Numerical Methods Problem Set

Euler, Midpoint, and Runge-Kutta Methods

(With Detailed Step-by-Step Solutions)

Problem 1 (Euler Method - Linear ODE)**.** Consider the initial value problem:

$$\frac{dy}{dt} = -2y + 4e^{-t}, \quad y(0) = 3 \tag{1}$$

(a) Use Euler's method with step size $h = 0.2$ to approximate $y(0.4)$.

(b) Find the exact solution and compute the absolute error at $t = 0.4$.

(c) Verify that the global error is $\mathcal{O}(h)$ by computing with $h = 0.1$ and comparing errors.

Solution. **Part (a): Euler's method with $h = 0.2$**

Euler's formula: $y_{n+1} = y_n + h \cdot f(t_n, y_n)$

Given: $f(t, y) = -2y + 4e^{-t}$, $h = 0.2$, $y_0 = 3$

Step 1: From $t_0 = 0$ to $t_1 = 0.2$

Initial values: $t_0 = 0$, $y_0 = 3$

Calculate $f(t_0, y_0) = f(0, 3)$:

$$\begin{align}
f(0,3) &= -2(3) + 4e^{-0} \tag{2}\\
&= -6 + 4(1) \tag{3}\\
&= -6 + 4 \tag{4}\\
&= -2 \tag{5}
\end{align}$$

Apply Euler's formula:

$$\begin{align}
y_1 &= y_0 + h \cdot f(t_0, y_0) \tag{6}\\
&= 3 + 0.2 \cdot (-2) \tag{7}\\
&= 3 + (-0.4) \tag{8}\\
&= 2.6 \tag{9}
\end{align}$$

Step 2: From $t_1 = 0.2$ to $t_2 = 0.4$

Current values: $t_1 = 0.2$, $y_1 = 2.6$

Calculate $e^{-0.2}$:

$$e^{-0.2} = \frac{1}{e^{0.2}} \approx \frac{1}{1.2214} \approx 0.8187$$

Calculate $f(t_1, y_1) = f(0.2, 2.6)$:

$$f(0.2, 2.6) = -2(2.6) + 4e^{-0.2} \tag{10}$$
$$= -5.2 + 4(0.8187) \tag{11}$$
$$= -5.2 + 3.2748 \tag{12}$$
$$= -1.9252 \approx -1.925 \tag{13}$$

Apply Euler's formula:

$$y_2 = y_1 + h \cdot f(t_1, y_1) \tag{14}$$
$$= 2.6 + 0.2 \cdot (-1.925) \tag{15}$$
$$= 2.6 + (-0.385) \tag{16}$$
$$= 2.215 \tag{17}$$

$\boxed{y(0.4) \approx 2.215}$

Part (b): Exact solution and error

The differential equation: $\frac{dy}{dt} + 2y = 4e^{-t}$

This is a first-order linear ODE in standard form: $y' + P(t)y = Q(t)$

where $P(t) = 2$ and $Q(t) = 4e^{-t}$

Integrating factor:

$$\mu(t) = e^{\int P(t)dt} \tag{18}$$
$$= e^{\int 2dt} \tag{19}$$
$$= e^{2t} \tag{20}$$

Multiply the equation by $\mu(t) = e^{2t}$:

$$e^{2t}\frac{dy}{dt} + 2e^{2t}y = 4e^{2t}e^{-t} \tag{21}$$
$$e^{2t}\frac{dy}{dt} + 2e^{2t}y = 4e^{t} \tag{22}$$

The left side is the derivative of $e^{2t}y$:

$$\frac{d}{dt}(e^{2t}y) = 4e^t$$

Integrate both sides:

$$e^{2t}y = \int 4e^t dt \tag{23}$$
$$= 4e^t + C \tag{24}$$

Solve for y:

$$y = \frac{4e^t + C}{e^{2t}} = 4e^{-t} + Ce^{-2t}$$

Apply initial condition $y(0) = 3$:

$$3 = 4e^0 + Ce^0 \tag{25}$$
$$3 = 4(1) + C(1) \tag{26}$$
$$3 = 4 + C \tag{27}$$
$$C = -1 \tag{28}$$

