourable issues are $EEEE + 6 . EEOO + OOOO$. The chance $= (\frac{1}{3})^4 + 6(\frac{1}{3})^2(\frac{2}{3})^2 + (\frac{2}{3})^4 = \frac{41}{81}$. Odds $= 41$ to $40$ in favour.

**287.** There are four orders in which $HHHM$ can occur: and the chance of each is $(\frac{3}{4})^3 \frac{1}{4}$. Hence chance $= \frac{27}{64}$.

**288.** $WWWLL$ can occur in 10 orders. In any order chance $= (\frac{3}{5})^3 (\frac{2}{5})^2$. $\therefore$ Total chance $= \frac{1080}{3125} = \frac{216}{625}$.

**289.** $W^3L^5D^2$ can occur in 2520 orders. In any order chance

$$= \frac{3^3 . 5^5 . 2^2}{10^{10}}.$$

$\therefore$ Total chance $= \frac{1701}{20000}$.

**290.** Chance of success at a given trial $= \frac{2}{3}$.

$$\text{Chance of } S^6 \text{ (1 order)} = (\tfrac{2}{3})^6.$$

$$\text{Chance of } S^5F \text{ (6 orders)} = 6 . \tfrac{1}{3} (\tfrac{2}{3})^5.$$

$$\text{Chance of } S^4F^2 \text{ (15 orders)} = 15 . (\tfrac{1}{3})^2 (\tfrac{2}{3})^4.$$

$$\text{Total chance} = \tfrac{496}{729}.$$

291. A bag contains four red balls and two others, each of which is equally likely to be red or white. Three times in succession a ball is drawn and replaced. Find the chance that all the drawings are red.

292. A bag contains a £5 note ($V$) a £10 note ($X$) and six pieces of blank paper ($O^6$). If a man is to draw them one by one and to go on drawing till he draws a blank, what is his expectation?

293. A box contains 10 pairs of gloves. $A$ draws out a single glove: then $B$ draws one: then $A$ draws a second: then $B$ draws a second. Shew that $A$'s chance of drawing a *pair* is the same as $B$'s and that the chance of neither drawing a pair is $290 \div 323$.

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**291.** The bag contains $R^6$ or $R^5W$ or $R^4W^2$ and the respective chances of these are $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$.

The chance of all the drawings being red is on the several hypotheses 1, $(\frac{5}{6})^3$, $(\frac{2}{3})^3$. Hence the chance

$$=\tfrac{1}{4}+\tfrac{1}{2}\left(\tfrac{5}{6}\right)^3+\tfrac{1}{4}\left(\tfrac{2}{3}\right)^3=\tfrac{265}{432}.$$

**292.** If the papers are not replaced he can only win by drawing $XO$ or $VO$, or $XVO$ or $VXO$. The chances of $XO$ or $VO$ are each $\frac{1}{8}\cdot\frac{6}{7}$. The chances of $XVO$ or $VXO$ are each $\frac{1}{8}\cdot\frac{1}{7}$.
$\therefore$ His expectation

$$=\tfrac{3}{28}\text{ of }£10+\tfrac{3}{28}\text{ of }£5+\tfrac{1}{28}\text{ of }£15=£2\tfrac{1}{7}.$$

**293.** $A$ will draw a pair provided

(i) $B$ does not draw the fellow to $A$'s first.

(ii) $A$ draws it.

$$\text{Chance}=\tfrac{18}{19}\cdot\tfrac{1}{18}=\tfrac{1}{19}.$$

$B$ will draw a pair provided

(i) $B$ does not draw the fellow to $A$'s first;

(ii) $A$ does not draw the fellow to $B$'s first;

(iii) $B$ draws it.

$$\text{Chance}=\tfrac{18}{19}\cdot\tfrac{17}{18}\cdot\tfrac{1}{17}=\tfrac{1}{19}.$$

$\therefore$ $A$'s chance and $B$'s are equal.

Both will draw a pair provided

(i) $B$ does not draw the fellow to $A$'s first;

294. A boy tries to jump a ditch: in jumping from the upper bank to the lower he succeeds five times out of six, in jumping from the lower to the upper he succeeds three times out of five. What is the chance that after four trials he leaves off on the same side on which he began?

295. Shew that the odds are eleven to three against a month selected at random containing portions of *six* different weeks.

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(ii) $A$ draws the fellow to his first;

(iii) $B$ draws the fellow to his first.

$$\text{Chance} = \tfrac{18}{19} \cdot \tfrac{1}{18} \cdot \tfrac{1}{17} = \tfrac{1}{323}.$$

$\therefore$ Chance that one at least draws a pair $= \frac{1}{19} + \frac{1}{19} - \frac{1}{323} = \frac{33}{323}$.

$\therefore$ Chance that neither draws a pair is $\frac{290}{323}$.

**294.** He must succeed an even no. of times in order to finish where he began. Let $x$ be the chance of success from the bank from which he starts and $y$ from the other bank.

Then chance of $FFFF$ $= (1-x)^4$;

$FFSS$, $FSSF$, $SSFF$ each $= (1-x)^2 xy$,

$FSFS$, $SFSF$ each $= (1-x)(1-y)xy$,

$SFFS$ $= (1-y)^2 xy$,

$SSSS$ $= x^2y^2$.

Hence the whole chance is

$$(1-x)^4 + xy(6 - 8x - 4y + 3x^2 + 3xy + y^2).$$

If he start from the upper bank $x = \frac{5}{6}$, $y = \frac{3}{5}$; and chance $= \cdot 439105$.

If he start from the lower bank $x = \frac{3}{5}$, $y = \frac{5}{6}$; and chance $= \cdot 596111$.

**295.** Chance that it has 31 days $= \frac{7}{12}$; then it must begin with Friday or Saturday. Chance $= \frac{2}{7}$.

Chance that it has 30 days $= \frac{4}{12}$; then it must begin with Saturday. Chance $= \frac{1}{7}$.

Total chance $= \frac{7}{12} \cdot \frac{2}{7} + \frac{4}{12} \cdot \frac{1}{7} = \frac{3}{14}$. Odds 11 : 3 against it.

296. $A$ walks at an uniform speed known to be greater than 3 and less than 4 miles an hour between 2 places 20 miles apart. An hour having elapsed since $A$'s departure, $B$ starts after him for the same place walking at the uniform speed of 4 miles an hour. Find the odds against $B$'s overtaking $A$, (1) on the hypothesis that within the given limits all distances per hour are equally likely, (2) on the hypothesis that within the limits all times per mile are equally likely.

297. Three different persons have each to name an integer not greater than $n$. Find the chance that the integers named will be such that every two are together greater than the third.

298. $A$ and $B$ play at chess, and $A$ wins on an average five games out of nine. Find $A$'s chance of winning a majority (1) out of three games, (2) out of four games, drawn games not being counted.

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**296.** $B$ takes 5 hours: he will overtake $A$ if $A$ takes more than 6 hours for the journey.

(i) Suppose $A$ travels at $3+x$ miles per hour. We must have $x<\frac{1}{3}$. The odds are 2 to 1 against it.

(ii) Suppose $A$ takes $20-x$ minutes per mile. $x$ may range from 0 to 5. But for $B$ to overtake $A$ we must have $x<2$. The odds are 3 to 2 against it.

**297.** If $f(n)$ be the no. of ways in which they can name integers subject to the required condition, $f(n)-f(n-1)$ will be the no. of ways in which one at least of them names the highest integer $n$. But all three name $n$ in one way; two only, in $3(n-1)$ ways; one only, in $\frac{3}{2}(n-1)(n-2)$ ways. Total $=1+\frac{3}{2}n(n-1)$. Thus we have

$$f(n)-f(n-1)=1+\tfrac{3}{2}n(n-1).$$

Give $n$ all values from 1 to $n$ and add, observing that $f(1)=1$; then

$$f(n)=n+\tfrac{1}{2}(n+1)n(n-1)=\tfrac{1}{2}(n^3+n).$$

But the total no. of ways in which they can each name three integers is $n^3$. $\therefore$ Chance $=\frac{1}{2}(n^3+n)\div n^3=(n^2+1)\div 2n^2$.

**298.** Omitting drawn games, chance of $AAA=(\frac{5}{9})^3$. Chance of $AAB$ in any order $=3(\frac{5}{9})^2(\frac{4}{9})$. $\therefore$ Chance of $A$ winning a majority out of 3 games $=(5^3+12.5^2)\div 9^3=\frac{425}{729}$.

299. If the odds on every game between two players are two to one in favour of the winner of the preceding game, what is the chance that he who wins the first game shall win at least two out of the next three?

300. $A$, $B$, $C$ play at a game in which each has a separate score, and the game is won by the player who first scores 3. If the chances are respectively $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, that any point is scored by $A$, $B$, $C$, find the respective chances of the three players winning the game.

---

Again chance of $AAAA = (\frac{5}{9})^4$. Chance of $AAAB$ in any order $= 4\,(\frac{5}{9})^3\,\frac{4}{9}$. $\therefore$ Chance of $A$ winning a majority in four games $= (5^4 + 16\,.\,5^3) \div 9^4 = \frac{875}{2187}$.

**299.** Suppose $A$ has won the first game. Consider the next 3 games.

$$\text{Chance of } AAA = (\tfrac{2}{3})^3,$$
$$AAB = (\tfrac{2}{3})^2\,\tfrac{1}{3},$$
$$ABA \text{ or } BAA, \text{ each} = (\tfrac{1}{3})^2\,.\,\tfrac{2}{3}.$$

Total chance $= (8 + 4 + 4) \div 27 = \frac{16}{27}$.

**300.** Take the more general case in which the respective chances of $A$, $B$, $C$ are $\alpha$, $\beta$, $\gamma$.

Then $A$'s chance of winning at the

$$\begin{aligned} &\text{3rd score} = \alpha^3, \\ &\text{4th} \quad ,, \quad = 3\alpha^3(\beta+\gamma), \\ &\text{5th} \quad ,, \quad = 6\alpha^3(\beta+\gamma)^2, \\ &\text{6th} \quad ,, \quad = 30\alpha^3\beta\gamma(\beta+\gamma), \\ &\text{7th} \quad ,, \quad = 90\alpha^3\beta^2\gamma^2. \end{aligned}$$

$A$'s total chance is $\therefore$

$$\alpha^3\{1 + 3(\beta+\gamma) + 6(\beta+\gamma)^2 + 30\beta\gamma(\beta+\gamma) + 90\beta^2\gamma^2\},$$

and $B$'s and $C$'s can be written down by symmetry.

Substituting $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$, $\gamma = \frac{1}{4}$, we find that $A$'s chance is $\frac{677}{1024}$ and $B$'s and $C$'s each $\frac{347}{2048}$.

301. The chance of one event happening is the square of the chance of a second event, but the odds against the first are the cube of the odds against the second. Find the chance of each.

302. With two dice having their faces numbered *alike* in any way whatever, the chance of throwing an even number can never be less than $\frac{1}{2}$.

303. If $mn$ men are formed into a solid column with $m$ men in front and $n$ in depth, find the chance that a given man will be on the outside of the column.

304. If $n$ men, among whom are $A$ and $B$, stand in a row, what is the chance that there will be exactly $r$ men between them?

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**301.** Let $x^2$ and $x$ be the chances. Then

$$\text{odds against first} \quad = 1 - x^2 : x^2,$$

$$\text{odds against second} = 1 - x : x.$$

$$\therefore \frac{1-x^2}{x^2} = \left(\frac{1-x}{x}\right)^3,$$

whence $x = \frac{1}{3}$. The chances are $\frac{1}{9}$ and $\frac{1}{3}$.

**302.** If $x$ faces be marked with odd nos. and $y$ with even nos. the chances of an even or odd sum in two throws are as $x^2 + y^2 : 2xy$. But $x^2 + y^2 > 2xy$, $\therefore$ the chance of an even sum exceeds $\frac{1}{2}$ whatever be the no. of faces; except when $x = y$, in which case the chance is $\frac{1}{2}$.

**303.** Total no. of men $= mn$. Total no. of outside men

$$= 2m + 2n - 4.$$

Chance $= 2(m + n - 2) \div mn$.

**304.** The total no. of arrangements is $\lfloor n$. The no. in which there will be $r$ men between $A$ and $B$ can be arrived at by first making a row of $r$ men ($P_r^{n-2}$ ways), then putting $A$ at one end and $B$ at the other (2 ways), and then permuting this block of $r + 2$ men with the remaining $n - r - 2$ men ($\lfloor n-r-1$ ways). $\therefore$ The chance is

$$2P_r^{n-2} \lfloor n-r-1 \div \lfloor n = 2(n-r-1) \div n(n-1).$$

305. If they stand in a ring instead of a row, the chance will be independent of $r$.

306. If $A$ agrees to pay $B$ a shilling for every man who stands between them, what is $B$'s expectation when the $n$ men are arranged at random in a row?

307. What is $B$'s expectation if they stand in a ring and count the shortest distance between them?

---

**305.** Let the block of $r+2$ men be constructed as in Qn. 304. After this block place the remaining $n-r-2$ men in order ($\underline{|n-r-2}$ ways). We thus get a line of which the ends may be joined to make a ring. But without restriction the ring may be made in $\underline{|n-1}$ ways. $\therefore$ The required chance is

$$2P_r^{n-2}\,\underline{|n-r-2} \div \underline{|n-1} = 2 \div (n-1).$$

The result is independent of $r$, shewing that all intervals are equally likely.

**305.** Second Solution. There must be two men at a distance $r$ from $A$. The chance that $B$ is one of them is plainly $2 \div (n-1)$. The case when $n = 2r+2$ should be noted. There is only one admissible position for $B$, viz. at the extremity of the diameter from $A$. The chance that he should be here is only $\frac{1}{n-1}$, but there are in this case *two* intervals of $r$ so that the expectation of an interval of $r$ is $\frac{2}{n-1}$ as in the other cases.

**306.** Assuming the result of Qn. 304, $B$'s expectation must be

$$\Sigma_{r=1}^{r=n-2}\left\{\frac{2(n-r-1)r}{n(n-1)}\right\} = \frac{n-2}{3}.$$

**307.** Assuming the result of Qn. 305, $B$'s expectation must be, if $n$ be odd,

$$\frac{2}{n-1}\left(1+2+3+\ldots+\frac{n-3}{2}\right) = \frac{n-3}{4}.$$

308. What is it, in the last case, if they count from $A$'s right to $B$?

309. A bag contains $m$ white and $n$ black balls. (1) If they are drawn out one by one, find the chance of first drawing a white and then a black, and so on alternately, until the balls remaining are all of one colour. (2) If $m$ balls are drawn at once, what is the chance that all the white balls will be drawn at the first trial?

310. A boy who on an average does four sums out of five correctly is given five sums to do. Shew that the odds are more than two to one against their being all right.

311. $A$ spins a teetotum with $a$ faces, numbered 1, 2, 3...$a$ respectively, $B$ spins one with $b$ faces, similarly numbered from 1 to $b$. Each spins once and the highest number wins. Shew that if $a > b$, $A$'s chance is $1 - \frac{b+1}{2a}$, $B$'s chance is $\frac{b-1}{2a}$, and the chance of a tie is $\frac{1}{a}$.

---

If $n$ be even (see note at end of Qn. 305) his expectation must be

$$\frac{2}{n-1}\left(1 + 2 + \ldots + \frac{n-4}{2}\right) + \frac{1}{n-1} \cdot \frac{n-2}{2} = \frac{(n-2)^2}{4(n-1)}.$$

**308.** Wherever $A$ is, $B$'s chance of being in the 1st, 2nd, 3rd, &c. place from him, each $= 1 \div (n-1)$. $\therefore$ $B$'s expectation is

$$\{1 + 2 + 3 + \ldots + (n-2)\} \div (n-1) = \tfrac{1}{2}(n-2).$$

**309.** The balls can be arranged in $C_{m,n}$ orders. Only one order will be favourable to each of the specified events. $\therefore$ The chance in each case is $\lfloor m \, \lfloor n \div \lfloor m+n$.

**310.** The chance that all are right is $(\frac{4}{5})^5 = \frac{1024}{3125}$. Odds 2101 : 1024; or more than 2 to 1 against.

**311.** Let $O$ denote a tie. The chance that $B$ spins $x$ is $\frac{1}{b}$, and then the respective chances of $A$, $O$, $B$ are

$$\frac{a-x}{a}, \; \frac{1}{a}, \; \frac{x-1}{a}.$$

Hence the total chances of $A$, $O$, $B$ are ($x$ varying from 1 to $b$)

$$\Sigma \frac{a-x}{ab}, \; \frac{1}{a}, \; \Sigma \frac{x-1}{ab},$$

312. If in the last exercise they are to spin again in case of a tie, $A$'s chance of winning will be $\frac{2a-b-1}{2a-2}$, and $B$'s chance will be $\frac{b-1}{2a-2}$.

313. Out of $2n$ tickets, numbered consecutively, three are drawn at random. Find the chance that the numbers on them are in A.P.

314. Out of $2n+1$ tickets, consecutively numbered, three are drawn at random. Find the chance that the numbers on them are in A.P.

315. $A$ and $B$ shoot at a mark, and $A$ hits it once in $n$ times, and $B$ once in $n-1$ times. If they shoot alternately, $A$ commencing, compare their chances of first hitting the mark.

316. If there be $m$ sorts of things and $n$ things of each sort, shew that the chance that $m-r$ things selected at random may be all different is

$$n^{m-r}\, C_r^{mn+r} \div C_m^{mn+r}.$$

---

that is
$$\frac{2a-b-1}{2a},\ \frac{1}{a},\ \frac{b-1}{2a}.$$

**312.** The case is the same as if ties are entirely disregarded. $A$'s and $B$'s chances are then as $2a-b-1 : b-1$, and their actual values must be

$$\frac{2a-b-1}{2a-2} \text{ and } \frac{b-1}{2a-2}.$$

**313, 314.** Assuming the results of Qn. 165, the chances must be $n(n-1) \div C_3^{2n}$ and $n^2 \div C_3^{2n+1}$, that is

$$3 \div (4n-2) \text{ and } 3n \div (4n^2-1).$$

**315.** Chance of failure at two consecutive shots (one $A$ and one $B$) is

$$\frac{n-1}{n} \cdot \frac{n-2}{n-1} = \frac{n-2}{n}.$$

$A$'s chance is

$$\frac{1}{n}\left\{1 + \frac{n-2}{n} + \left(\frac{n-2}{n}\right)^2 + \left(\frac{n-2}{n}\right)^3 + \dots\right\} = \frac{1}{2},$$

that is, their expectations are equal.

**316.** Out of $mn$ things there are $C_{m-r}^{mn}$ selections. For the favourable selections, $m-r$ different sorts may be determined in $C_r^m$ ways, and the things then selected in $n^{m-r}$ ways. Hence chance

$$= n^{m-r}\, C_r^m \div C_{m-r}^{mn} = n^{m-r}\, C_r^{mn+r} \div C_m^{mn+r}.$$

317. If $n$ things ($\alpha$, $\beta$, $\gamma$, &c.) be arranged in a row, subject to the condition that $\alpha$ comes before $\beta$, what is the chance that $\alpha$ comes *next* before $\beta$?

318. There are $n$ counters marked with odd numbers, and $n$ more marked with even numbers; if two are drawn at random shew that the odds are $n$ to $n-1$ against the sum of the numbers drawn being even.

319. If a head counts for *one* and a tail for *two*, shew that $3n$ is the most likely number to throw when $2n$ coins are tossed. Also shew that the chance of throwing $3(n+1)$ with $2(n+1)$ coins is less than the chance of throwing $3n$ with $2n$ coins in the ratio $2n+1 : 2n+2$.

320. A red card has been removed from a pack. Thirteen cards are then drawn and found to be all of one colour. Shew that the odds are 2 to 1 that this colour is black.

321. Two squares are marked at random on a chess-board. Shew that the odds are 17 to 1 against their being adjacent.

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**317.** As in Qn. 273, the chance is

$$2\,\underline{|n-1} \div \underline{|n} = 2 \div n.$$

**318.** The chances of $EE$, $EO$ in 2 orders, $OO$ are as

$$n\,.\,\overline{n-1}\ :\ 2n^2\ :\ n\,\overline{n-1}.$$

The first or third would give an even sum, the second an odd sum. Odds against an even sum are

$$2n^2\ :\ 2n(n-1) = n\ :\ n-1.$$

**319.** Chance of $x$ tails and $2n-x$ heads is $C_x^{2n}\left(\frac{1}{2}\right)^{2n}$. But $C_x^{2n}$ is greatest when $x=n$. $\therefore$ The most likely sum is $n$ heads and $n$ tails, counting for $3n$. The chance is $C_n^{2n}\left(\frac{1}{2}\right)^{2n}$. The chance of throwing $3(n+1)$ with $2(n+1)$ coins is $\therefore\ C_{n+1}^{2n+2}\left(\frac{1}{2}\right)^{2n+2}$.

And the ratio is $\quad C_{n+1}^{2n+2}\left(\frac{1}{2}\right)^{2n+2}\ :\ C_n^{2n}\left(\frac{1}{2}\right)^{2n}$

$$=\tfrac{1}{4}(2n+2)(2n+1)\ :\ (n+1)^2 = 2n+1\ :\ 2(n+1).$$

**320.** The cards are $B^{26}R^{25}$; chances of drawing 13 black or 13 red are as $C_{13}^{26} : C_{13}^{25} = 26 : 13 = 2 : 1$.

**321.** Two squares adjacent horizontally can be chosen in 56 ways; adjacent vertically in 56 ways; total 112 favourable selections out of $C_2^{64} = 2016$ possible selections. Chance $= \frac{1}{18}$. Odds 17 : 1 against.

322. If two squares of opposite colour are marked at random on a chess-board the chance that they are adjacent is $\frac{7}{64}$?

323. If two squares are marked at random on a chess-board the chance that they have contact at a corner is $\frac{7}{144}$?

324. If two squares of the same colour are marked at random on a chess-board the chance that they have contact at a corner is $\frac{49}{496}$?

325. Two squares are chosen at random on a chess-board: what is the chance that one is a castle's move from the other?

326. What is the chance that both are the same colour, and one a castle's move from the other?

327. Two squares are chosen at random on a chess-board: what is the chance that one is a knight's move from the other?

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**322.** The possible selections are now $32 \times 32 = 1024$.

$$\text{Chance} = 112 \div 1024 = \frac{7}{64}.$$

**323.** Two squares connected diagonally at a corner can be configured in two ways, and each configuration can be shunted into 49 positions. Total 98 favourable selections, out of 2016 possible selections. Chance $= \frac{98}{2016} = \frac{7}{144}$.

**324.** In this case the possible selections are $2C_2^{32} = 992$.

$$\text{Chance} = \frac{98}{992} = \frac{49}{496}.$$

**325.** Wherever the first is placed, there are 14 squares covered by a castle's move. But the second piece has 63 possible positions. Chance $= \frac{14}{63} = \frac{2}{9}$.

**326.** In this case there are 6 favourable positions out of 63. Chance $= \frac{2}{21}$.

**327.** An oblong of $3 \times 2$ squares can be shunted vertically and horizontally, so as to take $6 \times 7 = 42$ positions. It can also be placed with the longer side either vertical or horizontal; and either of its diagonals will mark a knight's move. Hence there are $42 \times 2 \times 2 = 168$ favourable selections of 2 squares among 2016 possible selections. Chance $= \frac{1}{12}$.

328. What is the chance that one is a bishop's move from the other?

329. What is the chance that one is a queen's move from the other?

330. What is the chance that one is a king's move from the other?

331. Four squares are chosen at random on a chess-board: what is the chance that they should form a knight's circuit? (N.B. A knight's circuit is a set of four squares such that a knight can move from one to another, ending where he began.)

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**328.** Two squares within a bishop's move of one another can be selected if with corner contact in $2 \,.\, 7^2$ ways (see Qn. 323). If one square intervene in $2 \,.\, 6^2$ ways; two squares intervening in $2 \,.\, 5^2$ ways, and so on. Hence the no. of favourable selections $= 2\,(1^2 + 2^2 + \ldots + 7^2) = 280$; and the no. of possible selections is 2016. Chance $= \frac{5}{36}$.

**329.** If one piece be on one of the centre four squares, there are 27 favourable positions for the second piece. If the first be on the next zone there are 25 favourable positions; if on the next 23, and if on the outermost zone 21. But the central square and successive zones contain respectively 4, 12, 20, 28 squares. Hence the chance that the second piece is within a queen's move of the first is

$$\frac{4}{64} \cdot \frac{27}{63} + \frac{12}{64} \cdot \frac{25}{63} + \frac{20}{64} \cdot \frac{23}{63} + \frac{28}{64} \cdot \frac{21}{63} = \frac{13}{36}.$$

The answer is, as we might expect, the sum of the results in Qns. 325 and 328.

**330.** By Qns. 321, 323 there are $112 + 98 = 210$ favourable selections of 2 squares among 2016 possible selections.

$$\text{Chance} = \tfrac{5}{48}.$$

**331.** A knight's circuit can be configured in 6 ways, viz. two forming squares sloping to the right or the left, two forming lozenges sloping to the right or to the left, and two forming lozenges vertically or horizontally. The first four can be shunted into $5 \times 5 = 25$ positions each, and the last two into $6 \times 4 = 24$ positions. Total 148 favourable ways among the $C_4^{64}$ possible selections. Chance $= \frac{37}{158844}$.

332. Two black squares and two white squares are chosen at random on a chess-board: what is the chance that they should form a knight's circuit?

333. Two boards are divided into squares coloured alternately like a chess-board. One has 9 rows of 16 squares each. The other has 12 rows of 12 squares. If two squares are chosen at random, shew that they are more likely to be adjacent on the square board than on the oblong one, in the ratio of 264 to 263?

334. In the last question if the squares were chosen at random of opposite colours, the square board would still have the same advantage over the oblong?

335. The two squares in Question 333 are more likely to have corner-contact on the square board than on the oblong one, in the ratio of 121 to 120?

336. If three squares be chosen at random on a chess-board, the chance that they are in the same line, excluding diagonal lines, is $\frac{2}{93}$?

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**332.** In this case the possible selections are $C_2^{32} . C_2^{32} = 246016$.

$$\text{Chance} = \frac{37}{61504}.$$

**333.** Two adjacent squares on a board $9 \times 16$ can be shunted into $8 . 16$ or $9 . 15$ positions according as they are vertical or horizontal. Total $128 + 135 = 263$ favourable selections among $C_2^{144}$ possible selections. On a board $12 \times 12$ the positions would be $2 \times 11 \times 12 = 264$ favourable selections among $C_2^{144}$ possible selections. There is a greater chance on the square board in the ratio $264 : 263$.

**334.** The favourable selections remain as in the last question; the possible selections are now $2C_2^{72}$ for either board: the ratio of the chances is unaltered.

**335.** For corner contact the configuration will take

$2 \times 11 \times 11 = 242$ positions on the square board,

$2 \times 8 \times 15 = 240$ on the oblong.

Ratio $121 : 120$.

**336.** There are 16 possible lines of 8 squares each.

$$\text{Chance} = 16C_3^8 \div C_3^{64} = \frac{2}{93}.$$

337. The chance that they should be in a diagonal line, is $\frac{7}{744}$?

338. If four squares be chosen at random on a chess-board, find the chance that they be at the corners of a square having its sides parallel to the sides of the board.

339. Find the chance that they be at the corners of a square having its sides parallel to the diagonals of the board.

340. What would be the chance in the last question if the four squares were taken at random of one colour?

341. In a single throw with four dice, what is the chance of throwing two doublets?

342. What is the chance of the throw containing a single doublet?

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**337.** There are 4 diagonals of 3 squares, and 4 each of 4, 5, 6, and 7 squares; and there are 2 diagonals of 8 squares. $\therefore$ The favourable positions are

$$= 4\,(C_3^3 + C_3^4 + C_3^5 + C_3^6 + C_3^7) + 2C_3^8 = 392.$$

And the chance $= 392 \div C_3^{64} = \frac{7}{744}$.

**338.** A square of $2 \times 2$ can be shunted into $7^2$ positions, $3 \times 3$ into $6^2$ positions, and so on. Hence the no. of favourable selections is $1^2 + 2^2 + 3^2 + \ldots + 7^2 = 140$.

$$\text{Chance} = 140 \div C_4^{64} = \frac{5}{22692}.$$

**339.** A square of $2 \times 2$ can now be shunted into only $6^2$ positions. A square of $3 \times 3$ into $4^2$ positions. A square of $4 \times 4$ into 4 positions; and no other square is possible. The favourable positions are therefore only 56, and

$$\text{Chance} = \frac{1}{11346}.$$

**340.** The chance will now be $56 \div 2C_4^{32} = \frac{7}{8990}$.

**341.** Two faces as $\alpha$, $\beta$ can be selected in 15 ways; $\alpha\alpha\beta\beta$ can then be arranged in 6 orders making 90 favourable throws.

$$\text{Chance} = 90 \div 6^4 = \frac{5}{72}.$$

**342.** $\alpha$ can be selected in 6 ways, $\beta\gamma$ in 10 ways; $\alpha\alpha\beta\gamma$ can then be arranged in 12 orders, making $6 \times 10 \times 12 = 720$ favourable throws. Chance $= \frac{5}{9}$.

343. If six dice are thrown together, what is the chance that the throw will be (1) three doublets, (2) two triplets, (3) a quartett and a doublet?

344. Compare the chances of throwing four with one die, eight with two dice, and twelve with three dice, having two trials in each case.

345. What is the probability of throwing not more than eight in a single throw with three dice?

346. There is a greater chance of throwing nine in a single throw with three dice than with two dice. Shew that the chances are as 25 to 24.

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**343.** Proceeding as in the last question the favourable throws are 1800, 300, 450: and the chances are

$$\tfrac{25}{648},\ \tfrac{25}{3888},\ \tfrac{25}{2592}.$$

**344.** The chances at the first throw are $\frac{1}{6}$, $\frac{5}{36}$, $\frac{25}{216}$. The chances of not failing in two trials are

$$1-(\tfrac{5}{6})^2,\ 1-(\tfrac{31}{36})^2,\ 1-(\tfrac{191}{216})^2,$$

which are as $216^2-180^2 : 216^2-186^2 : 216^2-191^2$,

or as $14256 : 12060 : 10175$.

**345.** The ways of throwing 3, 4, 5, 6, 7, 8 with three dice are the same as the ways of throwing 9 with 4 dice, for the 9 can be obtained by combining the throws 6, 5, 4, 3, 2, 1 of the 4th die with the respective throws of the 3 dice. The throws are

$$4(1, 1, 1, 6)+12(1, 1, 2, 5)+12(1, 1, 3, 4)$$
$$+12(1, 2, 2, 4)+12(1, 2, 3, 3)+4(2, 2, 2, 3)$$
$$=56 \text{ throws.}$$

$$\therefore \text{ Chance}=\frac{56}{6^3}=\frac{7}{27};$$

or we might have applied Prop. XXVIII. The no. of throws

$$=\text{coeff. of } x^5 \text{ in } \left(\frac{1-x^6}{1-x}\right)^4.$$

**346.** With 3 dice the throws are

$$(3, 3, 3)+3(1, 4, 4)+3(2, 2, 5)+6(2, 3, 4)$$
$$+6(1, 3, 5)+6(1, 2, 6)=25 \text{ throws.}$$

With 2 dice there are 4 throws.

The chances are as $\frac{25}{216} : \frac{4}{36} = 25 : 24$.

347. In a single throw with three dice, the chance that the sum turned up will be a multiple of three is one-third; and the chance that it will be a multiple of six is one-sixth.

348. A prize is to be won by $A$ as soon as he throws five with two dice, or by $B$ as soon as he throws ten with three dice. If they throw alternately, $A$ commencing, shew that their chances of winning are equal.

349. $A$ and $B$ have each a pair of dice: shew that the odds are nearly 19 to 1 against their making the same throw in one trial. And if they throw with three dice each, the odds are nearly 46 to 1.

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**347.** With each die the chance is $\frac{1}{3}$ that the number is of either of the forms $3x$, $3x+1$, $3x+2$. We must either have all three of the same form or one of each form. Therefore, however the first die falls, the chance that the others are suitable is

$$(\tfrac{1}{3})^2 + 2\,(\tfrac{1}{3})^2 = \tfrac{1}{3}.$$

This is therefore the chance that the sum is a multiple of 3.

Half the sums will be even and half odd. $\therefore$ Chance of a multiple of 6 is $\frac{1}{6}$.

**348.** By Prop. XXVIII the no. of ways of throwing 10 with 3 dice is the coeff. of $x^7$ in $(1-x^6)^3(1-x)^{-3}$, that is in

$$\{1 - 3x^6 + \&c.\}\left\{1 + \frac{2\,.\,3}{1\,.\,2}x + \ldots + \frac{8\,.\,9}{1\,.\,2}x^7 + \ldots\right\}.$$

$$\text{This coeff.} = \frac{8\,.\,9}{1\,.\,2} - 3\,\frac{2\,.\,3}{1\,.\,2} = 27.$$

$B$'s chance at any throw is therefore $\frac{27}{216} = \frac{1}{8}$.

$A$'s chance at any throw is $\frac{1}{9}$; therefore by Qn. 315 their expectations are equal.

**349.** (1) If $A$ throw doublets chance $= \frac{1}{6}$, and chance of $B$ throwing the same $= \frac{1}{36}$.

If $A$ throw not-doublets chance $= \frac{5}{6}$, and chance of $B$ making the same throw $= \frac{2}{36}$.

$\therefore$ Chance that they throw alike $= \frac{1}{6}\cdot\frac{1}{36} + \frac{5}{6}\cdot\frac{2}{36} = \frac{11}{216}$.

Odds 205 : 11 against.

350. The chance of throwing 14 is the same whether we throw with three dice or with five.

351. The chance of throwing 15 is less with three dice than with five as 360 to 651.

352. If we throw with two dice and count the difference of the two numbers which turn up, this is as likely as not to be *one-or-two*. Also the chance of *zero* is the same as the chance of *three*, and is the same as the chance of *four-or-five*.

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(2) With three dice $A$'s throw may be of the form $x, x, x$; $x, x, y$; or $x, y, z$.

Respective chances $= \frac{6}{216}, \frac{90}{216}, \frac{120}{216}$.

Respective chances of $B$ making the same throw

$$= \frac{1}{216}, \frac{3}{216}, \frac{6}{216}.$$

$\therefore$ Chance that they throw alike

$$= (6 + 270 + 720) \div 216^2 = 83 \div 3888.$$

Odds $3805 : 83 = 46 : 1$ nearly.

**350.** With 3 dice 14 can be thrown in 15 ways: with 5 dice in 540 ways (Prop. XXVIII).

The chances are as $\frac{15}{6^3} : \frac{540}{6^5}$, i.e. they are equal.

**351.** By Prop. XXVIII we find

$$15 \text{ with 3 dice in } \frac{13 \cdot 14}{1 \cdot 2} - 3 \cdot \frac{7 \cdot 8}{1 \cdot 2} + 3 = 10 \text{ ways},$$

$$15 \text{ with 5 dice in } \frac{11 \cdot 12 \cdot 13 \cdot 14}{1 \cdot 2 \cdot 3 \cdot 4} - 5\,\frac{5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4} = 651 \text{ ways}.$$

The chances are as

$$\frac{10}{6^3} : \frac{651}{6^5} = 360 : 651.$$

**352.** The chances of the several throws 0, 1, 2, 3, 4, 5 are respectively $\frac{3}{18}, \frac{5}{18}, \frac{4}{18}, \frac{3}{18}, \frac{2}{18}, \frac{1}{18}$.

$\therefore$ Chance of one or two $= \frac{9}{18} = \frac{1}{2}$.

$$\text{Chance of } 0 = \text{chance of } 3 = \frac{3}{18}$$

$$= \text{chance of 4 or 5}.$$

353. $A$, $B$, $C$, $D$ throw with two dice in succession, the highest throw winning a prize of one guinea. If two or more throw equal sums (higher than their competitors) they divide the prize. $A$ having thrown 9, what is the value of his expectation?

354. A die is loaded so that *six* turns up twice as often as *ace* and three times as often as any other face. Another die is loaded so that *six* turns up three times as often as *ace* and twice as often as any other face. If these dice be thrown together what is the chance of throwing *sixes*?

355. What is the chance of throwing at least one *six*?

356. What is the chance of throwing at least eleven?

357. What is the chance of throwing doublets?

358. What is the chance of throwing seven?

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**353.** The chances that any throw should be greater than, equal to, or less than 9, are $\frac{3}{18}$, $\frac{2}{18}$, $\frac{13}{18}$.

$A$'s chance of the whole guinea $= (\frac{13}{18})^3$.

$A$'s chance of half $= 3\,(\frac{13}{18})^2 \cdot \frac{2}{18}$,

one-third $= 3\,(\frac{13}{18})\,(\frac{2}{18})^2$,

one-fourth $= (\frac{2}{18})^3$.

His whole expectation in guineas is

$$(13^3 + 3\,.\,13^2 + 4\,.\,13 + 2) \div 18^3 = \frac{1379}{2916}.$$

**354** to **358.** The chances of the several throws are

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1st die | 3 | 2 | 2 | 2 | 2 | 6 ÷ 17, |
| 2nd die | 2 | 3 | 3 | 3 | 3 | 6 ÷ 20. |

$\therefore$ Chance of sixes $= \frac{6}{17} \cdot \frac{6}{20} = \frac{9}{85}$.

Of no six: $\frac{11}{17} \cdot \frac{14}{20} = \frac{77}{170}$.

$\therefore$ At least one six $= 1 - \frac{77}{170} = \frac{93}{170}$.

Chance of at least 11

$$= \frac{6}{17} \cdot \frac{3}{20} + \frac{2}{17} \cdot \frac{6}{20} + \frac{6}{17} \cdot \frac{6}{20} = \frac{33}{170}.$$

Chance of doublets

$$= \frac{5\,.\,3\,.\,2 + 6\,.\,6}{17\,.\,20} = \frac{33}{170}.$$

Chance of seven

$$= (3\,.\,6 + 2\,.\,3 + 2\,.\,3 + 2\,.\,3 + 2\,.\,3 + 6\,.\,2) \div 17\,.\,20 = \frac{27}{170}.$$

359. Shew that if each die is loaded so that *six* turns up oftener than any other face in the ratio 7 : 4, the odds are 41 to 40 in favour of a throw with two dice exceeding *seven.*

360. If each be so loaded that *six* turns up less often than any other face in the ratio 2 : 5 the odds will be 55 to 26 against the throw exceeding *seven.*

361. Eleven cards are marked with the letters of the word *Hammersmith*, and one is lost. Out of the remaining ten a card is drawn and is found to be *m*. What is the chance that the lost letter was *m*?

362. If in the last question the drawn card is replaced and again a card drawn at random is found to be *m*, what is the chance that the lost letter was *m*? And if the same operation be repeated again with the same result, what does the chance then become?

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**359, 360.** Let $\alpha$ be the chance of *six*, $\beta$ the chance of any other throw. Then

Chance of 6, 6 $= \alpha^2$.

Chance of 6, 5 ; 6, 4 ; 6, 3 ; 6, 2 ; $= 8\alpha\beta$.

Chance of 5, 5 ; 4, 4 ; $= 2\beta^2$.

Chance of 5, 4 ; 5, 3 ; $= 4\beta^2$.

Total chance of throw exceeding 7 is

$$\alpha^2 + 8\alpha\beta + 6\beta^2.$$

In Qn. 359 ; $\alpha = \frac{7}{27}$, $\beta = \frac{4}{27}$. Chance $= \frac{41}{81}$; odds 41 : 40 in favour.

In Qn. 360 ; $\alpha = \frac{2}{27}$, $\beta = \frac{5}{27}$. Chance $= \frac{26}{81}$; odds 55 : 26 against.

**361.** Lost card either *m*, or not-*m*.

*A priori* chances $\frac{3}{11}$ and $\frac{8}{11}$.

Consequent chances of observed event $\frac{2}{10}$ and $\frac{3}{10}$.

$\therefore$ *A posteriori* chances as 6 : 24 = 1 : 4. Chance that *m* was lost $= \frac{1}{5}$.

**362.** The consequent chances of the observed event now become $(\frac{2}{10})^2$ and $(\frac{3}{10})^2$, and in the subsequent case $(\frac{2}{10})^3$ and $(\frac{3}{10})^3$

In the one case the *à posteriori* chances are as 12 : 72, or as 1 : 6 ; in the other as 24 : 216, or as 1 : 9.

The chances become $\frac{1}{7}$ and $\frac{1}{10}$.

363. If in the last question the drawn card had not been replaced, what would the chances have been after each operation?

364. It is not known whether a set of dominoes goes to double eight or double nine, each being equally likely. Two dominoes are drawn and found to contain no number higher than eight. Shew that the odds are now three to two in favour of the set going only to double eight.

365. Two letters have fallen out of the word *Mississippi*; they are picked up and placed at random in the two blank spaces. Find the chance that the letters are in right order.

---

**363.** The consequent chances would have been $\frac{2}{10} \cdot \frac{1}{9}$ and $\frac{3}{10} \cdot \frac{2}{9}$.

The *à posteriori* chances as 1 to 8. Chance $= \frac{1}{9}$.

After the final operation three $m$'s having been drawn the chance of the lost card being $m$ becomes zero.

**364.** A set to double eight contains 45 dominoes, and a set to double nine contains 55.

*A priori* chances of the set terminating at 8, 8 or 9, 9 are $\frac{1}{2}$ and $\frac{1}{2}$.

Consequent chances of observed event

$$1 \text{ and } \frac{45 \cdot 44}{55 \cdot 54} = \frac{2}{3}.$$

$\therefore$ *A posteriori* chances as $\frac{1}{2}$ to $\frac{1}{3}$.

The odds are 3 : 2 in favour of the set going only to 8, 8.

**365.** We have $i^4 s^4 p^2 m$.

The chance that the two dropped letters are alike is

$$\frac{4 \cdot 3 + 4 \cdot 3 + 2 \cdot 1}{11 \cdot 10} = \frac{13}{55}.$$

Chance that they are different $= \frac{42}{55}$.

Chance that they are restored to their proper places

$$= \frac{13}{55} + \frac{1}{2} \cdot \frac{42}{55} = \frac{34}{55}.$$

366. A box contains four dice two of which are true, and the others are so loaded that with either of them the chance of throwing six is $\frac{1}{4}$ and the chance of throwing ace is $\frac{1}{12}$. I take two dice at random out of the box and throw them. If they turn up *sixes* find the chance (1) that both are loaded, (2) that one only is loaded, and (3) that neither is loaded.

367. What would be the chances in the last question, if *six and five* had turned up?

368. What would be the chances in the same question if *six and one* had turned up?

369. One card out of a pack has been lost. From the remainder of the pack thirteen cards are drawn at random, and are found to consist of two spades, three clubs, four hearts, and four diamonds. What are the respective chances that the missing card is a spade, a club, a heart, or a diamond?

---

**366.** They may be (1) both loaded, (2) one only loaded, (3) both true.

*A priori* chances $\frac{1}{6}$, $\frac{4}{6}$, $\frac{1}{6}$.

Consequent chances of observed event $\frac{1}{16}$, $\frac{1}{24}$, $\frac{1}{36}$.

*A posteriori* chances as $\frac{1}{16} : \frac{4}{24} : \frac{1}{36}$.

$\therefore$ The chances are $\frac{9}{37}$, $\frac{24}{37}$, $\frac{4}{37}$.

**367.** In this case the consequent chances of the observed event are $\frac{1}{12}$, $\frac{1}{24}+\frac{1}{36}=\frac{5}{72}$, $\frac{1}{18}$.

