ADVANCED METHODS OF NON-LIFE INSURANCE

Copulas

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Statistics for Finance and Insurance

Copulas

- Introduction
- Sklar's theorem
- Fundamental Copulas and Fréchet-Hoeffding bounds
- Gaussian and Student copula's
- Archimedean copula's
- Rank correlation
- Tail dependence
- Calibration and Identification.

Copulas are of interest to statisticians for two main reasons: First, as a way of studying scale-free measures of dependence; and secondly, as a starting point for constructing families of bivariate (multivariate) distributions . . .

Fisher, N.I.

Selected References

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Introduction

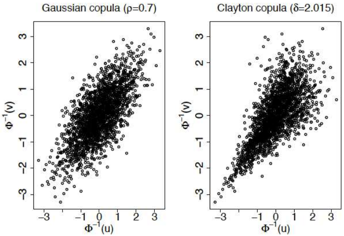
- Understanding relationships among multivariate outcomes is basic problem in statistical science.
- For long time, statistical modelling in actuarial science and finance basically relied on simplified assumptions.
- Multivariate normal distributions are appealing because association between any two random outcomes can be fully described knowing only
  - marginal distributions
  - (Pearson) correlation coefficient.
- There is need for examining alternatives to the normal distribution setup, certainly in actuarial science applications such as for lifetime random variables and long-tailed claims variables.
- Many multivariate distributions have been developed as immediate extensions of univariate distributions (bivariate Pareto, bivariate gamma, etc.).
  $\Rightarrow$ Drawbacks
  - different family needed for each marginal distribution
  - extensions to more than just bivariate case not clear
  - measures of association often appear in marginal distributions.
- Construction of multivariate distributions that does not suffer from these drawbacks is based on **copula** function.

Introduction

- Copula: from copulare $\equiv$ connect, join.
  $\Rightarrow$ Popular method for modeling the dependence of multivariate distributions.
- Can be combined with any set of marginal distributions.
- Decoupling the study of the marginal distribution of rvs of their dependence
  $\Rightarrow$ transform the rvs into uniformly distributed rvs
- Determination of multivariate distributions performed in two independent steps:
  1. determination of the marginal distributions using standard techniques for distributions of a single variable
  2. determination of the copula which specifies completely the dependence structure between the random variables
- $\exists$ few, if any, parametric joint distributions based on marginal cdfs from different families $\Rightarrow$ copula approach allows to construct such joint cdfs
- We know often much more about the marginal cdf of rvs than about their joint behavior

Introduction

- Copula contains all information about dependence between random variables.
- Any multivariate distributions can serve as a copula.
- Most traditional measures of dependence are based on correlation coefficient, which is restricted to multivariate elliptical distributions.
  Simulations from 2 models with standard normal margins and correlation 0.7.



  Copular representations of dependence are free of such limitations.
- Most traditional measures of dependence are measures of pairwise dependence. Copulas measure the dependence between all random variables

Revision I

The joint cumulative distribution function (cdf) of $\mathbf{X} = (X_1, \ldots, X_N)$ is defined by

$$F_{\boldsymbol{X}}(x_1, \ldots, x_N) = P\left(X_1 \leq x_1, \ldots, X_N \leq x_N\right)$$

The survival function of $\mathbf{X} = (X_1, \ldots, X_N)$ is defined by

$$S_{\boldsymbol{X}}(x_1, \ldots, x_N) = \overline{F}_{\boldsymbol{X}}(x_1, \ldots, x_N) = P\left(X_1 > x_1, \ldots, X_N > x_N\right)$$

- $N = 1 : \overline{F}_X(x) = 1 - F_X(x)$
- $N = 2 : \overline{F}_{\boldsymbol{X}}(x_1, x_2) = 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{\boldsymbol{X}}(x_1, x_2)$

A continuous uniform rv on the interval (a, b), $X \sim$**Uniform**(a, b) is a continuous rv with distribution function equal to $1/(b-a)$ on (a, b) and to 0 outside this interval:

$$f_X(x) = \frac{1}{b-a} \quad \text{and so} \quad F_X(x) = \frac{x-a}{b-a}, \quad a \leq x \leq b.$$

Integral transformation

Probability Integral Transformation (PIT)

We consider rv X with cdf F, where F is strictly increasing on some interval I, $F = 0$ to the left of I and $F = 1$ to the right of I. F^{-1} is well defined for $x \in I$.

1. Let $Y = F(X)$, then Y has a uniform distribution on $[0, 1]$.
2. Let U be uniform on $[0, 1]$ and let $Z = F^{-1}(U)$. then the cdf of Z is F.

Proof

1. $P(Y \leq y) = P(F(X) \leq y) = P(X \leq F^{-1}(y)) = F(F^{-1}(y)) = y$.
2. $P(Z \leq z) = P(F^{-1}(U) \leq z) = P(U \leq F(z)) = F(z)$.

Generating pseudorandom numbers using PIT I

PIT is very useful in **generating pseudorandom numbers** with a given cdf F!

Exercise 1

How to generate random variables from an exponential distribution (with cdf $F(t) = 1 - e^{-\lambda t}$) if we have access to a uniform random number generator?

Generating pseudorandom numbers using PIT II

Answer of exercise 1

- F^{-1} can be found by solving $x = 1 - e^{(-\lambda t)}$ for t
  $\Rightarrow t = -\log(1 - x)/\lambda$.
- If $U \sim U([0, 1])$, then $T = -\log(1 - U)/\lambda$ is an exponential rv with parameter λ .

Fréchet spaces I

Fréchet spaces offer the natural framework for studying dependence. These spaces gather together all probability distributions with fixed univariate marginals. Elements in a given Fréchet space only differ in their dependence structures, and not in their marginal behaviours.

Fréchet spaces

Let $F_1, F_2, \ldots, F_n$ be univariate cdfs. The Fréchet space $\mathcal{R}_n(F_1, \ldots, F_n)$ consists of all the n-dimensional (cdfs $F_{\boldsymbol{X}}$ of) random vectors $\boldsymbol{X}$ possessing $F_1, \ldots, F_n$ as marginal cdfs, that is,

$$F_i(x) = P(X_i \leq x) \qquad x \in \mathbb{R} \qquad i = 1, 2, \ldots, n.$$

The elements of $\mathcal{R}_n(F_1, \ldots, F_n)$ are bounded above by a special multivariate cdf, called the Fréchet upper bound.

Fréchet spaces II

Fréchet upper bound

Define the Fréchet upper bound as

$$W_n(\boldsymbol{x}) = \min\left\{F_1(x_1), F_2(x_2), \ldots, F_n(x_n)\right\}$$

Then the inequality

$$F_{\boldsymbol{X}}(\boldsymbol{x}) \leq W_n(\boldsymbol{x})$$

holds for all $\boldsymbol{x} \in \mathbb{R}$ and $\boldsymbol{X} \in \mathcal{R}_n(F_1, \ldots, F_n)$

Fréchet (1951): The Fréchet space is not empty (e.g. $\prod_{i=1}^n F_i(x_i)$ when $X_1, \ldots, X_n$ are independent), but it was not clear which other elements belong to the set. Dall'Aglio and Fréchet (1972) studied under which conditions there is just one cdf belonging to the Fréchet space.

Brief history I

- Sklar and Schweizer made progress in work on statistical metric spaces and communicated results to Fréchet. This led to exchange of letters.
- Sklar (1959) defined functions on the unit d-cube that link d-dimensional distributions to their one-dimensional margins for $d \geq 2$.
(Féron (1956) introduced similar functions for $d = 3$).
- The proof of Sklar's theorem was provided in Schweizer and Sklar (1983) or Nelsen (2006). Elegant proof is due to Rüschendorf (2009).
- All results concerning copulas were obtained in theory of probabilistic metric spaces. Around mid seventies, statistical community started to show interest (Schweizer and Wolff). Copulas allow us to define useful alternative dependence measures.
- Since interest in questions of statistical dependence increased, others came to the subject from different directions, e.g. Genest and MacKay (1986).
- At end of nineties, copulas became increasingly popular, e.g. Joe (1997) and Nelsen (1999).

Brief history II

- Discovery of notion of copulas in several applied fields like finance and insurance, e.g. Embrechts et al. (2002, 2003) and Li (2001).
- Since 2006 several insurance companies, banks and other financial institutions apply copulas as risk management tools.
- Reference on copula methods in finance is Cherubini, Luciano and Vecchiato (2004) and Cherubini, Mulinacci, Gobbi and Romagnoli (2011). Embrechts (2009) contains some references to the discussion concerning the pros and cons of coupla modeling in insurance and finance.
- Recent interesting book is Joe (2014), which covers advances that have taken place during the last 15 years (including vine copula modeling of high-dimensional data - outside scope of this course).

Brief history

- Quote Embrechts (2009): *The notion of copula is both natural as well as easy for looking at multivariate cdfs. But why do we witness such an incredible growth in papers published starting the end of the nineties (recall, the concept goes back to the fifties and even earlier, but not under that name)? Here I can give three reasons: finance, finance, finance. In the eighties and nineties we experienced an explosive development of quantitative risk management methodology within finance and insurance, a lot of which was driven by either new regulatory guidelines or the development of new products.*
- Different fields like hydrology, energy, biostatistics discovered importance of copulas for constructing more flexible multivariate models.
- Quote Schweizer (2007): *The era of i.i.d. is over: and when dependence is taken seriously, copulas naturally come into play. It remains for the statistical community at large to recognize this fact.*

Motivational Example I

We consider a bivariate digital option: pays one euro if two stocks (or indices) are above or below a pair of strike price levels.

- Assume product written on Nikkei 225 and S&P500 indexes which pays 1 euro at exercise date T if both are lower than given levels K_N and K_S.
- Price of digital put option in complete market setting is

$$DP = exp(-r(T-t))Q(K_N, K_S)$$

  where $Q(K_N, K_S)$ is joint risk-neutral probability that both Japanese and US market indexes are below corresponding strike prices.
- From market data we can estimate the marginal risk-neutral distributions of the Nikkei and S&P500 indexes.
- It would be great if we could write price as

$$DP = exp(-r(T-t))C(Q_N, Q_S)$$

  where $C(x, y)$ is bivariate function and Q_N is risk-neutral probability that Nikkei index will be below K_N at time T (similar definition for Q_S).

Motivational Example II

General requirements for $C(x, y)$
(in order to represent a joint probability function)

- Output in unit interval.
- If one of the two events has zero probability, then joint probability that both events occur must also be zero. So if $x = 0$ or $y = 0$, then $C(x, y) = 0$.
- If one event will occur for sure, then joint probability that both events will take place corresponds to probability that second event will be observed.
- If probabilities of both events increase, then joint probability should also increase (certainly not decrease).

Copula functions enable us to express a joint probability distribution as a function of the marginal ones. This opportunity rests on a fundamental finding, known as Sklar's theorem.

Copula: Definition I

Given bivariate cdf $F_{\boldsymbol{X}}$ with univariate marginal cdfs F_1 and F_2, we can associate 3 numbers with each pair $\boldsymbol{x} = (x_1, x_2)$

- $F_1(x_1) \in [0, 1]$
- $F_2(x_2) \in [0, 1]$
- $F_{\boldsymbol{X}}(\boldsymbol{x}) \in [0, 1]$.

Therefore, each pair $\boldsymbol{x}$ leads to point $(F_1(x_1), F_2(x_2))$ in unit square, and this pair in turn corresponds to number $F_{\boldsymbol{X}}(\boldsymbol{x}) \in [0, 1]$.