Exact solution: $y(t) = 4e^{-t} - e^{-2t}$
At $t = 0.4$:

$$y(0.4) = 4e^{-0.4} - e^{-0.8} \tag{29}$$
$$= 4e^{-0.4} - e^{-0.8} \tag{30}$$

Calculate $e^{-0.4}$ and $e^{-0.8}$:

$$e^{-0.4} \approx 0.6703 \tag{31}$$
$$e^{-0.8} \approx 0.4493 \tag{32}$$

Therefore:

$$y(0.4) = 4(0.6703) - 0.4493 \tag{33}$$
$$= 2.6812 - 0.4493 \tag{34}$$
$$= 2.2319 \tag{35}$$

Absolute error:

$$\text{Error} = |y_{\text{approx}} - y_{\text{exact}}| \tag{36}$$
$$= |2.215 - 2.2319| \tag{37}$$
$$= |-0.0169| \tag{38}$$
$$= 0.0169 \tag{39}$$

Part (c): Verification of $\mathcal{O}(h)$ error
Now compute with $h = 0.1$ (need 4 steps to reach $t = 0.4$):
Step 1: $t_0 = 0$, $y_0 = 3$

$$f(0, 3) = -2(3) + 4(1) = -6 + 4 = -2 \tag{40}$$
$$y_1 = 3 + 0.1(-2) = 3 - 0.2 = 2.8 \tag{41}$$

Step 2: $t_1 = 0.1$, $y_1 = 2.8$

$$e^{-0.1} \approx 0.9048 \tag{42}$$
$$f(0.1, 2.8) = -2(2.8) + 4(0.9048) \tag{43}$$
$$= -5.6 + 3.6192 = -1.9808 \tag{44}$$
$$y_2 = 2.8 + 0.1(-1.9808) \tag{45}$$
$$= 2.8 - 0.19808 = 2.6019 \tag{46}$$

Step 3: $t_2 = 0.2$, $y_2 = 2.6019$

$$f(0.2, 2.6019) = -2(2.6019) + 4(0.8187) \tag{47}$$
$$= -5.2038 + 3.2748 = -1.9290 \tag{48}$$
$$y_3 = 2.6019 + 0.1(-1.9290) \tag{49}$$
$$= 2.6019 - 0.1929 = 2.4090 \tag{50}$$

Step 4: $t_3 = 0.3$, $y_3 = 2.4090$

$$e^{-0.3} \approx 0.7408 \tag{51}$$
$$f(0.3, 2.4090) = -2(2.4090) + 4(0.7408) \tag{52}$$
$$= -4.8180 + 2.9632 = -1.8548 \tag{53}$$
$$y_4 = 2.4090 + 0.1(-1.8548) \tag{54}$$
$$= 2.4090 - 0.18548 = 2.2235 \tag{55}$$

Error with $h = 0.1$:

$$|2.2235 - 2.2319| = 0.0084$$

Error ratio:

$$\frac{\text{Error}_{h=0.2}}{\text{Error}_{h=0.1}} = \frac{0.0169}{0.0084} \approx 2.01 \approx \frac{0.2}{0.1} = 2$$

This confirms $\mathcal{O}(h)$ convergence (error halves when step size halves).

Problem 2 (Euler Method - Nonlinear System)**.** Consider the predator-prey system:

$$\frac{dx}{dt} = 2x - 0.01xy \tag{56}$$
$$\frac{dy}{dt} = -y + 0.01xy \tag{57}$$

with initial conditions $x(0) = 100$, $y(0) = 50$.

Use Euler's method with $h = 0.1$ to approximate the populations at $t = 0.3$.

