Lecture 3
Autoregressive Distributed Lag Models

9.6 Autoregressive Distributed Lag Models

- An autoregressive distributed lag (ARDL) model is one that contains both lagged x_t's and lagged y_t's

Eq. 9.52

$$y_t = \delta + \delta_0 x_t + \delta_1 x_{t-1} + \cdots + \delta_q x_{t-q} + \theta_1 y_{t-1} + \cdots + \theta_p y_{t-p} + v_t$$

– Two examples:

$$\text{ADRL}(1,1):\ INF_t = 0.3336 + 0.5593 INF_{t-1} - 0.6882 DU_t + 0.3200 DU_{t-1}$$

$$\text{ADRL}(1,0):\ INF_t = 0.3548 + 0.5282 INF_{t-1} - 0.4909 DU_t$$

- An ARDL model can be transformed into one with only lagged *x*'s which go back into the infinite past:

Eq. 9.53

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \beta_3 x_{t-3} + \cdots + e_t$$

$$= \alpha + \sum_{s=0}^{\infty} \beta_s x_{t-s} + e_t$$

  – This model is called an **infinite distributed lag model**

- Four possible criteria for choosing *p* and *q*:
  1. Has serial correlation in the errors been eliminated?
  2. Are the signs and magnitudes of the estimates consistent with our expectations from economic theory?
  3. Are the estimates significantly different from zero, particularly those at the longest lags?
  4. What values for *p* and *q* minimize information criteria such as the *AIC* and *SC*?

- The **Akaike information criterion** (***AIC***) is:

Eq. 9.54

$$\text{AIC} = \ln\left(\frac{SSE}{T}\right) + \frac{2K}{T}$$

where $K = p + q + 2$

- The Schwarz criterion (*SC*), also known as the Bayes information criterion (*BIC*), is:

Eq. 9.55

$$\text{SC} = \ln\left(\frac{SSE}{T}\right) + \frac{K\ln(T)}{T}$$

  – Because $K\ln(T)/T > 2K/T$ for $T \geq 8$, the *SC* penalizes additional lags more heavily than does the *AIC*

Eq. 9.56

- Consider the previously estimated ARDL(1,0) model:

$$\widehat{INF}_t = 0.3548 + 0.5282 INF_{t-1} - 0.4909 DU_t, \quad \text{obs} = 90$$
$$(se) \quad (0.0876) \quad (0.0851) \quad (0.1921)$$

