Lecture 4
Unit Root Tests for Stationarity

Chapter Contents

- The aim is to describe how to estimate regression models involving nonstationary variables
  - The first step is to examine the time-series concepts of **stationarity** (and **nonstationarity**) and how we distinguish between them.
  - **Cointegration** is another important related concept that has a bearing on our choice of a regression model

12.1

Stationary and Nonstationary Variables

- The change in a variable is an important concept
  - The change in a variable y_t, also known as its first difference, is given by $\Delta y_t = y_t - y_{t-1}$.
    - Δy_t is the change in the value of the variable y from period t - 1 to period t

FIGURE 12.1 U.S. economic time series

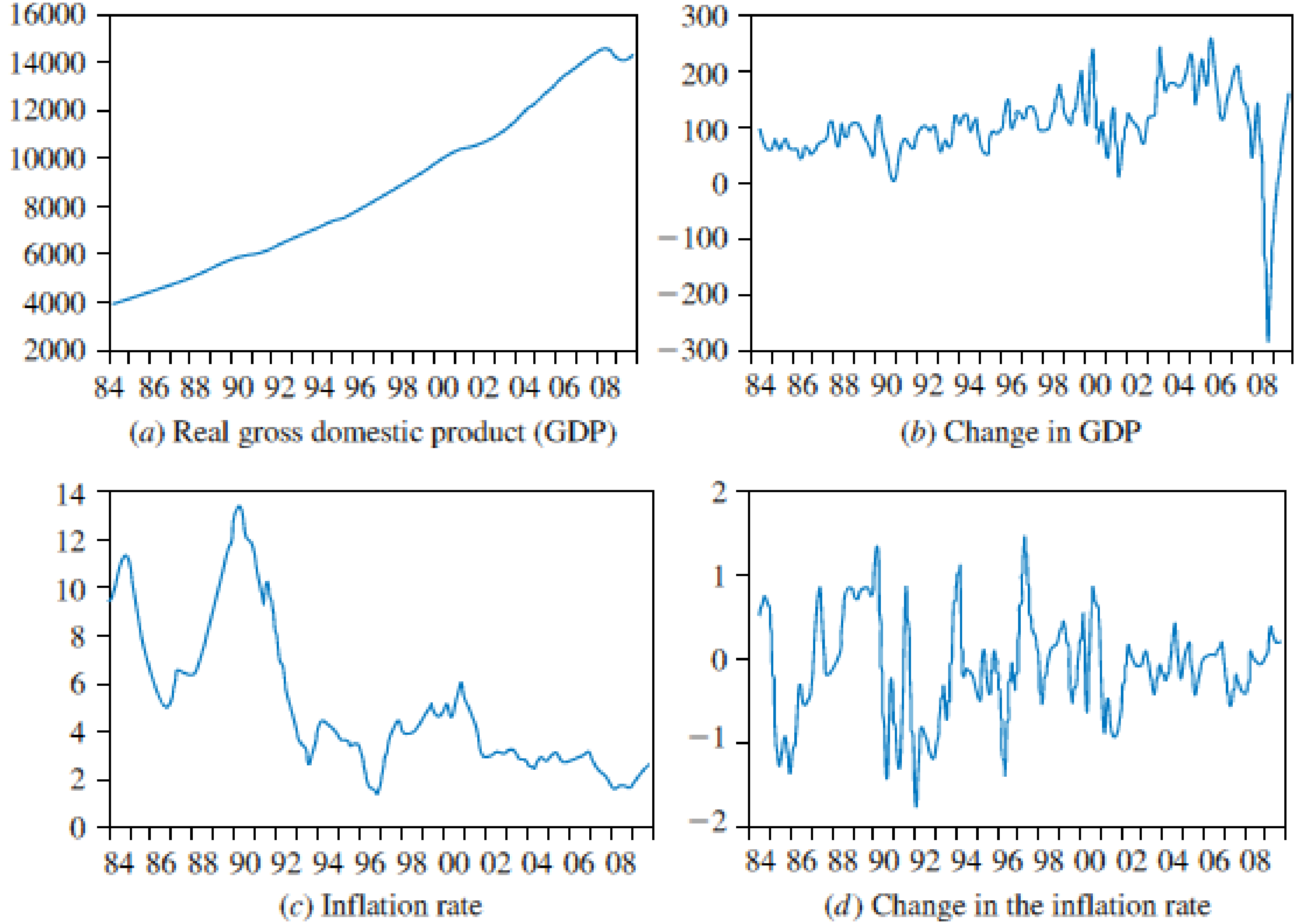
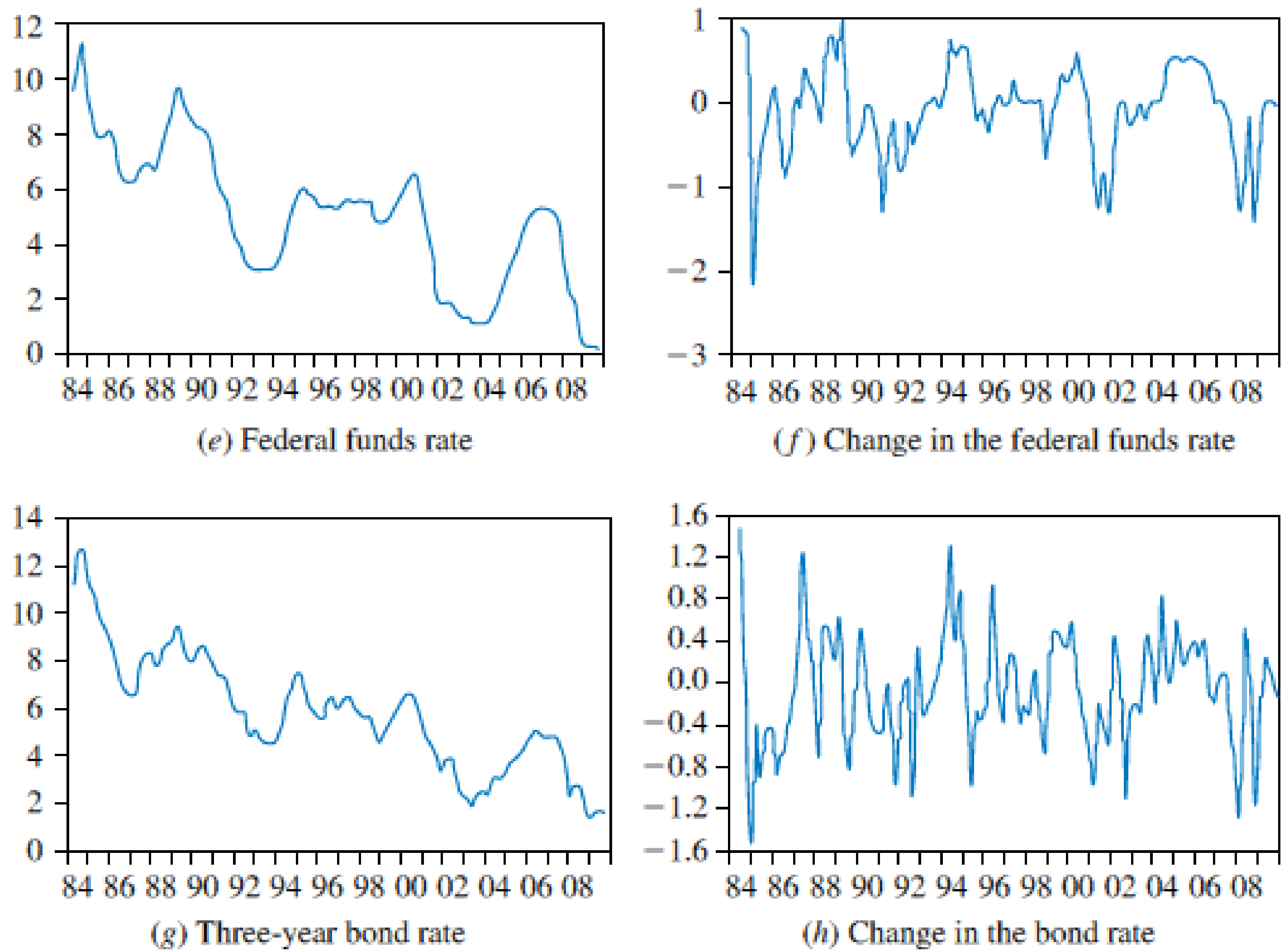


FIGURE 12.1 (Continued) U.S. economic time series



- Formally, a time series y_t is stationary if its mean and variance are constant over time, and if the covariance between two values from the series depends only on the length of time separating the two values, and not on the actual times at which the variables are observed

- That is, the time series y_t is stationary if for all values, and every time period, it is true that:

Eq. 12.1a $E(y_t) = \mu$ (constant mean)

Eq. 12.1b $\text{var}(y_t) = \sigma^2$ (constant variance)

