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THE HECKSCHER–OHLIN AND TRAVIS–VANEK THEOREMS UNDER UNCERTAINTY

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The Heckscher–Ohlin theorem is valid under uncertainty of the special benchmark sort modelled by Helpman–Razin. Their pessimism over its validity is vanquished by more structure: rational expectations are imposed on consumers and identical and constant relative aversion to income risk is assumed. Under these circumstances, with free commodity trade, the Heckscher–Ohlin theorem holds for the international exchange of equities.

1. Introduction

Recent work on the theory of international trade under uncertainty has called into question the basic theorems of the Heckscher–Ohlin model. Helpman and Razin (1978) have brilliantly demonstrated that with the presence of a Diamond stock market (1967), international trade in equities restores all save the Heckscher–Ohlin theorem. This paper will show that the Heckscher–Ohlin theorem and its higher dimension generalization, the Travis–Vanek theorem, are also restored by international trade in equities.

Section 2 briefly reviews the Helpman–Razin model. Section 3 then shows how the Heckscher–Ohlin and Travis–Vanek theorems immediately follow with rational expectations under a specialization of the usual certainty case restriction that indifference curves over commodity bundles be identical and homothetic. Formally, the usual case allows a utility indicator to be any monotonic increasing function of a homogeneous quasi-concave function which is identical across countries. The specialization imposed here is that 'any monotonic increasing function' is reduced to any positive power function (and, inessentially, any increasing linear transform of a positive power function). This condition is necessary and sufficient for homothetic indifference curves over commodities to translate into homothetic indifference curves over equities. Furthermore, to ensure that identical homothetic

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commodity preferences imply *identical* homothetic equity preferences, a sufficient condition is that the power function must be in the same degree across countries. In other words, the utility indicator must be identical and homogeneous up to a linear transform. This is equivalent to restricting behavior to display constant relative and identical aversion to income risk. Helpman–Razin overlooked this result in part because they failed to utilize the restrictions imposed by rational expectations, and in part because they considered the relative factor price definition of relative factor abundance, which does not sufficiently restrict preferences.

2. The Helpman–Razin model

Helpman and Razin adapt Diamond's stock market model for trade theory purposes. There are two stages of decision-making (or two periods). One is ex post, after all uncertainty is resolved. The other is ex ante, before values of random variables are known. The two-sector production model has technological uncertainty of the Hicks-neutral multiplicative variety. Resources are allocated ex ante, before the technological uncertainty is resolved. The Diamond stock market stipulates that, ex ante, firms produce and sell equities in a competitive market. Helpman–Razin offer a convenient interpretation. One unit of activity in sector j provides an ex post basket of outputs of commodity j, $(\theta_{j1},\dots,\theta_{jS})$, one output for each state of nature s: $s=1,\dots,S$. Thus, one unit of equity in sector j entitles the owner to the basket $(\theta_{j1},\dots\theta_{jS})$. Equities are produced by firms under the usual competitive production model conditions and subject to a given real price of equities (q_j). The real equity price is revealed by the stock market; it equals the stock market value of the industry divided by its activity level. Competitive firms in the industry take it as a parameter. Firms (and with constant returns the industries) hire resources to solve the program

$$\max_{L_j,K_j} q_j f_j(L_j,K_j) - wL_j - rK_j,$$

where $Z_j = f_j(L_j,K_j)$ is the neoclassical production function, w and r are input prices, and L_j,K_j are labor and capital inputs in j. This is very similar to profit maximization. For the general equilibrium, the allocation problem depends only on relative equity prices, and the production of equities can be depicted in the usual diagram.

On the demand side of the stock market, ex ante, consumers identical in tastes and endowments purchase equities out of income received from the sale of labor and capital services and of initial equity holdings. Ex post, the consumers hold endowments $(\theta_{1s}Z_1,\theta_{2s}Z_2)$ and seek to exchange them to optimize a quasi-concave utility function over commodity bundles $\psi(C_1,C_2)$.

