Lecture Notes

MODUEL-1

1. **1.1 Introduction to Matrices**: The matrix has a long history of application in solving linear equations and study the nature of system of linear equations

2. **Matrix:** A system of m.n real (or) complex numbers are arranged in the form of 'm' rows, each row consisting of 'n' columns an ordered set of 'numbers is called matrix .This is denoted by [ ] (or) ( ) (or) | |

3. Eg: $\begin{bmatrix} a_{11}a_{12}\ldots\ldots\ldots a_{1n} \\ a_{21}a_{12}\ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}a_{m2}\ldots\ldots a_{mn} \end{bmatrix}$ m x n= [aij]mxn where 1≤ i ≤ m, 1≤ j ≤ n.

4. Oder of a matrix: In a matrix number of rows and columns is called Oder of matrix this is denoted by m x n or m/n

1.1.2 Types of Matrices

5. **Square matrix:** In a matrix number of rows equals to number of columns is called A square matrix. Or A matrix of order n x n is called a square matrix of order n

   Eg: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is 2^{nd} order matrix

6. **Rectangular matrix:** In a matrix number of rows not equals to number of columns this type of matrix is called a rectangular matrix,

   Eg: $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2x3 matrix

7. **Row matrix:** A matrix of order 1xm is called a row matrix

   Eg: $[1 \quad 2 \quad 3]_{1x3}$

8. **Column matrix:** A matrix of order nx1 is called a column matrix

   Eg: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3x1}$

9. **Unit matrix:** if A= $[a_{ij}]_{nxn}$ such that a_{ij} = 1 for i = j and a_{ij} = 0 for i≠j, then A is called a unit matrx.

   Eg: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. **Null matrix :** it A = $[a_{ij}]_{mxn}$ such that $a_{ij} = 0$ ∀ i and j then A is called a zero matrix (or) null matrix

Eg: O = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3x3}$

1. **Principal Diagonal elements in a matrix:**

$A= [a_{ij}]_{nxn}$, the elements a_{ij} of A for which i = j. i.e. $(a_{11}, a_{22}....a_{nn})$ are called the principal diagonal elements of A

Eg: A= $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ diagonal elements are 1, 5, 9

2. **Diagonal matrix:** A square matrix all of whose elements except those in leading diagonal are zero is called diagonal matrix.

If $d_1, d_2..... d_n$ are diagonal elements of a diagonal matrix A, then A is written as A = diag $(d_1,d_2....d_n)$

E.g.: A = diag (3, 1, -2) = $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

3. **Scalar matrix:** A diagonal matrix whose leading diagonal elements are equal is called a scalar matrix.

E.g.: A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

4. **Equal Matrices:** Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if and only if
   (i) A and B are of the same type(order)
   (ii) aij = bij for every i & j

5. **The transpose of a matrix:** The matrix obtained from any given matrix A, by interchanging its rows and columns is called the transpose of A. It is denoted by A^1 (or) A^T.

If $A = [a_{ij}]_{m \times n}$ then the transpose of A is $A^1 = [b_{ji}]_{n \times m}$, where $b_{ji} = a_{ij}$ Also $(A^1)^1 = A$

Note: A^1 and B^1 be the transposes of A and B respectively, then

(i) $(A^1)^1 = A$

(ii) $(A+B)^1 = A^1+B^1$

(iii) $(KA)^1 = KA^1$, K is a scalar

(iv) $(AB)^1= B^1A^1$

6. **The conjugate of a matrix:** The matrix obtained from any given matrix A, on replacing its elements by corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$

Note: if $\bar{A}$ and $\bar{B}$ be the conjugates of A and B respectively then,

(i) $\overline{(\bar{A})}$ = A

(ii) $\overline{(A+B)} = \bar{A} + \bar{B}$

(iii) $(\overline{KA}) = \overline{KA}$ (K is a any complex number)

(iv) $(\overline{AB}) = \overline{AB}$

E.g.; if A=$\begin{bmatrix} 2 & -3i \\ 3+i & -3i \end{bmatrix}$ and $\bar{A} = \begin{bmatrix} 2 & 3i \\ 3-i & 3i \end{bmatrix}$

7. The conjugate Transpose of a matrix:

The conjugate of the transpose of the matrix A is called the conjugate transpose of A and is

denoted by A^{θ} Thus $A^{\theta} = (\bar{A})^1$

i.e. the $(i,j)^{th}$ element of A^{θ} conjugate complex of the $(j, i)^{th}$ element of A.

E.g.: if A = $\begin{bmatrix} 2 & -3i & -i \\ 3+i & -3i & 1 \end{bmatrix}$ then A^{θ} $=\begin{bmatrix} 2 & 3-i \\ 3i & 3i \\ i & 1 \end{bmatrix}$

14. (i) Upper Triangular matrix: A square matrix all of whose elements below the leading diagonal are zero is called an Upper triangular matrix.

E.g.: $\begin{bmatrix} 2 & 3 & 8 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ is an upper triangular matrix

(ii) **Lower triangular matrix:** A square matrix all of whose elements above the leading diagonal are zero is called a lower triangular matrix. i.e, $a_{ij=0}$ for i< j

E.g.: $\begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 6 & 5 & 2 \end{bmatrix}$

(iii) **Triangular matrix:** A matrix is said to be triangular matrix it is either an upper Triangular matrix or a lower triangular matrix

15. **Symmetric matrix:** A square matrix A =$[a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for every i and j thus, A is a symmetric matrix if A^T= A

E.g.: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

16. Skew – Symmetric: A square matrix A = $[a_{ij}]$ is said to be skew – symmetric if $a_{ij} = - a_{ji}$ for every i and j.

E.g.: $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew – symmetric matrix

Thus, A is a skew – symmetric iff $A= -A^1$ or $-A= A^1$

Note: Every diagonal element of a skew – symmetric matrix is necessarily zero.

Since $a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$

17. Multiplication of a matrix by a scalar.

Let 'A' be a matrix. The matrix obtains by multiplying every element of A by a scalar K, is called the product of A by K and is denoted by KA.

Thus: $A = [a_{ij}]$ then $KA = [ka_{ij}] = k[a_{ij}]$

18. Sum of matrices:

Let $A = [a_{ij}]_{mxn}$,$B = [b_{ij}]_{mxn}$ be two matrices. The matrix $C = [c_{ij}]_{mxn}$ where $c_{ij} = a_{ij}+b_{ij}$ is called the sum of the matrices A and B.

The sum of A and B is denoted by A+B.

Thus $[a_{ij}]_{mxn} + [b_{ij}]_{mxn} = [a_{ij}+b_{ij}]_{mxn}$ and $[a_{ij}+b_{ij}]_{mxn} = [a_{ij}]_{mxn} + [b_{ij}]_{mxn}$

19. The difference of two matrices: If A, B are two matrices of the same type then A+ (-B) is taken as A – B

20. Matrix multiplication: Let $A = \left[a_{ij}\right]_{mxn}$ $B = [b_{kj}]_{nxp}$ then the matrix $C = [c_{ij}]_{mxp}$ where c_{ij}

is called the product of the matrices A and B in that order and we write C = AB.

The matrix A is called the pre-factor & B is called the post – factor

Note: If the number of columns of A is equal to the number of rows in B then the matrices are said to be conformable for multiplication in that order.

21. Positive integral powers of a square matrix:

Let A be a square matrix. Then A^2 is defined A.A

Now, by associative law $A^3 = A^2.A = (AA)A= A(AA) = AA^2$

Similarly, we have $A^{m-1}A = A\,A^{m-1} = A^m$ where m is a positive integer

Note 1: Multiplication of matrices is distributive w.r.t. addition of matrices.

i.e, A (B+C) =AB + AC

(B+C)A =BA+CA

Note 2: If A is a matrix of order mxn then $AI_n = I_nA = A$

22. **Trace of A square matrix:** Let $A = [a_{ij}]_{mxn}$ the trace of the square matrix A is defined as $\sum_{i=1}^{n} a_{ii}$ and is denoted by 'tr A'

Thus $trA = \sum_{i=1}^{n} a_{ii} = a_{11}+a_{22}+ \ldots\ldots a_{nn}$

E.g.: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then trA = a+b+c

Properties:_ If A and B are square matrices of order n and λ is any scalar, then

(i) $tr(\lambda A) = \lambda\, tr A$

(ii) $tr(A+B) = trA + tr B$

(iii) $tr(AB) = tr(BA)$

23. Idempotent matrix: If A is a square matrix such that $A^2 = A$ then 'A' is called idempotent matrix

24. Nilpotent Matrix: If A is a square matrix such that $A^m=0$ where m is a +ve integer

then A is called nilpotent matrix.

Note: If m is least positive integer such that $A^m = 0$ then A is called nilpotent of index m

25. Involuntary: If A is a square matrix such that $A^2 = I$ then A is called involuntary matrix.

26. Orthogonal Matrix: A square matrix A is said to be orthogonal if $AA^T = A^TA = I$

27. Minors and cofactors of a square matrix

Let $A =[a_{ij}]_{n\times n}$ be a square matrix when form A the elements of i^{th} row and j^{th}

column is deleted the determinant of (n-1) rowed matrix [Mij] is called the minor

of a_{ij} of A and is denoted by $|M_{ij}|$

The signed minor $(-1)^{i+j}|M_{ij}|$ is called the cofactor of a_{ij} and is denoted by A_{ij}..

