Longstaff and Schwartz model (LSM) for American option pricing

Seminar in Epiricial Finance and Financial Econometrics

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Outline I

Inroduction

Interesting methods for option pricing discussed in the our paper

European Option	American Option
Black, Scholes and Merton (1973)	Schwartz (1977)
Cox, Ross and Rubinstein (1979)	Von and Roy (1999)
Schwartz (1977)	Longstaff and Schwartz (2001)
Boyle (1979)	Lee (2008)

Introduction

The Main focus of our work is to identify value as

$$V_0 = \max_{\tau \in \mathcal{T}} \mathbb{E}[\widehat{g}(S_\tau)] \tag{1}$$

- Where $\hat{g}$ is the measurable (discounted payoff) function.
- $(S_t)_{0 \leq t \leq T}$ is underlying stochastic process.
- $\tau \in \mathcal{T}([0, T])$ is optimal exercising point for American Option.

Introduction

We work along technique named as Least square Monte Carlo given as

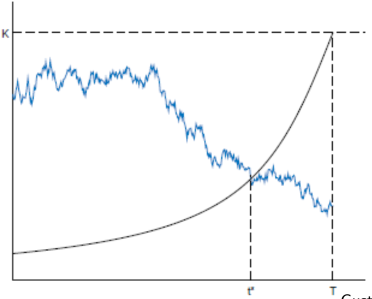
$$\mathbb{E}\left[\hat{g}\left(S^l_{t_{i+1}}\right)|\mathcal{F}_t\right] = \alpha_o + \alpha_1 f_1\left(S^l_{t_i}\right) + \cdots + \alpha_M f_M\left(S^l_{t_i}\right) \qquad (2)$$

- Conditional expectation, $\mathbb{E}\left[\hat{g}\left(S^l_{t_{i+1}}\right)|\mathcal{F}_t\right]$, represent the continuation value.
- where $\alpha_0, \alpha_1, ..., \alpha_M$ are regression coefficients from Least square technique.
- $f_1(X), f_2(X), ..., f_M(X)$ are special functions, which can span on any point in $\mathcal{L}^2$ space.

Monte Carlo approach

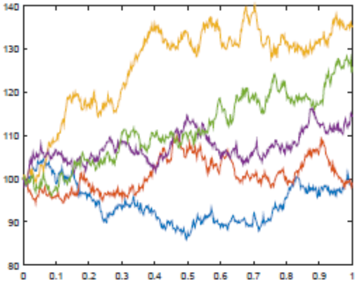
Monte Carlo simulation algorithm is basically sampling numerous paths for stochastic processes and taking average of discounted payoff from option from all the paths.

Figure 1: Price of American Option



Gustafsson, W. 2015

Figure 2: Price Simulation



Monte Carlo setting I

Discretization Method

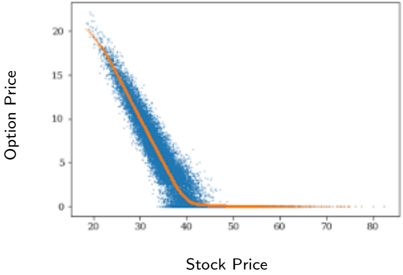
- To mitigate first issue with continuous time frame, we rely on estimations on only a finite set, $\mathcal{T} = \{0 = t_1, t_2, \ldots, t_K = T\}$ of time points with $t_1 < t_2 < \cdots < t_K$ and each point is $0 \leq t \leq T$.
- Option is priced as Bermundan style, can be only exercised on some discrete time points, but whenever time steps $K \to \infty$ then valuation process mimics pricing of American option.

Monte Carlo setting II

Least square Monte Carlo

- Using LSM to estimate continuation value as given in equation 2 has fair advantage in computational efficiency and simplicity.
- Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (1999).

Figure 3: Least square Regression at t_i



Sodhi, A. 2015

Regression based Monte Carlo

Longstaff and Schwartz (2001) vs Tsitsiklis and Van Roy (1996)

Framework

Underlying price process and Mathematical space is framed along common model assumptions.

We assume complete probability space $(\Omega, \mathcal{F}_t, \mathcal{P})$.

- Ω consists of all possible values of random process, namely ω.
- $\mathcal{F}_t$ is the sigma field containing all the relevant price information until the period t.
- $\mathcal{P}$ is the probability measure containing the probabilistic characteristics of sample paths. We are going to use Equivalent Martingale measure to drive our prices.

Framework

Pricing American-style derivative at time $t_k \in \mathcal{T}$ is based on idea of Snell envelope

Snell Envelope

At each simulated path, one required to make exercises decision based on current exercise premium against the present value of future cash generated, $C\left(\omega, t_j; t_k, T\right)$.