$\therefore$ The *à posteriori* chances are as

$$\frac{1}{6}\cdot\frac{1}{12} : \frac{4}{6}\cdot\frac{5}{72} : \frac{1}{6}\cdot\frac{1}{18} = 3 : 10 : 2.$$

The respective chances are therefore $\frac{3}{15}$, $\frac{10}{15}$, $\frac{2}{15}$.

**368.** The consequent chances now become

$$\frac{1}{24},\ \frac{1}{24}+\frac{1}{72}=\frac{1}{18},\ \frac{1}{18}.$$

$\therefore$ The *à posteriori* chances are as

$$\frac{1}{6}\cdot\frac{1}{24} : \frac{4}{6}\cdot\frac{1}{18} : \frac{1}{6}\cdot\frac{1}{18} = 3 : 16 : 4;$$

and the respective chances are $\frac{3}{23}$, $\frac{16}{23}$, $\frac{4}{23}$.

**369.** The pack may be $s^{12}c^{13}h^{13}d^{13}$ or $s^{13}c^{12}h^{13}d^{13}$, or $s^{13}c^{13}h^{12}d^{13}$ or $s^{13}c^{13}h^{13}d^{12}$.

The *à priori* chances are as $1 : 1 : 1 : 1$.

370. A man has left his umbrella in one of three shops which he visited in succession. He is in the habit of leaving it, on an average, once in every four times that he goes to a shop. Find the chance of his having left it in the first, second, and third shops respectively.

371. Eleven cricketers had to elect a captain. Each was as likely as not to vote for himself. Otherwise he voted at random. What are the odds that a man who polled five votes, voted for himself?

372. The odds are estimated as 2 to 1 that a man will write *rigorous* rather than *rigourous*. Out of the word which he writes a letter is taken at random and is found to be *u*. What are now the odds that he wrote the word in the former way?

---

The consequent chances of the observed event are as

$$12.11\times13.12.11\times13.12.11.10\times13.12.11.10$$
$$:13.12\times12.11.10\times13.12.11.10\times13.12.11.10$$
$$:13.12\times13.12.11\times12.11.10.9\times13.12.11.10$$
$$:13.12\times13.12.11\times13.12.11.10\times12.11.10.9$$
$$=11:10:9:9.$$

$\therefore$ The *à posteriori* chances are $\frac{11}{39}$, $\frac{10}{39}$, $\frac{9}{39}$, $\frac{9}{39}$.

**370.** If he left it in the first, events have happened whose likelihood was $\frac{1}{4}\times1\times1=\frac{1}{4}$. If in the second $\frac{3}{4}\times\frac{1}{4}\times1=\frac{3}{16}$. If in the third $\frac{3}{4}\times\frac{3}{4}\times\frac{1}{4}=\frac{9}{64}$.

The chances are therefore as 16 : 12 : 9.

**371.** The chance of a man voting for himself $=\frac{10}{20}$, for any assigned man $=\frac{1}{20}$.

The man voted for himself or not.

In the former case he got 4 votes out of the other 10.

$$\text{Chance}=210\left(\tfrac{1}{20}\right)^4\left(\tfrac{19}{20}\right)^6.$$

In the latter case he got 5 votes out of 10.

$$\text{Chance}=252\left(\tfrac{1}{20}\right)^5\left(\tfrac{19}{20}\right)^5.$$

Odds that he voted for himself are $19.210:252=95:6$.

**372.** *A priori* 2 : 1. Consequent chances $\frac{1}{8}:\frac{2}{9}$. *A posteriori* chances $\frac{2}{8}:\frac{2}{9}=9:8$ in favour of *rigorous*.

373. If two letters had been taken at random out of the word and found to be *both alike*, what would be the odds that the word was written in the former way?

374. The face of a die, which should have been marked ace, has been accidentally marked with one of the other five numbers. A six is thrown twice in two throws. What is the chance that the third throw will give a six?

375. Reference is made to a month which contains portions of six different weeks: what is the chance that it contains thirty-one days?

376. One card has been lost out of a pack. From the remainder 13 cards are drawn and are found to be all black. Find the chance that the lost card is red.

377. A bag contains seven cards inscribed with the letters of the word *s i n g i n g*. Another bag contains the letters of *m o r n i n g*, and a third bag the letters of *e v e n i n g*. From one of the bags two letters are drawn and are found to be $n$ and $g$. What is the chance that it was the first bag?

---

**373.** Consequent chances $\frac{2}{8.7} : \frac{3}{9.8} = 6 : 7$. *A posteriori* chances $12 : 7$.

**374.** *A priori* chances that the die has one or two sixes are as $4 : 1$. Consequent chances as $(\frac{1}{6})^2 : (\frac{1}{3})^2 = 1 : 4$.

$\therefore$ *A posteriori* chances are equal and chance that the next throw will be six $= \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{4}$.

**375.** It must contain either 30 or 31 days. *A priori* chances as $4 : 7$. Consequent chances of observed event are $\frac{1}{7}, \frac{2}{7}$. $\therefore$ *A posteriori* chances are as $4 : 14$; i.e. the chance that it contains 31 days is $\frac{7}{9}$.

**376.** We have either $r^{25}b^{26}$ or $r^{26}b^{25}$. The consequent chances of the observed event are as $C_{13}^{26}$ to $C_{13}^{25}$ or as $26 : 13 = 2 : 1$.

$\therefore$ There is a chance $\frac{2}{3}$ that the lost card was red.

**377, 378.** The word is either $i^2n^2g^2s$ or $n^2 m o r i g$ or $e^2n^2 v i g$.

The consequent chances of the event observed in Qn. 377 are as $4 : 2 : 2$; and of the event in Qn. 378 as $3 : 1 : 2$.

378. From one of the bags in the last question two letters are drawn and are *both alike*, what are the respective chances of the bag having been the first, the second or the third?

379. One letter each is drawn from two of the bags, and the two letters drawn are found to be both alike. What are the respective chances that the bag not drawn from was the first, the second or the third?

380. In six trials, on an average, $A$ hits the centre 3 times, the outer twice, and misses once. $B$ misses 3 times, hits the outer twice, and the centre once. A man who is equally likely to be $A$ or $B$ is observed to fire three shots, two of which hit the centre and one misses. Shew that the odds are now 3 to 1 that the man is $A$.

381. If two letters are taken at random out of *little lilies* shew that the odds are 5 to 1 against their being both alike.

---

The chances of the several bags are therefore in the former case $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, and in the latter $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{3}$.

**379.** The letters are drawn either from the 2nd and 3rd, 3rd and 1st or 1st and 2nd. The respective chances of the observed event are as 6 : 8 : 8. $\therefore$ The chances that the 1st, 2nd or 3rd bag was not drawn from are $\frac{3}{11}$, $\frac{4}{11}$, $\frac{4}{11}$.

**380.**

| | *Centre* | *Outer* | *Miss* |
|---|---|---|---|
| $A$'s chances are | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{6}$ |
| $B$'s chances are | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{2}$ |

The shooter is either $A$ or $B$.

Consequent chances of observed event are as

$$\left(\tfrac{1}{2}\right)^2 \tfrac{1}{6} : \left(\tfrac{1}{6}\right)^2 \tfrac{1}{2} = 3 : 1.$$

The *à posteriori* odds are therefore 3 : 1 that it is $A$.

**381.** We have $l^4 i^3 t^2 e^2 s$.

Favourable ways $4 \,.\, 3 + 3 \,.\, 2 + 2 \,.\, 1 + 2 \,.\, 1 = 22$.

Possible ways $12 \,.\, 11 = 132$.

Odds $= 110 : 22 = 5 : 1$ against.

382. If three of the letters are taken at random the odds are 6 to 5 against two (at least) being alike.

383. Four persons draw each a card from an ordinary pack. Find the chance (i) that one card is of each suit: (ii) that no two cards are of equal value: (iii) that one card is of each suit and no two of equal value.

384. Each of four persons draws a card from an ordinary pack. Find the chance that one card is of each suit, and that in addition, on a second drawing, each person shall draw a card of the same suit as before.

385. All the cards marked 1, 2, 3, 4, 5 are dealt into one heap, and all marked 6, 7, 8, 9, 10 into another. If a card be taken at random out of each heap, what is the chance that the sum of the numbers on them is eleven?

---

**382.** All three alike in $4 + 1 = 5$ ways.

Two alike and 1 different in $8C_2^4 + 9C_2^3 + 10\,(C_2^2 + C_2^2) = 95$. Total: 100.

Possible ways $= C_3^{12} = 220$.

Odds $= 120 : 100 = 6 : 5$ against.

**383.** The favourable ways are

$$\text{(i)}\ 13^4, \quad \text{(ii)}\ C_4^{13}4^4, \quad \text{(iii)}\ C_4^{13}\,\underline{|4}.$$

The chances are therefore

$$\frac{13^4}{C_4^{52}} = \frac{2197}{20825}, \quad \frac{256\,.\,C_4^{13}}{C_4^{52}} = \frac{2816}{4165}, \quad \frac{24C_4^{13}}{C_4^{52}} = \frac{264}{4165}.$$

**384.** Chance $= \dfrac{13^4}{C_4^{52}} \cdot \dfrac{12^4}{48\,.\,47\,.\,46\,.\,45}$

$$= \frac{6\,.\,13^3\,.\,12^4}{51\,.\,50\,.\,49\,.\,48\,.\,47\,.\,46\,.\,45}.$$

**385.** There are 16 ways of drawing $6 + 5$, 16 ways of $7 + 4$, and so on. Total: 80 favourable ways.

Possible ways $= 20 \times 20 = 400$. Chance $= \frac{1}{5}$.

386. If two cards be taken out of each heap what is the chance that the sum is twenty-two?

387. A bag contains $m$ black balls and $n$ white balls $(m>n)$. These are drawn out in succession. Shew that the chance that there shall never be an equal number of black and white balls drawn is $(m-n)\div(m+n)$.

388. A man undertakes to win two games consecutively out of three games, playing alternately with two opponents of unequal skill. Shew that it is to his advantage to begin by playing with the more skilful opponent.

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**386.** $2+20$ or $10+12$ can be drawn in $6^2$ ways,

| | | | |
|---|---|---|---|
| $3+19$ or $9+13$ | ,, | $16^2$ | ,, |
| $4+18$ or $8+14$ | ,, | $22^2$ | ,, |
| $5+17$ or $7+15$ | ,, | $32^2$ | ,, |
| $6+16$ | ,, | $38^2$ | ,, |

$$\text{Favourable ways} = 2 \,.\, 6^2 + 2 \,.\, 16^2 + 2 \,.\, 22^2 + 2 \,.\, 32^2 + 38^2 = 5044.$$

Possible ways $= 190 \times 190$.

Chance $= \frac{5044}{36100} = \frac{1261}{9025}$.

**387.** The black balls drawn must always be in excess of the white. Hence the first ball must be black and the remaining $m+n-1$ balls must be drawn in such an order that the white never exceed the black. By Prop. XXXVIII, the no. of favourable orders is $C_{m-1,\,n}(m-n)\div m$. But the total number of orders for the $m+n$ balls is $C_{m,\,n}$. $\therefore$ The chance is

$$\frac{(m-n)\,C_{m-1,\,n}}{mC_{m,\,n}} = \frac{m-n}{m+n}.$$

**388.** Let $x$ and $y$ be $A$'s chances of winning a game against $B$ and $C$ respectively. Then if he play with $B$, $C$, $B$ in order his chance of winning two consecutive games is

$$xy + (1-x)\,xy = xy\,(2-x).$$

If he play with $C$, $B$, $C$ it is $xy\,(2-y)$. The former is the greater if $x<y$; that is, he must begin with the player against whom he has the smaller chance.

389. When $A$ and $B$ play at chess $A$ wins on an average two games out of three. What are the respective chances of $A$ winning each of the following matches, $A$ undertaking (1) to win two games out of the first three, (2) to win two consecutive games out of the first four, (3) to win four games out of the first six?

390. Two men engage in a match to win two games out of three, the odds in every game being three to two in favour of the player who begins. If the winner of any game always begins the next game, shew that the odds on the match are 69 : 56 in favour of the player who begins the first game.

391. $A$ and $B$ play at chess. If $A$ has the first move the odds are eleven to six in favour of his winning the game, but if $B$ has the first move the odds are only nine to five in $A$'s favour. It is known that $A$ has won a game, what are the odds that he had first move?

392. In the last Question, if $A$ has the first move in the first game, and the loser of each game plays first in the next game, what is $A$'s chance of winning at least two out of the first three games?

---

**389.** Chance of (1) $= (\frac{2}{3})^2 + 2 \cdot \frac{1}{3} (\frac{2}{3})^2 = \frac{20}{27}$.
Chance of (2) $= (\frac{2}{3})^2 + (\frac{1}{3}) (\frac{2}{3})^2 + \frac{1}{3} (\frac{2}{3})^2 = \frac{20}{27}$.
Chance of (3) $= 15 \cdot (\frac{2}{3})^4 (\frac{1}{3})^2 + 6 (\frac{2}{3})^5 \frac{1}{3} + (\frac{2}{3})^6 = \frac{496}{729}$.

**390.** Let $A$ be the player who begins the first game, and $B$ his competitor. We have chances

$= (\frac{3}{5})^3$ of $AAA$,
$= (\frac{3}{5})^2 \cdot \frac{2}{5}$ of $AAB$, $ABB$, $BBB$,
$= \frac{3}{5} (\frac{2}{5})^2$ of $ABA$, $BAA$, $BBA$,
$= (\frac{2}{5})^3$ of $BAB$.

Hence the odds in favour of $A$ are

$$(3^3 + 3^2 2 + 2 \cdot 3 \cdot 2^2) : (2 \cdot 3^2 2 + 3 \cdot 2^2 + 2^3) = 69 : 56.$$

**391.** *A priori* $A$ is as likely as not to have had the first move. The consequent chances of the observed event are as $\frac{11}{17} : \frac{9}{14} = 154 : 153$. $\therefore$ The odds are 154 : 153 that $A$ had the first move.

**392.** $A$ will win if the games occur in the following orders:

$AA$; chance $= \frac{11}{17} \cdot \frac{9}{14}$.
$ABA$; chance $= \frac{11}{17} \cdot \frac{5}{14} \cdot \frac{11}{17}$.
$BAA$; chance $= \frac{6}{17} \cdot \frac{11}{17} \cdot \frac{9}{14}$.

Total chance $= \frac{1441}{2023}$.

393. A marksman's score is made up of *centres* counting 2 each and *outers* counting 1 each. He finds that he hits the centre once in five trials and he hits the outer ring as often as he misses altogether. What score may he expect to make in 5 shots, and what is the chance that he makes exactly this?

394. What is the chance that he makes more than the expected score, and what is the chance that he makes less?

395. There are ten counters in a bag marked with numbers. A person is allowed to draw two of them. If the sum of the numbers drawn is an odd number, he receives that number of shillings; if it is an even number, he pays that number of shillings. Is the value of his expectation greater when the counters are numbered from 0 to 9 or from 1 to 10?

---

**393.** His chances of centre, outer, and miss are $\frac{1}{5}$, $\frac{2}{5}$, $\frac{2}{5}$ respectively. On an average he scores $2+2=4$ in five shots. To score exactly this he must make either

$$C^2M^3 \text{ or } C^1O^2M^2 \text{ or } O^4M^1.$$

These can occur respectively in 10, 30, 5 orders. Therefore his chance is

$$10\left(\tfrac{1}{5}\right)^2\left(\tfrac{2}{5}\right)^3 + 30\,\tfrac{1}{5}\left(\tfrac{2}{5}\right)^4 + 5\left(\tfrac{2}{5}\right)^5 = \tfrac{144}{625}.$$

**394.** The scores that give him less than 4 are

$$C^1O^1M^3,\ C^1M^4,\ O^3M^2,\ O^2M^3,\ O^1M^4,\ M^5.$$

These can occur respectively in

$$20,\ 5,\ 10,\ 10,\ 5,\ 1 \text{ orders.}$$

$$\therefore \text{ Chance} = (25\,.\,2^4 + 26\,.\,2^5) \div 5^5 = \tfrac{1232}{3125}.$$

The chance that he scores more than 4 is therefore

$$1 - \tfrac{144}{625} - \tfrac{1232}{3125} = \tfrac{1173}{3125}.$$

**395.** An odd sum can be drawn in 25 ways, an even sum in 20. Respective chances $\frac{5}{9}$, $\frac{4}{9}$. $\therefore$ His expectation $=\left(\frac{5}{9}-\frac{4}{9}\right)$ of 2 counters $=\frac{2}{9}$ of the average value of a counter. But in the former case the average value is $\frac{9}{2}$, in the latter case $\frac{11}{2}$. $\therefore$ His expectations are respectively 1 shilling and $1\frac{2}{9}$ shillings.

396. A bag contains six shillings and two sovereigns. What is the value of one's expectation if one is allowed to draw till one draws a sovereign?

397. One of two bags contains ten sovereigns, and the other ten shillings. One coin is taken out of each and placed in the other. This is repeated ten times. What is now the expectation of each bag?

398. Four whist players cut for partners (i.e. each draws a card, the two highest to play together and the two lowest together), what is the chance that they will have to cut again?

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**396.** The sovereigns are equally likely to have any places in the series of shillings drawn. They will divide the shillings into three lots of equal expectation. Hence the expected value of each lot is 2 shillings. But one of these lots consists of the shillings which are drawn before the sovereign. Hence the whole expectation of the drawer is £1. 2*s.*

**397.** More generally let $A$ and $B$ denote the original contents of the bags, and $A_n$ and $B_n$ the expectations after the change has been made $n$ times. Then $A_n + B_n = A + B$, and

$$\begin{aligned} A_n &= \tfrac{9}{10}A_{n-1} + \tfrac{1}{10}B_{n-1} \\ &= \tfrac{4}{5}A_{n-1} + \tfrac{1}{10}(A+B) \end{aligned}$$

for all values of $n$. Give $n$ in succession all values from $n$ to 1; multiply the successive equations by 1, $\frac{4}{5}$, $(\frac{4}{5})^2$, $(\frac{4}{5})^3$ ... and add. Then we obtain

$$2A_n = A + B + (\tfrac{4}{5})^n (A - B).$$

Similarly, $$2B_n = A + B - (\tfrac{4}{5})^n (A - B).$$

In the particular case, $A = 40$ crowns, $B = 2$ crowns.

$$\therefore\ A_{10} = 21 + 19\,(\tfrac{4}{5})^{10},\quad B_{10} = 21 - 19\,(\tfrac{4}{5})^{10}.$$

**398.** They will have to cut again if there be four equal cards (13 ways); or three equal and one other,

$$(13 . 12 . 4 . 4 = 2496 \text{ ways});$$

or a pair with one superior and one inferior card

$$(C^{13}_3 . 4 . 6 . 4 = 27456 \text{ ways}).$$

399. The whole number of possible ways in which four whist players can have 13 cards each is 53644,737765,488792,839237,440000. Hence shew that if a party play a hundred millions of games the odds are more than ten billions to one against their ever having the same distribution of cards twice.

400. $A$ and $B$ play at cards. $A$'s skill : $B$'s :: 3 : 2. Nothing is known as to whether either has a confederate; if one has and the other has not, the odds on the dishonest player are doubled; $B$ wins 3 games successively, what is the chance he will win the next game?

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Total 29965 favourable ways. But the whole no. of ways is

$$C_4^{52} = 270725.$$

$$\text{Chance} = 29965 \div 270725 = 461 \div 4165.$$

**399.** Let $N$ represent the given number of ways in which the cards can be distributed, and let $n$ be the number of times they are dealt. The total no. of ways in which $n$ distributions can be made is $N^n$, but if no distribution is to occur twice it is $P_n^N$. Hence the chance of no repetition is

$$P_n^N \div N^n = \left(1 - \frac{1}{N}\right)\left(1 - \frac{2}{N}\right)\left(1 - \frac{3}{N}\right)\ldots\left(1 - \frac{n-1}{N}\right)$$

$$= 1 - \frac{n(n-1)}{2N} + \text{higher powers of } \frac{n}{N}.$$

Therefore the chance of some distribution occurring more than once is approximately

$$\frac{n^2}{2N} = \frac{1}{10{,}728947{,}553097};$$

or the odds are more than ten billions to one against it.

**400.** *A priori* four hypotheses are equally likely, viz. that $A$ only has a confederate, that neither has a confederate, that both have confederates, and that $B$ only has a confederate. The second and third may be treated together, so we have only to consider three hypotheses, viz. ($\alpha$) $A$ having advantage, ($\beta$) fair terms, ($\gamma$) $B$ having advantage.

*A priori* chances as 1 : 2 : 1.

401. What is the chance that two persons aged 33 and 57 should both be alive two years hence?

402. What is the chance that they should both be dead?

403. Find the chance that one and only one is dead.

404. If it be known that one and only one is alive, find the chance that the youngest is alive.

405. If at the age of 89 the expectation of life is 2·95 years what is the expectation at the age of 88?

406. What is the expectation at the age of 87?

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Consequent chances of observed event as $(\frac{1}{4})^3 : (\frac{2}{5})^3 : (\frac{4}{7})^3$.

$\therefore$ *A posteriori* chances of the three hypotheses are as

$$(\tfrac{1}{4})^3 : 2\,(\tfrac{2}{5})^3 : (\tfrac{4}{7})^3,$$

and the chance that $B$ wins the next game is

$$\{(\tfrac{1}{4})^4 + 2\,(\tfrac{2}{5})^4 + (\tfrac{4}{7})^4\} \div \{(\tfrac{1}{4})^3 + 2\,(\tfrac{2}{5})^3 + (\tfrac{4}{7})^3\}.$$

**401.** In this and the following questions it is understood that the table of mortality, given under Rule VIII of *Choice and Chance*, is to be used.

Of 252 people reaching 33, 247 reach 35. Of 171 reaching 57, 162 reach 59.

$$\therefore \text{ Chance} = \frac{247\,.\,162}{252\,.\,171} = \frac{13}{14}.$$

**402.** So $\frac{5}{252} \cdot \frac{9}{171} = \frac{5}{4788}$.

**403.** $\frac{247}{252} \cdot \frac{9}{171} + \frac{5}{252} \cdot \frac{162}{171} = \frac{337}{4788}$.

**404.** If one and only one be alive the odds are

$$9\,.\,247 : 5\,.\,162 = 247 : 90.$$

The chance is therefore $\frac{247}{337}$ that the younger man is alive.

**405, 406.** We have $R_{88}\,(E_x - \frac{1}{2}) = R_{89}\,(E_{x+1} + \frac{1}{2})$. But by the table $R_{87} : R_{88} : R_{89} = 9 : 7 : 5$.

$$\therefore\ 7\,(E_{88} - \tfrac{1}{2}) = 5\,(2{\cdot}95 + \tfrac{1}{2}),\ \ E_{88} = 2{\cdot}96.$$

So $$9\,(E_{87} - \tfrac{1}{2}) = 7\,(2{\cdot}96 + \tfrac{1}{2}),\ \ E_{87} = 3{\cdot}19.$$

407. Given that at the age of four the expectation of life is 52·1 shew that this is greater than the expectation either at three or five.

408. I promise to pay a boy's school fees for the next five years provided we both live. The fees are £50 per annum payable yearly in advance. If my age is 52 and the boy's age 13 what is the present value of the boy's expectation? (Interest at 5 per cent.)

409. Twelve months later we are both alive, for what immediate sum might I fairly compound the future payments?

410. What is the expectation of life at the age of 90, if among 100 persons reaching this age the deaths in successive years are 19, 17, 15, &c.?

411. Compare the chance of a person aged 20 living to be 60, with the chance of a person aged 60 living to be 80.

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**407.** We have $R_3 : R_4 : R_5 = 312 : 305 : 300$.

$\therefore\ 312\,(E_3 - \frac{1}{2}) = 305\,(E_4 + \frac{1}{2})$ and $300\,(E_5 + \frac{1}{2}) = 305\,(E_4 - \frac{1}{2})$.

But $E_4 = 52{\cdot}1$. $\therefore\ E_3 = 51{\cdot}92$ and $E_5 = 51{\cdot}96$.

**408.** Chance of second payment $= \frac{187}{191} \cdot \frac{286}{287}$.

Of third, $= \frac{183}{191} \cdot \frac{285}{287}$. Of fourth, $= \frac{179}{191} \cdot \frac{284}{287}$.

Of fifth, $= \frac{175}{191} \cdot \frac{283}{287}$.

$$\text{Present value} = \frac{£50}{191 \,.\, 287}\{191 \,.\, 287 + \tfrac{20}{21} \,.\, 187 \,.\, 286 + (\tfrac{20}{21})^2\, 183 \,.\, 285 + (\tfrac{20}{21})^3\, 179 \,.\, 284 + (\tfrac{20}{21})^4\, 175 \,.\, 283\} = £216{\cdot}83.$$

**409.** The expectation is now

$$\frac{£50}{187 \,.\, 286}\{(\tfrac{20}{21})\, 183 \,.\, 285 + (\tfrac{20}{21})^2\, 179 \,.\, 284 + (\tfrac{20}{21})^3\, 175 \,.\, 283\} = £129{\cdot}53.$$

**410.** Expectation $= \frac{19}{100} \cdot \frac{1}{2} + \frac{17}{100} \cdot \frac{3}{2} + \frac{15}{100} \cdot \frac{5}{2} + \ldots$ to 10 terms

$= \frac{670}{200} = 3{\cdot}35$.

**411.** $\dfrac{R_{60}}{R_{20}} : \dfrac{R_{80}}{R_{60}} = \frac{157}{278} : \frac{37}{157} = 24649 : 10286$.

412. A person is now aged 20, shew that the chance of his dying before he is 60 is nearly equal to the chance of his dying between 60 and 80.

413. Shew that the odds are 2 to 1 against a child 5 years old living to be 70.

414. Shew that the odds are 2 to 1 (nearly) in favour of a child 5 years old living to be 50.

415. A man emigrated leaving two brothers aged 51 and 61. Five years afterwards he hears that one is alive and one is dead. Find the odds in favour of the younger one being alive.

416. Three men are known to have been alive four years ago when their ages were 34, 59 and 70. Find the chance that they are all alive now.

417. Find the chance that two and only two are alive now.

418. Find the chance that one and only one is alive now.

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**412.** The table shews that of 278 persons reaching the age of 20, $20+26+33+42=121$ die before the age of 60, and $57+63=120$ between 60 and 80. Chances as 121 : 120.

**413.** $R_{70} \div R_5 = 100 \div 300 = \frac{1}{3}$.

Odds 2 to 1 against his surviving.

**414.** $1 - R_{50} \div R_5 = 1 - 199 \div 300 = \frac{1}{3}$ nearly.

Odds nearly 2 to 1 in favour.

**415.** The odds are as

$$R_{56}(R_{61} - R_{66}) : R_{66}(R_{51} - R_{56})$$
$$= 175(152 - 124) : 124(195 - 175) = 245 : 124.$$

**416.** Chance $= R_{38} \,.\, R_{63} \,.\, R_{74} \div R_{34} \,.\, R_{59} \,.\, R_{70}$

$$= 238 \,.\, 142 \,.\, 72 \div 250 \,.\, 162 \,.\, 100 = 16898 \div 28125.$$

**417.** Chance

$$= \frac{238 \,.\, 142 \,.\, 28 + 238 \,.\, 20 \,.\, 72 + 12 \,.\, 142 \,.\, 72}{250 \,.\, 162 \,.\, 100} = \frac{88231}{253125}.$$

**418.** Chance

$$= \frac{12 \,.\, 20 \,.\, 72 + 12 \,.\, 142 \,.\, 28 + 238 \,.\, 20 \,.\, 28}{250 \,.\, 162 \,.\, 100} = \frac{12392}{253125}.$$

419. If it be known that one and only one is alive find the chance that it is the youngest.

420. Find the chance that it is the eldest.

421. What is the chance that a hand of 5 cards contains at least two aces?

422. What is the chance that it contains exactly three aces?

423. What is the chance that it contains a pair?

424. What is the chance that it contains two pairs and no more?

425. What is the chance that it contains exactly three pairs?

426. What is the chance of having six pairs exactly in a hand of six cards?

427. What is the chance of having seven pairs in a hand of six cards?

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**419.** Odds are

$$(12 \,.\, 20 \,.\, 72 + 12 \,.\, 142 \,.\, 28) : 238 \,.\, 20 \,.\, 28 = 2031 : 4165 \text{ against.}$$

**420.** Odds are

$$(12 \,.\, 142 \,.\, 28 + 238 \,.\, 20 \,.\, 28) : 12 \,.\, 20 \,.\, 72 = 1414 : 135 \text{ against.}$$

**421.** Chance $= 1 - (C_5^{48} + 4C_4^{48}) \div C_5^{52} = \frac{2257}{54145}$.

**422.** Chance $= 4C_2^{48} \div C_5^{52} = \frac{94}{54145}$.

**423.** Chance of no pair $= C_5^{13} \,.\, 4^5 \div C_5^{52}$. This must be subtracted from unity. Result: $2053 \div 4165$.

**424.** The suitable hand may occur in $3C_3^{13} 6 \,.\, 6 \,.\, 4$ ways.

$$\text{Chance} = 428C_3^{13} \div C_5^{52} = 198 \div 4165.$$

**425.** The only suitable hand must contain three cards alike. No. of ways $= 3C_3^{13} \,.\, 4^3$.

$$\text{Chance} = 192C_3^{13} \div C_5^{52} = 88 \div 4165.$$

**426.** We must either have 4 cards alike and 2 different or 3 alike and 3 alike. The former hand occurs in $3C_3^{13} \,.\, 4^2$ ways, the latter in $C_2^{13} \,.\, 4^2$ ways.

$$\text{Chance} = 16\,(3C_3^{13} + C_2^{13}) \div C_6^{52} = 144 \div 195755.$$

**427.** We must have 4 cards alike and 2 alike. The no. of ways is $2C_2^{13} \,.\, 6$. Chance $= 12C_2^{13} \div C_6^{52} = 9 \div 195755$.

428. What is the chance of having a sequence of three cards and no more in a hand of five cards?

429. What is the chance that a hand of four cards should be a sequence?

430. $A$ has a hand of four cards containing three aces. $B$ has a hand of four cards out of the same pack. What is the chance that he has at least three cards alike?

431. Two aces, three kings and a queen having been removed from a pack, what is the chance that a hand of four cards should contain at least two alike?

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**428.** A sequence can either be extreme (beginning with 1 or ending with 13) or medial. An extreme sequence of 3 can be selected in $2 \,.\, 4^3$ ways, and there are then 36 cards from which the other two may be selected. Choice $= 2 \,.\, 4^3 C_2^{36}$. A medial sequence can be selected in $9 \,.\, 4^3$ ways, and there are 32 cards from which the other two can be selected. Choice $= 9 \,.\, 4^3 C_2^{32}$.

Hence chance $= (2C_2^{36} + 9C_2^{32})\, 4^3 \div C_5^{52} = 7632 \div 54145$.

**429.** The sequence can be formed in $10 \,.\, 4^4$ ways.

$$\text{Chance} = 10 \,.\, 4^4 \div C_4^{52} = 512 \div 54145.$$

**430.** Let $A$'s cards be $\alpha\alpha\alpha\beta$. Then $B$ can have such a hand as $\beta\beta\beta\mu$ in 45 ways and $\lambda\lambda\lambda\mu$, in $4 \times 11 \times 44 = 1936$ ways, and $\mu\mu\mu\mu$ in 11 ways. Total 1992 ways.

$$\text{Chance} = 1992 \div C_4^{48} = \tfrac{166}{16215},$$

giving odds over 96 : 1 against the event. But if $A$'s fourth card have not been seen there is a chance $\frac{1}{49}$ that it be an ace, and then $B$ can have $\lambda\lambda\lambda\mu$ in $4 \times 12 \times 44 = 2112$ ways and $\mu\mu\mu\mu$ in 12 ways. Total 2124 ways. Therefore $B$'s chance becomes

$$(\tfrac{1}{49} \text{ of } 2124 + \tfrac{48}{49} \text{ of } 1992) \div C_4^{48},$$

which reduces to $\frac{543}{52969}$, giving odds over 96 to 1 against the event.

**431.** Express the cards as $\alpha\alpha$, $\beta$, $\gamma\gamma\gamma$, $\delta\delta\delta\delta$, &c. ... $\mu\mu\mu\mu$.

First find the chance that the four cards are all different.

432. All the sevens having been removed from a pack what is the chance that a hand of six cards should be a sequence?

433. What would the chance be, if all the tens were removed instead of all the sevens?

434. Three cards out of a hand of six are seen to be *seven*, *eight*, *nine*. What is the chance that the whole hand forms a sequence?

435. What is the chance that it consists of three pairs?

---

They may be as $\alpha\beta\gamma\kappa$ in $6 \,.\, 40 = 240$ ways.

$$\left.\begin{matrix}\alpha\beta\\ \alpha\gamma\\ \beta\gamma\end{matrix}\right\} \kappa\lambda \text{ in } (2+6+3)\, C_2^{10} \,.\, 4^2 = 11 \,.\, 45 \,.\, 16 \text{ ways,}$$

$$\left.\begin{matrix}\alpha\\ \beta\\ \gamma\end{matrix}\right\} \kappa\lambda\mu \text{ in } (2+1+3)\, C_3^{10} \,.\, 4^3 = 6 \,.\, 120 \,.\, 64 \text{ ways,}$$

$\kappa\lambda\mu\nu$ in $C_4^{10} \,.\, 4^4 = 210 \,.\, 256$ ways.

The sum of these ways, divided by $C_4^{46}$, will give the chance that no two cards are alike $= \frac{7200}{10879}$.

And the complementary chance that two at least are alike is therefore $\frac{3679}{10879}$.

**432.** The sequences must be from 1 to 6 or from 8 to king. They can be made in $2 \,.\, 4^6$ ways. The chance is therefore $2 \,.\, 4^6 \div C_6^{48} = \frac{1024}{1533939}$.

**433.** In this case the sequences must be from 1 to 6, or 2 to 7, or 3 to 8, or 4 to 9, and can be made in $4 \,.\, 4^6$ ways. The result must be double of that in the last question $= \frac{2048}{1533939}$.

**434.** Besides the cards that have been seen there remain 49 cards out of which the other three might be selected in $C_3^{49}$ ways. The favourable ways must complete a sequence either from 4, 5, 6 or 7. Their number is $4 \,.\, 4^3$.

Hence the chance is $256 \div C_3^{49} = 32 \div 2303$.

436. What is the chance that it contains no pair?

437. If the three cards seen were *seven, eight, ten*, what would be the chance of the whole hand forming a sequence?

438. $A$ and $B$ have each a hand of five cards out of the same pack. If $A$'s cards form a sequence of five beginning from ace, what is the chance that $B$'s cards also form a sequence?

439. $A$ draws a black and a red card from a pack and $B$ draws two black. Shew that $A$'s chance of drawing a pair is greater than $B$'s in the ratio of 25 : 13.

440. Shew that in the last question $B$'s chance is the same whether $A$'s cards be replaced or not.

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**435.** The only favourable selection is another 7, 8, 9 out of the remaining 49 cards. This can be made in $3^3 = 27$ ways.

$$\text{Chance} = 27 \div C_3^{49} = 27 \div 18424.$$

**436.** Out of the 10 remaining sorts we must select 3 sorts and take one card of each.

$$\text{Chance} = C_3^{10} \,.\, 4^3 \div C_3^{49} = 960 \div 2303.$$

**437.** The sequence must be from 5, 6 or 7. There are therefore $3 \,.\, 4^3$ favourable selections.

$$\text{Chance} = 3 \,.\, 4^3 \div 18424 = 24 \div 2303.$$

**438.** $B$'s sequence can begin with ace in $3^5$ ways, with two in $3^4 \,.\, 4$ ways, and so on; and with 6, 7, 8, 9, each in $4^5$ ways. Total no.

$$= 3^5 + 3^4 \,.\, 4 + 3^3 \,.\, 4^2 + 3^2 \,.\, 4^3 + 3 \,.\, 4^4 + 4 \,.\, 4^5 = 7 \,.\, 4^5 - 3 \,.\, 3^5 = 6439.$$

$$\text{Chance} = 6439 \div C_5^{47} = 137 \div 32637.$$

**439.** Whatever card $A$ draws first, his chance of a second to pair with it is $\frac{2}{26}$. Whatever card $B$ draws first his chance of a second to pair with it is $\frac{1}{25}$. Ratio $= 25 : 13$.

**440.** If $A$'s cards are not replaced, $B$ has to draw two cards from 25 black ones, amongst which there are 12 pairs. His chance is $12 \div C_2^{25} = \frac{1}{25}$ as before.

441. Shew that the chance of throwing doublets with two dice one of which is true and the other loaded is the same as if both were true.

442. Shew that the chance of throwing seven is the same as if both were true.

443. There are three dice $A$, $B$, $C$, two of which are true and one is loaded so that in twelve throws it turns up *six* 3 times, *ace* once, and each of the other faces twice. Each of the dice is thrown three times and $A$ turns up 6, 6, 1; $B$ turns up 6, 5, 4; and $C$ turns up 3, 2, 1. What are now the respective chances that $A$, $B$ or $C$ is loaded?

444. If the dice in the last question are thrown once more and $A$ turns up 6, $B$ 1, and $C$ 4, what are now the respective chances that $A$, $B$ or $C$ is loaded?

445. If two dice are loaded *alike* in any way whatever, shew that the chance of throwing doublets is increased by the loading.

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**441.** For whatever face of the loaded die turn up there is still the same chance $\frac{1}{6}$ that the true die will turn up the same.

**442.** There is also the same chance $\frac{1}{6}$ that the true die will turn up the complementary number to make 7.

**443.** *A priori* it is equally likely that the loaded die is $A$, $B$ or $C$.

The consequent chances of the observed event are as

$$\tfrac{1}{4}\cdot\tfrac{1}{4}\cdot\tfrac{1}{12}\left(\tfrac{1}{6}\right)^6 : \tfrac{1}{4}\cdot\tfrac{1}{6}\cdot\tfrac{1}{6}\left(\tfrac{1}{6}\right)^6 : \tfrac{1}{6}\cdot\tfrac{1}{6}\cdot\tfrac{1}{12}\left(\tfrac{1}{6}\right)^6 = 9 : 12 : 4.$$

$\therefore$ The *à posteriori* chances are $\frac{9}{25}$, $\frac{12}{25}$, $\frac{4}{25}$.

**444.** We have now the *à priori* chances as $9 : 12 : 4$, and the consequent chances of the observed event are as

$$\tfrac{1}{4} : \tfrac{1}{12} : \tfrac{1}{6} = 3 : 1 : 2.$$

Hence the *à posteriori* chances are as $27 : 12 : 8$.

**445.** We know that if a magnitude be divided into a fixed number of parts, the sum of the squares of the parts is least when the parts are equal.

446. Shew that the chance of throwing *doublets-or-seven* is increased by the loading, except in one particular case in which it is unaltered.

447. In the case where two dice are loaded alike, so that the chance of throwing *doublets-or-seven* is unaltered, if $\frac{1}{6}+f$ be the chance of throwing doublets, the chance of throwing triplets with three such dice will be $\frac{1}{36}+\frac{1}{2}f$.

---

Let $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $x_6$ be the chances of 1, 2, 3, 4, 5, 6 respectively turning up when a loaded die is thrown, so that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1.$$

Then if there be two such dice loaded alike the chance of throwing doublets is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2;$$

this is a minimum when

$$x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = \tfrac{1}{6},$$

.e. when the die is true.

**446.** With the same dice the chance of throwing doublets-or-seven is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + 2(x_1x_6 + x_2x_5 + x_3x_4),$$

or $$(x_1 + x_6)^2 + (x_2 + x_5)^2 + (x_3 + x_4)^2.$$

This is a minimum when

$$x_1 + x_6 = x_2 + x_5 = x_3 + x_4 = \tfrac{1}{3}.$$

Therefore the chance of throwing doublets-or-seven is least when either the die is true, or when the sum of the chances of any two opposite faces is the same as in a true die.

**447.** Here we have

$$x_1 + x_6 = x_2 + x_5 = x_3 + x_4 = \tfrac{1}{3} \quad \therefore \ldots\ldots\ldots\ldots(1),$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = \tfrac{1}{6} + f \quad \ldots\ldots\ldots\ldots(2),$$

$$2(x_1x_6 + x_2x_5 + x_3x_4) = \tfrac{1}{6} - f \quad \ldots\ldots\ldots\ldots(3).$$

448. The chance of throwing three such dice so that two may turn up alike and the other different is $\frac{5}{12}+\frac{3}{2}f$.

449. The chance of throwing the same three dice so that all may turn up different is $\frac{5}{9}-2f$.

450. There are $1+2^n$ dice of which one is loaded so that the chance of throwing *six* with it is $\frac{1}{3}$; the rest are all true. If one of the dice selected at random be thrown $n$ times and always turns up *six*, shew that it is as likely as not to be the loaded one.

451. If in the $n$ throws there be only $r$ *sixes* the odds will be $5^{n-r}$ to $2^{n-r}$ against the die being the loaded one.

---

The chance of triplets $=\Sigma x^3$

$$=x_1^3+x_6^3+x_2^3+x_5^3+x_3^3+x_4^3$$

$$=(x_1+x_6)(x_1^2-x_1x_6+x_6^2)+\text{etc.}$$

or, in virtue of (1), (2), (3),

$$=\tfrac{1}{3}\left\{\tfrac{1}{6}+f-\tfrac{1}{2}\left(\tfrac{1}{6}-f\right)\right\}=\tfrac{1}{36}+\tfrac{1}{2}f.$$

**448.** The chance is $3\{x_1^2(1-x_1)+x_2^2(1-x_2)+\text{etc.}\}$

$$=3(\Sigma x^2-\Sigma x^3)=3\left(\tfrac{1}{6}+f\right)-3\left(\tfrac{1}{36}+\tfrac{1}{2}f\right)=\tfrac{5}{12}+\tfrac{3}{2}f.$$

**449.** The chance must be complementary to the sum of the chances in Qns. 447, 448

$$=1-\left(\tfrac{1}{36}+\tfrac{1}{2}f\right)-\left(\tfrac{5}{12}+\tfrac{3}{2}f\right)=\tfrac{5}{9}-2f.$$

**450.** The *à priori* chances that the selected die is true or loaded are as $2^n:1$. The consequent chances of the observed event are $(\frac{1}{6})^n$ and $(\frac{1}{3})^n$. Hence the *à posteriori* chances are as $2^n(\frac{1}{6})^n:(\frac{1}{3})^n$ a ratio of equality.

**451.** The respective chances that the true and the loaded die should produce $r$ sixes and $n-r$ not-sixes are $C_r^n(\frac{1}{6})^r(\frac{5}{6})^{n-r}$ and $C_r^n(\frac{1}{3})^r(\frac{2}{3})^{n-r}$.

Hence the *à posteriori* chances that the selected die is true or loaded are as

$$2^n\left(\tfrac{1}{6}\right)^r\left(\tfrac{5}{6}\right)^{n-r}:\left(\tfrac{1}{3}\right)^r\left(\tfrac{2}{3}\right)^{n-r}=5^{n-r}:2^{n-r}.$$

452. If there be two dice, loaded so that with either the chances of throwing 1, 2, 3, 4, 5, 6 are as $1-x : 1+2x : 1-x : 1+x : 1-2x : 1+x$, shew that the chances of throwing *more-than-seven*, *seven*, and *less-than-seven* are as $5+2x^2 : 2-4x^2 : 5+2x^2$.

453. Shew that with three dice loaded as in the last question the odds against throwing *more-than-ten* are as $18+x+5x^3$ to $18-x-5x^3$.

454. Shew that the most likely sum to be thrown with $2n$ dice is $7n$: and with $2n+1$ dice the most likely sums are $7n+3$ and $7n+4$.