If A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$|A| = a_{11}|M_{11}| + a_{12}|M_{12}| + a_{13}|M_{13}|$ (or)

$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

E.g.: Find Determinant of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ by using minors and co-factors.

Sol: A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

det A = $1\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix}$

$=1(-12-12)-1(-4-6)+3(-4+6)$

$= -24+10+6 = -8$

Similarly, we can find det A by using co-factors also.

Note:

1. If A is a square matrix of order n then, $|KA| = K^n|A|$ where k is a scalar.

2. If A is a square matrix of order n, then $|A| = |A^T|$

3: If A and B be two square matrices of the same order, then $|AB| = |A||B|$

28. Inverse of a Matrix: Let A be any square matrix, then a matrix B, if exists such that

AB = BA =I then B is called inverse of A and is denoted by A^{-1}.

Note 1: $(A^{-1})^{-1} = A$ 2: $I^{-1} = I$

29. Adjoint of a matrix:

Let A be a square matrix of order n. The transpose of the matrix got from A by replacing the elements of A by the corresponding co-factors is called the adjoint of A and is denoted by adj A.

Note:

1. For any scalar k, adj(kA) = k^{n-1} adj A
2. The necessary and sufficient condition for a square matrix to possess inverse is that $|A| \neq 0$
3. if $|A| \neq 0$ then $A^{-1} = \frac{adjA}{|A|}$

30. Singular and Non-singular Matrices:

A square matrix A is said to be singular if $|A| = 0$

If $|A| \neq 0$then 'A' is said to be non-singular.

Note: 1. only non-singular matrices possess inverses.

2. The product of non-singular matrices is also non-singular.

3. If A, B are invertible matrices of the same order, then

(i). $(AB)^{-1} = B^{-1}A^{-1}$

(ii). $(A^1)^{-1} = (A^{-1})^1$

1.1.3 Rank of a Matrix:

Let A be m x n matrix. If A is a null matrix, we define its rank to be '0'. If A is a non-

zero matrix, we say that r is the rank of A if

(i) Every $(r+1)^{th}$ order minor of A is '0' (zero) &

(ii) At least one r^{th} order minor of A which is not zero.

Note:

1. It is denoted by ρ (A)
2. Rank of a matrix is unique.
3. Every matrix will have a rank.
4. If A is a matrix of order mxn,

   Rank of A $\leq$ min (m,n)
5. If $\rho(A) = r$ then every minor of A of order r+1, or more is zero.
6. Rank of the Identity matrix I_n is n.
7. If A is a matrix of order n and A is non-singular then $\rho(A) = n$
8. The rank of a matrix is $\leq r$ if all minors of (r+1)th order are zero.
9. The rank of a matrix is $\geq r$, if there is at least one minor of order 'r' which is not equal to zero.

1. Find the rank of the given matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Sol: Given matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

$\rightarrow$ det A = 1(48-40)-2(36-28) +3(30-28)

= 8-16+6 = -2 $\neq$ 0

We have minor of order 3

$\rho(A) = 3$

2. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol: Given the matrix is of order 3x4

Its Rank $\leq$ min (3,4) = 3

Highest order of the minor will be 3.

Let us consider the minor $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

Determinant of minor is = 1(-49)-2(-56) +3(35-48)

= -49+112-39 = 24 $\neq$ 0.

Hence rank of the given matrix is '3'.

1.1.4 Elementary Transformations on a Matrix:

i). Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$

(ii). If i^{th} row is multiplied with k then it is denoted by $R_i \rightarrow K\,R_i$

(iii). If all the elements of i^{th} row are multiplied with k and added to the corresponding

elements of j^{th} row then it is denoted by $R_j \rightarrow R_j + KR_i$

Note: 1. The corresponding column transformations will be denoted by writing 'c'. i.e

$c_i \leftrightarrow c_j$, $c_i \rightarrow k\,c_j$ $c_j \rightarrow c_j + kc_i$

2. The elementary operations on a matrix do not change its rank.

Equivalence of Matrices: If B is obtained from A after a finite number of elementary transformations on A, then B is said to be equivalent to A and it is denoted as B~A.

Note: 1. If A and B are two equivalent matrices, then rank A = rank B.

2. If A and B have the same size and the same rank, then the two matrices are

equivalent.

1.1.5 Echelon form of a matrix:

A matrix is said to be in Echelon form, if

(i). Zero rows, if any exists, they should be below the non-zero row.

(ii). the first non-zero entry in each non-zero row is equal to '1'.

(iii). the number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note:

1. The number of non-zero rows in echelon form of A is the rank of 'A'.
2. The rank of the transpose of a matrix is the same as that of original matrix.
3. The condition (ii) is optional.

E.g.:

1. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

2. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ is a row echelon form.

PROBLEMS

1. Find the rank of the matrix A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ by reducing it to Echelon form.

Sol: Given A = A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Applying row transformations on A

$R_1 \leftrightarrow R_3$

A ~ $\begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$

$R_2 \rightarrow R_2 - 3R_1$, $R_3 \rightarrow R_3 - 2R_1$

~ $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$

$R_2 \rightarrow R_2/7$, $R_3 \rightarrow R_3/9$

~ $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_2$

~$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

This is the Echelon form of matrix A.

The rank of a matrix A = Number of non-zero rows =2

2. For what values of k the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$ has rank '3'.

Sol: The given matrix is of the order 4x4

If its rank is 3 ⇨ det A =0

$$⇨ \begin{vmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{vmatrix} = 0$$

By applying $R_2 \rightarrow 4R_2 - R_1$, $R_3 \rightarrow 4R_3 - kR_1$, $R_4 \rightarrow 4R_4 - 9R_1$

$$\sim \begin{vmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{vmatrix} = 0$$

$$\sim 4 \begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$$

⇨ 1[(8-4k)3]-1(8-4k) (4k+27)] = 0

⇨ (8-4k) (3-4k-27) = 0

⇨ (8-4k) (-24-4k) =0

⇨ (2-k) (6+k) =0

⇨ k =2 or k = -6

3. Find the rank of the matrix by using Echelon form $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

By applying$R_2 \rightarrow R_2 - 2R_1; R_3 \rightarrow R_3 - 4R_1\,; R_4 \rightarrow R_4 - 4R_1$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

By applying$R_2 \rightarrow \frac{R_2}{-1}; R_3 \rightarrow \frac{R_3}{-1}; R_4 \rightarrow \frac{R_4}{-3}$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 5 & 7 \end{bmatrix}$$

By applying $R_3 \rightarrow R_3 - R_2\,; R_4 \rightarrow R_4 - R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴A is in Echelon form

Rank of A= Number of non-zero rows

$\therefore \rho(A) = 2$

4. Find the rank of the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{0} & \mathbf{1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{3} & \mathbf{-3} & \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$ by reducing it to Echelon form.

Sol: Given $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

By applying$R_2 \rightarrow R_2 - 2R_1; R_3 \rightarrow R_3 - 3R_1\,; R_4 \rightarrow R_4 + R_1$

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix}$$

By applying $R_3 \rightarrow R_3 - R_2\,; R_4 \rightarrow R_4 + R_2$

$$A\sim\begin{bmatrix}1 & -2 & 0 & 1\\ 0 & 3 & 1 & -2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$\therefore$ A is in Echelon form

Rank of A= Number of non-zero rows

$\therefore \rho(A) = 2$

5. Find the rank of the matrix $\boldsymbol{A} = \begin{bmatrix}\mathbf{-1} & \mathbf{2} & \mathbf{1} & \mathbf{8}\\ \mathbf{2} & \mathbf{1} & \mathbf{-1} & \mathbf{0}\\ \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{7}\end{bmatrix}$ by reducing it to Echelon form.

Sol: Given $A = \begin{bmatrix}-1 & 2 & 1 & 8\\ 2 & 1 & -1 & 0\\ 3 & 2 & 1 & 7\end{bmatrix}$

By applying $R_1 \to -R_1$

$$A\sim\begin{bmatrix}1 & -2 & -1 & -8\\ 2 & 1 & -1 & 0\\ 3 & 2 & 1 & 7\end{bmatrix}$$

By applying $R_2 \to R_2 - 2R_1\,;\, R_3 \to R_3 - 3R_1$

$$A\sim\begin{bmatrix}1 & -2 & -1 & -8\\ 0 & 5 & 1 & 16\\ 0 & 8 & 4 & 31\end{bmatrix}$$

By applying $R_2 \to 8R_2$

$$A\sim\begin{bmatrix}1 & -2 & -1 & -8\\ 0 & 40 & 8 & 128\\ 0 & 8 & 4 & 31\end{bmatrix}$$

By applying $R_2 \leftrightarrow R_3$

$$A\sim\begin{bmatrix}1 & -2 & -1 & -8\\ 0 & 8 & 4 & 31\\ 0 & 40 & 8 & 128\end{bmatrix}$$

By applying $R_3 \to R_3 - 5R_2$

$$A\sim\begin{bmatrix}1 & -2 & -1 & -8\\ 0 & 8 & 4 & 31\\ 0 & 0 & -12 & -27\end{bmatrix}$$

$\therefore$ A is in Echelon form

Rank of A= Number of non-zero rows

$\therefore \rho(A) = 3$

6. Find the rank of the matrix $\boldsymbol{A} = \begin{bmatrix}\mathbf{1} & \mathbf{4} & \mathbf{3} & \mathbf{-2} & \mathbf{1}\\ \mathbf{-2} & \mathbf{-3} & \mathbf{-1} & \mathbf{4} & \mathbf{3}\\ \mathbf{-1} & \mathbf{6} & \mathbf{7} & \mathbf{2} & \mathbf{9}\\ \mathbf{-3} & \mathbf{3} & \mathbf{6} & \mathbf{6} & \mathbf{12}\end{bmatrix}$ by reducing it to Echelon form.