FIGURE 9.9 Correlogram for residuals from Phillips curve ARDL(1,0) model

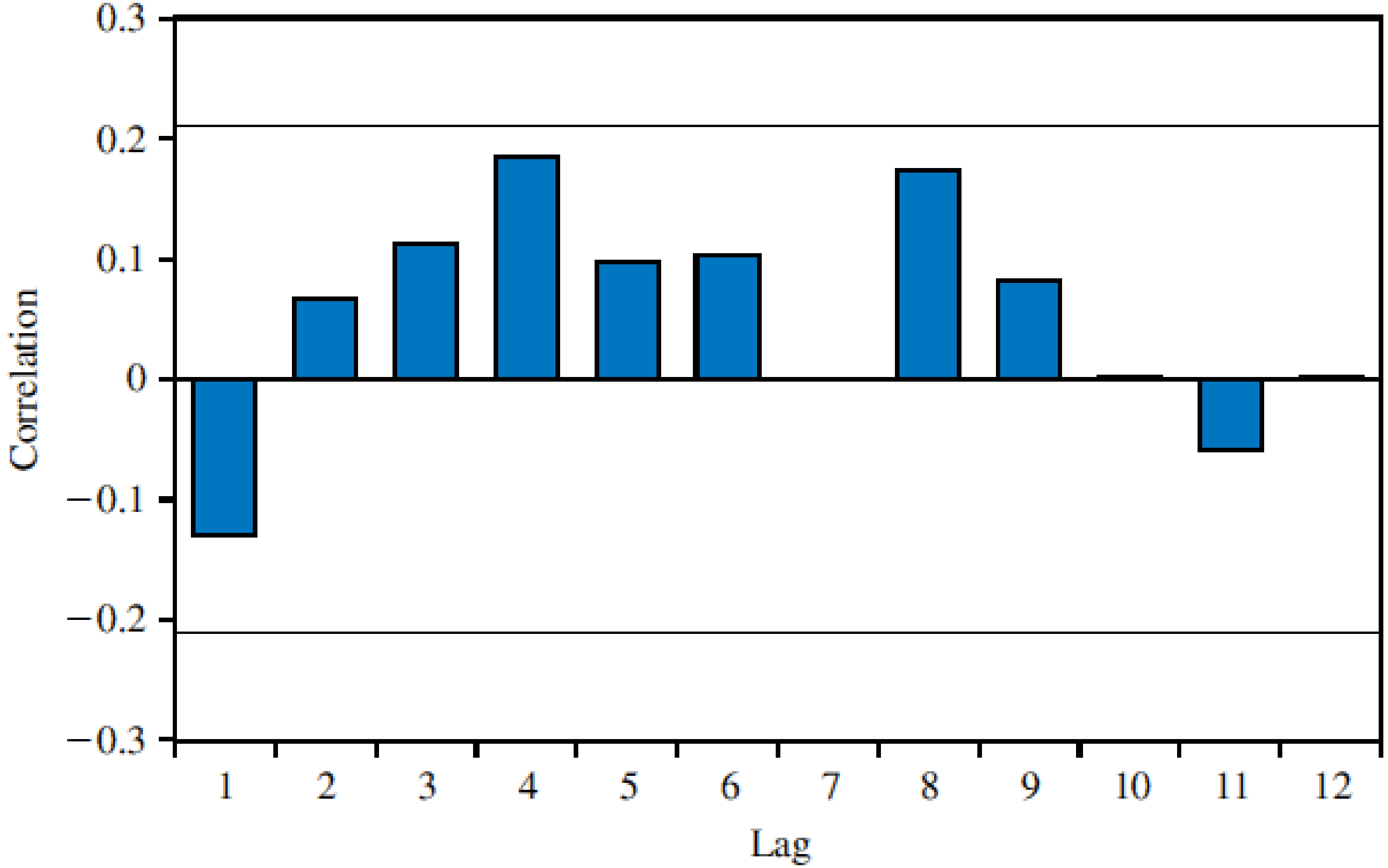

Table 9.3 *p*-values for LM Test for Autocorrelation

Lag	*p*-value
1	0.0421
2	0.0772
3	0.1563
4	0.0486
5	0.0287

Eq. 9.57

- For an ARDL(4,0) version of the model:

$$\begin{aligned}
\widehat{INF}_t &= 0.1001 + 0.2354 INF_{t-1} + 0.1213 INF_{t-2} + 0.1677 INF_{t-3} \\
(se) \quad & \quad (0.0983) \quad (0.1016) \qquad\quad (0.1038) \qquad\quad (0.1050) \\
& \qquad + 0.2819 INF_{t\text{-}4} - 0.7902 DU_t \\
& \qquad\quad (0.1014) \qquad\quad (0.1885) \qquad \text{obs} = 87
\end{aligned}$$

- Inflation is given by:

$$INF_t^E = 0.1001 + 0.2354 INF_{t-1} + 0.1213 INF_{t-2} + 0.1677 INF_{t-3} + 0.2819 INF_{t\text{-}4}$$

Table 9.4 *AIC* and *SC* Values for Phillips Curve ARDL Models

p	q	AIC	SC	p	q	AIC	SC
1	0	−1.247	−1.160	1	1	−1.242	−1.128
2	0	−1.290	−1.176	2	1	−1.286	−1.142
3	0	−1.335	−1.192	3	1	−1.323	−1.151
4	0	−1.402	−1.230	4	1	−1.380	−1.178
5	0	−1.396	−1.195	5	1	−1.373	−1.143
6	0	−1.378	−1.148	6	1	−1.354	−1.096