---

**452.** To the denominator 36, the chance that

$$\begin{array}{lll} \text{1st die gives } 2, & \text{and 2nd gives } 6 & = (1+2x)(1+x), \\ 3, & 6 \text{ or } 5 & = (1-x)(2-x), \\ 4, & 6, 5, 4 & = (1+x) \, . \, 3, \\ 5, & 6, 5, 4, 3 & = (1-2x)(4-x), \\ 6, & 6, 5, 4, 3, 2 & = (1+x)(5+x). \end{array}$$

Therefore by addition the chance that the throw is greater than 7 is $\frac{1}{12}(5+2x^2)$.

The complementary throws are less than 7, and the chance must be obtained by writing $-x$ for $x$ in the last result: that is the chance is still $\frac{1}{12}(5+2x^2)$.

Subtracting the sum of these from unity we obtain $\frac{1}{12}(2-4x^2)$ as the chance that seven is thrown.

**453.** The method of the last question can be extended to the solution of this.

**454.** The no. of ways in which a number $m$ can be thrown with $r$ dice is the coeff. of $x^m$ in the expansion of

$$(x+x^2+x^3+x^4+x^5+x^6)^r.$$

See Prop. XXVIII. But by the process of continued multiplication when $x+x^2+x^3+x^4+x^5+x^6$ is multiplied by itself, and again by itself, any number of times, the coefficients of the terms from either end must be the same and the middle coefficient or coefficients must be the highest. With $2n$ dice the mean throw is $7n$, and with $2n+1$ dice the mean throws will be $7n+3$ and $7n+4$. These will therefore be the throws most likely to occur.

455. If $n$ dice be thrown the chance of an even number of aces turning up is one-half of $1+(\frac{2}{3})^n$, and the chance that the number of aces is a multiple of 4 is one-fourth of the sum of the $n$th powers of the roots of the equation $(6x-5)^4=1$.

456. $A$ and $B$ play for a stake which is to be won by him who makes the highest score in four throws of a die. After two throws, $A$ has scored 12, and $B$ 9. What is $A$'s chance of winning?

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**455.** The chance of 0, 2, 4, 6, &c. aces is

$$\{5^n + C_2^n 5^{n-2} + C_4^n 5^{n-4} + \ldots\} \div 6^n$$
$$= \tfrac{1}{2}\{(5+1)^n + (5-1)^n\} \div 6^n = \tfrac{1}{2}\{1 + (\tfrac{2}{3})^n\}.$$

The chance of 0, 4, 8, 12, &c. aces is

$$\{5^n + C_4^n 5^{n-4} + C_8^n 5^{n-8} + \ldots\} \div 6^n$$
$$= \tfrac{1}{4}\{(5+1)^n + (5-1)^n + (5+\sqrt{-1})^n + (5-\sqrt{-1})^n\} \div 6^n$$

= one fourth of the sum of the $n$th powers of the four roots of the equation $(6x-5)^4=1$.

**456.** With two dice, the chances of throwing

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

are as 1 : 2 : 3 : 4 : 5 : 6 : 5 : 4 : 3 : 2 : 1.

Therefore

| If $A$ throw | $A$ wins in | Equal in | $B$ wins in |
|---|---|---|---|
| 2 | 1 . 6 = 6 | 1 . 4 = 4 | 1 . 26 = 26 |
| 3 | 2 . 10 = 20 | 2 . 5 = 10 | 2 . 21 = 42 |
| 4 | 3 . 15 = 45 | 3 . 6 = 18 | 3 . 15 = 45 |
| 5 | 4 . 21 = 84 | 4 . 5 = 20 | 4 . 10 = 40 |
| 6 | 5 . 26 = 130 | 5 . 4 = 20 | 5 . 6 = 30 |
| 7 | 6 . 30 = 180 | 6 . 3 = 18 | 6 . 3 = 18 |
| 8 | 5 . 33 = 165 | 5 . 2 = 10 | 5 . 1 = 5 |
| 9 | 4 . 35 = 140 | 4 . 1 = 4 | |
| 10 | 3 . 36 = 108 | | |
| 11 | 2 . 36 = 72 | | |
| 12 | 1 . 36 = 36 | | |
| | 986 | 104 | 206 |

$A$'s expectation is therefore $(986+52) \div 1296 = 173 \div 216$.

457. $A$ and $B$ throw for a stake, $A$ having a die whose faces are numbered 10, 13, 16, 20, 21, 25; and $B$ a die whose faces are numbered 5, 10, 15, 20, 25, 30. The highest throw to win, and equal throws to go for nothing. Prove that the odds are 17 to 16 in favour of $A$.

458. A die of any number of faces has $\mu$ of its faces marked 1, and $\nu$ marked $-1$, the rest being zero. It is repeatedly thrown until the sum of the numbers turned up no longer lies between the limits $m$ and $-n$. Shew that either extreme of these limits is *equally* likely to be reached if

$$\mu^{m+n} + \nu^{m+n} = 2\mu^m\nu^n.$$

459. $A$ and $B$ throw each with a pair of dice until one of them throws sixes. $A$'s dice are true, but one of $B$'s is loaded so that the chance of its turning up *six* is $\frac{6}{35}$. Shew that if $A$ has the first throw their expectations are equal.

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**457.**

| If $A$ throw | $A$ wins in | Equal in | $B$ wins in |
|---|---|---|---|
| 10 | 1 way | 1 way | 4 ways |
| 13 | 2 | 0 | 4 |
| 16 | 3 | 0 | 3 |
| 20 | 3 | 1 | 2 |
| 21 | 4 | 0 | 2 |
| 25 | 4 | 1 | 1 |
| | 17 | 3 | 16 |

Odds 17 : 16 in favour of $A$.

**458.** This is a particular case of the general proposition dealt with under Qn. 532 *infrà*. The chance of any point being scored to plus or minus are as $\mu : \nu$. "Plus" wins by getting $m$ points in excess of "Minus" and "Minus" wins by getting $n$ points in excess of "Plus."

By the proposition referred to, the expectation of Plus is

$$(\mu^{m+n} - \mu^m\nu^n) \div (\mu^{m+n} - \nu^{m+n}),$$

which becomes $\frac{1}{2}$ if

$$\mu^{m+n} + \nu^{m+n} = 2\mu^m\nu^n.$$

**459.** $A$'s chance of throwing sixes is $\frac{1}{36}$, $B$'s chance is $\frac{6}{35} \times \frac{1}{6} = \frac{1}{35}$. Therefore by Qn. 315 their expectations are equal.

460. $A$ and $B$ throw each with a pair of dice until one of them has thrown *six-and-one*. $A$'s dice are true, but one of $B$'s is loaded so that *six* turns up three times as often as any other face. Shew that the odds are 17 to 12 against $A$ if he begins, and 18 to 11 against $A$ if $B$ begins.

461. A bag contains $2n$ counters, of which half are marked with odd numbers and half with even numbers, the sum of all the numbers being $S$. A man is to draw two counters. If the sum of the numbers drawn be an odd number, he is to receive that number of shillings; if an even number, he is to pay that number of shillings. Shew that his expectation is worth (in shillings) $S \div n(2n-1)$.

462. If in the case of the last question there be $m+n$ counters, of which $m$ are marked with odd numbers, amounting to $M$, and $n$ with even numbers amounting to $N$, the man's expectation is worth

$$\frac{M+N-(m-n)(M-N)}{\frac{1}{2}(m+n)(m+n-1)}.$$

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**460.** At any throw $A$'s chance is $\frac{1}{18}$ and $B$'s is $\frac{3}{8}\cdot\frac{1}{6}+\frac{1}{6}\cdot\frac{1}{8}=\frac{1}{12}$. Therefore, if $x=\frac{17}{18}\times\frac{11}{12}$, if $A$ begin, $A$'s expectation is

$$\tfrac{1}{18}\{1+x+x^2+\ldots\},$$

and $B$'s is $\frac{17}{18}\cdot\frac{1}{12}\{1+x+x^2+\ldots\}$. The odds are 17 to 12 against $A$. But if $B$ begin, his expectation is $\frac{1}{12}\{1+x+x^2+\ldots\}$ and $A$'s is $\frac{11}{12}\cdot\frac{1}{18}\{1+x+x^2+\ldots\}$; or the odds are 18 to 11 against $A$.

**461.** An odd sum can be drawn in $n^2$ ways: an even sum in $2C_2^n = n(n-1)$ ways. The respective chances of an odd and even sum are therefore $\dfrac{n}{2n-1}$ and $\dfrac{n-1}{2n-1}$. His expectation is therefore to receive $\dfrac{2n}{2n-1}$ and to pay $\dfrac{2n-2}{2n-1}$ counters: i.e. to gain $\dfrac{2}{2n-1}$ counters. But the average value of a counter is $S \div 2n$. Therefore his expectation is $S \div n(2n-1)$.

**462.** Let $f$ denote $2 \div (m+n)(m+n-1) =$ the chance of any two specified counters being drawn. His expectation from drawing an odd and even counter is $fmn\left(\dfrac{M}{m}+\dfrac{N}{n}\right)=f(nM+mM)$.

463. $A$ says that $B$ says that $C$ says that $D$ says that a certain coin turned up head. If the odds are $a : b$ in favour of any person making a true report shew that the odds are $a^4+6a^2b^2+b^4$ to $4a^3b+4ab^3$ in favour of the coin having turned up head.

464. In the last question if the evidence came through a chain of $n$ witnesses, shew that the odds in favour of the event would be

$$(a+b)^n+(a-b)^n \text{ to } (a+b)^n-(a-b)^n.$$

465. All the combinations of $n$ letters, $r$ together, repetitions permissible, are written down. Find the chance that a combination selected at random will contain no repetition. $(n>r.)$

466. All the combinations of $n$ letters, $r$ together, repetitions permissible, are written down. Find the chance that a combination selected at random will contain all the $n$ letters. $(n<r.)$

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His expectation from two odd counters is

$$-fm(m-1)\frac{M}{m}=-f(m-1)M.$$

So from two even counters it is $-f(n-1)N$. Hence his total expectation is

$$f\{nM+mN-(m-1)M-(n-1)N\}$$
$$=\frac{M+N-(m-n)(M-N)}{\frac{1}{2}(m+n)(m+n-1)}.$$

**463, 464.** If the coin turned up head then an even number of persons made a false report and the rest spoke the truth; the odds in favour of the report are therefore as $E : O$, where

$$E=a^n+C_2^n a^{n-2}b^2+C_4^n a^{n-4}b^4+\dots$$
$$O=C_1^n a^{n-1}b+C_3^n a^{n-3}b^3+C_5^n a^{n-5}b^5+\dots$$

But $E+O=(a+b)^n$ and $E-O=(a-b)^n$,

$$\therefore\ E : O=\{(a+b)^n+(a-b)^n\} : \{(a+b)^n-(a-b)^n\}.$$

**465.** Possible combinations $=R_r^n$. Favourable combinations $=C_r^n$. Chance $=C_r^n\div R_r^n$.

**466.** Possible combinations $=R_r^n$. For favourable combinations we must take the $n$ letters and combine with them $r-n$ more selected with repetitions from the same. Choice $=R_{r-n}^n$,
$\therefore$ Chance $=R_{r-n}^n\div R_r^n=C_n^r\div R_n^r$.

467. All the combinations $n$ together of $mn$ letters being written down, a person selects $m$ of these combinations at random. Find the chance that the resulting combination will contain all the $mn$ letters.

468. A reading society of 13 members passes books in circular rotation from member to member in defined order. If for a new year the order be rearranged at random, what is the chance that no one will pass his books to the same member as in the previous year?

469. In a company of sixty members, each member votes for one of the members to fill an office. If the votes be regarded as given at random, what is the chance that some member shall get a majority of the whole number of votes? Also determine the chance of the same event on the hypothesis that every different result of the poll is equally likely to occur.

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**467.** The total no. of ways in which he can make his choice is $C_m^N = |\underline{N} \div |\underline{m}\, |\underline{N-m}$, where $N = C_n^{mn}$.

For a favourable issue each letter must occur once and only once. Now the $mn$ letters can be divided into $m$ indifferent parcels of $n$ each in $|\underline{mn} \div |\underline{m}\, (|\underline{n})^m$ ways. This is therefore the no. of favourable ways of selecting.

Hence, Chance $= |\underline{mn}\, |\underline{N-m} \div (|\underline{n})^m\, |\underline{N}$.

**468.** The no. of possible arrangements is $|\underline{12} = 479001600$. And by Prop. XXXIV the no. of favourable arrangements is

$$||\underline{12} - ||\underline{11} + ||\underline{10} - ||\underline{9} + \&c. = 162744944.$$

Hence,

Chance $= 162744944 \div 479001600 = \cdot 3397586646892202...$

**469.** (1) The odds are 59 to 1 against any specified vote being given to $A$. Hence $A$'s chance of getting exactly $30 + x$ votes is $C_{30+x}^{60} \left(\frac{1}{60}\right)^{30-x} \left(\frac{59}{60}\right)^{30+x}$. Giving $x$ all values from 1 to 30 $A$'s chance of a majority = the sum of the last 30 terms in the expansion of $(1 + 59)^{60} \div 60^{60} = \Sigma$ suppose, and the chance of someone getting a majority is $60\,\Sigma$.

(2) By Prop. XXVI the number of ways in which the votes can fall is $C_{59}^{119}$. In order that $A$ may have a majority we must give him 31 votes and distribute the others. The choice

470. A boy has done $n$ sums; the odds are $h : 1$ against any given sum being wrong; and the odds are $k : 1$ against the examiner seeing that any given wrong sum is wrong. If the examiner discovers that $r$ sums are wrong, shew that the most likely number to be wrong is the greatest integer in $(k+nk+rh+rhk)\div(h+k+hk)$.

471. If the odds against your getting full marks for any answer you send up be $9 : 1$, how many must you send up to make it as probable as not that you will get full marks for one at least?

472. A series is summed to the number of terms given by the throw (1) of a single die, (2) of a pair of dice. Find the expectation, in each case, when the series is $1+3+5+\ldots$.

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(by Prop. XXVI) is $C_{59}^{88}$. Hence the chance that $A$ has a majority is $C_{59}^{88} \div C_{59}^{119}$ and the chance that someone has a majority is 60 times as much $= 60 \,\underline{|88}\,\underline{|60} \div \underline{|119}\,\underline{|29}$.

**470.** *A priori* the chance that $n-x$ were right and $x$ wrong was $C_x^n h^{n-x} \div (h+1)^n$. The chance that out of $x$ wrong the examiner should detect exactly $r$ is $C_r^x k^{x-r} \div (k+1)^x$. If $F_x$ be the *à posteriori* chance that exactly $x$ were wrong, $F_x$ is proportional to

$$C_x^n C_r^x h^{n-x} k^{x-r} \div (h+1)^n (k+1)^x,$$

$$\therefore \frac{F_x}{F_{x-1}} = \frac{k}{h(k+1)} \cdot \frac{n-x+1}{x-r}.$$

$F_x > F_{x-1}$ if $x(hk+h+k) < hkr + hr + kn + k$. The chance is therefore greatest when $x$ is the greatest integer in the quotient

$$(hkr + hr + kn + k) \div (hk + h + k).$$

**471.** Suppose you send up $x$ questions. The chance that you get full marks for none is $(\frac{9}{10})^x$. This must be less than $\frac{1}{2}$, $\therefore (\frac{10}{9})^x > 2$. Hence

$$x > \frac{\log 2}{\log 10 - \log 9} = \frac{\cdot 3010300}{\cdot 0457575} = 7 \text{ nearly}.$$

**472 to 474.** Let the series be $a_1 + a_2 + a_3 + \ldots$. And let $s_1, s_2, s_3 \ldots$ denote the sum to 1, 2, 3 ... terms.

473. Find the expectation when the series is $1+4+9+16+\ldots$.

474. If the throw be made with two dice, the expectation will be equal to the sum of seven terms, if the series be such that the 8th term is equal to the 7th, the 9th to the 6th, and so on.

For example: let the series be $1.14+2.13+3.12+\ldots$.

475. Three customers order amongst them $3n$ copies of a pamphlet. In how many proportions may their orders have been given? and if all proportions are equally likely, what is the chance that the numbers ordered are unequal and in arithmetical progression?

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If only one die is thrown the expectation is obviously $\frac{1}{6}(s_1+s_2+s_3+s_4+s_5+s_6)$ or which is the same thing

$$\tfrac{1}{6}(6a_1+5a_2+4a_3+3a_4+2a_5+a_6).$$

If two dice are thrown the chances of throwing sums from 2 to 12 are as

$$1:2:3:4:5:6:5:4:3:2:1.$$

The expectation is therefore $\frac{1}{36}$th of

$$s_2+s_{12}+2(s_3+s_{11})+3(s_4+s_{10})+4(s_5+s_9)+5(s_6+s_8)+6s_7,$$

which may be written

$$s_7+\tfrac{1}{36}(a_{12}-a_3)+\tfrac{1}{12}(a_{11}-a_4)+\tfrac{1}{6}(a_{10}-a_5)+\tfrac{5}{18}(a_9-a_6)+\tfrac{5}{12}(a_8-a_7).$$

Hence in the series of Qn. 472 the expectation with one die is $\frac{1}{6}$ of $91=15\frac{1}{6}$; and with two dice it is

$$49+\tfrac{18}{36}+\tfrac{14}{12}+\tfrac{10}{6}+\tfrac{30}{18}+\tfrac{10}{12}=54\tfrac{5}{6}.$$

In the series of Qn. 473 the expectation of one die is

$$\tfrac{1}{6}\{6+20+36+48+50+36\}=32\tfrac{2}{3},$$

with two dice it is

$$140+\tfrac{135}{36}+\tfrac{105}{12}+\tfrac{75}{6}+\tfrac{225}{18}+\tfrac{75}{12}=183\tfrac{3}{4}.$$

The truth of Qn. 474 is obvious from the expression which we have found for the expectation. In the example $s_7=280$.

**475.** By Prop. XXV the no. of ways is $C_2^{3n-1}$.

If the numbers are to be in A. P. one man must take $n$ copies. The others can be divided into 2 unequal parcels in $n-1$ ways. The 3 lots can be assigned to the 3 men in $\underline{|3}$ ways. Therefore chance $=12(n-1)\div(3n-1)(3n-2)$.

476. Of three independent events the chance that the first *only* should happen is $a$; the chance of the second *only* is $b$; the chance of the third *only* is $c$. Shew that the independent chances of the three events are respectively

$$\frac{a}{a+x}, \quad \frac{b}{b+x}, \quad \frac{c}{c+x},$$

where $x$ is a root of the equation

$$(a+x)(b+x)(c+x)=x^2.$$

477. The theorem of the last question can be extended to the case of $n$ events, $x$ being then a root of the equation

$$(a+x)(b+x)(c+x)\ldots\text{to } n \text{ factors}=x^{n-1}.$$

478. If three numbers be named at random it is just as likely as not that every two of them will be greater than the third.

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**476.** Let $x$, $y$, $z$ be the independent chances of the three events, then

$$a = x(1-y)(1-z),$$
$$b = y(1-z)(1-x),$$
$$c = z(1-x)(1-y),$$

therefore

$$(1-x)(1-y)(1-z) = \frac{a(1-x)}{x} = \frac{b(1-y)}{y} = \frac{c(1-z)}{z} = s \text{ suppose},$$

then $$a - ax = sx,$$

$$\therefore \quad x = \frac{a}{a+s}, \quad y = \frac{b}{b+s}, \quad z = \frac{c}{c+s},$$

and $$s = (1-x)(1-y)(1-z)$$

$$= \frac{s}{a+s} \cdot \frac{s}{b+s} \cdot \frac{s}{c+s},$$

or $$(a+s)(b+s)(c+s) = s^2.$$

**477.** The more general case is established by precisely the same steps.

**478.** If each number is restricted not to exceed $n$, the chance is shewn in Qn. 297 to be $\frac{1}{2} + \frac{1}{2n^2}$. Make $n$ infinite and chance $= \frac{1}{2}$.

479. If the chance of any given person being ill at any given time is $\theta$, what is the most likely population of a parish in which there are $r$ persons ill? And what is the most likely number of persons to be ill at once in a parish of population $n$?

480. It is not known whether a set of dominoes goes to double six, double eight, or double nine. All three are equally likely. $n$ dominoes are drawn from the set and contain no number higher than six. What are now the respective chances that the set goes to double six, double eight, or double nine?

481. A floor is paved with tiles, each tile being a rhomboid whose breadth measured perpendicularly between two opposite sides is $a$, and perpendicularly between the other two opposite sides is $b$; and one of the diagonals is $d$. A stick of length $c$ is thrown upon the floor so as to fall

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**479.** Let there be a population $= x$, then the chance of the observed event is $C_r^x \theta^r (1-\theta)^{x-r} = F_x$ suppose.

Then
$$\frac{F_x}{F_{x-1}} = \frac{x(1-\theta)}{x-r},$$

$F_x$ is therefore greatest when
$$x - r = x - \theta x,$$
$$x = \frac{r}{\theta}.$$

(2) The chance that $x$ people are ill is $C_x^n \theta^x (1-\theta)^{n-x}$ and is greatest when
$$\frac{\theta}{1-\theta} \cdot \frac{n-x}{x} = 1 \qquad x = \theta n.$$

**480.** Following the steps of Qn. 364 noting that the three contingencies imply 28, 45, or 55 dominoes we have

$$\textit{à priori} \text{ chances as } 1 : 1 : 1,$$

consequent chances of observed event as
$$1 : C_n^{28} \div C_n^{45} : C_n^{28} \div C_n^{55}.$$

therefore the *à posteriori* chances are inversely as
$$C_n^{28} : C_n^{45} : C_n^{55}.$$

parallel to this diagonal. Shew that the chance that it lies entirely on one tile is $\left(1-\frac{c}{d}\right)^2$.

482. A circle, of diameter $c$, is thrown down on the same floor, shew that the chance that it lies on one tile is $\left(1-\frac{c}{a}\right)\left(1-\frac{c}{b}\right)$.

483. A ball of diameter $c$ inches is thrown at random against a trellis composed of bars of width $b$ inches crossing one another at right angles so as to leave openings of $a$ inches square: shew that the chance of the ball passing clear through the trellis is $(a-c)^2 \div (a+b)^2$.

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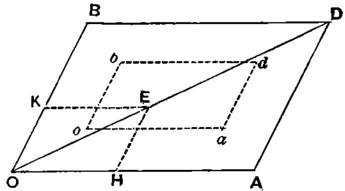
**481, 482.** Let the parallelogram $OADB$ represent the tile on which the nearer end of the stick falls, and let $OD$ be the diagonal whose length is $d$. From $OD$ cut off $DE=c$. On the diagonal $OE$ complete the parallelogram $OHEK$. The nearer end of the stick may fall anywhere on the tile, but that the stick may lie entirely on the tile it must fall on the rectangle $HK$. Hence the chance




$$=\frac{\text{area } OHEK}{\text{area } OADB}=\frac{OE^2}{OD^2}=\left(\frac{d-c}{d}\right)^2=\left(1-\frac{c}{d}\right)^2.$$

For the case of the circle, form a parallelogram $oadb$ within the other by lines parallel to the sides of the original parallelogram and at a perpendicular distance $=\frac{1}{2}c$ from them. For a favourable result the centre of the circle must lie on the area $oadb$, while its possible locus is the area $OADB$. Hence the chance

$$=\frac{\text{area } oadb}{\text{area } OADB}=\frac{(a-c)(b-c)}{ab}=\left(1-\frac{c}{a}\right)\left(1-\frac{c}{b}\right).$$

**483.** The whole area may be divided into squares of side $a+b$, consisting of the square opening, and the bar on two adjacent sides of it. The centre of the ball is equally likely to impinge upon any point on this square. But in order that it may pass

484. An indefinitely large piece of bread contains a spherical plum of diameter $b$. If the bread is cut up into cubical dice (each side $=a>b$), find the chance that the plum is cut.

485. A shot fired at a target of radius $a$ makes a hole of radius $b$. If the centre of the shot is equally likely to hit all points of the target, find the chance that the hole made is completely within the circumference of the target.

486. If a point be taken at random on a finite line it is as likely as not that one of the parts into which it divides the line will be at least three times as great as the other.

487. If two points be taken at random on the circumference of a circle, it is as likely as not that one of the arcs into which they divide the circumference will be at least three times as great as the other.

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through the opening it must be restricted to a square $(a-c)^2$ in the centre of the open square. Hence the

$$\text{chance} = (a-c)^2 \div (a+b)^2.$$

**484.** The centre of the plum is equally likely to be at any point within the cube of side $a$. But in order that it may not be cut it must lie within an inner cube of side $a-b$. Hence the chance that it is not cut is $(a-b)^3 \div a^3$, and the chance that it is cut is the complementary chance

$$1-\left(1-\frac{b}{a}\right)^3.$$

**485.** It must fall within a radius $a-b$.

$$\text{Chance} = (a-b)^2 : a^2.$$

**486.** Divide the line into four equal parts. If the point fall in either of the extreme quarters the condition is fulfilled: if in either of the mean quarters it is not fulfilled. The chances are equal.

**487.** Wherever the first point may be, think of the circle as cut there. The question then resolves itself into Qn. 486.

488. In either of the last two questions the chance that one part should be at least $m$ times the other is $\frac{2}{m+1}$.

489. If two points be taken at random on the circumference of a circle, the odds are 2 to 1 that their (rectilinear) distance apart will be greater than the radius of the circle.

490. The odds are also 2 to 1 that their distance apart will be less than $\sqrt{3}$ times the radius of the circle.

491. Their distance is as likely as not to exceed $\sqrt{2}$ times the radius of the circle.

492. If a chord equal to the radius be drawn in a circle, and taken as the base of a triangle whose vertex is a point taken at random on the circumference, the area of the triangle is as likely as not to exceed the area of an equilateral triangle whose sides are equal to the radius.

493. The odds are 5 to 1 against the area of the triangle being more than double that of the equilateral triangle.

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**488.** Divide the line into $m + 1$ parts. The condition is only fulfilled when the point falls on either of the extreme segments. $\therefore$ the chance is $2 \div (m + 1)$.

**489.** Describe a hexagon in the circle having one of its angular points at the first of the two points. The circle is now divided into six segments, and if the condition is to be fulfilled the second point must not fall on either of the two segments adjacent to the first point. Hence the odds are 2 to 1 in favour.

**490.** If the second point fall on either of the two furthest segments the distance will be greater than $\sqrt{3}$ radius. The odds are 2 to 1 against this.

**491.** Inscribe a square instead of a hexagon, and this proposition is obvious.

**492, 493.** Inscribe a hexagon in the circle. One side will be the given chord. If the vertex fall on the segment below this chord or either of the adjacent segments the area will be less than the triangle having its vertex at the centre. If it fall on

494. Two chords each equal to the radius are placed at random in a circle so as not to intersect. If they become the opposite sides of a quadrilateral, the area of the quadrilateral is as likely as not to be more than 3 times as great as its least possible area.

495. A point is taken at random in each of two adjacent sides of a square. Shew that the average area of the triangle formed by joining them is one eighth of the area of the square.

496. If the points be taken on two adjacent sides of a triangle the average area will be one fourth of the area of the triangle.

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the furthest segment the area will be more than the double. Hence the first event is as likely as not, and the odds against the second are 5 to 1.

**494.** Let one of the sides of the hexagon be the first chord, and consider it fixed. The quadrilateral will be least when the chords are adjacent and one of the sides of the quadrilateral vanishes. The area is then equal to that of the equilateral triangle on the chord. If the second chord assume the position of either of the segments *next but one* to the fixed chord the area will be three times this. If a perpendicular be let fall from the centre on the variable chord it will be seen that the favourable positions of this perpendicular occupy 120°, and the whole possible range is 240°. It is therefore as likely as not to be less than three times its least possible area.

**495, 496.** Let the adjacent sides be $OA \cdot OB$ and bisect them in $H$, $K$. Consider one point $P$ fixed in any position on $OA$. Divide $OB$ into $n$ equal segments ($n$ indefinitely large). Consider the two triangles formed when the second point is at the $r$th and the $(n-r)$th points of section. These two triangles are equal to the triangle $POB$, and their average value is the triangle $POK$. As this is true whatever be $r$, $POK$ must be the average value of all the triangles having $P$ for vertex, and we may consider the second point as fixed at $K$. Now treat $P$ as moveable on $OA$ and apply the same argument again. We thus

497. If $n$ points be taken at random on a finite line the average distance between any two consecutive points will be one $(n+1)$th of the line.

498. On opposite sides of a rectangle two points are taken at random and are joined by a straight line, dividing the rectangle into two quadrilaterals. Find the chance that the area of one is more than $n$ times the area of the other.

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find that for the average $P$ may be fixed at $H$. Hence $OHK$ represents the average area of the triangle, and this is $\frac{1}{8}$th of the square (Qn. 495) and $\frac{1}{4}$th of the triangle (Qn. 496).

**497.** Suppose $n+1$ points taken at random on a circle or other closed curve: they must divide the line into $n+1$ parts of equal expectation. The expectation of each is therefore one $(n+1)$th of the line. But the first point may be made to cut the closed curve, so that it becomes a finite line on which the other $n$ points are taken at random. Hence if $n$ points are taken at random on a finite line the expectation of the interval between two consecutive points is one $(n+1)$th of the line.

Second Solution (by Rev. T. C. Simmons). Let $AB$ be the finite line, and $P$, $Q$ any two consecutive random points. Suppose another random point $X$ introduced. Then since all orders of the $n+1$ points are equally likely the chance that $X$ falls between $P$ and $Q$ is $1 \div (n+1)$. But the chance that $X$ falls on $PQ$ must be $PQ \div AB$. Therefore (in the average)

$$PQ \div AB = 1 \div (n+1).$$

**498.** If a number be taken at random from each of the series

$$1, 2, 3 \ldots\ldots a, \text{ and } 1, 2, 3 \ldots\ldots b,$$

the chance that their sum shall be $x$, where $x$ is not greater than either $a$ or $b$, is $x \div ab$. $\therefore$ The chance that their sum shall not exceed $x$ is $(1+2+3+\ldots+x) \div ab = x(x+1) \div 2ab$.

Hence if a segment be cut off from each of two lines of lengths $a$ and $b$, the chance that the sum of the segments shall not exceed $s$ is $s^2 \div 2ab$.

In order that one quadrilateral may not exceed $\dfrac{1}{n+1}$ of the rectangle the random points must cut off at either end from the

499. A coin whose diameter is one third of a side of a tile is thrown on the pavement of Question 154. Find the chance that it falls entirely on one tile.

500. On a chess-board, in which the side of every square is $a$, there is thrown a coin of diameter $b$, so as to be entirely on the board, which includes a border of width $c$ outside the squares. Find the chance that the coin is entirely on one square. $(a > b > c.)$

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opposite sides segments whose sum shall not exceed $\frac{2}{n+1}$; the length of the sides being unity. The chance that this may occur at either end is $\left(\frac{2}{n+1}\right)^2 \div 2$. And as either end will serve, the total chance is the double of this, or $\frac{4}{(n+1)^2}$.

**499.** The areas of the squares and triangles are as $4 : \sqrt{3}$.

But there are twice as many triangles as squares. $\therefore$ The chances of the centre of the coin falling on a square or a triangle are as $2 : \sqrt{3}$.

If it fall on a square the favourable area is to the whole area as $(\frac{2}{3})^2 : 1$. If it fall on a triangle the favourable area is to the whole area as $\left(1 - \frac{1}{\sqrt{3}}\right)^2 : 1$.

Hence the whole chance of a favourable result is

$$\frac{2}{2+\sqrt{3}}\left(\frac{2}{3}\right)^2 + \frac{\sqrt{3}}{2+\sqrt{3}}\left(1 - \frac{1}{\sqrt{3}}\right)^2 = \frac{2}{9}(17\sqrt{3} - 28).$$

**500.** The area upon which the coin can fall is

$$(8a - b + 2c)^2.$$

The restriction that it must fall entirely on the board does not affect the favourable area which is $64(a-b)^2$. Hence chance is

$$(a-b)^2 \div (a - \tfrac{1}{8}b + \tfrac{1}{4}c)^2.$$

501. A purse contains 2 sovereigns and 10 shillings. The coins are to be drawn out one by one and I am to receive all the shillings that are drawn between the two sovereigns. What is my expectation?

502. A die is thrown until every face has turned up at least once. Shew that on an average $14\frac{7}{10}$ throws will be required.

503. A pack of cards is repeatedly cut until at least one card of every suit has been exposed. Shew that on an average it must be cut $8\frac{1}{3}$ times.

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**501.** If all the coins be drawn the sovereigns will divide the shillings into three series of equal expectation. One of these series is that which I am to receive. $\therefore$ Expectation $= 10 \div 3$ shillings.

**502.** If the die be thrown $n$ times the no. of ways is $6^n$.

Among which ace will be missing in $5^n$, ace and deuce in $4^n$, and so on. Hence by Prop. XIV, Cor. the no. of ways in which no face will be missing is

$$6^n - 6 \cdot 5^n + 15 \cdot 4^n - 20 \cdot 3^n + 15 \cdot 2^n - 6 \cdot 1^n;$$

and the chance of this is

$$1 - 6\left(\tfrac{5}{6}\right)^n + 15\left(\tfrac{4}{6}\right)^n - 20\left(\tfrac{3}{6}\right)^n + 15\left(\tfrac{2}{6}\right)^n - 6\left(\tfrac{1}{6}\right)^n;$$

or if $f_n$ be the chance of failing in $n$ throws to turn every face

$$f_n = 6\left(\tfrac{5}{6}\right)^n - 15\left(\tfrac{4}{6}\right)^n + 20\left(\tfrac{3}{6}\right)^n - 15\left(\tfrac{2}{6}\right)^n + 6\left(\tfrac{1}{6}\right)^n.$$

(Note that this reduces to unity if $n = 1, 2, 3, 4, 5$.)

And by Prop. XLIX success will be attained on an average in $s$ trials where

$$s = 1 + f_1 + f_2 + \dots = 1 + \frac{6 \cdot \frac{5}{6}}{1 - \frac{5}{6}} - \frac{15 \cdot \frac{4}{6}}{1 - \frac{4}{6}} + \frac{20 \cdot \frac{3}{6}}{1 - \frac{3}{6}} - \frac{15 \cdot \frac{2}{6}}{1 - \frac{2}{6}} + \frac{6 \cdot \frac{1}{6}}{1 - \frac{1}{6}}$$

$$= 1 + 30 - 30 + 20 - \tfrac{15}{2} + \tfrac{6}{5} = 14\tfrac{7}{10}.$$

**503.** Following the steps of Qn. 504 we get

$$f_n = 4\left(\tfrac{3}{4}\right)^n - 6\left(\tfrac{2}{4}\right)^n + 4\left(\tfrac{1}{4}\right)^n,$$

and $$s = 1 + 12 - 6 + \tfrac{4}{3} = 8\tfrac{1}{3}.$$

504. Cards are drawn one by one from a pack until at least one card of every suit has been drawn. Shew that on an average $7\frac{419}{630}$ cards must be drawn.

505. $A$ deals from a pack of cards till he has dealt a spade: $B$ takes the remainder of the pack and deals till he has dealt a spade: $C$ takes the remainder and deals till he has dealt a spade. Shew that their expectations (of the number of cards they will deal) are equal.

506. A person draws cards one by one from a pack until he has drawn $n$ hearts. Shew that his expectation is $\frac{53}{14}n$ cards.

507. A person draws cards one by one from a pack until he has drawn all the aces. How many cards may he expect to draw?

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**504.** With the notation of the preceding questions

$$f_n = (4C_n^{39} - 6C_n^{26} + 4C_n^{13}) \div C_n^{52},$$

$$f_1 = 4 \cdot \frac{39}{52} - 6 \cdot \frac{26}{52} + 4 \cdot \frac{13}{52},$$

$$f_2 = 4 \cdot \frac{39 \cdot 38}{52 \cdot 51} - 6 \cdot \frac{26 \cdot 25}{52 \cdot 51} + 4 \cdot \frac{13 \cdot 12}{52 \cdot 51},$$

$$f_3 = 4 \cdot \frac{39 \cdot 38 \cdot 37}{52 \cdot 51 \cdot 50} - 6 \cdot \frac{26 \cdot 25 \cdot 24}{52 \cdot 51 \cdot 50} + 4 \cdot \frac{13 \cdot 12 \cdot 11}{52 \cdot 51 \cdot 50},$$

and so on.

By summation we get

$$s = 1 + 4\left(\tfrac{53}{14} - 1\right) - 6\left(\tfrac{53}{27} - 1\right) + 4\left(\tfrac{53}{40} - 1\right) = 7\tfrac{419}{630}.$$

**505.** If the whole pack is arranged in all possible orders, the average intervals between the successive spades must be the same. Hence the successive dealers have equal expectations.

**506.** The 13 hearts divide the 39 not-hearts into 14 series of equal expectation. Expectation of each $= 39 \div 14$ cards. The drawer receives $n$ of these series in addition to $n$ hearts. His expectation $= n + \frac{39}{14}n = \frac{53}{14}n$.

**507.** The 4 aces will divide the 48 not-aces into 5 series of equal expectation, of which the drawer receives 4, in addition to the 4 aces. Expectation $= 4 + \frac{4}{5} \cdot 48 = 42\frac{2}{5}$.

508. A person draws cards one by one from a pack and replaces them till he has drawn two consecutive aces. How many cards may he expect to draw?

509. If he draw till he have drawn four consecutive spades how many cards may he expect to draw?

510. One throws a die until one has thrown two consecutive *sixes*, and then receives a shilling for every *six* he has thrown. What is the value of his expectation?

511. A pair of dice is to be thrown until doublets have turned up in every possible way. How many throws are to be expected?

512. A bag contains $m$ sovereigns and $n$ shillings. A man is allowed to draw coins one by one until he has drawn $p$ shillings. Shew that the value of his expectation is $\frac{mp}{n+1} + \frac{p}{20}$ pounds.

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**508.** Apply Prop. LIV. $\mu = \frac{1}{13}$, $\kappa = 2$. Result 182.

**509.** Here $\mu = \frac{1}{4}$, $\kappa = 4$. Result 340.

**510.** He may expect to make 42 throws. Of these the last two are certainly sixes, and the previous one is not six. These facts, however, do not affect the value of his expectation. An average of 42 throws will give an average of 7 shillings.

**511.** If the dice be thrown $n$ times the no. of ways is $36^n$.

Among which, aces will be missing in $35^n$, aces and deuces in $34^n$, and so on. Hence, as in Qn. 502,

$$f_n = 6\left(\tfrac{35}{36}\right)^n - 15\left(\tfrac{34}{36}\right)^n + 20\left(\tfrac{33}{36}\right)^n - 15\left(\tfrac{32}{36}\right)^n + 6\left(\tfrac{31}{36}\right)^n - \left(\tfrac{30}{36}\right)^n,$$

and $$s = 1 + 6 \cdot \tfrac{35}{1} - 15 \cdot \tfrac{34}{2} + 20 \cdot \tfrac{33}{3} - 15 \cdot \tfrac{32}{4} + 6 \cdot \tfrac{31}{5} - \tfrac{30}{6}$$
$$= 1 + 210 - 255 + 220 - 120 + 37\tfrac{1}{5} - 5 = 88\tfrac{1}{5}.$$

**512.** He certainly receives $p$ shillings; we have to add to this the expectation of sovereigns. Now the $n$ shillings divide the $m$ sovereigns into $n+1$ series of equal expectation; and the man receives $p$ of these series. Hence his expectation of sovereigns is $mp \div (n+1)$.

The total expectation in pounds is therefore

$$\frac{mp}{n+1} + \frac{p}{20}.$$

513. What would be the expectation in the last question if each coin drawn were scored to the drawer's credit and replaced before the next drawing?

514. If he were to draw until he had drawn $p$ pounds his expectation would be worth $\frac{np}{m+1}+20p$ shillings.

515. If an experiment is equally likely to succeed or fail, the chance that it will succeed exactly $n$ times in $2n$ trials is the ratio of the product of the first $n$ odd numbers to the product of the first $n$ even numbers.

516. If an experiment succeeds $n+1$ times for $n$ times that it fails, the chance that it will succeed exactly $n+1$ times in $2n+1$ trials is to the chance in the last question as $\{1-(2n+1)^{-2}\}^n$ to 1, or nearly as $4n+3$ to $4n+4$.

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**513.** The coins being replaced the sovereigns and shillings are always in the ratio of $m : n$. Therefore in any series of drawings the expectation of pounds must be to the expectation of shillings as $m : n$. If, therefore, he draw till he has $p$ shillings he must expect $mp \div n$ pounds. The value of his total expectation will therefore be

$$20mp \div n + p,$$

or
$$(20m+n)\,p \div n \text{ shillings.}$$

**514.** Interchanging $m$ and $n$ and interchanging pounds and shillings in Qn. 512 we get

$$\frac{np}{m+1} \text{ shillings} + p \text{ pounds} = \left(\frac{np}{m+1} + 20\,p\right) \text{ shillings.}$$

**515.** Chance $= C_{n,n} \div 2^{2n}$

$$= \frac{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,\ldots\,2n}{1\,.\,1\,.\,2\,.\,2\,.\,3\,.\,3\,\ldots\,n\,.\,n} \cdot \frac{1}{2^{2n}}$$

$$= \frac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{1\,.\,2\,.\,3\,\ldots\,n} \cdot \frac{1}{2^n} = \frac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{2\,.\,4\,.\,6\,\ldots\,2n}.$$

**516.** Chance $= \dfrac{(n+1)^{n+1}\,n^n}{(2n+1)^{2n+1}} \cdot C_{n,n+1}$ which is equal to the

517. If on an average 18 male children are born for 17 female, find the chance that among the next 35 births there are exactly 18 males.

518. If the odds are $m$ to $n$ (where $m>n$) in favour of an experiment succeeding, the most likely number of successes in $m+n$ trials is $m$; and $m+1$ is a more likely number than $m-1$ in the ratio of $mn+m$ to $mn+n$; but less likely than $m$ in the ratio of $m$ to $m+1$.

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result of the last question multiplied by

$$\frac{(n+1)^{n+1}n^n}{(2n+1)^{2n+1}} \cdot \frac{2n+1}{n+1} \times 2^{2n} = \left\{\frac{4n(n+1)}{(2n+1)^2}\right\}^n = \{1-(2n+1)^{-2}\}^n.$$

Expanding by the Binomial theorem we get a first approximation $(4n+3) \div (4n+4)$.

**517.** As in the last question the chance is

$$\left(\frac{306}{1225}\right)^{17} \cdot \frac{\lfloor 34}{\lfloor 17 \lfloor 17} = \frac{\lfloor 34}{\lfloor 17 \lfloor 17 \, . \, 4^{17}}\left(1-\frac{1}{1225}\right)^{17}$$

$$= \frac{1 \, . \, 3 \, . \, 5 \, \ldots\ldots \, 33}{2 \, . \, 4 \, . \, 6 \, \ldots\ldots \, 34} \times \frac{1208}{1225} \text{ nearly.}$$

**518.** Let $\mu_x$ denote the chance of exactly $x$ successes. Then

$$\mu_x = C_x^{m+n} m^x n^{m+n-x} \div (m+n)^{m+n},$$

and
$$\frac{\mu_{x+1}}{\mu_x} = \frac{m(m+n-x)}{n(x+1)},$$

$$\therefore \mu_{x+1} > \mu_x \text{ if } m(m+n-x) > n(x+1),$$

$$\text{i.e. if } (m+n)(m-x) > n,$$

$$\text{i.e. if } x = 0, 1, 2, \ldots (m-1).$$

And
$$\mu_{x+1} < \mu_x \text{ if } x = m, m+1, m+2, \ldots (m+n),$$

$$\therefore \mu_x \text{ is greatest when } x = m.$$

And
$$\mu_{m+1} : \mu_m = m : m+1,$$

$$\mu_m : \mu_{m-1} = n+1 : n,$$

$$\therefore \mu_{m+1} : \mu_{m-1} = mn+m : mn+n. \quad \text{Q. E. D.}$$

519. If the chance of a trial succeeding is to its chance of failing as $m : n$, the most likely event in $(m+n)r$ trials is $mr$ successes and $nr$ failures.

