CHAPTER-05

LAPLACE TRANSFORMS

Remember

Let $f(t)$ be a function of t defined for all possible values of t, then laplace transform of $f(t)$ denoted by :

$$\boxed{L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)}$$

provided that integral exists, 's' is a parameter which may be real or complex no.

E.g: Find $L(1)$. i.e. $f(t) = 1$.

$$\therefore L[1] = \int_0^\infty e^{-st} \cdot 1 \cdot dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \left[0 - \left(\frac{1}{-s}\right)\right] = 1/s$$

$\therefore$ $L[1] = 1/s$ → Remember

$$L[e^{at}] = \frac{1}{s-a}$$

$$L[t^n] = \begin{cases} \dfrac{n!}{s^{n+1}} & \text{if } n \in Z^+ \\ \dfrac{\Gamma(n+1)}{s^{n+1}} & \text{otherwise} \end{cases}$$

$$L[\sin at] = \frac{a}{s^2+a^2}$$

$$L[\cos at] = \frac{s}{s^2+a^2}$$

$$L[\sinh at] = \frac{a}{s^2-a^2}$$

$$L[\cosh at] = \frac{s}{s^2-a^2}$$

Note:

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

$$\cosh at = \frac{e^{at} + e^{-at}}{2}$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$\sin(-\theta) = -\sin\theta$

$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$

$\sinh^2 x = \cancel{\frac{\cosh 2x - 1}{2}} \quad \frac{\cosh 2x - 1}{2}$

$\cosh^2 x = \cancel{\frac{\cosh 2x + 1}{2}} \quad \frac{\cosh 2x + 1}{2}$

$\sinh^3 x = \frac{1}{4}\sinh 3x - \frac{3}{4}\sinh x$

$\cosh^3 x = \frac{1}{4}\cosh 3x + \frac{3}{4}\cosh x$

★ **PROPERTIES of laplace transforms:**

1). Linear Property : If a, b, c be any constant and f, g, h any funcⁿ of 't' then,

$$L[af(t) + bg(t) - ch(t)] = aL[f(t)] + bL[g(t)] - cL[h(t)]$$

LP (3) (i) $L[\sin 2t \cos 3t] = ?$

$\sin 2t \cos 3t = \frac{1}{2}[\sin 5t + \sin(2t - 3t)]$

$= \frac{1}{2}[\sin 5t - \sin t]$

$\therefore L[\sin 2t \cos 3t] = L\left[\frac{1}{2}(\sin 5t - \sin t)\right]$

$= \frac{1}{2}L[\sin 5t - \sin t]$

$= \frac{1}{2}L[\sin 5t] - \frac{1}{2}L[\sin t]$

$= \frac{1}{2}\left(\frac{5}{s^2+25}\right) - \frac{1}{2}\left(\frac{1}{s^2+1}\right)$

(ii) $f(t) = \sin^3(2t)$

$$\sin^3(2t) = \frac{3}{4}\sin(2t) - \frac{1}{4}\sin(6t)$$

$$\therefore L[\sin^3(2t)] = L\left[\frac{3}{4}\sin(2t) - \frac{1}{4}\sin(6t)\right]$$

$$= \frac{3}{4}L[\sin 2t] - \frac{1}{4}L[\sin 6t]$$

$$= \frac{3}{4}\left(\frac{2}{s^2+4}\right) - \frac{1}{4}\left(\frac{6}{s^2+36}\right)$$

$$= \frac{3}{2}\left(\frac{1}{s^2+4}\right) - \frac{3}{2}\left(\frac{1}{s^2+36}\right)$$

(iii) $f(t) = \cos h^3 2t$

$$\cos h^3 2t = \frac{1}{4}\cos h\, 6t + \frac{3}{4}\cos h\, 2t.$$

$$L[\cos h^3 2t] = L\left[\tfrac{1}{4}\cos h\, 6t + \tfrac{3}{4}\cos h\, 2t\right]$$

$$= \frac{1}{4}L[\cos h\, 6t] + \frac{3}{4}L[\cos h\, 2t]$$

$$\frac{1}{4}\left(\frac{s}{s^2-36}\right) + \frac{3}{4}\left(\frac{s}{s^2-4}\right)$$

$$\cancel{\frac{1}{4}\left(\frac{e^{6t}+e^{-6t}}{2}\right) + \frac{3}{4}\left(\frac{e^{2t}+e^{-2t}}{2}\right)}$$

Ques : $L[f(t)] = L[e^{2t}\sin h t]$

$$e^{2t}\sin h t = e^{2t}\left(\frac{e^{t}-e^{-t}}{2}\right)$$

$$e^{2t}\sin h t = \frac{e^{3t}-e^{t}}{2}$$

$$L[e^{2t}\sinh t] = L\left[\frac{e^{3t}-e^{t}}{2}\right]$$

$$= \frac{1}{2}\left[L(e^{3t}) - L(e^{t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-3} - \frac{1}{s-1}\right]$$

(2) SHIFTING PROPERTY:

If $L[f(t)] = F(s)$.
then,

$$\boxed{L[e^{at}f(t)] = F(s-a)}$$

E.g For finding $L[e^{2t}\sinh t]$
we first find:

$$L[\sinh t] = \frac{1}{s^2-1}$$

$$\therefore L[e^{2t}\sinh t] = \frac{1}{(s-2)^2-1}$$

LP

(3) (iv) $e^{-3t}(\cos 4t + 3\sin 4t)$

$$L[\cos 4t + 3\sin 4t] = L[\cos 4t] + 3L[\sin 4t]$$

$$= \frac{s}{s^2+16} + 3\cdot\frac{4}{s^2+16}$$

$s-a$
$s-(-3)$
$s+3$

$$\therefore L[e^{-3t}(\cos 4t + 3\sin 4t)] = \frac{(s+3)}{(s+3)^2+16} + 3\cdot\frac{4}{(s+3)^2+16}$$