Eq. 12.1c $\text{cov}(y_t, y_{t+s}) = \text{cov}(y_t, y_{t-s}) = \gamma_s$ (covariance depends on s, not t)

Table 12.1 Sample Means of Time Series Shown in Figure 12.1

	Sample periods	
Variable	1984:2 to 1996:4	1997:1 to 2009:4
Real GDP (a)	5813.0	11458.2
Inflation rate (c)	6.9	3.2
Federal funds rate (e)	6.4	3.5
Bond rate (g)	7.3	4.0
Change in GDP (b)	82.7	120.3
Change in the inflation rate (d)	−0.16	0.02
Change in the federal funds rate (f)	−0.09	−0.10
Change in the bond rate (h)	−0.10	−0.09

- The autoregressive model of order one, the AR(1) model, is a useful univariate time series model for explaining the difference between stationary and nonstationary series:

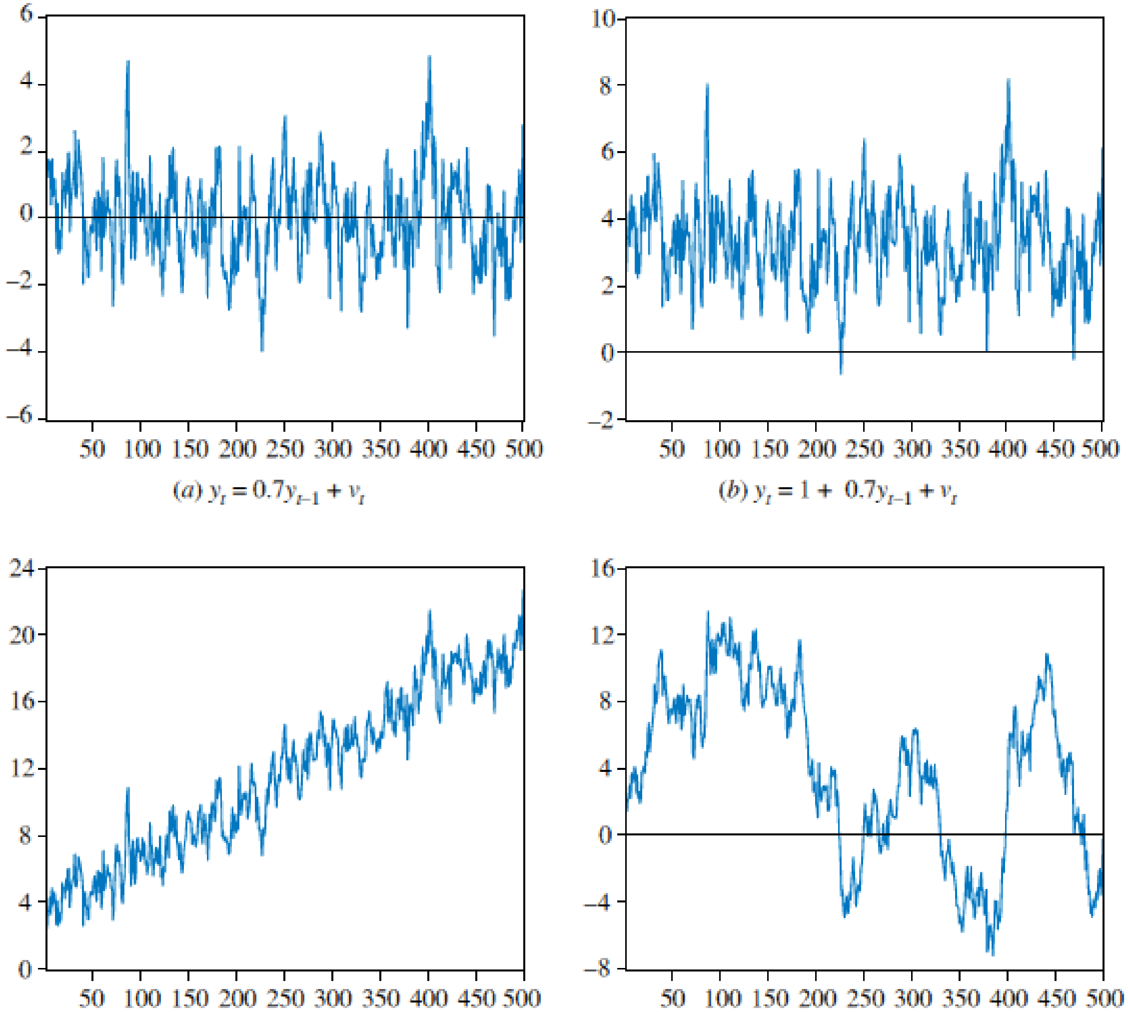
Eq. 12.2a

$$y_t = \rho y_{t-1} + v_t, \qquad |\rho| < 1$$

  – The errors v_t are independent, with zero mean and constant variance σ_v^2, and may be normally distributed

