Formulae

1. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
2. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
3. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
4. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$
7. $\tan(A+45) = \dfrac{1+\tan A}{1-\tan A}$
8. $\tan(A-45) = \dfrac{\tan A - 1}{1+\tan A}$
9. $\tan(45-A) = \dfrac{1-\tan A}{1+\tan A}$
10. $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$
11. $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$
12. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
13. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
14. $\sin 2A = 2\sin A\cos A$

    $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$
15. $\cos 2A = \cos^2 A - \sin^2 A$

    $= 1 - 2\sin^2 A$

    $= 2\cos^2 A - 1$
16. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
17. $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
18. $\sin 3A = 3\sin A - 4\sin^3 A$

    $\cos 3A = 4\cos^3 A - 3\sin A$
20. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
21. $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
22. $\cos A\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$
23. $\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
24. $\sin A\sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$
25. $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
26. $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
27. $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
28. $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
29. $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x-a} = na^{n-1}$ $\quad \lim\limits_{x\to a}\dfrac{x^m - a^m}{x^n - a^n} = \dfrac{m}{n}a^{m-n}$
30. $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$ $\quad \lim\limits_{\theta\to 0}\dfrac{\sin a\theta}{\theta} = a$
31. $\lim\limits_{\theta\to 0}\dfrac{\tan\theta}{\theta} = 1$ $\quad \lim\limits_{\theta\to 0}\dfrac{\tan a\theta}{\theta} = a$
32. $\lim\limits_{x\to 0}\dfrac{e^x - 1}{x} = \log_e e = 1$
33. $\lim\limits_{x\to 0}\dfrac{a^x - 1}{x} = \log_e a$
34. $\lim\limits_{x\to 0}\dfrac{\log(1+x)}{x} = 1$
35. $\lim\limits_{x\to 0}(1+x)^{1/x} = e$
36. $\lim (1+\frac{1}{x})^x = e$

37. $\frac{d}{dx}(x^n) = nx^{n-1}$

38. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

39. $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

40. $\frac{d}{dx}(e^x) = e^x$

41. $\frac{d}{dx}(a^x) = a^x \log_e a$

42. $\frac{d}{dx}(\log x) = \frac{1}{x}$

43. $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x \, dx = \sin x + c$

44. $\frac{d}{dx}(\cos x) = -\sin x \Rightarrow \int \sin x \, dx = -\cos x + c$

45. $\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x \, dx = \tan x + c$

46. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x . \cot x \Rightarrow \int \operatorname{cosec} x \tan x \, dx = -\operatorname{cosec} x + c$

47. $\frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x \, dx = \sec x + c$

48. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \Rightarrow \int \operatorname{cosec}^2 x \, dx = -\cot x + c$

49. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$$
$$= -\cos^{-1} x + c$$

50. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$$
$$= -\cot^{-1} x + c$$

51. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + c$

$\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

$= -\operatorname{cosec}^{-1}x + c$

52. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

53. $\frac{d}{dx}(e^x) = e^x$ $\quad \int e^x\,dx = e^x + c$

54. $\int a^x\,dx = \frac{a^x}{\log a} + c$

$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$

55. $\int \frac{1}{x}\,dx = \log x + c$

56. $\int 1\,dx = x + c$

57. $\int \frac{d}{dx}[f(x)]\,dx = f(x)$

58. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

59. $\int k.f(x)\,dx = k.\int f(x)\,dx$

60. $\frac{d}{dx}\sinh x = \cosh x \Rightarrow \int \cosh x = \sinh x + c$

61. $\frac{d}{dx}\cosh x = \sinh x \Rightarrow \int \sinh x = \cosh x + c$

62. $\frac{d}{dx}\tanh x = \sec^2 h x \Rightarrow \int \sec^2 h x = \tanh x + c$

63. $\frac{d}{dx}\coth x = -\operatorname{cosec}^2 h x \Rightarrow \int \operatorname{cosec}^2 h x = -\coth x + c$

64. $\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x \Rightarrow \int \operatorname{cosech} x \coth x = -\operatorname{cosech} x + c$

65. $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x \Rightarrow \int \operatorname{sech} x \tanh x = -\operatorname{sech} x + c$

66. $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}} \Rightarrow \int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1}x + c$

67. $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}} \Rightarrow \int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}x + c$

68. $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2} \Rightarrow \int \frac{1}{1-x^2}\,dx = \tanh^{-1}x + c$

69. $\frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2} \Rightarrow \int \frac{1}{1-x^2}\,dx = \coth^{-1}x + c$

70. $\frac{d}{dx}(\mathrm{sech}^{-1}x) = \frac{-1}{|x|\sqrt{1-x^2}} \Rightarrow \int \frac{1}{|x|\sqrt{1-x^2}}\,dx = -\mathrm{sech}^{-1}x + c$

71. $\frac{d}{dx}(\mathrm{cosech}^{-1}x) = \frac{-1}{|x|\sqrt{1+x^2}} \Rightarrow \int \frac{1}{|x|\sqrt{1+x^2}}\,dx = -\mathrm{cosech}^{-1}x + c$

72. $\int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$

73. $\int \frac{f'(x)}{f(x)} \cdot dx = \log|f(x)| + c$

74. $\int \cot x\,dx = \log|\sin x| + c$

75. $\int \tan x\,dx = -\log|\cos x| + c$

$= \log|\sec x| + c$

76. $\frac{d}{dx}(\log_a x) = \frac{1}{x \cdot \log a}$

77. $\log_b a = \frac{\log a}{\log b}$

78. $a^{\log_a x} = x$

79. $1+\sin 2x = (\cos x + \sin x)^2$

$1+\sin x = (\cos\frac{x}{2} + \sin\frac{x}{2})^2$

80. $1-\sin 2x = (\cos x - \sin x)^2$

$1-\sin x = (\cos\frac{x}{2} - \sin\frac{x}{2})^2$

81. $\sin(\sin^{-1}x) = x \quad \cos(\cos^{-1}x) = x \quad \tan(\tan^{-1}x) = x$

82. $\sin^{-1}(\sin\theta) = \theta \quad \cos^{-1}(\cos\theta) = \theta \quad \tan^{-1}(\tan\theta) = \theta$

83. $\sin^{-1}x = \mathrm{cosec}^{-1}\frac{1}{x} \quad \cos^{-1}x = \sec^{-1}\frac{1}{x} \quad \tan^{-1}x = \cot^{-1}\frac{1}{x}$