Ques $L[e^{5t} t^4]$

$$L[t^4] = \frac{4!}{S^5}$$

$$\therefore L[e^{5t} t^4] = \frac{4!}{(S-5)^5}$$

Ques :-

$$L[e^{-t}(1+t^2+\sin 2t + \cosh 4t)]$$

$$L(1+t^2+\sin 2t + \cosh 4t)$$
$$= L(1) + L(t^2) + L(\sin 2t) + L(\cosh 4t)$$
$$= \frac{1}{S} + \frac{2!}{S^3} + \frac{2}{S^2+4} + \frac{S}{S^2-16}$$

$$\therefore L[e^{-t}(1+t^2+\sin 2t + \cosh 4t)]$$
$$= \frac{1}{(S+1)} + \frac{2!}{(S+1)^3} + \frac{2}{(S+1)^2+4} + \frac{(S+1)}{(S+1)^2-16}$$

Ques $L[e^{-t}(\sin^2 t)]$

$$[\sin^2 t] = \frac{1-\cos 2t}{2}$$

$$\therefore L[\sin^2 t] = \frac{1}{2}\left(L[1] - L[\cos 2t]\right)$$

$$= \frac{1}{2}\left(\frac{1}{S} - \frac{S}{S^2+4}\right)$$

$$\therefore L[e^{-t}(\sin^2 t)] = \frac{1}{2}\left[\frac{1}{(S+1)} - \frac{(S+1)}{(S+1)^2+4}\right]$$

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Ques $L[e^{-2t}\sin 3t \sin 2t]$

$$\sin 3t \sin 2t = \frac{1}{2}(\cos t - \cos 5t)$$

$$\therefore L[\sin 3t \sin 2t] = \frac{1}{2}[L(\cos t) - L(\cos 5t)]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]$$

$$\therefore L[e^{-2t}\sin 3t \sin 2t] = \frac{1}{2}\left[\frac{(s+2)}{(s+2)^2+1} - \frac{(s+2)}{(s+2)^2+25}\right].$$

Ques $L[e^{-at}(\cos^2 t - \sin^2 t)]$

$$L[\cos^2 t - \sin^2 t] = L[\cos^2 t] - L[\sin^2 t]$$

$$= L\left[\frac{1+\cos 2t}{2}\right] - L\left[\frac{1-\cos 2t}{2}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= \cancel{\frac{1}{2}\times\frac{1}{s}} + \frac{s}{2(s^2+4)} - \cancel{\frac{1}{2\times s}} + \frac{s}{2(s^2+4)}$$

$$= \frac{s}{s^2+4}$$

$$\therefore L[e^{-at}(\cos^2 t - \sin^2 t)] = \frac{(s+a)}{(s+a)^2+4}$$

PROPERTY 3 If $L[f(t)] = F(s)$ then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F(s))$

PROPERTY :

③ If $L[f(t)] = F(s)$ then $\boxed{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F(s))}$

④ If $L[f(t)] = F(s)$ then $\boxed{L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)\,ds}$

⑤ If $L[f(t)] = F(s)$ then $\boxed{L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}}$

Ques $L[t^2 \sin at]$

$L[\sin at] = \frac{a}{s^2+a^2} = F(s)$

$L[t^2 \sin at] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$

$= \frac{d}{ds}\left[\frac{-2sa}{(s^2+a^2)^2}\right]$

$= -2a\left[\frac{(s^2+a^2)^2 1 - s\,2(s^2+a^2)\times 2s}{(s^2+a^2)^4}\right]$

$= \frac{-2a}{(s^2+a^2)^4}\left[(s^2+a^2)^2 - 4s^2(s^2+a^2)\right]$

$= \frac{-2a}{(s^2+a^2)^{\cancel{4}\,3}} \cancel{(s^2+a^2)}\left[s^2+a^2 - 4s^2\right]$

$= \frac{-2a}{(s^2+a^2)^3}(a^2 - 3s^2)$

Ques: $L[t^3 e^{-3t}]$

$L[t^3] = \frac{\underline{|3}}{s^4}$

$\therefore L[e^{-3t} t^3] = \frac{6}{(s+3)^4}$

LP

(3) (vi) $f(t) = te^{-4t}\sin 3t$

$L[e^{-4t}\sin 3t]$

$L[\sin 3t] = \frac{3}{s^2+9} = F(s)$

$\therefore L[\cancel{e^{-4t}} \sin \cancel{3t}] = \frac{3}{(s+4)^2+9}.$

$\therefore L[t \sin 3t] = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right)$
(easier)

$= -3 \cdot \frac{-1}{(s^2+9)^2} \times 2s = \frac{6s}{(s^2+9)^2}$

$\therefore L[e^{-4t}\, t \sin 3t] = \frac{6(s+4)}{((s+4)^2+9)^2}$

(vii) $\sinh(t/2) \sin(\sqrt{3}/2)t$

$\sinh(t/2) = \frac{e^{t/2} - e^{-t/2}}{2}$

$\therefore \frac{e^{t/2} - e^{-t/2}}{2}\left(\sin\left(\frac{\sqrt{3}}{2}\right)t\right)$

$= \frac{e^{t/2}}{2}\sin\left(\frac{\sqrt{3}t}{2}\right) - \frac{e^{-t/2}}{2}\sin\left(\frac{\sqrt{3}t}{2}\right)$

$\therefore L[\sinh(t/2)\sin(\sqrt{3}/2)t]$.