- Recall the model for Okun's Law:

Eq. 9.58

$$\begin{aligned} &DU_t = 0.5836 - 0.2020G_t - 0.1653G_{t-1} - 0.0700G_{t-2}, \quad \text{obs} = 96 \\ &(se) \quad (0.0472) \quad (0.0324) \quad (0.0335) \qquad (0.0331) \end{aligned}$$

FIGURE 9.10 Correlogram for residuals from Okun's law ARDL(0,2) model

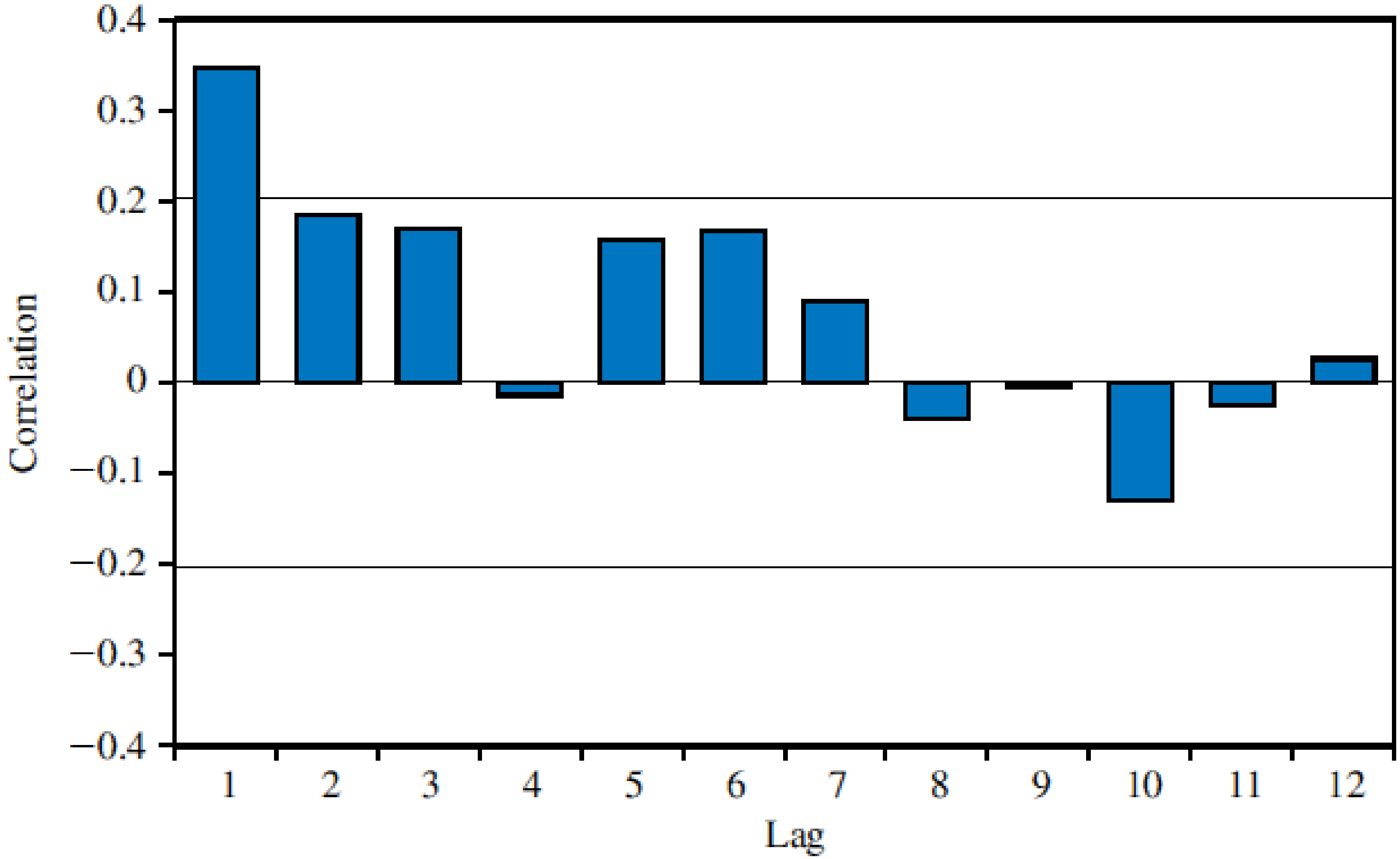

Table 9.5 *AIC* and *SC* Values for Okun's Law ARDL Models

(p, q)	AIC	SC	(p, q)	AIC	SC	(p, q)	AIC	SC
(0,1)	−3.436	−3.356	(1,1)	−3.588	−3.480	(2,1)	−3.569	−3.435
(0,2)	−3.463	−3.356	(1,2)	−3.568	−3.433	(2,2)	−3.548	−3.387
(0,3)	−3.442	−3.308	(1,3)	−3.561	−3.400	(2,3)	−3.549	−3.361

- Now consider this version:

Eq. 9.59

$$\begin{array}{l} \widehat{DU}_t = 0.3780 + 0.3501 DU_{t-1} - 0.1841 G_t - 0.0992 G_{t-1}, \quad \text{obs} = 96 \\ (se) \quad (0.0578) (0.0846) \qquad (0.0307) \qquad (0.0368) \end{array}$$

- An autoregressive model of order *p*, denoted AR(*p*), is given by:

Eq. 9.60

$$y_t = \delta + \theta_1 y_{t-1} + \theta_2 y_{t-2} + \cdots + \theta_p y_{t-p} + v_t$$

- Consider a model for growth in real GDP:

Eq. 9.61

$$G_t = 0.4657 + 0.3770 G_{t-1} + 0.2462 G_{t-2}$$
$$(se)\quad(0.1433)\quad(0.1000)\quad(0.1029) \qquad \text{obs} = 96$$