84. $\tan^{-1}x + \tan^{-1}y = \frac{x+y}{1-xy} \quad \tan^{-1}x - \tan^{-1}y = \frac{x-y}{1+xy}$

85. $\int \sec x\,dx = \log|\sec x + \tan x| + c$

$= \log|\tan(\pi/4 + x/2)| + c$

86. $\int \mathrm{cosec}\,x\,dx = \log|\mathrm{cosec}\,x - \cot x| + c$

$= \log|\tan x/2| + c$

87. $\int \tan^2 x\,dx = \tan x - x$

88. $\int \cot^2 x\,dx = -\cot x - x +$

89. $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

90. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

91. $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

92. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right| + c = \sinh^{-1}(x/a) + c$

93. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right| + c = \cosh^{-1}(x/a) + c$

94. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

95. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + a^2/2\,\sin^{-1}\left(\frac{x}{a}\right) + c$

96. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + a^2/2\,\log\left|x+\sqrt{x^2+a^2}\right| + c$

97. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{a}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$

From Relations and Functions (PBA) & ITF

1) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

2) $2\cos^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

3) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

4) $2\cos^{-1}x = \cos^{-1}(2x^2-1)$

5) $2\sin^{-1}x = \cos^{-1}(1-2x^2)$

6) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

7) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

8) $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$

9) $3\cos^{-1}x = \cos^{-1}(4x^3-3x)$

10) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

12-11-19

LAPLACE TRANSFORMS

If $f(t)$ is a real valued function defined for all $t \geq 0$ then Laplace transform of $f(t)$ denoted by $L[f(t)]$ is defined as

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= F(s)$$

· Linearity Property.

$$L[af_1(t) + bf_2(t) + cf_3(t)] = \int_0^\infty e^{-st} (af_1(t) + bf_2(t) + cf_3(t))\, dt$$

$$= a\int_0^\infty e^{-st} f_1(t)\, dt + b\int_0^\infty e^{-st} f_2(t)\, dt + c\int_0^\infty e^{-st} f_3(t)$$

$$= a\, L[f_1(t)] + b\, L[f_2(t)] + c.[L f_3(t)]$$

Laplace transform of some standard functions.

1. $L[1] = \frac{1}{s}$

Proof: $L[1] = \int_0^\infty e^{-st}(1)\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$

$$= \frac{-1}{s}(e^{-\infty} - e^{0})$$

$$= \frac{1}{s}.$$

2. $L[k] = \frac{k}{s}$ where k is any constant.

Scanned with CamScanner

$$L[e^{at}] = \frac{1}{s-a}$$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{-1}{(s-a)}\left[e^{-\infty} - e^{0}\right] = \frac{-1}{s-a}(-1)$$

$$= \frac{1}{s-a}$$

4. $L[e^{-at}] = \frac{1}{s+a}$

5. $L[\sinh at] = \frac{a}{s^2-a^2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$$L[\sinh at] = L\left[\frac{e^{at} - e^{-at}}{2}\right]$$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$$= \frac{1}{2}\left[L[e^{at}] - L[e^{-at}]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right] = \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$

6. $L[\cosh at] = \frac{s}{s^2-a^2}$

$$L[\cosh at] = L\left[\frac{e^{at} + e^{-at}}{2}\right] = \frac{1}{2}\left[L[e^{at}] + L[e^{-at}]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

- $e^{ix} = \cos x + i \sin x$
- $\sqrt{n} = (n-1)!$
- $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

7. Consider $L[e^{iat}] = \frac{1}{s-ia}$

÷ and × by $s+ia$

$$= \frac{1}{s-ia} \times \frac{s+ia}{s+ia}$$

$$= \frac{s+ia}{s^2 - i^2a^2} = \frac{s+ia}{s^2+a^2}$$

$$L[\cos at + i \sin at] = \frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2}$$

$\therefore L[\cos at] = \frac{s}{s^2+a^2}$ and $L[\sin at] = \frac{a}{s^2+a^2}$

8. $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$

$$L[t^n] = \int_0^\infty e^{-st} t^n \, dt$$

put $st = x$
$s\,dt = dx$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \cancel{s} \cdot \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \cdot x^n \, dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) = \frac{n!}{s^{n+1}}$$

	$f(t)$	$L[f(t)]$
1.	$L(1)$	$\frac{1}{s}$
2.	$L(k)$	$\frac{k}{s}$
3.	$L[e^{at}]$	$\frac{1}{s-a}$
4.	$L[e^{-at}]$	$\frac{1}{s+a}$
5.	$L[\sinh at]$	$\frac{a}{s^2-a^2}$
6.	$L[\cosh at]$	$\frac{s}{s^2-a^2}$
7.	$L[\cos at]$	$\frac{s}{s^2+a^2}$
8.	$L[\sin at]$	$\frac{a}{s^2+a^2}$
9.	$L[t^n]$	$\frac{\Gamma(n+1)}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$

- $\sin^2\theta = \frac{1-\cos 2\theta}{2}$
- $\cos(A+B) = \cos A\cos B - \sin A \sin B$
- $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

Q. $L[\cosh^2 3t]$

$$= L\left[\left(\frac{e^{3t}+e^{-3t}}{4}\right)^2\right]$$

$$= \frac{1}{4}\left[L\left[e^{6t}+e^{-6t}+2\right]\right]$$

$$= \frac{1}{4}\left[L[e^{6t}] + L[e^{-6t}] + L[2]\right]$$

$$= \frac{1}{4}\left[\frac{1}{s-6} + \frac{1}{s+6} + \frac{2}{s}\right]$$

Q. $L[\sin^2(2t+1)]$

$$= L\left[\frac{1-\cos 2(2t+1)}{2}\right]$$

$$= \frac{1}{2} L\left[1-\cos(4t+2)\right]$$

$$= \frac{1}{2} L\left[1-(\cos 4t \cos 2 - \sin 4t \sin 2)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \left(\cos 2\left(\frac{s}{s^2+16}\right) - \sin 2\left(\frac{4}{s^2+16}\right)\right)\right]$$

Q. $L[\cos 2t \sin 3t]$

$$= L\left[\tfrac{1}{2}(\sin 5t - \sin(-t))\right]$$

check $= L\left[\tfrac{1}{2}(\sin 5t + \sin t)\right]$

$- \sin t$

$$= \frac{1}{2}\left(L[\sin 5t] + L[\sin t]\right)$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