  – The errors are sometimes known as “shocks” or “innovations”

FIGURE 12.2 Time-series models



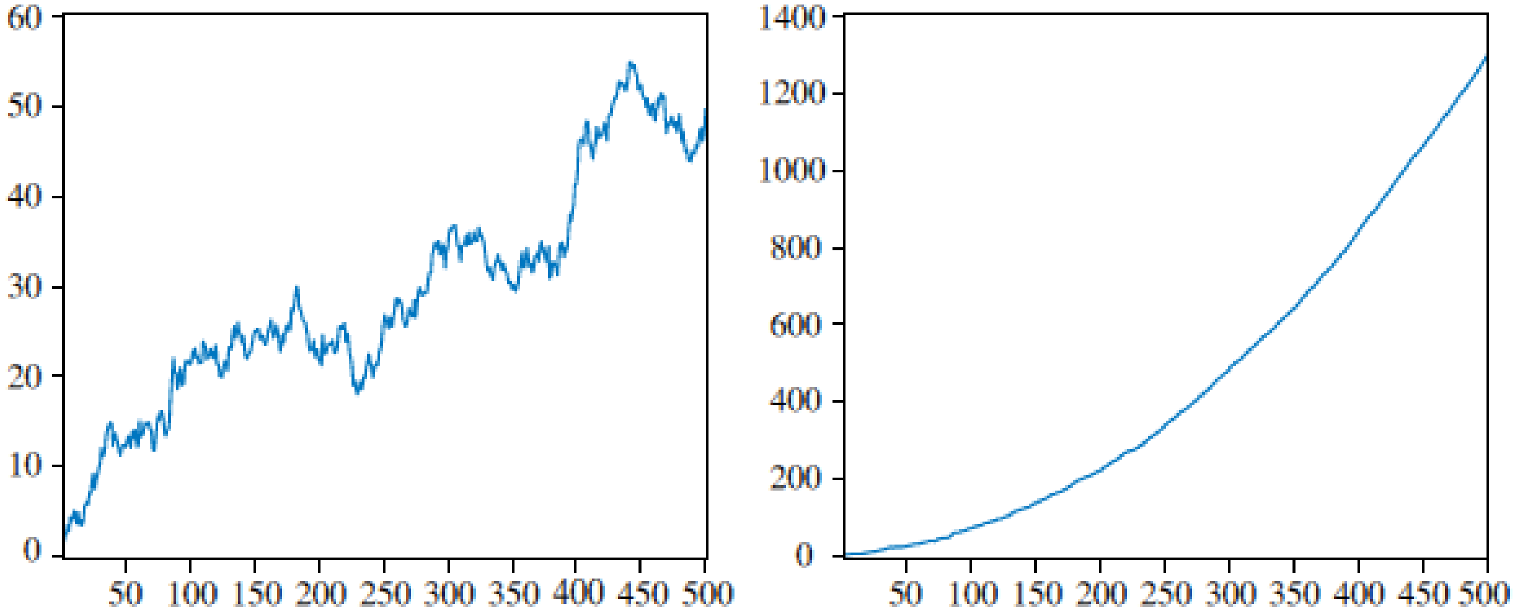
(a) $y_t = 0.7y_{t-1} + v_t$

(b) $y_t = 1 + 0.7y_{t-1} + v_t$

(c) $y_t = 1 + 0.01t + 0.7y_{t-1} + v_t$

(d) $y_t = y_{t-1} + v_t$

FIGURE 12.2 (Continued) Time-series models



(e) $y_t = 0.1 + y_{t-1} + v_t$

(f) $y_t = 0.1 + 0.01t + y_{t-1} + v_t$

- Real-world data rarely have a zero mean
  - We can introduce a nonzero mean μ as:

$$(y_t - \mu) = \rho(y_{t-1} - \mu) + v_t$$

  - Or

Eq. 12.2b

$$y_t = \alpha + \rho y_{t-1} + v_t, \qquad |\rho| < 1$$

- With $\alpha = 1$ and $\rho = 0.7$:

$$E(y_t) = \mu = \alpha/(1-\rho) = 1/(1-0.7) = 3.33$$

- An extension to Eq. 12.2a is to consider an AR(1) model fluctuating around a linear trend: $(\mu + \delta t)$
  - Let the "de-trended" series $(y_t - \mu - \delta t)$ behave like an autoregressive model:

$$(y_t - \mu - \delta t) = \rho(y_{t-1} - \mu - \delta(t-1)) + v_t, \qquad |\rho| < 1$$