This correspondence, which assigns value of joint cdf to each ordered pair of values of marginal cdfs, is indeed a function. Such functions are called copulas.

The word *copula* was first employed in a statistical sense by Sklar (1959) in theorem which now bears his name (see below). Knowing the word *copula* as a grammatical term for a word or expression that links a subject and predicate, he felt that this would make an appropriate name for a function that links a multidimensional distribution to its one-dimensional margins, and used it as such.

Copula: Definition II

Definition 1

A *bivariate copula* *C is a function mapping the unit square $[0,1]^2$ to the unit interval $[0,1]$ that is non-decreasing and right-continuous, and satisfies the following conditions*

1. $\lim_{u_i \to 0} C(u_1, u_2) = 0$ *for* $i = 1, 2$
2. $\lim_{u_1 \to 1} C(u_1, u_2) = u_2$ *and* $\lim_{u_2 \to 1} C(u_1, u_2) = u_1$
3. *C is supermodular (2-increasing), that is, the inequality*

$$C(v_1, v_2) - C(u_1, v_2) - C(v_1, u_2) + C(u_1, u_2) \geq 0$$

   is valid for any $u_1 \leq v_1$ and $u_2 \leq v_2$.

Essentially, copula is a joint cdf for a bivariate random vector with standard uniform marginals.

Sklar's theorem I

This theorem elucidates role that copulas play in the relationship between multivariate cdfs and their univariate marginals. Sklar's idea was to separate a multivariate cdf into a part that describes the dependence structure (copula) and parts that describe the marginal behaviour only.

Theorem 1

Let $F_{\boldsymbol{X}} \in \mathcal{R}_2(F_1, F_2)$ have continuous marginal cdfs F_1 and F_2. Then there exists a unique copula C such that for all $\boldsymbol{x} \in \mathbb{R}^2$

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = F_{\boldsymbol{X}}\left(x_1, x_2\right) = C\left(F_1(x_1), F_2(x_2)\right). \tag{1}$$

Conversely, if C is a copula and F_1 and F_2 are cdfs then the function $F_{\boldsymbol{X}}$ defined above is a bivariate cdf with margins F_1 and F_2.

If margins are not all continuous, C is uniquely determined only on $\mathrm{Ran}(F_1) \times \mathrm{Ran}(F_2)$ (where $\mathrm{Ran}(F_j)$ denotes the range of the cdf).

Sklar's theorem II

Proof.

Since the F_i are continuous, PIT guarantees that both $F_1(X_1)$ and $F_2(X_2)$ are $U(0,1)$. Let C be the joint cdf for the couple $(F_1(X_1), F_2(X_2))$

$$\begin{aligned} C(u_1, u_2) &= P\left(F_1(X_1) \leq u_1, F_2(X_2) \leq u_2\right) \\ &= P\left(X_1 \leq F_1^{-1}(u_1), X_2 \leq F_2^{-1}(u_2)\right) \\ &= F_{\boldsymbol{X}}\left(F_1^{-1}(u_1), F_2^{-1}(u_2)\right) \end{aligned}$$

Letting $u_j = F_j(x_j)$ for $j = 1, 2$ we see that

$$F_{\boldsymbol{X}}(x_1, x_2) = C\left(F_1(x_1), F_2(x_2)\right)$$

$\square$

Sklar's theorem III

Hence, C couples the marginal cdfs to the joint distribution. The dependence structure is entirely described by C and dissociated from the marginals. Thus, the manner in which X_1 and X_2 move together is captured by the copula, regardless of the scale in which the variable is measured.

$F_{\boldsymbol{X}}$ is decomposed into C which specifies the **dependence** structure among (X_1, X_2) **and the** marginal cdfs F_1, F_2 which characterize the univariate **marginal distributions**.

The rvs $F_1(X_1)$ and $F_2(X_2)$ are often called the ranks of X_1 and X_2. This makes the copula the joint cdf of the ranks.

Note that we have obtained an expression for copula C in the proof:

$$C(\boldsymbol{u}) = F_{\boldsymbol{X}}\left(F_1^{-1}(u_1), F_2^{-1}(u_2)\right) \qquad \boldsymbol{u} \in [0,1]^2$$

Conditional distributions derived from copulas

Conditional distributions can be derived from $F_X(x_1, x_2) = C\left(F_1(x_1), F_2(x_2)\right)$. We state some results concerning partial derivatives of copulas (proofs: Denuit et al.)

Proposition 2

Let C be a copula. For any $u_2 \in [0,1]$ the partial derivative $\frac{\partial}{\partial u_1} C(u_1, u_2)$ exists almost everywhere and for each (u_1, u_2) where it exists we have

$$0 \leq \frac{\partial}{\partial u_1} C(u_1, u_2) \leq 1.$$

Similarly, for any $u_1 \in [0,1]$ the partial derivative $\frac{\partial}{\partial u_2} C(u_1, u_2)$ exists almost everywhere for each (u_1, u_2) where it exists we have

$$0 \leq \frac{\partial}{\partial u_2} C(u_1, u_2) \leq 1.$$

(Almost everywhere means everywhere except perhaps on countable set of points.)

Conditional distributions derived from copulas

- Functions $u_1 \mapsto \frac{\partial}{\partial u_2} C(u_1, u_2)$ and $u_2 \mapsto \frac{\partial}{\partial u_1} C(u_1, u_2)$ are defined and non-decreasing almost everywhere on $[0, 1]$.
- Hence the partial derivatives of C exist and resemble cdfs. We will now relate these partial derivatives to conditional distributions.
- Let us first define $C_{2|1}$ and $C_{1|2}$ as

$$\begin{aligned} C_{2|1}(u_2|u_1) &= \frac{\partial}{\partial u_1} C(u_1, u_2) \\ C_{1|2}(u_1|u_2) &= \frac{\partial}{\partial u_2} C(u_1, u_2) \end{aligned}$$

Proposition 3

Given a random couple $\boldsymbol{X}$ with $F_{\boldsymbol{X}}(x_1, x_2) = C\left(F_1(x_1), F_2(x_2)\right)$, then

$$\begin{aligned} P(X_2 \leq x_2 | X_1 = x_1) &= C_{2|1}(F_2(x_2)|F_1(x_1)) \\ P(X_1 \leq x_1 | X_2 = x_2) &= C_{1|2}(F_1(x_1)|F_2(x_2)) \end{aligned}$$

hold for any $\boldsymbol{x} \in \mathbb{R}^2$.

Conditional distributions derived from copulas

Proof.

Let (U_1, U_2) be a random couple with joint cdf C.

$$\begin{aligned} P(U_2 \leq u_2 | U_1 = u_1) &= \lim_{\Delta u_1 \to 0} \frac{P(u_1 \leq U_1 \leq u_1 + \Delta u_1, U_2 \leq u_2)}{P(u_1 \leq U_1 \leq u_1 + \Delta u_1)} \\ &= \lim_{\Delta u_1 \to 0} \frac{C(u_1 + \Delta u_1, u_2) - C(u_1, u_2)}{\Delta u_1} \\ &= C_{2|1}(u_2 | u_1) \end{aligned}$$

Conditial cdf of X_2 given that $X_1 = x_1$ is then obtained by replacing u_i with $F_i(x_i)$ for $i = 1, 2$ and the result follows. □

pdfs associated with copulas

Proposition 4

If the marginal cdfs F_1 and F_2 are continuous with respective pdfs f_1 and f_2, then the joint pdf of $\boldsymbol{X}$ can be written as

$$f_{\boldsymbol{X}}(x_1, x_2) = c\left(F_1(x_1), F_2(x_2)\right) f_1(x_1) f_2(x_2) \qquad \boldsymbol{x} \in \mathbb{R}^2 \tag{2}$$

where the copula density c is given by

$$c(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1, u_2) \qquad \boldsymbol{u} \in [0,1]^2. \tag{3}$$

Hence joint pdf can be written as product of marginal pdfs and copula density
$\Rightarrow$ copula density encodes all information about the dependence among the X_i.
Factor $c\left(F_1(x_1), F_2(x_2)\right)$ distorts independence to induce actual dependence structure: joint pdf $f_{\boldsymbol{X}}$ is obtained from independence pdf $f_1(x_1)f_2(x_2)$ reweighted at $\boldsymbol{x}$ by $c\left(F_1(x_1), F_2(x_2)\right)$.
If X_i are independent, then $c \equiv 1$ (see next slide) and joint pdf $f_{\boldsymbol{X}}$ factors into product of marginals f_1 and f_2.

Fundamental Copulas

Let us now examine several elementary examples to illustrate the decomposition

$$\begin{aligned} F_{\boldsymbol{X}}(x_1, x_2) &= C(F_1(x_1), F_2(x_2)) \qquad \forall \boldsymbol{x} \in \mathbb{R}^2 \\ C(\boldsymbol{u}) &= F_{\boldsymbol{X}}(F_1^{-1}(u_1), F_2^{-1}(u_2)) \qquad \forall \boldsymbol{u} \in [0,1]^2 \end{aligned}$$

Definition 2

Consider independent rvs X_1 and X_2 with respective cdfs F_1 and F_2. Their joint cdf is $F_{\boldsymbol{X}}(\boldsymbol{x}) = F_1(x_1)F_2(x_2)$. The underlying independence copula is the copula of 2 *independent Uniform(0,1) rvs:*

$$C^{ind}(u_1, u_2) = F_1(F_1^{-1}(u_1))F_2(F_2^{-1}(u_2)) = u_1 u_2$$

(3) $\Rightarrow c^{\text{ind}}(u_1, u_2) = 1$ on $[0,1]^2 \Rightarrow$ uniform density on the interval $[0,1]^2$.

Definition 3

The (co-)monotonicity copula or Fréchet upper bound copula is the copula with perfect positive dependence of the rvs. Assume $U \sim$ Uniform(0,1), then

$$\begin{aligned} C^M(u_1, u_2) &= P\left(U \leq u_1, U \leq u_2\right) = P\left(U \leq \min(u_1, u_2)\right) \\ &= F_U(\min(u_1, u_2)) = \min(u_1, u_2) \end{aligned}$$

Corresponds to unit mass spread over main diagonal $u_1 = u_2$ of unit square.

Definition 4

The counter-monotonicity copula or Fréchet lower bound copula is the copula with perfect negative dependence of the rv.

$$\begin{aligned} C^{CM}(u_1, u_2) &= P\left(U \leq u_1, 1 - U \leq u_2\right) = P\left(1 - u_2 \leq U \leq u_1\right) \\ &= \max(u_1 + u_2 - 1, 0) \end{aligned}$$

Corresponds to unit mass spread over secondary diagonal $u_1 = 1 - u_2$ of unit square.

The Fréchet-Hoeffding bounds I

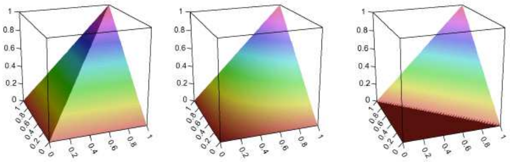
Theorem 5

Following inequality holds for every copula C

$$C^{CM}(u,v) \leq C(u,v) \leq C^{M}(u,v)$$

Graph of each copula is continuous surface over unit square that contains the skew quadrilateral whose vertices are $(0,0,0),(1,0,0),(0,1,0)$ and $(1,1,1)$. Surface is bounded from below by 2 triangles that make up surface of C^{M} and from above by 2 triangles that make up surface C^{CM}.