In Helpman–Razin, ψ is concave, which excludes the case of risk preference. Exchange equilibrium determines the ex post relative commodity price P_s and thus numeraire income $P_s\theta_{1s}Z_1+\theta_{2s}Z_2$. Each state of nature s implies a different outcome for relative price and income given the real equity holdings (i.e. resource allocation). Consumers have knowledge of the distribution of the relative price, but not its dependence on activity levels Z. Forming the indirect utility function V, the arguments of which are relative price and income, outcomes may be ranked over selection of equity holdings (Z_1,Z_2) given the state of nature. Imposing the expected utility axioms, the consumers' ex ante objective function (the assets utility function) is the expectation of indirect utility:

$$U(Z_1,Z_2\,|\,\phi_P)=\sum_{s=1}^{S}\pi_s V(P_s,P_s\theta_{1s}Z_1+\theta_{2s}Z_2), \tag{1}$$

where π_s = probability of state s, and the ϕ_P under $U(\cdot)$ denotes a given distribution of the relative price. Given concavity of ψ, U is concave in Z_1,Z_2. The ingenious simplification of Helpman–Razin is to note that the ex ante equity purchase problem of consumers combined with the value maximization equity sale problem of firms yields a stock market analysis very much like the usual commodity allocation problem in the certainty case.

In fig. 1, Z_1 and Z_2 are units of activity in industries 1 and 2. $\bar{Z}_1$ units of activity in industry 1 will yield $\theta_{1s}\bar{Z}_1$ units of output of good 1 in state of the

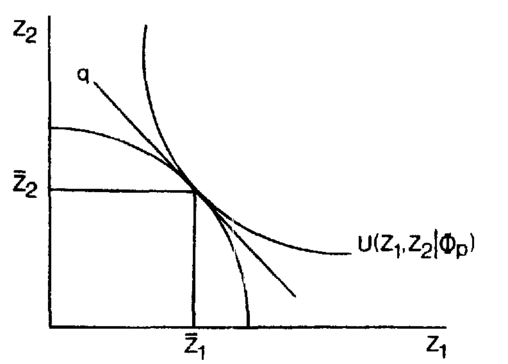

Fig. 1

world s. Z_1 and Z_2 are also the number of equities produced in industries 1 and 2. $\bar{Z}_1$ units of equity will yield $P_s\theta_{1s}\bar{Z}_1$ dollars worth of income in state of the world s, using good 2 as the numeraire. For this claim to uncertain income, consumers pay $q\bar{Z}_1$ ex ante where q is the relative equity price using good 2 as the numeraire. Portfolio and firm value maximization agree in the stock market equilibrium of fig. 1. The transformation locus of fig. 1 will, for our purposes, be taken to reflect the Heckscher–Ohlin production model.

3. The Heckscher–Ohlin and Travis–Vanek theorems

For the $2 \times 2 \times 2$ certainty case, the Heckscher–Ohlin theorem states that a country will increase production of and export the commodity relatively intensive in its relatively abundant factor. With the quantity definition of relative factor abundance, the possible offsetting effect of demand conditions is neutralized with the assumption of identical homothetic preferences. In fig. 2, relative supply functions are drawn for two countries. Z_1 and Z_2 are certain outputs. With good 1 relatively capital intensive and country A relatively capital abundant, the Rybczynski theorem guarantees that A's relative supply function lies to the right of B's. Identical homothetic preferences guarantee that a single relative demand function represents both countries' preferences. Isolation equilibria must lie at $P_B > P_A$. With the introduction of free trade, and thus a common commodity price, the Heckscher–Ohlin production and trade prediction follows.

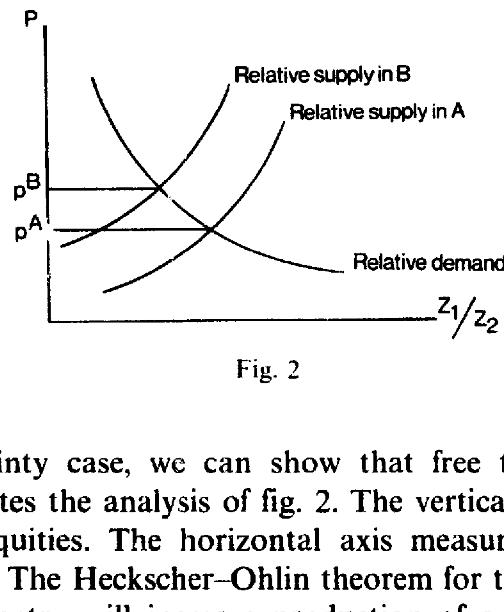

Fig. 2

For the uncertainty case, we can show that free trade in equities and commodities recreates the analysis of fig. 2. The vertical axis measures q, the relative price of equities. The horizontal axis measures relative supply of equities as in fig. 1. The Heckscher–Ohlin theorem for the case of uncertainty then follows: a country will increase production of and export that equity which is relatively intensive in its relatively abundant factor.