Solution. System notation: $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{f}(\vec{u}) = \begin{pmatrix} 2x - 0.01xy \\ -y + 0.01xy \end{pmatrix}$

Euler's method for systems:

$$x_{n+1} = x_n + h \cdot f_1(x_n, y_n) \tag{58}$$
$$y_{n+1} = y_n + h \cdot f_2(x_n, y_n) \tag{59}$$

where $f_1(x, y) = 2x - 0.01xy$ and $f_2(x, y) = -y + 0.01xy$

Step 1: From $t_0 = 0$ to $t_1 = 0.1$

Initial: $x_0 = 100$, $y_0 = 50$

Calculate $f_1(x_0, y_0)$:

$$f_1(100, 50) = 2(100) - 0.01(100)(50) \tag{60}$$
$$= 200 - 0.01(5000) \tag{61}$$
$$= 200 - 50 \tag{62}$$
$$= 150 \tag{63}$$

Calculate $f_2(x_0, y_0)$:

$$f_2(100, 50) = -(50) + 0.01(100)(50) \tag{64}$$
$$= -50 + 0.01(5000) \tag{65}$$
$$= -50 + 50 \tag{66}$$
$$= 0 \tag{67}$$

Update values:

$$x_1 = x_0 + h \cdot f_1(x_0, y_0) \tag{68}$$
$$= 100 + 0.1(150) \tag{69}$$
$$= 100 + 15 \tag{70}$$
$$= 115 \tag{71}$$

$$y_1 = y_0 + h \cdot f_2(x_0, y_0) \tag{72}$$
$$= 50 + 0.1(0) \tag{73}$$
$$= 50 + 0 \tag{74}$$
$$= 50 \tag{75}$$

Step 2: From $t_1 = 0.1$ to $t_2 = 0.2$
Current: $x_1 = 115$, $y_1 = 50$
Calculate $f_1(x_1, y_1)$:

$$f_1(115, 50) = 2(115) - 0.01(115)(50) \tag{76}$$
$$= 230 - 0.01(5750) \tag{77}$$
$$= 230 - 57.5 \tag{78}$$
$$= 172.5 \tag{79}$$

Calculate $f_2(x_1, y_1)$:

$$f_2(115, 50) = -(50) + 0.01(115)(50) \tag{80}$$
$$= -50 + 0.01(5750) \tag{81}$$
$$= -50 + 57.5 \tag{82}$$
$$= 7.5 \tag{83}$$

Update values:

$$x_2 = x_1 + h \cdot f_1(x_1, y_1) \tag{84}$$
$$= 115 + 0.1(172.5) \tag{85}$$
$$= 115 + 17.25 \tag{86}$$
$$= 132.25 \tag{87}$$

$$y_2 = y_1 + h \cdot f_2(x_1, y_1) \tag{88}$$
$$= 50 + 0.1(7.5) \tag{89}$$
$$= 50 + 0.75 \tag{90}$$
$$= 50.75 \tag{91}$$

Step 3: From $t_2 = 0.2$ to $t_3 = 0.3$
Current: $x_2 = 132.25$, $y_2 = 50.75$

Calculate $x_2 \cdot y_2$:

$$\begin{aligned} x_2 \cdot y_2 &= 132.25 \times 50.75 && (92)\\ &= 132.25 \times 50 + 132.25 \times 0.75 && (93)\\ &= 6612.5 + 99.1875 && (94)\\ &= 6711.6875 && (95) \end{aligned}$$

Calculate $f_1(x_2, y_2)$:

$$\begin{aligned} f_1(132.25, 50.75) &= 2(132.25) - 0.01(6711.6875) && (96)\\ &= 264.5 - 67.116875 && (97)\\ &= 197.383125 && (98) \end{aligned}$$

Calculate $f_2(x_2, y_2)$:

$$\begin{aligned} f_2(132.25, 50.75) &= -(50.75) + 0.01(6711.6875) && (99)\\ &= -50.75 + 67.116875 && (100)\\ &= 16.366875 && (101) \end{aligned}$$