3D plots of C^{M}, C^{Ind} and C^{CM}



The Fréchet-Hoeffding bounds II

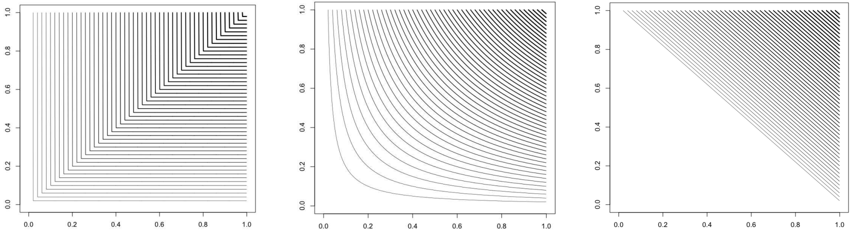
Level curves or contour plot of the copula $C(x,y)$ are the set of points of $[0,1]^2$ such that $C(x,y) = \alpha$ with constant α:

$$\{(x,y)|C(x,y) = \alpha\}$$

The level curves of the counter-monotonic and monotonic copula are

$$\begin{array}{lll} \{x,y & | & \max(x+y-1,0) = \alpha\} \qquad \alpha \in [0,1] \\ \{x,y & | & \min(x,y) = \alpha\} \qquad \alpha \in [0,1] \end{array}$$

Level plots of C^{M}, C^{Ind} and C^{CM}



Example: Marshall-Olkin I

Consider survivalship of two firms, whose default or survival time is denoted as X_1 and X_2 and let them be subject to three shocks: two idiosyncratic ones and the latter common to both firms. Les us assume that the shocks follow three independent poisson processes with parameters $\lambda_1, \lambda_2, \lambda \geq 0$, where they respectively indicate whether the shocks effect only component 1, only component 2 or both. This means that the times of occurrence of the shocks, denoted respectively as Z_1, Z_2, Z are independent exponential random variables with parameters $\lambda_1, \lambda_2, \lambda$. Their cdfs denoted as G_1, G_2, G are

$$G_1 = 1 - exp(-\lambda_1 z_1), \qquad G_2 = 1 - exp(-\lambda_2 z_2), \qquad G = 1 - exp(-\lambda z)$$

Since both remaining lifetimes are subject to the same shock, the age-at-death rvs are $X_1 = \min(Z_1, Z)$ and $X_2 = \min(Z_2, Z)$.
Probability that X_i survives beyond x_i (for $i = 1, 2$) is

$$\begin{aligned}
\overline{F}_1(x_1) &= P(X_1 > x_1) = P(Z_1 > x_1, Z > x_1) \\
&= \overline{G}_1(x_1)\overline{G}(x_1) = exp(-(\lambda_1 + \lambda)x_1) \\
\overline{F}_2(x_2) &= exp(-(\lambda_2 + \lambda)x_2)
\end{aligned}$$

The probability that both survive beyond x_1 and x_2 respectively is

$$\begin{aligned}
\overline{F}(x_1, x_2) &= P(X_1 > x_1, X_2 > x_2) \\
&= P(min(Z_1, Z) > x_1, min(Z_2, Z) > x_2) \\
&= P(Z_1 > x_1)P(Z_2 > x_2)P(Z > max(x_1, x_2)) \\
&= exp(-\lambda_1 x_1)exp(-\lambda_2 x_2)exp(-\lambda \max(x_1, x_2)) \\
&= exp(-(\lambda_1 + \lambda)x_1 - (\lambda_2 + \lambda)x_2 + \lambda \min(x_1, x_2)) \\
&= \overline{F}_1(x_1)\overline{F}_2(x_2) \min(exp(\lambda x_1), exp(\lambda x_2))
\end{aligned}$$

Therefore the joint cdf of $\boldsymbol{X}$

$$F(x_1, x_2) = F_1(x_1) + F_2(x_2) - 1 + \overline{F}_1(x_1)\overline{F}_2(x_2) \min\left(exp(\lambda x_1), exp(\lambda x_2)\right)$$

- To get bivariate distribution with exponential marginals, we took X_1, X_2 and Z exponentially distributed. This construction does not easily carry over to other distributions.
- Dependence is induced by common factor Z involved in both X_1 and X_2. Strenght of dependence is driven by λ. Dependence parameter appears both in the marginal and in the joint distributions, which makes interpretation more difficult.

Underlying copula is not directly apparent. Therefore, let

$$\alpha_1 = \frac{\lambda}{\lambda_1 + \lambda} \quad \Rightarrow \quad exp(\lambda x_1) = \overline{F}_1(x_1)^{-\alpha_1}$$
$$\alpha_2 = \frac{\lambda}{\lambda_2 + \lambda} \quad \Rightarrow \quad exp(\lambda x_2) = \overline{F}_2(x_2)^{-\alpha_2}$$

Then

$$\begin{aligned} \overline{F}(x_1, x_2) &= \overline{F}_1(x_1)\overline{F}_2(x_2)\min(\overline{F}_1(x_1)^{-\alpha_1}, \overline{F}_2(x_2)^{-\alpha_2}) \\ &= \min\left(\left[\overline{F}_2(x_2)\right]\left[\overline{F}_1(x_1)\right]^{1-\alpha_1}, \left[\overline{F}_1(x_1)\right]\left[\overline{F}_2(x_2)\right]^{1-\alpha_2}\right) \end{aligned}$$

We see that this joint survival probability can be written in terms of the one of X_1 and X_2, so that Sklar's representation for

$$F(x_1, x_2) = 1 - \overline{F}_1(x_1) - \overline{F}_2(x_2) + \overline{F}(x_1, x_2) = C(F_1(x_1), F_2(x_2))$$

holds with

$$C(u_1, u_2) = 1 - (1 - u_1) - (1 - u_2) + \min\left([(1 - u_2)][1 - u_1]^{1-\alpha_1}, [1 - u_1][1 - u_2]^{1-\alpha_2}\right) \quad (4)$$

The Gaussian Copula I

- Consider bivariate standard Gaussian (X, Y) random variable with mean $\boldsymbol{\mu} = (0, 0)$, correlation ρ and shape matrix

  $$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

  Denote its bivariate normal cdf by $\Phi_\rho(x, y)$.
- X and Y posses univariate standard Gaussian distributions with cdf Φ.
- Following Sklar's Theorem we define Gaussian copula with parameter ρ

  $$C_\rho^{\text{Gauss}}(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$$

- Note that C_ρ^{Gauss} is copula of any bivariate non-standard Gaussian random variable and many non Gaussian random variables as well.
- There are more bivariate random variables having Gaussian margins but do not posses Gaussian dependence structure (Gaussian copula).

The Gaussian Copula II

- Gaussian copula is given by

$$C_\rho^{\text{Gauss}}(u_1,u_2) = \int_{-\infty}^{\phi^{-1}(u_1)} \int_{-\infty}^{\phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{s_1^2 - 2\rho s_1 s_2 + s_2^2}{2(1-\rho^2)}\right) ds_1 ds_2$$

- Pdf of the Gaussian copula is given by

$$\begin{aligned} c_\rho^{\text{Gauss}}(u_1,u_2) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{\xi_1^2 - 2\rho\xi_1\xi_2 + \xi_2^2}{2(1-\rho^2)}\right) \frac{d}{du_1}\Phi^{-1}(u_1) \frac{d}{du_2}\Phi^{-1}(u_2) \\ &\overset{(*)}{=} \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{\xi_1^2 - 2\rho\xi_1\xi_2 + \xi_2^2}{2(1-\rho^2)}\right) \exp\left(\frac{\xi_1^2+\xi_2^2}{2}\right) \end{aligned}$$

  where $\xi_1 = \Phi^{-1}(u_1)$ and $\xi_2 = \Phi^{-1}(u_2)$

  $^{(*)}$ Note that $\frac{d}{du_i}\Phi^{-1}(u_i) = \frac{1}{\phi(\Phi^{-1}(u_i))} = \sqrt{2\pi}\exp(\frac{\xi_i^2}{2})$.

- $C_{\rho=0}^{\text{Gauss}} = C^{\text{Ind}}$, $C_{\rho=1}^{\text{Gauss}} = C^{\text{M}}$ and $C_{\rho=-1}^{\text{Gauss}} = C^{\text{CM}}$

  $\Rightarrow$ Gaussian copula interpolates between 3 fundamental dependency structures via one single parameter ρ (comprehensive copula)

  Strength of dependence increases with ρ:

  $C_\rho^{\text{Gauss}}(u_1,u_2) \leq C_{\rho'}^{\text{Gauss}}(u_1,u_2)$ for all $(u_1,u_2) \in [0,1]^2$ when $\rho \leq \rho'$.

The Student t-distribution

- Univariate t-distribution with ν degrees of freedom: $Y_1 \sim t_\nu$

$$Y_1 = \frac{X_1}{\sqrt{\xi/\nu}} = \frac{\sqrt{\nu}X_1}{\sqrt{Z_1^2 + \ldots + Z_\nu^2}}$$

  where $X_1, Z_1, \ldots Z_\nu \sim$ Normal(0,1) and $\xi \sim \chi^2_\nu$, X_1 independent on $\mathbf{Z}$
- Bivariate t-distribution with mean $\boldsymbol{\mu}$ and shape matrix $\frac{\nu}{\nu-2}\boldsymbol{\Sigma}$:

$$(Y_1, Y_2) = \boldsymbol{\mu} + \left(\frac{X_1}{\sqrt{\xi/\nu}}, \frac{X_2}{\sqrt{\xi/\nu}}\right)$$

  where $\mathbf{X} = (X_1, X_2) \sim$ Normal$(\mathbf{0}, \boldsymbol{\Sigma})$
- If $\rho = 0$, then Y_1 and Y_2 are still not independent since some dependence is introduced via ξ

The Student Copula I

- Student copula is copula derived from multivariate Student distribution

$$C^{\text{Student}}_{\rho,\nu}(u_1, u_2) = T_{\rho,\nu}\left(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2)\right)$$

where $t_\nu \equiv$ univariate Student distribution with ν df and $T_{\rho,\nu} \equiv$ bivariate Student distribution with ν df and $0 \leq \rho \leq 1$.

- Student copula

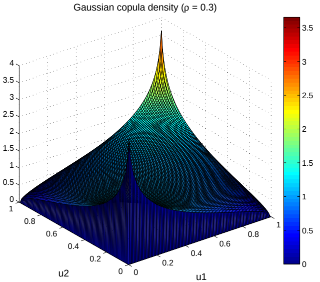
$$C^{\text{Student}}_{\rho,\nu}(u_1, u_2) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{ds_1 ds_2}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{s_1^2 - 2\rho s_1 s_2 + s_2^2}{\nu(1-\rho^2)}\right)^{\frac{-(\nu+2)}{2}}$$

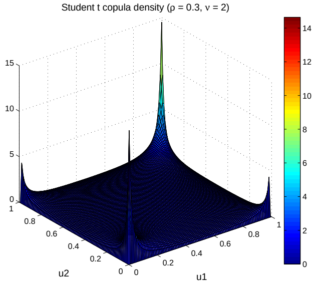
- Student distribution $\rightarrow$ Normal distribution when $\nu \rightarrow +\infty$

$$\Rightarrow C^{\text{Student}}_{\rho,\nu} \rightarrow C^{\text{Gaussian}}_{\rho} \quad \text{as } \nu \rightarrow +\infty$$

The Gaussian and Student Copula I

Gaussian copula density (left) and Student copula density (right)





Definition 5

- *A random vector* $\mathbf{X}$ *has a meta-Gaussian distribution if* $\mathbf{X}$ *has a Gaussian copula. (Univariate marginal cdfs of* $\mathbf{Y}$ *can be any distributions.)*
- *A random vector* $\mathbf{Y}$ *has a meta-*t_ν *distribution if* $\mathbf{Y}$ *has a Student copula*

Meta distributions

- Converse statement of Sklar's theorem provides a powerful technique for constructing multivariate distributions with arbitrary margins and copulas.
- If we start with copula C and margins $F_1, \ldots, F_d$, then

$$F(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_d(x_d))$$

  defines a multivariate df with margins $F_1, \ldots, F_d$.
- For example build a distribution with Gauss copula but arbitrary margins.
- Such a model is sometimes called a meta-Gaussian distribution.
- We extend the meta terminology to other distributions, e.g. meta-t_v distribution, meta-Clayton distribution.