520. A candidate who correctly solves on an average $n-1$ questions out of $n$ goes in for an examination in two parts. In each part $n$ questions are proposed and he is required to solve $n-1$ of them. Shew that if the two parts were merged in one and he were required to solve $2(n-1)$ questions out of the $2n$, his chance of passing would be improved in the ratio of 4 to 5 very nearly. [Accurately as $4n^2-4n+1$ to $5n^2-5n+1$.]

521. A raffling match is composed of five persons each throwing three times with seven coins, the one turning up the greatest number of heads to be the winner. A player having turned up 20 heads, it is required to find his chance of winning.

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**519.** With the same notation as in the last question we find

$$\mu_{x+1} > \mu_x \text{ if } (m+n)(mr-x) > n,$$
$$\therefore\ \mu_x \text{ is greatest when } x = mr.$$

**520.** In the first case, Chance

$$= [C_1^n (n-1)^{n-1} + (n-1)^n]^2 \div n^{2n}.$$

In the second case, Chance

$$= [C_2^{2n} (n-1)^{n-2} + C_1^{2n} (n-1)^{n-1} + (n-1)^n] \div n^{2n}.$$

$$\text{Ratio } = (n+n-1)^2 : n(2n-1) + 2n(n-1) + (n-1)^2$$
$$= 4n^2 - 4n + 1 : 5n^2 - 5n + 1 = 4 : 5 \text{ nearly.}$$

**521.** Let $\alpha$ denote $(\frac{1}{2})^{21} = \cdot 000000476837$. Then the chance of an assigned person throwing 21 is $\alpha$, of his throwing 20 is $21\alpha$, of his throwing less than 20 is $1-22\alpha$.

The first person having thrown 20 his expectation of winning is

$1$ if all the others throw less than 20;
$\frac{1}{2}$ if one throws 20, and the others less;
$\frac{1}{3}$ if two throw 20, and the others less;
$\frac{1}{4}$ if three throw 20, and the other less;
$\frac{1}{5}$ if all throw 20.

522. $A$ and $B$ play at a game which is such that no two consecutive games can be drawn. Their skill is such that when a game may be drawn the probability of its being drawn is $q$. Prove that the chance of $r$ games being drawn out of $p$ is $C_r^{p-r} q^r (1-q)^{p-2r} \{p-(q+1)r+1\} \div (p-2r+1)$.

523. In a game of mingled chance and skill, which cannot be drawn, the odds are 3 to 1 that any game will be decided by superiority of play and not by luck. $A$ plays three games with $B$, and wins two. Prove that the odds are 3 to 1 in favour of $A$ being the better player. If also $B$ beat $C$ two games out of three, prove that the chance of $A$ winning the first three games he plays with $C$ is $\frac{103}{352}$.

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Hence his chance is

$$(1-22a)^4 + 2 \,.\, 21a\,(1-22a)^3 + 2\,(21a)^2(1-22a)^2 \\ + (21a)^3(1-22a) + \tfrac{1}{5}(21a)^4 \\ = 1 - 46a + 1014a^2 - 11155a^3 + 49082{\cdot}2a^4.$$

**522.** Let $a$ represent a game not drawn and $\delta$ a drawn game.

I. If the last game is $\delta$, all the arrangements are found by permuting $r-1$ blocks of $\delta a$ with the remaining $p-2r+1$ $a$'s; i.e. in $C_{r-1}^{p-r}$ ways. And (the chance of the $a$'s which follow the $\delta$'s being unity) the chance of each such arrangement is $q^r(1-q)^{p-2r+1}$. Hence chance $= C_{r-1}^{p-r} q^r (1-q)^{p-2r+1}$.

II. If the last game is $a$, all the arrangements are found by permuting $r$ blocks of $\delta a$ with the remaining $p-2r$ $a$'s; i.e. in $C_r^{p-r}$ ways. And the chance of each such arrangement is $q^r(1-q)^{p-2r}$. Hence chance $= C_r^{p-r} q^r (1-q)^{p-2r}$.

Therefore the total chance is

$$C_r^{p-r} q^r (1-q)^{p-2r} \left\{\frac{r(1-q)}{p-2r+1} + 1\right\}$$
$$= C_r^{p-r} q^r (1-q)^{p-2r} (p - qr - r + 1) \div (p - 2r + 1).$$

**523.** [There is an ambiguity in the expression "that any game will be decided by superiority of play and not by luck." Strictly we should suppose that there were the chance $\frac{3}{4}$ that the better player would win independently of luck; or the chance $\frac{1}{4}$

524. My chance of winning a game against $A$ is $\alpha$; against $B$ it is $\beta$; against $C$ it is $\gamma$; and so on. One of these players in disguise enters into play with me and I win the first $n$ games, shew that the chance of my winning the next game is $(\alpha^{n+1}+\beta^{n+1}+\gamma^{n+1}+\ldots)\div(\alpha^n+\beta^n+\gamma^n+\ldots)$.

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that the game should be decided by luck, in which case either party would be equally likely to win. The odds would then be 7 to 1 that the better player should win. But here the phrase appears to mean that the odds are 3 to 1 that the superior player wins. Compare Qn. 651.]

We begin with the two contingencies that $A$ is the better player, or that $B$ is. The consequent chances of the observed event (viz., that $A$ has won two games out of 3) are as $3 \cdot 3 \cdot 1$ to $1 \cdot 1 \cdot 3 = 3$ to 1.

Next $C$ is introduced, and we have the double event that $A$ has beaten $B$ twice out of 3 times, and $B$ has beaten $C$ twice out of 3 times. *A priori* the order of merit of the three players may equally be

$$ABC,\ ACB,\ BCA,\ BAC,\ CAB,\ CBA,$$

the consequent chances of the observed double event are as

$$9 : 3 : 3 : 3 : 3 : 1.$$

Hence the chance that $A$ is a better player than $C$ is

$$(9+3+3)\div 22 = \tfrac{15}{22}.$$

$\therefore$ The chance of $A$ winning the three games is

$$\tfrac{15}{22}\cdot\left(\tfrac{3}{4}\right)^3+\tfrac{7}{22}\cdot\left(\tfrac{1}{4}\right)^3=\tfrac{103}{352}.$$

**524.** *A priori* the chances that the disguised man is $A$, $B$, $C$, &c. are equal. On the several hypotheses the chances of the observed event are $\alpha^n$, $\beta^n$, $\gamma^n$, &c.

Therefore the *à posteriori* chances of the several hypotheses are as $\alpha^n : \beta^n : \gamma^n :$ &c.

And the actual chance of the first hypothesis $A$ is

$$\frac{\alpha^n}{\alpha^n+\beta^n+\gamma^n+\text{\&c.}}.$$

525. At a chess tournament the players are supposed to be divisible into 2 classes, the odds on a member of the first in a game with a member of the second being 2 to 1. The second class is twice as numerous as the first. A player is observed to win a game, and a bet of 7 to 10 is made that he belongs to the first class. Shew that this is fair if there are 18 players.

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On this hypothesis, the chance that I win the next game is

$$\frac{\alpha^{n+1}}{\alpha^n+\beta^n+\gamma^n+\&c.}.$$

And similarly for the other hypotheses.

Therefore the total chance of my winning the $n+1^{\text{th}}$ game is

$$\frac{\alpha^{n+1}+\beta^{n+1}+\gamma^{n+1}+\dots}{\alpha^n+\beta^n+\gamma^n+\dots}.$$

**525.** Let $H$, $K$ denote the respective hypotheses that the observed man is of the 1st or 2nd class. The *à priori* chances of $H$, $K$ are as $1 : 2$.

Let the total no. of players be $3n$. Then on the hypothesis $H$, there are $n-1$ players of the 1st class, and $2n$ of the second, with whom the observed man may play. With a player of the 1st class his chance is $\frac{1}{2}$, of the 2nd class $\frac{2}{3}$. Therefore the chance of the observed event is

$$\frac{n-1}{3n-1}\cdot\frac{1}{2}+\frac{2n}{3n-1}\cdot\frac{2}{3}=\frac{11n-3}{6(3n-1)}.$$

Similarly on the second hypothesis the chance of the observed event is

$$\frac{n}{3n-1}\cdot\frac{1}{3}+\frac{2n-1}{3n-1}\cdot\frac{1}{2}=\frac{8n-3}{6(3n-1)}.$$

Therefore the *à posteriori* chances of $H$ and $K$ are as

$$11n-3 : 16n-6.$$

If there are 18 players, $n=6$, and this ratio becomes $7 : 10$.

526. A player has reckoned his chance of success in a game to be $\alpha$, but he considers that there is an even chance that he has made an error in his calculation affecting the result by $\beta$ (either in excess or defect). Shew that this consideration does not affect his chance of success in a single game, but increases his chance of winning a series of games.

527. $A$ and $B$ play at draughts and bet £1 even on every game, the odds are $\mu : 1$ in favour of the player who has the first move. If it be agreed that the winner of each game have the first move in the next, shew that the advantage of having the first move in the first game is £$\mu - 1$.

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**526.** Let $\mu$ be the chance of the error in excess, equal to the chance of the error in defect. Then his chance of winning a series of $n$ games is

$$(1-2\mu)\,\alpha^n + \mu\,(\alpha+\beta)^n + \mu\,(\alpha-\beta)^n$$
$$= \alpha^n \left\{1 + 2\mu C_2^n \left(\frac{\beta}{\alpha}\right)^2 + 2\mu C_4^n \left(\frac{\beta}{\alpha}\right)^4 + \&c.\right\}.$$

Hence if $n > 1$ his chance is increased.

**527.** It is assumed that they go on playing indefinitely.

Let $E$ be the expectation of the first player; then since what one gains the other loses, the second player's expectation is $-E$.

At the first game the first player either wins £1 and remains first player or he loses £1 and becomes second player. And the chances of these two events are as $\mu : 1$,

$$\therefore\ E = \frac{\mu\,(1+E) + (-1-E)}{\mu+1} = \frac{\mu-1}{\mu+1}\,(1+E);$$

whence $$E = \frac{\mu-1}{2},\quad -E = -\frac{\mu-1}{2};$$

and the first player's advantage is $\mu - 1$ pounds. Q. E. D.

But if the no. of games be limited, let $E_n$ be the first player's expectation when $n$ more games are to be played. Then

$$E_0 = 0 \text{ and } E_n = \frac{\mu-1}{\mu+1}\,(1+E_{n-1}) = k\,(1+E_{n-1}).$$

528. Seven clubs compete annually for a challenge cup, which is to become the property of any club which wins it three consecutive years. Assuming all the clubs to be equally good, what is the chance that last year's winner (not having won the previous year) will ultimately own the cup?

529. What is the chance of a club that has won the last two years?

530. If there be $n$ clubs and the cup have to be won $k$ years in succession, what is the chance of a club that has won the last $r$ years ultimately owning the cup?

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Whence we find

$$E_1 = k, \quad E_2 = k + k^2, \quad E_3 = k + k^2 + k^3,$$

and so on;

$$E_n = \frac{k}{1-k}(1-k^n) = \frac{\mu-1}{2}\left\{1 - \left(\frac{\mu-1}{\mu+1}\right)^n\right\}.$$

And the first player's advantage is the double of this

$$= (\mu-1)\left\{1 - \left(1 - \frac{2}{\mu+1}\right)^n\right\} = (\mu-1)\left\{\frac{2n}{\mu+1} - \&c.\right\}.$$

**528** and **529**. Let $E_1$ be the expectation of a club that has won one year: then each of the other clubs has an expectation $\frac{1}{6}(1-E_1)$. Also let $E_2$ be the expectation of a club that has won the last two years. Then

$$E_1 = \frac{1}{7}E_2 + \frac{6}{7}\left(\frac{1-E_1}{6}\right),$$

$$E_2 = \frac{1}{7}E_3 + \frac{6}{7}\left(\frac{1-E_1}{6}\right).$$

But $E_3 = 1$. Hence solving the equations we have

$$E_1 = \tfrac{3}{19} \text{ and } E_2 = \tfrac{5}{19}.$$

**530.** With the notation of Qn. 528 we have

$$E_x = \frac{1}{n}E_{x+1} + \left(\frac{n-1}{n}\right)\left(\frac{1-E_1}{n-1}\right);$$

whence $$\frac{E_x}{n^x} = \frac{E_{x+1}}{n^{x+1}} + \frac{1-E_1}{n^{x+1}}.$$

531. Two players of equal skill engage in a match to play at draughts until one of them has won $k$ games *in succession.* Shew that after $A$ has won $r$ games in succession, the odds are $2^k+2^r-2$ to $2^k-2^r$ in favour of his winning the match.

532. $A$ and $B$ agree to play at chess until one has won $k$ games more than the other. If they play equally well, when one has won $r$ games more than the other, the odds are $k+r : k-r$ in favour of his winning the match. But if $A$ plays $\mu$ times as well as $B$, then when $A$ has won $r$ games the odds are $\mu^{2k}-\mu^{k-r} : \mu^{k-r}-1$ in favour of his winning the match.

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Giving $x$ all values from $r$ to $k-1$ (substituting $E_k=1$) and adding, we get

$$\frac{E_r}{n^r}=\frac{1}{n^k}+\frac{1-E_1}{n^k}\cdot\frac{n^{k-r}-1}{n-1},$$

or

$$n^{k-r}E_r=1+(1-E_1)\frac{n^{k-r}-1}{n-1}.$$

Also writing 1 for $r$,

$$n^{k-1}E_1=1+(1-E_1)\frac{n^{k-1}-1}{n-1};$$

whence

$$1-E_1=\frac{(n^{k-1}-1)(n-1)}{n^k-1};$$

and substituting this in the previous equation we find

$$E_r=(n^r-n^{r-1}+n^{k-1}-1)\div(n^k-1).$$

**531.** This is a particular case of the last question. $n=2$.

$$E_r=\frac{2^r-2^{r-1}+2^{k-1}-1}{2^k-1}=\frac{2^r+2^k-2}{2\,.\,2^k-2};$$

and the odds are therefore $2^r+2^k-2 : 2^k-2^r$.

NOTE. It is assumed that drawn games are not counted, and that the interposition of a drawn game does not vitiate a succession of victories.

**532** to **535.** Take the general case in which $A$'s chance of scoring a point is $\mu$ times $B$'s chance. And suppose that $A$ wins if he score $m$ in excess of $B$ before $B$ scores $n$ in excess of $A$.

533. $A$'s chance of scoring any point being better than $B$'s chance in the ratio of five to four, $A$ engages to score 14 in excess of $B$ before $B$ shall have scored three in excess of $A$. Shew that the odds are very slightly (about 331 to 330) in favour of $A$ winning the match.

---

Let $A_x$ denote $A$'s expectation when he has scored $x$ in excess of $B$. Then $A_m = 1$, $A_{-n} = 0$.

When $A$ has scored $x$ in excess of $B$, the issue of the next point must either increase his excess to $x+1$ or reduce it to $x-1$, and the odds are $\mu : 1$ in his favour;

$$\therefore \quad A_x = (\mu A_{x+1} + A_{x-1}) \div (\mu + 1);$$

whence $$\mu (A_{x+1} - A_x) = A_x - A_{x-1}.$$

Therefore $\mu^x (A_{x+1} - A_x)$ is constant for all values of $x$, and

$$\therefore \quad A_{x+1} - A_x = \mu^{-x} (A_1 - A_0);$$

give $x$ all values $y$, $y+1$, $y+2$ ... to $x-1$ and add, then

$$A_x - A_y = \frac{\mu^x - \mu^y}{\mu^{x+y}} \cdot \frac{\mu (A_1 - A_0)}{\mu - 1}.$$

And therefore, whatever be $x$, $y$, $u$, $v$,

$$\frac{A_x - A_y}{A_u - A_v} = \frac{\mu^x - \mu^y}{\mu^u - \mu^v} \cdot \frac{\mu^{u+v}}{\mu^{x+y}}.$$

Put $u = m$, $y = -n$, $v = -u$, and we have

$$A_x = \frac{\mu^{x+n} - 1}{\mu^{m+n} - 1} \cdot \mu^{m-x} \text{ and } A_0 = \frac{\mu^n - 1}{\mu^{m+n} - 1} \cdot \mu^m.$$

For the case of Qn. 532, $m = n = k$ and

$$A_r = (\mu^{2k} - \mu^{k-r}) \div (\mu^{2k} - 1),$$

or the odds are $\mu^{2k} - \mu^{k-r}$ to $\mu^{k-r} - 1$ in favour of $A$.

If $\mu = 1$, dividing each term by $\mu - 1$, the ratio becomes

$$k + r : k - r.$$

In Qn. 533, $m = 14$, $n = 3$, $\mu = \frac{5}{4}$,

$$A_0 = \{1 - (\tfrac{4}{5})^3\} \div \{1 - (\tfrac{4}{5})^{17}\} = \frac{\cdot 48800}{\cdot 97748},$$

or the odds are about 331 to 330 in $A$'s favour.

534. $A$ and $B$ play at a game which cannot be drawn, and the odds are seven to five in favour of $A$ winning any assigned game. They agree to play until either $A$ shall win by scoring 10 points ahead of $B$, or $B$ shall win by scoring two points ahead of $A$. Shew that the odds are about 176 to 175 in favour of $B$ winning the match.

535. $A$ and $B$ play at a game which cannot be drawn. $A$ undertakes to win $rn$ games in excess of $B$, before $B$ has won $n$ in excess of $A$. Shew that, if the match is fair, $A$'s chance of winning a single game must exceed $B$'s chance in the ratio

$$\sqrt[n]{2-\frac{1}{2^r-\frac{1}{2}r}} : 1 \text{ very nearly.}$$

536. $A$ and $B$ play a set of games, in which $A$'s chance of winning a single game is $p$, and $B$'s chance $q$. Find

(i) the chance that $A$ wins $m$ out of the first $m+n$,

(ii) the chance that when $A$ has won $m$ games, $m+n$ have been played,

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In Qn. 534, $m=10$, $n=2$, $\mu=\frac{7}{5}$,

$$A_0=\{1-(\tfrac{5}{7})^2\}\div\{1-(\tfrac{5}{7})^{12}\}=\frac{\cdot 4898}{\cdot 9824},$$

or the odds are about 176 to 175 in $B$'s favour.

In Qn. 535, $m=rn$, and since $A_0=\frac{1}{2}$ we must have $\mu=\sqrt[n]{x}$ where $x$ is given by the equation

$$x^{r+1}-2x^r+1=0,$$

$x$ has evidently a value very slightly differing from 2, say $2-y$. Then $y$ is given by the equation

$$1-2^r y+2^{r-1}ry^2-\&c.=0,$$

and for an approximation we have

$$y=1\div(2^r-\tfrac{1}{2}r);$$

$$\therefore\ \mu=\sqrt[n]{2-\frac{1}{2^r-\frac{1}{2}r}}.$$

**536.** (i) Chance $=C_{m,n}p^m q^n$.

(ii) It is assumed that any number of games is *à priori* as likely as any other number.

(iii) the chance that $A$ wins $m$ games before $B$ wins $n$ games.

537. A number of persons $A, B, C, D$ ... play at a game, their chances of winning any particular game being $\alpha, \beta, \gamma, \delta$ ... respectively. The match is won by $A$ if he gains $a$ games in succession; by $B$ if he gains $b$ games in succession; and so on. The play continues till one of these events happens. Shew that their chances of winning the match are proportional to

$$\frac{(1-\alpha)\,\alpha^a}{1-\alpha^a}, \quad \frac{(1-\beta)\,\beta^b}{1-\beta^b}, \quad \frac{(1-\gamma)\,\gamma^c}{1-\gamma^c}, \text{ \&c.}$$

and (ii) that the average no. of games in the match will be

$$1 \div \left\{\frac{(1-\alpha)\,\alpha^a}{1-\alpha^a} + \frac{(1-\beta)\,\beta^b}{1-\beta^b} + \frac{(1-\gamma)\,\gamma^c}{1-\gamma^c} + \dots\right\}.$$

---

Suppose the no. were $m+x$, then the chance of the observed event would be $C_{m,x}p^m q^x$. The chance of $m+n$ is therefore

$$C_{m,n}\,q^n \div \Sigma\,\{C_{m,x}\,q^x\},$$

$x$ varying from 0 to $\infty$.

But the sum may be written

$$1 + \frac{m+1}{1}\,q + \frac{(m+1)(m+2)}{1\,.\,2}\,q^2 + \dots$$

$$= (1-q)^{-(m+1)} = p^{-(m+1)}.$$

Therefore the chance $= C_{m,n}\,p^{m+1}\,q^n$.

(iii) $A$ wins at the $(m+x)$th game if he wins $m-1$ out of the first $m+x-1$ and then wins the $(m+x)$th, $x$ varying from 0 to $n-1$.

$\therefore$ Chance $= \Sigma\,\{C_{x,m-1}\,p^m q^x\}$ = the sum of the first $n$ terms in the expansion of $p^m(1-q)^{-m}$.

**537.** Suppose that they continue playing for a very large number of games, $=N$ suppose. Whenever a winning sequence occurs a match will be finished. But by Prop. LIV, $A$'s winning sequence will occur on an average once in $\mathfrak{a}^{-1}$ games, putting

$$\mathfrak{a} = \frac{(1-\alpha)\,\alpha^a}{1-\alpha^a}, \quad \mathfrak{b} = \frac{(1-\beta)\,\beta^b}{1-\beta^b}, \quad \mathfrak{c} = \frac{(1-\gamma)\,\gamma^c}{1-\gamma^c} \text{ \&c.}$$

538. A man possessed of $a+1$ pounds plays even wagers for a stake of 1 pound. Find the chance that he is ruined at the $(a+2x+1)^{\text{th}}$ wager and not before.

539. If a man playing for a constant stake, win $2n$ games and lose $n$ games, the chance that he is never worse off than at the beginning and never better off than at the end is $(n^2+n+2) \div (4n^2+6n+2)$.

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Hence $A$ may expect to win $N\mathfrak{a}$ matches in a course of $N$ games. So $B$ may expect to win $N\mathfrak{b}$ matches; and so on. $\therefore$ The expectations of $A$, $B$, $C$ ... are proportional to $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, ... .

Moreover (ii) in the course of the $N$ games the total no. of matches will be, on an average $N\mathfrak{a} + N\mathfrak{b} + N\mathfrak{c} + \dots$ . Hence the average no. of games in the match will be

$$1 \div (\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \dots).$$

**538.** He must lose $a+x$ and win $x$ out of the first $a+2x$ games, his losses never exceeding his gains by more than $a$, and then he must lose the $(a+2x+1)$th game.

The $a+2x$ games may occur in $C_{x,\,a+x}$ orders and among these, by Prop. XXXIX, $C_{x+a+1,\,x-1}$ are unfavourable. Hence

$$\text{Chance} = \{C_{x,\,a+x} - C_{x+a+1,\,x-1}\}\left(\tfrac{1}{2}\right)^{a+2x+1}$$
$$= \frac{a+1}{x} C_{x-1,\,a+x+1}\left(\tfrac{1}{2}\right)^{a+2x+1}.$$

**539.** Represent the gains and losses as in the diagram of Prop. XXXV. Classify the routes according as the $n$th gain is made after 0, 1, 2 ... or $n-1$ losses:—say after $x$ losses. Then we have $J_{n-1,\,x}$ favourable routes prior to the $n$th gain, and if we think of the routes in reversed order we shall see that there are $J_{n,\,n-x}$ favourable routes subsequent to the $n$th gain. Hence the total no. of favourable routes is

$$\Sigma_0^{n-1}\{J_{n-1,\,x}\, J_{n,\,n-x}\} = \Sigma\,\{(C_{n-1,\,x} - C_{n,\,x-1})\,(C_{n,\,n-x} - C_{n+1,\,n-x-1})\}$$
$$= \Sigma\,\{C_{n-1,\,x}\, C_{n,\,n-x} - C_{n-1,\,x}\, C_{n+1,\,n-x-1} - C_{n,\,x-1}\, C_{n,\,n-x} + C_{n,\,x-1}\, C_{n+1,\,n-x-1}\}$$
$$= C_{2n,\,n} - 2C_{2n+1,\,n-1} + C_{2n+2,\,n-2}.$$

540. If he win $2n+1$ games and lose $n+1$ games, the chance is
$$n \div (4n+6).$$

541. Prove that at birth the expectation of life is
$$\tfrac{1}{2} + R_1 + R_2 + R_3 + \dots.$$

542. Prove that the expectation at the age of $x$ years will be
$$\tfrac{1}{2} + (R_{x+1} + R_{x+2} + R_{x+3} + \dots) \div R_x.$$

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But the whole no. of possible routes is $C_{2n,n}$. Therefore the chance is
$$1 - 2 \cdot \frac{n}{2n+1} + \frac{n(n-1)}{(2n+2)(2n+1)} = \frac{n^2+n+2}{4n^2+6n+2}.$$

**540.** Proceeding as in the last question, dividing the route into 3 portions of (1) $n$ gains and $x$ losses, (2) one gain, (3) $n$ gains and $n+1-x$ losses, we have the total no. of favourable routes, $\Sigma \{J_{n,x} \, J_{n,n-x+1}\}$
$$= \Sigma \left\{\frac{(n-x+1)x}{(n+1)^2} C_{n,x} \, C_{n,n-x+1}\right\}$$
$$= \Sigma \{C_{n+1,x-1} \, C_{n+1,n-x}\} = C_{2n+3,n-1}.$$
$$\therefore \text{ Chance} = \frac{C_{2n+3,n-1}}{C_{2n+1,n+1}} = \frac{n}{4n+6}.$$

**541.** If all the deaths in each year occurred at the end of the year we should have
$$E = \delta_1 + 2\delta_2 + 3\delta_3 + 4\delta_4 + \dots$$
$$= 1 + \delta_2 + 2\delta_3 + 3\delta_4 + \dots.$$

If all the deaths occurred at the beginning of each year we should have
$$E = \delta_2 + 2\delta_3 + 3\delta_4 + \dots.$$

Taking the mean we have
$$E = \tfrac{1}{2} + \delta_2 + 2\delta_3 + 3\delta_4 + \dots$$
$$= \tfrac{1}{2} + R_1 + R_2 + R_3 + \dots.$$

**542.** If a man have completed $x$ years his chances of his dying in the $(x+1)$th, $(x+2)$th, $(x+3)$th, &c. years are respectively
$$\frac{\delta_{x+1}}{R_x}, \quad \frac{\delta_{x+2}}{R_x}, \quad \frac{\delta_{x+3}}{R_x}, \text{ \&c.}$$

543. Thirteen persons meet at dinner, their ages being $n$, $n+1$, $n+2$, and so on, to $n+12$. Shew that the chance that they will all be alive a year hence is $R_{n+13} \div R_n$, and express the chance that they will all be alive five years hence.

544. Three persons born on the same day keep their 21st birthday together. They agree to keep in like manner their 31st, 41st, 51st, &c., as long as they are all alive. How many such celebrations are there likely to be?

545. An aunt aged 70 has a nephew born to-day and she promises to make him three annual payments of £100 commencing on his 15th birthday, subject to their both being alive. What is the chance that the nephew will be entitled to all the 3 payments? And what is the chance that he will receive none of them?

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Hence following the steps of Qn. 541, we have

$$E = \tfrac{1}{2} + (R_{x+1} + R_{x+2} + R_{x+3} + \ldots) \div R_x.$$

**543.** The respective chances of their living a year are

$$\frac{R_{n+1}}{R_n}, \quad \frac{R_{n+2}}{R_{n+1}}, \quad \frac{R_{n+3}}{R_{n+2}}, \text{ \&c. } \frac{R_{n+13}}{R_{n+12}}.$$

The chance of all being alive a year hence is the continued product $= R_{n+13} \div R_n$.

**544.** The chances of their keeping the successive anniversaries are $\left(\frac{R_{31}}{R_{21}}\right)^3$, $\left(\frac{R_{41}}{R_{21}}\right)^3$, $\left(\frac{R_{51}}{R_{21}}\right)^3$, and so on.

The expectation is therefore

$$\{(R_{31})^3 + (R_{41})^3 + (R_{51})^3 + \text{\&c.}\} \div (R_{21})^3$$
$$= \{(276)^3 + (256)^3 + (229)^3 + (195)^3 + (152)^3 + (93)^3 + (32)^3 + (3{\cdot}24)^3\} \div (276)^3.$$

**545.** He will receive all three payments if both live 17 years. Chance

$$= \frac{R_{87}}{R_{70}} \cdot \frac{R_{17}}{R_0} = \frac{9}{100} \cdot \frac{283}{400} = \frac{2547}{40000}.$$

He will receive none, *unless* both live 15 years. Chance

$$= 1 - \frac{R_{85}}{R_{70}} \cdot \frac{R_{15}}{R_0} = 1 - \frac{15}{100} \cdot \frac{285}{400} = \frac{1429}{1600}.$$

546. What is the present value of the aunt's gift? *Questions of Present Value are to be solved approximately, interest being calculated at 5 per cent.*

547. If aunt and nephew are both alive 5 years hence what will then be the value of the nephew's expectation?

548. If out of 100 persons who reach the age of 87 years, 19 die in the first year, 17 the next year, 15 the next and so on in A. P., shew that the expectation of life at 87 is 3·35 years, at 88 it is $3{\cdot}0\dot{1}8\dot{5}$, and at 89 it is 2·6875.

549. On the hypothesis of the last question what is the present value of an annuity of £100 per annum on the life of a person aged 87: the first payment being due a year hence?

550. On the same hypothesis what annual premium should a person pay for an insurance of £1000, the first payment being made on his 87th birthday?

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**546.** Let $\beta$ denote $20 \div 21$. Then the present value will be in pounds

$$(15\,.\,285\beta^{15} + 12\,.\,284\beta^{16} + 9\,.\,283\,.\,\beta^{17}) \div 400$$
$$= £5{\cdot}1408 + 3{\cdot}9031 + 2{\cdot}7868 = £11{\cdot}8307.$$

**547.** The present value will be increased in the ratio $(20:21)^5$, and also in the ratio $R_5R_{75} : R_0R_{70}$.

Result $£11{\cdot}8307 \times 2{\cdot}578 = £30{\cdot}5$ nearly.

**548.** Here we have $R_{87+x} = (10-x)^2 \div 100$. The expectation of anniversaries is $(81 + 64 + 49 + \dots) \div 100 = 2{\cdot}85$. To this we must add $\frac{1}{2}$ for the expectation of the fraction of a year. $\therefore\ E_{87} = 3{\cdot}35$.

So $\quad E_{88} = \frac{1}{2} + (64 + 49 + 36 + \dots) \div 81 = 3{\cdot}0\dot{1}8\dot{5}$.

And $\quad E_{89} = \frac{1}{2} + (49 + 36 + 25 + \dots) \div 64 = 2{\cdot}6875$.

**549.** As the annuity will not be paid for a fraction of a year the expectation is only $81 + 64 + 49 + \dots$ and the present value is in pounds

$$81\beta + 64\beta^2 + 49\beta^3 + \dots + 4\beta^8 + \beta^9 = £248{\cdot}4.$$

**550.** The present value of £1000 payable at death is in pounds

$$190\beta^{\frac{1}{2}} + 170\beta^{\frac{3}{2}} + 150\beta^{\frac{5}{2}} + \dots = £854{\cdot}58.$$

551. $A$ who is aged 87 is entitled to the reversion of an annuity of £100 a year on his own life after the death of $B$ who is just three years older than himself. What is the present value of $A$'s expectation on the hypothesis of the last three questions?

552. What is the value of his expectation on the hypothesis that out of 9 persons who reach the age of 87, two die in the first year and one every year afterwards till all are dead?

553. If (on another planet) the conditions of life are such that out of every 95 persons who reach the age of 5 years one dies every year till all are dead, what is the chance that a person aged $m$ years will outlive another person whose age is $n$ years? $(n > m > 5.)$

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The P. V. of the annual premiums must be the same. But by Qn. 549 the P. V. of an annuity of £100 is £248·4.

$\therefore$ 248·4 : 854·58 = £100 : premium required; or annual premium = £344·06.

**551.** $A$ will receive the $x$th payment provided he attains $87 + x$ years and $B$ does not attain $90 + x$.

$$\text{Chance} = \frac{R_{87+x}}{R_{87}}\left(1 - \frac{R_{90+x}}{R_{90}}\right) = \frac{(10-x)^2\, x\,(14-x)}{4900}.$$

Note however that for the last three payments $B$'s death is certain. Hence P. V. of $A$'s expectation is 1-49th of

$$1053\beta + 1536\beta^2 + 1617\beta^3 + 1440\beta^4 + 1125\beta^5 + 768\beta^6$$
$$+ 441\beta^7 + 196\beta^8 + 49\beta^9 = £218·785.$$

**552.** We have $R_{87} = 9$, $R_{88+x} = 7 - x$, and P. V. = 100-45ths of

$$7\beta + 12\beta^2 + 15\beta^3 + 16\beta^4 + 15\beta^5 + 10\beta^6 + 5\beta^7 = £219·38.$$

**553.** We have $R_x = 100 - x$,

$$R_x + R_{x+1} = 199 - 2x, \quad R_x - R_{x+1} = 1,$$

while $x < 100$. Afterwards $R_x = 0$. Therefore the chance

$$= \frac{(199-2m) + (197-2m) + (195-2m) + \ldots \text{ to } (100-n) \text{ terms}}{2\,(100-m)\,(100-n)}$$

$$= \frac{(100-m)^2 - (n-m)^2}{2\,(100-m)\,(100-n)} = \frac{100 - 2m + n}{2\,(100-m)} = \frac{1}{2} + \frac{1}{2}\cdot\frac{n-m}{100-m}.$$

554. If out of $n$ persons who reach the age of $N$ years one dies every year till all are dead, shew that the expectation of life at the age of $N+r$ is $\frac{1}{2}(n-r)$.

555. On the hypothesis of the last question the chance that three persons aged $N$, $N+a$, $N+2a$ respectively should all be alive $a$ years hence is $(n-3a)\div n$.

556. On the same hypothesis find the present value of an annuity of £1 on the life of a person aged $N$ years, £$t$ being the interest on £1 for 1 year.

557. On the same hypothesis find the present value of an annuity of £1 to continue during the joint lives of two persons now aged $N$ and $N+a$.

558. What would be the present value of the annuity in the last question if it were to continue until both persons were dead?

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**554.** We have $R_{N+x}=n-x$. Therefore by Qn. 542 the expectation is

$$\tfrac{1}{2}+\{(n-r-1)+(n-r-2)+(n-r-3)+\ldots\}\div(n-r)$$
$$=\tfrac{1}{2}(n-r).$$

**555.** Chance $=R_{N+3a}\div R_N=(n-3a)\div n$.

**556.** If $\tau=1\div(1+t)$, we have for the present value

$$\{(n-1)\tau+(n-2)\tau^2+(n-3)\tau^3+\ldots+\tau^{n-1}\}\div n$$
$$=\Sigma_1^{n-1}\{x\tau^{n-x}\}\div n=(nt-t-1+\tau^{n-1})\div nt^2.$$

**557.** Chance of receiving $x$th payment

$$=\frac{R_{N+x}}{R_N}\cdot\frac{R_{N+a+x}}{R_{N+a}}=\frac{(n-x)(n-x-a)}{n(n-a)}.$$

It will be convenient for summation to write $n-x$ for $x$.

Hence P. V. $=\Sigma_{a+1}^{n-1}\{x(x-a)\tau^x\}\div n(n-a)$.

**558.** The annuity is equivalent to two annuities on their several lives, deducting an annuity on their joint lives.

Hence the P. V. is

$$\frac{\Sigma_1^{n-1}\{x\tau^{n-x}\}}{n}+\frac{\Sigma_1^{n-a-1}\{x\tau^{n-a-x}\}}{n-a}-\frac{\Sigma_{a+1}^{n-1}\{x(x-a)\tau^x\}}{n(n-a)}.$$

559. If out of $n^2$ persons who reach the age of $N$ years, $2n-1$ die in the first year, $2n-3$ in the second, $2n-5$ in the third, and so on, shew that the expectation of life at the age of $N+r$ is $\frac{n-r}{3}+\frac{1}{6(n-r)}$.

560. On the hypothesis of the last question, shew that the chance that $p$ persons whose ages are $N, N+1, N+2, \ldots N+p-1$, will live another year is $\left(1-\frac{p}{n}\right)^2$, and the chance that they will all live for $r$ years is $\{C_r^{n-p} \div C_r^n\}^2$.

561. A bag contains $2m$ balls of which $2n$ are white. A second contains $3m$ balls of which $3n$ are white. If two balls are to be drawn from each bag, which bag is the more likely to give both white? and which is the more likely to give at least one white?

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**559.** We have $R_{N+x}=(n-x)^2$. Therefore by Qn. 542 the expectation is

$$\tfrac{1}{2}+\{(n-r-1)^2+(n-r-2)^2+(n-r-3)^2+\ldots\}\div(n-r)^2$$

$$=\tfrac{1}{2}+\frac{(n-r-1)(2n-2r-1)}{6(n-r)}=\frac{n-r}{3}+\frac{1}{6(n-r)}.$$

**560.** The chance that all live a year is

$$R_{N+1}R_{N+2}\ldots R_{N+p}\div R_N R_{N+1}\ldots R_{N+p-1}$$

$$=R_{N+p}\div R_N=(n-p)^2\div n^2.$$

So the chance that they all live $r$ years is

$$R_{N+r}R_{N+r+1}\ldots R_{N+p+r-1}\div R_N R_{N+1}\ldots R_{N+p-1}$$

$$=\left\{\frac{\lfloor n-r \, \lfloor n-p}{\lfloor n-p-r \, \lfloor n}\right\}^2=\left\{\frac{C_r^{n-p}}{C_r^n}\right\}^2.$$

**561.** Chances of both white

$$\frac{2n(2n-1)}{2m(2m-1)} \text{ and } \frac{3n(3n-1)}{3m(3m-1)}$$

are as $\quad 6mn-3m-2n+1 : 6mn-2m-3n+1.$

The *second* is the greater since $m>n$.

Chances of at least one white

$$1-\frac{2(m-n)(2m-2n-1)}{2m(2m-1)},$$

562. Each of two bags contains $m$ sovereigns and $n$ shillings. If a man draw a coin out of each bag he is more likely to draw two sovereigns than if all the coins were in one bag and he drew two.

563. There are 3 balls in a bag, and each of them may with equal probability be white, black, and red. A person puts in his hand and draws a ball. It is white. It is then replaced. Find the chance (i) of all the balls being white, (ii) that the next drawing will give a red ball.

---

and $$1 - \frac{3(m-n)(3m-3n-1)}{3m(3m-1)}.$$

The *first* will be the greater because

$$\frac{2m-2n-1}{2m-1} < \frac{3m-3n-1}{3m-1}.$$

**562.** We have

$$\left(\frac{m}{m+n}\right)^2 > \frac{2m(2m-1)}{2(m+n)(2m+2n-1)}$$

because $$\frac{m}{m+n} > \frac{2m-1}{2m+2n-1}.$$

**563.** The following combination of balls are possible:

$$www, \quad wwb, \quad wwr, \quad wbb, \quad wrr, \quad wbr$$

occurring respectively in the following numbers of different orders:

$$1, \quad 3, \quad 3, \quad 3, \quad 3, \quad 6.$$

The probabilities of each producing the observed result are

$$1, \quad \tfrac{2}{3}, \quad \tfrac{2}{3}, \quad \tfrac{1}{3}, \quad \tfrac{1}{3}, \quad \tfrac{1}{3}.$$

The *à posteriori* probabilities of the respective combinations are therefore as

$$1 : 2 : 2 : 1 : 1 : 2.$$

And their actual values are

$$\tfrac{1}{9}, \quad \tfrac{2}{9}, \quad \tfrac{2}{9}, \quad \tfrac{1}{9}, \quad \tfrac{1}{9}, \quad \tfrac{2}{9}.$$

Hence the odds are 8 to 1 against all the balls being white.

The chances of each of these combinations giving a red ball at the next drawing are

$$0, \quad 0, \quad \tfrac{1}{3}, \quad 0, \quad \tfrac{2}{3}, \quad \tfrac{1}{3}.$$

Hence the chance that the next ball drawn is red

$$= \tfrac{2}{9}\cdot\tfrac{1}{3} + \tfrac{1}{9}\cdot\tfrac{2}{3} + \tfrac{2}{9}\cdot\tfrac{1}{3} = \tfrac{6}{27} = \tfrac{2}{9}.$$

564. A bag contains $a$ black balls and $b$ white balls. A second bag contains $a+c$ black balls and $b-c$ white balls. $m$ balls are drawn from one and $n$ from the other, and are found to be all black. Shew that the odds are $C_{c,\,a-m} : C_{c,\,a-n}$ in favour of the $m$ balls having been drawn from the first bag.

565. A bag contains counters, one marked 1, two marked 4, three marked 9, &c. A person draws out a counter marked at random, and is to receive as many shillings as the number marked on it. Prove that the value of his expectation varies as the number of counters in the bag.

566. Counters ($n$) marked with consecutive numbers are placed in a bag, from which a number of counters ($m$) are to be drawn at random. Shew that the expectation of the sum of the numbers drawn is the arithmetic mean between the greatest and least sums which can be indicated by the number of counters ($m$) to be drawn.

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**564.** The *à priori* chances are equal of drawing

$$\left.\begin{array}{r} m \text{ from first} \\ n \text{ from second} \end{array}\right\} : \left\{\begin{array}{l} n \text{ from first} \\ m \text{ from second.} \end{array}\right.$$

The consequent chances of drawing only black balls are

$$\frac{C^a_m C^{a+c}_n}{C^{a+b}_m C^{a+b}_n} \text{ and } \frac{C^a_n C^{a+c}_m}{C^{a+b}_n C^{a+b}_m}.$$

Hence the *à posteriori* chances are as

$$C^a_m C^{a+c}_n : C^a_n C^{a+c}_m$$

$$= \underline{|a-n}\,\underline{|a+c-m} : \underline{|a-m}\,\underline{|a+c-n}$$

$$= C^{a+c-m}_c : C^{a+c-n}_c.$$

**565.** The expectation is

$$\frac{1+2\,.\,2^2+3\,.\,3^2+\dots}{1+2+3+\dots} = 1+2+3+\dots$$

= no. of counters in the bag.

**566.** Let the $n$ counters be marked

$$a,\ a+1,\ a+2,\ \dots\ a+n-1,$$

and let the sum of these numbers be $s$; and let $l = a+n-1$.

If all the possible combinations of $m$ numbers out of the $n$, be written down in a column each of the numbers will appear

567. There are $n$ tickets in a bag numbered 1, 2, 3, ... $n$. A man draws two tickets at once, and is to receive a number of sovereigns equal to the product of the numbers drawn. What is his expectation?

---

equally. The sum will therefore be the same as if each counter were marked $\frac{s}{n}$. The expectation is therefore $\frac{ms}{n}$.

But the greatest number that can be represented by $m$ counters is

$$l + (l - 1) + (l - 2) + \dots \text{ to } m \text{ terms.}$$

And the least number is

$$a + (a + 1) + (a + 2) + \dots \text{ to } m \text{ terms.}$$

The arithmetic mean between these is $\frac{m}{2}(a + l) = \frac{ms}{n}$, the expression which we have already found for the expectation.