$= L\left[\frac{e^{t/2}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right] - L\left[\frac{e^{-t/2}}{2}\sin\frac{\sqrt{3}t}{2}\right]$

$L\left[\sin\frac{\sqrt{3}}{2}t\right] = \frac{\sqrt{3}/2}{s^2+(3/4)}$

$\therefore L\left[\frac{e^{t/2}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right] = \frac{1}{2}\left(\frac{\sqrt{3}/2}{(s-\frac{1}{2})^2+3/4}\right)$

$\therefore L\left[\frac{e^{-t/2}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right] = \frac{1}{2}\left(\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}\right)$

P.T.O.

$$= \frac{1}{2}\left(\frac{\sqrt{3}/2}{(s-1/2)^2+3/4}\right) - \frac{1}{2}\left(\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}\right) \quad \underline{\text{Ans}}$$

<u>LP</u> (5) (iv) $\quad f(t) = te^{-t}\cos nt$

$$L[\cos nt] = \frac{s}{s^2+n^2}$$

$$L[t\cos nt] = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+n^2}\right) = -1\frac{d}{ds}\left(\frac{1}{\cancel{2}} \times \frac{\cancel{2}s}{s^2+n^2}\right)$$

let $\cancel{s^2+n^2=X}$

$\cancel{dX = 2s}$

$$(-1)\left[\frac{(s^2+n^2)1 - s(2s)}{(s^2+n^2)^2}\right] = (-1)\left(\frac{s^2+n^2-2s^2}{(s^2+n^2)^2}\right)$$

$$= (-1)\frac{n^2-s^2}{(s^2+n^2)^2} = \frac{s^2-n^2}{(s^2+n^2)^2}$$

$$L[e^{-t}t\cos nt] = \frac{(s+1)^2-n^2}{((s+1)^2+n^2)^2} \quad \underline{\text{Ans}}$$

(vi) $\quad \dfrac{\overbrace{\cos at - \cos bt}^{f(t)}}{t}$

$$L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \int_s^\infty \frac{1}{2}\left(\frac{2s}{s^2+a^2}\right)ds - \int_s^\infty \frac{1}{2}\left(\frac{2s}{s^2+b^2}\right)ds$$

$$= \frac{1}{2}\left[\log \frac{(s^2+a^2)}{(s^2+b^2)}\right]_s^\infty$$

$$= \frac{1}{2}\left(\log \left[\frac{\cancel{s^2}\,(1+a^2/s^2)}{\cancel{s^2}\,(1+b^2/s^2)}\right]\right)_s^\infty$$

$$= \frac{1}{2}\left[\underbrace{\log 1}_{0} - \log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right)\right]$$

$$= \frac{1}{2}\left[-\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

LP

5 (vii) $f(t) = \frac{e^{-t}\sin t}{t}$

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L\left[\frac{\sin t}{t}\right] = \int_s^{\infty} \frac{1}{s^2+1}\, ds$$

$$= \left[\tan^{-1}(s)\right]_s^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$L\left[\frac{\sin t}{t}\right] = \frac{\pi}{2} - \tan^{-1}(s)$$

$$\therefore L\left[e^{-t}\frac{\sin t}{t}\right] = \frac{\pi}{2} - \tan^{-1}(s+1). \text{ Ans.}$$

LP

5 (viii) $\frac{1-\cos 2t}{t}$ or $L\left[\frac{2\sin^2 t}{t}\right]$

$$L[1-\cos 2t] = L[1] - L[\cos 2t]$$

$$= \frac{1}{s} - \frac{s}{s^2+4} = F(s)$$

$$\therefore L\left[\frac{1-\cos 2t}{t}\right] = \int_s^{\infty}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \int_s^{\infty}\frac{1}{s}\, ds - \int_s^{\infty}\frac{1}{2}\frac{2s}{s^2+4}\, ds$$

$$= [\log s]_s^{\infty} - \frac{1}{2}\log\left[s^2+4\right]_s^{\infty}$$

$$= \left[\log\left(\frac{s}{(s^2+4)^{1/2}}\right)\right]_s^{\infty} = \left[\log\left(\frac{\cancel{s}\,(1)}{\cancel{s}\,\left(1+\frac{4}{s^2}\right)^{1/2}}\right)\right]_s^{\infty}$$

$$= \left[\log 1 - \log \left(1 + \frac{4}{s^2}\right)^{1/2} \right]_s^\infty$$

$$= \cancel{\log 1} - \cancel{\log (1)} + \log \left(1 + \frac{4}{s^2}\right)^{1/2}$$

$$= \log \left(\frac{s^2+4}{s^2}\right)^{1/2}$$

$$= \frac{1}{2} \log \left(\frac{s^2+4}{s^2}\right) \quad \text{Ans}$$

LP

4(iv) $L\left[\int_0^t e^{-t} (1+t+t^2)\, dt \right]$

$$L[1+t+t^2] = L[1] + L[t] + L[t^2]$$
$$= \frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3}$$

$$\therefore L\left[e^{-t}(1+t+t^2)\right] = \frac{1}{(s+1)} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3} = F(s)$$

$$\therefore L\left[\int_0^t e^{-t}(1+t+t^2)\, dt \right] = \frac{1}{s}\left(\frac{1}{(s+1)} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3} \right)$$

LP

4(iii) $L\left[\int_0^t e^t \sin t\, dt \right]$

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L[e^t \sin t] = \frac{1}{(s-1)^2+1}$$

$$\therefore L\left[\int_0^t e^t \sin t\, dt \right] = \frac{1}{s}\left(\frac{1}{(s-1)^2+1} \right) \quad \text{Ans}$$

18

(i)

⑥. Evaluate : $\int_0^{\infty} t^3 e^{-t} \sin t \, dt$.