FIGURE 9.11 Correlogram for residuals from AR(2) model for GDP growth

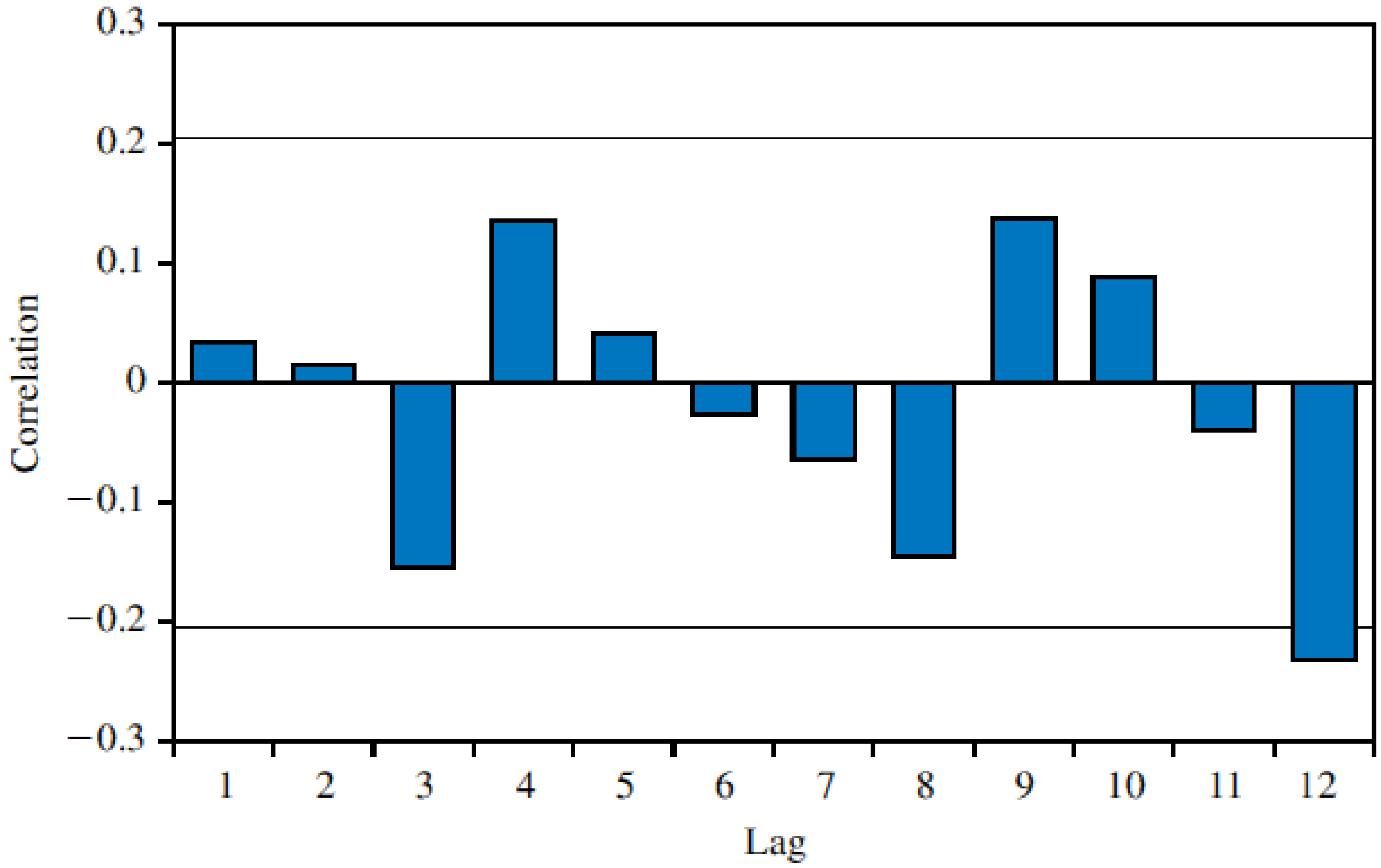

Table 9.6 *AIC* and *SC* Values for AR Model of Growth in U.S. GDP

Order (p)	1	2	3	4	5
AIC	−1.094	−1.131	−1.124	−1.133	−1.112
SC	−1.039	−1.049	−1.015	−0.997	−0.948

9.7 Forecasting

- We consider forecasting using two different models:
    1. AR model
    2. ARDL model

- Consider an AR(2) model for real GDP growth:

Eq. 9.62

$$G_t = \delta + \theta_1 G_{t-1} + \theta_2 G_{t-2} + v_t$$

- The model to forecast G_{T+1} is:

$$G_{T+1} = \delta + \theta_1 G_T + \theta_2 G_{T-1} + v_{T+1}$$

- The growth values for the two most recent quarters are:

$$G_T = G_{2009Q3} = 0.8$$

$$G_{T\text{-}1} = G_{2009Q2} = \text{-}0.2$$

- The forecast for G_{2009Q4} is:

Eq. 9.63