Archimedean Copula

Definition 6

An Archimedean Copula with a strict generator is a copula of the form

$$C(u_1, u_2) = \phi^{-1}\left(\phi(u_1) + \phi(u_2)\right) \tag{5}$$

where ϕ is the generator of the copula and satisfies

1. *ϕ is a continuous, strictly decreasing and convex function mapping $[0, 1]$ onto $[0, +\infty]$*
2. $\phi(0) = +\infty$
3. $\phi(1) = 0$

- Archimedean representation: reduce the study of a multivariate copula to a single univariate function.
- Satisfy Exchangeability condition: $C(u_1, u_2) = C(u_2, u_1)$.
- ϕ and $\alpha\phi$ generate same copula for any positive constant α, so that ϕ is only defined up to positive factor.

Frank Copula I

Definition 7

The Frank Copula is the Archimedean copula with generator

$$\phi(t) = -\log\left(\frac{\exp(-\theta t) - 1}{\exp(-\theta) - 1}\right) \qquad \textit{and} \qquad -\infty < \theta < +\infty$$

This generator satisfies assumptions 1-3 of former slide.
Hence, the bivariate Frank copula equals

$$C^{\mathrm{Fr}}(u_1, u_2) = \frac{-1}{\theta}\log\left(1 + \frac{(\exp(-\theta u_1) - 1)(\exp(-\theta u_2) - 1)}{\exp(-\theta) - 1}\right) \qquad (6)$$

It is interesting to study the limits of the Frank copula

- $\theta = 0$: $\lim_{\theta\to 0} C^{\mathrm{Fr}}(u_1, u_2) \to u_1 u_2 = C^{\mathrm{ind}}(u_1, u_2)$
- $\theta \to -\infty$: $\lim_{\theta\to -\infty} C^{\mathrm{Fr}}(u_1, u_2) = \max(u_1 + u_2 - 1, 0) = C^{\mathrm{CM}(u_1,u_2)}$
- $\theta \to +\infty$: $\lim_{\theta\to +\infty} C^{\mathrm{Fr}}(u_1, u_2) = \min(u_1, u_2) = C^{\mathrm{M}(u_1,u_2)}$

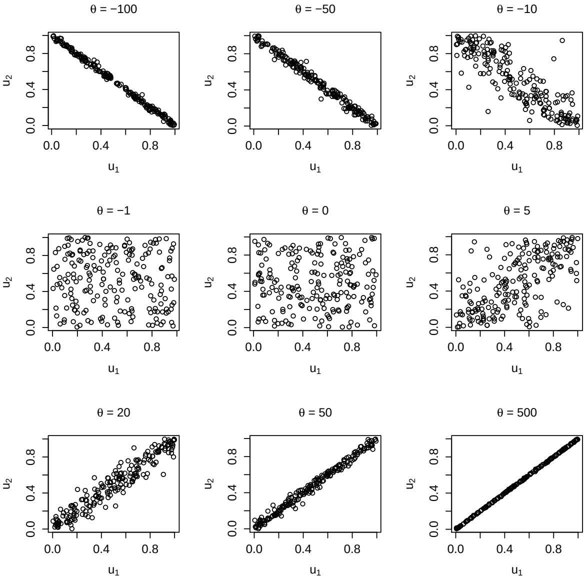


Figure 1: Random sample of size 200 generated from Frank copulas.

Clayton Copula I

Definition 8

The Clayton Copula is the Archimedean copula with generator

$$\phi(t) = \frac{t^{-\theta} - 1}{\theta} \qquad \textit{and} \quad \theta > 0$$

Hence, the bivariate Clayton copula equals

$$C^{\mathrm{Cl}}(u_1, u_2) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta} \tag{7}$$

Limits of the Clayton copula

- $\theta \to 0$: $\lim_{\theta \to 0} C^{\mathrm{Cl}}(u_1, u_2) = u_1 u_2 = C^{\mathrm{ind}}(u_1, u_2)$
- $\theta \to +\infty$: $\lim_{\theta \to +\infty} C^{\mathrm{Cl}}(u_1, u_2) = C^{\mathrm{M}}(u_1, u_2)$

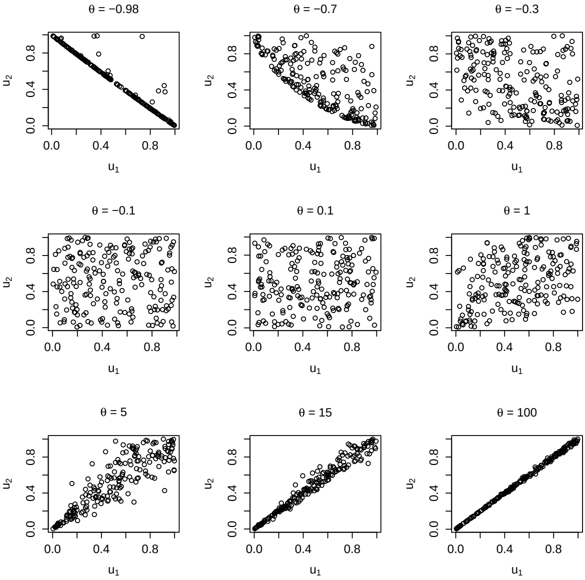


Figure 2: Random sample of size 200 generated from Clayton copulas.

Gumbel Copula I

Definition 9

The Gumbel Copula is the Archimedean copula with generator

$$\phi(t) = (-\log(t))^{\theta} \qquad \textit{and} \quad \theta \geq 1$$

Hence, the bivariate Gumbel copula equals

$$C^{\text{Gu}}(u_1, u_2) = exp\left(-\left\{(-\log(u_1))^{\theta} + (-\log(u_2))^{\theta}\right\}^{1/\theta}\right) \tag{8}$$

Limit of the Gumbel copula

- $\theta = 1$: $\lim\limits_{\theta \to 1} C^{\text{Gu}}(u_1, \ldots, u_N) = C^{\text{ind}}(u_1, \ldots, u_N)$
- $\theta \to +\infty$: $\lim_{\theta \to +\infty} C^{\text{Gu}}(u_1, \ldots, u_N) = C^{\text{M}}(u_1, \ldots, u_N)$

Gumbel copula can not have negative dependence

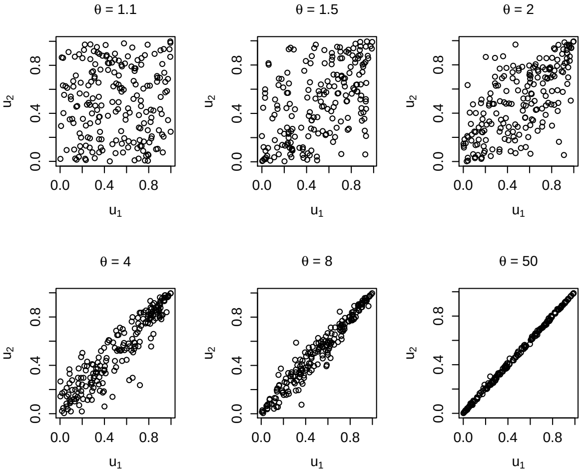


Figure 3: Random sample of size 200 generated from Gumbel copulas.

In applications it is useful that the different copula families have different properties, since this increases the likelihood of finding a copula that fits the data.

Comparing different copulas I

- We show 2000 simulated points from four copulas:
  1. Gaussian copula with parameter $\rho = 0.7$
  2. Gumbel copula with parameter $\theta = 2$
  3. Clayton copula with parameter $\theta = 2.2$
  4. t copula with parameters $\nu = 4$ and $\rho = 0.71$

  Parameters of copulas have been chosen so that all distributions have a linear correlation around 70%.

Comparing different copulas II

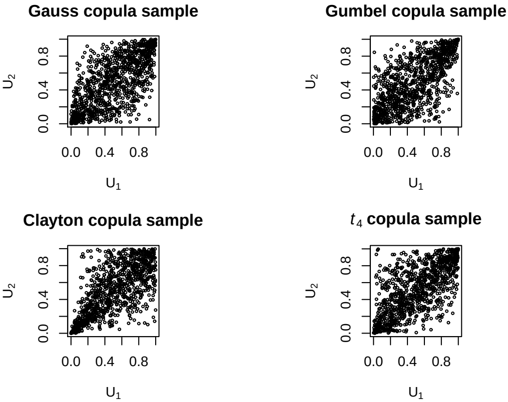
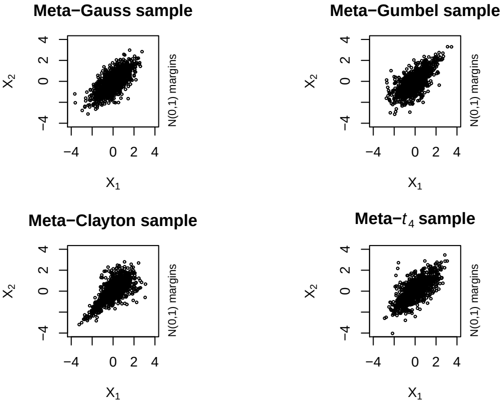


Figure 4: Random sample of size 2000 generated from various copulas.

Comparing different copulas III

- We transform these points componentwise using the quantile function of the standard normal distribution to get realizations from four different meta distributions with standard normal margins.



Exercise I

Exercise 2

Find the inverse generators of the bivariate Frank, Clayton and Gumbel Copula and construct the copulas by using their generators and inverse generators.