- $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
- $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
- $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

classmate

Date

Page

Q. $L[(3t+4)^3 + 5^t] = L[27t^3 + 64 + 108t^2 + 144t + e^{\log 5^t}]$

$$= \frac{27(3!)}{s^4} + \frac{64}{s} + \frac{108(2!)}{s^3} + \frac{144(1)}{s^2} + L[e^{t.\log 5}]$$

$$= \frac{27(3!)}{s^4} + \frac{64}{s} + \frac{108(2!)}{s^3} + \frac{144}{s^2} + \frac{1}{s-\log 5}.$$

-11-19 First Shifting Theorem

Thm: If $L[f(t)] = F(s)$ then $L[e^{at}f(t)] = F(s-a)$

Ex: $L[e^{-3t}(\cos 4t + 3\sin 4t)]$

$$= L[e^{-3t}\cos 4t] + 3\,L[e^{-3t}\sin 4t]$$

$$= L[\cos 4t]\Big|_{s\to s+3} + 3\,L[\sin 4t]\Big|_{s\to s+3}$$

$$= \frac{s}{s^2+16}\Big|_{s\to s+3} + 3.\frac{4}{s^2+16}\Big|_{s\to s+3}$$

$$= \frac{s+3}{(s+3)^2+16} + 3.\frac{4}{(s+3)^2+16} = \frac{s+3+12}{(s+3)^2+16} = \frac{s+15}{(s+3)^2+16}.$$

- $\sin 3x = 3\sin x - 4\sin^3 x$
- $\Gamma\frac{1}{2} = \sqrt{\pi}$

Date

Page

Q. Find laplace transform of $e^{4t}\sin^3 3t$

$$L[e^{4t}\sin^3 3t]$$

$$= L[\sin^3 3t]\Big|_{s\to s-4}$$

$$= L\left(\frac{3\sin 3t - \sin 9t}{4}\right)\Big|_{s\to s-4}$$

$$= \frac{1}{4}\left[3.\frac{3}{s^2+9} - \frac{9}{s^2+81}\right]\Big|_{s\to s-4}$$

$$= \frac{9}{4}\left[\frac{1}{(s-4)^2+9} - \frac{1}{(s-4)^2+81}\right]$$

Q. Find $L[e^t\sqrt{t}]$

$$= L[t^{1/2}]\Big|_{s\to s-1}$$

$$= \frac{\Gamma 3/2}{s^{3/2}}\Big|_{s\to s-1}$$

$$= \frac{\cancel{\Gamma 3+}\ \frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}}\Big|_{s\to s-1}$$

$$= \frac{\sqrt{\pi}}{2(s-1)^{3/2}}$$

Q. Thm : If $L[f(t)] = F(s)$ then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(L[f(t)])$

$$= (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Q. $L[t^2 \cos at]$

$$= (-1)^2 \frac{d^2}{ds^2}\left(L[\cos at]\right)$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left[\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2(-2s) - (a^2-s^2)\,2(s^2+a^2)(2s)}{(s^2+a^2)^4}$$

$$= \frac{(s^2+a^2)(-2s) - (a^2-s^2)(4s)}{(s^2+a^2)^3}$$

$$= \frac{-2s^3 - 2sa^2 - 4sa^2 + 4s^3}{(s^2+a^2)^3}$$

$$= \frac{2s^3 - 6a^2 s}{(s^2+a^2)^3} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

Q. $L[t \sin^2 3t]$

$$= (-1)^1 \frac{d}{ds} L[\sin^2 3t]$$

$$= (-1) \frac{d}{ds} L\left[\frac{1-\cos 6t}{2}\right]$$

$$= (-1) \frac{d}{ds} \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+36}\right)$$

$$= \frac{-1}{2}\left[\frac{d}{ds}\left(\frac{1}{s}\right) - \frac{d}{ds}\left(\frac{s}{s^2+36}\right)\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} - \frac{(s^2+36)(1) - s(2s)}{(s^2+36)^2}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} - \frac{s^2+36-2s^2}{(s^2+36)^2}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} - \frac{-s^2+36}{(s^2+36)^2}\right]$$

$$= \frac{1}{2s^2} + \frac{36-s^2}{(s^2+36)^2}$$

Q. $L[t^3 e^{-3t}] = L[e^{-3t}\, t^3]$

$$= L[t^3]\Big|_{s\to s+3}$$

$$= \frac{3!}{s^4}$$

$$= \frac{3!}{(s+3)^4}$$

Q. $L\left[\sinh(t/2) \cdot \sin \frac{\sqrt{3}}{2} t\right]$

$$= L\left[\left(\frac{e^{t/2} - e^{-t/2}}{2}\right) \cdot \sin \frac{\sqrt{3}}{2} t\right]$$

$$= \frac{1}{2} L\left[e^{t/2} \sin \frac{\sqrt{3}}{2} t - e^{-t/2} \sin \frac{\sqrt{3}}{2} t\right]$$

$$= \frac{1}{2}\left[L\left[\sin \frac{\sqrt{3}}{2} t\right]\Big|_{s\to s-1/2} - L\left[\sin \frac{\sqrt{3}}{2} t\right]\Big|_{s\to s+1/2}\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{3}/2}{s^2+3/4}\Big|_{s\to s-1/2} - \frac{\sqrt{3}/2}{s^2+3/4}\Big|_{s\to s+1/2}\right]$$

classmate

Date

Page

$$= \frac{1}{2}\left[\frac{\sqrt{3}/2}{(s-1/2)^2+3/4} - \frac{\sqrt{3}/2}{(s+1/2)^2+3/4}\right]$$

Q. $L\left[t^2 e^{-2t}\cos t\right]$

$$= L\left[e^{-2t}\,(t^2\cos t)\right]$$

$$= L\left[t^2\cos t\right]\Big|_{s\to s+2} \quad \to ①$$

$$L\left[t^2\cos t\right] = (-1)^2\frac{d^2}{ds^2}L[\cos t]$$

$$= (-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{d}{ds}\left[\frac{s^2+1-s(2s)}{(s^2+1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right]$$

$$= \frac{(s^2+1)^2(-2s) - (1-s^2)\,(2.(s^2+1))\,(2s)}{(s^2+1)^4}$$

$$= \frac{-2s(s^2+1) - (1-s^2)(4s)}{(s^2+1)^3}$$

$$= \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}$$

$$= \frac{2s^3-6s}{(s^2+1)^3}$$

put in ①

$$\therefore \Rightarrow L\left[t^2e^{-2t}\cos t\right] = \frac{2s^3-6s}{(s^2+1)^3}\Bigg|_{s\to s+2} = \frac{2(s+2)^3-6(s+2)}{((s+2)^2+1)^3}$$