Sol: Given $A = \begin{bmatrix}1 & 4 & 3 & -2 & 1\\ -2 & -3 & -1 & 4 & 3\\ -1 & 6 & 7 & 2 & 9\\ -3 & 3 & 6 & 6 & 12\end{bmatrix}$

By applying $R_2 \to R_2 + 2R_1;\, R_3 \to R_3 + R_1\,;\, R_4 \to R_4 + 3R_1$

$$A\sim\begin{bmatrix}1&4&3&-2&1\\0&5&5&0&5\\0&10&10&0&10\\0&15&15&0&15\end{bmatrix}$$

By applying $R_3 \to R_3 - 2R_3\,;R_4 \to R_4 - 3R_2$

$$A\sim\begin{bmatrix}1&4&3&-2&1\\0&5&5&0&5\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

∴ A is in Echelon form

Rank of A= Number of non-zero rows

$\therefore \rho(A) = 2$

1.2. Normal Form:

Every mxn matrix of rank r can be reduced to the form $\begin{bmatrix}I_r&0\\0&0\end{bmatrix}$ (or) $\begin{bmatrix}I_r\\0\end{bmatrix}$ (or) $[I_r \quad 0]$ by a finite number of elementary transformations, where $I_r$is the r – rowed unit matrix.

Normal form another name is "canonical form"

1. By reducing the matrix $\begin{bmatrix}1&2&3&4\\2&1&4&3\\3&0&5&-10\end{bmatrix}$ into normal form, find its rank.

Sol: Given A = $\begin{bmatrix}1&2&3&4\\2&1&4&3\\3&0&5&-10\end{bmatrix}$

By applying $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix}1&2&3&4\\0&-3&-2&5\\0&-6&-4&-22\end{bmatrix}$$

By applying $R_3 \to R_3/-2$

$$A \sim \begin{bmatrix}1&2&3&4\\0&-3&-2&5\\0&3&2&11\end{bmatrix}$$

By applying $R_3 \to R_3+R_2$

$$A \sim \begin{bmatrix}1&2&3&4\\0&-3&-2&5\\0&0&0&6\end{bmatrix}$$

By applying $C_2\to C_2 - 2C_1$, $C_3\to C_3-3C_1$, $C_4\to C_4-4C_1$

$$A \sim \begin{bmatrix}1&0&0&0\\0&-3&-2&5\\0&0&0&6\end{bmatrix}$$

By applying $C_3 \to 3\,C_3 -2C_2$, $C_4\to 3C_4-5C_2$

$$A \sim \begin{bmatrix}1&0&0&0\\0&-3&0&0\\0&0&0&18\end{bmatrix}$$

By applying $C_2\to C_2/-3$, $C_4\to C_4/18$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By applying $C_4 \leftrightarrow C_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This is in normal form $[I_3\ 0]$

Hence Rank of A is '3'.

1.3. SYSTEM OF HOMOGENEOUS AND NON-HOMOGENEOUS LINEAR EQUATIONS:

An equation of the form $a_1x_1 + a_2x_2 + \ldots\ldots\ldots\ldots + a_nx_n = b$ where $x_1, x_2, \ldots\ldots, x_n$ are unknowns and $a_1, a_2, \ldots\ldots\ldots, a_n, b$ are constants is called linear equation in n unknowns .

Definition: Consider the system of m linear equations in n unknowns $x_1, x_2 \ldots\ldots, x_n$ as given below:

$$a_{11}\ \ x_1 + a_{1_2}x_2 + \ldots\ldots\ldots\ldots + a_{1_n}x_n = b_1$$
$$a_{2_1}x_1 + a_{2_2}x_2 + \ldots\ldots\ldots\ldots + a_{2_n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{i_1}x_1 + a_{i_2}x_2 + \ldots\ldots\ldots\ldots + a_{i_n}x_n = b_i$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m_1}x_1 + a_{m_2}x_2 + \ldots\ldots\ldots\ldots + a_{m_n}x_n = b_m$$

The number a_{ij}'s are known as coefficient and $b_1, b_2, \ldots\ldots\ldots\ldots b_m$ are constants. An ordered n-tuple $(x_1, x_2 \ldots\ldots\ldots, x_n)$ satisfying all the equations simultaneously is called a solution of system.

Homogeneous system:

If all $b_i = 0$ for $i = 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots, m$.

Non-Homogeneous system:

If all $b_i \neq 0$ i.e.at least one $b_i \neq 0$.

Matrix Representation:

The above system of linear non-Homogeneous equations can be written in Matrix form as AX=B

$$\begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_m \end{bmatrix}$$

Augmented Matrix:

It is denoted by [A/B] or [A B] is obtained by Augmenting A by the column B.

$$\therefore [A/B] = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots\dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots\dots & \dots \\ a_{m1} & a_{m2} & \dots\dots & a_{mn} & b_m \end{bmatrix}$$

By reducing [A /B] into its row echelon form the existence and uniqueness of solution AX = B exists.

NOTE:

Given a system, we do not know in general whether it has a solution or not. If there is at least one solution, then the system is said to be consistent. If does not have any solution then the system is inconsistent.

1. CONSISTENT: A system is said to be consistent if it has at least one solution

a) If $\rho(AB) = \rho(A) = r = n$ (total number of unknowns) then the system is consistent and it has unique solution.

b) If $\rho(AB) = \rho(A) = r < n$ (total number of unknowns) then the system is consistent and it has Infinitely many solutions.

NOTE: To obtain solutions set (n – r) variables any arbitrary value & solve for remaining unknowns.

2. INCONSISTENT: If $\rho(AB) \neq \rho(A)$ then the system is inconsistent and it has no solution.

For Non-Homogeneous System, the system AX = B is consistent i.e. it has a solution.

The system is inconsistent i.e., it has no solution.

NOTE: Find the rank A and rank [A /B] by reducing the augmented matrix [A /B] to Echelon form by elementary row operations. Then the matrix A will be reduced to Echelon form.

This procedure is illustrated through the following examples.

Example 1: Find whether the following equations are consistent, if so solve them

x + y + 2z= 4; 2x – y + 3z = 9; 3x – y - z = 2.

Solution: The given equations can be written in the matrix form as

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

i.e .AX = B

The Augmented matrix [A/B] = $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R3 \rightarrow R3 - 3R1$

[A/B] = $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$

[A/B] = $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$

Since Rank of A = 3 & Rank of [A/B] = 3

Since the number of non- zero rows of matrix A are 3

Since the number of non- zero rows of matrix [A/B] is 3

∴ Rank of A = Rank of [A/B]

i.e. $\rho(A) = \rho(AB)$

The given system is consistent. So, it has a solution.

Since Rank of A = Rank of [A/B] = Number of unknowns (n)

∴ The given system has a unique solution.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

⇒ x + y + 2z= 4 → (1)

-3y – z = 1→ (2)

-17z = -34→ (3)

From (3) z = 2

Substituting z = 2 in eq(2), we get y = -1

Substituting z = 2 & y = -1 in eq(1), we get x = 1.

∴ x = 1, y = -1, z = 2 is the solution.

Example 2: Find whether the following system of equations is consistent. If so solve them.

$$x + 2y + 2z = 2;\ 3x - 2y - z = 5;\ 2x - 5y + 3z = -4;\ x + 4y + 6z = 0.$$

Solution: The given equations can be written in the matrix form as AX = B

i.e., $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$

The Augmented matrix [A/B] = $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{bmatrix}$

Applying $R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 2R_1$; $R4 \to R_4 - R_1$

[A/B] = $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$

Applying $R_3 \to 8R_3 - 9R_2$; $R_4 \to 4R_4 + R_2$, we get

[A/B] = $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 55 & -55 \\ 0 & 0 & 9 & -9 \end{bmatrix}$

Applying $R_3 \to R_3/55$; $R_4 \to R_4/9$

[A/B] $\approx \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

Applying $R_4 \to R_4 - R_3$

[A/B] $\approx \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Since Rank of A = 3 & Rank of [A/ B] = 3

∴ Rank of A = Rank of [A /B]

i.e. $\rho(A) = \rho(AB)$

The given system is consistent, so it has a solution.

Since Rank of A = Rank of [A /B] = Number of unknowns (n)

∴ The given system has a unique solution.