Exercise II

Answer of exercise 2

1. Frank Copula

$$\begin{aligned}
\phi(u) &= -\log\left(\frac{e^{-\theta u}-1}{e^{-\theta}-1}\right) \qquad -\infty<\theta<\infty \\
y &= -\log\left(\frac{e^{-\theta\left(\phi^{-1}(y)\right)}-1}{e^{-\theta}-1}\right) \qquad u=\phi^{-1}(y) \\
e^{-y}\left(e^{-\theta}-1\right)+1 &= e^{-\theta\left(\phi^{-1}(y)\right)} \\
\frac{\log\left(e^{-y}(e^{-\theta}-1)+1\right)}{-\theta} &= \phi^{-1}(y)
\end{aligned}$$

Therefore

$$\begin{aligned}
C(u_1,u_2) &= \phi^{-1}\left(\phi(u_1)+\phi(u_2)\right) \\
&= -\frac{1}{\theta}\log\left(e^{-(\phi(u_1)+\phi(u_2))}(e^{-\theta}-1)+1\right)
\end{aligned}$$

Exercise III

Moreover

$$\begin{aligned}\phi(u_1)+\phi(u_2) &= -\left(\log\left[\frac{e^{-\theta u_1}-1}{e^{-\theta}-1}\right]+\log\left[\frac{e^{-\theta u_2}-1}{e^{-\theta}-1}\right]\right)\\ \Rightarrow e^{-(\phi(u_1)+\phi(u_2))} &= \left[\frac{e^{-\theta u_1}-1}{e^{-\theta}-1}\right]\left[\frac{e^{-\theta u_2}-1}{e^{-\theta}-1}\right]\\ &= \frac{\left(e^{-\theta u_1}-1\right)\left(e^{-\theta u_2}-1\right)}{\left(e^{-\theta}-1\right)^2}\end{aligned}$$

Hence

$$C(u_1,u_2) = -\frac{1}{\theta}\log\left[\frac{\left(e^{-\theta u_1}-1\right)\left(e^{-\theta u_2}-1\right)}{e^{-\theta}-1}+1\right]$$

Exercise IV

2. Clayton Copula

$$
\begin{aligned}
\phi(u) &= \frac{u^{-\theta}-1}{\theta} \qquad \theta > 0 \\
y &= \frac{\left(\phi^{-1}(y)\right)^{-\theta}-1}{\theta} \qquad u = \phi^{-1}(y) \\
(y\theta+1)^{-\frac{1}{\theta}} &= \phi^{-1}(y)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
C(u_1, u_2) &= \phi^{-1}\left(\phi(u_1)+\phi(u_2)\right) \\
&= \left[\theta\left(\phi(u_1)+\phi(u_2)\right)+1\right]^{-\frac{1}{\theta}} \\
&= \left[\theta\left(\frac{u_1^{-\theta}-1}{\theta}+\frac{u_2^{-\theta}-1}{\theta}\right)+1\right]^{-\frac{1}{\theta}} \\
&= \left[u_1^{-\theta}+u_2^{-\theta}-2+1\right]^{-\frac{1}{\theta}}
\end{aligned}
$$

Exercise V

3. Gumbel Copula

$$
\begin{aligned}
\phi(u) &= (-\log(u))^{\theta} \qquad \theta \geq 1 \\
y &= \left(-\log\left[\phi^{-1}(y)\right]\right)^{\theta} \qquad u = \phi^{-1}(y) \\
e^{\left(-y^{\frac{1}{\theta}}\right)} &= \phi^{-1}(y)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
C(u_1, u_2) &= \phi^{-1}\left(\phi(u_1) + \phi(u_2)\right) \\
&= e^{\left(-(\phi(u_1)+\phi(u_2))^{\frac{1}{\theta}}\right)} \\
&= e^{\left(-\left[(-\log(u_1))^{\theta}+(-\log(u_2))^{\theta}\right]^{\frac{1}{\theta}}\right)}
\end{aligned}
$$

Survival copula I

Definition 10

If C is a copula, then so is the survival copula $\overline{C}$ associated with C

$$\overline{C}(u_1, u_2) = C(1-u_1, 1-u_2) + u_1 + u_2 - 1.$$

- Computed at $(1-u_1, 1-u_2)$ it represents probability of two unit uniform rvs with copula C being larger than u_1 and u_2 respectively.
- Note that

$$\begin{aligned} P(U_1 > u_1, U_2 > u_2) &= 1 - u_1 - u_2 + C(u_1, u_2) \\ P(U_1 > u_1, U_2 > u_2) &\neq \overline{C}(u_1, u_2) \\ P(U_1 > u_1, U_2 > u_2) &= \overline{C}(1-u_1, 1-u_2) \end{aligned}$$

Survival copula II

- Since $\overline{C}$ is copula, it holds that

$$C^{CM}(u_1, u_2) \leq \overline{C}(u_1, u_2) \leq C^{M}(u_1, u_2).$$

  Furthermore: $\overline{C}^{CM} = C^{CM}, \quad \overline{C}^{ind} = C^{ind}, \quad \overline{C}^{M} = C^{M}.$

- Restating Sklar's theorem

$$\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) = 1 - F_1(x_1) - F_2(x_2) + F_{\boldsymbol{X}}(\boldsymbol{x}) = \overline{C}(\overline{F}_1(x_1), \overline{F}_2(x_2))$$

Example: Marshall-Olkin The construction of the Marhall-Olkin survival copula (see equation (4)) leads to following copula family

$$\begin{aligned} C_{\alpha_1,\alpha_2}(u_1, u_2) &= \min(u_1^{1-\alpha_1} u_2, u_1 u_2^{1-\alpha_2}) \\ &= \begin{cases} u_1^{1-\alpha_1} u_2 & u_1^{\alpha_1} \geq u_2^{\alpha_2} \\ u_2^{1-\alpha_2} u_1 & u_1^{\alpha_1} < u_2^{\alpha_2} \end{cases} \end{aligned}$$

Exercise I

Exercise 3

Find the Marshall-Olkin conditional copula $C_{2|1}(u_2|u_1)$ and the Marshall-Olkin copula density.

Exercise II

Answer of exercise 3

$$
\begin{aligned}
C_{\alpha_1,\alpha_2}(u_1,u_2) &= \min(u_1^{1-\alpha_1}u_2, u_1u_2^{1-\alpha_2}) \\
&= \begin{cases} u_1^{1-\alpha_1}u_2 & u_1^{\alpha_1} \geq u_2^{\alpha_2} \\ u_2^{1-\alpha_2}u_1 & u_1^{\alpha_1} < u_2^{\alpha_2} \end{cases}
\end{aligned}
$$

$$
\frac{\partial}{\partial u_1}C_{\alpha_1,\alpha_2}(u_1,u_2) = \begin{cases} (1-\alpha_1)u_1^{-\alpha_1}u_2 & u_1^{\alpha_1} \geq u_2^{\alpha_2} \\ u_2^{1-\alpha_2} & u_1^{\alpha_1} < u_2^{\alpha_2} \end{cases}
$$

$$
\frac{\partial}{\partial u_1 \partial u_2}C_{\alpha_1,\alpha_2}(u_1,u_2) = \begin{cases} (1-\alpha_1)u_1^{-\alpha_1} & u_1^{\alpha_1} \geq u_2^{\alpha_2} \\ (1-\alpha_2)u_2^{-\alpha_2} & u_1^{\alpha_1} < u_2^{\alpha_2} \end{cases}
$$

Example I

- Well-known generalization of the important univariate Pareto distribution is the bivariate Pareto distribution with survivor function given by

$$\overline{F}(x_1, x_2) = \left(\frac{x_1 + \kappa_1}{\kappa_1} + \frac{x_2 + \kappa_2}{\kappa_2} - 1\right)^{-\alpha} \qquad x_1, x_2 \geq 0, \alpha, \kappa_1, \kappa_2 > 0$$

- Marginal survivor functions are given by $(i = 1, 2)$

$$\overline{F}_i(x) = \left(\frac{\kappa_i}{\kappa_i + x}\right)^{\alpha}$$

- Survival copula is given by

$$\overline{C}(u_1, u_2) = (u_1^{-1/\alpha} + u_2^{-1/\alpha} - 1)^{-\alpha}$$

- This is the Clayton copula with parameter $\theta = 1/\alpha$.

Dual and co-copula I

Definition 11

The co-copula C^* and dual $\tilde{C}$ are defined as

$$\begin{aligned} C^*(u_1, u_2) &= 1 - C(1-u_1, 1-u_2) \\ \tilde{C}(u_1, u_2) &= u_1 + u_2 - C(u_1, u_2) \end{aligned}$$

- Neither C^* nor $\tilde{C}$ is a copula.
- They represent following probabilities

$$\begin{aligned} P(X_1 > x_1 \text{ or } X_2 > x_2) &= C^*(\overline{F}_1(x_1), \overline{F}_2(x_2)) \\ P(X_1 \leq x_1 \text{ or } X_2 \leq x_2) &= \tilde{C}(F_1(x_1), F_2(x_2)). \end{aligned}$$

- $(C^*)^* = C$
- $(C^{CM})^* = \min(u_1 + u_2, 1), \quad (C^{ind})^* = u_1 + u_2 - u_1 u_2, \quad (C^M)^* = \max(u_1, u_2)$

Invariance theorem I

For strictly monotone transformations of rvs copulas are either invariant or change in predictable ways.

Theorem 6

Let X_1 and X_2 be continuous rvs with copula C. Let g_1 and g_2 be continuous monotone functions.

- *If g_1 and g_2 are strictly increasing then $(g_1(X_1), g_2(X_2))$ has copula C.*
- *If g_1 and g_2 are strictly decreasing then $(g_1(X_1), g_2(X_2))$ has copula $\overline{C}$.*

- Form of copula of $(g_1(X_1), g_2(X_2))$ independent of particular choice of g_1, g_2
- Dependence of X_1 and X_2 captured by the copula, whatever the scale in which they are measured
- **Example:** (X_1, X_2) and $(\log X_1, \log X_2)$

Invariance theorem II

Proof.

Let $Y = g_1(X_1)$ and $Y_2 = g_2(X_2)$

$$\begin{aligned} F_{\boldsymbol{Y}}(y_1, y_2) &= P\left(g_1(X_1) \leq y_1, g_2(Y_2) \leq y_2\right) \\ &= P\left(X_1 \leq g_1^{-1}(y_1), X_2 \leq g_2^{-1}(y_2)\right) \\ &= F_{\boldsymbol{X}}(g_1^{-1}(y_1), g_2^{-1}(y_2)) \end{aligned} \tag{9}$$

We thus have

$$F_{Y_j}(y_j) = F_{X_j}(g_j^{-1}(y_j)) \quad \text{and so} \quad F_{Y_j}^{-1}(u_j) = g_j(F_{X_j}^{-1}(u_j)) \;\forall j \tag{10}$$

$$\begin{aligned} C_Y(u_1, u_2) &= F_Y\left(F_{Y_1}^{-1}(u_1), F_{Y_2}^{-1}(u_2)\right) \\ &= F_X(g_1^{-1}(F_{Y_1}^{-1}(u_1)), g_2^{-1}(F_{Y_2}^{-1}(u_2))) && \text{(Eq (9))} \\ &= F_X(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2)) && \text{(Eq (10))} \\ &= C_X(u_1, u_2) \end{aligned}$$

□

Multivariate copulas I

Results of preceding sections for bivariate case can be extended to multivariate copulas.

Definition 12

n-dimensional copula C is function mapping $[0,1]^n$ to unit interval $[0,1]$ which is non-decreasing and right-continuous and satisfies

1. $C(u_1,\ldots,u_{i-1},0,u_{i+1},\ldots,u_n)=0 \qquad \forall i=1,\ldots,n$
2. $C(1,\ldots,1,u_i,1,\ldots,1)=u_i \qquad \forall i=1,\ldots,n$
3. $\forall \boldsymbol{\alpha},\boldsymbol{\beta}\in[0,1]^n$ *with* $\alpha_i<\beta_i$ *for* $i=1,\ldots,n$

$$\Delta_{\alpha_1,\beta_1}\Delta_{\alpha_2,\beta_2}\ldots\Delta_{\alpha_n,\beta_n}C(\boldsymbol{u})\geq 0$$

$\forall \boldsymbol{u}\in[0,1]^n$ *and with*

$$\Delta_{\alpha_i,\beta_i}F_{\boldsymbol{X}}(\boldsymbol{x})=F_{\boldsymbol{X}}(x_1,\ldots,x_{i-1},\beta_i,x_{i+1},\ldots,x_n)-F_{\boldsymbol{X}}(x_1,\ldots,x_{i-1},\alpha_i,x_{i+1},\ldots,x_n)$$

Multivariate copulas II

Note that condition 3 ensures that

$$P(\boldsymbol{\alpha} \leq \boldsymbol{X} \leq \boldsymbol{\beta}) \geq 0 \quad \text{for any} \quad \boldsymbol{\alpha} \leq \boldsymbol{\beta} \in \mathbb{R}^n.$$

When $F_{\boldsymbol{X}}$ is differentiable, this condition is equivalent to

$$\frac{\partial^n}{\partial x_1 \ldots \partial x_n} F_{\boldsymbol{X}} \geq 0$$

Every cdf $F_{\boldsymbol{X}}$ or a random vector $\boldsymbol{X} \in \mathcal{R}^n(F_1, \ldots, F_n)$ can be represented as

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_n(x_n)), \qquad \boldsymbol{x} \in \mathbb{R}^n$$

in terms of a copula C. If marginals $F_1, \ldots, F_n$ are continuous then C is unique and given by

$$C(\boldsymbol{u}) = F_{\boldsymbol{X}}\left(F^{-1}(u_1), \ldots, F_n^{-1}(u_n)\right), \qquad \boldsymbol{u} \in [0,1]^n$$

C is then cdf of random vector $(F_1(x_1, \ldots, F_n(x_n)))$ which has $U([0,1])$ marginals.