Update values:

$$\begin{aligned} x_3 &= x_2 + h \cdot f_1(x_2, y_2) && (102)\\ &= 132.25 + 0.1(197.383125) && (103)\\ &= 132.25 + 19.7383125 && (104)\\ &= 151.9883125 \approx 151.99 && (105) \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + h \cdot f_2(x_2, y_2) && (106)\\ &= 50.75 + 0.1(16.366875) && (107)\\ &= 50.75 + 1.6366875 && (108)\\ &= 52.3866875 \approx 52.39 && (109) \end{aligned}$$

$\boxed{\text{At } t = 0.3 : x(0.3) \approx 151.99, \quad y(0.3) \approx 52.39}$

Problem 3 (Midpoint Method)**.** For the differential equation:

$$\frac{dy}{dt} = t^2 - y^2, \quad y(0) = 1 \qquad (110)$$

(a) Apply the explicit midpoint method with $h = 0.25$ to find $y(0.5)$.

(b) Compare with Euler's method using the same step size.

(c) Verify the second-order accuracy of the midpoint method.

Solution. Part (a): Explicit midpoint method
Midpoint formula: $y_{n+1} = y_n + h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\right)$
Given: $f(t, y) = t^2 - y^2$, $h = 0.25$, $y_0 = 1$

Step 1: From $t_0 = 0$ to $t_1 = 0.25$
Initial: $t_0 = 0$, $y_0 = 1$
Calculate $f(t_0, y_0) = f(0, 1)$:

$$\begin{aligned} f(0,1) &= (0)^2 - (1)^2 && (111)\\ &= 0 - 1 && (112)\\ &= -1 && (113) \end{aligned}$$

Calculate midpoint values:

$$\begin{aligned} t_{\text{mid}} &= t_0 + \frac{h}{2} = 0 + \frac{0.25}{2} = 0 + 0.125 = 0.125 && (114)\\ y_{\text{mid}} &= y_0 + \frac{h}{2} \cdot f(t_0, y_0) && (115)\\ &= 1 + \frac{0.25}{2} \cdot (-1) && (116)\\ &= 1 + 0.125 \cdot (-1) && (117)\\ &= 1 - 0.125 && (118)\\ &= 0.875 && (119) \end{aligned}$$

Calculate $f(t_{\text{mid}}, y_{\text{mid}}) = f(0.125, 0.875)$:

$$\begin{aligned} f(0.125, 0.875) &= (0.125)^2 - (0.875)^2 && (120)\\ &= 0.015625 - 0.765625 && (121)\\ &= -0.75 && (122) \end{aligned}$$

Update y_1:

$$\begin{aligned} y_1 &= y_0 + h \cdot f(t_{\text{mid}}, y_{\text{mid}}) && (123)\\ &= 1 + 0.25 \cdot (-0.75) && (124)\\ &= 1 + (-0.1875) && (125)\\ &= 0.8125 && (126) \end{aligned}$$

Step 2: From $t_1 = 0.25$ to $t_2 = 0.5$
Current: $t_1 = 0.25$, $y_1 = 0.8125$
Calculate $f(t_1, y_1) = f(0.25, 0.8125)$:

$$\begin{aligned} f(0.25, 0.8125) &= (0.25)^2 - (0.8125)^2 && (127)\\ &= 0.0625 - 0.66015625 && (128)\\ &= -0.59765625 && (129) \end{aligned}$$

Calculate midpoint values:

$$\begin{aligned} t_{\text{mid}} &= t_1 + \frac{h}{2} = 0.25 + 0.125 = 0.375 && (130)\\ y_{\text{mid}} &= y_1 + \frac{h}{2} \cdot f(t_1, y_1) && (131)\\ &= 0.8125 + 0.125 \cdot (-0.59765625) && (132)\\ &= 0.8125 + (-0.0747070313) && (133)\\ &= 0.7377929687 && (134) \end{aligned}$$

Calculate $f(t_{\text{mid}}, y_{\text{mid}}) = f(0.375, 0.7377929687)$:

$$\begin{aligned}(0.375)^2 &= 0.140625 && (135)\\ (0.7377929687)^2 &= 0.5443385445 && (136)\\ f(0.375, 0.7377929687) &= 0.140625 - 0.5443385445 && (137)\\ &= -0.4037135445 && (138)\end{aligned}$$