H.W Q $L[t^2 \sin at]$, Q. $L[t e^{-t} \cos nt]$

Q. Evaluate $\int_0^\infty t e^{-3t} \cos 2t \, dt$

WKT $L[f(t)] = \int_0^\infty e^{-st} f(t) \, dt$

Here, $s=3$

$$\therefore \int_0^\infty e^{-3t} (t \cos 2t) \, dt$$

$$= L[t \cos 2t] \Big|_{s=3}$$

$$= (-1)^1 \frac{d}{ds} L[\cos 2t] \Big|_{s=3}$$

$$= -1 \frac{d}{ds} \left[\frac{s}{s^2+4}\right] \Big|_{s=3}$$

$$= -1 \left(\frac{s^2+4 - s(2s)}{(s^2+4)^2}\right) \Big|_{s=3}$$

$$= \frac{s^2-4}{(s^2+4)^2} \Big|_{s=3}$$

$$= \frac{9-4}{(9+4)^2}$$

$$= \frac{5}{169}$$

Q. Evaluate $\int_0^\infty t^3 e^{-t} \sin t \, dt$

$$= \int_0^\infty e^{-t} (t^3 \sin t) \, dt$$

$(s=1)$

$$= L[t^3 \sin t]\Big|_{s=1}$$

$$= (-1)^3 \frac{d^3}{ds^3} L[\sin t]\Big|_{s=1}$$

$$= (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)\Big|_{s=1}$$

$$= -\frac{d^2}{ds^2}\left[\frac{-1}{(s^2+1)^2}\cdot(2s)\right]\Big|_{s=1}$$

$$= -\frac{d^2}{ds^2}\left[\frac{-2s}{(s^2+1)^2}\right]\Big|_{s=1}$$

$$= -\frac{d}{ds}\left[\frac{(s^2+1)^2(-2) - (-2s)\,2\cdot(s^2+1)(2s)}{(s^2+1)^4}\right]\Big|_{s=1}$$

$$= -\frac{d}{ds}\left[\frac{\cancel{-2s^2} - 2(s^2+1) + 8s^2}{(s^2+1)^3}\right]\Big|_{s=1}$$

$$= -\frac{d}{ds}\left[\frac{-2s^2 - 2 + 8s^2}{(s^2+1)^3}\right]\Big|_{s=1}$$

$$= -\frac{d}{ds}\left[\frac{6s^2-2}{(s^2+1)^3}\right]\Big|_{s=1}$$

$$= -\left[\frac{(s^2+1)^3(12s) - (6s^2-2)\,3(s^2+1)^2(2s)}{(s^2+1)^6}\right]\Big|_{s=1}$$

$$= -\left[\frac{(s^2+1)(12s) - 6s(6s^2-2)}{(s^2+1)^4}\right]\Big|_{s=1}$$

$$= -\left[\frac{12s^3 + 12s - 36s^3 + 12s}{(s^2+1)^4}\right]\Big|_{s=1}$$

$$= -\left[\frac{24s - 24s^3}{(s^2+1)^4}\right]\Big|_{s=1}$$

$$= 0$$

- Laplace transform of $\frac{f(t)}{t}$

Thm: If $L[f(t)] = F(s)$ and $\lim_{t\to 0} \frac{f(t)}{t}$ exists

then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,.ds$

Q Find Laplace transform of:

Q1. $L\left[\frac{\sin at}{t}\right] = \int_s^\infty L(\sin t)\,ds$

$= \int_s^\infty \frac{a^2}{s^2+a^2}\,ds$

$= a.\frac{1}{a}\left[\tan^{-1}\frac{s}{a}\right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}\frac{s}{a}$

$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$

$= \cot^{-1}\frac{s}{a}$

Q2. $L\left[\frac{1-\cos 2t}{t}\right] = \int_s^\infty L(1-\cos 2t)\,ds$

$= \int_s^\infty L(1) - L(\cos 2t)\,ds$

$= \int_s^\infty \left(\frac{1}{s} - \frac{s^2}{s^2+a^2}\right) ds$

$= [\log s]_s^\infty - \left[\frac{2s}{2s^2+a^2}\right] .ds$

Scanned with CamScanner

$$= \left[\log \frac{s}{\sqrt{s^2+4}}\right]_s^{\infty} = \log \frac{s}{s\sqrt{1+4/s^2}} - \cancel{\log \frac{}{}\,1}$$

$$= \cancel{\tfrac{1}{s}}\left[\log 1 - \log \frac{1}{\sqrt{1+4/s^2}}\right] = \frac{-\log s}{\sqrt{s^2+4}} = \frac{\log \sqrt{s^2+4}}{s}$$

Q. Evaluate $\int_0^{\infty} \frac{\cos at - \cos bt}{t}\, dt$

$$= \int_0^{\infty} e^{-st} f(t)\,.\,dt = L[f(t)]$$

$$= \int_0^{\infty} e^{-0t}\left[\frac{\cos at - \cos bt}{t}\right] dt$$

$$= L\left[\frac{\cos at - \cos bt}{t}\right]_{s=0}$$

$$= \int_0^{\infty} \cos at - \cos bt \; ds$$

$$= \int_0^{\infty} \frac{s}{2s^2+a^2} - \frac{2s}{2s^2+b^2}$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]$$

$$= \frac{\log(1+a^2/s^2)}{\log(1+b^2/s^2)} = 0 - \log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$= \frac{1}{2}\log\frac{b^2}{a^2} = \frac{1}{2}\log\left(\frac{b}{a}\right)^2 = \log\left(\frac{b}{a}\right)$$

$$\therefore \int_0^\infty \frac{\cos 4t - \cos 5t}{t}\, dt = \log \frac{5}{4}$$

Q Evaluate $\int_0^\infty e^{-t} \frac{\sin^2 t}{t}\, dt$

$$= \int_0^\infty e^{-st} \frac{\sin^2 t}{t}\, dt$$

$s = 1$

$$= \int_0^\infty L\left[\frac{\sin^2 t}{t}\right]_{s=1}$$

$$= \int_s^\infty L[\sin^2 t]$$

$$= \int_s^\infty \frac{1}{2} L[1 - \cos 2t]$$

$$= \frac{1}{2} \int_s^\infty L(1) - L[\cos 2t]$$

$$= \frac{1}{2} \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2} \log(s^2+4)\right]$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2} \log(s^2+4)\right]$$

$$= \frac{1}{2}\left[\log s - \log \sqrt{s^2+4}\right] = \frac{1}{2} \log \frac{s}{\sqrt{s^2+4}}$$

$$= \frac{1}{2} \log \frac{1}{\sqrt{1 + 4/s^2}}$$

$$= \log (s)^{1/4} = \log \frac{\sqrt{s^2+4}}{s}$$

$$= \frac{1}{2} \log \sqrt{5} = \frac{1}{4} \log 5$$

Q. $L\left[\frac{1-e^{-at}}{t}\right] = \int_0^\infty e^{-0t}\left(\frac{e^{-3t}-e^{-6t}}{t}\right).dt = \log 2$

• Laplace transform of derivatives

* If $L[f(t)] = F(s)$

1. $L[f'(t)] = s\,L[f(t)] - f(0)$

2. $L[f''(t)] = s^2[L[f(t)]] - sf(0) - f'(0)$

3. $L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$.