Multivariate copulas III

Furthermore

$$\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) = C(\overline{F}_1(x_1), \ldots, \overline{F}_n(x_n))$$

The pdf associated with C is given by

$$c(\boldsymbol{u}) = \frac{\partial^n}{\partial u_1 \ldots \partial u_n} C(\boldsymbol{u})$$

The pdf of $\boldsymbol{X}$ with copula C and marginals $F_1, \ldots, F_n$

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \left(\prod_{i=1}^{n} f_i(x_i) \right) c(F_1(x_1), \ldots, F_n(x_n))$$

As in bivariate case, $f_{\boldsymbol{X}}$ is obtained by reweighting pdf corresponding to independence using copula density c

Multivariate copulas IV

Invariance of copula to any **strictly increasing transformation** of the variables:

Theorem 7

Consider n continuous rv. $X_1, \ldots, X_n$ with copula C_X, then if $g_1(X_1), \ldots, g_n(X_n)$ are strictly increasing and continuous, $Y_1 = g_1(X_1), \ldots, Y_n = g_n(X_n)$ have the same copula $C_X = C_Y$.

The Fréchet upper bound copula is defined by

$$C^M(\boldsymbol{u}) = \min(u_1, \ldots, u_n)$$

However, multivariate extension of Fréchet lower bound copula C^{CM}

$$\max\left(\sum_{i=1}^{n} u_i - (n-1), 0\right)$$

does **not** satisfy conditions for being a copula!
Other copulas are easily extended.

Survival function I

Definition 13

We denote by $\overline{\overline{C}}$ the **survival function** *for n random variables with joint distribution C, i.e. if $(U_1, \ldots, U_n)'$ has distribution function C, then*

$$\overline{\overline{C}}(u_1, \ldots, u_n) = P(U_1 > u_1, \ldots, U_n > u_n)$$

Recall that **survival copula** of two rvs with copula C is given by

$$\overline{C}(u_1, u_2) = C(1 - u_1, 1 - u_2) + u_1 + u_2 - 1$$

whereas the joint survival function for 2 $U(0,1)$ rvs with joint distribution function C is given by

$$\overline{\overline{C}}(u, v) = P(U_1 > u_1, U_2 > u_2) = \overline{C}(1 - u_1, 1 - u_2) = C(u_1, u_2) + 1 - u_1 - u_2$$

Survival function II

Definition 14

If C_1 and C_2 are copulas, C_1 is smaller than C_2 (written $C_1 \prec C_2$) if

$$C_1(\boldsymbol{u}) \leq C_2(\boldsymbol{u}) \quad \text{and} \quad \overline{\overline{C}}_1(\boldsymbol{u}) \leq \overline{\overline{C}}_2(\boldsymbol{u})$$

for all $\boldsymbol{u} \in [0,1]^n$.

- Note that in bivariate case

$$\begin{aligned} \overline{\overline{C}}_1(u,v) \leq \overline{\overline{C}}_2(u,v) \quad &\Leftrightarrow \quad 1 - u - v + C_1(u,v) \leq 1 - u_1 - u_2 + C_2(u,v) \\ &\Leftrightarrow \quad C_1(u,v) \leq C_2(u,v). \end{aligned}$$

Dependence measures I

Definition 15

$M_{X,Y} = M_C$ is measure of concordance *between rvs X and Y (with copula C) $\Leftrightarrow$*

1. *it is defined for every pairs of rvs (completeness)*
2. *it is a relative (normalized) measure, i.e. $M_{X,Y} \in [-1, 1]$*
3. *it is symmetric, i.e. $M_{X,Y} = M_{Y,X}$*
4. *if X and Y are independent, then $M_{X,Y} = 0$*
5. $M_{-X,Y} = M_{X,-Y} = -M_{X,Y}$
6. *if $\{(X_n, Y_n)\}$ is sequence of continuous rvs with copula C_n and* $\lim_{n\to+\infty} C_n(x,y) = C(x,y), \forall (x,y) \in [0,1]^2$ *then* $\lim_{n\to+\infty} M_{X_n,Y_n} = M_{X,Y}$.
7. *it respects concordance order: if $C1 \prec C2$, then $M_{C_1} \leq M_{C_2}$*

Definition implies invariance with respect to increasing transformations: If $g()$ and $h()$ are strictly increasing functions on Ran X and Ran Y respectively then $M_{g(X),h(Y)} = M_{X,Y}$.

Dependence measures II

- We will focus on 3 dependence measures:
  - Pearson linear correlation
  - Rank correlation
  - Tail dependence
- These dependence measures yield a scalar measurement of the strenght of the dependence for a pair of rvs although the nature and properties of the measures are very different.
- Rank correlation and tail dependence are copula based dependence measures. They are functions of the copula only and can thus be used in the parametrization of copulas.
- Linear correlation plays central role in actuarial science, but it is important to realize that the concept is only really a natural one in the context of multivariate normal or more generally elliptical models.

Dependence measures III

Definition 16

The Pearson (or linear) correlation coefficient *between* X_1 *and* X_2 *is defined by*

$$\rho = \frac{cov(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} = \frac{E[X_1X_2] - E[X_1]E[X_2]}{\sigma_{X_1}\sigma_{X_2}}$$

- ρ satisfies axioms 1-5 and 7 of concordance measure definition.
- ρ not invariant under non linear changes of scale
  - $\rho(\log X_1, \log X_2) \neq \rho(X_1, X_2)$
- ρ is invariant under strictly increasing **linear** transformation: $\rho(\alpha X + \beta, \gamma Y + \delta) = \text{sign}(\alpha\gamma)\rho(X, Y), \ \alpha, \gamma \neq 0$
- If $|\rho(X_1, X_2)| = 1$ then this is equivalent to saying that X_1 and X_2 are perfectly linearly dependent, meaning $X_2 = \alpha + \beta X_1$ almost surely.

Dependence measures IV

- If X_1 and X_2 are independent, then $\rho(X_1, X_2) = 0$ but it is important to be clear that the converse is false: the uncorrelatedness of X_1 and X_2 does not in general imply their independence.
- Correlation is only defined when the variances of X_1 and X_2 are finite. This restriction to finite-variance models is not ideal for a dependence measure and can cause problems when we work with heavy-tailed distributions. For example, actuaries who model losses in different business lines with infinite-variance distributions may not describe the dependence of their risks using correlation.
- Sensitive to outliers.

Linear correlation coefficient as measure of dependence shows some shortcomings and may be even misleading when moving away from from elliptical models. We therefore look at two popular alternatives:

- Kendall's tau
- Spearmann's rho.

Dependence measures V

Definition 17

The rank *of the observation X_i in a sample of size M is the number of observations which are less than or equal to X_i:*

$$rank(X_i) = \sum_{j=1}^{M} \mathbb{1}_{X_j \leq X_i}$$

- **invariance** of the rank to any **strictly monotonic transformation**
  $\Rightarrow \text{rank}(F_{X_i}(X_i)) = \text{rank}(X_i)$
  $\Rightarrow$ rank correlations depend only on copula, not on univariate marginals
- We focus on rank statistics that measure statistical associations between pairs of variables. These statistics are called rank correlations, examples are Kendall's tau and Spearman's rho.

Dependence measures VI

Definition 18

Assume $\mathbf{X} = (X_1, X_2) \sim F_X$ *and* $(X_1^{(1)}, X_2^{(1)})$ *and* $(X_1^{(2)}, X_2^{(2)})$ *two independent bivariate random vectors generated by* F_X*. Then* $(X_1^{(1)}, X_2^{(1)})$ *and* $(X_1^{(2)}, X_2^{(2)})$ *are a concordant pair if the ranking of* $X_1^{(1)}$ *relative to* $X_1^{(2)}$ *is the same as the ranking of* $X_2^{(1)}$ *relative to* $X_2^{(2)}$*, i.e. if*

$$(X_1^{(1)} - X_1^{(2)})(X_2^{(1)} - X_2^{(2)}) > 0$$

concordance $\equiv$ large value of X_1 associated with large values of X_2

Definition 19

$(X_1^{(1)}, X_2^{(1)})$ *and* $(X_1^{(2)}, X_2^{(2)})$ *are a discordant pair if*

$$(X_1^{(1)} - X_1^{(2)})(X_2^{(1)} - X_2^{(2)}) < 0$$

Kendall's tau I

Definition 20

Kendall's tau is the probability of a concordant pair minus the probability of a discordant pair:

$$\begin{aligned}\rho_\tau(X_1, X_2) &= P\left((X_1^{(1)} - X_1^{(2)})(X_2^{(1)} - X_2^{(2)}) > 0\right) \\ &\quad - P\left((X_1^{(1)} - X_1^{(2)})(X_2^{(1)} - X_2^{(2)}) < 0\right) \\ &= E\left[sign\left((X_1^{(1)} - X_1^{(2)})(X_2^{(1)} - X_2^{(2)})\right)\right]\end{aligned}$$

$$\rho_\tau(X_1, X_2) = 4 \int \int C_X(u,v) dC_X(u,v) - 1$$

Invariance of ρ_τ to any monotonically increasing transformation:

$$\rho_\tau(g(X_1), h(X_2)) = \rho_\tau(X_1, X_2)$$

where g and h are increasing functions.
Assume $g \equiv F_{X_1}$ and $h \equiv F_{X_2} \Rightarrow \rho_\tau$ depends only on the copula!

Kendall's tau II

Definition 21

The *sample Kendall's tau* *of a bivariate sample of size* M*,*
$\mathbf{X}^{(j)} = (X_1^{(j)}, X_2^{(j)}),\ j = 1, \ldots M$*, is*

$$\hat{\rho}_\tau(X_1, X_2) = \left(\begin{array}{c} M \\ 2 \end{array}\right)^{-1} \sum_{1 \leq i \leq j \leq M} sign\left((X_1^{(i)} - X_1^{(j)})(X_2^{(i)} - X_2^{(j)})\right)$$

where

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!}$$

Less sensitive to outliers than Pearson correlation coefficient.

Spearman's rho I

Definition 22

The Spearman's rho is the Pearson correlation coefficient of $F_{X_1}(X_1)$ and $F_{X_2}(X_2)$:

$$\rho_S(X_1, X_2) = Corr(F_{X_1}(X_1), F_{X_2}(X_2))$$

$$\rho_S(X_1, X_2) = 12 \int \int (C_X(u, v) - uv) du dv$$

- Distribution of $(F_{X_1}(X_1), F_{X_2}(X_2)) \equiv$ copula of (X_1, X_2)
  $\Rightarrow \rho_S$ depends only on the copula.
- Less sensitive to outliers than Pearson correlation.