We have $\begin{bmatrix} 1 & 2 & 2 \\ 0 & -8 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

$\Rightarrow$ x + 2y + 2z = 2 $\rightarrow (1)$

-8y - 7z = -1 $\rightarrow (2)$

z = -1 $\rightarrow (3)$

From (3) z = -1

Substituting z = -1 in eq(2), we get y = 1

Substituting z = -1 & y = 1 in eq(1), we get x = 2.

$\therefore$ x = 2, y = 1, z = -1 is the solution.

Example3: Show that the equations x + y + z = 4, 2x + 5y -2z = 3, x + 7y – 7z = 5 are not

consistent.

Solution: The given equations can be written in the matrix form as AX = B

Where A = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}$,X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ B = $\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

The Augmented matrix [A/B] = $\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - R_1$

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - 2R_2$, we get

$$[A /B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

We can see $\rho(AB)$ = 3, since the number of non- zero rows is 3

Applying the same row operations on A, we get from above

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Here the number of non- zero rows is 2 so the rank of A = 2

Here we have $\rho(AB) \neq \rho(A)$.

$\therefore$ The given system is not consistent.

Example 4: Discuss for what values of λ, μ the simultaneous equations x + y + z = 6;

x + 2y + 3z =10; x + 2y + λz = μ have

i) no solution

ii) a unique solution

iii) an infinite number of solutions.

Solution: The given equations can be written in the matrix form as AX = B

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = B$$

We have the Augmented matrix $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$ we get

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix}$$

Case i): Let $\lambda \neq 3$ then rank of A =3 and rank of [A/B]=3, So that they have same

rank. Then the system of equations is consistent. Here the number of unknowns are 3 which is same as the rank of A. The system of equations will have a unique solution. This is true for any value of μ.

Thus if $\lambda \neq 3$ and μ has any value, the given system of equations will have a

Unique solution.

Case ii): Suppose $\lambda = 3$ & $\mu \neq 10$, then we can see that rank of A = 2 and rank of [A/B] =3.

Since the ranks of A and [A/B] are not equal, we say that the system of equations have no solution (inconsistent).

Case iii): Suppose $\lambda = 3 \& \mu = 10$. Then we have rank of A = rank of [A/B] = 2

$\therefore$The given system of equations will be consistent.

But here the number of unknowns = 3 > rank of A

Hence the system has infinitely many solutions.

Homogeneous System of Linear Equations:

Definition:

Consider the system of m homogeneous linear equations in n unknowns $x_1, x_2 \ldots\ldots, x_n$ as given below:

$$a_{1_1}x_1 + a_{1_2}x_2 + \ldots\ldots\ldots\ldots + a_{1_n}x_n = 0$$

$$a_{2_1}x_1 + a_{2_2}x_2 + \ldots\ldots\ldots\ldots + a_{2_n}x_n = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{i_1}x_1 + a_{i_2}x_2 + \ldots\ldots\ldots\ldots + a_{i_n}x_n = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m_1}x_1 + a_{m_2}x_2 + \ldots\ldots\ldots\ldots + a_{m_n}x_n = 0$$

Matrix Representation:

The above system of linear Homogeneous equations can be written in Matrix form as AX = 0

$$\begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ 0 \end{bmatrix}$$

Here A is called coefficient Matrix. It is clear that $x_1 = 0 = x_2 = x_3 = \ldots\ldots\ldots\ldots = xn$ is a solution.

This is called Trivial solution of AX = 0.

Definitions:

1. The system AX = 0 is always consistent since X = 0 is a Solution of AX = 0. This solution is
   called a trivial solution of the system. The Trivial solution is called zero solution.
2. Given AX = 0, we try to decide whether it has a solution X≠0. Such a solution, if exists,
   is called a non -Trivial solution.

Note: 1. If A is non-Singular matrix i.e. (det A ≠0) then the linear system AX = 0 has only
the zero solution.

2. The system AX = 0 possesses a non – zero solution ⇔ A is singular matrix.

Working Rule for Finding the Solutions of The Equation AX = 0:

1. i) If r = n (number of variables) ⇒ the system of equations has only Trivial solution (Zero solution)

   ii) If r< n ⇒ the system of equations has an infinite number of Non – Trivial solutions,

   we shall have n – r linearly independent solutions.

   To obtain infinite solutions, set (n – r) variables any arbitrary value and solve for the remaining Unknowns.

2. If the number of equations is less than number of variables, the solution is always other than a Trivial Solution.

   If the number of equations is equal to number of variables, the necessary and Sufficient condition for the solutions other than a Trivial solution is that the determinant of the coefficient matrix is zero.

Example 1: Solve completely the system of equations:

x +2y + 3z = 0; 3x + 4y + 4z = 0; 7x + 10y + 12z =0.

Solution: Taking A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and O= $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We get the system of equations as AX = 0

Consider A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 7R_1$, we get

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

This is in Echelon form. Number of non -zero rows is 3.

∴The rank of A = 3

∴ No. of variables = 3

∴ Number of non-zero solutions is = n - r =3 - 3 = 0.

∴ x = 0, y = 0, z = 0 is the only solution.

Example2: Solve completely the system of equations:

$$x + y + w = 0;\ y + z = 0;\ x + y + z + w = 0;\ x + y + 2z = 0.$$

Solution: The given equations can be written in the matrix form as AX = 0

$$\text{Where } A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Consider} \quad A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$, we get

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

Applying $R_4 \to R_4 - 2R_3$, we get

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

∴ Rank of A =4 and Number of variables = 4

∴ There is no non -zero solution

Hence x = y = z = w = 0 is the only solution.

Example 3: Find all the solutions of system of equations.

$$x + 2y - z = 0;\ 2x + y + z = 0;\ x - 4y + 5z = 0.$$

Solution: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -4 & 5 \end{bmatrix}$ .Then det A = 1 (5 +4) -2(10-1)-1(-8 - 1)

=9 -18+9 = 0

∴ The rank of A = 2 < 3(no. of variables)

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$, we get

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{bmatrix}$$ Applying $R_3 \to R_3 - 2R_2$, we get

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the rank of A = 2

∴ Number of non – zero solutions = n – r = 3 - 2 = 1.

From the above matrix, -3y + 3z = 0

$\Rightarrow y = z$

Let $z = k$. Then $y = k$

From the above matrix, $x + 2y - z = 0$

$\Rightarrow x = z - 2y$

$\Rightarrow x = k - 2k = -k$

$\therefore$ The solutions are given by $x = -k, y = k, z = k$.

Example 4: Solve completely the system of equations:

$x + y - 2z + 3w = 0$; $x - 2y + z - w = 0$; $4x + y - 5z + 8w = 0$; $5x - 7y + 2z - w = 0$.

Solution: The given equations can be written in the matrix form as AX = 0

$$AX = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & -2 & 1 & -1 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 5R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix}$$

Applying $R_4 \to R_4/4$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in Echelon form. We have

Rank of A = The number of non -zero rows in this echelon form = 2(r) and number of unknowns = 4

Since r <n. The given system has infinite number of non – trivial solutions.

$\therefore$ Number of independent solutions = 4 – 2 = 2 Now we shall assign arbitrary values to 2 variables and the remaining 2 variables shall be found in terms of these. The given system is equivalent to

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives the equations $x + y - 2z + 3w = 0$; $-3y + 3z - 4w = 0$.

Taking z = k_1 and w = k_2 We see that x = $k_1 - \frac{5}{3}k_2$, Y = $k_1 - \frac{4}{3}k_2$, z = k_1, w = k_2 constitutes the general solution of the given system.

Example 5: show that the only real number λ for which the system

x+ 2y + 3z = λx; 3x + y + 2z =λy; 2x + 3y + z =λz has non -zero solution is 6 and

solve them when λ =6.

Solution: The given equations can be written in the matrix form as AX = 0

Where A = $\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$0 = $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here the number of variables = n = 3

The given system of equations possesses a non-zero (non-trivial) solution, if

rank of A < no. of unknowns i.e., Rank of A < 3

For this we must have detA = 0

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get $\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix}$ =0

$$\Rightarrow (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ we get

$$(6-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (6-\lambda)\,[(-2-\lambda)(-1-\lambda)+1] = 0$

$\Rightarrow (6-\lambda)\,(\lambda^2 + 3\lambda + 3) = 0$

$\Rightarrow \lambda = 6$ is the only real value and other values are complex.

When $\lambda = 6$, the given system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to 5R_2 + 3R_1$, $R_3 \to 5R_3 + 2R_1$, we get

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_2$, we get

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ -5x + 2y + 3z = 0

$\Rightarrow$ -19y + 19z = 0

$\Rightarrow$y = z

Since Rank of A < Number of unknowns. i.e., r = 2 < n =3

$\therefore$ The given system has infinite number of non-trivial solutions.

Set n - r = 3 -2 = 1. Let z = k $\Rightarrow$ y = k and from the above equations

-5x +2k + 3k = 0

$\Rightarrow$ x = k

$\therefore$ x = k, y = k, z = k is the solution.

1.4.1. Eigen Values & Eigen Vectors:

Definition: Characteristic vector of a matrix:

Let A= [a_{ij}] be an n x n matrix. A non-zero vector X is said to be a Characteristic Vector of A if there exists a scalar such that AX=λX.