The trick is to show that under the identical homogeneous restriction the relative demand function of fig. 2 is the same for both countries in isolation and for the world in free trade. First note that the identical technology assumption of the Helpman–Razin version of the H–O model includes the uncertain portion. Thus, θ_{1s} is the shift parameter in industry 1 for state s in all countries. The assets utility function of consumers is $U(Z_1, Z_2 \mid \phi_P)$. Holding the price distribution constant implies a fixed assets production (Z_1^0, Z_2^0), so only $U(Z_1^0, Z_2^0 \mid \phi_P)$ is consistent with closed economy equilibrium. A change in production will change the price distribution and imply another family of asset indifference curves with a complex relation to the first family. As Helpman–Razin noted, this shifting of indifference curves appears to rule out the Heckscher–Ohlin theorem under any but very restrictive conditions.

Fortunately there is an envelope relation on the set of assets utility functions (one for each price distribution) which will establish the Heckscher–Ohlin theorem under the identical homogeneous commodity preferences restriction. Define the rational expectations (closed economy equilibrium) assets utility function:

$$W(Z_1,Z_2)\equiv U(Z_1,Z_2,\phi_P(Z_1,Z_2)). \tag{2}$$

$W(Z_1,Z_2)$ stipulates that, in equilibrium, rational consumers must know and act on the correct relative price distribution, though they are not aware of its dependence on Z_1,Z_2. In any state $s, P_s(Z_1,Z_2)$ is the actual ex post market clearing price based on activity of Z_1,Z_2. $\phi_P(Z_1,Z_2)$ is the price distribution based on Z_1,Z_2. If consumers assume any other relative price distribution, systematic error will occur in their price forecasts. Rational consumers will revise their expectations until such errors are eliminated, and the assumed distribution is $\phi_P(Z_1,Z_2)$. Since $W(Z_1,Z_2)$ is defined on the basis of closed economy equilibrium, it is immediately clear that

$$W(Z_1,Z_2)=\sum_{s=1}^{S}\pi_s\psi(\theta_{1s}Z_1,\theta_{2s}Z_2)$$

for each country in autarky equilibrium and for the world in free trade equilibrium. Also, the quasi-concavity and homogeneity properties of ψ carry through to W. Finally, we have the following lemma:

1. The indifference curve $W(Z_1,Z_2)=\bar{W}$ is the outer envelope of the set of indifference curves $U(Z_1,Z_2\,|\,\phi_P)=\bar{W}$, where each member of the set has a different price distribution.
2. With $\psi(C_1,C_2)=F(G(C_1,C_2))$ being any homogeneous increasing function $F(\cdot)$ of a quasi-concave increasing homogeneous function $G(\cdot)$, $U(Z_1,Z_2\,|\,\phi_P)$ is homogeneous and convex (concave) as ψ is homogeneous of degree greater (less) than one.

With $\psi(C_1,C_2)$ being homogeneous and identical in both countries, the rational expectations indifference curves $W(Z_1,Z_2)$ are identical and homogeneous. In fig. 2, a single rational expectations (or closed economy equilibrium) relative assets demand function is drawn which represents the tastes of both countries. It is downward sloping due to the quasi-concavity of W and independent of scale due to the homogeneity of W. The Heckscher–Ohlin theorem compares two autarky equilibria with a world free trade (closed world economy) equilibrium. With rational expectations, the diagram shows that the free trade relative equity price must lie between the autarky relative equity prices which in turn have the Heckscher–Ohlin ranking. The consumer's relative assets demand functions are not shown on fig. 2. At each of the three equilibria, given the price distributions implied by Z_1/Z_2, a relative assets demand function is defined which will intersect the rational expectations equilibrium relative assets demand function at the equilibrium relative equity price. The only restriction on their shape is that they cannot cross the rational expectations equilibrium assets demand function more than once and their elasticity must be algebraically greater in the neighborhood of the intersection.