4. $L[y'] = s\,L[y] - y(0)$

Evaluate.

Q. $L[f'(t)]$ if $f(t) = e^{-5t}\sin t$

Date
Page

19-11-19

- Laplace transform of Integrals.

Thm: If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

Q. Find $L\left[\int_0^t t^2 e^{3t}\,dt\right] = \frac{L[e^{3t}.t^2]}{s}$

$$= \frac{1}{s}\left[\frac{2}{s^3}\Big|_{s\to s-3}\right]$$

$$= \frac{1}{s}\left[\frac{2}{(s-3)^3}\right]$$

Q. Find $L\left[\int_0^t e^t \sin t\,dt\right] = \frac{L[e^t \sin t]}{s}$

$$= \frac{1}{s}\left[\frac{1}{s^2+1}\Big|_{s\to s-1}\right]$$

$$= \frac{1}{s}\left[\frac{1}{(s-1)^2+1}\right]$$

Q. Find $L\left[\int_0^t e^{-t}(1+t+t^2)\,dt\right]$

$$= \frac{L[e^{-t}(1+t+t^2)]}{s} = \frac{1}{s}\left[L(1+t+t^2)\Big|_{s\to s+1}\right]$$

$$= \frac{1}{s}\left[\left(\frac{1}{s}+\frac{1}{s^2}+\frac{2}{s^3}\right)\Big|_{s\to s+1}\right]$$

$$= \frac{1}{s}\left[\frac{1}{s+1}+\frac{1}{(s+1)^2}+\frac{2}{(s+1)^3}\right]$$

Q Find $L\left[e^{-4t}\int_0^t t\sin 3t\,dt\right]$

$$= L\left[\int_0^t t\sin 3t\,dt\right]\Bigg|_{s\to s+4}$$

$$= \frac{L[t\sin 3t]}{s}\Bigg|_{s\to s+4}$$

$$= \frac{1}{s}\left[(-1)\frac{d}{ds}\frac{3}{s^2+9}\right]\Bigg|_{s\to s+4}$$

$$= \left[-\frac{3}{s}\left(\frac{-2s}{(s^2+9)^2}\right)\right]\Bigg|_{s\to s+4}$$

$$= \left(\frac{6}{(s^2+9)^2}\right)\Bigg|_{s\to s+4} = \frac{6}{((s+4)^2+9)^2}$$

- INVERSE LAPLACE TRANSFORMS.

If $L[f(t)] = F(s)$ then $f(t)$ is called inverse Laplace transform of $F(s)$ i.e.,

$$L^{-1}(F(s)) = f(t).$$

Standard Results:

1. $L[1] = \frac{1}{s}$ $\qquad L^{-1}\left[\frac{1}{s}\right] = 1$

2. $L[e^{at}] = \frac{1}{s-a}$ $\qquad L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

3. $L[e^{-at}] = \frac{1}{s+a}$ $\qquad L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

Scanned with CamScanner

4. $L[\sin at] = \frac{a}{s^2+a^2}$ $\qquad L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$.

5. $L[\cos at] = \frac{s}{s^2+a^2}$ $\qquad L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

6. $L[\sinh at] = \frac{a}{s^2-a^2}$ $\qquad L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$.

7. $L[\cosh at] = \frac{s}{s^2-a^2}$ $\qquad L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cosh at$

8. $L[t^n] = \frac{n!}{s^{n+1}}$ or $\frac{\Gamma n+1}{s^{n+1}}$ $\qquad L^{-1}\left[\frac{n!}{s^{n+1}}\right] = \frac{t^n}{n!}$ or $\frac{t^n}{\Gamma n+1}$

- Linearity property

$$L^{-1}[a F(s) + b G(s)] = a L^{-1}[F(s)] + b L^{-1}[G(s)].$$

Ex: $$L^{-1}\left[\frac{3(s^2-1)^2}{2s^5}\right] = \frac{3}{2} L^{-1}\left[\frac{s^4-2s^2+1}{s^5}\right]$$

$$= \frac{3}{2} L^{-1}\left[\frac{1}{s} - \frac{2}{s^3} + \frac{1}{s^5}\right]$$

$$= \frac{3}{2}\left[1 - 2\frac{t^2}{2!} + \frac{t^4}{4!}\right]$$

$$= \frac{3}{2}\left(1 - t^2 + \frac{t^4}{24}\right)$$

classmate
Date
Page

- Inverse Laplace transform by completing the square

Ex: $L^{-1}\left[\frac{s+5}{s^2-6s+13}\right]$

$= L^{-1}\left[\frac{s+5}{s^2-6s+9+4}\right]$

$= L^{-1}\left[\frac{s+5}{(s-3)^2+2^2}\right]$

$= L^{-1}\left[\frac{(s-3)+8}{(s-3)^2+2^2}\right] = L^{-1}\left[\frac{s-3}{(s-3)^2+2^2}\right] + L^{-1}\left[\frac{8}{(s-3)^2+2^2}\right]$

$= e^{3t}\cos 2t + 8\,\frac{\sin 2t}{2}.\, e^{3t}$

$= e^{3t}\,(\cos 2t + 4\sin 2t)$

Q. Find $L^{-1}\left[\frac{7s+4}{4s^2+4s+9}\right]$

$= L^{-1}\left[\frac{7s+4}{4\left(s^2+s+\frac{9}{4}\right)}\right] = \frac{1}{4}\left[L^{-1}\left[\frac{7s+4}{s^2+s+\frac{1}{4}+2}\right]\right]$

$= \frac{1}{4}\, L^{-1}\left[\frac{7s+4}{(s+1/2)^2+(\sqrt{2})^2}\right] = \frac{7}{4}\, L^{-1}\left[\frac{s+4/7+1/2-1/2}{(s+1/2)^2+(\sqrt{2})^2}\right]$