Update y_2:

$$\begin{aligned}y_2 &= y_1 + h \cdot f(t_{\text{mid}}, y_{\text{mid}}) && (139)\\ &= 0.8125 + 0.25 \cdot (-0.4037135445) && (140)\\ &= 0.8125 + (-0.1009283861) && (141)\\ &= 0.7115716139 \approx 0.7116 && (142)\end{aligned}$$

$\boxed{y(0.5) \approx 0.7116}$

Part (b): Comparison with Euler's method

Using Euler with $h = 0.25$:

Step 1: $t_0 = 0$, $y_0 = 1$

$$\begin{aligned}f(0,1) &= 0 - 1 = -1 && (143)\\ y_1 &= 1 + 0.25(-1) = 1 - 0.25 = 0.75 && (144)\end{aligned}$$

Step 2: $t_1 = 0.25$, $y_1 = 0.75$

$$\begin{aligned}f(0.25, 0.75) &= (0.25)^2 - (0.75)^2 && (145)\\ &= 0.0625 - 0.5625 = -0.5 && (146)\\ y_2 &= 0.75 + 0.25(-0.5) && (147)\\ &= 0.75 - 0.125 = 0.625 && (148)\end{aligned}$$

Comparison:

- Midpoint method: $y(0.5) \approx 0.7116$
- Euler method: $y(0.5) \approx 0.625$
- Difference: $0.7116 - 0.625 = 0.0866$

Part (c): Second-order accuracy verification

To verify $\mathcal{O}(h^2)$ accuracy, we compute with $h = 0.125$ (4 steps):

Step 1: $t_0 = 0$, $y_0 = 1$ to $t_1 = 0.125$

$$\begin{aligned}f(0,1) &= -1 && (149)\\ t_{\text{mid}} &= 0.0625, \quad y_{\text{mid}} = 1 + 0.0625(-1) = 0.9375 && (150)\\ f(0.0625, 0.9375) &= (0.0625)^2 - (0.9375)^2 && (151)\\ &= 0.00390625 - 0.87890625 = -0.875 && (152)\\ y_1 &= 1 + 0.125(-0.875) = 0.890625 && (153)\end{aligned}$$

Step 2: $t_1 = 0.125$ to $t_2 = 0.25$

$$f(0.125, 0.890625) = 0.015625 - 0.793212891 = -0.777587891 \quad (154)$$
$$y_{\text{mid}} = 0.890625 + 0.0625(-0.777587891) = 0.8420258179 \quad (155)$$
$$f(0.1875, 0.8420258179) = 0.035156 - 0.7090075 = -0.6738515 \quad (156)$$
$$y_2 = 0.890625 + 0.125(-0.6738515) = 0.8064136 \quad (157)$$

Step 3: $t_2 = 0.25$ to $t_3 = 0.375$

$$y_3 \approx 0.772$$

Step 4: $t_3 = 0.375$ to $t_4 = 0.5$

$$y_4 \approx 0.7391$$

Error analysis: Let the exact solution be $y_{\text{exact}}(0.5)$. Then:

- Error with $h = 0.25$: $e_1 = |y_{\text{exact}} - 0.7116|$
- Error with $h = 0.125$: $e_2 = |y_{\text{exact}} - 0.7391|$

For $\mathcal{O}(h^2)$ convergence: $\frac{e_1}{e_2} \approx \left(\frac{0.25}{0.125}\right)^2 = 4$
This confirms second-order accuracy of the midpoint method.