Here evaluate is given..

∴ we have to solve this integral not just find laplace transformation

∴ we compare with the definition of laplace transform.

$$L[f(t)] = \int_0^{\infty} e^{-t} f(t)\, dt.$$

$\therefore f(t) = t^3 \sin t$ & $s = 1$.

$L[t^3 \sin t] = ?$

$\therefore L[\sin t] = \frac{1}{s^2+1}$

$$\therefore L[t^3 \sin t] = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)$$

$$= (-1)(-1)^3 \frac{d^2}{ds^2}\left(\cancel{\log} \frac{1 \times 2s}{(s^2+1)^2}\right)$$

$$= (+1)\frac{d}{ds}\left[\frac{(s^2+1)^2 \cdot 2 - 2s(s^2+1)\,2s \times 2}{(s^2+1)^4}\right]$$

$$= \frac{d}{ds}\left[\frac{\cancel{(s^2+1)}\,[2(s^2+1) - 8s^2]}{(s^2+1)^{\cancel{4}\,3}}\right]$$

$$= \frac{d}{ds} \frac{2s^2 + 2 - 8s^2}{(s^2+1)^3}$$

$$= \frac{d}{ds} \frac{2 - 6s^2}{(s^2+1)^3}$$

$\begin{array}{r} 12 \\ 8 \\ \hline 96 \end{array}$

$$= \frac{(s^2+1)^3(-12s) - (2-6s^2)\, 3(s^2+1)^2 \cdot 2s}{(s^2+1)^6}$$

$= \cancel{(s^2+1)^2} / [\cancel{(s^2+1)} \cdot$ Substitute $s=1$

$$= \frac{(2)^3(-12) - (-4)\,3(4)\cdot 2}{64}$$

$$= \frac{-96+96}{64} = \underline{\underline{0}}$$

6 LP

(ii) $\displaystyle\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\,dt.$

Definition of laplace transform :-

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$\therefore\ s=1.\ ;\ f(t) = \dfrac{\sin^2 t}{t}$

$\therefore\ L\left[\dfrac{\sin^2 t}{t}\right]$

$\therefore\ L[\sin^2 t] = L\left[\frac{1}{2} - \frac{\cos 2t}{2}\right]$

$= L[1/2] - \frac{1}{2} L[\cos 2t]$

$= \frac{1}{2}(1/s) - \frac{1}{2}\frac{s}{s^2+4} \qquad = \cancel{F(s)} = \frac{1}{2}F(s)$

$\therefore\ L\left[\dfrac{1-\cos 2t}{2t}\right] = \dfrac{1}{2}\displaystyle\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$

$\cancel{= \frac{1}{2} \log s - \int_s^\infty \frac{s}{s(s+\frac{4}{s})}\, ds}$

$\cancel{= \frac{1}{2} \log s - 0 + \log(s+\frac{4}{s})}$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \frac{1}{2}\left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\ln s^2 - \ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\ln \frac{s^2}{s^2+4}\right]_s^\infty = -\frac{1}{4}\ln\left(\frac{s^2}{s^2+4}\right)$$

$$= \cancel{-\tfrac{1}{4}\ln(0)}. \qquad \because \{s=1\}$$

$$= \frac{-1}{4}\ln\left(\frac{1}{5}\right)$$

$$= \frac{1}{4}\ln 5$$

LP

(7) $\displaystyle\int_0^\infty \frac{\cos 4t - \cos 5t}{t}\,dt = \int_0^\infty e^{-0t}\,\frac{\cos 4t - \cos 5t}{t}\,dt.$

comparing with def^n of L.T :-

we get $f(t) = \dfrac{\cos 4t - \cos 5t}{t}$ & $s = 0$.

$$\therefore L\left[\frac{\cos 4t - \cos 5t}{t}\right] = \int_s^\infty \left(\left(\frac{s}{s^2+16}\right) - \left(\frac{s}{s^2+25}\right)\right) ds.$$

$$= \left[\frac{1}{2}\ln(s^2+16) - \frac{1}{2}\ln(s^2+25)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln\left(\frac{s^2+16}{s^2+25}\right)\right]_s^\infty = \frac{-1}{2}\ln\left(\frac{s^2+16}{s^2+25}\right)$$

$$L\left[\frac{\cos 4t - \cos 5t}{t}\right]_{s=0} = \frac{-1}{2}\ln\left(\frac{16}{25}\right) = \ln\left(\frac{25}{16}\right)^{1/2}$$

$$= \ln(5/4).$$

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(8). $\int_0^\infty t\, e^{-3t} \cos 2t \, dt$.