Note: If AX=λX (X≠0), then we say 'λ' is the Eigen value (or) characteristic root of 'A'.

E.g.: Let $A = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix} \quad X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$AX = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1. \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$AX = 1.X$$

Here Characteristic vector of A is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and Characteristic root of A is "1".

Note: We notice that an Eigen value of a square matrix A can be 0. But a zero vector cannot be an Eigen vector of A.

Method of finding the Eigen vectors of a matrix:

Let A = [a_{ij}] be a nxn matrix. Let X be an Eigen vector of A corresponding to the Eigen value λ.

Then by definition AX = λX.

AX = λIX

AX –λIX = 0

(A-λI)X = 0 ------- (1)

This is a homogeneous system of n equations in n unknowns. Will have a non-

zero solution X if and only |A-λI| = 0

(A-λI) is called characteristic matrix of A and |A-λI| is a polynomial in λ

of degree n and is called the characteristic polynomial of A

∴ |A-λI|=0 is called the characteristic equation

Solving characteristic equation of A, we get the roots, $\lambda_1, \lambda_2, \ldots \ldots \lambda_n$ these are

called the characteristic roots or Eigen values of the matrix.

Corresponding to each one of these n Eigen values, we can find

the characteristic vectors.

Procedure to find Eigen values and Eigen vectors:

Let $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$ be a given square matrix

Characteristic matrix of A is (A-λI)

$$(A - \lambda I) = \begin{bmatrix} a_{11-\lambda} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22-\lambda} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn-\lambda} \end{bmatrix}$$

Then the characteristic polynomial is $|A - \lambda I|$

$$|A - \lambda I| = \begin{vmatrix} a_{11-\lambda} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22-\lambda} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn-\lambda} \end{vmatrix}$$

The characteristic equation is $|A - \lambda I|$=0 we solve the $|A - \lambda I|$=0

we get n roots; these are called Eigen values or latent values or proper values.

Let each one of these Eigen values say λ their Eigen vector X

corresponding the given value λ is obtained by solving Homogeneous system and

determining the non-trivial solution

$$\begin{bmatrix} a_{11-\lambda} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22-\lambda} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn-\lambda} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ .. \\ 0 \end{bmatrix}$$

PROBLEMS

1. Find the Eigen values and the corresponding Eigen vectors of $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Sol: Let $A=\begin{bmatrix}8 & -4\\ 2 & 2\end{bmatrix}$

The characteristic equation of matrix A is $|A-\lambda I|$=0

$$\begin{vmatrix}8-\lambda & -4\\ 2 & 2-\lambda\end{vmatrix}=0$$

$$\Rightarrow \quad (8-\lambda)(2-\lambda)+8=0$$

$$\Rightarrow \quad 16+\lambda^2-10\lambda+8=0$$

$$\Rightarrow \quad \lambda^2-10\lambda+24=0$$

$$\Rightarrow \quad (\lambda-6)(\lambda-4)=0$$

$\Rightarrow \quad \lambda = 4,6$ are the eigen values of matrix A.

Consider the system $\begin{bmatrix}8-\lambda & -4\\ 2 & 2-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$

Eigen vector corresponding to λ=4:

Put λ=4 in the above system, we get

⇨ $\begin{bmatrix}4 & -4\\ 2 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

⇨ $4x_1-4x_2=0 ------(1)$

⇨ $2x_1-2x_2=0 ------(2)$

From equations (1) and (2) we have $x_1=x_2=K$

Eigen vector X=$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}K\\ K\end{bmatrix}=K\begin{bmatrix}1\\ 1\end{bmatrix}$

$\begin{bmatrix}1\\ 1\end{bmatrix}$ is a Eigen vector of matrix A corresponding to the Eigen value λ=4

Eigen vector corresponding to λ=6

Put λ=6 in the above system, we get

$$\begin{bmatrix}2 & -4\\ 2 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$\Rightarrow\ 2x_1-4x_2=0$

$\Rightarrow\ 2x_1-4x_2=0$

from the above equations we have $x_1 = 2x_2$

Say $x_2=k$ then $x_1=2k$

Eigen vector $X_2=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}2k\\ k\end{bmatrix}=k\begin{bmatrix}2\\ 1\end{bmatrix}$

$\begin{bmatrix}2\\ 1\end{bmatrix}$ is a Eigen vector of matrix A corresponding to the Eigen value λ=6

2. Find the Eigen values and the corresponding Eigen vectors of the matrix

$$\begin{bmatrix}\mathbf{2} & \mathbf{0} & \mathbf{1}\\ \mathbf{0} & \mathbf{2} & \mathbf{0}\\ \mathbf{1} & \mathbf{0} & \mathbf{2}\end{bmatrix}$$

Sol: Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

The characteristic equation of matrix A is $|A-\lambda I|=0$

i.e. $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)(2-\lambda)^2 - 0 + 1[0-(2-\lambda)] = 0$

$\Rightarrow (2-\lambda)^3 + (\lambda-2) = 0$

$\Rightarrow (\lambda-2)(-\lambda^2+4\lambda-3) = 0$

$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$

$\therefore$ λ=1, 2, 3

The Eigen values of A are 1, 2, 3

To find the Eigen vector consider the system $(A-\lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector corresponding to λ=1:

$$\begin{bmatrix} 2-1 & 0 & 1 \\ 0 & 2-1 & 0 \\ 1 & 0 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_3 = 0$

$\Rightarrow x_2 = 0$

$\Rightarrow x_1 = -x_3$

$\Rightarrow$ Let $x_3 = k$ then $x_1 = -k$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is the Eigenvector to the corresponding Eigen value λ=1.

Eigen vector corresponding to λ=2:

$$\begin{bmatrix} 2-2 & 0 & 1 \\ 0 & 2-2 & 0 \\ 1 & 0 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $x_1 = 0$ and $x_3 = 0$ and we can take any arbitrary value for $x_2 = k$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The Eigen vector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Eigen vector corresponding to λ=3:

$$\begin{bmatrix} 2-3 & 0 & 1 \\ 0 & 2-3 & 0 \\ 1 & 0 & 2-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

⇨ $$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

⇨ $-x_1 + x_3 = 0$

⇨ $-x_2 = 0$

$x_2 = 0$ and $x_1 = x_3 = k$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \mathrm{k} \\ 0 \\ \mathrm{k} \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

3. Find the Eigen values and the corresponding Eigen vectors of the matrix

$$\begin{bmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic equation of matrix A is |A-λI|=0

i.e. |A- λ I| $= \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$

⇨ $(6-\lambda)[(3-\lambda)^2 - 1] + 2[-2(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0$

⇨ $(6-\lambda)[\lambda^2 - 6\lambda + 8] + 2[2\lambda - 4] + 2[2\lambda - 4] = 0$

⇨ $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\lambda = 2, 2, 8$

The Eigen values of A is 2, 2, 8

The Eigen vector of matrix A corresponding to $\boldsymbol{\lambda} = \mathbf{2}$;

To find the Eigen vector consider the system $(A - \lambda\mathrm{I})\mathrm{X} = 0$

$\Rightarrow \quad (A - 2\mathrm{I})\mathrm{X} = 0$

$$\Rightarrow \begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \ 4x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 + x_2 - x_3 = 0$

$2x_1 - x_2 + x_3 = 0$

By observing that three equations are identical

Therefore, we have to take two arbitrary constants for any two variables

Let $x_1 = k_1, x_2 = k_2$

From third equation $2k_1 - k_2 + x_3 = 0$

$x_3 = -2k_1 + k_2$

$$\mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ -2k_1 + k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ -2k_1 \end{bmatrix} + \begin{bmatrix} 0 \\ k_2 \\ k_2 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The Eigen vector of matrix A corresponding to $\boldsymbol{\lambda} = \mathbf{8}$;

To find the Eigen vector consider the system $(A - \lambda\mathrm{I})\mathrm{X} = 0$

$\Rightarrow \quad (A - 8\mathrm{I})\mathrm{X} = 0$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \ -2x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 - 5x_2 - x_3 = 0$

$2x_1 - x_2 - 5x_3 = 0$

By solving above three equations we get $x_1 = 2, x_2 = -1, x_3 = 1$

$$\mathrm{X}_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

The Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

4. Find the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation of matrix A is |A-λI|=0

i.e. |A- λ I| $= \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

⇨ $(8-\lambda)[(7-\lambda)(3-\lambda)-16]+6[-6(3-\lambda)+8]+2[24-2(7-\lambda)]=0$

⇨ $(8-\lambda)[\lambda^2-10\lambda+5]+6[6\lambda-10]+2[2\lambda-10]=0$

⇨ $\lambda^3-18\lambda^2+45\lambda=0$

$\lambda = 0, 3, 15$

The Eigen values of A is 0, 3, 15

The Eigen vector of matrix A corresponding to$\boldsymbol{\lambda} = \mathbf{0}$;

To find the Eigen vector consider the system $(A-\lambda\mathrm{I})\mathrm{X} = 0$

$(A-0\mathrm{I})\mathrm{X} = 0$

⇨ $\begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

⇨ $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

⇨ $8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$

By solving above any of the two equations we get $x_1 = 1, x_2 = 2, x_3 = 2$

Eigen vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

The Eigen vector of matrix A corresponding to$\boldsymbol{\lambda} = \mathbf{3}$;