Spearman's rho II

Definition 23

The sample Spearman's rho of a bivariate sample of size M,
$\mathbf{X}^{(j)} = (X_1^{(j)}, X_2^{(j)}),\ j = 1, \ldots M$, *is*

$$\hat{\rho}_S(X_1, X_2) = \quad \frac{12}{M(M^2-1)} \sum_{i=1}^{M} \left(rank(X_1^{(i)}) - \frac{M+1}{2} \right) \left(rank(X_2^{(i)}) - \frac{M+1}{2} \right)$$

If there are ties, then ranks are average among tied observations

Properties:

- ρ_τ and ρ_S are measures of concordance.
- comonotonic rvs $\Leftrightarrow \rho_\tau = 1 = \rho_S$
- counter-monotonic rvs $\Leftrightarrow \rho_\tau = -1 = \rho_S$
- ρ_τ and ρ_S: invariant under monotonically increasing transformations
- For given marginal cdfs, any rank correlations in $[-1, 1]$ can be reached

Exercise I

Exercise 4

Suppose that $X \sim U(0,1)$ and $Y = X^2$. What can you tell about Pearson correlation, Spearman rank correlation and Kendall's tau between X and Y?

Exercise II

Answer of exercise 4

Because X is nonnegative in this example, Y is a monotonic transformation of X.

- Therefore any rank correlation between X and Y will equal that rank correlation between X and itself and so will be equal to 1. Hence, Spearman's rho and Kendall's tau between X and Y will both equal 1.
- Pearson correlation equals 1 only if the random variables are perfectly linearly related, which is not true for X and Y. Therefore Pearson correlation between X and Y will be less than 1.

Exercise III

Exercise 5

Evaluate Spearman's rho for Marshall-Olkin copula.

Exercise IV

Answer of exercise 5

Note that

$$\iint_{[0,1]^2} uv\, dudv = \int_0^1 u \left[\frac{v^2}{2}\right]_0^1 du = \int_0^1 \frac{u}{2}\, du = \frac{1}{4}.$$

We can hence rewrite the expression for the Spearman correlation from the slides as

$$\begin{aligned}\rho_S(X,Y) &= 12\iint_{[0,1]^2} \left(C_X(u,v) - uv\right)\, dudv \\ &= 12\iint_{[0,1]^2} C_X(u,v)\, dudv - 3.\end{aligned}$$

The Marshall-Olkin copula is defined as

$$C_{\alpha_1,\alpha_2}(u,v) = \min\{u^{1-\alpha_1}v, uv^{1-\alpha_2}\} = \begin{cases} u^{1-\alpha_1}v & \text{if } u^{\alpha_1} \geq v^{\alpha_2} \\ uv^{1-\alpha_2} & \text{if } u^{\alpha_1} \leq v^{\alpha_2}. \end{cases}$$

Exercise V

$$
\begin{aligned}
\rho_S(X,Y) &= 12\int_0^1 \left(\int_0^{u^{\frac{\alpha_1}{\alpha_2}}} u^{1-\alpha_1} v \, dv + \int_{u^{\frac{\alpha_1}{\alpha_2}}}^1 u v^{1-\alpha_2} \, dv \right) du - 3 \\
&= 12\int_0^1 u^{1-\alpha_1} \frac{u^{2\frac{\alpha_1}{\alpha_2}}}{2} \, du + 12\int_0^1 u \left[\frac{v^{2-\alpha_2}}{2-\alpha_2} \right]_{u^{\frac{\alpha_1}{\alpha_2}}}^1 du - 3 \\
&= 6\left[\frac{u^{2-\alpha_1+2\frac{\alpha_1}{\alpha_2}}}{2-\alpha_1+2\frac{\alpha_1}{\alpha_2}} \right]_0^1 + \frac{12}{2-\alpha_2}\int_0^1 u \, du - \frac{12}{2-\alpha_2}\int_0^1 u^{\frac{\alpha_1}{\alpha_2}(2-\alpha_2)+1} \, du - 3 \\
&= \frac{6\alpha_2}{2\alpha_2 - \alpha_1\alpha_2 + 2\alpha_1} + \frac{6}{2-\alpha_2} + \frac{12}{2-\alpha_2}\frac{1}{\frac{\alpha_1}{\alpha_2}(2-\alpha_2)+2} - 3 \\
&= \frac{6\alpha_2}{2\alpha_1 - \alpha_1\alpha_2 + 2\alpha_2} + \frac{6}{2-\alpha_2} - \frac{12\alpha_2}{(2-\alpha_2)(2\alpha_1 - \alpha_1\alpha_2 + 2\alpha_2)} - 3 \\
&= \frac{12\alpha_2 - 6\alpha_2^2 + 12\alpha_1 + 12\alpha_2 - 6\alpha_1\alpha_2 - 12\alpha_2 - 12\alpha_1 - 12\alpha_2 + 6\alpha_1\alpha_2}{(2-\alpha_2)(2\alpha_1 - \alpha_1\alpha_2 + 2\alpha_2)} \\
&\quad + \frac{6\alpha_1\alpha_2 + 6\alpha_2^2 - 3\alpha_1\alpha_2^2}{(2-\alpha_2)(2\alpha_1 - \alpha_1\alpha_2 + 2\alpha_2)} \\
&= \frac{3\alpha_1\alpha_2(2-\alpha_2)}{(2-\alpha_2)(2\alpha_1 - \alpha_1\alpha_2 + 2\alpha_2)} = \frac{3\alpha_1\alpha_2}{2\alpha_1 - \alpha_1\alpha_2 + 2\alpha_2}.
\end{aligned}
$$

Table 1: ρ_τ and ρ_S for Archimedean copulas

Family	Kendall's tau	Spearman's rho
Independence	0	0
Clayton	$\frac{\theta}{\theta+2}$	Complicated
Gumbel	$1-\frac{1}{\theta}$	No closed-form
Frank	$1-\frac{4}{\theta}(D_1(-\theta)-1)$	$1-\frac{12}{\theta}(D_2(-\theta)-D_1(-\theta))$

where the Debye function $D_k(.)$ is defined as

$$D_k(x)=\frac{k}{x^k}\int_0^x\frac{t^k}{e^t-1}dt,\ k=1,2 \quad \text{and} \quad D_k(-x)=D_k(x)+\frac{kx}{k+1}$$

limited range for $\theta \Rightarrow$

- $\rho_\tau^{\text{Cl}}\geq 0$ and $\rho_\tau^{\text{Gu}}\geq 0 \Rightarrow$ Clayton and Gumbel families allow only for non-negative dependence
- Frank family allows for both negative and positive dependence

ρ_τ and ρ_S for Gaussian and Student copula I

Theorem 8

Let $\mathbf{X} = (X_1, \ldots, X_N)$ *have a meta-Gaussian distribution with continuous marginal cdfs and copula* $C^{Gauss}(\cdot\,; \boldsymbol{\Omega}), (\boldsymbol{\Omega})_{ij} = \rho_{ij}$, *then*

$$\rho_\tau(X_i, X_j) = \frac{2}{\pi} \arcsin(\rho_{ij})$$

$$\rho_S(X_i, X_j) = \frac{6}{\pi} \arcsin\left(\frac{\rho_{ij}}{2}\right) \approx \rho_{ij}$$

Theorem 9

Let $\mathbf{X} = (X_1, \ldots, X_N)$ *have a meta-*t_ν *distribution with continuous marginal cdfs and copula* $C^{Student}(\cdot\,; \boldsymbol{\Omega}, \nu)$, *then*

$$\rho_\tau(X_i, X_j) = \frac{2}{\pi} \arcsin(\rho_{ij})$$

ρ_τ and ρ_S for Gaussian and Student copula II

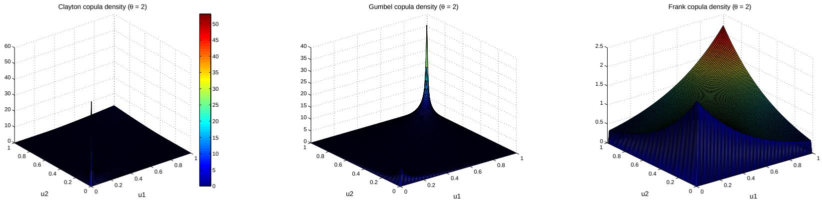


Figure 5: Clayton copula density (left), Gumbel copula density (center) and Frank copula density (right).

Opinion of Embrechts et al. (1999): *One should choose a model for the dependence structure that reflects more detailed knowledge of the risk management problem at hand instead of summarizing dependence with a single number like (linear or rank) correlation*. One such measure is **tail dependence**.

Tail dependence I

- Deviation from normality in actuarial science and finance often caused by fat-tails.
- Dependence of the distributions in the tails: important issue for risk management.
- More probable that extreme events occur simultaneously or independently?

$\Rightarrow$ Measure "tail dependence"

Definition 24

Assume $\mathbf{X} = (\mathbf{X_1}, \mathbf{X_2})$ is a bivariate vector with marginal cdfs F_{X_1} and F_{X_2} and with copula C. The coefficient of upper tail dependence of X_1 and X_2 is defined by

$$\lambda_u = \lim_{q \to 1^-} P\left(X_2 > F_{X_2}^{-1}(q) | X_1 > F_{X_1}^{-1}(q)\right)$$

provided the limit $\lambda_u \in [0, 1]$ exists.

Tail dependence II

We are concerned with probability of observing an unusually large loss for one policy given that an unusually large loss has occurred for another policy.
λ_u: "limiting conditional probability that both margins exceed a certain quantile level q given that one margin does"

- $\lambda_u > 0$: extreme events tend to occur simultaneously with probability λ_u (asymptotically dependent in the upper tail).
- $\lambda_u = 0$: extreme events tend to occur essentially independently (asymptotically independent).

$$\begin{aligned}
\lambda_u &= \lim_{q\to 1^-} \frac{P\left(X_2 > F_{X_2}^{-1}(q), X_1 > F_{X_1}^{-1}(q)\right)}{P\left(X_1 > F_{X_1}^{-1}(q)\right)} \\
&= \lim_{q\to 1^-} \frac{P\left(F_{X_2}(X_2) > q, F_{X_1}(X_1) > q\right)}{P\left(F_{X_1}(X_1) > q\right)} \\
&= \lim_{q\to 1^-} \frac{1 - P\left(F_{X_2}(X_2) \le q\right) - P\left(F_{X_1}(X_1) \le q\right) + P\left(F_{X_2}(X_2) \le q, F_{X_1}(X_1) \le q\right)}{1 - P\left(F_{X_1}(X_1) \le q\right)} \\
&= \lim_{q\to 1^-} \frac{1 - 2q + C(q,q)}{1 - q}
\end{aligned}$$

$\Rightarrow$ Tail dependence: pure copula property, independent on the marginals

Tail dependence III

Definition 25

The coefficient of lower tail dependence of X_1 and X_2 is defined by

$$\lambda_\ell = \lim_{q \to 0^+} P\left(X_2 \leq F_{X_2}^{-1}(q) | X_1 \leq F_{X_1}^{-1}(q)\right)$$

provided the limit $\lambda_\ell \in [0, 1]$ exists.