A more intuitive argument relating the two sorts of demand functions at equilibrium is instructive. At any of the three closed equilibria, ex post relative commodity prices in state s are

$$P_s = \frac{\psi_1(\theta_{1s}Z_1, \theta_{2s}Z_2)}{\psi_2(\theta_{1s}Z_1, \theta_{2s}Z_2)}.$$

Equilibrium relative equity prices are

$$q = \frac{W_1(Z_1, Z_2)}{W_2(Z_1, Z_2)} = \frac{\sum_{s=1}^{S} \pi_s \psi_1(\theta_{1s}Z_1, \theta_{2s}Z_2)\theta_{1s}}{\sum_{s=1}^{S} \pi_s \psi_2(\theta_{1s}Z_1, \theta_{2s}Z_2)\theta_{2s}}$$

along the rational expectations relative assets demand function of fig. 2. The consumer's assets relative demand function is derived from

$$q = \frac{U_1(Z_1, Z_2 \mid \phi_P)}{U_2(Z_1, Z_2 \mid \phi_P)} = \frac{\sum_{s=1}^{S} \pi_s V_I(P_s, P_s\theta_{1s}Z_1 + \theta_{2s}Z_2)P_s\theta_{1s}}{\sum_{s=1}^{S} \pi_s V_I(P_s, P_s\theta_{1s}Z_1 + \theta_{2s}Z_2)\theta_{2s}},$$

where V_I is the marginal utility of income. Substituting in the true ex post expression for P_s and noting that $V_I = \psi_2$, and $V_I P_s = \psi_1$, we see that with correct expectations, the consumer's relative assets demand function agrees with (intersects) the rational expectations assets demand function.

The usual restriction on tastes in the certainty case is that indifference curves in commodity space be identical and homothetic. Formally, the utility indicator $\psi(C_1, C_2)$ must be able to be written as $F(G(C_1, C_2))$, where $G(C_1, C_2)$ is quasi-concave, homogeneous and identical across countries, and $F(G)$ is any monotonic increasing function, possibly different across countries. For the uncertainty case, for W to have identical and scale-free indifference curves across countries, it is sufficient to restrict $F(G)$ to the class of positive power functions and impose identical F across countries.[1]

It remains to prove the lemma.

Proof of 1. Let activity levels be Z_1^0, Z_2^0. In ex post commodity space, the realized utility in state s is $\psi_s^0 = \psi(\theta_{1s}Z_1^0, \theta_{2s}Z_2^0)$. The assumed realized utility in state s is $V(P_s, P_s\theta_{1s}Z_1^0 + \theta_{2s}Z_2^0) = V_s^0$. Let $P_s(Z_1^0, Z_2^0)$ be the realization in state s of the market clearing relative price based on activity of (Z_1^0, Z_2^0). For $P_s \neq P_s(Z_1^0, Z_2^0)$, $\psi_s^0 \leqq V_s^0$ since an infeasible (but beneficial) exchange is implicitly assumed by consumers. For $P_s = P_s(Z_1^0, Z_2^0)$, $\psi_s^0 = V_s^0$. By definition

$$U(Z_1, Z_1 \mid \phi_P) = \sum_{s=1}^{S} \pi_s V(P_s, P_s\theta_{1s}Z_1 + Z_2)$$

and

$$W(Z_1, Z_2) = \sum_{s=1}^{S} \pi_s \psi(\theta_{1s}Z_1, \theta_{2s}Z_2),$$

so we have $U(Z_1, Z_2 \mid \phi_P) \geqq W(Z_1, Z_2)$, with equality where $\phi_P = \phi_P(Z_1, Z_2)$. Since W and U are quasi-concave, $W(Z_1, Z_2) = \bar{W}$ must be the outer envelope of a family of asset indifference curves $U(Z_1, Z_2 \mid \phi_P) = \bar{W}$.

