Lecture 2
Estimation with Serially Correlated Errors

- Suppose we proceed with least squares estimation without recognizing the existence of serially correlated errors. What are the consequences?
  1. The least squares estimator is still a linear unbiased estimator, but it is no longer best
  2. The formulas for the standard errors usually computed for the least squares estimator are no longer correct
     - Confidence intervals and hypothesis tests that use these standard errors may be misleading

- It is possible to compute correct standard errors for the least squares estimator:
  - **HAC (heteroskedasticity and autocorrelation consistent) standard errors**, or **Newey-West standard errors**
    - These are analogous to the heteroskedasticity consistent standard errors

- Let's reconsider the Phillips Curve model:

Eq. 9.29

$$\widehat{INF} = 0.7776 - 0.5279DU$$
$$(0.0658) \quad (0.2294) \qquad \text{(incorrect se)}$$
$$(0.1030) \quad (0.3127) \qquad \text{(HAC se)}$$

- The t and p-values for testing H_0: $\beta_2 = 0$ are:

$$t = \frac{-0.5279}{0.2294} = -2.301 \qquad p = 0.0238 \qquad \text{(from LS standard errors)}$$

$$t = \frac{-0.5279}{0.3127} = -1.688 \qquad p = 0.0950 \qquad \text{(from HAC standard errors)}$$

- Return to the Lagrange multiplier test for serially correlated errors where we used the equation:

Eq. 9.30

$$e_t = \rho e_{t-1} + v_t$$

  - Assume the v_t are uncorrelated random errors with zero mean and constant variances:

Eq. 9.31

$$E(v_t) = 0 \quad \text{var}(v_t) = \sigma_v^2 \quad \text{cov}(v_t, v_s) = 0 \ \text{ for } t \neq s$$

- Eq. 9.30 describes a **first-order autoregressive model** or a **first-order autoregressive process** for e_t
  - The term **AR(1) model** is used as an abbreviation for first-order autoregressive model
  - It is called an **autoregressive** model because it can be viewed as a regression model
  - It is called **first-order** because the right-hand-side variable is e_t lagged one period

9.5.2a
Properties of an AR(1) Error

- We assume that:

Eq. 9.32

$$-1 < \rho < 1$$

- The mean and variance of e_t are:

Eq. 9.33

$$E(e_t) = 0 \quad \text{var}(e_t) = \sigma_e^2 = \frac{\sigma_v^2}{1-\rho^2}$$

- The covariance term is:

Eq. 9.34

$$\text{cov}(e_t, e_{t-k}) = \frac{\rho^k \sigma_v^2}{1-\rho^2}, \qquad k > 0$$

- The correlation implied by the covariance is:

Eq. 9.35

$$\begin{aligned}\rho_k &= \operatorname{corr}(e_t, e_{t-k}) \\ &= \frac{\operatorname{cov}(e_t, e_{t-k})}{\sqrt{\operatorname{var}(e_t)\operatorname{var}(e_{t-k})}} \\ &= \frac{\operatorname{cov}(e_t, e_{t-k})}{\operatorname{var}(e_t)} \\ &= \frac{\dfrac{\rho^k \sigma_v^2}{(1-\rho^2)}}{\dfrac{\sigma_v^2}{(1-\rho^2)}} \\ &= \rho^k\end{aligned}$$

Eq. 9.36

- Setting $k = 1$:

$$\rho_1 = \mathrm{corr}(e_t, e_{t-1}) = \rho$$

  – ρ represents the correlation between two errors that are one period apart
    - It is the **first-order autocorrelation** for *e*, sometimes simply called the autocorrelation coefficient
    - It is the population autocorrelation at lag one for a time series that can be described by an AR(1) model
    - r_1 is an estimate for ρ when we assume a series is AR(1)

- For an AR(1) model, we have:

$$\hat{\rho}_1 = \hat{\rho} = r_1 = 0.549$$

- For longer lags, we have:

$$\hat{\rho}_2 = \hat{\rho}^2 = (0.549)^2 = 0.301$$
$$\hat{\rho}_3 = \hat{\rho}^3 = (0.549)^3 = 0.165$$
$$\hat{\rho}_4 = \hat{\rho}^4 = (0.549)^4 = 0.091$$
$$\hat{\rho}_5 = \hat{\rho}^5 = (0.549)^5 = 0.050$$

- Our model with an AR(1) error is:

Eq. 9.38

$$y_t = \beta_1 + \beta_2 x_t + e_t \quad \text{with} \quad e_t = \rho e_{t-1} + v_t$$

with $-1 < \rho < 1$

– For the v_t, we have:

Eq. 9.39

$$E(v_t) = 0 \quad \text{var}(v_t) = \sigma_v^2 \quad \text{cov}(v_t, v_{t-1}) = 0 \quad \text{for } t \neq s$$

- With the appropriate substitutions, we get:

Eq. 9.40

$$y_t = \beta_1 + \beta_2 x_t + \rho e_{t-1} + v_t$$

– For the previous period, the error is:

Eq. 9.41

$$e_{t-1} = y_{t-1} - \beta_1 - \beta_2 x_{t-1}$$

– Multiplying by ρ:

Eq. 9.42

$$\rho e_{t-1} = e_t y_{t-1} - \rho\beta_1 - \rho\beta_2 x_{t-1}$$

- Substituting, we get:

Eq. 9.43

$$y_t = \beta_1(1-\rho) + \beta_2 x_t + \rho y_{t-1} - \rho\beta_2 x_{t-1} + v_t$$

- We have the model:

Eq. 9.46

$$y_t = \beta_1(1-\rho) + \beta_2 x_t + \rho y_{t-1} - \rho\beta_2 x_{t-1} + v_t$$

  – Suppose now that we consider the model:

Eq. 9.47

$$y_t = \delta + \theta_1 y_{t-1} + \delta_0 x_t + \delta_1 x_{t-1} + v_t$$

    • This new notation will be convenient when we discuss a general class of **autoregressive distributed lag (ARDL) models**
      – Eq. 9.47 is a member of this class

Eq. 9.48

- Note that Eq. 9.47 is the same as Eq. 9.47 since:

$$\delta = \beta_1(1-\rho) \quad \delta_0 = \beta_2 \quad \delta_1 = -\rho\beta_2 \quad \theta_1 = \rho$$

  – Eq. 9.46 is a restricted version of Eq. 9.47 with the restriction $\delta_1 = -\theta_1\delta_0$ imposed

ASSUMPTION FOR MODELS WITH A LAGGED DEPENDENT VARIABLE

TSMR2A In the multiple regression model $y_t = \beta_1 + \beta_2 x_{t2} + \cdots + \beta_K x_K + v_t$
Where some of the x_{tk} may be lagged values of y, v_t is uncorrelated with all x_{tk} and their past values.

- Applying the least squares estimator to Eq. 9.47 using the data for the Phillips curve example yields:

Eq. 9.49

$$\widehat{INF}_t = 0.3336 + 0.5593 INF_{t-1} - 0.6882 DU_t + 0.3200 DU_{t-1}$$
$$(se) \quad (0.0899) \quad (0.0908) \qquad (0.2575) \qquad (0.2499)$$

- We have described two ways of overcoming the effect of serially correlated errors:
  1. Estimate the model using least squares with *HAC* standard errors
  2. Use least squares to estimate the model with a lagged *x* and a lagged *y*, but without the restriction implied by an AR(1) error specification