$$\begin{aligned}\hat{G}_{T+1} &= \hat{\delta} + \hat{\theta}_1 G_T + \hat{\theta}_2 G_{T-1} \\ &= 0.46573 + 0.37700 \times 0.8 + 0.24624 \times (-0.2) \\ &= 0.7181\end{aligned}$$

- For two quarters ahead, the forecast for G_{2010Q1} is:

Eq. 9.64

$$\begin{aligned}\hat{G}_{T+2} &= \hat{\delta} + \hat{\theta}_1 G_{T+1} + \hat{\theta}_2 G_T \\ &= 0.46573 + 0.37700 \times 0.71808 + 0.24624 \times 0.8 \\ &= 0.9334\end{aligned}$$

- For three periods out, it is:

Eq. 9.65

$$\begin{aligned}\hat{G}_{T+3} &= \hat{\delta} + \hat{\theta}_1 G_{T+2} + \hat{\theta}_2 G_{T+1} \\ &= 0.46573 + 0.37700 \times 0.93343 + 0.24624 \times 0.71808 \\ &= 0.9945\end{aligned}$$

- Summarizing our forecasts:
  - Real GDP growth rates for 2009Q4, 2010Q1, and 2010Q2 are approximately 0.72%, 0.93%, and 0.99%, respectively