**567.** The number of ways of drawing is $\frac{1}{2}n(n-1)$. The expectation has this denominator and has for numerator the sum of

$$\begin{aligned} &n\{1 + 2 + 3 + \dots + (n-1)\} \\ &+ (n-1)\{1 + 2 + 3 + \dots + (n-2)\} \\ &+ (n-2)\{1 + 2 + 3 + \dots + (n-3)\} + \&\text{c. to } n \text{ terms} \end{aligned}$$

$$= \Sigma_{x=1}^{x=n} \frac{x^3 - x^2}{2} = \tfrac{1}{2}(s^2 - S),$$

where $s$ denotes the sum of the first $n$ natural numbers, and $S$ the sum of their squares. The sum may be written

$$\frac{n^2(n+1)^2}{8} - \frac{n(n+1)(2n+1)}{12} = \frac{n(n+1)(n-1)(3n+2)}{24}.$$

Therefore the expectation is

$$(n + 1)(3n + 2) \div 12.$$

568. What would be the expectation in the last exercise if three tickets were drawn and their continued product taken?

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**568.** Let $s$ be the sum of the numbers and $S$ the sum of their squares. The denominator will now be

$$\tfrac{1}{6}n(n-1)(n-2),$$

and the numerator will be the sum of all such products as $\alpha\beta\gamma$ where $\alpha$, $\beta$, $\gamma$ are all different and take all values from 1 to $n$.

We may sum these products thus:

By Qn. 567 the sum of all such products as $\alpha\beta$ is $\frac{1}{2}(s^2 - S)$.

Exclude from these the products which contain the number $x$. Their sum is $x(s-x)$.

Hence the sum of all such products as $\alpha\beta$, excluding the number $x$, is

$$\tfrac{1}{2}(s^2 - S) - x(s-x).$$

Let all these products be multiplied by $x$, and we obtain the sum of all such products as $\alpha\beta\gamma$ which contain $x$, viz.

$$\frac{x}{2}(s^2 - S) - x^2(s-x).$$

Give $x$ all values from 1 to $n$ and we obtain the sum of *all* products such as $(\alpha\beta\gamma)$ taken 3 times over. Hence the sum of such products is

$$\frac{1}{3}\left[\frac{s}{2}(s^2 - S) - sS + s^2\right] = \frac{s}{6}(s^2 - 3S + 2s)$$

$$= \frac{n(n+1)}{12}\left\{\frac{n^3(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{2} + n(n+1)\right\}$$

$$= \frac{n^2(n+1)^2}{48}(n-1)(n-2).$$

and therefore the expectation is

$$n(n+1)^2 \div 8.$$

569. A bag contains $m$ counters marked with odd numbers, and $n$ counters marked with even numbers. If $r$ counters be drawn at random the chance that the sum of the numbers drawn be even is $\frac{1}{2}(1+\mu)$, and that it be odd $\frac{1}{2}(1-\mu)$, where $\mu$ is the coefficient of $x^r$ in the expansion of $(1-x)^m(1+x)^n \div C_r^{m+n}$.

570. A bag contains a number of counters known not to exceed $m$, numbered in order from 1 upwards. A person draws $r$ times, replacing the counters drawn, and the highest number drawn is found to be $n$. What is the chance that the number of counters does not exceed $p$?

571. A bag contains balls, some of which are white, and I am entitled to receive a shilling for every ball I draw as long as I continue to draw white balls only; the balls drawn not being replaced. But an additional ball, not

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**569.** If $F_e$ denote the chance of an even sum and $F_0$ the chance of an odd sum, we have

$$F_e : F_0 = C_0^m C_r^n + C_2^m C_{r-2}^n + C_4^m C_{r-4}^n + \text{\&c.} : C_1^m C_{r-1}^n + C_3^m C_{r-3}^n + \text{\&c.}$$

Therefore

$$\frac{F_e - F_0}{F_e + F_0} = \frac{C_0^m C_r^n - C_1^m C_{r-1}^n + C_2^m C_{r-2}^n - \text{\&c.}}{C_0^m C_r^n + C_1^m C_{r-1}^n + C_2^m C_{r-2}^n + \text{\&c.}}$$

$$= \text{coeff. of } x^r \text{ in } (1-x)^m(1+x)^n \div C_r^{m+n} = \mu.$$

But $$F_e + F_0 = 1, \quad \therefore \; F_e - F_0 = \mu,$$

$$F_e = \tfrac{1}{2}(1+\mu), \qquad F_0 = \tfrac{1}{2}(1-\mu).$$

**570.** The bag may contain $n, n+1, n+2, \ldots$ or $m$ counters. If it contain $n+x$ the chance that the highest number drawn should be $n$, (i.e. that none should exceed $n$, but that some should exceed $n-1$) is $\{n^r - (n-1)^r\} \div (n+x)^r$ which varies as $(n+x)^{-r}$.

The chance that the no. in the bag is within the range $n$ to $p$ is therefore

$$\Sigma_{x=0}^{x=p-n}\{(n+x)^{-r}\} \div \Sigma_{x=0}^{x=m-n}\{(n+x)^{-r}\}$$

$$= \frac{n^{-1} + (n+1)^{-1} + (n+2)^{-1} + \ldots + p^{-1}}{n^{-1} + (n+1)^{-1} + (n+2)^{-1} + \ldots + m^{-1}}.$$

**571.** Let there be originally $m$ white balls and $n$ not-white. The latter will divide the series of $m$ white balls into $n+1$ lots of white, having been introduced into the bag, I claim as a compensation to replace every white ball I draw. Shew that this is a fair equivalent.

572. There are $m$ white balls and $m$ black ones: $m$ balls are placed in one bag, and the remaining $m$ in a second bag, the number of white and black in each being unknown. If one ball be drawn from each bag, find the chance that they are of the same colour.

573. In the last exercise, if $m=4$, and a ball of the same colour has been drawn from each, find the chance that a second drawing will give balls of the same colour: (i) if the balls drawn at first have been replaced, and (ii) if they have not been replaced.

---

equal expectation. $\therefore$ Expectation of each $= m \div (n+1)$. But one of the lots consists of the balls drawn before the first not-white. Hence my expectation is $m \div (n+1)$, and the additional not-white ball would reduce it to $m \div (n+2)$. But when the additional ball is added and I replace every ball drawn, my expectation is

$$\frac{m}{m+n+1} + \left(\frac{m}{m+n+1}\right)^2 + \left(\frac{m}{m+n+1}\right)^3 + \dots \text{ to inf.}$$

$$= m \div (n+1) \text{ as at first.}$$

**572.** Think of the balls placed in a line in all possible orders, the two halves of the line representing the two bags. As the balls take all possible orders, the chances will not be affected if we consider the balls at the ends of the row as the two balls drawn. If they be both of the same colour the intermediate balls can be arranged in $C_{m,m-2}$ ways; but if of opposite colour in $C_{m-1,m-1}$ ways. The odds against their being of the same colour are therefore $C_{m-1,m-1} : C_{m,m-2} = m : m-1$ and the required chance is $(m-1) \div (2m-1)$.

The idea of this solution is due to Mr C. H. Brooks.

**573.** Suppose a white ball has been drawn from each, then the original division must have been

$$wb^3 \mid w^3b, \quad \text{or} \quad w^2b^2 \mid w^2b^2, \quad \text{or} \quad w^3b \mid wb^3,$$

and the *à priori* chances of these were as $16 : 36 : 16 = 4 : 9 : 4$.

574. A bag contains $m$ white and $n$ black balls, and from it balls are drawn one by one till a white ball is drawn. $A$ bets $B$ at each drawing, $x$ to $y$, that a black ball is drawn. Prove that the value of $A$'s expectation at the beginning of the drawing is $\frac{ny}{m+1} - x$.

575. From a bag containing $m$ gold and $n$ silver coins, a coin is drawn at random, and then replaced; and this operation is performed $p$ times. Find the chance that all the gold coins will be included in the coins thus drawn.

---

The consequent chances of the observed event ($w^2$) were then as 3 : 4 : 3.

Hence *à posteriori* the chances of the three possible ways of original division are as 12 : 36 : 12; and their actual values are $\frac{1}{5}, \frac{3}{5}, \frac{1}{5}$.

(i) If the balls first drawn have been replaced, the three possible ways of the original division give the chances of two balls alike $\frac{3}{8}, \frac{4}{8}, \frac{3}{8}$; and the total chance is

$$\frac{1}{5} \cdot \frac{3}{8} + \frac{3}{5} \cdot \frac{4}{8} + \frac{1}{5} \cdot \frac{3}{8} = \frac{9}{20}.$$

(ii) If the balls have not been replaced, we still have chances $\frac{1}{5}, \frac{3}{5}, \frac{1}{5}$ that the bags contain

$$b^3 \mid w^2 b, \quad \text{or} \quad w b^2 \mid w b^2, \quad \text{or} \quad w^2 b \mid b^3,$$

and the respective chances of two alike are $\frac{3}{9}, \frac{5}{9}, \frac{3}{9}$.

Therefore the total chance is

$$\frac{1}{5} \cdot \frac{3}{9} + \frac{3}{5} \cdot \frac{5}{9} + \frac{1}{5} \cdot \frac{3}{9} = \frac{7}{15}.$$

**574.** He receives $y$ for every game he wins and by Qn. 571, his expectation is $ny \div (m+1)$. When the first white ball is drawn he loses $x$. Therefore his net expectation is

$$\frac{ny}{m+1} - x.$$

**575.** The total number of orders in which $p$ coins can be drawn is $(m+n)^p$.

Of these, any one gold coin will be absent from $(m+n-1)^p$ ways.

576. A bag contains a £10 note ($X$), two £5 notes ($V^2$) and three pieces of blank paper ($O^3$). I find that I succeed twice out of three times, by the aid of the sense of touch, in drawing a bank note. What ought I to give for the privilege of drawing one piece of paper from the bag?

577. On the principle of Prop. LIX. what should I give for the expectation in the last question when my available funds are (1) twenty pounds or (2) five hundred pounds?

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Any two gold coins will be absent from $(m+n-2)^p$ ways and so on. Hence (Prop. XIV. Cor.) the no. of ways in which no gold coin will be wanting is

$$(m+n)^p - C_1^m\,(m+n-1)^p + C_2^m\,(m+n-2)^p - \&c.$$

and the chance is therefore

$$1 - C_1^m\,(1-a)^p + C_2^m\,(1-2a)^p - C_3^m\,(1-3a)^p + \dots$$

where $a = 1 \div (m+n)$.

**576.** As I cannot distinguish between $X$ and $V$, if I succeed in drawing a note my expectation is £$20 \div 3$. But my chance of drawing a note is $\frac{2}{3}$. Hence the value of the drawing is £$40 \div 9$.

**577.** On an average he may expect in 9 trials to get $X$ twice, $V$ four times and $O$ three times. Hence if he pay £$x$ for each trial, his property will be multiplied by

$$\left(1-\frac{x}{n}\right)^3\left(1+\frac{5-x}{n}\right)^4\left(1+\frac{10-x}{n}\right)^2.$$

This must be unity. Therefore when $n=20$, we have

$$\left(1-\frac{x}{20}\right)^3\left(\frac{5}{4}-\frac{x}{20}\right)^4\left(\frac{3}{2}-\frac{x}{20}\right)^2 = 1,$$

$$\text{or } \left(1-\frac{x}{20}\right)^3\left(1-\frac{x}{25}\right)^4\left(1-\frac{x}{30}\right)^2 = \left(\frac{4}{5}\right)^4\left(\frac{2}{3}\right)^2.$$

This gives approximately $x = 4{\cdot}11$.

When $n=500$, we have

$$\left(1-\frac{x}{500}\right)^3\left(1-\frac{x}{505}\right)^4\left(1-\frac{x}{510}\right)^2 = \left(\frac{100}{101}\right)^4\left(\frac{50}{51}\right)^2,$$

which gives approximately $x = 4{\cdot}3$.

578. Nine black balls and 5 white balls are drawn, one by one, from a bag. Find the chance that throughout the process there shall never have been more white than black drawn.

579. If $mn$ balls have been distributed into $m$ bags, $n$ into each, what is the chance that two specified balls will be found in the same bag? And what does the chance become when $r$ bags have been examined and found not to contain either ball?

580. A bag contains $m$ white balls and $n$ black balls, and balls are to be drawn from it so long as they are all drawn of the same colour. If this be white, $A$ pays $B$ $x$ shillings for the first, $rx$ for the second, $\frac{r(r+1)}{2}x$ for the third, $\frac{r(r+1)(r+2)}{\lfloor 3}x$ for the fourth, and so on; but if black, $B$ pays $A$ $y$ shillings for the first, $ry$ for the second, and so on. Find the value of $A$'s expectation at the beginning of the drawing.

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**578.** By Prop. XXXVIII, the no. of unfavourable ways is $5 \div (9+1) = \frac{1}{2}$ of the whole. $\therefore$ Chance $= \frac{1}{2}$.

**579.** Let the specified balls be $A$ and $B$. In whatever bag $A$ may be, there are $n-1$ balls in the same bag, and the chance that $B$ should be one of them is $(n-1) \div (mn-1)$. When $r$ bags have been examined there remain only $mn - rn - 1$ possible places for $B$ and the chance becomes $(n-1) \div (mn - rn - 1)$.

**580.** $A$'s expectation of paying is

$$\frac{mx}{m+n} + \frac{m(m-1)rx}{(m+n)(m+n-1)} + \&c.,$$

the general ($z$th) term being

$$xC_{z-1,\,r-1} \cdot C_{m-z,\,n} \div C_{m,\,n},$$

and the sum $= xC_{m-1,\,n+r} \div C_{m,\,n}$.

Similarly his expectation of receiving is

$$yC_{n-1,\,m+r} \div C_{m,\,n}.$$

So his net expectation is

$$\frac{yC_{n-1,\,m+r} - xC_{m-1,\,n+r}}{C_{m,\,n}}.$$

581. If two milestones be selected at random on a road $n$ miles long, shew that their average distance apart will be $\frac{1}{3}(n+2)$.

582. A town is surrounded by a road $n$ miles long. If two milestones be selected at random, find their average distance apart measured the shortest way along the road.

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**581.** Consider the $n+1$ milestones. Two of them selected at random will divide the other $n-1$ into three sets of equal expectation. Therefore on an average there will be $\frac{1}{3}(n-1)$ milestones between the two selected ones, and therefore

$$\tfrac{1}{3}(n-1)+1=\tfrac{1}{3}(n+2) \text{ miles.}$$

SECOND SOLUTION. Consider the miles with one mile added at each end. The $n+1$ milestones may then be regarded merely as points of division in the series of $n+2$ miles. Two of these points selected at random will divide the series of $n+2$ miles into three parts of equal expectation. Therefore the average distance between the two points is $\frac{1}{3}(n+2)$.

COROLLARY. If it be permissible to select the same milestone twice, the number of ways is increased in the ratio $n:(n+1)$, but the aggregate of the distances is not increased. Therefore the average distance is diminished in the ratio $(n+1):n$. That is,

$$\text{average}=\tfrac{1}{3}n(n+2)\div(n+1).$$

**582.** If $n$ be odd, wherever the first stone be selected, the average distance of the other will be

$$\left\{1+2+3+\ldots+\frac{n-1}{2}\right\}\div\frac{n-1}{2}=\frac{n+1}{4}.$$

If $n$ be even, excluding the case when two opposite stones are chosen the average will be

$$\left\{1+2+3+\ldots+\left(\frac{n}{2}-1\right)\right\}\div\left(\frac{n}{2}-1\right)=\frac{n}{4}.$$

The excluded case occurs once in $n-1$ times and gives a distance $\frac{n}{2}$.

$$\therefore\ \text{Average}=\left\{(n-2)\frac{n}{4}+\frac{n}{2}\right\}\div(n-1)=n^2\div 4(n-1).$$

583. If the road form a regular hexagon six miles long with a milestone at every angle, find the average distance measured in a straight line from one milestone to the other.

584. If two squares be taken at random on a chess-board as the opposite corners of a rectangle formed of complete squares, find the average number of squares in the rectangle.

585. Two squares are marked at random on a chess-board. If we are to move from one to the other by stepping from every square to an adjacent one (not diagonally), what is the average number of steps required?

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**583.** We have simply to take the average of 5 distances, of which two are 1 mile, two are $\sqrt{3}$ miles, and one is 2 miles.

$$\text{Average} = \tfrac{2}{5}(2 + \sqrt{3}).$$

**584.** The question excludes rectangles consisting of a single line of squares, as the extreme squares of such a line could not be termed "opposite corners."

Consider the general case of a board of $n \times n$ squares.

The two extreme squares in the base of the rectangle must divide the remaining $n-2$ squares of their row into three parts of equal expectation. Therefore the average of intervening squares is $\frac{1}{3}(n-2)$, and including the extreme squares the average base of the rectangle is $\frac{1}{3}(n-2) + 2 = \frac{1}{3}(n+4)$. So the average height must be $\frac{1}{3}(n+4)$, and as the height and base vary independently the average value of the product or area is

$$(n+4)^2 \div 9.$$

In the ordinary board $n = 8$, and the result becomes 16.

**585.** Consider the general case of a board of $n \times n$ squares.

If the two marked squares are in one row or column of squares there must be on an average $\frac{1}{3}(n-2)$ squares between them, and the average number of steps will be

$$\tfrac{1}{3}(n-2) + 1 = \tfrac{1}{3}(n+1).$$

If the two marked squares be not in the same row or column they will define a rectangle as in the last question, whose average

586. What will the average be reduced to if diagonal steps are admissible?

587. Two squares are marked at random on a chess-board and a knight moves (by as short a route as possible) from one to the other. Shew that if one of the squares be at a corner of the board the average number of moves is $3\frac{31}{63}$; but if one of the squares be one of the four central ones the average will be only $2\frac{26}{63}$.

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length and breadth is $\frac{1}{3}(n+4)$, and the average number of steps will be $\frac{2}{3}(n+4)-2=\frac{2}{3}(n+1)$.

But wherever the first square may be the chances that the second should be, or not be, in the same row or column are as

$$(2n-2):(n-1)^2=2:(n-1).$$

Hence the total average no. of steps

$$=\frac{2}{3}+\frac{2}{3}(n-1)=2n\div 3.$$

In the ordinary board $n=8$, and the average number of steps is $5\frac{1}{3}$.

**586.** It is easily shewn that on a board of $n\times n$ squares there are $2x(n-x)(2n-x)$ selections requiring $x$ steps each.

Hence the average no. of steps required is (from $x=1$ to $x=n-1$)

$$\frac{\Sigma\{2n^2x^2-3nx^3+x^4\}}{\Sigma\{2n^2x-3nx^2+x^3\}}=\frac{7n^2+2}{15n}.$$

Hence on an ordinary board, putting $n=8$, the average is $3\frac{3}{4}$.

**587.** This can only be solved by trial. Thus from the corner square

| | | | |
|---|---|---|---|
| 2 | squares can be reached in | | 1 move |
| 9 | ,, | ,, | 2 |
| 20 | ,, | ,, | 3 |
| 21 | ,, | ,, | 4 |
| 10 | ,, | ,, | 5 |
| 1 | ,, | ,, | 6 |
| 63 | squares in an aggregate of | | 220 moves |

$$\therefore \text{ Average}=3\frac{31}{63}.$$

588. If a knight be placed at random on a chess-board the odds are 3 to 1 against his having 8 possible moves; and his expectation is $5\frac{1}{4}$ moves.

589. A rectangular parallelepiped $a$ feet long, $b$ wide, and $c$ high, is formed of wire, and is divided by wires into $abc$ cubes, of one foot each; by how many routes can a fly travel along the wires from one corner to the opposite corner without travelling more than $a+b+c$ feet?

590. How many joints will there be in the parallelepiped? And if two of the joints be marked at random what will be their average distance apart, measured as shortly as possible along the wires?

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So from one of the centre squares 8 squares can be reached in 1 move; 26 in 2; 24 in 3; 5 in 4. Total 63 squares with an aggregate of 152 moves. Average $= 2\frac{26}{63}$.

**588.** We need only consider a quarter of the board $= 16$ squares. From each of the 4 inmost squares he has 8 moves. From the 4 adjacent to these he has 6. From the corner square he has 2; from the 2 adjacent to this he has 3; from the remaining 5 he has 4. Hence

$$\text{Average} = 5\tfrac{1}{4}.$$

And the odds are 3 to 1 against his having 8 moves.

**589.** He has to take $a$ steps in one direction, $b$ in another, and $c$ in another in any order.

$$\text{No. of orders} = \underline{|a+b+c} \div \underline{|a}\,\underline{|b}\,\underline{|c}.$$

**590.** There are $(a+1)(b+1)$ joints on the ground and the same number at each of the heights 1, 2, 3 ... $c$ feet. Hence the total no. is

$$N = (a+1)(b+1)(c+1).$$

If in marking the two joints it were admissible to take the same joint twice, their average distance apart would be by Qn. 581 Cor.

$$\begin{array}{lll} \frac{1}{3}a\,(a+2) \div (a+1) & \text{in direction} & a, \\ \frac{1}{3}b\,(b+2) \div (b+1) & \quad ,, & b, \\ \frac{1}{3}c\,(c+2) \div (c+1) & \quad ,, & c. \end{array}$$

591. What is the chance that two joints marked at random are connected by one straight line of wire?

592. A pyramidal pile of cannon balls is formed as follows:—the base is an equilateral triangle with $n$ balls along each side: on this balls are placed forming another triangle with $n-1$ along each side: and so on until the pile is crowned by a single ball. Find the chance that an assigned ball will be on one of the sloping faces of the pyramid.

593. Find the chance that two assigned balls will touch one another.

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Excluding the case when the same joint is taken twice the no. of ways is decreased in the ratio $N : N-1$. But the aggregate of distances remains unchanged, since the excluded distances are zero; therefore the average is increased in the ratio $N-1 : N$. Hence the total average is

$$\frac{1}{3}\frac{N}{N-1}\left\{\frac{a(a+2)}{a+1}+\frac{b(b+2)}{b+1}+\frac{c(c+2)}{c+1}\right\}.$$

**591.** Wherever the first joint be taken there are $a$ joints in a line with it in direction $a$, $b$ joints in direction $b$, and $c$ joints in direction $c$.

$$\text{Chance} = (a+b+c) \div (N-1).$$

**592.** The base contains $\frac{1}{2}n(n+1)$ balls. The whole pile contains $\frac{1}{2}\Sigma\{x(x+1)\}$, from $x=1$ to $x=n$,

$$= \tfrac{1}{6}n(n+1)(n+2) = s, \text{ suppose.}$$

The balls which are not on one of the sloping sides form an interior pyramid whose edge contains $n-3$ balls. The no. of balls in it is therefore

$$\tfrac{1}{6}(n-3)(n-2)(n-1).$$

Hence the required chance is

$$1-\frac{(n-1)(n-2)(n-3)}{n(n+1)(n+2)} = \frac{9n^2-9n+6}{n^3+3n^2+2n}.$$

**593.** A ball in the interior of the pile has contact with 6 balls in its own plane, 3 in the plane above and 3 in the plane below, $=12$ contacts.

594. Find the chance that two assigned balls will be at the same height from the ground.

595. Find the chance that two assigned balls shall be in a line parallel to one of the edges.

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A ball in one of the faces, but not at an edge, has contact with only 9 balls.

A ball on an edge, but not at a vertex, has 6 contacts, and a ball at a vertex has 3 contacts.

There are therefore

| | | | |
|---|---|---|---|
| $\alpha = \frac{1}{6}(n-2)(n-3)(n-4)$ | balls with | 12 | contacts |
| $\beta = \frac{1}{2}(n-2)(n-3) \times 4$ | ,, | 9 | ,, |
| $\gamma = (n-2) \times 6$ | ,, | 6 | ,, |
| $\delta = 4$ | ,, | 3 | ,, |

The chance that the first ball should be in the group $\alpha$ is $\alpha \div s$, and then the chance that the second should touch it is $12 \div (s-1)$. And so for $\beta$, $\gamma$, $\delta$. Hence the whole chance of contact is

$$(12\alpha + 9\beta + 6\gamma + 3\delta) \div s(s-1) = 72 \div (n+2)(n^2+4n+6).$$

**594.** The balls are in $n$ horizontal planes. Counting from the vertex the $x$th plane contains $\frac{1}{2}x(x+1)$ balls. The chance that both balls should be in the $x$th plane is therefore

$$\frac{(x^2+x)(x^2+x-2)}{4s(s-1)} = \frac{(x-1)x(x+1)(x+2)}{4s(s-1)}.$$

Giving $x$ all values from 1 to $n$ and adding, we find the total chance,

$$(n-1)n(n+1)(n+2)(n+3) \div 20s(s-1) = 9(n+3) \div 5(n^2+4n+6).$$

**595.** Let $P$ and $Q$ be two opposite edges of the pyramid, i.e. two edges which do not meet. If the pyramid be placed so that these two edges are horizontal the whole pile of balls will form a

596. Find the average distance apart of two balls which are known to be in a line parallel to one of the edges.

597. If the centres of the balls in any layer be at a height $b$ above the centres in the next lower layer, and if the lowest layer be sunk into the ground up to their centres, find the average height of all the balls above the ground.

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series of $n$ rectangular laminæ containing $1 \, . \, n$, $2(n-1)$, $3(n-2)$, ... $n \, . \, 1$ balls respectively.

Suppose the first of the assigned balls be in the rectangle of $p \, . \, q$ balls. Then there are $s-1$ possible positions for the second ball, of which $p+q-2$ are in lines parallel to $P$ or $Q$. But $p+q=n+1$. Hence (wherever the first ball be) the chance that the two balls are in a line parallel to $P$ or $Q$ is $(n-1)\div(s-1)$. And there is the same chance that they are in a line parallel to one of the constituents of either of the other two pairs of edges. Hence the whole chance is $3(n-1)\div(s-1)$, $s$ having the value given in Qn. 592.

**596.** Let them be in a line parallel to the edge $P$. Then if they be in the rectangle $p \, . \, q$ the no. of possible positions is $\frac{1}{2}q\,p\,(p-1)$ and the average distance between them is $\frac{1}{3}(p+1)$. Hence the average required is

$$\tfrac{1}{3}\Sigma\{q\,p\,(p-1)(p+1)\}\div\Sigma\{q\,p\,(p-1)\}.$$

But $$q=n+1-p=(n-1)-(p-2).$$

We can write down the summation at sight, from $p=1$ to $p=n$, and in the result we find

$$\text{Average}=\tfrac{1}{5}(n+3).$$

**597.** The $x$th layer from the apex of the pyramid contains $\frac{1}{2}x(x+1)$ balls: their height is $(n-x)\,b$. Therefore average height

$$=\Sigma\{x(x+1)(n-x)\}\,b\div\Sigma\{x(x+1)\}=\tfrac{1}{4}(n-1)\,b.$$

598. If two lengths be taken at random each greater than $b$ and less than $a$, the chance that their sum shall be less than $2b+c$ is $\frac{1}{2}c^2 \div (a-b)^2$: and the chance that their difference shall exceed $c$ is $(a-b-c)^2 \div (a-b)^2$. Express also the chance that the sum of their squares shall exceed $a^2$.

599. If a point taken at random within a triangle be joined to the two extremities of the base, the chance that the triangle thus formed will bear to the whole triangle a ratio greater than $n-1$ to $n$ is $1 \div n^2$.

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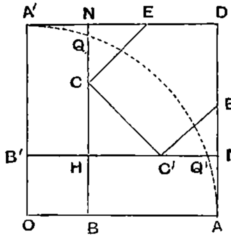
**598.** Construct the figure, making

$$OA = OA' = a, \quad OB = OB' = b, \quad HC = HC' = c;$$

$CE$, $C'E'$ being drawn parallel to the diagonal of the square, and the circular quadrant having its centre at $O$.

The two random lengths may be represented by the perpendiculars upon $OA$, $OA'$ from a point taken at random on the area $HNDN'$.

If the random point fall in the area $HCC'$ the sum of the random lines will be less than $2b+c$.




Chance = Area $HCC' \div$ Area $HNDN'$

$$= \tfrac{1}{2}c^2 \div (a-b)^2.$$

Again if the random point fall on either of the areas $NCE$, $N'C'E'$ the difference of the random lines will exceed $c$.

Chance = 2 Area $NCE \div$ Area $HNDN'$

$$= (a-b-c)^2 \div (a-b)^2.$$

Again if the random point fall on the area $DNQQ'N'$ the sum of the squares on the random lines will exceed $a^2$.

Chance = Area $DNQQ'N' \div$ Area $HNDN'$

$$= \frac{a^2}{(a-b)^2}\left\{1 - \frac{\pi}{4} + \sin^{-1}\frac{b}{a} - \frac{b}{a}\left(2 - \sqrt{1 - \frac{b^2}{a^2}}\right)\right\}.$$

**599.** Parallel to the base of the triangle draw a straight line at a vertical distance below the vertex equal to one-$n$th of

600. Within the area of a triangle another triangle is formed with its sides parallel to the sides of the original triangle. Shew that if all possible triangles are equally likely the average area is one-tenth of the area of the original triangle.

601. An omnibus makes $m$ journeys and carries a total of $n$ passengers. If each passenger is equally likely to take any journey, find the chance that on a given journey there will be no passengers.

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the altitude of the triangle. If the random point fall above this line the required condition is fulfilled. That is, the point must fall upon the area of the smaller triangle.

$$\text{Chance} = \frac{\text{area of smaller triangle}}{\text{area of whole triangle}} = \frac{1}{n^2}.$$

**600.** Let $\Delta$ be the area of the original triangle and $\mu^2\Delta$ the area of one of the random triangles. This triangle is capable of displacement while its vertex travels over an area $= (1-\mu)^2\,\Delta$. Hence the no. of random triangles of this area varies as $(1-\mu)^2$.
$\therefore$ the average area of all the triangles is

$$\Sigma\,\{(\mu - \mu^2)^2\,\Delta\} \div \Sigma\,\{(1-\mu)^2\},$$

the summation extending from $\mu = 0$ to $\mu = 1$.

If we assume the variation of $\mu$ to take place by an indefinitely large no. ($n$) of indefinitely small increments, and ultimately make $n$ infinite, we get the result. (Or with a knowledge of the integral calculus we may write it down at sight.)

$$\text{Average} = \tfrac{1}{10}\,\Delta.$$

[See note to Qn. 659.]

**601.** The chance that the first passenger should not take the given journey is $1 - \dfrac{1}{m}$, and so for each of the others. Hence the chance that none take it is $\left(1 - \dfrac{1}{m}\right)^n$.

602. Find the chance that on $r$ given journeys there will be no passengers.

603. Shew that the expectation of a person who is to receive £1 for every journey until the first passenger is carried, will be (in pounds)

$$(1^n + 2^n + 3^n + \dots \text{to } m-1 \text{ terms}) \div m^n.$$

604. If the omnibus make 10 journeys and carry 15 passengers, what is the chance that there will be at least one passenger on every journey?

605. If 20 omnibuses an hour pass my house, what is the chance that in the next five minutes none will pass?

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**602.** The chance for each passenger is now $\frac{m-r}{m}$ and the chance required is $\left(1 - \frac{r}{m}\right)^n$.

Note that it is implied that the omnibus can carry all the passengers at once if required.

**603.** Chance that he wins on 1st journey $= \left(\frac{m-1}{m}\right)^n$, by Qn. 601; chance that he *also* wins on second journey $= \left(\frac{m-2}{m}\right)^n$ and so on. Hence his expectation

$$= \{1^n + 2^n + 3^n + \dots + (m-1)^n\} \div m^n.$$

**604.** Chance that the omnibus is empty on any given journey $= (\frac{9}{10})^{15}$; on any two given journeys $= (\frac{8}{10})^{15}$; and so on.

Hence by Prop. XIV, Cor. 3, the chance that it shall never be empty is

$$1 - \frac{10}{1} \cdot \left(\frac{9}{10}\right)^{15} + \frac{10 \cdot 9}{1 \cdot 2}\left(\frac{8}{10}\right)^{15} - \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} \cdot \left(\frac{7}{10}\right)^{15} + \&\text{c}.$$

$$= 1 - 10\,\frac{9^{15} + 1^{15}}{10^{15}} + 45 \cdot \frac{8^{15} + 2^{15}}{10^{15}} - 120\,\frac{7^{15} + 3^{15}}{10^{15}} + 210\,\frac{6^{15} + 4^{15}}{10^{15}} - \frac{252}{2^{15}}.$$

**605.** By Prop. LI, chance $= e^{-\frac{5}{3}} = \left(\frac{1}{e}\right)^{\frac{5}{3}}$.

606. If 100 persons an hour cross a certain bridge, each taking a minute to traverse it, what is the chance that at a given instant no one will be on the bridge?

607. If there pass over the bridge in an hour $a$ persons who traverse it in $\alpha$ minutes each, $b$ persons who traverse it in $\beta$ minutes each, $c$ persons who traverse it in $\gamma$ minutes each, and so on, shew that the chance that at a given instant there is no one on the bridge is

$$1 \div \sqrt[60]{e^{a\alpha+b\beta+c\gamma+\dots}}.$$

608. A railway company carries $n$ passengers an average of 10 miles each for every one passenger who is fatally injured. What is the chance that a passenger makes a given journey of 100 miles in safety?

609. In reading a French book I have to refer to a dictionary for two words a page on an average; what is the chance that I shall read the next page I come to without referring to a dictionary?

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**606.** That no one be on the bridge at a given instant requires that no one shall have entered the bridge in the previous 60 seconds. But on an average a person enters the bridge every 36 seconds.

Hence by Prop. LI, chance $= \left(\frac{1}{e}\right)^{\frac{60}{36}} = \left(\frac{1}{e}\right)^{\frac{5}{3}}$.

**607.** As in the previous question, chance that none of the $a$ persons be on the bridge is $\left(\frac{1}{e}\right)^{\frac{a\alpha}{60}}$, for one comes on an average in $\frac{60}{a}$ minutes and he takes $\alpha$ minutes to traverse the bridge.

Hence the required chance

$$= \left(\frac{1}{e}\right)^{\frac{a\alpha+b\beta+c\gamma+\dots}{60}} = 1 \div \sqrt[60]{e^{a\alpha+b\beta+c\gamma+\dots}}.$$

**608.** The fatal accident occurs once in $10n$ passenger-miles. The chance that it should occur in a given 100 passenger-miles is, by Prop. LI, $\left(\frac{1}{e}\right)^{100 \div 10n} = \left(\frac{1}{e}\right)^{\frac{10}{n}}$.

**609.** By Prop. LI, chance $= \left(\frac{1}{e}\right)^{1 \div \frac{1}{2}} = \left(\frac{1}{e}\right)^{2}$.

610. A rider on a bicycle has a fall on an average once in 10 miles. What is the chance that he will perform a given journey of 15 miles without a fall?

611. If all the permutations (1, 2, 3 or more together) of the 26 letters of the alphabet be written down, and one of them be selected at random, the chance that this one contains $a$ is very nearly $\frac{25}{26}$.

612. The chance that the selected permutation contains all the letters of the alphabet is very nearly $\frac{227}{617}$.

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**610.** By Prop. LI, chance $=\left(\frac{1}{e}\right)^{15\div 10}=\left(\frac{1}{e}\right)^{\frac{3}{2}}$.

**611.** The total no. of permutations of $n$ letters is

$$\underline{|n}\left\{1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\text{\&c. to } n+1 \text{ terms}\right\}.$$

The chance that one of the permutations of 26 letters should not contain $a$ is therefore

$$\frac{\underline{|25}}{\underline{|26}}\left\{1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\ldots+\frac{1}{\underline{|25}}\right\}\div\left\{1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\ldots+\frac{1}{\underline{|26}}\right\}.$$

The difference between the two series in brackets is $\frac{1}{\underline{|26}}$, or less than $\frac{1}{10\,\underline{|25}}$ of either of them. Hence their ratio is very nearly of equality. Therefore the chance of $a$ not appearing is $\frac{1}{26}$ and the chance that the permutation contains $a$ is $\frac{25}{26}$ very nearly.

**612.** All the letters appear in $\underline{|n}$ permutations. Hence the chance that all appear is

$$1\div\left\{1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\ldots+\frac{1}{\underline{|26}}\right\}=\frac{1}{e}\text{ nearly, }=\frac{227}{617}\text{ nearly.}$$

613. If a man can throw a pair of dice 10 times in a minute, how many hours must he expect to play before he throws double sixes five times in succession?

614. Shew that when a whist player has dealt more than $C_{12}^{51} \log_e 2$ times it is more likely than not that he has held at some time all the trumps.

615. Two players of equal skill play for 36 nights, stopping each night as soon as one of them has won three games in succession. The winner of each game begins the next game and this gives him an advantage over his opponent in the ratio 3 : 2. Shew that their expectation is 196 games. But if the loser of each game were to begin the next game their expectation would be 351 games.

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**613.** By Prop. LIV he must expect to make an average of $\frac{36}{35}(36^5-1)$ trials and the no. of hours for which he must play is

$$36(36^5-1) \div 21000 = 103656{\cdot}3.$$

**614.** Chance at any deal $= 1 \div C_{12}^{51} = 1 \div C$ suppose. Then when he has dealt $x$ times, the chance that he has never held all the trumps $= \left(1-\frac{1}{C}\right)^x = e^{-\frac{x}{C}}$; because $C$ being an extremely high number $\left(1-\frac{1}{C}\right)^C$ is not distinguishable from $e^{-1}$. For an even chance we must have

$$e^{-\frac{x}{C}} = \tfrac{1}{2} \quad \text{or} \quad e^{\frac{x}{C}} = 2$$

and so $$x = C\log_e 2 > 110039{,}000000.$$

**615.** Consider the play of a single night. After the first game, every game may be recorded either as a sequence ($s$) or as a change ($c$) and the chance of $s$ is always $\frac{3}{5}$. Two consecutive $s$'s will give the three consecutive games, which terminate the play. Therefore by Prop. LIV we must expect after the first game $(1+\frac{3}{5}) \div \frac{9}{25}$ games $= 4\frac{4}{9}$. Hence the total average to be expected each night is $5\frac{4}{9}$ games, or 196 games in 36 nights.

In the second case the chance of $s$ is $\frac{2}{5}$. Hence we must expect after the first game $(1+\frac{2}{5}) \div \frac{4}{25} = 8\frac{3}{4}$ games per night. $\therefore$ Expectation for 36 nights $= 36 \times 9\frac{3}{4} = 351$ games.

616. A die is thrown $m$ times: shew that the chance that every face has turned up at least once is $(\frac{1}{6})^m \underline{|m}$ times the coefficient of $x^m$ in $(e^x - 1)^6$. For instance if it be thrown 10 times the chance is $38045 \div 139968$.

617. If $x > y$, both being even or both odd, so that $x + y = 2s$; and if $x$ be expressed as the sum of $y$ integers, all the ways of so expressing it being equally likely, the chance of all the $y$ integers being odd is $\Pi^s_y \div \Pi^x_y$.

618. A pack of cards consists of $p$ suits of $q$ cards each, numbered from 1 up to $q$. A card is drawn and turned up: and $r$ other cards are drawn at random. Find the chance that the card first drawn is the highest of its suit among all the cards drawn.

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**616.** At every throw a score of 1 is added to one of the categories 1, 2, 3, 4, 5, 6. The $m$ throws have to be distributed amongst these six. Without limitation the no. of ways is $6^m$. With the limitation that no category be left blank the no. of ways is, by Prop. XXII, $\underline{|m}$ times coefficient of $x^m$ in $(e^x - 1)^6$. Hence the chance required is $\underline{|m} \times$ coefficient of $x^m$ in $(e^x - 1)^6 \div 6^m$.

**617.** The total no. of ways in which the distribution can be made is $\Pi^x_y$.

The number of ways in which each term shall be an odd number may be arrived at as follows.

First distribute into the $y$ parcels $x + y = 2s$ things in pairs; i.e. two things counting as one. The number of distributions is $\Pi^s_y$. Then take one thing out of each parcel. This will leave $x$ things altogether with every term odd. Hence the chance is

$$\Pi^s_y \div \Pi^x_y.$$

**618.** Call the suit of the first card "hearts," and suppose the $r$ cards consist of $x$ hearts and $r - x$ not-hearts. The chance that the first card is the highest heart drawn is $1 \div (x + 1)$. And $x$ may have any value from 0 to $r$.

$$\therefore \text{ Chance} = \Sigma \left\{C^{q-1}_x C^{pq-q}_{r-x} \div (x+1)\, C^{pq-1}_r\right\}$$

$$= \Sigma \left\{C^q_{x+1} C^{pq-q}_{r-x}\right\} \div qC^{pq-1}_r = \left\{C^{pq}_{r+1} - C^{pq-q}_{r+1}\right\} \div qC^{pq-1}_r$$

$$= \frac{p}{r+1}\left\{1 - \frac{C^{pq-q}_{r+1}}{C^{pq}_{r+1}}\right\}.$$

619. A pack of $n$ different cards is laid face downwards on a table. A person names a card. That and all the cards above it are shewn to him, and removed. He names another; and the process is repeated until there are no cards left. Find the chance that, in the course of the operation, a card was named which was (at the time) at the top of the pack.

620. If the birth rate is double the death rate, shew that the annual multiplier is given by the equation

$$\frac{R_1}{\mu}+\frac{R_3}{\mu^3}+\frac{R_5}{\mu^5}+\ldots=\frac{1-R_2}{\mu^2}+\frac{1-R_4}{\mu^4}+\frac{1-R_6}{\mu^6}+\ldots.$$

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**619.** Let $\beta_r$ denote the chance when $r$ cards remain. Obviously $\beta_0=0$, $\beta_1=1$, $\beta_2=\frac{1}{2}$, $\beta_3=\frac{2}{3}$.

When $r$ cards remain there is an equal chance of any one of them being named.

$$\therefore\ \beta_r=\{1+\beta_{r-2}+\beta_{r-3}+\ldots+\beta_2+\beta_1+\beta_0\}\div r,$$

or
$$r\beta_r=1+\beta_1+\beta_2+\ldots+\beta_{r-2}.$$

$$\therefore\ (r+1)\,\beta_{r+1}-r\beta_r=\beta_{r-1},$$

or
$$(r+1)\,(\beta_{r+1}-\beta_r)=-(\beta_r-\beta_{r-1}).$$

$\therefore\ \underline{|r+1}\,(\beta_{r+1}-\beta_r)\,(-1)^r$ is constant for all values of $r$. But when $r=0$, it becomes unity. $\therefore$ It is always unity. Hence

$$\beta_{r+1}-\beta_r=\frac{(-1)^r}{\underline{|r+1}}.$$

Give $r$ all values from 0 to $n-1$ and add. Then

$$\beta_n=\frac{1}{\underline{|1}}-\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}-\ldots-\frac{(-1)^n}{\underline{|n}}=1-\frac{\underline{\|n}}{\underline{|n}}.$$

**620.** We have $B=2D$, and

$$\frac{D}{N}=\frac{B}{N}-(\mu-1).$$

Hence
$$\frac{B}{N}=2\,(\mu-1).$$

621. A man writes a number of letters greater than eight, and directs the same number of envelopes. If he puts one letter into each envelope at random, the chance that all go wrong is (to six decimal places) ·367879.

622. A class of twelve men is arranged in order of merit: find the chance that no name is in the place it would have occupied if the class had been in alphabetical order.

623. A list is to be published in three classes. The odds are $m$ to 1 that the examiners will decide to arrange each class in order of merit, but if they are not so arranged, the names in each will be arranged in alphabetical order. The list appears, and the names in each class are observed to be in alphabetical order, the numbers in the several classes being $a$, $b$, and $c$. What is the chance that the order in each class is also the order of merit?