$\mathcal{L}^2 - space$

Our method, LSM, is only suited for derivatives who's continuation value is in $\mathcal{L}^2 - space$, i.e. is square-integrable or finite variance function.

Main Task

Generate the projection of continuation value which is

$$F\left(\omega, t_j; t_k, T\right) = E_Q\left[\sum_{t=k+1}^{t_j} \exp\left(-\int_{t_k}^{t_j} r(w,s)ds\right) C\left(\omega, t_j; t_k, T\right) | \mathcal{F}_{t_k}\right] \quad (3)$$

and by using backward dynamic programming, as introduced by Carriere (1996), we use the projection to evaluate the value function at each time step until t_0.

Idea behind Least Square Monte Carlo I

Least square technique

Looking back at equation 2

$$E\left[\hat{g}\left(S^l_{t_{i+1}}\right)|\mathcal{F}_t\right] = \alpha_o + \alpha_1 f_1\left(S^l_{t_i}\right) + \cdots + \alpha_M f_M\left(S^l_{t_i}\right)$$

We define Longstaff and Schwartz, where above functions are replaced by basis of $\mathcal{L}^2-$ space

$$B_1, \ldots, B_M : \mathbb{R}^d \to \mathbb{R}$$

Choices for B

Hermite, Laugerre, Legendre, Chebyshev, Gegenbauer, Jacobi polynomials,

Fourier or trigonometric series as well as Higher order polynomials.

Idea behind Least Square Monte Carlo II

We use such basis and find coefficients $\alpha_0, \alpha_1, \ldots \alpha_M \in \mathbb{R}$

$$\hat{F}\left(\omega, t_j; t_k, T\right) = \alpha_0 + \sum_{m=1}^{M} \alpha_m B_m \tag{4}$$

so that for all in-the-money paths, we have

$$\min_{\alpha_1, \alpha_2, \ldots, \alpha_M \in \mathbb{R}} \sum_{l=1}^{l \leq N} \left(\left(F\left(\omega_l, t_j; t_k, T\right) - \hat{F}\left(\omega_l, t_j; t_k, T\right) \right) \right)^2 \tag{5}$$

We can use $\hat{F}\left(\omega, t_j; t_k, T\right)$ in place of $F\left(\omega, t_j; t_k, T\right)$ and compare it with immediate exercise value $\hat{g}\left(S_{t_k}\right)$, where exercising point remains stopping time for each backward recursion.

Algorithm

At maturity T, we have $\mathrm{V}\left(\omega_i; t_k\right) = g\left(S_T^i\right)$ for $i = 1, ..., N$.

Using Backward induction, from time point $K-1$ to 1, we make the comparison for each simulated, in-the-money, ith-path and define our discounted returns from options as

$$\mathrm{V}\left(\omega_i; t_k\right) = \begin{cases} g\left(S_{t_k}^i\right) & \text{if } g\left(S_{t_k}^i\right) \geq \hat{F}\left(\omega_i; t_k\right) \\ \hat{\mathrm{V}}\left(\omega_i; t_{k+1}\right) & \text{otherwise} \end{cases} \tag{6}$$

Where $\hat{\mathrm{V}}\left(\omega_i; t_k\right)$ is path-wise discounted option value at time step t_k.

Finally at t_0 we have our value estimate as

$$V_N^M = \frac{1}{N} \sum_{i=1}^{N} \hat{\mathrm{V}}\left(\omega_i; t_1\right) \tag{7}$$

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Application of Least square Monte Carlo I

LSM algorithm has specific purpose of pricing American style derivatives, which can be exercised before maturity. In standard case, we assume the price process S_t follows geometric Brownian motion (GBM)

$$dS = rSdt + \sigma Sdz \tag{8}$$

with constant drift (risk neutral) rate r, volatility σ and z is standard Brownian motion.

Application of Least square Monte Carlo II

Like original model, we take first 3 weighted Laugerre basis functions in our regression.

$$L_n(X) = \exp\left(-\frac{X}{2}\right) \frac{e^X}{n!} \frac{d^n}{dX^n} \left(X^n e^{-X}\right) \tag{9}$$

We use antithetic Monte Carlo with $N = 100000$ and Time steps $K = 50$

- Results were almost same as given in original paper

  We test with different Initial Prices, Volatilities and Maturities

- Standard error was also low, however there are many ways with which we can further reduce the error

Application of Least square Monte Carlo III

- For comparison of we take values from Finite Difference Method.

As we will discuss later, LSM do provide results which are lower than many other methods. Indeed, it produces error by valuing option lower than the fair value.

There are choice of M and N can reduce the difference between estimated and actual value.

1st Convergence Property

Clement et. al. (2002) discuss the convergence results for LSM in details.

Now we elaborate two important points given by the writers.