Or:

Eq. 12.2c

$$y_t = \alpha + \rho y_{t-1} + \lambda t + v_t$$

Eq. 12.3a

- Consider the special case of ρ = 1:

$$y_t = y_{t-1} + v_t$$

  – This model is known as the random walk model
    - These time series are called **random walks** because they appear to wander slowly upward or downward with no real pattern
    - the values of sample means calculated from subsamples of observations will be dependent on the sample period
      – This is a characteristic of nonstationary series

- We can understand the "wandering" by recursive substitution:

$$y_1 = y_0 + v_1$$

$$y_2 = y_1 + v_2 = (y_0 + v_1) + v_2 = y_0 + \sum_{s=1}^{2} v_s$$

$$\vdots$$

$$y_t = y_{t-1} + v_t = y_0 + \sum_{s=1}^{t} v_s$$

- The term $\sum_{s=1}^{t} v_s$ is often called the **stochastic trend**
  - This term arises because a stochastic component v_t is added for each time *t*, and because it causes the time series to trend in unpredictable directions

- Recognizing that the v_t are independent, taking the expectation and the variance of y_t yields, for a fixed initial value y_0:

$$E(y_t) = y_0 + E(v_1 + v_2 + \ldots + v_t) = y_0$$

$$\text{var}(y_t) = \text{var}(v_1 + v_2 + \ldots + v_t) = t\sigma_v^2$$

  – The random walk has a mean equal to its initial value and a variance that increases over time, eventually becoming infinite

- Another nonstationary model is obtained by adding a constant term:

Eq. 12.3b

$$y_t = \alpha + y_{t-1} + v_t$$

  - This model is known as the **random walk with drift**

- A better understanding is obtained by applying recursive substitution:

$$y_1 = \alpha + y_0 + v_1$$

$$y_2 = \alpha + y_1 + v_2 = \alpha + (\alpha + y_0 + v_1) + v_2 = 2\alpha + y_0 + \sum_{s=1}^{2} v_s$$

$$\vdots$$

$$y_t = \alpha + y_{t-1} + v_t = t\alpha + y_0 + \sum_{s=1}^{t} v_s$$

- The term $t\alpha$ a **deterministic trend** component
  - It is called a deterministic trend because a fixed value α is added for each time t
  - The variable y wanders up and down as well as increases by a fixed amount at each time t

- The mean and variance of y_t are:

$$E(y_t) = t\alpha + y_0 + E(v_1 + v_2 + v_3 + \ldots + v_t) = t\alpha + y_0$$

$$\text{var}(y_t) = \text{var}(v_1 + v_2 + v_3 + \ldots + v_t) = t\sigma_v^2$$

- We can extend the random walk model even further by adding a time trend:

Eq. 12.3c

$$y_t = \alpha + \delta t + y_{t-1} + v_t$$

12.2 Spurious Regressions

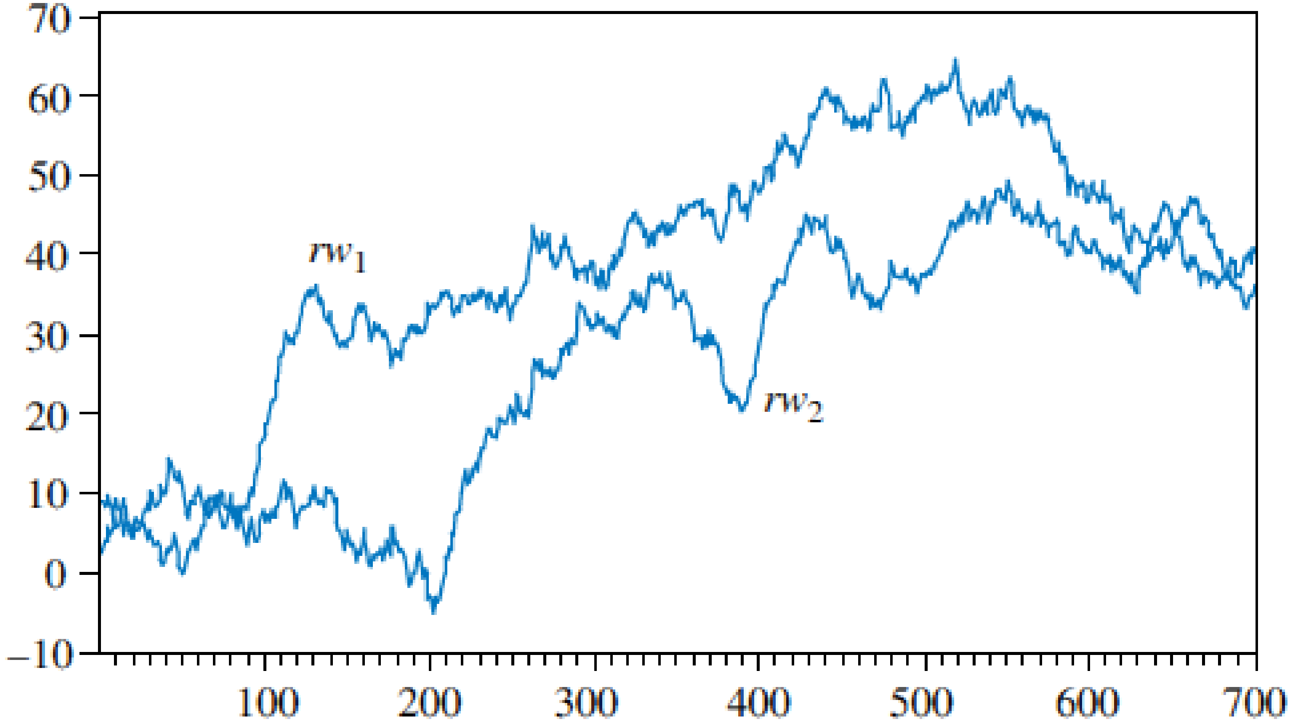
- The main reason why it is important to know whether a time series is stationary or nonstationary before one embarks on a regression analysis is that there is a danger of obtaining apparently significant regression results from unrelated data when nonstationary series are used in regression analysis
  - Such regressions are said to be **spurious**

- Consider two independent random walks:

$$
\begin{aligned}
rw_1:&\quad y_t = y_{t-1} + v_{1t} \\
rw_2:&\quad x_t = x_{t-1} + v_{2t}
\end{aligned}
$$

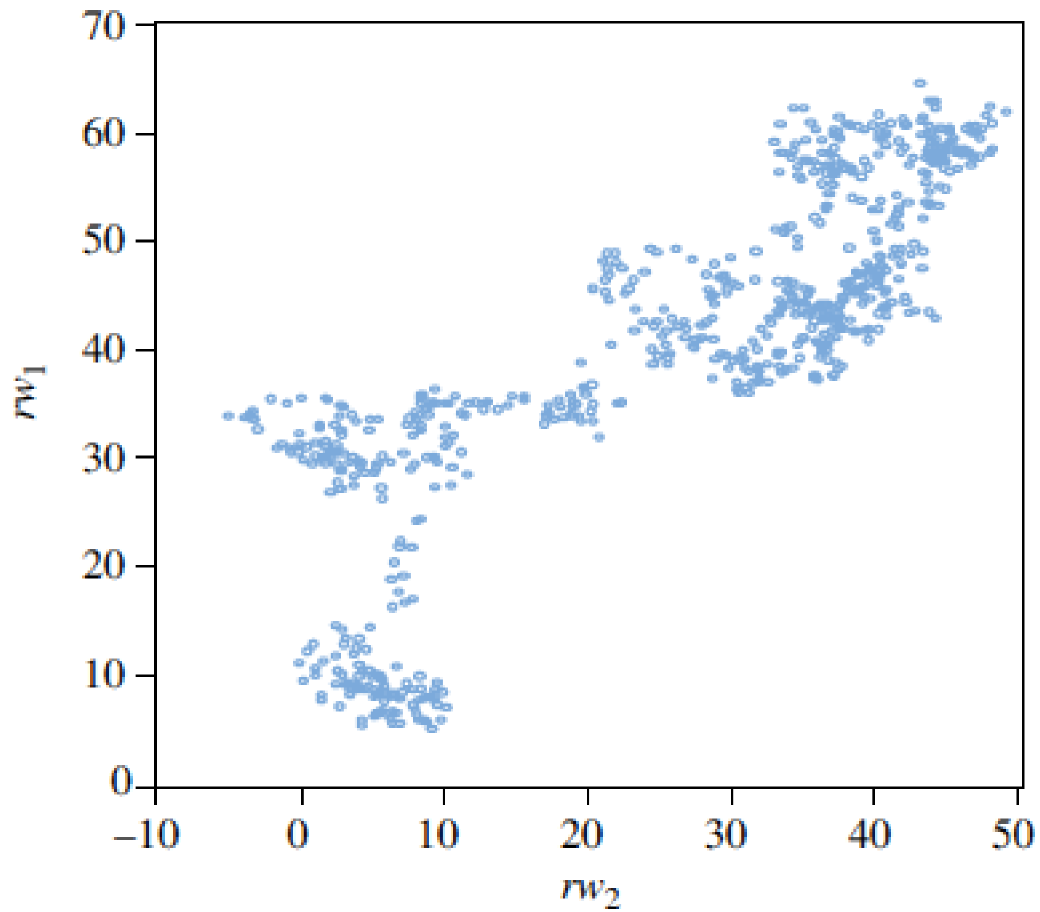
  - These series were generated independently and, in truth, have no relation to one another
  - Yet when plotted we see a positive relationship between them