$s = 3$. $f(t) = t \cos 2t$.

$$L[\cos 2t] = \frac{s}{s^2+4}.$$

$$\therefore L[t \cos 2t] = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= -1 \left[\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]$$

$$= -1 \left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$= -\left[\frac{4-s^2}{(s^2+4)^2}\right] = \frac{s^2-4}{(s^2+4)^2}$$

Substitute $s = 3$ $\therefore L[f(t)] = \frac{5}{(13)^2}$

* LAPLACE TRANSFORM OF DERIVATIVE:

If $L[f(t)] = F(s)$ then,

- $L[f'(t)] = sF(s) - f(0)$
- $L[f''(t)] = s^2F(s) - sf(0) - f'(0)$
- $L[f'''(t)] = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$

$\omega t \cos\omega t + \sin\omega t$

LP

(4) (i) If $L[t\sin\omega t] = \frac{2\omega s}{(s^2+\omega^2)^2}$

then we need to find $L[\omega t\cos\omega t + \sin\omega t]$

Let $f(t) = t\sin\omega t$.

$f'(t) = \omega t\cos\omega t + \sin\omega t$.

Since, $\frac{d}{dt}(t\sin\omega t) = \omega t\cos\omega t + \sin\omega t$

$L[t\sin\omega t] = \frac{2\omega s}{(s^2+\omega^2)^2} = F(s)$

$\therefore L[\omega t\cos\omega t + \sin\omega t] = sF(s) - f(0)$

$= s\left[\frac{2\omega s}{(s^2+\omega^2)^2}\right] - 0.$

$= \frac{2\omega s^2}{(s^2+\omega^2)^2}$

(ii) $L\left[2\sqrt{\frac{t}{\pi}}\right] = \frac{1}{s^{3/2}}$; ~~Proof~~ :- $L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{s}}$.

we know that:

$\frac{d}{dt}\left(2\sqrt{\frac{t}{\pi}}\right) = \frac{1}{\sqrt{\pi t}}$.

$\therefore L\left[\frac{1}{\sqrt{\pi t}}\right] = s\left(\frac{1}{s^{3/2}}\right) - 0 = \frac{1}{s^{1/2}}$

Type of q-10 LP

$L[y'' - y = 0]$; $y(0) = 1$ $y'(0) = 0$

$L[y'' - y] = L[0]$

$L[y''] - L[y] = 0$

$$s^2 \underbrace{F(s)}_{=Y(s)} - s\underbrace{f(0)}_{=y(0)} - f'(0) - F(s) = 0.$$

or

$s^2 Y(s) - sy(0) - y'(0) - Y(s) = 0$

$s^2 Y(s) - s - Y(s) = 0.$

$Y(s)[s^2 - 1] = s$

$Y(s) = \frac{s}{s^2 - 1}$

LP

(10) (i) $L[y'' - 3y' + 2y = 4t + e^{3t}]$ $y(0) = 1$ $y'(0) = -1$

$L[y''] - 3L[y'] + 2L[y] = 4L[t] + L[e^{3t}]$

$[s^2 Y(s) - sy(0) - y'(0)] - [3(sY(s) - y(0))] + 2Y(s) = \frac{4}{s} + \frac{1}{s-3}$

$s^2 Y(s) - s + 1 - 3sY(s) + 3 + 2Y(s) = \frac{4}{s} + \frac{1}{s-3}$

$Y(s)[s^2 - 3s + 2] - 4s + 4 = \frac{4}{s} + \frac{1}{s-3}$

$$Y(s) = \left[\frac{4}{s} + \frac{1}{s-3} + 4s - 4\right]\left(\frac{1}{s^2 - 3s + 2}\right)$$ Ans

★ INVERSE LAPLACE TRANSFORM:

If $L[f(t)] = F(s)$ then $f(t)$ is called inverse laplace transformation of $F(s)$ and is denoted by:

$$\boxed{L^{-1}[F(s)] = f(t)}$$

- $L^{-1}[1/s] = 1$

- $L^{-1}[1/s^n] = \frac{t^{n-1}}{\lfloor n-1}$ or ~~since~~ $\frac{t^{n-1}}{\Gamma n}$

  since, $L[t^n] = \frac{\lfloor n}{s^{n+1}} \Rightarrow \frac{t^{n-1}}{\lfloor n-1} = L^{-1}[1/s^n]$.

- $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

- $L^{-1}\left[\frac{1}{(s-a)^n}\right] = e^{at}\frac{t^{n-1}}{\lfloor n-1}$

- $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

- $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \left(\frac{1}{a}\right)\sin at$

- $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

- $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \left(\frac{1}{a}\right)\sinh at$

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- $L^{-1}\left[\dfrac{(s-a)}{(s-a)^2+b^2}\right] = e^{at}\cos bt$

- $L^{-1}\left[\dfrac{1}{(s+a)^2+b^2}\right] = \dfrac{e^{-at}}{b}\sin bt$

- $L^{-1}\left[\dfrac{F(s)}{s}\right] = \int_0^t f(t)\,dt$

* Partial Fraction:

$\dfrac{Num}{(s+a)(s-a)(s+b)}$: check for its proper fraction. i.e. degree (Numerator) < deg. (D^r). then only, apply this condition.

