Lecture 2. Estimation of Dynamic Models

Narek Ohanyan

American University of Armenia

January 30, 2024

Outline

Estimation of Distributed Lag models

Estimation with Serially Correlated Errors

Estimation of Autoregressive Distributed Lag Models

Assumptions of the Distributed Lag Model

Suppose that we have a distributed lag model

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \cdots + \beta_q x_{t-q} + e_t$$

Under these assumptions, the OLS estimators are the best linear unbiased estimators (BLUE).

Assumptions of the Distributed Lag model

1. The true model is linear in the parameters and correctly specified
2. y and x are stationary random variables, and e_t is independent of current, past and future values of x
3. $\mathrm{E}(e_t) = 0$
4. $\mathrm{Var}(e_t) = \sigma^2$
5. $\mathrm{Cov}(e_t, e_s) = 0$ for all $t \neq s$
6. $e_t \sim \mathcal{N}(0, \sigma^2)$

Consequences of autocorrelation in the error term

The assumption $\mathrm{Cov}(e_t, e_s) = 0$ implies that there is no serial correlation in the error term and is likely to be violated in time-series data.

If errors e_t are serially correlated, then

- the OLS estimators are still unbiased, but they are no longer the best linear unbiased estimators (~~BLUE~~)
- the standard errors of the OLS estimators are incorrect
- the OLS estimators are no longer normally distributed, and the usual t and F tests are invalid

Hence, we need to check for serial correlation in the error term.

Testing for serial correlation

Serial correlation in the error term may be tested using a Lagrange multiplier (LM) test.

Suppose that we have a distributed lag model

$$y_t = \alpha + \beta x_t + e_t$$

and we suspect serial correlation in e_t, that is $e_t = \rho e_{t-1} + \varepsilon_t$.

We can substitute the second equation into the first to obtain

$$y_t = \alpha + \beta x_t + \rho e_{t-1} + \varepsilon_t$$

Also noting that we have $y = \widehat{\alpha} + \widehat{\beta} x + \widehat{e}$, we get

$$\widehat{e}_t = (\alpha - \widehat{\alpha}) + \left(\beta - \widehat{\beta}\right) x_t + \rho \widehat{e}_{t-1} + \varepsilon_t$$

Lagrange multiplier test for serial correlation

The Lagrange multiplier (LM) test is based on the following regression

$$\widehat{e}_t = \gamma_0 + \gamma_1 x_t + \rho \widehat{e}_{t-1} + \varepsilon_t$$

where $\gamma_0 = \alpha - \widehat{\alpha}$ and $\gamma_1 = \beta - \widehat{\beta}$.

The test statistic is

$$\mathrm{LM} = T \times R^2$$

where R^2 is the R-squared from the above regression.

If H_0 is true, the LM test statistic is asymptotically distributed as $\chi^2_{(1)}$.

The test may be also easily extended to test for higher-order serial correlation.

Outline

Estimation of Distributed Lag models

Estimation with Serially Correlated Errors

Estimation of Autoregressive Distributed Lag Models

Standard errors in the presence of serial correlation

Serial correlation in the error term results in the OLS estimators being *inefficient* and standard errors being *incorrect*.

Recall that the variance of the OLS slope estimator is

$$\mathrm{Var}(\widehat{\beta}) = \sum_{t=1}^{T} w_t^2 \mathrm{Var}(e_t) + \sum_{t=1}^{T} \sum_{s \neq t} w_t w_s \mathrm{Cov}(e_t, e_s)$$

where

$$w_t = \frac{x_t - \overline{x}}{\sum_{s=1}^{T} (x_s - \overline{x})^2}$$

Hence, if $\mathrm{Cov}(e_t, e_s) \neq 0$, the variance of the OLS slope estimator is incorrect.

HAC standard errors

It is possible to obtain correct standard errors in the presence of serial correlation.

The correct standard errors are called **heteroskedasticity and autocorrelation consistent (HAC)** standard errors or **Newey-West** standard errors.