- A 95% interval forecast for *j* periods into the future is given by:

$$\hat{G}_{T+j} \pm t_{(0.975,df)}\hat{\sigma}_j$$

where $\hat{\sigma}_j$ is the standard error of the forecast error and *df* is the number of degrees of freedom in the estimation of the AR model

- The first forecast error, occurring at time T+1, is:

$$u_1 = G_{T+1} - \hat{G}_{T+1} = \left(\delta - \hat{\delta}\right) + \left(\theta_1 - \hat{\theta}_1\right) G_T + \left(\theta_2 - \hat{\theta}_2\right) G_{T-1} + v_{T+1}$$

- Ignoring the error from estimating the coefficients, we get:

Eq. 9.66

$$u_1 = v_{T+1}$$

- The forecast error for two periods ahead is:

Eq. 9.67

$$u_2 = \theta_1 \left(G_{T+1} - \hat{G}_{T+1} \right) + v_{T+2} = \theta_1 u_1 + v_{T+2} = \theta_1 v_{T+1} + v_{T+2}$$

- The forecast error for three periods ahead is:

Eq. 9.68

$$u_3 = \theta_1 u_2 + \theta_2 u_1 + v_{T+3} = \left(\theta_1^2 + \theta_2 \right) v_{T+1} + \theta_1 v_{T+2} + v_{T+3}$$

- Because the v_t's are uncorrelated with constant variance σ_v^2, we can show that:

$$\sigma_1^2 = \operatorname{var}(u_1) = \sigma_v^2$$

$$\sigma_2^2 = \operatorname{var}(u_2) = \sigma_v^2 \left(1 + \theta_1^2\right)$$

$$\sigma_3^2 = \operatorname{var}(u_3) = \sigma_v^2 \left(\left(\theta_1^2 + \theta_2\right)^2 + \theta_1^2 + 1\right)$$

Table 9.7 Forecasts and Forecast Intervals for GDP Growth

Quarter	Forecast $\hat{G}_{T+j}$	Standard Error of Forecast Error ($\hat{\sigma}_j$)	Forecast Interval $(\hat{G}_{T+j} \pm 1.9858 \times \hat{\sigma}_j)$
2009Q4 ($j = 1$)	0.71808	0.55269	(−0.379, 1.816)
2010Q1 ($j = 2$)	0.93343	0.59066	(−0.239, 2.106)
2010Q2 ($j = 3$)	0.99445	0.62845	(−0.254, 2.242)

- Consider forecasting future unemployment using the Okun's Law ARDL(1,1):

Eq. 9.69

$$DU_t = \delta + \theta_1 DU_{t-1} + \delta_0 G_t + \delta_1 G_{t-1} + v_t$$

- The value of *DU* in the first post-sample quarter is:

Eq. 9.70

$$DU_{T+1} = \delta + \theta_1 DU_T + \delta_0 G_{T+1} + \delta_1 G_T + v_{T+1}$$

  – But we need a value for G_{T+1}

- Now consider the *change* in unemployment
  - Rewrite Eq. 9.70 as:

$$U_{T+1} - U_T = \delta + \theta_1 \left(U_T - U_{T-1} \right) + \delta_0 G_{T+1} + \delta_1 G_T + v_{T+1}$$

  - Rearranging:

Eq. 9.71

$$\begin{aligned} U_{T+1} &= \delta + \left(\theta_1 + 1 \right) U_T - \theta_1 U_{T-1} + \delta_0 G_{T+1} + \delta_1 G_T + v_{T+1} \\ &= \delta + \theta_1^* U_T + \theta_2^* U_{T-1} + \delta_0 G_{T+1} + \delta_1 G_T + v_{T+1} \end{aligned}$$

- For the purpose of computing point and interval forecasts, the ARDL(1,1) model for a change in unemployment can be written as an ARDL(2,1) model for the level of unemployment
  - This result holds not only for ARDL models where a dependent variable is measured in terms of a change or difference, but also for pure AR models involving such variables