1. For a Markovian underlying process $X \in (0, \infty)$, LSM algorithm provides us with low price the option value $V(X)$ however its approaches closer to actual value for large M, such as

$$V(X) \geq \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \hat{\mathrm{V}}\left(\omega_i; t_k\right) \tag{10}$$

1st Convergence Property

Given that our Benchmark Price: 4.492 from Finite Difference Method show that LSM is indeed an low estimate.

Importantly, Price is approaching with increasing basis functions.

Table 1: Price of American Put Option

Basis Functions and Order of Polynomials	1	2	3	4	5
Weighted Laguerre	3.958	4.078	4.278	4.465	4.467
Probabilistic Hermite	4.412	4.458	4.468	4.474	4.469
High Order Polynomials	3.952	4.468	4.461	4.461	4.472

2nd Convergence Property

1. For American Put option, LSM estimated $\widehat{F_M}(\omega; t_k)$ and also continuous conditional expectation $F(\omega; t_k)$, if both satisfy

$$\int_0^\infty e^{-X} F^2(\omega; t_k)\, dX < \infty$$

$$\int_0^\infty e^{-X} \hat{F}_X^2(\omega; t_k)\, dX < \infty$$

then for any arbitrary small value ϵ there $\exists M$ such that

$$\lim_{N \to \infty} \Pr\left[\left|V(X) - \frac{1}{N}\sum_{i=1}^{N} \hat{\mathrm{V}}(\omega_i; t_k)\right| > \epsilon\right] = 0 \tag{11}$$

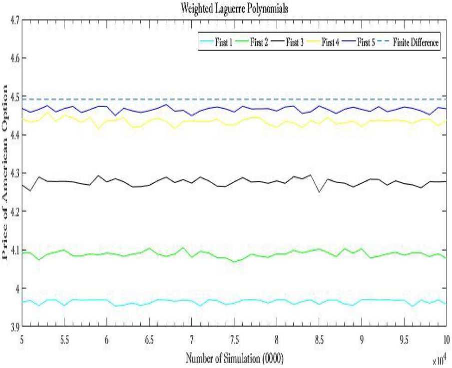
Clement, Protter and Lamberton (2002)

As proved by Clement et. al. (2002). large order of basis functions indeed help LSM price to converge in probability to actual option value such

$$\lim_{N\to\infty} V_N^M(X) \xrightarrow{p} V(X)$$

but one need to be careful since for optimal and efficient results depends on size of M and N.

Figure 4: Convergence with Laguerre Polynomials



Accuracy of our estimation

Least square is Prone different errors.

- **Source of Error**
  - Dynamic Progamming and discretization Error
  - Model Selection Error
  - Functional Form Error

Increasing time steps K

- Dynamic Programming and Discretization

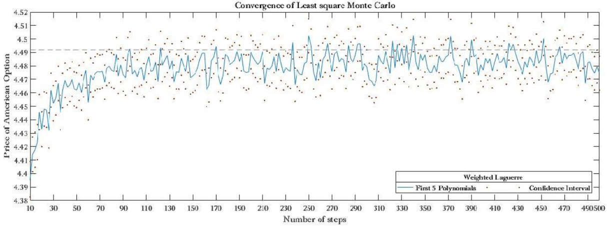


Figure 5: Time steps

Model Selection I

- Choice of Basis

Choices for B

Hermite, Legendre, Chebyshev, Gegenbauer, Jacobi polynomials, Fourier or trigonometric series and Higher order polynomials.

- Choice of size M

$$\hat{F}(\omega, t) = \alpha_0 + \alpha_1 S + \alpha_2 S^2 \tag{12}$$

Model Selection II

Table 2: Bias v/s Variance Tadeoff

Simulations and Order of Polynomials		1	2	3	4	5
Weighted Laguerre	50000	3.9952 0.0015	4.0580 0.0059	4.2634 0.0088	4.4448 0.0112	4.4750 0.0117
	75000	3.9952 0.0012	4.0577 0.0049	4.2233 0.0070	4.4201 0.0093	4.4897 0.0101
	100000	3.9956 0.0010	4.0561 0.0040	4.2456 0.0063	4.4351 0.0080	4.4765 0.0087
Probabilistic Hermite	50000	4.4212 0.0129	4.4633 0.0124	4.4863 0.0124	4.4853 0.0127	4.4996 0.0127
	75000	4.4188 0.0105	4.4670 0.0101	4.4802 0.0101	4.4884 0.0104	4.4848 0.0101
	100000	4.4118 0.0091	4.4832 0.0092	4.4938 0.0090	4.4840 0.0088	4.4935 0.0086
High Order Polynomials	50000	4.4038 0.0129	4.4642 0.0125	4.4878 0.0128	4.4822 0.0123	4.5015 0.0123
	75000	4.4196 0.0105	4.4821 0.0102	4.4904 0.0104	4.4797 0.0105	4.4922 0.0103
	100000	4.4020 0.0092	4.4814 0.0088	4.4820 0.0090	4.4785 0.0087	4.4770 0.0091

Model Selection III

First two Hermite and higher order polynomial the regression provides slightly better results (by around 0.5 cent) then the function of 5 weighted Laguerre basis.