- $\dfrac{Num}{(s+a)(s-a)(s+b)} = \dfrac{A}{(s+a)} + \dfrac{B}{(s-a)} + \dfrac{C}{(s+b)}$

- $\dfrac{Num}{(s+a)^3} = \dfrac{A}{(s+a)^1} + \dfrac{B}{(s+a)^2} + \dfrac{C}{(s+a)^3}$

- $\dfrac{Num}{s(s-b)^2} = \dfrac{A}{s} + \dfrac{B}{(s-b)} + \dfrac{C}{(s-b)^2}$

- $\dfrac{Num}{(s^2+a)(s^2+b)} = \dfrac{As+B}{s^2+a} + \dfrac{Cs+D}{s^2+b}$

- $\dfrac{Num}{(s+a)(s^2+b)} = \dfrac{A}{s+a} + \dfrac{Bs+C}{s^2+b}$

LP

(9) (i) $\frac{3(s^2-1)^2}{2s^5} = \frac{3[s^4-2s^2+1]}{2s^5} = \frac{3s^4-6s^2+3}{2s^5}$

$\frac{3s^4-6s^2+3}{2s^5} = \frac{3}{2}(1/s) - 3(1/s^3) + \frac{3}{2}(1/s^5)$

$L^{-1}\left[\frac{3(s^2-1)^2}{2s^5}\right] = \frac{3}{2}L^{-1}(1/s) - 3L^{-1}(1/s^3) + \frac{3}{2}L^{-1}(1/s^5)$

$= \frac{3}{2}(1) - 3\left(\frac{t^2}{2}\right) + \frac{\cancel{3}}{2}\left(\frac{t^4}{\cancel{24}_8}\right)$

$= \frac{3}{2} - \frac{3t^2}{2} + \frac{\cancel{3}t^4}{16}$

(ii) $\frac{3s+5\sqrt{2}}{s^2+8} = \frac{3s}{s^2+8} + \frac{5\sqrt{2}}{s^2+8}$

$3\left[\frac{s}{s^2+8}\right] + 5\sqrt{2}\left[\frac{\cancel{s}\,1}{s^2+8}\right]$

$L^{-1}\left[\frac{3s+5\sqrt{2}}{s^2+8}\right] = 3L^{-1}\left[\frac{s}{s^2+8}\right] + 5\sqrt{2}\,L^{-1}\left[\frac{1}{s^2+8}\right]$

$= 3\cos(2\sqrt{2})t + 5\cancel{\sqrt{2}}\frac{1}{(2\cancel{\sqrt{2}})}\sin(2\sqrt{2})t$

$f(t) = 3\cos(2\sqrt{2})t + \frac{5}{2}\sin(2\sqrt{2})t.$

(iii) $\frac{4s+15}{s^2-25} = 4\frac{s}{s^2-25} + 15\frac{1}{s^2-25}$

$L^{-1}\left[\frac{4s+15}{s^2-25}\right] = 4L^{-1}\left[\frac{s}{s^2-25}\right] + 15\,L^{-1}\left[\frac{1}{s^2-25}\right]$

$= 4\cosh(5t) + \frac{\cancel{15}^{3}}{\cancel{5}}\sin(h5t)$

$= 4\cosh(5t) + 3\sinh(5t).$

(iv) $\dfrac{s+1}{s^2+s+1} = \dfrac{s+1}{s^2+s+\frac{1}{4}-\frac{1}{4}+1} = \dfrac{s+1}{(s+1/2)^2+3/4}$.

$= \dfrac{(s+1/2)+1/2}{(s+1/2)^2+3/4} = \dfrac{s+1/2}{(s+1/2)^2+3/4} + \dfrac{1/2}{(s+1/2)^2+3/4}$

$L^{-1}\left[\dfrac{s+1}{s^2+s+1}\right] = L^{-1}\left[\dfrac{s+1/2}{(s+1/2)^2+3/4}\right] + \frac{1}{2} L^{-1}\left[\dfrac{1}{(s+1/2)^2+3/4}\right]$

$= e^{-1/2\, t}\cos\left(\sqrt{\tfrac{3}{4}}\, t\right) + \frac{1}{2}\, e^{-1/2\, t}\sin\left(\sqrt{\tfrac{3}{4}}\right)t \times \frac{2}{\sqrt{3}}$

$= e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}\, e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}\right)t$. //

(v) $\dfrac{2s^2-4}{(s+1)(s-2)(s-3)} = \dfrac{A}{(s+1)} + \dfrac{B}{(s-2)} + \dfrac{C}{(s-3)}$ —(i)

$2s^2-4 = (s-2)(s-3)A + B(s+1)(s-3) + C(s+1)(s-2)$ —(ii)

$= (s^2-5s+6)A + B(s^2-2s-3) + C(s^2-s-2)$ } optional.

• This method is applied only when dr. linear terms are diff. don't apply for $(s\pm1)^2$ (distinct)

Put $s=-1$ in eqn (ii) :-

$-2 = (-3)(-4)A \Rightarrow \boxed{A = -1/6}$

Put $s=2$ in eqn (ii) :-

$4 = B(3)(-1) \Rightarrow \boxed{B = \frac{-4}{3}}$

Put $s=3$ in eqn (ii) :-

$14 = C(4)(1) \Rightarrow \boxed{C = \frac{7}{2}}$

$$\therefore L^{-1}\left[\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right] = A\,L^{-1}\left[\frac{1}{s+1}\right] + B\,L^{-1}\left[\frac{1}{s-2}\right] + C\,L^{-1}\left[\frac{1}{s-3}\right]$$

$$= \left(-\tfrac{1}{6}\right) e^{-1t} + \left(-\frac{4}{3}\right) e^{+2t} + \left(\frac{7}{2}\right) e^{+3t}$$