$= \frac{7}{4}\left(L^{-1}\left[\frac{s+1/2}{(s+1/2)^2+(\sqrt{2})^2}\right] + \frac{1}{14}\, L^{-1}\left[\frac{1}{(s+1/2)^2+(\sqrt{2})^2}\right]\right)$

$= \frac{7}{4}\left[e^{-t/2}\cos\sqrt{2}t + \frac{1}{14}\,\frac{e^{-t/2}\sin\sqrt{2}t}{\sqrt{2}}\right]$

Q. $L^{-1}\left[\dfrac{s+1}{s^2+s+1}\right]$

• Inverse laplace transform by method of partial fractions.

i) $\dfrac{1}{(x+a)(x+b)} = \dfrac{A}{(x+a)} + \dfrac{B}{(x+b)}$

ii) $\dfrac{1}{(x+a)^2(x+b)} = \dfrac{A}{(x+a)} + \dfrac{B}{(x+a)^2} + \dfrac{C}{(x+b)}$

iii) $\dfrac{1}{ax^2+bx+c} = \dfrac{Ax+B}{ax^2+bx+c}$

Q. Find $L^{-1}\left[\dfrac{4s+15}{s^2-25}\right]$

$= L^{-1}\left[\dfrac{4s+15}{(s-5)(s+5)}\right]$

$= L^{-1}\left[\dfrac{1}{(s+5)}\left(\dfrac{-5}{-10}\right) + \dfrac{1}{(s-5)}\left(\dfrac{35}{10}\right)\right]$

$= \dfrac{1}{2} L^{-1}\left[\dfrac{1}{s+5}\right] + \dfrac{7}{2} L^{-1}\left[\dfrac{1}{s-5}\right]$

$= \dfrac{1}{2} e^{-5t} + \dfrac{7}{2} e^{5t}$

Q. $L^{-1}\left[\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right]$

$= L^{-1}\left[\dfrac{1}{(s+1)}\left(\dfrac{-2}{12}\right) + \dfrac{1}{(s-2)}\left(\dfrac{4}{-3}\right) + \dfrac{1}{(s-3)}\left(\dfrac{14}{4}\right)\right]$

$= \dfrac{-1}{6} e^{-t} - \dfrac{4}{3} e^{2t} + \dfrac{7}{2} e^{3t}$

Q. Find $L^{-1}\left[\frac{21s-33}{(s-2)^3}\right]$

$\Rightarrow$ ~~1st~~ Consider $\frac{21s-33}{(s-2)^3}$

$$= \frac{A}{(s-2)} + \frac{B}{(s-2)^2} + \frac{C}{(s-2)^3}$$

$$\frac{21s-33}{\cancel{(s-2)^3}} = \frac{(s-2)^2A + (s-2)B + C}{\cancel{(s-2)^3}}$$

$$21s-33 = (s-2)^2A + (s-2)B + C$$

put $s=2$, $\boxed{91 = C}$

put $s=0$, $-33 = 4A - 2B + 91$

$4A - 2B = -42 \rightarrow (1) \Rightarrow 2A - B = -21$

put $s=1$, $-12 = A - B + 9$

$A - B = -21 \rightarrow (2)$.

$$\begin{aligned} A - \cancel{B} &= -\cancel{21} \\ 2A - \cancel{B} &= -\cancel{21} \\ - \quad \cancel{+} & \quad \cancel{+} \end{aligned}$$

$\boxed{A = 0}$ $\therefore$ $\boxed{B = 21}$

$\therefore L^{-1}\left[\frac{21s-33}{(s-2)^3}\right] = L^{-1}\left[\frac{21}{(s-2)^2} + \frac{9}{(s-2)^3}\right]$

$= 21e^{2t}.t + 9.e^{2t}t^2$

10-11-19

Q. Find $L^{-1}\left[\dfrac{5s+3}{(s+1)(s^2+2s+5)}\right]$

Consider,

$$\frac{5s+3}{(s+1)(s^2+2s+5)} = \frac{A}{s+1} + \frac{Bx+C}{s^2+2s+5}$$

$$\frac{5s+3}{(s+1)(s^2+2s+5)} = \frac{(s^2+2s+5)A + (Bs+C)(s+1)}{(s+1)(s^2+2s+5)}$$

$$5s+3 = (s^2+2s+5)A + (Bs+C)(s+1)$$

$1-2+5$

put $s=-1$

$$-2 = 4A$$

$$\therefore A = \frac{-1}{2}$$

put $s=0$

$$3 = 5A + C$$

$$C = 3+\frac{5}{2} = \frac{11}{2}$$

put $s=1$

$$8 = 8A + 2B + 2C$$

$$8 = -4 + 2B + 11$$

$$2B = 8+4-11$$

$$B = \frac{1}{2}$$

$$\therefore L^{-1}\left[\frac{5s+3}{(s+1)(s^2+2s+5)}\right] = \frac{-1}{2} L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{\frac{1}{2}s+\frac{11}{2}}{s^2+2s+5}\right]$$

H.W

$$= \frac{-1}{2} L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2} L^{-1}\left[\frac{s+1+10}{(s+1)^2+2^2}\right]$$

$$= \frac{-1}{2} e^{-t} + \frac{1}{2} L^{-1}\left[\frac{s+1}{(s+1)^2+2^2} + 10 \cdot \frac{1}{(s+1)^2+2^2}\right]$$

$$= \frac{-1}{2} e^{-t} + \frac{1}{2} e^{-t}\cos 2t + 5 e^{-t} \frac{\sin 2t}{2}$$

$$= \frac{e^{-t}}{2}\left[\cos 2t + 5\sin 2t - 1\right]$$

Q. Find $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$

Consider,

$$\frac{1}{s^2(s^2+a^2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+a^2}$$

$$1 = s(s^2+a^2)A + (s^2+a^2)B + s^2(Cs+D)$$

put $s=0$

$$1 = a^2 . B$$

$$\therefore B = \frac{1}{a^2}$$

put $s=1$

$$1 = (1+a^2)A + (1+a^2)\frac{1}{a^2} + C + D$$

$$1 = (1+a^2)A + \frac{1}{a^2} + 1 + C + D$$

$$(1+a^2)A + C + D = \frac{-1}{a^2} \longrightarrow (1)$$

put $s=2$

$$1 = (4+a^2)2A + (4+a^2)\frac{1}{a^2} + 4(2C+D)$$

$$1 = (8+2a^2)A + \frac{4}{a^2} + 1 + 8C + 4D.$$

$$(8+2a^2)A + 8C + 4D = \frac{-4}{a^2} \rightarrow (2)$$

put $s=-1$

$$1 = -1(1+a^2)A + (1+a^2)\frac{1}{a^2} - C + D$$

$$1 = -(1+a^2)A + \frac{1}{a^2} + 1 - C + D$$

$$\therefore -(1+a^2)A - C + D = \frac{-1}{a^2} \rightarrow (3).$$

Solving (1) & (3). Adding.