The HAC standard errors are computed as

$$\mathrm{Var}(\widehat{\beta}) = \sum_{t=1}^{T} w_t^2 \mathrm{Var}(e_t) + 2 \sum_{k=1}^{M} \left(1 - \frac{k}{M+1}\right) \sum_{t=k+1}^{T} w_t w_{t-k} \mathrm{Cov}(e_t, e_{t-k})$$

where M is the number of lags used in the calculation of the HAC standard errors.

A rule of thumb for choosing M is

$$M = 0.75\, T^{1/3}$$

Outline

Estimation of Distributed Lag models

Estimation with Serially Correlated Errors

Estimation of Autoregressive Distributed Lag Models

Autoregressive Distributed Lag Models

Recall that serial correlation in the error term results in the OLS estimators being *inefficient*.

Suppose that we have a distributed lag model

$$\begin{aligned} y_t &= \alpha + \beta_0 x_t + e_t \\ e_t &= \rho e_{t-1} + \varepsilon_t \end{aligned}$$

Substituting the second equation into the first yields

$$y_t = \delta + \rho y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + \varepsilon_t$$

where $\delta = \alpha\left(1 - \rho\right)$ and $\beta_1 = -\rho\beta_0$.

Hence, the model with autoregressive errors is a special case of an $ARDL(1,1)$ model.

Assumptions of the Autoregressive Distributed Lag Model

Suppose that we have an autoregressive distributed lag model

$$y_t = \alpha + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \beta_0 x_t + \beta_1 x_{t-1} + \cdots + \beta_q x_{t-q} + e_t$$

Under these assumptions, the OLS estimators are not BLUE but are **consistent** and **asymptotically efficient**.

Assumptions of the Autoregressive Distributed Lag model

1. The true model is linear in the parameters and correctly specified
2. y and x are stationary random variables, and e_t is independent of y_{t-1}, x_t and their past values
3. $\mathrm{E}(e_t) = 0$
4. $\mathrm{Var}(e_t) = \sigma^2$
5. $\mathrm{Cov}(e_t, e_s) = 0$ for all $t \neq s$
6. $e_t \sim \mathcal{N}(0, \sigma^2)$

Consistency and asymptotic efficiency

An estimator is **consistent** if it converges in probability to the true value of the parameter as the sample size increases.

An estimator is **asymptotically efficient** if no other consistent estimator has a smaller asymptotic variance.

The OLS estimators of the ARDL model are consistent and asymptotically efficient, i.e.

$$\widehat{\theta} \xrightarrow{p} \theta \quad \text{and} \quad \mathrm{Avar}(\widehat{\theta}) \leq \mathrm{Avar}(\widetilde{\theta}) \quad \text{as} \quad T \to \infty$$

for any other consistent estimator $\widetilde{\theta}$, where θ the the true values of the parameters $\theta = (\alpha, \phi_1, \ldots, \phi_p, \beta_0, \beta_1, \ldots, \beta_q)$ and $\mathrm{Avar}(\cdot)$ is the asymptotic variance.

The OLS estimators are also **asymptotically normally distributed**. That is $\widehat{\theta} \xrightarrow{d} \mathcal{N}(\theta, \Sigma)$.

Lag order selection

The orders of the Autoregressive Distributed Lag models p and q are usually unknown.

Several criteria have been proposed for selecting the lag orders.

- Has serial correlation in the errors been eliminated?
  - The assumptions of the model are not satisfied if the errors are serially correlated
- Are the estimates significantly different from zero?
  - If the estimates of some lags are close to zero, then we can drop those lags
- Information criteria: AIC, BIC, etc.
  - How well does the model fit the data and how many parameters are used?

Information criteria

The most commonly used information criteria are the AIC and BIC.

The **Akaike information criterion (AIC)** is defined as

$$\mathrm{AIC} = \ln\left(\frac{SSE}{T}\right) + \frac{2k}{T}$$

where SSE is the sum of squared residuals, T is the number of observations, and $k = p + q + 2$.

The **Bayesian information criterion (BIC)** is defined as

$$\mathrm{BIC} = \ln\left(\frac{SSE}{T}\right) + \frac{k\ln(T)}{T}$$

Because $\ln(T) > 2$ for $T \geq 8$, the BIC penalizes additional lags more heavily than the AIC.