To find the Eigen vector consider the system $(A-\lambda\mathrm{I})\mathrm{X} = 0$

$(A-3\mathrm{I})\mathrm{X} = 0$

⇨ $\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 = 0$$

By solving above any of the two equations we get $x_1 = -2, x_2 = -1, x_3 = 2$

$$\text{Eigen vector } X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

The Eigen vector of matrix A corresponding to$\boldsymbol{\lambda} = \mathbf{15}$;

To find the Eigen vector consider the system $(A - \lambda\mathrm{I})\mathrm{X} = 0$

$$(A - 15\mathrm{I})\mathrm{X} = 0$$

$$\Rightarrow \begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0$$

By solving above any of the two equations we get $x_1 = 2, x_2 = -2, x_3 = 1$

$$\text{Eigen vector } X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

The Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \quad \mathrm{X}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

1.4.2 Properties of Eigen Values:

1. The sum of the Eigen values of a matrix A is same as trace of the matrix A.
   Proof: To prove this theorem, we consider third order square matrix

$$A= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The characteristic polynomial of A is

$$= |A - \lambda I|$$

$$= \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22-\lambda} & a_{23} \\ a_{31} & a_{32} & a_{33-\lambda} \end{vmatrix}$$

$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda\,(a_{21}a_{12} + a_{13}a_{31} + \cdots \ldots) + (a_{11}a_{23}a_{32} + a_{12}a_{23}a_{32} + \cdots)$

We have

$$\text{Sum of the roots} = \frac{-\lambda^2\ coefficient}{\lambda^3\ coefficient}$$

$$= \frac{-(a_{11}+a_{22}+a_{33})}{-1}$$

$$= a_{11} + a_{22} + a_{33}$$

= Trace of A

Hence the result.

2. The product of the Eigen values of a matrix A is equal to its determinant.

Proof: Let $\lambda_1, \lambda_2 \ldots \ldots . . \lambda_n$ be the Eigen values of $A_{n\times n}$ .Then

$$|A - \lambda I| = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots \ldots \ldots . (\lambda - \lambda_n) = 0$$

Put $\lambda = 0$

$$|A| = (-1)^n(-\lambda_1)(-\lambda_2) \ldots \ldots \ldots . (-\lambda_n)$$

$$= (-1)^n(-1)^n(\lambda_1\lambda_2 \ldots \ldots \ldots . \lambda_n)$$

$$= (-1)^{2n}(\lambda_1\lambda_2 \ldots \ldots \ldots . \lambda_n)$$

$$|A| = \lambda_1\lambda_2 \ldots \ldots \ldots . \lambda_n$$

Hence the result.

3. If $\boldsymbol{\lambda}$ is an Eigen value of A corresponding to the Eigen vector X then $\boldsymbol{\lambda^n}$ is the Eigen
value of $\boldsymbol{A^n}$ corresponding to the Eigen vector X.

Proof:

Since λ is an Eigen value of A corresponding to the Eigen vector X, we have

$$AX = \lambda X \text{ ----------------------------- (1)}$$

Pre multiplying equation (1) by A

$$A(AX) = A\,(\lambda X)$$

$$(AA)X = \lambda\,(A X)$$

$$A^2X = \lambda\,(\lambda X)$$

$$A^2X = \lambda^2X$$

Hence λ^2 is Eigen value of A^2 with X itself as the corresponding Eigen Vector.

Thus, the theorem is true for $n = 2$.

Let the result be true for $n = k$.

Then $A^kX = \lambda^kX$

Pre multiplying this by A

$$A\,(A^kX) = A(\lambda^kX)$$

$$A^{k+1}X = \lambda^k(AX)$$

$$= \lambda^k(\lambda X) \qquad (\text{Since } \boldsymbol{AX = \lambda X})$$

$$= \lambda^{k+1}X$$

$$A^{k+1}X = \lambda^{k+1}X$$

Which implies that λ^{k+1} is Eigen value of A^{k+1} with X itself as the corresponding Eigen Vector.

Hence by the principle of mathematical induction, the theorem is true for all positive integers n.

4. A square matrix A and it's transpose $\boldsymbol{A^T}$ have the same Eigen values.

Proof:

We have $(A-\lambda I)^T = A^T - \lambda I^T$

$= A^T - \lambda I$

$|(A-\lambda I)^T| = |A^T - \lambda I|$ or

$|A-\lambda I| = |A^T - \lambda I|$ [Since $|\boldsymbol{A^T}| = |\boldsymbol{A}|$]

$\therefore$ $|A-\lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$

i.e., λ is an Eigen value of A if and only if λ is an Eigen value of A^T.

Thus, the Eigen values of A and A^T are same.

5. If $\boldsymbol{\lambda}$ is an Eigen value of the matrix A corresponding to the Eigen vector X then $\boldsymbol{k+\lambda}$

is an Eigen value of the matrix $\boldsymbol{A+kI}$.

Proof:

Let λ be an Eigen value of the matrix A corresponding to the Eigen vector X then

$AX = \lambda X$ ----------------------------- (1)

Now $(A+kI)\,X = AX + kIX$

$= \lambda X + kX$ (Since $\boldsymbol{AX} = \boldsymbol{\lambda X}$)

$= (\lambda + k)X$

$(A+kI)\,X = (\lambda+k)X$ -----------------(2)

From Equation (2) we see that that the scalar $k+\lambda$ is an Eigen value of the matrix $A+kI$ and X is a corresponding Eigen vector.

6. If $\boldsymbol{\lambda_1, \lambda_2 \dots \dots . . \lambda_n}$ are the Eigen values of a matrix A, then $\boldsymbol{k\lambda_1, k\lambda_2 \dots \dots . . k\lambda_n}$ are the Eigen values of the matrix KA, where k is a non-zero scalar.

Proof:

Let A be a square matrix of order n.

Then $|kA - \lambda kI| = |k(A-\lambda I)|$

$= k^n|A-\lambda I|$

Since $k \neq 0$, therefore $|kA - \lambda kI| = 0$ if and only if $|A-\lambda I| = 0$

i.e., $k\lambda$ is an Eigen value of kA if and only if λ is an Eigen value of A.

Thus $k\lambda_1, k\lambda_2 \dots \dots . . k\lambda_n$ are the Eigen Values of the Matrix kA if $\lambda_1, \lambda_2 \dots \dots . . \lambda_n$ are the Eigen values of the matrix A.

7. If $\boldsymbol{\lambda_1, \lambda_2 \ldots \ldots . . \lambda_n}$ are the Eigen Values of A, then $\boldsymbol{(\lambda_1 - k)(\lambda_2 - k) \ldots \ldots \ldots . (\lambda_n - k)}$ are

   the Eigen Values of the matrix $(\boldsymbol{A - kI})$, where k is a non-zero scalar.

   Proof:

   Since $\lambda_1, \lambda_2 \ldots \ldots . . \lambda_n$ are the Eigen values of A, then the characteristic polynomial of A is

$$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots \ldots \ldots . (\lambda_2 - \lambda) \text{ ------------------------------(1)}$$

   Thus, the Characteristic polynomial of $(A - kI)$ is

$$|A - kI - \lambda I| = |A - (k + \lambda)I|$$

$$= [\lambda_1 - (\lambda + k)]\ \ [\lambda_2 - (\lambda + k)] \ldots\ldots [\lambda_n - (\lambda + k)]$$ (Since from (1))

$$= [(\lambda_1 - k) - \lambda]\ \ [(\lambda_2 - k) - \lambda] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [(\lambda_n - k) - \lambda]$$

   This shows that the Eigen values of $(A - kI)$ are$(\lambda_1 - k)(\lambda_2 - k) \ldots \ldots \ldots . (\lambda_n - k)$.

8. Prove that the Eigen values of $\boldsymbol{A^{-1}}$ are the reciprocals of the Eigen values of A.

   (Or)

   If $\boldsymbol{\lambda}$ is an Eigen value of a non-singular matrix A corresponding to the Eigen vector X. Then $\boldsymbol{\lambda^{-1}}$ is an Eigen value of $\boldsymbol{A^{-1}}$ and corresponding Eigen vector X itself.

   Proof:

   Since A is non-singular and product of the Eigen values is equal to $|A|$, it follows that none of the Eigen values of A is 0.

   If λ is an Eigen value of the non-singular matrix A and X is the corresponding Eigen vector,

   $\lambda \neq 0$ and $AX = \lambda X$

   Pre multiplying this with A^{-1} , we get

$$A^{-1}\,(AX) = A^{-1}(\lambda X)$$

$$(A^{-1}\,A)X = \lambda(A^{-1}\,X)$$

$$IX = \lambda(A^{-1}\,X)$$

(1) $$X = \lambda(A^{-1}\,X)$$

$$\lambda^{-1}X = A^{-1}\,X$$

$$A^{-1}\,X = \lambda^{-1}X$$ Since $\boldsymbol{\lambda \neq 0}$

   Hence by the definition, it follows that λ^{-1} is an Eigen value of A^{-1} and X is the corresponding Eigen vector.