$$2D = -\frac{2}{a^2}$$

$$\boxed{D = \frac{-1}{a^2}}$$

Subtracting (1) $-$ (3)

$$2(1+a^2)A + 2C = 0.$$

$$(1+a^2)A + C = 0$$

$$C = -(1+a^2)A$$

$\therefore$ From (2),

$$(8+2a^2)A + 8C + 4D = -\frac{4}{a^2}$$

$$(8+2a^2)A - 8(1+a^2)A - \frac{4}{a^2} = \frac{-4}{a^2}$$

$$\cancel{8A} + 2Aa^2 - \cancel{8A} - 8a^2A = 0$$

$$\boxed{A=0}$$

$$\Rightarrow \boxed{C=0}$$

classmate

Date

Page

$$\therefore L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = L^{-1}\left[\frac{1/a^2}{s^2} - \frac{1/a^2}{(s^2+a^2)}\right]$$

$$= \frac{1}{a^2} L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{a^2} L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \frac{1}{a^2} . t - \frac{1}{a^2} \frac{\sin at}{a}$$

$$= \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]$$

Q. Find $L^{-1}\left[\frac{2s^2-1}{(s^2+1)(s^2+4)}\right]$

$$\frac{2s^2-1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$2s^2-1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$= As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + Cs + D$$

~~put $s=0$~~

$\cancel{-1 = 4B + D}$

$$2s^2-1 = (A+C)s^3 + (B+D)s^2 + (4A+C)s + 4B+D$$

Equating like terms.

$(A+C) = 0 \rightarrow$ (1)

$(B+D) = 2 \rightarrow$ (3)

$(4A+C) = 0 \rightarrow$ (2)

From (1) & (2) $A=0$, $C=0$

$4B+D = -1 \rightarrow$ (4)

Solving ③ and ④

$-3B = 3$

$\boxed{B = -1}$, $\boxed{D = 3}$

$\therefore L^{-1}\left[\frac{2s^2-1}{(s^2+1)(s^2+4)}\right]$

$= L^{-1}\left[\frac{-1}{s^2+1} + \frac{3}{s^2+4}\right]$

$= -\sin t + \frac{3\sin 2t}{2}$

Q. Find $L^{-1}\left[\frac{s}{s^4+4a^4}\right]$

WKT, $a^2+b^2 = (a+b)^2 - 2ab$

$s^4+4a^4 = (s^2)^2 + (2a^2)^2$

$= (s^2+2a^2)^2 - 4s^2a^2$

$= \cancel{s^2+2a^2+4s^2a}$

$= (s^2+2a^2+2as)(s^2+2a^2-2as)$

$\therefore \frac{s}{s^4+4a^4} = \frac{s}{(s^2+2a^2+2as)(s^2+2a^2-2as)}$

After solving by partial fractions,

$A = 0$ $B = \frac{1}{4a}$, $C = 0$, $D = \frac{1}{4a}$

$\Rightarrow \frac{1}{4a} \cdot L^{-1}\left[\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right]$

$(s^2+1)^2 - s^2$

$(s^2+1+s)(s^2-s+1)$

H.W.

$$= L^{-1}\left[\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right]\cdot\frac{1}{4a}$$

$$= \left[e^{at}\frac{\sin at}{a} - e^{-at}\frac{\sin at}{a}\right]\cdot\frac{1}{4a}$$

Q. $L^{-1}\left[\frac{s}{s^4+s^2+1}\right]$

$s^4+s^2+1 = (s^2)^2 + s^2+1+2s^2-2s^2$

$= (s^2+s+1)(s^2-s+1)$

$\therefore \frac{s}{(s^4+s^2+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2-s+1}$

$s = As^3 - s^2A + sA + Bs^2 - Bs + B + s^3C + s^2C + Cs + s^2D + sD + D$

$s = (A+C)s^3 + (-A+C+B+D)s^2 + (A-B+C+D)s + B+D$

$A+C=0 \quad \rightarrow (1)$

$-A+C+B+D=0 \quad \rightarrow (2)$

$B+D=0 \quad \rightarrow (3)$

$A-B+C+D=1 \quad \rightarrow (4)$

put (3) in (2), $-A+C=0$

$-A+C=0$ (1) $\Rightarrow A+C=0$

$\Rightarrow A=0,\ C=0$

put (1) in (4), $-B+D=1$

$B+D=0$

$2D=1$

$\Rightarrow D=\frac{1}{2},\quad B=\frac{-1}{2}$

$$\Rightarrow L^{-1}\left[\frac{-1/2}{s^2+s+1}+\frac{1/2}{s^2-s+1}\right]$$

$$=\frac{1}{2}L^{-1}\left[\frac{1}{s^2-s+1+\frac{1}{4}-\frac{3}{4}}-\frac{1}{s^2+s+1+\frac{1}{4}-\frac{3}{4}}\right]$$

$$=\frac{1}{2}L^{-1}\left[\frac{1}{(s-1/2)^2-(\sqrt{3}/2)^2}-\frac{1}{(s+1/2)^2-(\sqrt{3}/2)^2}\right].$$

$$=\frac{1}{2}\left[\frac{e^{+t/2}\sin\frac{\sqrt{3}}{2}t}{\sqrt{3}/2}-\frac{e^{-t/2}\sin\frac{\sqrt{3}}{2}t}{\sqrt{3}/2}\right]$$

$$=\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\left[e^{t/2}-e^{-t/2}\right]$$

Q. $$L^{-1}\left[\frac{s^3}{s^4-a^4}\right]=L^{-1}\left[\frac{s^3}{(s^2-a^2)(s^2+a^2)}\right]$$

$$=L^{-1}\left[\frac{s^3}{(s+a)(s-a)(s^2+a^2)}\right]$$

$$\Rightarrow L^{-1}\left[\frac{A}{s+a}+\frac{B}{s-a}+\frac{Cs+D}{s^2+a^2}\right]$$

Scanned with CamScanner

21-11-19

- Applications of Laplace transforms for DIFFERENTIAL EQUATIONS.