Although, last two methods saves from bias but comes at a price of higher variance. standard error is increasing at faster rate except for weighted Laguerre, which is more constrained among all three methods.

Model Selection IV

Model selection techniques

- Mallow's C_p
- AIC and BIC criterion
- Adjusted R^2
- Control variate

Parametric v/s Non-Parametric Models I

- **Local Averaging**

  Nadaraya-Watson and K-NN

- **Local Modelling**

  Local Polynomials Kernel estimate

- **Global Modelling**

  Neural Network

Parametric v/s Non-Parametric Models II

Lee's (2008) Kernel Method

At maturity T, we have $\mathrm{V}(\omega_i; t_k) = g\left(S_T^i\right)$ for $i = 1, ..., N$.

Using Backward induction, from time point $K-1$ to 1, Calculate discounted payoff value $\hat{g}\left(S_{t_k}^i\right)$ and estimate continuation value $\hat{F}(\omega_i; t_k)$ for each path:

$$\hat{F}(\omega_i; t_k) = \sum_{j=1}^{N} \frac{\mathcal{K}_{\text{Gaus}}\left(S_{t_k}^i, S_{t_k}^j\right)}{\sum_{l=1}^{N} \mathcal{K}_{\text{Gaus}}\left(S_{t_k}^i, S_{t_k}^l\right)} \hat{\mathrm{V}}(\omega_j; t_k) \tag{13}$$

we make the comparison for each simulated path and update path value as

$$\mathrm{V}(\omega_i; t_k) = \begin{cases} \hat{F}(\omega_i; t_k) & \text{otherwise} \\ \hat{g}\left(S_{t_k}^i\right) & \text{if } \hat{g}\left(S_{t_k}^i\right) \geq \hat{F}(\omega_i; t_k) \text{ and } \hat{g}\left(S_{t_k}^i\right) \geq 0 \end{cases} \tag{14}$$

Parametric v/s Non-Parametric Models III

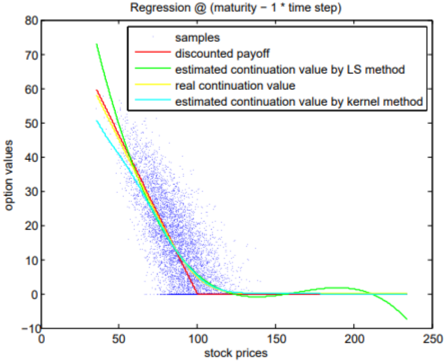


Figure 6: Original LSM v/s Lee (2015)

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[1] Songyin, 2015

References I

Abramowitz, M., Stegun, I. (1970). Handbook of Mathematical Functions. New York: Dover Publication.

Black, F., Scholes, M. (1973). The pricing of Options and Corporate Liabilities. Journal of Political Economy, 637-654.

Boyle, P. P. (1977). Options: A monte carlo approach. Journal of financial economics, 323-338.

Carriere, J. (1996). Valuation of Early-Exercises Price of Options Using Simulations and Nonparametric Regression. Insurance: Mathematics and Economics, 5, 363-384.

References II

Clement, E., Protter, P., Lamberton, D. (2002). An Analysisof a Least Squares Regression Method for American Option Pricing. Finance and Stochastic 6, 4, 449-471.

Glasserman, P., Yu, B. (2004). Number of paths versus number of basis functions in American option pricing. The Annals of Applied Probability, 14(4), 2090-2119.

Longstaff, F. A., Schwartz, E. S. (2001). Valuing American options by simulation: a simple least-squares approach. The review of financial studies, 14(1), 113-147.

References III

Lee, J. (2008). Recent advances in American option pricing using simulations.

Moreno, M., Navas, J. F. (2003). On the robustness of least squares Monte Carlo (LSM) for pricing American derivatives. Review of Derivatives Research, 6(2), 107-128.

Stentoft, L. (2004). Convergence of the least squares Monte Carlo approach to American option valuation. Management Science, 50(9), 1193-1203.

Tang, S. (2015). American-style Option Pricing and Improvement of Regression based Monte Carlo. KAISERSLAUTERN: TECHNISCHE UNIVERSITÄT KAISERSLAUTERN.

References IV

Tsitsiklis, J., Van Roy, B. (1999). Optimal stopping of Markov Processes: Hilbert Space Theory, Aproximation algorithms, and Application to pricing high dimensional financial derivative. IEEE Transaction on Automatic control, 44, 1840-1851.