9. If $\boldsymbol{\lambda}$ is an Eigen value of a non-singular matrix A, then $\frac{|\boldsymbol{A}|}{\boldsymbol{\lambda}}$ is an Eigen value of the matrix $adj\ A$.

Proof:

   Since λ is an Eigen value of a non-singular matrix, therefore $\lambda \neq \mathbf{0}$.

   Also λ is an Eigen value of A implies that there exist a non-zero vector X such that

$$AX = \lambda X \text{ ------------------(1)}$$

Pre multiplying (1) by $adj\ A$

$$(adj\ A)\ AX = (adj\ A)\ \lambda X$$

$$[(adj\ A)\ A]X = \lambda[(adj\ A)\ X]$$

$$|A|\ I\ X = \lambda(adj\ A)X \qquad \text{[Since } (\boldsymbol{adj\ A})\boldsymbol{A} = |\boldsymbol{A}|\ \boldsymbol{I}\text{]}$$

$$|A|\ X = \lambda(adj\ A)X$$

$$\frac{|A|}{\lambda} X = (adj\ A)X$$

$$(adj\ A)X = \frac{|A|}{\lambda} X$$

Since X is a non-zero vector, therefore from the relation It is clear that $\frac{|A|}{\lambda}$ is an Eigen value of the matrix $adj\ A$.

10. If $\boldsymbol{\lambda}$ is an Eigen value of an orthogonal matrix, then $\frac{\boldsymbol{1}}{\boldsymbol{\lambda}}$ is also an Eigen value.

Proof:

We know that if λ is an Eigen value of a matrix A, then $\frac{1}{\lambda}$ is an Eigen value of A^{-1}.

Since A is an orthogonal matrix, therefore

$A^{-1} = A^T$

$\frac{1}{\lambda}$ is an Eigen value of A^T.

But the matrices A and A^T have the same Eigen values. Since the determinants $|A - \lambda I|$ and $|A^T - \lambda I|$are same.

Hence $\frac{1}{\lambda}$ is also an Eigen value of A.

11. The Eigen values of a diagonal matrix is its diagonal elements.

Proof: Let

$$D = \begin{bmatrix} a_{11} & 0 & 0 & \dots\dots & 0 \\ 0 & a_{22} & 0 & \dots\dots & 0 \\ 0 & 0 & a_{33} & \dots\dots & 0 \\ . & . & . & \dots\dots & . \\ . & . & . & \dots\dots & . \\ 0 & 0 & 0 & \dots\dots & a_{nn} \end{bmatrix}$$

The characteristic equation of D is

$$|D - \lambda I| = 0$$

$$\begin{vmatrix} a_{11} - \lambda & 0 & 0 & \dots\dots & 0 \\ 0 & a_{22} - \lambda & 0 & \dots\dots & 0 \\ 0 & 0 & a_{33} - \lambda & \dots\dots & 0 \\ . & . & . & \dots\dots & . \\ . & . & . & \dots\dots & . \\ 0 & 0 & 0 & \dots\dots & a_{nn-\lambda} \end{vmatrix} = 0$$

$$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) \dots\dots\dots.(a_{nn} - \lambda) = 0$$

$\Rightarrow \lambda = a_{11}, a_{22}, a_{33} \dots\dots\dots\dots a_{nn}$ are the Eigen values of D.

12. The Eigen values of a triangular matrix are the diagonal elements.

Proof:

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots\dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots\dots & a_{2n} \\ 0 & 0 & a_{33} & \dots\dots & a_{3n} \\ . & . & . & \dots\dots & . \\ . & . & . & \dots\dots & . \\ 0 & 0 & 0 & \dots\dots & a_{nn} \end{bmatrix}$$ be the upper triangular matrix.

The Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \dots\dots & a_{1n} \\ 0 & a_{22}-\lambda & a_{23} & \dots\dots & a_{2n} \\ 0 & 0 & a_{33}-\lambda & \dots\dots & a_{3n} \\ . & . & . & \dots\dots & . \\ . & . & . & \dots\dots & . \\ 0 & 0 & 0 & \dots\dots & a_{nn-\lambda} \end{vmatrix} = 0$$

$$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)\dots\dots\dots.(a_{nn}-\lambda) = 0$$

$$\Rightarrow \lambda = a_{11}, a_{22}a_{33}\dots\dots\dots\dots a_{nn}$$

Hence the Eigen values are the diagonal elements.

13. Eigen values of two similar matrices are same.

Proof:

Let A and B be two similar matrices so that

$$B = P^{-1}AP$$

To prove that A and $P^{-1}AP$ have the same Eigen values.

Let λ be Eigen value of B then,

$$|B - \lambda I| = 0$$

$$|P^{-1}AP - \lambda P^{-1}P| = 0$$

$$|P^{-1}P(A - \lambda I)| = 0$$

$$|P^{-1}||P|\,|A - \lambda I| = 0$$

$$|A - \lambda I| = 0 \quad \boldsymbol{since\ |P^{-1}||P| = 1}$$

$$|B - \lambda I| = 0 \Rightarrow |A - \lambda I| = 0$$

Hence A and B have the same Eigen values.

14. If A and B are square matrices and if A is invertible then the matrices $\boldsymbol{A^{-1}B}$ and $\boldsymbol{BA^{-1}}$ have the same Eigen values.

Proof:

Given A is invertible, then A^{-1} exists.

Now $A^{-1}B = A^{-1}B\,I$

$= A^{-1}B\,(A^{-1}A)$

$= A^{-1}(BA^{-1})A$

$\Rightarrow A^{-1}B = A^{-1}(BA^{-1})A$ ---------------- (1)

By using the above property, matrices BA^{-1} and $A^{-1}(BA^{-1})A$ have the same Eigen values.

Now by (1) the matrices $A^{-1}B$ and BA^{-1} have the same Eigen values.

15. The Eigen values of a real symmetric matrix are always real.

Proof:

Let λ be an Eigen value of a real symmetric matrix A and let X be the corresponding Eigen vector. Then $AX = \lambda X$ ------------------(1)

Taking the Conjugate of (1)

$$\bar{A}\,\bar{X} = \bar{\lambda}\,\bar{X}$$

Taking the transpose on both sides

$$(\bar{A}\,\bar{X})^T = (\bar{\lambda}\,\bar{X}\ )^T$$

$$(\bar{X})^T\,(\bar{A}\,)^T = \bar{\lambda}\,(\bar{X})^T$$

Since A is symmetric, we have

$\bar{A} = A$ and $A^T = A$

$$\therefore (\bar{X})^T\,A = \bar{\lambda}\,(\bar{X})^T$$

Post multiplying by X, we get

$$(\bar{X})^T\,A\,X = \bar{\lambda}\,(\bar{X})^T X \text{ ------------(2)}$$

Pre multiplying (1) with $(\bar{X})^T$, we get

$$(\bar{X})^T AX = (\bar{X})^T \lambda\,X \text{ ------------(3)}$$

(2)-(3)

$$\Rightarrow (\lambda - \bar{\lambda}\,)\,(\bar{X})^T\,X = 0$$

$$\Rightarrow (\lambda - \bar{\lambda}\,) = 0 \qquad \boldsymbol{since}\ (\bar{\boldsymbol{X}})^{\boldsymbol{T}}\,\boldsymbol{X} \neq \boldsymbol{0}$$

$$\Rightarrow \lambda = \bar{\lambda}$$

$\Rightarrow$ λ is real

Hence the result.

16. If $\boldsymbol{\lambda}$ is an Eigen value of A, then prove that the Eigen value of $\boldsymbol{B} = \boldsymbol{a_0}\boldsymbol{A}^{\boldsymbol{2}} + \boldsymbol{a_1}\boldsymbol{A} + \boldsymbol{a_2}\boldsymbol{I}$ is $\boldsymbol{a_0}\boldsymbol{\lambda}^{\boldsymbol{2}} + \boldsymbol{a_1}\boldsymbol{\lambda} + \boldsymbol{a_2}$.

Proof:

If X be the Eigen vector corresponding to the Eigen value λ , then

$$AX = \lambda\,X \text{ ------------------(1)}$$

Pre multiplying by A on both sides of (1)

$$A(AX) = A(\lambda\,X)$$

$$(AA)X = \lambda\,(A\,X)$$

$$A^2X = \lambda\,(\lambda\,X) \qquad \text{(Since } \boldsymbol{AX} = \boldsymbol{\lambda}\,\boldsymbol{X}\text{)}$$

$$A^2X = \lambda^2X$$

Hence λ^2 is Eigen value of A^2 with X itself as the corresponding Eigen Vector.

We have

$$B = a_0A^2 + a_1A + a_2I$$

$$BX = (a_0A^2 + a_1A + a_2I)X$$

$$BX = a_0A^2X + a_1AX + a_2X$$

$$BX = a_0\lambda^2X + a_1\lambda X + a_2X$$

$$BX = (a_0\lambda^2 + a_1\lambda + a_2)X$$

This shows that $a_0\lambda^2 + a_1\lambda + a_2$ is an Eigen value of B and the corresponding Eigen Vector of B is X.