WKT,

$$L(y') = sL(y) - y(0)$$

$$L(y'') = s^2 L(y) - s(y(0)) - y'(0)$$

$$L(y''') = s^3 L(y) - s^2 y(0) - s y'(0) - y''(0)$$

Q Solve $(D^2 + 4D + 4)y = e^{-t}$ with $y(0) = y'(0) = 0$

$$y'' + 4y' + 4y = e^{-t}$$

Apply Laplace transform

$$L[y'' + 4y' + 4y] = L[e^{-t}]$$

$$[s^2 L(y) - s y(0) - y'(0)] + 4[sL(y) - y(0)] + 4\overset{L(y)}{\cancel{y}} = \frac{1}{s+1}$$

$$L(y)(s^2 + 4s + 4) = \frac{1}{s+1}$$

$$L(y) = \frac{1}{(s+1)(s+2)^2}$$

$$y = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2}\right]$$

After calculation, $A = 1$, $B = -1$, $C = -1$

classmate

Date

Page

$$= L^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}\right]$$

$$= e^{-t} - e^{-2t} - e^{-2t}.t$$

Q. $y'' - 3y' + 2y = 4t + e^{3t}$ $\quad y(0) = 1 \quad y'(0) = -1$

$$L[y'' - 3y' + 2y] = L[4t] + L[e^{3t}]$$

$$[s^2 L(y) - s y(0) - y'(0)] - 3[sL(y) - y(0)] + 2L(y) = \frac{4}{s^2} + \frac{1}{s-3}$$

$$L(y)(s^2 - 3s + 2) - s + 1 + 3 = \frac{4}{s^2} + \frac{1}{s-3}$$

$$L(y)(s^2 - 3s + 2) = \frac{4}{s^2} + \frac{1}{s-3} + s - 4$$

$$L(y) = \frac{4(s-3) + s^2 + (s-4)s^2(s-3)}{s^2(s-3)(s^2-3s+2)}$$

$$L(y) = \frac{4(s-3) + s^2 + s^2(s-3)(s-4)}{s^2(s-3)(s-2)(s-1)}$$

Consider $\dfrac{4(s-3) + s^2 + s^2(s-3)(s-4)}{s^2(s-1)(s-2)(s-3)}$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-2} + \frac{E}{s-3}$$

$$= \frac{s(s-1)(s-2)(s-3)A + (s-1)(s-2)(s-3)B + s^2(s-2)(s-3)C + s^2(s-1)(s-3)D + s^2(s-1)(s-2)E}{s^2(s-1)(s-2)(s-3)}$$

Scanned with CamScanner

put $s=0$

$$-12 = -6B$$

$$\boxed{B=2}$$

put $s=1$

$$-8+1+6 = 2C$$

$$\boxed{C=-1/2}$$

put $s=2$

$$-4+4+8 = -4D$$

$$\boxed{D=-2}$$

put $s=3$

$$9 = 18E$$

$$\boxed{E=1/2}$$

put $s=-1$

$$-16+1+20 = 24A - 24(2) + 12\left(\frac{-1}{2}\right) + 8(-2) + 6\left(1/2\right)$$

$$5 = 24A - 48 - 6 - 16 + 3$$

$$\boxed{A=3}$$

$$\therefore y = L^{-1}\left[\frac{3}{s} + \frac{2}{s^2} - \frac{1/2}{s-1} - \frac{2}{s-2} + \frac{1/2}{s-3}\right]$$

$$y = 3 + 2t - \frac{1}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}$$

Q. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}\sin t \qquad x(0)=0, \quad x'(0)=1$

$$L[x''] + 2L[x'] + 5L[x] = \frac{1}{(s+1)^2+1}$$

$$[s^2L(x) - sx(0) - x'(0)] + 2[sL[x] - x(0)] + 5L[x] = \frac{1}{(s+1)^2+1}$$

After solving, $A=C=0$, $B=1/3$, $D=2/3$

and $x = \frac{1}{3}e^{-t}(\sin t + \sin 2t)$

Q $y''' + y'' = e^t + t + 1 \qquad y(0) = y'(0) = y''(0) = 0$

$L(y''') + L(y'') = L(e^t) + L(t) + L(1)$

$s^3 L(y) - s^2 y(0) - s y'(0) - y''(0) + s^2 L(y) - s y(0) - y'(0) = \frac{1}{s-1} + \frac{1}{s^2} + \frac{1}{s}$

$L(y) \cancel{(s^3 + s^2)} = \frac{s^2 + s - 1 + s(s-1)}{s^2(s-1)\, s^2(s+1)}$

$y = L^{-1}\left[\frac{2s^2 - s - 1 + s}{s^4(s-1)(s+1)}\right]$

Consider,

$\frac{2s^2-1}{s^4(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E}{s-1} + \frac{F}{s+1}$

$2s^2 - 1 = s^3(s-1)(s+1)A + s^2(s-1)(s+1)B + s(s-1)(s+1)C + (s-1)(s+1)D + s^4(s+1)E + s^4(s-1)F.$

put $s=0$

$-1 = -D \qquad \boxed{D=1}$

put $s=1$

$1 = 2E \qquad \boxed{E = 1/2}$

put $s=-1$

$1 = -2F \qquad \boxed{F = -1/2}$

put $s=2$

$7 = 24A + 12B + 6C + 3 + 48\left(\frac{1}{2}\right) + 16\left(\frac{-1}{2}\right)$

$24A + 12B + 6C = -12$

$4A + 2B + C = -2 \rightarrow ①$

put $s=-2$

$7 = -24A + 12B - 6C + 3 - 48\left(\frac{-1}{2}\right) - 16\left(\frac{1}{2}\right)$

$-4A + 2B - C = -2 \rightarrow ②$

classmate

Date ____
Page ____

put $s=3$

$$17 = 216A + 72(B) + 24C + 8 + 324(1/2) + 162(-1/2)$$

$$17 = 216 + 72B + 24C + 89$$

$$216A + 72B + 24C = -72$$

$$9A + 3B + C = -3 \rightarrow (3)$$

(1) + (2) $\Rightarrow$

$$4B = -4 \qquad \boxed{B = -1}$$

put in (3) $\qquad 9A + C = 0$

(1) − (2) $\Rightarrow$ $8A + 2C = 0$

on solving
$$8A + 2C = 0$$
$$18A + 2C = 0$$
$$- \quad - \quad -$$
$$-10A = 0$$
$$A = 0, \quad C = 0.$$

$$\therefore \ y = L^{-1}\left[\frac{-1}{s^2} + \frac{1}{s^4} - \frac{1/2}{s+1} + \frac{1/2}{s-1}\right]$$

$$y = -t + \frac{t^3}{3!} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}.$$