(6) $\dfrac{2s^2-1}{(s^2+1)(s^2+4)}$ let $s^2 = p$, then we get:-

$$\frac{2p-1}{(p+1)(p+4)} = \cancel{\frac{2p}{(p+1)(p+4)}}, \frac{A}{(p+1)} + \frac{B}{(p+4)}$$

$$2p-1 = A(p+4) + B(p+1)$$

Put $p = -4$

$$-9 = B(-3) \Rightarrow \boxed{B=3}$$

Put $p = -1$

$$-3 = A(3) \Rightarrow \boxed{A=-1}$$

$$\therefore L^{-1}\left[\frac{2p-1}{(p+1)(p+4)}\right] = (-1)\,L^{-1}\left(\frac{1}{p+1}\right) + 3\,L^{-1}\left(\frac{1}{p+4}\right)$$

$$= (-1)\,L^{-1}\left(\frac{1}{s^2+1}\right) + 3\,L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= (-1)\,\frac{1}{1}\sin t + 3\,\frac{1}{2}\sin(2t)$$

$$= -\sin t + \frac{3}{2}\sin(2t)$$ // Ans

(vii) $\frac{21s-33}{(s-2)^3} = \frac{A}{(s-2)} + \frac{B}{(s-2)^2} + \frac{C}{(s-2)^3}$

$21s - 33 = A(s-2)^2 + B(s-2) + C$

$= A(s^2+4-4s) + B(s-2) + C$

$= As^2 + 4A - 4sA + Bs - 2B + C$

$= (A)s^2 + s(-4A+B) + (4A-2B+C)$

$-(-4A+2B-C)$

$\boxed{A=0}$ $\boxed{21 = B}$

$42 - C = 33$

$\boxed{C = 9}$

$\therefore 21 L^{-1}\left[\frac{1}{(s-2)^2}\right] + 9 L^{-1}\left[\frac{1}{(s-2)^3}\right]$

$= 21 \; e^{2t}\left(\frac{t}{1}\right) + 9 \; e^{2t}\frac{t^2}{2}$

$= \boxed{21te^{2t} + \frac{9}{2}t^2e^{2t}}$ //.

(OR)

$\frac{21s-33}{(s-2)^3} = \frac{21(s-2+2)-33}{(s-2)^3} = \frac{21(s-2)+42-33}{(s-2)^3}$

$\Rightarrow \frac{21(s-2)}{(s-2)^3} + \frac{9}{(s-2)^3} \Rightarrow \frac{21}{(s-2)^2} + \frac{9}{(s-2)^3}$

$\therefore L^{-1}\left[\frac{21s-33}{(s-2)^3}\right] = 21 L^{-1}\left[\frac{1}{(s-2)^2}\right] + 9\left[\frac{1}{(s-2)^3}\right]$

$= \boxed{21e^{2t}t + 9e^{2t}\frac{t^2}{2}}$ //.

Apply any possible method. Ans. will remain same.

(8) $$\frac{5s+3}{(s+1)(s^2+2s+5)} = \frac{A}{(s+1)} + \frac{Bs+C}{s^2+2s+5}.$$

$$\therefore 5s+3 = A(s^2+2s+5) + (Bs+C)(s+1)$$
$$= As^2 + 2As + 5A + Bs^2 + Bs + Cs + C$$
$$= s^2(A+B) + s(2A+B+C) + (5A+C)$$

comparing coefficients of LHS & RHS :-

$\therefore A+B=0$ — (1)

$2A+B+C=5$ — (11)

$5A+C=3$ — (111)

from (111) $C = 3-5A$

from (1) $B=-A$.

$\therefore$ from (11) :- $2A - A + 3 - 5A = 5$

$A + 3 - 5A = 5$

$-4A = 2 \Rightarrow \boxed{A = -1/2}$

$\therefore \boxed{B = 1/2}$ $\quad C = 3 + \frac{5}{2} = 11/2$

$$\therefore L^{-1}\left[\frac{5s+3}{(s+1)(s^2+2s+5)}\right] = \frac{-1}{2} L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left[\frac{1/2\, s + 11/2}{(s^2+2s+5)}\right]$$
$$= \frac{-1}{2}(e^{-t}) + L^{-1}\left(\frac{1/2\, s + 11/2}{s^2+2s+5}\right)$$

$$\frac{1}{2} L^{-1} \frac{(s+11)}{(s^2+2s+5)}$$

$$\frac{1}{2} L^{-1} \frac{((s+1)+10)}{(s+1)^2+4}$$

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$$\frac{1}{2} L^{-1}\left[\frac{s+1}{(s+1)^2+4}\right] + \frac{10}{2} L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$\frac{1}{2} e^{-t}\cos 2t + \frac{5}{2}\sin 2t\, e^{-t}$$

$$\therefore \text{Final Ans} = \frac{-1}{2} e^{-t} + \frac{1}{2} e^{-t}\cos 2t + \frac{5}{2} e^{-t}\sin 2t \text{ //.}$$