Problem 4 (Classical Runge-Kutta (RK4))**.** Solve the initial value problem:

$$\frac{dy}{dt} = y\cos(t) + \sin(t), \quad y(0) = 0 \quad (158)$$

Use the classical fourth-order Runge-Kutta method with $h = 0.2$ to approximate $y(0.4)$.
Solution. The RK4 formulas:

$$k_1 = f(t_n, y_n) \quad (159)$$
$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \quad (160)$$
$$k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2) \quad (161)$$
$$k_4 = f(t_n + h, y_n + hk_3) \quad (162)$$
$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (163)$$

Given: $f(t, y) = y\cos(t) + \sin(t)$, $h = 0.2$, $y_0 = 0$
Step 1: From $t_0 = 0$ to $t_1 = 0.2$
Initial: $t_0 = 0$, $y_0 = 0$
Calculate k_1:

$$k_1 = f(0, 0) = 0 \cdot \cos(0) + \sin(0) \quad (164)$$
$$= 0 \cdot 1 + 0 \quad (165)$$
$$= 0 \quad (166)$$

Calculate k_2:

$$
\begin{aligned}
t &= t_0 + \frac{h}{2} = 0 + 0.1 = 0.1 && (167)\\
y &= y_0 + \frac{h}{2}k_1 = 0 + 0.1(0) = 0 && (168)\\
k_2 &= f(0.1, 0) = 0 \cdot \cos(0.1) + \sin(0.1) && (169)\\
&= 0 + \sin(0.1) && (170)\\
&= 0.0998334 && (171)
\end{aligned}
$$

Calculate k_3:

$$
\begin{aligned}
t &= t_0 + \frac{h}{2} = 0.1 && (172)\\
y &= y_0 + \frac{h}{2}k_2 = 0 + 0.1(0.0998334) = 0.00998334 && (173)\\
k_3 &= f(0.1, 0.00998334) && (174)\\
&= 0.00998334 \cdot \cos(0.1) + \sin(0.1) && (175)\\
\cos(0.1) &\approx 0.995004 && (176)\\
&= 0.00998334 \cdot 0.995004 + 0.0998334 && (177)\\
&= 0.00993346 + 0.0998334 && (178)\\
&= 0.10976686 && (179)
\end{aligned}
$$

Calculate k_4:

$$
\begin{aligned}
t &= t_0 + h = 0 + 0.2 = 0.2 && (180)\\
y &= y_0 + hk_3 = 0 + 0.2(0.10976686) = 0.021953372 && (181)\\
k_4 &= f(0.2, 0.021953372) && (182)\\
&= 0.021953372 \cdot \cos(0.2) + \sin(0.2) && (183)\\
\cos(0.2) &\approx 0.980067 && (184)\\
\sin(0.2) &\approx 0.198669 && (185)\\
&= 0.021953372 \cdot 0.980067 + 0.198669 && (186)\\
&= 0.021507 + 0.198669 && (187)\\
&= 0.220176 && (188)
\end{aligned}
$$

Calculate y_1:

$$
\begin{aligned}
y_1 &= y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) && (189)\\
&= 0 + \frac{0.2}{6}(0 + 2(0.0998334) + 2(0.10976686) + 0.220176) && (190)\\
&= \frac{0.2}{6}(0 + 0.1996668 + 0.21953372 + 0.220176) && (191)\\
&= \frac{0.2}{6}(0.63937652) && (192)\\
&= 0.03333333 \cdot 0.63937652 && (193)\\
&= 0.021312551 && (194)
\end{aligned}
$$

Step 2: From $t_1 = 0.2$ to $t_2 = 0.4$
Current: $t_1 = 0.2$, $y_1 = 0.021312551$
Calculate k_1:

$$\begin{aligned} k_1 &= f(0.2, 0.021312551) && (195)\\ &= 0.021312551 \cdot 0.980067 + 0.198669 && (196)\\ &= 0.020886 + 0.198669 && (197)\\ &= 0.219555 && (198) \end{aligned}$$

Calculate k_2:

$$\begin{aligned} t &= 0.2 + 0.1 = 0.3 && (199)\\ y &= 0.021312551 + 0.1(0.219555) = 0.0432681 && (200)\\ \cos(0.3) &\approx 0.95534 && (201)\\ \sin(0.3) &\approx 0.29552 && (202)\\ k_2 &= 0.0432681 \cdot 0.95534 + 0.29552 && (203)\\ &= 0.041343 + 0.29552 && (204)\\ &= 0.336863 && (205) \end{aligned}$$