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Substituting in the known formula between $B$ and $N$, we get

$$\frac{R_1}{\mu}+\frac{R_2}{\mu^2}+\frac{R_3}{\mu^3}+\ldots=\frac{1}{\mu^2-1}=\frac{1}{\mu^2}+\frac{1}{\mu^4}+\frac{1}{\mu^6}+\ldots.$$

$$\frac{R_1}{\mu}+\frac{R_3}{\mu^3}+\frac{R_5}{\mu^5}+\ldots=\frac{1-R_2}{\mu^2}+\frac{1-R_4}{\mu^4}+\frac{1-R_6}{\mu^6}+\ldots \quad \text{Q.E.D.}$$

**621.** Let $n$ be the number of letters. Then by Prop. XXXI the no. of ways in which all can go wrong is $||\underline{n}$. Therefore the chance is $||\underline{n} \div |\underline{n}$. If $n>8$ this quotient is ·367879.

**622.** The result must be $||\underline{12} \div |\underline{12} = {\cdot}367879$.

**623.** The chance that the order of merit is the same as the alphabetical order in all three classes is $1 \div |\underline{a}\,|\underline{b}\,|\underline{c}$.

*A priori* the chances of alphabetical order and order of merit were as $1 : m$.

The respective chances that these hypotheses should produce the observed event are as

$$1 : 1 \div |\underline{a}\,|\underline{b}\,|\underline{c}.$$

Therefore the *à posteriori* chances are as

$$1 : m \div |\underline{a}\,|\underline{b}\,|\underline{c},$$

or as $|\underline{a}\,|\underline{b}\,|\underline{c} : m$.

624. $A$ goes to hall $p$ times in $q$ consecutive days and sees $B$ there $r$ times. What is the most probable number of times that $B$ was in hall in the $q$ days?

Ex. Suppose $p=4$, $q=7$, $r=3$.

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Therefore the chance that the examiners decided to issue the lists in order of merit is now

$$\frac{m}{m+\underline{|a}\,\underline{|b}\,\underline{|c}},$$

and the chance that though they put it in alphabetical order, it is yet the order of merit, is

$$\frac{\underline{|a}\,\underline{|b}\,\underline{|c}}{m+\underline{|a}\,\underline{|b}\,\underline{|c}} \times \frac{1}{\underline{|a}\,\underline{|b}\,\underline{|c}} = \frac{1}{m+\underline{|a}\,\underline{|b}\,\underline{|c}}.$$

Hence the total chance that the lists are in order of merit is

$$\frac{m+1}{m+\underline{|a}\,\underline{|b}\,\underline{|c}}.$$

**624.** It must be assumed that *à priori* $B$ was equally likely to go to hall

$$1,\ 2,\ 3,\ \ldots\ r\ \ldots \text{ or } q \text{ times.}$$

But the only hypotheses that would produce the observed event are that he goes

$$r,\ r+1,\ r+2,\ \ldots \text{ or } q \text{ times.}$$

If he goes $r+x$ times, the chance that he meets $A$ exactly $r$ times is $C^p_r\, C^{q-p}_x \div C^q_{r+x}$.

Therefore the *à posteriori* chances that $B$ was there $x$ times is proportional to $C^{q-p}_x \div C^q_{r+x}$. We have to determine $x$ so as to make this a maximum. If $x$ have two successive values $v$ and $v+1$, the corresponding values of this quotient are in the ratio

$$1 : \frac{(r+v+1)(q-p-v)}{(v+1)(q-r-v)},$$

625. If $n$ witnesses concur in reporting an event of which they received information from another person, the chance that the report is true will be $(p^{n+1}+q^{n+1})\div(p^n+q^n)$ where $p=1-q$ is the chance of the correctness of a report made by any single person.

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which is a ratio of equality if

$$v=(q+1)\frac{r}{p}-(r+1),$$

i.e. if

$$r+v=(q+1)\frac{r}{p}-1.$$

If this quantity is an integer then there is an equal likelihood of $B$ having been at hall $(q+1)\frac{r}{p}-1$ or $(q+1)\frac{r}{p}$ times. If however it be a fraction, the most likely number of times will be the greatest integer in $(q+1)\frac{r}{p}$.

Ex. If $p=4$, $q=7$, $r=3$, $B$ is equally likely to have been 5 or 6 times.

**625.** If the event happened, either the first person spoke the truth and the other $n$ truly reported him (of which the chance is $p^{n+1}$) or else the first spoke falsely and the others falsely reported him (of which the chance is $q^{n+1}$).

But if the event did not happen then either the first person spoke the truth and the other $n$ falsely reported him (of which the chance is $pq^n$) or else the first spoke falsely and the others truly reported him (of which the chance is $p^nq$).

Therefore if *à priori* the event was equally likely to happen or not, *à posteriori* the odds are

$$p^{n+1}+q^{n+1} : p^nq+pq^n$$

in favour of its having happened. And the chance that it happened is

$$\frac{p^{n+1}+q^{n+1}}{p^{n+1}+p^nq+pq^n+q^{n+1}}=\frac{p^{n+1}+q^{n+1}}{(p+q)(p^n+q^n)}=\frac{p^{n+1}+q^{n+1}}{p^n+q^n}.$$

626. $A$ judges correctly 5 times out of 6, and $B$ 3 times out of 4. The same four £5 notes have been submitted to them both independently. $A$ declares that 3 of the notes are good without specifying them. $B$ declares that only one is good. How much ought one to give for the notes?

627. There are two clerks in an office, each of whom goes out for an hour in the afternoon, one may start at any time between two and three o'clock, the other at any time between three and four, and all times within these limits are equally likely. Find the chance that they are not out together.

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**626.** There are five different hypotheses: ($\alpha$) that all the notes are good; ($\beta$) that three only are good; ($\gamma$) that two only are good; ($\delta$) that one only is good and ($\epsilon$) that all are bad.

On the first hypothesis $A$ must have judged 3 correctly, and $B$ only 1 correctly. The chance of this, to denominator $(24)^4$, is $4 . 5^3 \times 4 . 3 = 6000$.

On the hypothesis ($\beta$) $A$ must either have judged all correctly or only two correctly, $B$ must have judged either two correctly or none correctly. Chance $= (625 + 75)(1 + 27) = 19600$.

In like manner we get

$$\begin{aligned} &\text{Chance on hypothesis } (\gamma) = 15600 \\ &\quad ,, \qquad ,, \qquad ,, \quad\; (\delta) = 8208 \\ &\quad ,, \qquad ,, \qquad ,, \quad\; (\epsilon) = 2160. \end{aligned}$$

Therefore, if *à priori* all the hypotheses were equally likely, the expectation is

$$\frac{6000 \times 20 + 19600 \times 15 + 15600 \times 10 + 8208 \times 5}{6000 + 19600 + 15600 + 8208 + 2160},$$

which gives the value £11·85 very nearly.

If, however, each note was *à priori* equally likely to be good or bad the *à priori* chances of ($\alpha$) ($\beta$) ($\gamma$) ($\delta$) ($\epsilon$) would be as $1 : 4 : 6 : 4 : 1$. The *à posteriori* chances would then be as $6000 : 78400 : 93600 : 32832 : 2160$, and the expectation would be £11·25.

**627.** Let $R$ denote the return of the first clerk, and $S$ the

628. The reserved seats in a concert-room are numbered consecutively from 1 to $m+n+r$. I send for $m$ consecutive tickets for one concert and $n$ consecutive tickets for another concert. What is the chance that I shall find no number common to the two sets of tickets?

629. Two persons are known to have passed over the same route in opposite directions within a period of time $m+n+r$, the one occupying time $m$, and the other time $n$: find the chance that they will have met.

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start of the second. Then both $R$ and $S$ must fall between three and four o'clock; and for either, all instants in the hour are equally likely. Therefore it is an even chance that $R$ fall before or after $S$. The required chance is therefore $\frac{1}{2}$.

**628.** The series of $m$ tickets may begin with any of the first $n+r+1$ tickets; i.e. they may be chosen in $n+r+1$ ways. So the series of $n$ may be chosen in $m+r+1$ ways. Total choice $=(m+r+1)(n+r+1)$.

But if there is to be no ticket in common the choice is the no. of arrangements of a series of $m$ tickets, a series of $n$ tickets and $r$ indifferent tickets; i.e. of $r+2$ things of which $r$ are alike. The choice is therefore $(r+1)(r+2)$.

Hence chance $=(r+1)(r+2) \div (m+r+1)(n+r+1)$.

**629.** If each unit of time is divided into $\omega$ intervals, $\omega$ ultimately to be infinite, the question becomes identical with the last, $\omega m$, $\omega n$, $\omega r$ being written for $m$, $n$, $r$. Therefore chance that they meet not

$$= \frac{(\omega r+1)(\omega r+2)}{(\omega m+\omega r+1)(\omega n+\omega r+1)}$$

$$= r^2 \div (m+r)(n+r) \text{ ultimately.}$$

And the chance that they meet

$$= (mn+mr+nr) \div (m+r)(n+r).$$

630. $A$ writes to $B$ requiring an answer within $n$ days. It is known that $B$ will be at the address on some one of these days, any one equally likely. It is a $p$-days' post between $A$ and $B$. If one in every $q$ letters is lost in transit, find the chance that $A$ receives an answer in time. $(n > 2p.)$

631. There are $n$ vessels containing wine, and $n$ vessels containing water. Each vessel is known to hold $a, a+1, a+2, \ldots$ or $a+m-1$ gallons. Find the chance that the mixture formed from them all will contain just as much wine as water.

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**630.** The day on which $B$ is at home must neither be among the first $p$ nor among the last $p$ of the $n$ days. Chance $= (n-2p) \div n$.

We have to combine with this the chance that both letters travel safely. Chance $= (q-1)^2 : q^2$.

Therefore the required chance $= (n-2p)(q-1)^2 \div nq^2$.

**631.** The no. of ways in which the contents of $n$ vessels can occur is $m^n$.

The no. of ways in which the contents shall amount to $r$ gallons is the coefficient of $x^r$ in

$$(x^a + x^{a+1} + x^{a+2} + \ldots + x^{a+m-1})^n,$$

that is, the coefficient of $x^{r-na}$ in

$$(1 + x + x^2 + \ldots + x^{m-1})^n.$$

Let the result of the involution be

$$A_0 + A_1 x + A_2 x^2 + \ldots + A_p x^p, \text{ where } p = mn - n.$$

Then the chance that the two sets of $n$ vessels should give the same contents is

$$(A_0^2 + A_1^2 + A_2^2 + \ldots + A_p^2) \div m^{2n}.$$

But $A_0 = A_p$, $A_1 = A_{p-1}$, &c., therefore the chance may be written

$$(A_0 A_p + A_1 A_{p-1} + A_2 A_{p-2} + \ldots) \div m^{2n}.$$

632. A vessel is filled with three liquids whose specific gravities in descending order of magnitude are $S_1$, $S_2$, $S_3$. All volumes of the several liquids being equally likely, prove that the chance of the specific gravity of the mixture being greater than $S$ is

$$\frac{(S_1-S)^2}{(S_1-S_2)(S_1-S_3)}, \text{ or } 1-\frac{(S-S_3)^2}{(S_2-S_3)(S_1-S_3)},$$

according as $S$ lies between $S_1$ and $S_2$, or between $S_2$ and $S_3$.

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Now the quantity in the bracket is the coefficient of the middle term in the expansion of

$$(A_0+A_1x+A_2x^2+\ldots+A_px^p)^2,$$

or

$$\left(\frac{1-x^m}{1-x}\right)^{2n}.$$

Therefore the chance that the quantities of wine and water should be equal is the coefficient of the middle term of

$$\{(1-x^m)\div m(1-x)\}^{2n}.$$

**632.** Take an equilateral triangle $ABC$ whose height shall represent the capacity of the vessel. Then the trilinear coordinates of any point within the triangle will represent the volumes of the three liquids with which the vessel is filled. (See the note at the end of the volume.) And the point $(\alpha, \beta, \gamma)$ will give a specific gravity $S_1\alpha+S_2\beta+S_3\gamma$. The points giving a specific gravity greater than $S$ must lie on the area bounded by the straight line

$$S_1\alpha+S_2\beta+S_3\gamma=S,$$

or

$$(S_1-S)\,\alpha+(S_2-S)\,\beta+(S_3-S)\,\gamma=0,$$

where $S_1-S$ is positive, $S_3-S$ is negative and $S_2-S$ may be either positive or negative. Let $P$, $Q$, $R$ be the points in which this line cuts $BC$, $CA$, $AB$ (produced if necessary). If $S>S_2$, $Q$ and $R$ lie on the unproduced sides and the triangle $AQR$ is the area representing specific gravities greater than $S$. If $S<S_2$, $P$ and $Q$ lie on the unproduced sides, and the triangle $CPQ$

633. A man drinks in random order $n$ glasses of wine and $n$ glasses of water (all equal); shew that the odds are $n$ to 1 against his never having drunk throughout the process more wine than water.

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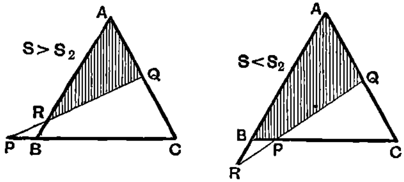


is the area representing specific gravities less than $S$. Therefore the chances required are respectively

$$\frac{\triangle AQR}{\triangle ABC} \text{ and } 1-\frac{\triangle CPQ}{\triangle CAB},$$

that is

$$\frac{AQ \cdot AR}{AC \cdot AB} \text{ and } 1-\frac{CP \cdot CQ}{CB \cdot CA}.$$

But

$$\frac{AQ}{AC} = \text{the } \gamma \text{ coordinate of } Q$$

$$= \frac{S_1 - S}{S_1 - S_3}.$$

Similarly

$$\frac{AR}{AB} = \frac{S_1 - S}{S_1 - S_2}, \quad \frac{CP}{CB} = \frac{S - S_3}{S_2 - S_3}, \quad \frac{CQ}{CA} = \frac{S - S_3}{S_1 - S_3}.$$

Hence we have the expressions given in the question.

**633.** By Prop. XXXVI $\frac{n}{n+1}$ths of the possible orders will fail to fulfil the condition. So the odds are $n$ to 1 against its being fulfilled.

634. If $n$ men and their wives go over a bridge in single file, in random order, subject only to the condition that there are to be never more men than women gone over, prove that the chance that no man goes over before his wife is $(n+1)\,2^{-n}$.

635. A dinner party consists of $n$ gentlemen and their wives. When $n-1$ gentlemen have taken $n-1$ ladies down to dinner, no gentleman taking his own wife, and all possible arrangements being equally likely, the odds are $\underline{||n}$ to $\underline{||n-1}$ against the gentleman and lady remaining being husband and wife.

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**634.** If no man is to go over before his wife we can first arrange the wives in $\underline{|n}$ orders. Then the husband of the last wife can be placed in one way (viz. at the end of the row); then the husband of the second in 3 ways; then the husband of the third in 5 ways; and so on. Hence there are

$$\underline{|n}\,.\,1\,.\,3\,.\,5\,\ldots\,(2n-1),$$

favourable arrangements. But the total number of arrangements subject only to the condition that there are never to be more men than women is (by Prop. XXXVI) $\underline{|2n} \div (n+1)$. Hence the chance

$$=\underline{|n}\,.\,1\,.\,3\,.\,5\,\ldots\,(2n-1)\times(n+1)\div\underline{|2n}$$

$$=\underline{|n+1}\div(2\,.\,4\,.\,6\,\ldots\,2n)=(n+1)\div 2^n.$$

**635.** If no gentleman is to take his own wife, $n$ gentlemen can go out in $\underline{|n}$ ways and their wives in $\underline{||n}$ ways. Total arrangements $=\underline{|n}\,\underline{||n}$.

If the last gentleman only is to have his own wife, this couple can be selected in $n$ ways, and the rest can go out in $\underline{|n-1}\,.\,\underline{||n-1}$ ways. Total arrangements $=\underline{|n}\,\underline{||n-1}$.

The odds are therefore $\underline{||n} : \underline{||n-1}$ against the latter event.

636. A coin is tossed $m+n$ times, where $m>n$. Shew that the chance of there being at least $m$ consecutive heads is $(n+2)\div 2^{m+1}$.

637. If a coin be tossed $n$ times, the chance that there will not be $\kappa$ consecutive heads is the coefficient of $x^n$ in the expansion of

$$\frac{1}{2^n}\cdot\frac{1+x+x^2+\ldots+x^{\kappa-1}}{1-x-x^2-\ldots-x^{\kappa}}.$$

638. A person throws up a coin $n$ times: for every sequence of $m$ throws, heads or tails, for all possible values of $m$, he is to receive $2^m-1$ shillings; prove that the value of his expectation is $3n(n+3)$ pence.

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**636.** Since $m>n$ there cannot be more than one sequence of $m$ or more heads. The chance that such a sequence begins at the first throw is $(\frac{1}{2})^m$. The chance that it begins at the second is $(\frac{1}{2})^{m+1}$. That it begins at the third is likewise $(\frac{1}{2})^{m+1}$, and so on to the $(n+1)$th inclusive since we must have one tail and $m$ heads in succession. Hence the whole chance is

$$(\tfrac{1}{2})^m+n(\tfrac{1}{2})^{m+1}=(n+2)(\tfrac{1}{2})^{m+1}.$$

**637.** By Prop. LIII the required chance is the coefficient of $x^n$ in

$$\left[1-\left(\frac{x}{2}\right)^k\right]\div\left[1-x+\left(\frac{x}{2}\right)^{k+1}\right],$$

or writing $2x$ for $x$ it is the coefficient of $x^n$ in

$$\frac{1}{2^n}\{1-x^k\}\div\{1-2x+x^{k+1}\},$$

i.e. in
$$\frac{1}{2^n}\,\frac{1+x+x^2+\ldots+x^{k-1}}{1-x-x^2-\ldots-x^k}.$$

**638.** Let $E_n$ denote his original expectation and $E_r$ his expectation when he begins again with only $r$ throws to make. He must begin with a sequence of 1, 2, 3 ... or $n$ throws. The

639. $A$ having a single penny, throws for a stake of a penny with $B$, who has at least 8 pence. Shew that the chance that $A$ loses his penny at the 17th throw and not before is $1430 \div 2^{17}$.

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chance that he begins with a sequence of $x$ throws exactly is $(\frac{1}{2})^x$ when $x < n$, but the double of this when $x = n$. Therefore

$$E_n = \Sigma_1^n \left[(\tfrac{1}{2})^x \{2^x - 1 + E_{n-x}\}\right] + \frac{2^n - 1}{2^n}$$

$$= \Sigma \{1 - (\tfrac{1}{2})^x + (\tfrac{1}{2})^x E_{n-x}\} + 1 - \frac{1}{2^n}$$

$$= n + \tfrac{1}{2} E_{n-1} + \tfrac{1}{4} E_{n-2} + \tfrac{1}{8} E_{n-3} + \text{\&c.}$$

Write $n-1$ for $n$ and divide by 2

$$\tfrac{1}{2} E_{n-1} = \frac{n-1}{2} + \tfrac{1}{4} E_{n-2} + \tfrac{1}{8} E_{n-3} + \text{\&c.}$$

Therefore by subtraction

$$E_n - E_{n-1} = \frac{n+1}{2}.$$

For $n$ write $n$, $n-1$, $n-2$, &c. and add, noting that $E_1 = 1$; then

$$E_n = \tfrac{1}{4} n (n+3).$$

Or, in pence, his expectation is $3n(n+3)$.

**639.** Out of the first 16 games he must lose 8 and win 8 without his losses ever exceeding his gains. By Prop. XXXVIII the number of possible orders is

$$\frac{1}{9} \frac{\lfloor 16}{\lfloor 8 \lfloor 8} = 1430.$$

The chance is therefore $1430 \times (\frac{1}{2})^{16}$. Then he must lose the 17th game, of which the chance is $\frac{1}{2}$. Therefore the required probability is $1430 \div 2^{17}$.

640. If a player repeatedly stake the same sum in a series of either $2n$ or $2n-1$ even wagers, the chance that throughout the play he is never worse off than he was at the beginning is expressed by the ratio of the product of the first $n$ odd numbers to the product of the first $n$ even numbers.

641. A bag contains $m+n+r$ sovereigns of which $n$ are of the Victorian Jubilee issue, $m$ of the previous issue, and $r$ of the subsequent issue. I am allowed to draw them one by one as long as those drawn are all of one issue. What is my expectation?

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**640.** Let $m$ denote the number of games, whether $2n$ or $2n-1$, and let $p$ denote $\frac{m}{2}$ or $\frac{m-1}{2}$ as $m$ is even or odd.

The chance that he loses $x$ games and wins $m-x$ without even being worse off than at the beginning is by Prop. XXXVIII

$$(\tfrac{1}{2})^m C_x^m \frac{m-2x+1}{m-x+1} = (\tfrac{1}{2})^m \{C_x^m - C_{x-1}^m\},$$

and $x$ varies from 0 to $p$. Hence by summation the total chance $= (\tfrac{1}{2})^m C_p^m$

$$= (\tfrac{1}{2})^{2n} \frac{\lfloor 2n}{\lfloor n \lfloor n} \quad \text{or} \quad (\tfrac{1}{2})^{2n-1} \frac{\lfloor 2n-1}{\lfloor n \lfloor n-1},$$

according as $m$ is even or odd. Either expression

$$= \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n \lfloor n}$$

$$= \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n}.$$

**641.** If my right were confined to the Jubilee issue I should expect one of the $m+r+1$ series, into which the $m+r$ non-Jubilee coins would divide the $n$ Jubilee coins. My expectation would therefore be $n \div (m+r+1)$. My expectation from the other issues is similarly estimated. Hence my total expectation is

$$\frac{m}{n+r+1} + \frac{n}{m+r+1} + \frac{r}{m+n+1}.$$

642. If in the last question there were $n$ coins of each of $p$ different issues, what would my expectation be?

643. A person throws a die until some face has turned up $k$ times in succession. How many throws must he expect to make?

644. From a point $P$ within the rectangle $OADB$, perpendiculars $OM$, $ON$ are let fall on $OA$, $OB$. Shew that if $P$ be taken at random within the rectangle, $\mathcal{E}(OM.ON)=\frac{1}{4}OA.OB$. But if $P$ be restricted to lie on the diagonal $OD$, then $\mathcal{E}(OM.ON)=\frac{1}{3}OA.OB$; and if on the diagonal $AB$ then $\mathcal{E}(OM.ON)=\frac{1}{6}OA.OB$.

NOTE. *We use the symbol $\mathcal{E}(X)$ to denote the expectation or average value of a variable quantity $X$. If $X$ can take $n$ different values all equally likely and $\Sigma(X)$ denote their sum it follows that $\Sigma(X)=n\mathcal{E}(X)$.*

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**642.** As in the last question my expectation from any particular issue would be $n \div (pn-n+1)$ and my total expectation would be $pn \div (pn-n+1)$.

**643.** After the first throw the chance of a sequence is always $\frac{1}{6}$. In order to have the same face $k$ times in succession we must have $k-1$ consecutive sequences. By Prop. LIV the average no. of throws required for this is $\frac{6}{5}(6^{k-1}-1)$. Including the first we must therefore expect

$$1+\tfrac{6}{5}(6^{k-1}-1)=\tfrac{1}{5}(6^k-1) \text{ throws.}$$

NOTE. We may obtain the result from Qn. 537 (ii) by putting

$$\alpha=\beta=\gamma=\ldots=\tfrac{1}{6},$$

and $$a=b=c=\ldots=k.$$

**644.** (i) It is obvious that $\mathcal{E}(OM)=\frac{1}{2}OA$; and wherever $M$ may be $\mathcal{E}(ON)=\frac{1}{2}OB$. $\therefore\ \mathcal{E}(OM.ON)=\frac{1}{4}OA.OB$.

(ii) If the diagonal $OD$ be divided into $n$ indefinitely small elements $P$ is equally likely to fall at any one of the points of partition. There are therefore equal chances that the rectangle

645. A man plays continuously, always staking one-tenth of his fund in an even wager. Shew (i) that after 70 gains and 70 losses he will have lost more than half his fund; and (ii) that it will require 304 gains to balance 275 losses.

646. Shew the fallacy of the following argument: "*Three persons $A$, $B$, $C$ blind-folded, place themselves at random in a straight line; required the*

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about $OP$ shall be 0, $\left(\frac{1}{n}\right)^2$, $\left(\frac{2}{n}\right)^2$, $\left(\frac{3}{n}\right)^2$ or ... or $\left(\frac{n}{n}\right)^2$ of the original triangle.

$$\therefore \frac{\mathcal{E}\,(OM\,.\,ON)}{OA\,.\,OB} = \frac{1^2+2^2+\ldots+n^2}{n^2\,(n+1)} = \frac{2n+1}{6n},$$

or when $n$ is indefinitely increased

$$\mathcal{E}\,(OM\,.\,ON) = \tfrac{1}{3}OA\,.\,OB.$$

(iii) Since the expectation of each of the rectangles about the diagonal $OD$ is $\frac{1}{3}OA\,.\,OB$; the expectation of each of the complements must be $\frac{1}{6}OA\,.\,OB$. But by interchanging the position of $O$ and $A$ one of these complements is seen to be equivalent to the rectangle formed in the third case. Therefore in this case $\mathcal{E}\,(OM\,.\,ON) = \frac{1}{6}OA\,.\,OB$.

**645.** (i) His fund will be multiplied by $(\cdot 99)^{70}$, i.e. it will be divided by $(1\cdot\dot{0}\dot{1})^{70}$. The logarithm of this divisor is ·305536 which is greater than log 2. Hence he will have lost more than half his fund.

(ii) Suppose $x$ gains balance $y$ losses then

$$\left(\tfrac{11}{10}\right)^x \cdot \left(\tfrac{9}{10}\right)^y = 1, \text{ or } \left(\tfrac{11}{10}\right)^x = \left(\tfrac{10}{9}\right)^y,$$

$$\therefore x\,(\log 11 - \log 10) = y\,(\log 10 - \log 9),$$

$$413927x = 457575y,$$

or

$$x : y = 304 : 275 \text{ nearly}.$$

**646.** The chance that $B$ is to the right of $A$ is $\frac{1}{2}$. When *chance that both B and C will place themselves to the right of A. The chance that B is to the right of A is* $\frac{1}{2}$; *the chance that C is to the right of A is also* $\frac{1}{2}$; $\therefore$ *the chance that both are to the right of* $A = \frac{1}{4}$." Obviously the correct result is $\frac{1}{3}$.

647. A table of logarithms is constructed to 10 places of decimals. In what proportion of the entries shall we expect less than one cypher and in what proportion more than one cypher?

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however he is so placed the chance that $C$ should be to the right is more than $\frac{1}{2}$. For the further $A$ goes to the left the more likely $C$ is to be on his right. If $A$ and $B$ have taken positions at random they divide all the unoccupied positions into three series of equal expectation. The expectation of vacant places to $A$'s right is therefore double of the expectation of vacant places to his left, and the chance of $C$ placing himself on $A$'s right is not $\frac{1}{2}$ but $\frac{2}{3}$. Hence the fallacy.

**647.** We have no reason to expect one digit to appear more frequently than any other. Therefore we may expect among $10^{10}$ entries, $9^{10}$ with no cypher, $10 \,.\, 9^9$ with one cypher, and $10^{10} - 19 \,.\, 9^9$ with more than one cypher. These numbers are approximately as 36 : 40 : 27. We may therefore expect an excess of cyphers in 3 entries for every 4 entries with defect of cyphers.

NOTE. It was evident *à priori* that the average must be departed from more often by defect than by excess, as the defect could only be by unity while the excess might be by any number short of 10. So generally if an experiment succeed on an average $r$ times in $n$ trials, $r$ being less than $\frac{1}{2}n$, the successes in any given series of $n$ trials are more likely to be below $r$ than above $r$.

E.g. If a die be thrown 600 times, the number of aces is more likely to be under 100 than over 100. The next question further illustrates this principle.

648. I offer a boy $4n$ oranges, apples, pears and plums, in any proportion he likes. If all proportions be equally probable, the chance that he takes exactly $n$ oranges is less than $27 \div 64n$, and the oranges are more likely to be defect that in excess of $n$, in the ratio greater than 37 : 27.

649. If a train consisting of $p$ carriages, each of which will hold $q$ men, contains $pq - m$ men, find the chance that another man $B$ getting in, and being equally likely to take any vacant place will travel in the same carriage with a given passenger $A$.

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**648.** He can take 0 oranges in $R^3_{4n}$ ways,

$$\begin{array}{llllll} ,, & ,, & 1 & ,, & ,, \; R^3_{4n-1} & ,, \\ ,, & ,, & 2 & ,, & ,, \; R^3_{4n-2} & ,, \end{array}$$

and so on.

Chance of exactly $n$ oranges $= R^3_{3n} \div R^4_{4n}$

$$= \frac{3(3n+1)(3n+2)}{(4n+1)(4n+2)(4n+3)} < \frac{27}{64n}.$$

Chance of more than $n$ oranges $= R^4_{3n-1} \div R^4_{4n}$.

Chance of less than $n = 1 - R^4_{3n} \div R^4_{4n}$.

Therefore the chances of defect and excess are in the ratio

$$(1 - R^4_{3n}) : R^4_{3n-1} = (37n^2 + 42n + 11) : (27n^2 + 27n + 6)$$
$$> 37 : 27.$$

**649.** There are $pq$ places to be occupied by $A$, $B$, $pq - m - 1$ other men and $m - 1$ blanks. All arrangements are equally probable, and the chance is not affected by our attributing a different individuality to each of the blanks.

All selections of $q - 1$ persons or blanks to complete $A$'s carriage are equally likely. But there are $C^{pq-1}_{q-1}$ such possible selections, of which $C^{pq-2}_{q-1}$ exclude $A$. Therefore the chance of $B$ not travelling with $A$ is $C^{pq-2}_{q-1} \div C^{pq-1}_{q-1} = (pq - q) \div (pq - 1)$. And the complementary chance is $(q - 1) \div (pq - 1)$.

650. Shew that with $n$ dice the no. of ways of throwing a sum $s$ is

$$C^{s-1}_{n-1} - C^n_1 C^{s-7}_{n-1} + C^n_2 C^{s-13}_{n-1} - \&c.$$

651. In a game of mixed chance and skill the odds are $k:1$ that any given game shall be won by the superior player. $A$, $B$, $C$ are unequal players. $A$ plays $m+1$ games with $B$ and wins $m$ of them. He plays $n+1$ games with $C$ and wins $n$ of them. Shew that the odds are

$$(1+k)(1+k^{m+n})+k^m+k^{n+1} : (1+k)(1+k^{m+n})+k^{m+1}+k^n$$

that $B$ will win the first game he plays with $C$.

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**650.** By Prop. XXV the $s$ units can be distributed among the $n$ dice in $C^{s-1}_{n-1}$ ways.

Among these, a given die will have more than 6 units in $C^{s-7}_{n-1}$ ways; two given dice will have more than 6 each in $C^{s-13}_{n-1}$ ways; and so on. Therefore by Prop. XIV the number of distributions in which each die will represent from 1 to 6 is

$$C^{s-1}_{n-1} - C^n_1 C^{s-7}_{n-1} + C^n_2 C^{s-13}_{n-1} - \&c.$$

The result might also have been obtained by writing $s$, $n$, 1, 6 for $n$, $r$, $q$, $z$, in the result of Prop. XXVIII.

**651.** The order of merit of the players may be

$$ABC,\ ACB,\ BCA,\ BAC,\ CAB,\ CBA,$$

and the consequent chances of the observed double event are as

$$k^{m+n} : k^{m+n} : 1 : k^n : k^m : 1.$$

Therefore the chance that $B$ is superior to $C$ is

$$(k^{m+n}+k^n+1) \div (2k^{m+n}+k^m+k^n+2).$$

Therefore $B$'s chance of beating $C$ in a single game is

$$\frac{(k^{m+n}+k^n+1)\,k+k^{m+n}+k^m+1}{(1+k)(2k^{m+n}+k^m+k^n+2)},$$

and therefore the odds are

$$(1+k)(1+k^{m+n})+k^m+k^{n+1} : (1+k)(1+k^{m+n})+k^{m+1}+k^n,$$

in favour of $B$, if $m > n$.

652. Shew that the no. of squares visible on an enlarged chess-board with $n$ squares along each side is the sum of the first $n$ square numbers, and the no. of rectangles visible is the sum of the first $n$ cube numbers.

653. If from a random point on the diameter of a semicircle a perpendicular be erected to meet the arc its average length will be $\pi \div 4$ of the radius. But if from a random point on the arc a perpendicular be let fall on the diameter, its average length will be $2 \div \pi$ of the radius.

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**652.** (i) A square consisting of $k \times k$ squares of the board may be shunted into $(n-k+1)^2$ positions. Hence the total no. of squares visible is $\Sigma (n-k+1)^2$ where $k$ has all values from $n$ to 1. Sum $= 1^2 + 2^2 + \ldots + n^2$.

(ii) Let the board be held with a diagonal vertical and classify all rectangles according to the highest square which they contain.

Consider the rectangles which have as their highest square that which is $h$th and $k$th from the lower boundaries. The lowest square may be any contained in the rectangle $h \times k$. Hence there are $hk$ such rectangles and the total number of rectangles is therefore $\Sigma (h \,.\, k)$ where $h$ and $k$ each take the values $1, 2, 3 \ldots n$. Therefore the number is $\frac{1}{4} n^2 (n+1)^2$ which is known to be the sum of $1^3 + 2^3 + \ldots + n^3$.

**653.** (i) Let the diameter $AB$ $(=2r)$ be divided into $n$ equal elements of which one is $DD'$. Erect perpendiculars $DR$, $D'R'$ to meet the arc in $RR'$. Then ultimately when $n$ is indefinitely increased $\Sigma \{RD \,.\, DD'\} =$ area of semicircle, or $\frac{2r}{n} \Sigma (RD) = \frac{\pi r^2}{2}$. Whence $\mathcal{E}(RD) = \frac{\pi r}{4}$.

(ii) Let the arc be divided into $n$ equal elements of which one is $PP'$. Let fall perpendiculars $PN$, $P'N'$ on the diameter. Also on $AB$ describe a square $ABFG$ and about the square describe a circle. Let $AP$, $AP'$ be produced to meet this circle in $QQ'$; so that $AQ = AP + PB$.

654. In a circle the average length of a random chord is a third proportional to the semi-circumference and the diameter.

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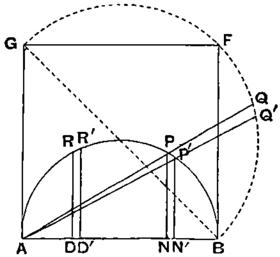


Since the elementary arcs as $PP'$ are equal they subtend equal angles at $A$. Therefore $\angle\, QAQ' = \pi \div 2n$. Hence ultimately

$$\Sigma \left\{\frac{\pi}{2n} \cdot \frac{AQ^2}{2}\right\} = \text{area } ABQFG = (\pi + 2)\, r^2,$$

$$\therefore\ \mathcal{E}\,(AQ^2) = 4r^2 + \frac{8r^2}{\pi}.$$

But $AQ^2 = (AP + PB)^2 = 4r^2 + 4r \,.\, PN$,

$$\therefore\ \mathcal{E}\,(4r \,.\, PN) = \frac{8r^2}{\pi} \text{ or } \mathcal{E}\,(PN) = \frac{2r}{\pi}.$$

**654.** A random chord is a chord drawn from any point on the circumference in any direction. But if all directions are equally likely, since equal angles stand upon equal arcs, all points on the circumference are equally likely for the further extremity of the chord. Hence the random chord may be defined as joining two random points. But the average length of the chords will not be affected if we confine our attention to chords in a particular direction, say those at right angles to the diameter $AB$. Hence by Qn. 653 (ii)

$$\mathcal{E}\,(\text{chord}) = 2\mathcal{E}\,(PN) = 4r \div \pi,$$

which is a third proportional to $\pi r$ and $2r$.

655. If it be stipulated that the chord must be greater than the chord of a quadrant, its average length will be multiplied by $\sqrt{2}$.

656. $P$, $Q$ are random points on the circumference of a circle, on opposite sides of a fixed chord $AB$ which subtends an angle $2\alpha$ at the centre. Shew that the average area of the quadrilateral $APBQ$ is

$$\frac{AB^2}{4}\left\{\frac{1}{\alpha}+\frac{1}{\pi-\alpha}\right\}.$$

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**655.** Let $\mu r$ be the expectation when the chord must be greater than that of a quadrant.

Then in the figure of Qn. 653, we have

$$\mathcal{E}\,(AQ)=\mu r\sqrt{2} \text{ and } \mathcal{E}\,(AP)=4r\div\pi.$$

But $$\mathcal{E}\,(AQ)=\mathcal{E}\,(AP+PB)=2\mathcal{E}\,(AP),$$

$$\therefore\ \mu r\sqrt{2}=8r\div\pi,$$

$$\mu=4\sqrt{2}\div\pi,$$

that is, the average length is multiplied by $\sqrt{2}$.

**656.** Let the angle $\alpha$ between the chord and the tangent at $A$ be divided into $n$ equal parts, $n$ being ultimately increased indefinitely. Then we have

$$\Sigma\,(AP)^2\,\alpha\div n=2 \text{ area of segment}$$

$$=\tfrac{1}{2}AB^2\operatorname{cosec}^2\alpha\,(\alpha-\sin\alpha\cos\alpha);$$

$$\therefore\ \mathcal{E}\,(AP)^2=\tfrac{1}{2}AB^2\,(\operatorname{cosec}^2\alpha-\cot\alpha\div\alpha)=\mathcal{E}\,(BP)^2.$$

But Area of triangle $APB=\frac{1}{4}\,(AB^2-AP^2-BP^2)\tan\alpha$;

$$\therefore\ \mathcal{E}\,(\text{Area } APB)=\frac{AB^2}{4}\left(\frac{1}{\alpha}-\cot\alpha\right).$$

So $$\mathcal{E}\,(\text{Area } AQB)=\frac{AB^2}{4}\left(\frac{1}{\pi-\alpha}+\cot\alpha\right);$$

$$\therefore\ \mathcal{E}\,(\text{Area } APQB)=\frac{AB^2}{4}\left(\frac{1}{\alpha}+\frac{1}{\pi-\alpha}\right).$$

657. If two points be taken at random on a straight line of length $a$ the chance that the distance between them exceeds a given length $b$ is

$$(a-b)^2 \div a^2.$$

658. If a point be taken at random on the area of a circle (or of a sector of a circle) its average distance from the centre is two-thirds of the radius. And if it be taken on the annulus between two concentric circles of radii $a$ and $b$ (or on any sector of this annulus), the average distance is

$$\tfrac{2}{3}(a^2+ab+b^2) \div (a+b).$$

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**657.** On $AB=a$ describe a square $ABCD$ and draw the diagonal $AC$. Let the square be divided into an indefinitely large no. of equal elements by lines parallel to $AB$. Let $HK$ be one of these lines, meeting $AC$ in $P$. We may take $HK$ to represent the line which is to be divided and $P$ to be the first point of division, all possible positions of $HK$ giving all possible positions of $P$. From $AB$, $AD$, cut off $AM$, $AN$, each $=b$; and through $MN$ draw lines $MM'$, $NN'$ parallel to $AC$, meeting $AC$, $CD$ in $M'$, $N'$. Considering the aggregate of possible positions of $HK$ the second point of division $Q$ may lie anywhere on the square, and all positions are equally likely. But for a favourable result $Q$ must lie on one of the triangles $MBM'$, $NDN'$ which together make $(a-b)^2$. Hence the chance is $(a-b)^2 \div a^2$.

**658.** (i) Let $O$ be the centre of the circle; $OA=a$ the radius. Let the circle be divided into an indefinitely large no. of concentric rings of equal width $=\delta$. The chance that a random point $P$ should be at a distance $x$ from $O$ varies as $x$, and therefore

$$\mathcal{E}\,(OP) = \Sigma\,(x^2) \div \Sigma\,(x),$$

where $x$ takes all the values 1, 2, 3 ... $(a \div \delta)$. Hence, ultimately,

$$\mathcal{E}\,(OP) = \tfrac{2}{3}a.$$

(ii) If $\mathcal{E}_a$ denote the expectation when $P$ may fall anywhere within the circle $a$, $\mathcal{E}_b$ the expectation when it must fall within the circle $b$, and $\mathcal{E}_{a-b}$ the expectation when it must fall upon the annulus, then since the chances of its falling on these areas

659. A random point $P$ is taken on the area of a triangle $ABC$. $AP$ is produced to meet $BC$ in $D$, shew that $D$ is equally likely to fall on any point in $BC$.

660. In the last question

$$\mathcal{E}(PBC) = \mathcal{E}(PCA) = \mathcal{E}(PAB) = \tfrac{1}{3}ABC.$$

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are as $a^2 : b^2 : a^2 - b^2$, we have

$$(a^2 - b^2)\,\mathcal{E}_{a-b} + b^2\mathcal{E}_b = a^2\mathcal{E}_a.$$

$$\therefore\ \mathcal{E}_{a-b} = \frac{a^2\mathcal{E}_a - b^2\mathcal{E}_b}{a^2 - b^2} = \frac{2}{3}\,\frac{a^3 - b^3}{a^2 - b^2}$$

$$= \tfrac{2}{3}(a^2 + ab + b^2) \div (a + b).$$

Corollary. By precisely the same reasoning we have

$$\mathcal{E}\,(OP^n) = \Sigma\,(x^{n+1}) \div \Sigma\,(x) = 2a^n \div (n + 2).$$

**659.** Consider any two equal small elements $XX'$ and $YY'$ in $BC$. $D$ will fall on $XX'$ or $YY'$ if $P$ falls within the triangle $AXX'$ or $AYY'$. But these triangles are equal and therefore the chances are equal. Hence $D$ is equally likely to fall on any point in $BC$.

**660.** Since $D$ is equally likely to fall anywhere on $BC$ we have $\quad \mathcal{E}\,(BD) = \mathcal{E}\,(DC). \quad \therefore\ \mathcal{E}\,(ABD) = \mathcal{E}\,(ADC).$

But the areas $APB$, $APC$ are proportional to $ABD$, $ADC$.

Hence $\qquad \mathcal{E}\,(APB) = \mathcal{E}\,(APC).$

Similarly $\qquad \mathcal{E}\,(APB) = \mathcal{E}\,(BPC).$

Hence the triangles $APB$, $APC$, $BPC$ are of equal expectation, and therefore the expectation of each $= \frac{1}{3}ABC$.

Cor. If $\alpha$, $\beta$, $\gamma$ be the trilinear coordinates of a random point in a triangle the expectations of $\alpha$, $\beta$, $\gamma$ are inversely as the sides and therefore directly as the altitudes of the triangle.

661. Two points $P$, $Q$, equidistant from the base $BC$, are taken at random on the area of a triangle. Shew that if all possible positions of the pair $P$, $Q$ are equally likely $\mathcal{E}(PQ)=\frac{1}{4}BC$. But if all positions of $P$ are equally probable, and then all possible positions of $Q$ equally probable, $\mathcal{E}(PQ)=\frac{2}{9}BC$.

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**661.** Let the triangle $ABC$ be ruled by lines parallel to the base $BC$ into elements of uniform width $\delta$, where $\delta$ is to be ultimately decreased indefinitely.

Let the altitude $= m\delta$, so that $m$ is the no. of elements.

The lengths of the elements counting from the vertex downwards are as 1, 2, 3 ... $m$ to $m$.