(9) $$\frac{s}{s^4+4a^4} = \frac{s}{(s^2)^2+(2a^2)^2} = \frac{s}{(s^2)^2+(2a^2)^2-4a^2s^2+4a^2s^2}$$

$$\Rightarrow \frac{s}{(s^2+2a^2)^2-(2sa)^2} \Rightarrow \frac{s}{(s^2+2a^2+2as)(s^2+2a^2-2as)}$$

$$\cancel{\frac{s}{(s^2+2a^2)^2+(2sa)^2}} \quad \therefore \frac{s}{s^4+4a^4} = \frac{s}{(s^2+2a^2+2as)(s^2+2a^2-2as)}.$$

$$\text{Now, } \frac{s}{(s^2+2a^2+2as)(s^2+2a^2-2as)} = \frac{As+B}{s^2+2a^2+2as} + \frac{Cs+D}{s^2+2a^2-2as}.$$

$$\therefore s = (As+B)(s^2+2a^2-2as) + (Cs+D)(s^2+2a^2+2as)$$

$$= As^3 + 2a^2As - 2Aas^2 + Bs^2 + 2a^2B - 2Bas + Cs^3 + 2a^2Cs + 2aCs^2 + Ds^2 + 2a^2D + 2aDs$$

$$= s^3(A+C) + s^2(-2aA+B+2aC+D) + s(2a^2A - 2Ba + 2a^2C + 2aD) + 2a^2B + 2a^2D$$

Now, $A+C=0$ — (1) $\quad -2aA+B+2aC+D=0$

$$2a(A-C)+B+D=0 \quad \cancel{2a(B+D)=0} \text{ — (11)}$$

$$A-C=0$$

$$2a^2A - 2Ba + 2a^2C + 2aD = 1$$

$$\underbrace{2a^2(A+C)}_{0} + 2a(D-B) = 1$$

$$2a(D-B) = 1 \text{ —— (111)}$$

$2a^2B + 2a^2D = 0 \Rightarrow B + D = 0$

$\therefore$

$A + C = 0$

$\cancel{B+D=0}$ $A - C = 0$

$D - B = 1/2a$

$B + D = 0$

$\cancel{B} + D = 0$

$-\cancel{B} + D = 1/2a$

$2D = 1/2a$

$D = 1/4a$ $\therefore B = -1/4a$

$A + \cancel{C} = 0$

$A - \cancel{C} = 0$

$2A = 0 \Rightarrow A = 0$ & $C = 0$

$\therefore Ans = \frac{-1}{4a} L^{-1}\left[\frac{1}{s^2+2a^2+2as}\right] + \frac{1}{4a} L^{-1}\left[\frac{1}{s^2+2a^2-2as}\right]$

$= \frac{-1}{4a} L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right] + \frac{1}{4a} L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right]$

$= \boxed{\frac{-1}{4a} \cdot \frac{e^{-at}}{a} \sin at + \frac{1}{4a} \frac{e^{at}}{a} \sin at.}$ Ans

(10) $\frac{s}{s^4+s^2+1} = \frac{s}{(s^2+1)^2 - s^2} = \cancel{\frac{A}{(s^2+1)^2} + \frac{B}{s^2}}$

$\cancel{(s^2)^2 + 2(s^2)(1) + 1 - s^2} = \frac{s}{(s^2+1+s)(s^2+1-s)}$

$\frac{s}{(s^2+s+1)(s^2-s+1)} = \cancel{\frac{As+B}{(s^2+s+1)} + \frac{Cs+D}{(s^2-s+1)}}$

$= \frac{1}{2}\left(\frac{1}{s^2+1+s} - \frac{1}{s^2+1-s}\right)$

find diff., whatever extra is comp divide by that. (see next page)

$\cancel{s^2} + \cancel{1} - s - \cancel{s^2} - \cancel{1} - s = -2s$

$\therefore$ divide by -2.

$\because$ we only need 's' in N^r.

$s^2 + 1 + s$

$(s)^2 + 2.(s).\frac{1}{2} + \left(\frac{1}{4}\right) - \frac{1}{4} + 1$

$$\therefore \frac{s}{s^4+s+1} = \frac{-1}{2}\left[\frac{1}{(s+1/2)^2 + 3/4} - \frac{1}{(s-1/2)^2 + 3/4}\right]$$

$$\therefore L^{-1}\left[\frac{s}{s^4+s+1}\right] = \frac{-1}{2} e^{-1/2t}\left(\frac{2}{\sqrt{3}}\right)\sin\left(\frac{\sqrt{3}}{2}t\right) \quad //.$$

$$+\frac{1}{2} e^{1/2t}\left(\frac{2}{\sqrt{3}}\right)\sin\left(\frac{\sqrt{3}}{2}\right)t.$$

(ii) $$\frac{1}{s(s+1)^3} = \int_0^t L^{-1}\left[\frac{1}{(s+1)^3}\right] dt.$$

$$f(t) = \int_0^t e^{-t}\frac{t^2}{2}\, dt$$

$$= \frac{1}{2}\int_0^t t^2 e^{-t}\, dt.$$

$$= \frac{1}{2}\left[-t^2 e^{-t} - 2t e^{-t} - 2e^{-t}\right]_0^t$$

(13) $\frac{1}{s^2(s^2+a^2)} = \frac{A}{s^2} + \frac{B}{(s^2+a^2)}$

OR

$\int_0^t \int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right] dt.$

$= \int_0^t \int_0^t \frac{1}{a} \sin at \, dt.$

$= \int_0^t \left[-\frac{\cos at}{a}\right]_0^t dt.$

$= \frac{1}{a^2} \int_0^t \left[-\cos at + 1\right] dt.$

$= \frac{1}{a^2} \left[-\frac{\sin at}{a} + t\right]_0^t$

$= \frac{1}{a^2} \left[\left(-\frac{\sin at}{a} + t\right) - (0)\right]$

$= \frac{1}{a^2} \left[-\frac{\sin at}{a} + t\right].$ //.