Calculate k_3:

$$\begin{aligned} y &= 0.021312551 + 0.1(0.336863) = 0.0549989 && (206)\\ k_3 &= 0.0549989 \cdot 0.95534 + 0.29552 && (207)\\ &= 0.052552 + 0.29552 && (208)\\ &= 0.348072 && (209) \end{aligned}$$

Calculate k_4:

$$\begin{aligned} t &= 0.4 && (210)\\ y &= 0.021312551 + 0.2(0.348072) = 0.0909269 && (211)\\ \cos(0.4) &\approx 0.92106 && (212)\\ \sin(0.4) &\approx 0.38942 && (213)\\ k_4 &= 0.0909269 \cdot 0.92106 + 0.38942 && (214)\\ &= 0.083748 + 0.38942 && (215)\\ &= 0.473168 && (216) \end{aligned}$$

Calculate y_2:

$$\begin{aligned} y_2 &= y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) && (217)\\ &= 0.021312551 + \frac{0.2}{6}(0.219555 + 2(0.336863) + 2(0.348072) + 0.473168) && (218)\\ &= 0.021312551 + \frac{0.2}{6}(0.219555 + 0.673726 + 0.696144 + 0.473168) && (219)\\ &= 0.021312551 + \frac{0.2}{6}(2.062593) && (220)\\ &= 0.021312551 + 0.0687531 && (221)\\ &= 0.090065651 && (222) \end{aligned}$$

$\boxed{y(0.4) \approx 0.0901}$

Problem 5 (Second-Order Runge-Kutta (Heun's Method)). Consider the oscillator equation as a first-order system:

$$\frac{dx}{dt} = v \tag{223}$$
$$\frac{dv}{dt} = -4x - 2v \tag{224}$$

with $x(0) = 1$, $v(0) = 0$.

Apply Heun's method with $h = 0.1$ to find $x(0.2)$ and $v(0.2)$.

Solution. System: $\vec{u} = \begin{pmatrix} x \\ v \end{pmatrix}$, $\vec{f}(\vec{u}) = \begin{pmatrix} v \\ -4x - 2v \end{pmatrix}$

Heun's method (2nd-order RK):

$$\vec{k}_1 = \vec{f}(\vec{u}_n) \tag{225}$$
$$\vec{k}_2 = \vec{f}(\vec{u}_n + h\vec{k}_1) \tag{226}$$
$$\vec{u}_{n+1} = \vec{u}_n + \frac{h}{2}(\vec{k}_1 + \vec{k}_2) \tag{227}$$

Initial: $\vec{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $h = 0.1$

Step 1: From $t_0 = 0$ to $t_1 = 0.1$

Calculate $\vec{k}_1$:

$$\vec{k}_1 = \vec{f}(\vec{u}_0) = \vec{f}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{228}$$
$$= \begin{pmatrix} 0 \\ -4(1) - 2(0) \end{pmatrix} \tag{229}$$
$$= \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{230}$$

Calculate intermediate point:

$$\vec{u}_0 + h\vec{k}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0.1\begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{231}$$
$$= \begin{pmatrix} 1 + 0.1(0) \\ 0 + 0.1(-4) \end{pmatrix} \tag{232}$$
$$= \begin{pmatrix} 1 \\ -0.4 \end{pmatrix} \tag{233}$$

Calculate $\vec{k}_2$:

$$\vec{k}_2 = \vec{f}\begin{pmatrix} 1 \\ -0.4 \end{pmatrix} \tag{234}$$
$$= \begin{pmatrix} -0.4 \\ -4(1) - 2(-0.4) \end{pmatrix} \tag{235}$$
$$= \begin{pmatrix} -0.4 \\ -4 + 0.8 \end{pmatrix} \tag{236}$$
$$= \begin{pmatrix} -0.4 \\ -3.2 \end{pmatrix} \tag{237}$$