Proof of 2. Let $\psi(C_1, C_2) = F(G(C_1, C_2)) = [G(C_1, C_2)]^k$, where $G(\cdot)$ is a concave homogeneous (of degree one without loss of generality) function. Homogeneity of $\psi(C_1, C_2)$ implies that the ex post demand functions are homogeneous of degree one in income: $C_i(P_s, \lambda I_s) = \lambda C_i(P_s, I_s)$, $i = 1, 2$. The indirect utility function is $V(P_s, I_s) = V(C_1(P_s, I_s), C_2(P_s, I_s))$. Evidently $V(P_s, \lambda I_s) = \lambda^k V(P_s, I_s)$. With $I_s = P_s\theta_{1s}Z_1 + \theta_{2s}Z_2$, V is homogeneous of degree k in (Z_1, Z_2). Since the assets utility function is

$$U(Z_1, Z_2 \mid \phi_P) = \sum_{s=1}^{S} \pi_s V(P_s, I_s),$$

it is also homogeneous of degree k in (Z_1, Z_2). Evidently V (and hence U) is convex (concave) in income (hence activities) as $k > (<)\ 1$.

The restriction that $F(G)$ be a positive power function is sufficient for homothetic preferences over commodities to imply homothetic preferences over equities, but also nearly necessary. For F to reflect well-behaved

[1]For the case where G is Cobb–Douglas, F can differ across countries, so identical behavior towards risk is not always necessary.

preferences, it must be continuous. The necessary restriction on F which permits the theorem is that it be any positive linear transform of a positive power function, and linear transformations are irrelevant to behavior under the expected utility axioms. $\psi(C_1, C_2)$ is homothetic if and only if $\psi(C_1, C_2) = F(G(C_1, C_2))$, where G is quasi-concave homogeneous and F is monotonic increasing. $W(Z_1, Z_2)$ is homothetic if and only if $W(Z_1, Z_2) = H(J(Z_1, Z_2))$ where analogous restrictions apply to H and J:

$$W(Z_1, Z_2) = \sum_{s=1}^{S} \pi_s \psi(\theta_{1s} Z_1, \theta_{2s} Z_2) = \sum_{s=1}^{S} \pi_s F(G(\theta_{1s} Z_1, \theta_{2s} Z_2).$$

With the continuity condition on F holding, it can be arbitrarily well approximated as an nth order polynomial in G:

$$F(G) = a + b_1 G + b_2 G^2 + \ldots + b_n G^n.$$

Only when all but one of the b_i are zero can $W(Z_1, Z_2)$ be written as a monotonic transformation of a homogeneous function.[2]

The lemma evidently generalizes readily to higher dimensions. Thus, we can demonstrate that the Travis–Vanek theorem generalizes to trade in equities.

For the certainty case of n goods, r factors, $n \geq r$, the Travis–Vanek theorem states that in free trade factor price equalization equilibrium, embodied factor trade relative to world endowments will follow the ordering of home factor endowments relative to world factor endowments. It follows from identical homothetic preferences, implying that home consumption (hence embodied factor consumption) is proportional to world consumption (hence embodied factor consumption = world endowments). If A is the unit factor requirements matrix at free trade prices,

$$F^{A} - AC^{A} \equiv T, \tag{3}$$

where T = embodied factor trade, C^{A} = country A's consumption vector, and F^{A} = country A's factor endowment vector. With identical homothetic preferences,

$$C^{A} = b(C^{A} + C^{B}) = b(X^{A} + X^{B}), \tag{4}$$

where b is a fraction, and X denotes production. Substituting in (3) we

[2]In an earlier version of this paper I committed the possibly not uncommon error of defining homotheticity following Lancaster: $F(x)$ is homothetic iff $F(\lambda x) = f(\lambda)F(x)$. Kats (1970) shows this definition does not include all radial expansion path functions and, moreover, the only admissible form for $f(\lambda)$ is λ^k. Thus, this approach reduces to that in the text.

obtain

$$F^A - bA(X^A + X^B) = T = F^A - b(F^A + F^B). \tag{5}$$

The Travis–Vanek theorem follows when (5) is multiplied by a diagonal matrix with $1/F_i^A + F_i^B$ on the diagonal for all i. Evidently the proposition that identical homogeneous commodity preferences imply identical homothetic rational expectations asset preferences carries the Travis–Vanek theorem through to the case of uncertain technology, provided there is free trade in equities and commodities. Simply reinterpret C and X as purchase and production of equities. Factor trade embodied in the international exchange of equities will follow the ordering of home factor endowments relative to foreign factor endowments.

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