If $HK$ be one of the elements the no. of pairs of points on $HK$ varies as $HK^2$; and the average distance between a pair of points on this line is $\frac{1}{3}HK$ (by Qn. 497). Hence

$$\mathcal{E}(PQ)=\frac{1}{3}\frac{\Sigma(HK^3)}{\Sigma(HK^2)}=\frac{BC}{3m}\cdot\frac{1^3+2^3+\ldots+m^3}{1^2+2^2+\ldots+m^2},$$

or increasing $m$ indefinitely,

$$\mathcal{E}(PQ)=\tfrac{1}{4}\,.\,BC.$$

But if $P$ be first taken at random, the chance that it lies on $HK$ varies as $HK$; and $Q$ must be taken on the same line. The average distance of $P$ and $Q$ on this line is (as before) $\frac{1}{3}PQ$, and we have

$$\mathcal{E}(PQ)=\frac{1}{3}\frac{\Sigma(HK^2)}{\Sigma(HK)}=\frac{BC}{3m}\cdot\frac{1^2+2^2+\ldots+m^2}{1+2+\ldots+m},$$

or increasing $m$ indefinitely

$$\mathcal{E}(PQ)=\tfrac{2}{9}BC.$$

NOTE. The distinction here made is very important. To take a random point on a given area and then another random point on the same is generally the same thing as to take a random pair of points. But it is not the same thing if some relation have to be fulfilled between the two points. In the case before us if all pairs are equally likely $P$ is more likely to be at $B$ than at $H$ in the ratio $BC : HK$, because the no. of possible positions for $Q$ is as $BC : HK$. So in Qn. 600 if all positions and magnitudes of the

662. A random point $P$ is taken in the base $BC$ of a triangle $ABC$, and another random point $Q$ anywhere on the area of the triangle. Shew that $PQ$ produced is equally likely to cut $AB$ or $AC$.

663. A random point $P$ is taken on a side of a parallelogram and another random point $Q$ anywhere on the area. Shew that the chance of $PQ$ (produced if necessary) cutting a given diagonal (not produced) is $\frac{3}{4}$.

664. On each of three assigned faces of a tetrahedron a random point is taken: shew that the chance of the plane thus determined cutting any given edge of the fourth face is $\frac{1}{2}$: and the chance of its cutting the fourth face is $\frac{3}{4}$.

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random triangle are equally likely the vertex of the new triangle is more likely to fall at $D$ than at $E$ if $D$ be an assigned point further from the base than $E$. If we assumed that all positions of the new vertex were equally probable and then all altitudes equally probable the expectation would be one-18th instead of one-10th of the original area.

**662.** $PQ$ will cut $AB$ or $AC$ according as $Q$ lies on the triangle $APB$ or $APC$. The chances are therefore as the areas of these triangles and therefore as the bases $CP$, $PA$. But these bases are of equal expectation. Therefore $PQ$ is equally likely to cut $AB$ or $AC$.

**663.** Let $ABCD$ be the parallelogram, and let $P$ be any point on $BC$. Consider the chance of $PQ$ cutting $AC$. By Qn. 662 if $Q$ were confined to the area $ABC$ the chance would be $\frac{1}{2}$. But $Q$ is taken at random over the double of this area, the whole of the new area being favourable. Therefore the chance of $PQ$ cutting $AC$ is $\frac{3}{4}$. And the argument holds on whatever side $P$ be taken and whichever diagonal be selected.

$$\therefore \text{ Required chance} = \tfrac{3}{4}.$$

Corollary. The chance of cutting either diagonal and not the other is $\frac{1}{4}$. The chance of cutting both is $\frac{1}{2}$.

**664.** Let $OABC$ be the tetrahedron, the assigned faces being those which meet in $O$. Let $A'$, $B'$ be the random points on $OBC$

665. A fixed point $O$ outside a closed area or on its perimeter is joined to a random point $P$ within the area. Two other random points $H$, $K$ are taken within the area. Find the chance that $H$, $K$ are on opposite sides of $OP$. (N.B. If the boundary of the area be re-entrant or inflected the point $O$ must be assumed to be external to a string tightly wrapped round the figure.)

666. A line of length $c$ is divided into $n$ segments by $n-1$ random points. Find the chance that no segment is less than a given length $a$, where $c>na$. (Say $c-na=ma$.)

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and $OCA$, and let $CA'$, $CB'$ be produced to meet $OB$, $OA$ in $K$, $H$. (The student is requested to make the figure for himself.) Let $P$ be the random point in the face $OAB$; then if $P$ lie outside the triangle $OHK$ the plane $PA'B'$ will cut $OC$; but if $P$ lie inside $OHK$ it will cut $CA$, $CB$. But by Qn. 659 $H$ is equally likely to be any point on $OA$, and so $K$ any point on $OB$. Therefore the average area $OHK$ is one-fourth of $OAB$. $\therefore$ The chance that the plane cuts the edges $CA$, $CB$ is $\frac{1}{4}$. So the chance that it cut the edges $CA$, $AB$ is $\frac{1}{4}$; and it cannot cut $CA$ without cutting either $CB$ or $AB$. Hence the chance of its cutting $CA$ is $\frac{1}{2}$.

(ii) The chance of the plane cutting any two of the three edges $BC$, $CA$, $AB$ being $\frac{1}{4}$, the chance that it cuts *some* two is $\frac{3}{4}$. This is therefore the chance that it cuts the face $ABC$.

**665.** $P$, $H$, $K$ are all random points subject to the same conditions and $OP$, $OH$, $OK$ lie within an angle not exceeding $180°$ at $O$. Therefore one and only one of the three will have two of the points on opposite sides of it. The chance that this one is $OP$ is $\frac{1}{3}$.

**666.** Suppose the whole line divided into $\omega c$ equal elements, $\omega$ ultimately to be indefinitely great. Then the given length will contain $\omega a$ of the same elements. The points of partition can be selected in $C_{n-1}^{\omega c-1}$ ways. But if we first distribute $\omega a$ elements to

667. In the last question find the chance that $r$ of the segments shall be less than $a$ and $n-r$ greater than $a$.

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each segment we have then an excess of $\omega ma$ elements to distribute as increments of the $n$ segments. We can make this distribution (Prop. XXVI) in $C_{\omega ma, n-1}$ ways. This is therefore the no. of favourable ways.

$$\therefore \text{ Chance} = \frac{C_{\omega ma, n-1}}{C_{\omega c-n, n-1}} = \frac{(\omega ma+1)(\omega ma+2)\dots(\omega ma+n-1)}{(\omega c-n+1)(\omega c-n+2)\dots(\omega c-1)},$$

and when $\omega$ is made infinite this becomes

$$(ma)^{n-1} \div c^{n-1} = m^{n-1} \div (m+n)^{n-1}.$$

**667.** If the line be divided into $r$ segments less than $a$, and $n-r$ segments greater than $a$, all possible orders of these segments must be equally likely. And there are $\underline{|n}$ possible orders, of which $\underline{|r}\,\underline{|n-r}$ will have the $r$ small elements together at the beginning of the line. We may therefore assume that the $r$ small elements are at the beginning of the line provided we multiply the final result by $\underline{|n} \div \underline{|r}\,\underline{|n-r} = C^n_r$.

Let then the first $r$ segments be $a-x$, $a-y$, $a-z$, $a-u$, $a-v$, &c. If to each of the remaining $n-r$ segments we first assign a length $a$, we have vacant $\omega(ma+x+y+z+\dots)$ elements. These may be distributed as increments of the $n-r$ equal segments in

$$C_{n-r-1, \omega(ma+x+y+z+\dots)} \text{ ways.}$$

Give $\omega x$ all values from 0 to $\omega a$ and we have (Introduction, Table I., 4),

$$C_{n-r, \omega(ma+a+y+z+\dots)} - C_{n-r, \omega(ma+y+z+\dots)-1} \text{ distributions.}$$

Then give $y$ all values from 0 to $\omega a$, and we have

$$C_{n-r+1, \omega(ma+2a+z+\dots)} - 2C_{n-r+1, \omega(ma+a+z+\dots)-1}$$
$$+ C_{n-r+1, \omega(ma+z+\dots)-2} \text{ distributions.}$$

668. In a circle whose circumference is $c$ there are placed at random $n$ equal chords, each subtending an arc $a$. Find the chance that none of the chords will intersect, where $c > na$. (Say $c - na = ma$.)

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Then give $z$ all values from 0 to $\omega a$; and so on; and we ultimately obtain

$$C_{n-1,\,\omega(ma+ra)} - C_1^r\, C_{n-1,\,\omega(ma+ra-a)-1}$$
$$+ C_2^r\, C_{n-1,\,\omega(ma+ra-2a)-2} - \&c. \text{ to } r+1 \text{ terms.}$$

Dividing this by the whole no. of possible distributions, which is $C_{\omega c-n,\,n-1}$ and increasing $\omega$ indefinitely we have

$$\left(\frac{m+r}{m+n}\right)^{n-1} - C_1^r\left(\frac{m+r-1}{m+n}\right)^{n-1} + C_2^r\left(\frac{m+r-2}{m+n}\right)^{n-1} - \&c.$$

And the whole chance required is $C_r^n$ times this; or, reversing the series,

$$\frac{m^{n-1} - C_1^r(m+1)^{n-1} + C_2^r(m+2)^{n-1} - \&c. \pm (m+r)^{n-1}}{\pm(m+n)^{n-1}}.$$

**668.** Let the whole circumference be divided into $\omega c$ elements. Then each of the chords subtends an arc of $\omega a$ elements.

We may regard one of the random chords as fixed.

If the chords are not to intersect the no. of arrangements of the rest is the no. of orders of $n-1$ groups of $\omega a$ elements and $\omega ma$ single elements, $= C_{n-1,\,\omega ma}$ ways.

But the no. of possible positions for the $n-1$ chords is $R_{n-1}^{\omega c}$. Hence

$$\text{Chance} = \frac{C_{n-1,\,\omega ma}}{R_{n-1}^{\omega c}} = \frac{(\omega ma+1)(\omega ma+2)\ldots \text{ to } n-1 \text{ factors}}{\omega c(\omega c+1)\ldots \text{ to } n-1 \text{ factors}}.$$

And increasing $\omega$ indefinitely

$$\text{Chance} = \left(\frac{ma}{c}\right)^{n-1} = \left(\frac{m}{m+n}\right)^{n-1}.$$

669. In the last question find the chance that there shall be one and only one intersection.

670. A circular slate is handed to two blind men and each draws a chord across it. Find the chance that the chords intersect.

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**669.** Proceeding as before, let one of the chords intersect the fixed chord so as to overlap to the extent of $\omega x$ elements. As the second chord may overlap either to the right or the left the chance is $2 \div \omega c$. The two chords together subtend $\omega(2a - x)$ elements of arc. If there is to be no other intersection the remaining $\omega(c - 2a + x)$ elements must be arranged in $n - 2$ groups of $\omega a$ elements each, and $\omega(ma + x)$ single elements.

$$\text{Chance} = C_{n-2,\,\omega ma+\omega x} \div R^{\omega c}_{n-2},$$

and $\omega x$ can take all values from 0 to $\omega a$.

Hence the whole chance required is

$$\Sigma\left\{\frac{2}{\omega c} \cdot \frac{C_{n-2,\,\omega ma+\omega x}}{R^{\omega c}_{n-2}}\right\}$$

$$= \frac{2}{\omega c} \cdot \frac{C_{n-1,\omega ma+\omega a} - C_{n-1,\,\omega ma-1}}{R^{\omega c}_{n-2}},$$

and when $\omega$ is indefinitely increased this becomes

$$\frac{(ma+a)^{n-1} - (ma)^{n-1}}{\frac{1}{2}c^{n-1}} = \frac{(m+1)^{n-1} - m^{n-1}}{\frac{1}{2}(m+n)^{n-1}}.$$

**670.** As in Qn. 654, a chord drawn in random direction from a random point on the circumference is the same thing as a line joining two random points on the circumference.

Let $A$, $B$, $C$, $D$ be four random points. Then the three pairs of chords $AB, CD$; $AC, BD$; $AD, BC$ are all equally likely. But only one pair gives an intersection. Hence chance $= \frac{1}{3}$.

671. If $2n$ points be given on the circumference of a circle (or other closed curve without cusps or inflections), in how many ways can they be joined two and two so as to form $n$ chords?

672. In how many ways can the chords in the last question be drawn so that each may intersect all the rest? And in how many ways so that the no. of intersections may be one less than the maximum?

673. If a random chord is the line joining two random points on a curve, find the respective chances that three random chords will give 0, 1, 2 or 3 points of intersection, and shew that the average expectation is *one* such point.

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**671.** From the first given point we can draw a chord in $2n-1$ ways; then from the next in $2n-3$ ways; from the next in $2n-5$ ways; and so on. Hence the no. of ways is

$$1\,.\,3\,.\,5\,\ldots\,(2n-1) = \underline{|2n} \div 2^n\,\underline{|n}.$$

**672.** For the first case each point must be joined in the point diametrically opposite, and therefore there is only one way.

For the second case we have two chords which do not intersect, but all the other chords must cut these two. This can only be when there are $n-2$ points on each side of the two non-intersecting chords, these latter joining two consecutive points to the two points diametrically opposite. The two chords can be selected in $n$ ways, and then the other $n-2$ chords can only be drawn in one way. $\therefore$ Result $n$ ways.

**673.** Three random chords join six random points two and two. But six points can be joined two and two in 15 different ways (Qn. 671). And of these, 5 ways give no intersections; 6 ways give 1; 3 ways give 2, and 1 way gives three. Hence the no. of sets of 3 chords giving 0, 1, 2, or 3 intersections must be as 5 : 6 : 3 : 1. The respective chances are therefore $\frac{5}{15}$, $\frac{6}{15}$, $\frac{3}{15}$, $\frac{1}{15}$, and the expectation is

$$\tfrac{6}{15} + \tfrac{3}{15}\,.\,2 + \tfrac{1}{15}\,.\,3 = 1.$$

674. If $n$ random chords be drawn as in the last question, the expectation of intersections is $\frac{1}{6}n(n-1)$.

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**674.** Consider a chord which divides the curve into two arcs in the ratio $1-x$ to $1+x$.

The chance that a random chord will intersect this is

$$2(1-x)(1+x) \div 4 = \tfrac{1}{2}(1-x^2),$$

and the chance that it will not intersect is $\frac{1}{2}(1+x^2)$.

Hence the chance that $n$ random chords will make exactly $r$ intersections with the fixed chord is

$$C^n_r (1-x^2)^r (1+x^2)^{n-r} \div 2^n.$$

And the expectation of the number of intersections along the fixed chord is the sum from $r=0$ to $r=n$, of

$$\frac{rC^n_r (1-x^2)^r (1+x^2)^{n-r}}{2^n} = \frac{nC^{n-1}_{r-1} (1-x^2)^r (1+x^2)^{n-r}}{2^n}.$$

The sum is, by the Binomial Theorem,

$$n(1-x^2)\{(1-x^2)+(1+x^2)\}^{n-1} \div 2^n = \tfrac{1}{2}n(1-x^2).$$

Now if we give $x$ all possible values, viz. ($\omega$ being indefinitely great)

$$\frac{0}{\omega}, \frac{1}{\omega}, \frac{2}{\omega}, \dots \frac{\omega}{\omega},$$

we make our fixed chord into a random chord. Hence the expectation of the no. of intersections which an $(n+1)$th random chord will add to those already made by $n$ random chords is

$$\frac{\frac{1}{2}n\Sigma(1-x^2)}{\omega+1} = \frac{(4\omega-1)n}{12\omega} = \frac{n}{3} \text{ ult.}$$

Or, if $I_n$ denote the no. of intersections to be expected from $n$ chords

$$I_{n+1} - I_n = \tfrac{1}{3}n.$$

But $I_1 = 0$, $\therefore$ $I_2 = \frac{1}{3}$; $I_3 = 1$, (as in Qn. 672); and generally

$$I_n = \tfrac{1}{6}n(n-1).$$

675. If $n$ random chords be drawn, the chance that there shall be no intersection is $2^n \div \underline{|n+1}$.

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Another Solution. If there be $n$ chords there must be $2n$ points, and these can be joined two and two in

$$\alpha = 1 \,.\, 3 \,.\, 5 \ldots (2n-1) \text{ ways.}$$

This is therefore the no. of sets of chords that can be made with the $2n$ points.

Join the $2n$ points in all possible ways. We then have

$$\beta = \frac{n}{12}(2n-1)(2n-2)(2n-3) \text{ points of intersection.}$$

[For we have $2n$ lines with no intersections on them; $2n$ with $1(2n-3)$ intersections; $2n$ with $2(2n-4)$ intersections, and so on; finally $2n$ with $n(n-2)$ intersections, and $n$ with $(n-1)^2$ intersections.] Now any intersection lies on two lines, and therefore appropriates 4 points, and the no. of sets in which it occurs must be the no. of ways in which the remaining $2n-4$ points can be joined two and two, viz. $\gamma = 1 \,.\, 3 \,.\, 5 \ldots (2n-5)$. Therefore if all the sets possible be formed the figure will be superposed upon itself $\gamma$ times, and the total no. of intersections belonging to the $\alpha$ sets will be $\beta \times \gamma$. Therefore the average no. of intersections on each set is $\beta\gamma \div \alpha = \frac{1}{6}n(n-1)$.

**675.** Let $f_n$ be the no. of ways in which $2n$ points on the curve can be joined two and two without intersections. The chord through any point $A$ must pass between the remaining $2n-2$ points, so as to leave an even number on either side, say $2x$ on the left and $2n-2x-2$ on the other. The rest of the chords can then be drawn in $f_x f_{n-x-1}$ ways. And $x$ may have any value from 0 to $n-1$. Therefore

$$f_n = f_{n-1} + f_{n-2}f_1 + f_{n-3}f_2 + \cdots + f_{n-1}$$
$$= \text{coeff. of } x^{n-1} \text{ in } (1 + f_1 x + f_2 x^2 + \ldots)^2.$$

Therefore $\quad (1 + f_1 x + f_2 x^2 + \ldots)^2 = f_1 + f_2 x + f_3 x^2 + \ldots.$

676. The chance that there shall be one and only one intersection is

$$2^n n(n-1) \div \underline{|n+2}.$$

677. If a straight line be broken at random into three parts the odds are 3 to 1 against their making the sides of a triangle; but if the line be first broken into two parts and then the longer portion be broken into two parts, the odds are 2 to 1.

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Writing $X = 1 + f_1 x + f_2 x^2 + \ldots$ we have

$$X^2 = \frac{X-1}{x}; \text{ whence } X = \frac{1-\sqrt{1-4x}}{2x};$$

$$\therefore f_n = Y_n \div (n+1).$$

But the total no. of ways in which the $n$ chords can be drawn is

$$1.3.5\ldots(2n-1) = Y_n \underline{|n} \div 2^n.$$

Therefore the chance that there is no intersection is

$$2^n \div \underline{|n+1}.$$

**676.** The two chords which make the intersection must divide the remaining $2n-4$ points into four groups, each containing an even number. Hence if we draw one of the intersecting chords through $A$ the number of ways of drawing the other chords will be $f_u f_v f_w f_x$ for every way in which $n-2$ can be partitioned into $u+v+w+x$. And the original point $A$ can be taken in $2n$ ways, but each arrangement will then be counted 4 times over. Therefore the no. of ways of drawing the $n$ chords is the coeff. of $x^{n-2}$ in

$$(1 + f_1 x + f_2 x^2 + \ldots)^4 . n \div 2 = n(f_{n+1} - 2f_n) \div 2.$$

Thence proceeding as in the last question we find the

$$\text{Chance} = 2^n n(n-1) \div \underline{|n+2}.$$

**677.** Make an equilateral triangle $ABC$ of altitude equal to the given straight line. Bisect the sides in $A'$, $B'$, $C'$. The trilinear coordinates of any point $P$ within the triangle will

678. If a straight line be broken at random into three parts the odds are about 10 : 9 in favour of the square on the *middle* part being greater than the rectangle contained by the other two parts, and about 7 : 5 in favour of the square on the *mean* part being greater than the rectangle contained by the other two parts.

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represent segments into which the line may be divided. (See at the end of the volume the note on Qn. 632.) If no two are to be greater than the third $P$ must lie within $\triangle A'B'C'$. This area is one-fourth of the whole. $\therefore$ Chance $= \frac{1}{4}$. Odds are 3 to 1 against.

(ii) Let the perpendicular from $P$ on $BC$ represent the shorter of the two parts into which the line is first divided. $P$ must therefore lie on the area $BC'B'C$. But the favourable area is $A'B'C'$ as before.

$$\therefore \text{ Chance} = \tfrac{1}{3}. \quad \text{Odds are 2 to 1 against.}$$

**678.** Proceeding as in Qn. 677, the points fulfilling the required condition are bounded by the curve $\beta^2 = \alpha\gamma$. But this is the circle touching $BA$, $BC$ at $A$ and $C$. $\therefore$ The chance that $\beta^2 < \alpha\gamma$ is the ratio of the segment on $AC$ to the triangle $ABC$

$$= \frac{4\pi\sqrt{3} - 9}{27} = \frac{12{\cdot}7656}{27}.$$

Hence the odds in favour of the square being greater than the rectangle are

$$14{\cdot}2344 : 12{\cdot}7656 = 10 : 9 \text{ nearly.}$$

(ii) If $\beta$ be the *mean* part, i.e. intermediate in magnitude between $\alpha$ and $\gamma$, all possible fractures will be represented by points on the triangles $AOC'$ and $COA'$, ($AA'$, $CC'$ intersecting in $O$).

The points fulfilling the required condition are bounded by the same circle as before, and this circle passes through $O$.

679. $A$ is to have the larger and $B$ the smaller of two parcels into which $2n$ things, (or $2n+1$ things) are divided at random. Find the ratio of the expectations (i) when the things are indifferent and all possible numbers (zero included) in either parcel are equally likely; and (ii) when the things are all different and every possible selection in either parcel is equally likely.

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$\therefore$ The chance that $\beta^2 < \alpha\gamma$ is the ratio of the segments on $OA$, $OC$ to the triangles $AOC'$, $COA'$,

$$= \frac{4\pi\sqrt{3} - 18}{9} = \frac{3{\cdot}7656}{9}.$$

Hence the odds in favour of the square being greater than the rectangle are $5{\cdot}2344 : 3{\cdot}7656 = 7 : 5$ nearly.

**679.** When the things are indifferent $A$ is certain of $n$ things out of $2n$, and he has an equal expectation with $B$ of the remaining $n$ things. Therefore $A$'s expectation is $\frac{3}{2}n$ and $B$'s is $\frac{1}{2}n$. Ratio $3 : 1$.

Out of $2n+1$ things $A$ is certain of $n+1$, and he has an equal expectation with $B$ of the remaining $n$ things. Their respective expectations are therefore $\frac{3}{2}n+1$ and $\frac{1}{2}n$. Ratio $3n+2 : n$.

(ii) When the things are all different the no. of ways in which $2n$ things can be divided so that $B$ has not more than $n$ is

$$C_0^{2n} + C_1^{2n} + C_2^{2n} + \ldots + C_n^{2n} = \tfrac{1}{2}(2^{2n} + Y_n),$$

$B$'s expectation is therefore

$$\Sigma_0^n \{xC_x^{2n}\} \div \tfrac{1}{2}(2^{2n} + Y_n) = n2^{2n} \div (2^{2n} + Y_n),$$

$A$'s expectation is therefore

$$n(2^{2n} + 2Y_n) \div (2^{2n} + Y_n).$$

$$\text{Ratio} = (2^{2n-1} + Y_n) : 2^{2n-1}.$$

Similarly the no. of ways in which $2n+1$ things can be divided so that $B$ has not more than $n$ is $2^{2n}$, and $B$'s expectation is

$$\Sigma_0^n \{xC_x^{2n+1}\} \div 2^{2n} = \tfrac{1}{2}(2n+1)(2^{2n} - Y_n) \div 2^{2n},$$

$A$'s expectation is therefore

$$\tfrac{1}{2}(2n+1)(2^{2n} + Y_n) \div 2^{2n}.$$

$$\text{Ratio} = (2^{2n} + Y_n) : (2^{2n} - Y_n).$$

680. A stick is broken at random into three parts. I am to take the largest piece, my wife the next larger, and my child the smallest. Shew that our expectations are as 11 : 5 : 2.

681. A lump of dough is divided at random into three parts, and each of these is made into a circular cake, all the cakes equally thick. Shew that the chance that the diameters will make the sides of a triangle is $\pi \div 3\sqrt{3}$.

682. The value of a rough diamond varying as the square of its weight, shew that if a diamond of value $V$ be broken at random into three parts, the

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**680.** With the construction of Qn. 677, let $O$ be the intersection of $AA'$, $BB'$, $CC'$. Let the coordinate $\alpha$ of any point $P$ denote my share, $\beta$ the wife's share and $\gamma$ the child's share. Then since $\alpha > \beta > \gamma$, $P$ must lie on the area $AOC'$. Our respective expectations are the average values of $\alpha$, $\beta$, $\gamma$ while $P$ moves over this area. But these must be the coordinates of the centre of gravity of $AOC'$. Now the coordinates of $O$ are $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$, and of the middle point of $AC'$ $\frac{3}{4}, \frac{1}{4}, 0$. Hence the coordinates of the C. G. are proportional to 11 : 5 : 2. These are therefore the ratios of our expectations.

**681.** Let the altitude of the same triangle represent the volume of the dough, so that the coordinates of $P$ represent the volumes of the three cakes. For the prescribed conditions we must have

$$\sqrt{\alpha} + \sqrt{\beta} > \sqrt{\gamma}, \quad \sqrt{\gamma} + \sqrt{\alpha} > \sqrt{\beta}, \quad \sqrt{\beta} + \sqrt{\gamma} > \sqrt{\alpha}.$$

The limiting positions are therefore defined by the equation

$$\sqrt{\alpha} \pm \sqrt{\beta} \pm \sqrt{\gamma} = 0,$$

which represents the circle inscribed in the triangle. Thus we have possible positions of $P$ over the whole triangle, favourable positions over the inscribed circle.

$$\text{Chance} = \text{ratio of areas} = \pi \div 3\sqrt{3}.$$

**682.** With the same construction, the height of the triangle representing the weight of the unbroken diamond which we may value of the three parts together lies between $\frac{1}{3}V$ and $V$; the most likely value is $\frac{1}{2}V$; and the chance that the value is less than $\mu V$ is

$$\frac{\pi}{3\sqrt{3}}(6\mu-2), \text{ or } \frac{\pi-3(\theta-\cos\theta\sin\theta)}{3\sqrt{3}}(6\mu-2),$$

according as $\mu<\frac{1}{2}$ or $\mu>\frac{1}{2}=\frac{1}{2}+\frac{1}{6}\tan^2\theta$, suppose.

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take as unity, the coordinates of $P$ will represent the weights of the several portions, so that

$$\alpha+\beta+\gamma=1.$$

If the value of the broken parts be $\mu V$ we must have

$$\alpha^2+\beta^2+\gamma^2=\mu.$$

This defines the boundary of the favourable area, and it represents a circle intercepting a length $\frac{2}{3}\sqrt{6\mu-2}$ on the line $\beta=\gamma$. This must be a diameter, and therefore the area is $(6\mu-2)\pi\div 9$. If $\mu<\frac{1}{2}$ the circle lies entirely within the triangle, and we have

$$\text{Chance}=\frac{\odot}{\triangle}=\frac{\pi(6\mu-2)}{3\sqrt{3}}.$$

If however $\mu>\frac{1}{2}$ the circle cuts the triangle, and the favourable area is only that which is common to circle and triangle. The intercept on each side is $2\sqrt{2\mu-1}\div\sqrt{3}$. The area of the circle (say $\pi r^2$) has to be reduced by the area of the three external segments each $=r^2(\theta-\sin\theta\cos\theta)$ where $\mu=\frac{1}{2}+\frac{1}{6}\tan^2\theta$. Therefore the chance is

$$(\pi-3\theta+3\sin\theta\cos\theta)(6\mu-2)\div 3\sqrt{3}.$$

The least possible value of $\mu$ is obtained when $6\mu-2=0$, i.e. $\mu=\frac{1}{3}$. This is when the diamond is broken into three equal parts. The circumference of the circle is proportionate to the chance of each particular value of $\mu$, and it goes on increasing till it becomes the inscribed circle when $\mu=\frac{1}{2}$. After this the chance is represented only by the arcs of the circle within the triangle, and it continually decreases until the circle becomes the circumscribed circle, when $\mu=1$ and the chance vanishes.

683. If three magnitudes are chosen at random between the limits $m$ and $n$, the odds are $2:1$ that their sum will lie between the limits $2m+n$ and $m+2n$.

684. A bag contains $n$ coins, some of which are sovereigns and the rest worthless imitations, all numbers of each being equally likely; $r$ coins are drawn and found to be sovereigns. What is the expectation of the value of the remainder?

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**683.** Let $m=n+c$. Each magnitude then lies between $n$ and $n+c$ and we have to consider the chance of their sum lying between $3n+c$ and $3n+2c$. But we need only think of the excess of each magnitude above $n$. Each excess varies from 0 to $c$ and we must consider the chance of the sum of the three excesses lying between $c$ and $2c$.

Let the three excesses be represented by coordinates $x$, $y$, $z$ referred to rectangular axes in space. All the possible cases will be represented by points within the cube bounded by the planes of reference and the planes $x=c$, $y=c$, $z=c$; and all points within this cube will be equally likely. But the favourable points are bounded by the planes $x+y+z=c$ and $x+y+z=2c$ which cut off opposite pyramids each equal to a sixth of the cube. The favourable volume is therefore four sixths of the whole, or the odds are 2 to 1.

**684.** Originally there may have been

$$r,\ r+1,\ r+2,\ \dots \text{ or } n$$

sovereigns, and *à priori* all these cases are equally likely. The consequent chances of the observed event are as

$$C_{r,0} : C_{r,1} : C_{r,2} \dots : C_{r,n-r}$$

*à posteriori* the chances are as these numbers. And the values of the remainders in pounds are respectively

$$0,\ 1,\ 2\ \dots\ n-r.$$

Hence the expectation is

$$\frac{C_{r,1}+2C_{r,2}+\dots+(n-r)\,C_{r,n-r}}{1+C_{r,1}+C_{r,2}+\dots+C_{r,n-r}}=\frac{(r+1)\,C_{r+2,n-r-1}}{C_{r+1,n-r}}.$$

$$=(n-r)(r+1)\div(r+2).$$

685. A die of $p$ faces is being repeatedly thrown (i) until $r$ specified faces have all turned up or (ii) until $r$ different faces (unspecified) have turned up. Shew that the no. of throws which we must expect to make is (i) $pH_r$ and (ii) $p(H_p - H_{p-r})$ where $H_n$ denotes the sum to $n$ terms of the Harmonic progression $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$.

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**685.** (i) Let $E_r^p$ denote the no. of throws we must expect to make with a die of $p$ faces if we are to go on until every one of $r$ assigned faces has appeared. Then after the first throw the expectation is $E_r^p - 1$. But if the first throw has given one of the assigned faces the expectation must now be $E_{r-1}^p$, and if some other face has turned up the expectation must be $E_r^p$. But the chances of these two contingencies are as $r : p - r$;

$$\therefore\ rE_{r-1}^p + (p-r)\,E_r^p = p\,(E_r^p - 1),$$

whence $E_r^p - E_{r-1}^p = p \div r$.

Give $r$ the successive values 1, 2, 3 ... $r$ and add, noting that $E_1^p = p$ and $E_0^p = 0$, and we get $E_r^p = pH_r$.

(ii) Let $x$ be the no. of throws to be expected if we go on till $r$ different faces have appeared. Then by (i) we must expect to make $pH_{p-r}$ *more* throws in order that all the faces may appear. But again by (i) the expectation if we go on till all the faces have appeared is $pH_p$. Therefore $x + pH_{p-r} = pH_p$,

or $$x = p\,(H_p - H_{p-r}).$$

Second Solution. The results may be written down at sight on consideration of the rule (Prop. XLIX Cor.) that $n$ trials must be expected when the chance of success at each trial is $1 \div n$. We must therefore expect $p \div r$ trials to get one of the $r$ specified faces; then $p \div (r-1)$ trials to get another of them; then $p \div (r-2)$ trials to get a third, and so on for the $r$ faces. Hence the expectation (i) is $pH_r$. Similarly when the faces are unspecified one throw gives one face; we must expect $p \div (p-1)$ throws to get a different face; $p \div (p-2)$ throws to get a third face; and so on. Hence the expectation (ii) is $p\,(H_p - H_{p-r})$.

686. Shew that the no. of orders in which $3n$ letters, $n$ alike and $n$ alike and $n$ alike, can be written down without two letters alike coming together is

$$2\{C_{n+1}^{2n+1}+k_0k_1C_{n+1}^{2n}+k_1k_1C_{n+1}^{2n-1}+k_1k_2C_{n+1}^{2n-2}+\ldots \text{ to } n+1 \text{ terms}\}$$

where $k_r$ denotes $C_r^{n-1}$.

687. If $n$ red beads, $n$ white beads, and $n$ black beads be strung at random to make a necklace, the chance that no two beads of the same colour come together is

$$6\{C_{n-1}^{2n-1}+k_1k_1C_{n-1}^{2n-3}+k_2k_2C_{n-1}^{2n-5}+\&c.\}\,(\underline{|n})^3\div\underline{|3n}.$$

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**686.** Let the letters be $\alpha^n\beta^n\gamma^n$. First let $\alpha^n\beta^n$ be written down in any order. Suppose there are $x$ contacts of $\alpha$ with $\alpha$ or $\beta$ with $\beta$: then there will be $2n-x-1$ contacts of $\alpha$ with $\beta$. Now consider the $\gamma$'s; $x$ of them must be interposed in the contacts of $\alpha$ with $\alpha$ and $\beta$ with $\beta$. The remaining $n-x$ must be placed singly in $n-x$ of the other contacts or at the ends of the row. Choice $=C_{n-x}^{2n-x+1}=C_{n+1}^{2n-x+1}$. But by Qns. 193, 194 the no. of ways in which the $\alpha^n\beta^n$ can be arranged with $2n-x-1$ contacts of $\alpha$ with $\beta$ is $2k_yk_y$ if $x=2y$, or $2k_yk_{y-1}$ if $x=2y-1$. We must give $x$ all values from 0 to $n$ (for if there were more than $n$ contacts of $\alpha$ with $\alpha$ or $\beta$ with $\beta$ we should not have enough of $\gamma$'s to separate them). Hence the total no. of arrangements is

$$2\{k_0k_0C_{n+1}^{2n+1}+k_0k_1C_{n+1}^{2n}+k_1k_1C_{n+1}^{2n-1}+\&c. \text{ to } n+1 \text{ terms}\}.$$

**687.** Regard one of the $\alpha$'s as an initial point. The ring can be completed in $\underline{|3n-1}\div\underline{|n}\,\underline{|n}\,\underline{|n-1}=\underline{|3n}\div 3(\underline{|n})^3$ ways. First make a ring of $n$ $\alpha$'s. Choose $n-x$ of the intervals (Choice $=C_x^n$) and into them distribute $n$ $\beta$'s with no blank lots (Choice $=C_x^{n-1}$). There are now $2n-2x$ contacts of $\alpha$, $\beta$, and therefore $2x$ contacts of $\alpha$, $\alpha$ or $\beta$, $\beta$. Into each of these place $\gamma$. There are then $n-2x$ left, and for them choose $n-2x$ out of the $2n-2x$ contacts of $\alpha$ and $\beta$. (Choice $=C_n^{2n-2x}$.) Thus we have

$$C_x^nC_x^{n-1}C_n^{2n-2x}=2k_xk_xC_{n-1}^{2n-2x-1},$$

favourable arrangements in the case when the $\alpha$'s and $\beta$'s without

688. If a coin be repeatedly thrown, the chance that heads and tails are equal at the $2n$th throw is $Y_n \div 4^n$, and the chance that they are equal for the first time at the $2n$th throw is $Y_n \div 4^n (2n-1)$.

689. The chance that in $2n$ throws the heads and tails are never equal is

$$Y_n \div 4^n.$$

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the $\gamma$'s form $2n-2x$ sequences. But $x$ can have the values 0, 1, 2, 3, ... subject to the limit that $n-2x$ is positive or zero. Hence the total number of arrangements is

$$2\{C_{n-1}^{2x-1} + k_1k_1\, C_{n-1}^{2x-3} + k_2k_2\, C_{n-1}^{2x-5} + \&c.\},$$

and the chance is as stated in the question.

**688.** The coin being thrown $2n$ times can fall in $2^{2n}$ ways. But $n$ heads and $n$ tails can be arranged in $Y_n$ orders. Hence chance that heads and tails are equal is $Y_n \div 2^{2n}$.

(ii) If equality occurs for the first time at the $2n$th throw, if head fall first we must then have $n-1$ heads and $n-1$ tails, in which the tails never exceed the heads; and then one tail. By Prop. XXXVI the no. of ways is $Y_{n-1} \div n$. And there will be an equal no. of ways beginning with tail. Hence the total no. is $2Y_{n-1} \div n = Y_n \div (2n-1)$ and the chance is $Y_n \div 2^{2n}(2n-1)$.

**689.** The no. of ways in which equality has occurred in a series of $2n$ throws can be analysed according to the point at which equality *first* occurred.

In virtue of Qn. 688 there are

$$2^{2n-2x} \times 2Y_{x-1} \div x$$

ways in which equality first occurred at the $2x$th throw. Give $x$ the values 1, 2, 3 ... $n$ and add. We thus get the no. of ineligible ways, viz.:

$$2\left\{4^{n-1} + \tfrac{1}{2}Y_1 4^{n-2} + \tfrac{1}{3}Y_2 4^{n-3} + \dots + \frac{1}{n}Y_{n-1}\right\}$$

$$= \text{coeff. of } x^n \text{ in the product of}$$

$$1 - \sqrt{1-4x} = 2\{x + \tfrac{1}{2}Y_1x^2 + \tfrac{1}{3}Y_2x^3 + \dots\},$$

690. The chance that in $2n$ throws the heads and tails have been equal once and once only is $Y_n \div 4^n$.

691. The chance that the heads and tails are equal for the $p$th time at the $2n$th throw is $pR^n_{n-p} \div n2^{2n-p}$.

---

and

$$(1-4x)^{-1} = 1 + 4x + (4x)^2 + (4x)^3 + \dots .$$

But this coefficient is $4^n - Y_n$. Hence there are $4^n - Y_n$ ineligible ways and therefore $Y_n$ eligible ways, i.e. ways in which equality never occurs. Hence Chance $= Y_n \div 2^{2n}$.

**690.** Suppose equality occurs after $2x$ throws and at no other time. Then as in Qn. 688 the first $2x$ throws can be made in $2Y_{x-1} \div x$ ways, and by Qn. 689 the remaining $2n - 2x$ throws can be made in $Y_{n-x}$ ways. Hence the whole series can occur in $2Y_{x-1}Y_{n-x} \div x$ ways. And $x$ must have all values from 1 to $n$.

But $\Sigma\{2Y_{x-1}Y_{n-x} \div x\}$ is the coefficient of $x^n$ in the product of

$$1 - \sqrt{1-4x} = 2\{x + \tfrac{1}{2}Y_1x^2 + \tfrac{1}{3}Y_2x^3 + \dots\},$$

and

$$(1-4x)^{-\frac{1}{2}} = 1 + Y_1x + Y_2x^2 + Y_3x^3 + \dots$$

$$\therefore\ \Sigma\{2Y_{x-1}Y_{n-x} \div x\} = Y_n,$$

and the chance is $Y_n \div 4^n$.

**691.** Suppose that equality occurs at intervals of $2\alpha, 2\beta, 2\gamma, \dots$ throws, where the no. of the variables $\alpha, \beta, \gamma \dots$ is $p$ and

$$\alpha + \beta + \gamma + \dots = n.$$

Subject to their sum being $n$, the variables $\alpha, \beta, \gamma \dots$ can severally take any values $1, 2, 3 \dots$ .

As in Qn. 688 the no. of ways in which the throws can occur is

$$2^p Y_{\alpha-1}Y_{\beta-1}Y_{\gamma-1} \dots \div \alpha\beta\gamma \dots .$$

But this is the coefficient of $x^n$ in the expansion of

$$2^p\{x + \tfrac{1}{2}Y_1x^2 + \tfrac{1}{3}Y_2x^2 + \dots\}^p = \{1 - \sqrt{1-4x}\}^p.$$

Hence the no. of ways is $2^p pR^n_{n-p} \div n$ and the chance is

$$pR^n_{n-p} \div n2^{2n-p}.$$

692. The chance that in $2n$ throws the heads and tails shall have been equal exactly $p$ times is $R_{n-p}^{n+1} \div 2^{2n-p}$.

693. A die of $k$ faces is thrown $nk$ times, and $n$ aces have turned up. Shew that the chance that the ratio (one ace to $k-1$ not-aces) has never previously occurred in the course of the play is $(k-1) \div (nk-1)$.

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**692.** Let $N(p)$ denote the number of orders in which $2n$ throws can be made so that the heads and tails may be equal exactly $p$ times.

Comparing Qns. 689, 690 we see that if the $p$th equality occur at the $2x$th throw (where $x < n$) the no. of ways of concluding the play with no further equality or with one more equality are the same.

Hence $N(p)$ and $N(p+1)$ only differ by the no. of ways in which the $p$th equality can occur at the $2n$th throw. And by Qn. 691, this number is $pR_{n-p}^{n}\, 2^p \div n$.

Hence $$N(p+1) - N(p) = pR_{n-p}^{n} 2^p \div n.$$

Now by Qns. 689, 690, we have

$$N(0) = 2R_n^n,$$
$$N(1) = 2R_n^n = 2R_{n-1}^{n+1}.$$
$$\therefore\ N(2) = 2R_n^n - 2R_{n-1}^n \div n = 4R_{n-2}^{n+1}.$$

So $$N(3) = 8R_{n-3}^{n+1},$$
$$N(4) = 16R_{n-4}^{n+1},$$
$$\dots$$

and generally $$N(p) = 2^p R_{n-p}^{n+1}.$$
$$\therefore\ \text{Chance} = R_{n-p}^{n+1} \div 2^{2n-p}.$$

**693.** Let $C_x$ (written for shortness instead of $C_x^{kx}$) denote the no. of ways in which the ratio is reached at the $xk$th throw and let $K_x$ denote the no. of ways in which it is reached *for the first time* at the $xk$th throw.

Then if we analyse the ways included in $C_n$ according to the throw at which the ratio first appeared, we have

$$C_n = \Sigma_1^n \{K_x C_{n-x}\}.$$

694. The chance that the ratio (one ace to $k-1$ not-aces) should occur at the $nk$th throw and not before is

$$C_{n-1}^{nk-2}(k-1)^{nk-n} \div nk^{nk-1}.$$

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Hence, giving $n$ the values 1, 2, 3 ... in succession, we get

$$C_1 = K_1,$$

$$C_2 = K_2 + C_1K_1,$$

$$C_3 = K_3 + C_1K_2 + C_2K_1,$$

....................

$$C_n = K_n + C_1K_{n-1} + \ldots + C_{n-1}K_1;$$

$n$ simple equations to determine $K_1, K_2, \ldots K_n$. Solving we get

$$\frac{K_1}{C_1} = 1, \quad \frac{K_2}{C_2} = \frac{k-1}{2k-1}, \quad \frac{K_3}{C_3} = \frac{k-1}{3k-1}, \text{ \&c.}$$

and generally

$$\frac{K_n}{C_n} = \frac{k-1}{nk-1}.$$

This is the chance required.

**694.** The chance that in $nk$ throws there should be exactly $n$ aces is $(k-1)^{nk-n}C_n^{nk} \div k^{nk}$.

And then (by Qn. 693) the chance that the ratio should not have occurred previously is $(k-1) \div (nk-1)$.