FIGURE 12.3 Time series and scatter plot of two random walk variables



(a) Time series

FIGURE 12.3 (Continued) Time series and scatter plot of two random walk variables



(*b*) Scatter plot

- A simple regression of series one (rw_1) on series two (rw_2) yields:

$$\widehat{rw_{1t}} = 17.818 + 0.842\, rw_{2_t}, \qquad R^2 = 0.70$$
$$(t) \qquad\qquad (40.837)$$

  – These results are completely meaningless, or spurious
    - The apparent significance of the relationship is false

- When nonstationary time series are used in a regression model, the results may spuriously indicate a significant relationship when there is none
  - In these cases the least squares estimator and least squares predictor do not have their usual properties, and *t*-statistics are not reliable
  - Since many macroeconomic time series are nonstationary, it is particularly important to take care when estimating regressions with macroeconomic variables

12.3

Unit Root Tests for Stationarity

- There are many tests for determining whether a series is stationary or nonstationary
  - The most popular is the Dickey–Fuller test

- The AR(1) process $y_t = \rho y_{t-1} + v_t$ is stationary when $|\rho| < 1$
  - But, when $\rho = 1$, it becomes the nonstationary random walk process
  - We want to test whether ρ is equal to one or significantly less than one
    - Tests for this purpose are known as **unit root tests for stationarity**

- Consider again the AR(1) model:

Eq. 12.4

$$y_t = \rho y_{t-1} + v_t$$

  – We can test for nonstationarity by testing the null hypothesis that $\rho = 1$ against the alternative that $|\rho| < 1$
    - Or simply $\rho < 1$

- A more convenient form is:

Eq. 12.5a

$$
\begin{aligned}
y_t - y_{t-1} &= \rho y_{t-1} - y_{t-1} + v_t \\
\Delta y_t &= (\rho - 1) y_{t-1} + v_t \\
&= \gamma y_{t-1} + v_t
\end{aligned}
$$

– The hypotheses are:

$$
\begin{aligned}
H_0 : \rho = 1 \quad &\Leftrightarrow \quad H_0 : \gamma = 0 \\
H_1 : \rho < 1 \quad &\Leftrightarrow \quad H_1 : \gamma < 0
\end{aligned}
$$

- The second Dickey–Fuller test includes a constant term in the test equation:

Eq. 12.5b

$$\Delta y_t = \alpha + \gamma y_{t-1} + v_t$$

  - The null and alternative hypotheses are the same as before

- The third Dickey–Fuller test includes a constant and a trend in the test equation:

Eq. 12.5c

$$\Delta y_t = \alpha + \gamma y_{t-1} + \lambda t + v_t$$

  - The null and alternative hypotheses are $H_0: \gamma = 0$ and $H_1: \gamma < 0$ *(same as before)*

- To test the hypothesis in all three cases, we simply estimate the test equation by least squares and examine the *t*-statistic for the hypothesis that $\gamma = 0$
  - Unfortunately this *t*-statistic no longer has the *t*-distribution
  - Instead, we use the statistic often called a τ (*tau*) statistic

Table 12.2 Critical Values for the Dickey–Fuller Test

Model	1%	5%	10%
$\Delta y_t = \gamma y_{t-1} + v_t$	−2.56	−1.94	−1.62
$\Delta y_t = \alpha + \gamma y_{t-1} + v_t$	−3.43	−2.86	−2.57
$\Delta y_t = \alpha + \lambda t + \gamma y_{t-1} + v_t$	−3.96	−3.41	−3.13
Standard critical values	−2.33	−1.65	−1.28

Note: These critical values are taken from R. Davidson and J. G. MacKinnon (1993), *Estimation and Inference in Econometrics*, New York: Oxford University Press, p. 708.