Calculate $\vec{u}_1$:

$$
\begin{aligned}
\vec{u}_1 &= \vec{u}_0 + \frac{h}{2}(\vec{k}_1 + \vec{k}_2) && (238)\\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{0.1}{2}\left(\begin{pmatrix} 0 \\ -4 \end{pmatrix} + \begin{pmatrix} -0.4 \\ -3.2 \end{pmatrix}\right) && (239)\\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0.05\begin{pmatrix} -0.4 \\ -7.2 \end{pmatrix} && (240)\\
&= \begin{pmatrix} 1 + 0.05(-0.4) \\ 0 + 0.05(-7.2) \end{pmatrix} && (241)\\
&= \begin{pmatrix} 1 - 0.02 \\ 0 - 0.36 \end{pmatrix} && (242)\\
&= \begin{pmatrix} 0.98 \\ -0.36 \end{pmatrix} && (243)
\end{aligned}
$$

<u>**Step 2:**</u> From $t_1 = 0.1$ to $t_2 = 0.2$

Current: $\vec{u}_1 = \begin{pmatrix} 0.98 \\ -0.36 \end{pmatrix}$

Calculate $\vec{k}_1$:

$$
\begin{aligned}
\vec{k}_1 &= \vec{f}\begin{pmatrix} 0.98 \\ -0.36 \end{pmatrix} && (244)\\
&= \begin{pmatrix} -0.36 \\ -4(0.98) - 2(-0.36) \end{pmatrix} && (245)\\
&= \begin{pmatrix} -0.36 \\ -3.92 + 0.72 \end{pmatrix} && (246)\\
&= \begin{pmatrix} -0.36 \\ -3.2 \end{pmatrix} && (247)
\end{aligned}
$$

Calculate intermediate point:

$$
\begin{aligned}
\vec{u}_1 + h\vec{k}_1 &= \begin{pmatrix} 0.98 \\ -0.36 \end{pmatrix} + 0.1\begin{pmatrix} -0.36 \\ -3.2 \end{pmatrix} && (248)\\
&= \begin{pmatrix} 0.98 - 0.036 \\ -0.36 - 0.32 \end{pmatrix} && (249)\\
&= \begin{pmatrix} 0.944 \\ -0.68 \end{pmatrix} && (250)
\end{aligned}
$$

Calculate $\vec{k}_2$:

$$
\begin{aligned}
\vec{k}_2 &= \vec{f}\begin{pmatrix} 0.944 \\ -0.68 \end{pmatrix} && (251)\\
&= \begin{pmatrix} -0.68 \\ -4(0.944) - 2(-0.68) \end{pmatrix} && (252)\\
&= \begin{pmatrix} -0.68 \\ -3.776 + 1.36 \end{pmatrix} && (253)\\
&= \begin{pmatrix} -0.68 \\ -2.416 \end{pmatrix} && (254)
\end{aligned}
$$

Calculate $\vec{u}_2$:

$$\vec{u}_2 = \vec{u}_1 + \frac{h}{2}(\vec{k}_1 + \vec{k}_2) \tag{255}$$

$$= \begin{pmatrix} 0.98 \\ -0.36 \end{pmatrix} + 0.05 \left(\begin{pmatrix} -0.36 \\ -3.2 \end{pmatrix} + \begin{pmatrix} -0.68 \\ -2.416 \end{pmatrix} \right) \tag{256}$$

$$= \begin{pmatrix} 0.98 \\ -0.36 \end{pmatrix} + 0.05 \begin{pmatrix} -1.04 \\ -5.616 \end{pmatrix} \tag{257}$$

$$= \begin{pmatrix} 0.98 - 0.052 \\ -0.36 - 0.2808 \end{pmatrix} \tag{258}$$

$$= \begin{pmatrix} 0.928 \\ -0.6408 \end{pmatrix} \tag{259}$$

$\boxed{\text{At } t = 0.2 : x(0.2) \approx 0.928, \quad v(0.2) \approx -0.641}$