Hence the required chance is

$$\frac{(k-1)^{nk-n+1}C_n^{nk}}{(nk-1)\,k^{nk}} = \frac{(k-1)^{nk-n}C_{n-1}^{nk-2}}{nk^{nk-1}}.$$

*In the Questions which follow a knowledge of the Integral Calculus is assumed.*

695. If *à priori* a man's skill (i.e. the chance of his succeeding in a given experiment) is equally likely to have any value from 0 to 1, shew that after he has succeeded $p$ times in $n$ trials his skill must be estimated at

$$(p+1) \div (n+2).$$

696. If $A$ be a fixed point on the circumference of a circle, and $P$, $Q$ random points within the circle, find $\mathfrak{E}(AP)$ and $\mathfrak{E}(PQ)$.

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**695.** If the skill be $x$, the chance of $p$ successes in $n$ trials is $C^n_p x^p (1-x)^{n-p}$.

Hence after the observed event the chance that the skill was $x$ is proportional to $x^p (1-x)^{n-p}$, all values of $x$ from 0 to 1 being equally likely.

$\therefore$ Chance of success at the next trial is

$$\int_0^1 x^{p+1} (1-x)^{n-p} dx \div \int_0^1 x^p (1-x)^{n-p} dx.$$

In Eulerian Integrals this becomes

$$\begin{aligned} & B(p+2,\ n-p+1) \div B(p+1,\ n-p+1) \\ & = \Gamma(p+2)\,\Gamma(n+2) \div \Gamma(n+3)\,\Gamma(p+1) \\ & = (p+1) \div (n+2). \end{aligned}$$

But the chance of success at the next trial is the *à posteriori* estimate of the man's skill.

$$\therefore \text{ Skill} = (p+1) \div (n+2).$$

**696.** (i) Let $AR = r$ be a chord of the circle making an angle $\theta$ with the diameter $AB$. Let $R'$ be a point on the circumference indefinitely near to $R$.

The chance of $P$ lying on $ARR'$ is $r^2 d\theta \div \pi a^2$, $\theta$ varying from 0 to $\frac{1}{2}\pi$. But if $P$ lie on $ARR'$ the expectation of $AP$ is $\frac{2}{3}r$. (Qn. 658) Hence

$$\mathcal{E}(AP) = \int_0^{\frac{1}{2}\pi} \tfrac{2}{3} r^3 d\theta \div \pi a^2 = \frac{16a}{3\pi} \int \cos^3 \theta d\theta = \frac{32a}{9\pi}.$$

697. Find also $\mathcal{E}(AP^{2n})$ and $\mathcal{E}(AP^{2n-1})$. Also $\mathcal{E}(PQ^{2n})$ and $\mathcal{E}(PQ^{2n-1})$.

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(ii) Let $P$ be at a distance $x$ from the centre (Chance $\propto xdx$) and let $Q$ be nearer to the centre (chance $= x^2 \div a^2$).

Then by (i) $\mathcal{E}(PQ) = 32x \div 9\pi$. Hence, generally,

$$\mathcal{E}(PQ) = \frac{32}{9\pi}\int_0^a x^4 dx \div \int_0^a x^3 dx = \frac{128a}{45\pi}.$$

**697.** (i) If $P$ lie on the elementary area $ARR'$ (by Qn. 658, Cor.), the expectation of $AP^m$ is $2r^m \div (m+2)$. Hence, following the steps of Qn. 696, we have

$$\mathcal{E}(AP^m) = \int_0^{\frac{\pi}{2}} \frac{2r^{m+2}d\theta}{(m+2)\pi a^2} = \frac{2^{m+3}a^m}{(m+2)\pi}\int \cos^{m+2}\theta d\theta = \gamma a^m \text{ suppose.}$$

And (ii)

$$\mathcal{E}(PQ^m) = \gamma\int_0^a x^{m+3}dx \div \int_0^a x^3 dx = \frac{4\gamma a^m}{m+4}.$$

By the Integral Calculus the value of $\gamma$ is known: and we obtain

$$\mathcal{E}(AP^m) = \frac{2^{m+2}a^m}{(m+2)\sqrt{\pi}} \cdot \frac{\Gamma\{(m+3)\div 2\}}{\Gamma\{(m+4)\div 2\}},$$

and

$$\mathcal{E}(PQ^m) = \frac{2^{m+3}a^m}{(m+2)\sqrt{\pi}} \cdot \frac{\Gamma\{(m+3)\div 2\}}{\Gamma\{(m+6)\div 2\}}.$$

Or if $m$ be an integer we can use the simpler forms for the integral of $\sin^m\theta$, viz.

$\pi Y_n \div 2^{2n+1}$ when $m = 2n$,

and $2^{2n} \div Y_n(2n+1)$ when $m = 2n+1$.

Thus we get

$$\mathcal{E}(AP^{2n}) = a^{2n}Y_{n+1} \div 2(n+1),$$
$$\mathcal{E}(PQ^{2n}) = a^{2n}Y_{n+1} \div (n+1)(n+2).$$

And

$$\mathcal{E}(AP^{2n-1}) = 32(4a)^{2n-1} \div \pi(n+1)(2n+1)Y_{n+1},$$
$$\mathcal{E}(PQ^{2n-1}) = 128(4a)^{2n-1} \div \pi(n+1)(2n+1)(2n+3)Y_{n+1}.$$

698. Two arrows are sticking on a circular target; what is the chance that their distance is greater than the radius of the target?

699. An indefinitely large plane area is ruled with parallel equidistant lines. An ellipse whose major axis is less than the distance between the lines is thrown down on the area. Shew that the chance of the ellipse falling on one of the lines is the ratio of its perimeter to the perimeter of the circle which touches two successive lines.

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**698.** Let $r$ be the radius of the target: and let the first arrow be at a distance $2r\cos\theta$ from the centre. The chance of this $\propto \cos\theta$, and $\theta$ varies from $\frac{1}{2}\pi$ to $\frac{1}{3}\pi$.

With centre at the first arrow draw another circle of radius $r$. Then the chance that the second arrow should be at a distance from the first less than $r$ is represented by

$$\frac{\text{common area of } \odot\text{'s}}{\text{area of either } \odot} = \frac{\theta - \cos\theta\sin\theta}{\frac{1}{2}\pi}.$$

Therefore the required chance

$$= 1 - \frac{\int(\theta\cos\theta - \cos^2\theta\sin\theta)\,d\theta}{\frac{1}{2}\pi\int\cos\theta d\theta} = 1 - \left[\frac{\theta\sin\theta + \cos\theta + \frac{1}{3}\cos^3\theta}{\frac{1}{2}\pi\sin\theta}\right]_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}$$

$$= \frac{2+\sqrt{3}}{6}\left(\frac{13}{\pi} - 2\sqrt{3}\right) = \tfrac{35}{88} \text{ nearly.}$$

**699.** Let $p$ be the perpendicular from the centre of the ellipse upon a tangent parallel to the ruled lines; and let $\theta$ be the angle which this perpendicular makes with the major axis of the ellipse. All values of $\theta$ are equally likely, and for any value of $\theta$ the chance of the disc lying across a line is $p \div a$ where $2a$ is the distance between consecutive ruled lines. Hence integrating between $\theta = 0$ and $\theta = \frac{1}{2}\pi$ the whole chance is

$$\int p d\theta \div \int a d\theta = 2p d\theta \div a\pi = 2s \div a\pi,$$

(Todhunter's *Int. Calc.*, Art. 90) where $s$ is the length of a quadrant of the ellipse. That is, the chance is the ratio of the perimeter of the ellipse to the circumference of the circle of radius $a$.

700. A fixed point $O$ within a triangle is joined to a random point $P$ within the triangle. Two other random points $H$, $K$ are taken within the triangle. Find the chance that $H$, $K$ are on opposite sides of $OP$.

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**700.** Let $ABC$ be the triangle, and let $\alpha$, $\beta$, $\gamma$ be the ratios of the triangles $OBC$, $OCA$, $OAB$ to the whole triangle $ABC$; so that $\alpha+\beta+\gamma=1$.

Let $AO$, $BO$, $CO$ be produced to meet the opposite sides in $A'$, $B'$, $C'$.

The triangle is now divided into 6 compartments. Suppose $P$ falls in the compartment $BOA'$; the chance of this is the ratio of the areas

$$BOA' : ABC = \alpha\gamma : \beta+\gamma.$$

Let $OP$ meet $BA'$, $AB'$ in $M$, $N$; and let $CM=x$, $CN=y$. Then $x$, $y$ are connected by the equation

$$\alpha bx + \beta ay = xy.$$

Write $x=\beta az$. $\therefore\ y = \alpha bz \div (z-1)$.

By Qn. 659, since $P$ is equally likely to fall anywhere in $OA'B$, $M$ is equally likely to fall anywhere in $A'B$; i.e. $x$ varies equably between the values $a\beta \div (\beta+\gamma)$ and $a$.

Now the chance that $H$, $K$ should be on opposite sides of $OP$ is

$$2\frac{xy}{ab}\left(1-\frac{xy}{ab}\right) = \frac{2\alpha\beta z^2}{z-1} - \frac{2\alpha^2\beta^2 z^4}{(z-1)^2} = Z,$$

and for the whole compartment $OA'B$ we must integrate between the limits $x = a\beta \div (\beta+\gamma)$ and $x=a$, i.e. between the limits $z=\frac{1}{\beta}$ and $z=\frac{1}{\beta+\gamma}=\frac{1}{1-\alpha}$.

$$\therefore\ \text{Chance} = \frac{\alpha\gamma}{\beta+\gamma}\int Zdx \div \int dx = \frac{\alpha\gamma}{\beta+\gamma}\int Zdz \div \int dz = \alpha\beta\int Zdz.$$

By interchanging $\alpha$ and $\beta$ we find the chance for the compartment $OB'A$ to be the same integral between the limits

$z=\frac{1}{\alpha}$ and $z=\frac{1}{1-\beta}$. Hence for these two compartments together the chance is

$$2\alpha^2\beta^2\int\frac{z^2dz}{z-1}-2\alpha^3\beta^3\int\frac{z^4dz}{(z-1)^2},$$

the integration being taken between the limits $\frac{1}{\beta}$ and $\frac{1}{1-\alpha}$ and also between $\frac{1}{\alpha}$ and $\frac{1}{1-\beta}$.

Now $$\int\frac{z^2dz}{z-1}=\frac{z^2}{2}+z+\log(z-1),$$

and $$\int\frac{z^4dz}{(z-1)^2}=\frac{z^3}{3}+z^2+3z-\frac{1}{z-1}+4\log(z-1).$$

Hence writing $A_r$ for $\alpha^{-r}-(1-\alpha)^{-r}$ and similarly $B_r$ and $C_r$, the chance becomes

$$2\alpha^2\beta^2\{\tfrac{1}{2}(A_2+B_2)+(A_1+B_1)+2\log(1+\gamma/\alpha\beta)\}$$
$$-2\alpha^3\beta^3\{\tfrac{1}{3}(A_3+B_3)+(A_2+B_2)+4(A_1+B_1)+8\log(1+\gamma/\alpha\beta)\},$$

with two similar expressions for the other two compartments.

The whole chance will therefore be the sum of a logarithmic and a non-logarithmic function, the former being

$$4\alpha^2\beta^2\gamma^2\left\{\frac{1-4\beta\gamma}{\alpha^2}\log\left(1+\frac{\alpha}{\beta\gamma}\right)+\frac{1-4\gamma\alpha}{\beta^2}\log\left(1+\frac{\beta}{\alpha\gamma}\right)\right.$$
$$\left.+\frac{1-4\alpha\beta}{\gamma^2}\log\left(1+\frac{\gamma}{\alpha\beta}\right)\right\},$$

and the latter being

$$2\alpha^2(\beta^2+\gamma^2)(\tfrac{1}{2}A_2+A_1)-2\alpha^3(\beta^3+\gamma^3)(\tfrac{1}{3}A_3+A_2+4A_1),$$

and two similar functions of $B$ and $C$.

The last expression reduces to

$$\tfrac{1}{3}-10\alpha^2+\tfrac{100}{3}\alpha^3-40\alpha^4+16\alpha^5+24\alpha\beta\gamma(\alpha-2\alpha^2),$$

and adding the two similar functions of $\beta$ and $\gamma$ we obtain

$$\tfrac{1}{3}+4\alpha\beta\gamma(1-2\alpha^2-2\beta^2-2\gamma^2).$$

This added to the logarithmic function already obtained will give the chance required.

NOTE I. If $O$ be the centroid of the triangle the chance becomes $(20 \log 4 + 93) \div 243 = \cdot 4969$.

NOTE II. If $O$ be on the boundary of the triangle we have $\alpha = 0$ or $\beta = 0$ or $\gamma = 0$, and the chance reduces to $\frac{1}{3}$, which agrees with the result of Qn. 665.

# NOTES.

**Page ix.** When we simply desire to get a rough estimate of the value of some chance the simple approximation

$$Y_n = 4^n \div \sqrt{\pi n},$$

will often serve as well as the more general approximation (when $m$ and $n$ are both very large)

$$C_{m,n} = (m+n)^{m+n+\frac{1}{2}} \div \sqrt{2\pi}\, m^{m+\frac{1}{2}} n^{n+\frac{1}{2}},$$

which involves the somewhat abstruse limit

$$\lfloor n = e^{-n} n^n \sqrt{2n\pi}$$

when $n$ is indefinitely great.

The $Y$ notation enables us to express in a concise form the values of the definite integral of $\sin^n x$.
We may write

$$\int_0^{\frac{1}{2}\pi} \sin^n x dx = \frac{\pi}{2}\, Y_{\frac{1}{2}n} \div 2^n \qquad (n \text{ even})$$

$$= 2^{n+1} \div (n+1)\, Y_{\frac{1}{2}(n+1)} \qquad (n \text{ odd}),$$

whence, whether $n$ be even or odd,

$$\int_0^{\frac{1}{2}\pi} \sin^n x dx \text{ lies between } \sqrt{\frac{\pi}{2n}} \text{ and } \sqrt{\frac{\pi}{2(n+1)}}.$$

**Page xiii.** The example given in Art. 9 illustrates a mistake which beginners are apt to fall into. Let there be 8 chairs arranged in order, on which a party of 5 persons sit down at random, leaving 3 chairs unoccupied. If I am one of the 5 persons I am equally likely to sit 1st, 2nd, 3rd, 4th or 5th in the party.

The chance that I am (say) 3rd is therefore $\frac{1}{5}$. But I am also equally likely to take any of the 8 chairs. The chance that I take any particular one, say the 4th, is $\frac{1}{8}$. Now the tyro is inclined to multiply these chances together and to assume that the chance of my being 3rd in order and at the same time occupying the 4th chair is 1/40. But this is wrong. The multiplication of chances is based on the principle defined in *Choice*, Rule I., and in the enunciation of that rule the words " when it is done in any way " have to be carefully observed. In the case before us when my rank among the 5 persons is determined there is no longer an even chance of my taking any chair. The rule therefore will not apply. Compare Qn. 646.

**Pages xiv. to xvii.** Though $|\underline{n}$ has no meaning in elementary algebra except when $n$ is an integer, yet the quotient $|\underline{n+r} \div |\underline{n}$ is interpretable, being the product of $r$ factors, each of which has its proper meaning whatever $n$ may be, whether positive or negative, fractional or integral. So $P^n_r$, $C^n_r$, $R^n_r$ are all capable of interpretation provided $r$ be an integer.

The question then arises, will the summations registered in the tables hold good when $m$, $n$, $a$, $b$, &c. are fractional or negative?

A little discrimination must be exercised in answering this question. But it is plain that in the case of most frequent occurrence in which the formula of summation simply anticipates the result of the processes by which algebraical fractions or other functions are added together, the form of the result cannot be affected by the meaning of the symbols employed. If, for example (Table I. 29), we write

$$1 + 2a + 3a^2 + \ldots + na^{n-1} = \frac{1 - a^n(1 + n - an)}{1 - 2a + a^2},$$

the equation must be verified by multiplying each member by $1 - 2a + a^2$, and this process of verification will be identically the same whether $a$ represent an integer or not.

Again, if we take the case of Table I. 30, and multiply throughout by $\underline{|b} \div \underline{|a}$ we have

$$\Sigma\left\{\frac{\underline{|a+x}}{\underline{|a}} \div \frac{\underline{|b+x}}{\underline{|b}}\right\} = \frac{\dfrac{\underline{|a+n+1}}{\underline{|a}} \div \dfrac{\underline{|b+n}}{\underline{|b}} - b}{a-b+1}.$$

Every term in this equation is now interpretable, whether $a$ and $b$ be integers or not, and the summation only expresses the result of the algebraical process of bringing the $n+1$ fractions (which constitute the $n+1$ terms) to a common denominator and adding them together. This is independent of the meaning of $a$ and $b$. Therefore we may say that, whether $a$ and $b$ be integral or fractional, positive or negative, the summation holds good; so that to $n+1$ terms

$$1 + \frac{a+1}{b+1} + \frac{(a+1)(a+2)}{(b+1)(b+2)} + \frac{(a+1)(a+2)(a+3)}{(b+1)(b+2)(b+3)} + \&c.$$

$$= \left\{\frac{(a+1)(a+2)(a+3)\ldots(a+n+1)}{(b+1)(b+2)(b+3)\ldots(b+n)} - b\right\} \div (a-b+1).$$

Since $a$ and $b$ may be fractions we may write $a/c$ for $a+1$, and $b/c$ for $b+1$: so we get

$$1 + \frac{a}{b} + \frac{a(a+c)}{b(b+c)} + \frac{a(a+c)(a+2c)}{b(b+c)(b+2c)} + \&c. \text{ to } n+1 \text{ terms}$$

$$= \left\{\frac{a(a+c)(a+2c)\ldots(a+nc)}{b(b+c)(b+2c)\ldots(b+nc-c)} - b + c\right\} \div (a-b+c),$$

where $a$, $b$, $c$ may be fractional or negative.

**Page xxx. footnote.** Perhaps Laplace was not quite serious in this application of the rule. He gives 1826214 : 1 as the odds, by the rule of succession, that the sun will rise to-morrow, but he adds,

> Mais ce nombre est incomparablement plus fort pour celui qui connaissant par l'ensemble des phénomènes le principe régulateur des jours et des saisons, voit que rien dans le moment actuel ne peut en arrêter le cours. *Essai Philosophique*, p. 23 of Ed. of 1840.

**Page xxxi.** A man's skill in regard to a given experiment must have some value between 0 and 1; but it is quite arbitrary and most unreasonable to suppose that all values between these limits are equally likely. On the contrary, we can hardly conceive conditions under which a moderate skill would not be much more likely than either extreme; that is, if we are equally likely to meet with a man who succeeds 9 times out of 10 in a given experiment and another who succeeds once in 10 times, we should expect that there would be *many more* men who would succeed 4 or 5 or 6 times out of 10. The assumption is equivalent to the assumption that out of 11 millions of men each trying the experiment 10 times we should find a million failing every time, a million succeeding once only, a million succeeding twice, and so on.

Consider another magnitude which necessarily varies from 0 to 1. Let there be a bag containing a very large number $n$ of shot, and let some be taken out of the bag. If $x$ be the number taken out, the ratio $x/n$ may have any value from 0 to 1. If all quantities of shot are equally likely to be taken out, the case corresponds to the assumption which we are discussing, that all values of skill from 0 to 1 are equally likely. But if all combinations of shot are equally likely the case will correspond to what would seem to be a much more reasonable hypothesis. In this case the chance of the skill being of any value $x$ between 0 and 1 would vary as $C_{nx, n(1-x)}$, vanishing when $x=0$ or $x=1$, and attaining a maximum when $x=\frac{1}{2}$. Or if $y$ be the chance of the skill having the value $x$ we should depict $x$ and $y$ as the abscissa and ordinate of any point on a bell-shaped curve having as its base an intercept $=1$ on the axis of $x$.

The two sides of the bell are equal and symmetrical. But we can conceive very few experiments except purely mechanical ones (such as the drawing of balls from a bag and the like) in which such symmetry would correspond to the conditions of the case. Most experiments are capable of being classed as difficult or easy. In the former more men would succeed than fail, in the latter more would fail than succeed. If $y = \phi(x)$ express the chance that a man's skill should be $x$, the function must be such that $y$ will have positive values from $x = 0$ to $x = 1$, but it will not *generally* be the case that $\phi(x) = \phi(1 - x)$.

Whatever function $y$ may be of $x$ there must exist a corresponding law of succession (so-called). If the experiment has been observed to succeed $p$ times and to fail $q$ times, the expectation that it will succeed at the next trial is (Qn. 695)

$$\int_0^1 \phi(x)\, x^{p+1}(1-x)^q\, dx \div \int_0^1 \phi(x)\, x^p (1-x)^q\, dx.$$

A very reasonable hypothesis to assume is

$$\phi(x) \equiv x^a (1-x)^b,$$

where $a$ and $b$ are constants dependent on the nature of the experiment in question. If $a > b$ the experiment would be one in which success is more likely than failure. If $b > a$ the experiment is a more difficult one. In either case the most likely skill would be $a \div (a + b)$. The law of succession would take a form nearly as simple as that which is based upon the assumption that all degrees of skill were equally likely. After observing $p$ successes and $q$ failures, the expectation of success at the next trial is the same as if under the old law of succession one had observed $a + p$ successes and $b + q$ failures. That is, the odds in favour of success at the next trial would be $(a + p + 1) : (b + q + 1)$.

**172.** Proposed by Cayley in 1867. The proposer's elaborate solution is given in *Ed. Times, Reprint*, vii. 87, 88.

**284.** Published by Huygens in 1657.

**306.** The summation in the text is quite unnecessary. $A$ and $B$ being placed at random must divide the remaining $n - 2$ men into three rows of equal expectation. Hence the expectation of the number between $A$ and $B$ is $(n - 2) \div 3$.

**307.** This also is capable of a direct solution almost at sight.

Suppose $A$'s place fixed.

If $n$ be odd, whether $B$ be on the semicircumference to the right or to the left of $A$, he will divide the remaining $\frac{1}{2}(n-3)$ places of that semicircumference into two series of equal expectation.

$$\therefore \text{ Expectation} = \tfrac{1}{4}(n-3).$$

If $n$ be even, there is the chance $1 \div (n-1)$ that $B$ has the place diametrically opposite to $A$, and the chance $(n-2) \div (n-1)$ that some other man (say $C$) has it.

In the former case, $\frac{1}{2}(n-2)$ places intervene between $A$ and $B$, and we have the expectation $\frac{1}{2}(n-2) \div (n-1)$.

In the latter case, $B$ will divide the remaining $\frac{1}{2}(n-4)$ places, on either side, into two series of equal expectation; so we have the expectation $\frac{1}{4}(n-2)(n-4) \div (n-1)$.

Hence totally, when $n$ is even,

$$\text{Expectation} = \tfrac{1}{4}(n-2)^2 \div (n-1).$$

**463.** Mathematically we can make no distinction between truthfulness and sound judgment in stating the chance that a person makes a correct statement. It is convenient to say that his veracity is $\frac{3}{4}$ when we mean that on an average out of 4 statements which he makes 3 are correct. But the word veracity is used in a very wide sense and is applicable to such exercises of judgment as occur in Qn. 626.

When a man's veracity is given as a proper fraction it can only be applied in cases where there is a simple and direct issue "Yes" or "No," otherwise very fallacious results are likely to be attained. We may illustrate this by a consideration of the following adaptation of a classical problem:

A lottery of 10001 tickets has been drawn. $A$, whose veracity is ·99, says that No. $\alpha$ is the prize winner. $B$, whose veracity is ·97, says that No. $\beta$ has won the prize. Find the respective chances that $\alpha$, $\beta$, or some other ticket has won.

A hasty student would be very apt to solve the question as follows:

*A priori* the chances of $\alpha$, $\beta$, or some other are as $1 : 1 : 9999$. The consequent chances of the observed event (viz. that $A$ declares for $\alpha$ and $B$ for $\beta$) are as

$$\cdot 99 \times \cdot 03 \; : \; \cdot 97 \times \cdot 01 \; : \; \cdot 01 \times \cdot 03,$$

the terms of the ratios being the chances, (i) that $A$ has spoken truly and $B$ falsely, (ii) that $B$ has spoken truly and $A$ falsely, (iii) that both have spoken falsely.

Hence the *à posteriori* chances are proportional to

$$\cdot 0297 \; : \; \cdot 0097 \; : \; \cdot 0003 \times 9999.$$

The chances that $\alpha$ and $\beta$ have the prize are therefore respectively ·009772 and ·003191, the chance of any other assigned ticket being ·000098.

But this is all wrong. The chance that $A$ names the correct ticket is ·99, the chance that he names some wrong one is ·01, but this chance is distributed over 10000 tickets. The chance that he names any particular wrong one is ·000001. So the chance that $B$ names any particular wrong one is ·000003. The ratios of the "consequent chances" ought therefore to have been stated as

$$\cdot 99 \times \cdot 000003 \; : \; \cdot 97 \times \cdot 000001 \; : \; \cdot 000001 \times \cdot 000003$$

$$= \cdot 297 \; : \; \cdot 097 \; : \; \cdot 0000003,$$

and the *à posteriori* chances are therefore proportional to

$$\cdot 297 \; : \; \cdot 097 \; : \; \cdot 0029997.$$

Thus the chances that $\alpha$ and $\beta$ have the prize are respectively ·748 and ·244, whilst the chance of any other assigned ticket is only ·00000075.

**497.** This enables us to write down at sight the result of a problem by Laplace solved by integration in Todhunter's *History*, Art. 990. The problem is as follows:

Suppose there are $n$ candidates for an office and that an elector arranges them in order of merit: let $a$ denote the maximum merit (possible): required the mean value of the merit of a candidate whom the elector places $r$th on the list.

Take a line of length $a$. The merits of the candidates may be marked by $n$ random points on the line. These will divide the line into $n+1$ intervals of equal expectation, of which $n-r+1$ fill the space from zero to the $r$th candidate's mark. Hence the expectation is $a(n-r+1) \div (n+1)$.

**502, 503, 504, 506, &c.** It is necessary to distinguish very clearly between the number of times an experiment must be repeated in order that there may be an *even chance* of success, and the *average* number of times the experiment must be repeated (or the number of times we must expect to repeat it) in order to attain success. The former problem is often referred to as the problem of duration of play. It was discussed by Montmort in 1708, and by De Moivre in 1718. The latter problem is much more interesting and unique. It is fundamentally discussed in Choice and Chance, Prop. XLIX. To take the simplest case: if a coin is to be tossed and to turn up head, in *one* throw there is an "even chance" of success, but *two* throws are to be "expected" for every head turned up. So if it be required to throw ace with a die of $n$ faces there will be an even chance of success in $r$ throws where

$$r = \log 2 \div \{\log n - \log (n-1)\},$$

but the average number of throws to attain success is $n$.

The two results are very nearly coincident when the question refers to the drawing of a specified card or ball without replacing those that are drawn. For instance, if we are to draw cards one by one from a pack of 52 cards in the hope of drawing the ace of spades it is plain that we have an *even chance* of success in 26 drawings. It is equally plain that as the ace of spades divides the other 51 cards into two series of equal expectation we must expect on an average to draw $51/2 + 1 = 26\frac{1}{2}$ times.

If, however, the object is to draw *all* the spades we have an even chance of success in $x$ drawings where

$$\left(1-\frac{x}{40}\right)\left(1-\frac{x}{41}\right)\left(1-\frac{x}{42}\right)\dots\left(1-\frac{x}{52}\right)=\tfrac{1}{2},$$

whilst the average number of drawings is plainly $49\frac{3}{14}$.

**502, 503.** The methods of Qn. 685 might be adopted. Thus in Qn. 503 the first cut gives one suit. The chance of another suit at any trial is then 3/4. We must therefore expect 4/3 cuts to obtain a second suit. So we must expect 4/2 and 4/1 cuts respectively to obtain the third and fourth suits. Hence

$$\text{Expectation} = 1+\tfrac{4}{3}+\tfrac{4}{2}+\tfrac{4}{1}=8\tfrac{1}{3}.$$

**504.** The more general case may be solved as follows. Let the pack consist of four suits of $n$ cards each,

1 draw gives 1 suit; call it $\alpha$.

$4n-1$ cards remain of which $3n$ are not $\alpha$.

These $3n$ cards divide the remaining $n-1$ cards into $3n+1$ groups of equal expectation. Hence we must expect

$$\frac{n-1}{3n+1}+1=\frac{4n}{3n+1}$$

draws to give a second suit; call it $\beta$. There remain

$$12n^2 \div (3n+1)-1,$$

and of which $2n$ are *not* $\alpha$, $\beta$.

These $2n$ cards divide the rest into $2n+1$ groups of equal expectation. Hence we must expect

$$12n^2 \div (3n+1)(2n+1)$$

draws to give a third suit; call it $\gamma$. There remain

$$24n^3 \div (3n+1)(2n+1)-1$$

cards of which $n$ are *not* $\alpha$, $\beta$, $\gamma$.

These $n$ cards divide the rest into $n+1$ groups of equal expectation. Hence we must expect

$$24n^3 \div (3n+1)(2n+1)(n+1)$$

draws to give the fourth suit.

Hence altogether the number of cards which we expect to draw is

$$1 + \frac{4n}{3n+1} + \frac{12n^2}{(3n+1)(2n+1)} + \frac{24n^3}{(3n+1)(2n+1)(n+1)}.$$

But the still more general case may be solved, thus :—

Cards are drawn one by one from a pack consisting of $m$ suits of $n$ cards each until at least one card of every suit has been drawn. How many cards may be expected to be drawn?

When $r$ suits have been drawn, let $N_{r+1}$ denote the additional number of cards to be expected to be drawn to expose an $(r+1)$th suit. Then it is easily shewn that

$$\frac{N_{r+1}}{N_r} = \frac{mn - rn + n}{mn - rn + 1}.$$

But $N_1 = 1$, $\therefore\ N_2 = mn \div (mn - n + 1)$,

$N_3 = m(m-1)n^2 \div (mn - n + 1)(mn - 2n + 1)$, and so on.

Hence the number of cards required is $N_1 + N_2 + \ldots + N_m$

$$= 1 + mn\left\{1 - 1 \div \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right)\left(1 + \frac{1}{3n}\right)\ldots\left(1 + \frac{1}{mn - n}\right)\right\}.$$

**511.** Or thus. We expect to throw doublets in 6 throws: then to throw different doublets in $36 \div 5$ throws: then in $36 \div 4$ : then in $36 \div 3$, and so on.

$$\therefore \text{ Expectation} = 36\left\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6}\right\} = 88\tfrac{1}{5}.$$

**532 to 535.** These questions differ from the historical Problem of Points, because here either player's success depends not upon the absolute number of points he makes, but upon the excess of that number above the number made by his competitor.

The celebrated Problem of Points stated simply and generally was the following:

$A$'s chance of scoring any point is $p$. $B$'s chance is $q$. They play until $A$ wins by scoring $m$ points or $B$ by scoring $n$ points: what are their respective chances of winning?

The earliest writers however stated it as follows:

$A$ and $B$ are playing a game in which $A$'s chance of scoring a point is $p$ and $B$'s chance is $q$. They abandon the game when $A$ is short of $m$ points and $B$ is short of $n$ points. In what proportion ought they to divide the stakes?

The two statements of the problem are plainly equivalent, as the stakes must equitably be divided in proportion to the players' chances of winning.

Three different formulæ for $A$'s expectation can be written down at sight, as it makes no difference whether the play stop ($\alpha$) when $A$ has scored $m$ points, $B$ having scored any number less than $n$, or whether it be continued ($\beta$) until $B$ has scored $n$ points, $A$ having scored $m$ or more, or whether it be continued ($\gamma$) until $m+n-1$ points have been scored.

If $A$ and $B$ denote the expectations of the two players we write down on the consideration ($\alpha$)

$$A = p^m \{1 + R_1^m q + R_2^m q^2 + R_3^m q^3 + \dots \text{ to } n \text{ terms}\},$$

$$B = q^n \{1 + R_1^n p + R_2^n p^2 + R_3^n p^3 + \dots \text{ to } m \text{ terms}\}.$$

On the consideration ($\beta$) we write down

$$A = p^m q^n \{R_m^n + R_m^{n+1} p + R_m^{n+2} p^2 + \dots \text{ in inf.}\},$$

$$B = p^m q^n \{R_n^m + R_n^{m+1} q + R_n^{m+2} q^2 + \dots \text{ in inf.}\}.$$

On the consideration ($\gamma$) we write down

$$A = \text{first } m \text{ terms of } (p+q)^{m+n-1}$$

$$B = \text{last } n \text{ terms of } (p+q)^{m+n-1}.$$

This last formula ($\gamma$) was communicated by John Bernoulli to Montmort in 1710: the first formula ($\alpha$) appears to be due to Montmort. (Todhunter, *History*, p. 98.) The connection between the first and second forms is seen if we write down $A$ in the form ($\alpha$) and $B$ in the form ($\beta$). We then have

$$A + B = p^m \{1 + R_1^m q + R_2^m q^2 + \dots \text{ in inf.}\}$$
$$= p^m (1-q)^{-m} = 1.$$

**557.** St John's Coll., Camb., Fellowship Examination, 1866.

**572, 573.** St John's Coll., Camb., May, 1867.

**579.** St John's Coll., Camb., May, 1868.

**619.** By Ferrers. *Ed. Times*, *Reprint*, xi. 70.

**623.** St John's Coll., Camb., May, 1869.

**624.** St John's Coll., Camb., June, 1865.

**631.** St John's Coll., Camb., May, 1861.

**632.** The principle (again assumed in Qn. 677) that if a magnitude is divided at random into three parts, the parts may be represented by perpendiculars from a random point within it upon the three sides of an equilateral triangle whose altitude represents the undivided magnitude, needs careful justification. It is obvious enough that every point within the triangle corresponds to a different fracture of the magnitude, and that every fracture of the magnitude corresponds to a different point within the triangle, but it is still necessary for the student to assure himself that the points representing an equable distribution of fractures are equably distributed over the area. This is easily seen when we consider a magnitude capable of only a finite number of fractures, e.g. we may consider a row of 8 elements to be divided at random into three parts. If we take all possible fractures, viz. 8, 0, 0; 7, 1, 0; 6, 2, 0; 6, 1, 1; &c., and mark the corresponding points on the triangle, we find that we mark once and once only the angular points of 256 small triangles: and these are all equal, and fill up the space of the original triangle. The same will be true if for 8 we substitute any number $n$; and it must remain true when $n$ is infinite. Hence the chance of any particular series of fractures is always proportional to the area upon which the corresponding point may lie on the triangle.

**646.** An equivalent question was proposed in the examination for a Fellowship at St John's Coll. in October, 1865. Thus:

At a game of bowls $A$ plays with one ball and $B$ with two. Assuming equal skill in the two players required their respective chances in one trial. Wherein consists the error of the following solution? Suppose $A$ to have delivered his ball: the chance that $B$ will succeed with his first ball is $\frac{1}{2}$: the chance that $B$ will fail with his first but succeed with his second is $\frac{1}{2} \times \frac{1}{2}$; therefore $B$'s chance is $\frac{3}{4}$.

We answer, that as $B$'s first ball is farther off than $A$'s ball, $A$'s ball cannot be assumed to be in an average position: the chance that $B$'s second ball will beat it is therefore less than $\frac{1}{2}$. The plausible but unsound solution was given by Montmort in 1714.

**650.** More generally if a die have $p$ faces the number of ways of throwing $s$ with $n$ such dice (or in $n$ throws with one such die) is

$$C^{s-1}_{n-1} - C^n_1 C^{s-p-1}_{n-1} + C^n_2 C^{-2p-1}_{n-1} - \&c.$$

And if the die be thrown repeatedly, the number of ways in which the sum $s$ can be attained by any number of throws is

$$C_{s,s-1} - C^s_1 C_{s-p,s-1} + C^s_2 C_{s-2p,s-1} - \&c.$$

This follows from Prop. XXVI as the former result from Prop. XXV. We have only to think of $s$ throws, making the convention that any of them may be blanks.

It is not easy to pass from the number of ways in which the sum $s$ may be attained to the chance that the stated sum should be attained. By an independent method however I find the chance to be

$$\frac{(p+1)^{s-1}}{p^s}\left\{1 - \frac{s}{\mu} + \frac{s(s-2p-1)}{1\,.\,2\mu^2} - \frac{s(s-3p-1)(s-3p-2)}{1\,.\,2\,.\,3\mu^3} + \&c.\right\},$$

where $\mu$ denotes $(p+1)^{p+1} \div p^p$ and the series in the bracket is to be carried to $n+1$ terms, where $n$ is the greatest integer in $s \div (p+1)$.

This agrees with the result obtained by Woolhouse in 1864 and by Sprague in 1866, *Ed. Times, Reprint*, i. 77 and v. 84.

**653.** The same results are obtained by the Integral Calculus, *Ed. Times, Reprint*, xii. 101.

**656.** From this result the student who has read the Integral Calculus will easily deduce the average area of a quadrilateral or of a triangle inscribed at random in a given circle.

(i) Let $\square$ be any quadrilateral in the circle $\odot$; then we have

$$\mathcal{E}\,(\square \text{ on diagonal } AB) = \tfrac{1}{4}AB^2\pi \div \alpha\,(\pi - \alpha)$$
$$= \odot \sin^2 \alpha \div \alpha\,(\pi - \alpha).$$

But the number of $\square$'s on diagonal $AB$

$$\propto \text{arc } APB \times \text{arc } AQB \propto \alpha\,(\pi - \alpha)$$

$\therefore$ $\alpha$ varying from 0 to $\frac{1}{2}\pi$

$$\mathcal{E}\,(\square) = \odot \int \sin^2 \alpha d\alpha \div \int \alpha\,(\pi - \alpha)\,d\alpha$$
$$= 3 \odot \div \pi^2.$$

This result was given in 1867, *Ed. Times, Reprint*, vii. 31.

(ii) Let $\triangle$ be any triangle inscribed in the circle; then

$$\mathcal{E}\,(\triangle \text{ on base } AB) = \{\alpha\mathcal{E}\,(APB) + (\pi - \alpha)\,\mathcal{E}\,(AQB)\} \div \pi$$
$$= \odot \sin^2 \alpha\,\{2 + (\pi - 2\alpha)\cot \alpha\} \div \pi^2,$$

$\therefore$ $\alpha$ varying from 0 to $\frac{1}{2}\pi$,

$$\mathcal{E}\,(\triangle) = \odot \int \{2\sin^2 \alpha + (\pi - 2\alpha)\cos \alpha \sin \alpha\}\,d\alpha \div \pi^2 \int \alpha d\alpha$$
$$= 5 \odot \div 2\pi^2.$$

**664.** An elaborate solution of this question by the Integral Calculus was given by Capt. Clarke in 1868, *Ed. Times, Reprint*, x. 54.

**673.** A solution by Miller, 1864, occupies two pages of *Ed. Times, Reprint*, ii. 92.

**678.** These results are given by Wolstenholme, *Math. Problems*, 1878. Solutions equivalent to those in the text were also published by Simmons, *Association française pour l'avancement des Sciences*, 1894.

**680.** The expectations are as $1+\frac{1}{2}+\frac{1}{3}:\frac{1}{2}+\frac{1}{3}:\frac{1}{3}$. The general case of partition into $n$ parts may be stated as follows:

If a magnitude (denoted by unity) be divided at random into $n$ parts, and the parts be then arranged in ascending order of magnitude, the expectation of the $r$th part is $(H_n - H_{n-r}) \div n$, where $H_n$ denotes the sum of the reciprocals of the first $n$ natural numbers.

It will be simplest to think of the magnitude as represented by a straight line of length unity.

Let $x$ be the smallest of the $n$ parts into which the line is divided. Assigning the length $x$ to each of the parts there will remain a length $1-nx$ to make the increments of the remaining $n-1$ parts. The chance that $x$ is the smallest part is proportional to the number of ways in which $n-2$ points of partition can be placed on a line of length $1-nx$. $\therefore$ Chance $\propto (1-nx)^{n-2}$.

Now $x$ may have any value from 0 to $1/n$. Therefore

$$\mathcal{E}(x) = \int (1-nx)^{n-2}\, x\,dx \div \int (1-nx)^{n-2}\, dx,$$

between the stated limits, i.e. $\mathcal{E}(x) = 1 \div n^2$.

Hence if $E_r^n$ denote the expectation of the $r$th part we write $E_1^n = 1 \div n^2$, whatever be $n$.

Now the expectations of the several parts after the lowest may be regarded as made up of two portions, viz. (1) the expectation of the lowest, and (2) the expectation of an increment due to the partition of the residue $1 - nE_1^n = (n-1) \div n$ amongst the $n-1$ parts, the increments being themselves arranged in ascending order of magnitude.

But these $n-1$ increments must be proportional to

$$E_1^{n-1} : E_2^{n-1} : E_3^{n-1} : \&c.$$

Therefore $$E_r^n = E_1^n + \frac{n-1}{n} \cdot E_{r-1}^{n-1}.$$

Hence by continual substitution

$$E_r^n = E_1^n + \frac{n-1}{n}\left[E_1^{n-1} + \frac{n-2}{n-1}\left\{E_1^{n-2} + \frac{n-3}{n-2}\left(E_1^{n-3} + \&c.\right)\right\}\right],$$

or $$E_r^n = \frac{1}{n^2} + \frac{1}{n(n-1)} + \frac{1}{n(n-2)} + \ldots \text{ to } r \text{ terms}$$

$$= \frac{1}{n}(H_n - H_{n-r}).$$

**695.** The skill here spoken of is relative to a particular experiment which must either succeed or fail. There is no question of approximation to success, and therefore no consideration of the magnitude of error. The question is whether the man hits a mark or not; if he succeeds he succeeds, if he fails he fails. And the skill thus determined belongs only to the particular experiment in question, not to experiments in general. If $A$'s skill is said to be double of $B$'s in essaying to hit a target a foot in diameter it does not follow that $A$'s skill ought to be double of $B$'s in the essay to hit a target 2 feet in diameter.

A much more difficult subject is reached when a man aims at a mark and consideration is given to the measure of the distance by which he misses it. Herschel discussed this question. Assuming that for a given marksman the chance of any single shot landing between distances $r$ and $r+dr$ from the centre of the target is proportional to $e^{-k^2r^2}rdr$ he deduced the following formula: Let $a$ be the radius of the circle within which, out of a very large number of shots, a man lands half his total; then his chance of missing in any single shot a concentric circle of radius $r$ is $(\frac{1}{2})^{r^2/a^2}$ (Simmons, *Ed. Times Reprint*, lxviii).

**696.** Comparing this with Qn. 658 we have the solution of the following question proposed in St John's Coll. Camb. in 1858.

A large quantity of pebbles lie scattered uniformly over a circular field: compare the labour of collecting them one by one;—i. at the centre ($O$) of the field;—ii. at a point ($A$) on the circumference.

Ratio $= \mathcal{E}(OP) : \mathcal{E}(AP) = 3\pi : 16$.

The value of $\mathcal{E}(PQ^m)$ was given by Simmons in 1894, *Ed. Times*, *Reprint*, liv. 119. He points out that the result is true for all values of $m$ from $-1$ to $+\infty$.

**698.** I take the question *verbatim* from Todhunter, *Int. Calc.*, p. 267. Observe that nothing is said about the arrows having been originally aimed at the centre of the target. We are therefore not concerned with any theory of error. We can only assume that all positions on the target are equally likely for either arrow. The question is in fact the same as if instead of two arrows two hailstones had struck the target. Compare the note on Qn. 695.

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