- To carry out a one-tail test of significance, if τ_c is the critical value obtained from Table 12.2, we reject the null hypothesis of nonstationarity if $\tau \leq \tau_c$
  - If $\tau > \tau_c$ then we do not reject the null hypothesis that the series is nonstationary

- An important extension of the Dickey–Fuller test allows for the possibility that the error term is autocorrelated
  - Consider the model:

Eq. 12.6

$$\Delta y_t = \alpha + \gamma y_{t-1} + \sum_{s=1}^{m} a_s \Delta y_{t-s} + v_t$$

where

$$\Delta y_{t-1} = (y_{t-1} - y_{t-2}), \quad \Delta y_{t-2} = (y_{t-2} - y_{t-3}), \quad \ldots$$

- The unit root tests based on Eq. 12.6 and its variants (intercept excluded or trend included) are referred to as **augmented Dickey–Fuller tests**
  - When $\gamma = 0$, in addition to saying that the series is nonstationary, we also say the series has a unit root
  - In practice, we always use the augmented Dickey–Fuller test

Table 12.3 AR processes and the Dickey-Fuller Tests

AR processes: $\|\rho\| < 1$	Setting $\rho = 1$	Dickey Fuller Tests
$y_t = \rho y_{t-1} + u_t$	$y_t = y_{t-1} + u_t$	Test with no constant and no trend
$y_t = \alpha + \rho y_{t-1} + v_t$ $\alpha = \mu(1-\rho)$	$y_t = y_{t-1} + v_t$ $\alpha = 0$	Test with constant and no trend
$y_t = \alpha + \rho y_{t-1} + \lambda t + v_t$ $\alpha = (\mu(1-\rho) + \rho\delta)$ $\lambda = \delta(1-\rho)$	$y_t = \delta + y_{t-1} + v_t$ $\alpha = \delta$ $\lambda = 0$	Test with constant and trend

- The Dickey-Fuller testing procedure:
  - First plot the time series of the variable and select a suitable Dickey-Fuller test based on a visual inspection of the plot
    - If the series appears to be wandering or fluctuating around a sample average of zero, use test equation (12.5a)
    - If the series appears to be wandering or fluctuating around a sample average which is nonzero, use test equation (12.5b)
    - If the series appears to be wandering or fluctuating around a linear trend, use test equation (12.5c)

- As an example, consider the two interest rate series:
  - The federal funds rate (F_t)
  - The three-year bond rate (B_t)
- Following procedures described in Sections 12.3 and 12.4, we find that the inclusion of one lagged difference term is sufficient to eliminate autocorrelation in the residuals in both cases

- The results from estimating the resulting equations are:

$$\widehat{\Delta F_t} = 0.173 - 0.045F_{t-1} + 0.561\Delta F_{t-1}$$
$$(tau) \qquad (-2.505)$$

$$\widehat{\Delta B_t} = 0.237 - 0.056B_{t-1} + 0.237\Delta B_{t-1}$$
$$(tau) \qquad (-2.703)$$

  – The 5% critical value for *tau* (τ_c) is -2.86
  – Since -2.505 > -2.86, we do not reject the null hypothesis of non-stationarity.

- Recall that if y_t follows a random walk, then $\gamma = 0$ and the first difference of y_t becomes:

$$\Delta y_t = y_t - y_{t-1} = v_t$$

  – Series like y_t, which can be made stationary by taking the first difference, are said to be **integrated of order one**, and denoted as **I**(**1**)
    - Stationary series are said to be integrated of order zero, **I**(**0**)

  – In general, the order of integration of a series is the minimum number of times it must be differenced to make it stationary

- The results of the Dickey–Fuller test for a random walk applied to the first differences are:

$$\Delta\widehat{(\Delta F)}_t = -0.447(\Delta F)_{t-1}$$
$$(tau) \qquad (-5.487)$$

$$\Delta\widehat{(\Delta B)}_t = -0.701(\Delta B)_{t-1}$$
$$(tau) \qquad (-7.662)$$

- Based on the large negative value of the *tau* statistic (-5.487 < -1.94), we reject the null hypothesis that ΔF_t is nonstationary and accept the alternative that it is stationary
  - We similarly conclude that ΔB_t is stationary (-7.662 < -1.94)