Homework 1 solutions

ECON312 Time Series Analysis

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Assignment 1

Consider an $ARMA(1,1)$ process given by

$$y_t = \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \qquad \varepsilon_t \sim WN(0, \sigma^2)$$

with $|\phi| < 1$ and $\sigma^2 > 0$.

1. Find a representation of y_t in terms of ε_t, ε_{t-1}, ε_{t-2}, ...

   Solution

$$\begin{aligned}
y_t &= \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \\
&= \phi(\phi y_{t-2} + \varepsilon_{t-1} + \theta \varepsilon_{t-2}) + \varepsilon_t + \theta \varepsilon_{t-1} \\
&= \phi^2 \left(\phi y_{t-3} + \varepsilon_{t-2} + \theta \varepsilon_{t-3}\right) + \phi \varepsilon_{t-1} + \theta \phi \varepsilon_{t-2} + \varepsilon_t + \theta \varepsilon_{t-1} \\
&= \phi^3 \left(\phi y_{t-4} + \varepsilon_{t-3} + \theta \varepsilon_{t-4}\right) + \phi^2 \varepsilon_{t-2} + \theta \phi^2 \varepsilon_{t-3} + \phi \varepsilon_{t-1} + \theta \phi \varepsilon_{t-2} + \varepsilon_t + \theta \varepsilon_{t-1} \\
&\vdots \\
&= \phi^{j+1} y_{t-j-1} + \phi^j \theta \varepsilon_{t-j-1} + \phi^{j-1} \left(\phi + \theta\right) \varepsilon_{t-j} + \ldots + \phi \left(\phi + \theta\right) \varepsilon_{t-2} + \left(\phi + \theta\right) \varepsilon_{t-1} + \varepsilon_t
\end{aligned}$$

   We have $\lim_{j \to \infty} \phi^j = 0$ since $|\phi| < 1$, hence

$$y_t = \varepsilon_t + (\phi + \theta) \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j-1}$$

2. Find the mean and variance of y_t

   Solution

The mean of y_t is

$$\mathrm{E}(y_t) = \mathrm{E}(\varepsilon_t) + (\phi + \theta) \sum_{j=0}^{\infty} \phi^j \mathrm{E}(\varepsilon_{t-j-1}) = 0$$

The variance of y_t is

$$\begin{aligned}
\mathrm{Var}(y_t) &= \mathrm{Var}(\varepsilon_t) + (\phi + \theta)^2 \sum_{j=0}^{\infty} \phi^{2j} \mathrm{Var}(\varepsilon_{t-j-1}) \\
&= \sigma^2 + (\phi + \theta)^2 \sum_{j=0}^{\infty} \phi^{2j} \sigma^2 \\
&= \sigma^2 \left(1 + \frac{(\phi + \theta)^2}{1 - \phi^2} \right)
\end{aligned}$$

3. Find the first autocovariance of y_t

   Solution

   The first autocovariance of y_t is

$$\begin{aligned}
\mathrm{Cov}(y_t, y_{t-1}) &= \mathrm{Cov}\big(\varepsilon_t + (\phi + \theta)\,\varepsilon_{t-1} + \phi\,(\phi + \theta)\,\varepsilon_{t-2} + \phi^2\,(\phi + \theta)\,\varepsilon_{t-3} + \ldots, \\
&\quad \varepsilon_{t-1} + (\phi + \theta)\,\varepsilon_{t-2} + \phi\,(\phi + \theta)\,\varepsilon_{t-3} + \phi^2\,(\phi + \theta)\,\varepsilon_{t-4} + \ldots\big) \\
&= (\phi + \theta)\,\sigma^2 + \phi\,(\phi + \theta)^2\,\sigma^2 + \phi^3\,(\phi + \theta)^2\,\sigma^2 + \ldots \\
&= \sigma^2 \left((\phi + \theta) + \phi\,(\phi + \theta)^2 + \phi^3\,(\phi + \theta)^2 + \ldots \right) \\
&= \sigma^2 \left((\phi + \theta) + \phi\,(\phi + \theta)^2 \left(1 + \phi^2 + \phi^4 + \ldots \right) \right) \\
&= \sigma^2 \left((\phi + \theta) + \frac{\phi\,(\phi + \theta)^2}{1 - \phi^2} \right)
\end{aligned}$$

4. Find the first autocorrelation of y_t

   Solution

   The first autocorrelation of y_t is

$$\rho(1) = \frac{\mathrm{Cov}(y_t, y_{t-1})}{\mathrm{Var}(y_t)} = \frac{(1 - \phi^2)\,(\phi + \theta) + \phi\,(\phi + \theta)^2}{(1 - \phi^2) + (\phi + \theta)^2} = \frac{(\phi + \theta)\,(1 + \phi\theta)}{1 + 2\phi\theta + \phi^2}$$