17. Prove that the two Eigen vectors corresponding to the two different Eigen Values are linearly independent.

Proof:

Let A be a square matrix. Let $X_1$and X_2 be the two Eigen vectors of A corresponding to two distinct Eigen values λ_1 and λ_2 .Then

$$AX_1 = \lambda_1 X_1 \text{And } AX_2 = \lambda_2 X_2 \text{ ---------------------(1)}$$

Now we shall prove that the Eigen vectors $X_1$and X_2 are linearly independent.

Let us assume that the $X_1$and X_2 are linearly dependent.

Then for two scalars $k_1$and k_2 not both zeroes such that $k_1X_1 + k_2 X_2 = 0$ -----(2)

Multiplying both sides of (2) by A, we get

$$A\ (\ k_1X_1 + k_2 X_2) = A(0)$$

$$A\ (\ k_1X_1) + A(k_2 X_2) = 0$$

$$k_1(AX_1) + k_2\ (AX_2)\ = 0$$

$$k_1(\lambda_1X_1) + k_2\ (\lambda_2\ X_2) = 0 \text{ -------------------- (3)} \quad \text{(Since from (1))}$$

(3)- $\lambda_2\ (1)$ $\Rightarrow\ k_1(\lambda_1 - \lambda_2)X_1 = 0$

$\Rightarrow k_1 = 0$ Since $\boldsymbol{\lambda_1 \neq \lambda_2} \,\&\, \boldsymbol{X_1 \neq 0}$

Similarly, $k_2\ = 0$

But this contradicts our assumption that k_1, k_2 are not zeroes. Hence our assumption that

X_1 & X_2 are Linearly independent is wrong.

Hence the statement is true.

5. Find the Eigen values and Eigen vectors of the matrix A and it's inverse,

where A = $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

Sol: Given A = $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of A is given by |A-λI| = 0

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow\ (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\Rightarrow\ \lambda = 1,2,3$$

Eigen values of the matrix are 1,2,3

Eigen vector corresponding to λ=1: $(A - \lambda I)X_1 = 0$

$$\Rightarrow \begin{bmatrix} 1-1 & 3 & 4 \\ 0 & 2-1 & 5 \\ 0 & 0 & 3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \ 3x_2 + 4x_3 = 0$

$x_2 - 5x_3 = 0$

$2x_3 = 0$

$\therefore \ x_3 = 0$, $x_2 = 0$ and let $x_1 = k$, $\quad X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Eigen vector corresponding to λ=2: $(A - 2\mathrm{I})X_2 = 0$

$$\Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \ -x_{1+}3x_2 + 4x_3 = 0$

$\Rightarrow 5x_3 = 0$, i.e., $x_3 = 0$

$-x_{1+}3x_2 = 0$

$x_1 = 3x_2$

Let $x_2 = \mathrm{k}$ then $x_1 = 3\mathrm{k}$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3\mathrm{k} \\ \mathrm{k} \\ 0 \end{bmatrix} = k\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Eigen vector corresponding to λ=3: $(A - 3\mathrm{I})X_3 = 0$

$$\Rightarrow \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \ -2x_{1+}3x_2 + 4x_3 = 0$

$\Rightarrow -x_2 + 5x_3 = 0$

Let $x_3 = \mathrm{k}$ then $x_2 = 5\mathrm{k}$

$-2x_1 + 3(5\mathrm{k}) + 4\mathrm{k} = 0$

$x_1 = \frac{19}{2}k$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{19}{2}k \\ 5\mathrm{k} \\ \mathrm{k} \end{bmatrix} = \frac{k}{2}\begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix} \qquad X_3 = \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$$

Eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

Eigen values of A^{-1} are $1, \frac{1}{2}, \frac{1}{3}$

We know that Eigen vectors of A^{-1} are same as Eigen vectors of A.

6. Find the Eigen values of $3A^3 + 5A^2 - 6A + 2I$ where $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Sol: The characteristic equation of A is given by $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda)(-2-\lambda) = 0$$

$$\Rightarrow \lambda = 1,3,-2$$

Eigen values of the matrix are 1,3, - 2

We know that if λ is the eigen value of matrix A and f(A) is a polynomial in A

then the Eigen value of f(A) is f(λ)

Let f(A) = $3A^3 + 5A^2 - 6A + 2I$

Then Eigen values of f(A) are f(1), f(3) and f(-2)

f(1) = 3(1)3+5(1)2-6(1)+2(1) = 4

f(3) = 3(3)3+5(3)2-6(3)+2(1) = 110

f(-2) = 3(-2)3+5(-2)2-6(-2)+2(1) = 10

Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are 4,110,10

1.5. Cayley - Hamilton Theorem:

Statement: Every square matrix satisfies its own characteristic equation

1. Show that the matrix A = $\begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation

   Hence

   find A^{-1}.

Sol: The Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$C_2 \rightarrow C_2+C_3$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$$

The characteristic equation is $\lambda^3 - \lambda^2 + \lambda - 1 = 0$

By Cayley – Hamilton theorem, we have A^3-A^2+A-I=0------(1)

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Matrix A satisfies Cayley-Hamilton theorem.

To find A^{-1}, Multiplying equation (1) with A^{-1} we get A^2 – A + I =A^{-1}

$$A^{-1} = A^2 - A + I$$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

2. Using Cayley - Hamilton Theorem find the inverse and A^4 of the matrix

Where $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation is given by |A-λI|=0

$$\Rightarrow \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

The characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley – Hamilton theorem we have A^3-5A^2+7A-3I=0…..(1)

To find A^{-1}, Multiplying equation (1) with A^{-1} we get $A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$

$$A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \quad A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \quad A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$$

$$A^{-1} = \frac{1}{3}\left\{\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - 5\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

To find A^4 Multiply equation (1) with A, we get

$$A^4 - 5A^3 + 7A^2 - 3A = 0$$

$$A^4 = 5A^3 - 7A^2 + 3A$$

$$A^4 = \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

1.6. Diagonalization of a matrix:

A square matrix is said to be diagonalizable if there exists an invertible matrix 'P' such that $P^{-1}AP = D$ is known as Diagonal matrix

Modal and Spectral matrices:

The matrix P in the above result which diagonalizable the square matrix A is called modal matrix of A and the resulting diagonal matrix D is known as spectral matrix.

Calculation of powers of a matrix:

We can obtain the power of a matrix by using diagonalization

Let A be the square matrix then a non-singular matrix P can be found such that

$$D = P^{-1}AP$$

$$D^2 = (P^{-1}AP)\,(P^{-1}AP) = P^{-1}A^2P$$

Similarly $D^3 = P^{-1}A^3P$

In general $D^n = P^{-1}A^nP$……..(1)

To obtain A^n, Pre-multiply (1) by P and post multiply by P^{-1}

Then $PD^nP^{-1} = P(P^{-1}A^nP)\,P^{-1}$

$= (PP^{-1})\,A^n\,(PP^{-1}) = A^n$

$\therefore\ A^n = PD^nP^{-1}$

NOTE: The diagonal elements in the diagonal matrix are Eigen values of the given matrix.

PROBLEMS

1. Determine the modal matrix P of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Verify that $P^{-1}AP$ is a diagonal matrix.

Sol: The characteristic equation of A is | A-λI | = 0

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & 3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

i.e, $(\lambda - 5)(\lambda + 3)^2 = 0$

Thus the Eigen values are λ=5, λ=-3 and λ=-3

when λ=5

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -7x_1 + 2x_2 - 3x_3 = 0$$

$$2x_1 - 4x_2 - 6x_3 = 0$$

$$-x_1 - 2x_2 - 5x_3 = 0$$

By solving above we get $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly, for the given Eigen value λ=-3 we can have two linearly independent Eigen vectors

$$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Modal matrix $P = [X_1, X_2, X_3]$

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now $|P| = 1(-1) - 2(2) + 3(-1) = -8 \neq 0$

∴ P is invertible matrix so matrix A is diagonalizable.

$$P^{-1} = \frac{adjP}{|P|} = \frac{-1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$D = P^{-1}AP = \frac{-1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$D = P^{-1}AP = \frac{-1}{8}\begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Hence $P^{-1}AP$ is a diagonal matrix.

2. Find a matrix P which transforms the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} & \mathbf{3} \end{bmatrix}$ to diagonal form.

Hence calculate $\boldsymbol{A^4}$.

Sol: Characteristic equation of A is given by | A-λI | = 0

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

i.e., $(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\lambda = 1, 2, 3$

Thus, the eigen values of matrix are 1, 2, 3.

If x_1, x_2, x_3 be the components of an Eigen vector corresponding to the Eigen value λ, we have

$$[A-\lambda I]X = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When λ=1

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$i.e, 0.x_1 + 0.x_2 - x_3 = 0$

$x_1 + x_2 + x_3 = 0$

$2\,x_1 + 2x_2 + 2x_3 = 0$

By solving above equations $x_1 = 1, x_2 = -1, x_3 = 0$

Eigen vector is $[1, -1, 0]^T$

For λ=2, λ=3 we can obtain Eigen vector $[-2,1,2]^T$ and $[-1,1,2]^T$

Modal matrix $P = [X_1, X_2, X_3]$

$$P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{adjP}{|P|} = \frac{-1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

Now $P^{-1}AP = \frac{-1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} & \mathbf{3} \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

$$A^4 = PD^4P^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\left\{\frac{-1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}\right\}$$

$$A^4 = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$