λ_ℓ: "limiting conditional probability that both margins is less than or equal to a certain quantile level q given that one margin does"

$$\begin{aligned} \lambda_\ell &= \lim_{q \to 0^+} \frac{P\left(X_2 \leq F_{X_2}^{-1}(q), X_1 \leq F_{X_1}^{-1}(q)\right)}{P\left(X_1 \leq F_{X_1}^{-1}(q)\right)} \\ &= \lim_{q \to 0^+} \frac{P\left(F_{X_2}(X_2) \leq q, F_{X_1}(X_1) \leq q\right)}{P\left(F_{X_1}(X_1) \leq q\right)} \\ &= \lim_{q \to 0^+} \frac{C(q,q)}{q} \end{aligned}$$

$\Rightarrow$ Tail dependence: pure copula property, independent on the marginals

Examples I

- Independence copula C^{ind}: $\lambda_u = \lambda_\ell = 0$
- Monotonicity copula C^M: $\lambda_u = \lambda_\ell = 1$
- Counter-monotonicity copula C^{CM}: $\lambda_u = \lambda_\ell = 0$
- Gaussian copula C_ρ^{Gauss}: $\lambda_u = \lambda_\ell = 0$ except if $\rho = 1$
- Student copula $C_{\rho,\nu}^{\text{Student}}$: $\lambda_u = \lambda_\ell = \lambda$:

$$\lambda = 2t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right), \quad \text{if } \rho > -1$$

  $\Rightarrow$ upper and lower tail dependence (even for $\rho = 0$).
- Clayton copula: $\lambda_u = 0 \quad \text{and} \quad \lambda_\ell = 2^{-1/\theta}$
- Gumbel copula:$\lambda_u = 2 - 2^{1/\theta} \quad \text{and} \quad \lambda_\ell = 0$
- Frank copula: $\lambda_u = \lambda_\ell = 0$

Examples II

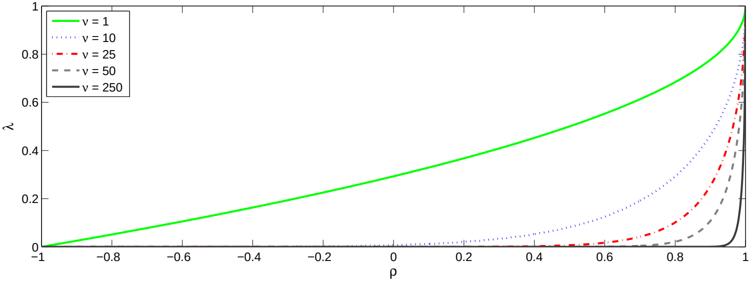


Figure 6: Tail dependence coefficient for $C^{\text{Student}}_{\rho,\nu}$.

Copula specification/calibration I

- Assume $\mathbf{X}^{(i)} = \left(X_1^{(i)}, \ldots, X_N^{(i)}\right), \; i = 1, \ldots, M \equiv$ i.i.d sample.
- Choose adequate copula family (dependence, tail dependence).

Methods to estimate copula C:

- **Maximum likelihood estimate**

$$
\begin{aligned}
L &= \log(f_{\boldsymbol{X}}(x_1, \ldots, x_N)) = \log\left(c\left(F_{X_1}(x_1), \ldots, F_{X_N}(x_N)\right) f_{X_1}(x_1) \ldots f_{X_N}(x_N\right. \\
\Rightarrow L &= L_C + \sum_{i=1}^{N} L_i
\end{aligned}
$$

where

$$
\begin{aligned}
L_C &= \sum_{j=1}^{M} \log[c(F_{X_1}(X_1^{(j)}), \ldots F_{X_N}(X_N^{(j)}))] \\
L_i &= \sum_{j=1}^{M} \log(f_{X_i}(X_i^{(j)})), \quad i = 1, \ldots, N
\end{aligned}
$$

Copula specification/calibration II

Assume the parametric models $F_{X_1}(\cdot\,;\boldsymbol{\theta}_1), \ldots, F_{X_N}(\cdot\,;\boldsymbol{\theta}_N)$ and $C(\cdot\,;\boldsymbol{\theta}_C)$:

$$\begin{aligned} L(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N, \boldsymbol{\theta}_C) = \quad & \sum_{j=1}^{M} \log\left(c(F_{X_1}(X_1^{(j)}; \boldsymbol{\theta}_1), \ldots, F_{X_N}(X_N^{(j)}; \boldsymbol{\theta}_N); \boldsymbol{\theta}_C)\right) \\ & + \sum_{i=1}^{N} \sum_{j=1}^{M} \log\left(f_{X_i}(X_i^{(j)}; \boldsymbol{\theta}_i)\right) \end{aligned}$$

Potential pitfalls with MLE:

1. for $N \gg$, maximizing L might be numerically challenging $\Rightarrow$
   - use of starting value for $(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N, \boldsymbol{\theta}_C)$ close to MLEs
   - use pseudo-maximum likelihood estimates
2. parametric models required for both the marginal cdfs and the copula $\Rightarrow$ if one or several marginal(s) not well fitted by a parametric model, biases in both $(\hat{\boldsymbol{\theta}}_1, \ldots, \hat{\boldsymbol{\theta}}_N)$ and $\hat{\boldsymbol{\theta}}_C$
   $\Rightarrow$ use semi-parametric (pseudo-)maximum likelihood estimates

Copula specification/calibration III

- **Pseudo maximum likelihood estimate (PMLE)**
  2 steps approach:
  1. estimation of the marginal cdf for each margin: $\hat{F}_{X_i}, i = 1, \ldots, N$
  2. estimation of $\boldsymbol{\theta}_C$ by maximizing L_C where the marginal cdfs are replaced by their estimates:

$$\max_{\boldsymbol{\theta}_C} \sum_{j=1}^{M} \log \left(c(\hat{F}_{X_1}(X_1^{(j)}), \ldots, \hat{F}_{X_N}(X_N^{(j)}); \boldsymbol{\theta}_C) \right) \tag{11}$$

$\Rightarrow$ 2 steps avoid high dimensional optimization

We can opt for parametric or non-parametric approach in Step 1:

- **Parametric pseudo-maximum likelihood estimate** estimation of $\boldsymbol{\theta}_i, i = 1, \ldots, N$ by maximizing L_i:

$$\max_{\boldsymbol{\theta}_i} \sum_{j=1}^{M} \log \left(f_{X_i}(X_i^{(j)}; \boldsymbol{\theta}_i) \right) \quad \text{and} \quad \hat{F}_{X_i}(\cdot) = F_{X_i}(\cdot\,; \hat{\boldsymbol{\theta}}_i)$$

Copula specification/calibration IV

- **Semi-parametric pseudo-maximum likelihood estimate** estimate $\hat{F}_{X_i}$ from the empirical cdf of $X_i^{(1)}, \ldots, X_i^{(M)}$:

$$\hat{F}_{X_i}(y) = \frac{\sum_{j=1}^{M} \mathbb{1}_{X_i^{(j)} \leq y}}{M+1}$$

Note: divide by $M+1$ instead of M such that $\hat{F}_{X_i}\left(\max_j \left\{X_i^{(j)}\right\}\right) = \frac{M}{M+1}$ instead of 1. If some of the $u_i = 1$, then usually $c_X(u_1, \ldots, u_N; \boldsymbol{\theta}_C) = \infty$

Definition 26

$\hat{F}_{X_i}(X_i^{(j)}), i = 1, \ldots N$ and $j = 1, \ldots M$ are called uniform-transformed variables since $\hat{F}_{X_i}(X_i^{(j)}) \approx$ Uniform(0,1)

Copula specification/calibration V

Definition 27

The empirical cdf of $\mathbf{X} = (X_1, \ldots, X_N)$ *is defined by*

$$\hat{F}_X(y_1, \ldots, y_N) = \frac{\sum_{j=1}^{M} \mathbb{1}_{X_i^{(j)} \leq y_i; i=1,\ldots,N}}{M}$$

where $\mathbf{X}^{(i)} = \left(X_1^{(i)}, \ldots, X_N^{(i)}\right), i = 1, \ldots, M$ *is a sample of size* M *of N-dimensional random vectors*

Definition 28

The empirical copula is the multivariate empirical cdf of the uniform-transformed variables.

$\Rightarrow$ useful to check goodness of fit of parametric copula models

Gaussian copula calibration

Problem in Step 2: maximizing (11) can be challenging if $\boldsymbol{\theta}_C$ is high-dimensional (e.g. Gaussian, t-copula: $N(N-1)/2$ correlation parameters) $\Rightarrow$

- assume some structure for the correlation (equi-correlation, ...)
- starting value of the correlation matrix inferred from rank correlations (Gaussian or t-copula)

Estimate $(\boldsymbol{\Omega})_{ij}$ by $\rho_S(X_i, X_j)$:

$$\begin{aligned}\rho_\tau(X_i, X_j) &= \frac{2}{\pi}\arcsin(\Omega_{i,j}) \\ \rho_S(X_i, X_j) &= \frac{6}{\pi}\arcsin(\Omega_{i,j}/2) \approx \Omega_{i,j}\end{aligned}$$

- final estimate
- starting value for MLE or PMLE

Student copula calibration I

Estimate $(\boldsymbol{\Omega})_{ij}$ by equating $\rho_\tau(X_i, X_j)$ and $\hat{\rho}_\tau(X_i, X_j)$:

$$(\hat{\boldsymbol{\Omega}}^{\star\star})_{ij} = \sin\left(\frac{\pi}{2}\hat{\rho}_\tau(X_i, X_j)\right)$$

$\Rightarrow \hat{\boldsymbol{\Omega}}^{\star\star}$

- symmetric
- $(\hat{\boldsymbol{\Omega}}^{\star\star})_{ii} = 1$
- might not be positive definite (i.e. all eigenvalues positive)

$\Rightarrow$

- if $\hat{\boldsymbol{\Omega}}^{\star\star}$ positive definite: $\hat{\boldsymbol{\Omega}} = \hat{\boldsymbol{\Omega}}^{\star\star}$

Student copula calibration II

- otherwise:

$$\hat{\mathbf{\Omega}}^{\star\star} = \mathbf{O}\text{diag}(\lambda_\ell)\mathbf{O}^\mathsf{T}$$

where $\mathbf{O}$ is orthogonal and whose columns are eigenvectors of $\hat{\mathbf{\Omega}}^{\star\star}$ and $\lambda_1, \ldots, \lambda_N$ are the eigenvalues

$$\hat{\mathbf{\Omega}}^{\star} = \mathbf{O}\text{diag}(\max(\epsilon, \lambda_\ell))\mathbf{O}^\mathsf{T}, \quad 0 < \epsilon \ll 1$$

$\Rightarrow \hat{\mathbf{\Omega}}^{\star}$

  - symmetric
  - positive definite
  - $(\hat{\mathbf{\Omega}}^{\star})_{ii}$ might not equal 1

$\Rightarrow$ multiply the ith row and the ith column of $\hat{\mathbf{\Omega}}^{\star}$ by $\left((\hat{\mathbf{\Omega}}^{\star})_{ii}\right)^{-1/2}, i = 1, \ldots, N$

$\Rightarrow$ This leads to $\hat{\mathbf{\Omega}}$ satisfying all necessary conditions.

Estimate ν by maximizing L_C (i.e. Eq (11)) where $\mathbf{\Omega}$ is replaced by $\hat{\mathbf{\Omega}}$

Archimedean copula identification I

Use procedure developed by Genest and Rivest (1993).
[Genest C. and Rivest L.-P. Statistical Inference Procedures for Bivariate Archimedean Copulas (1993). *Journal of the American Statistical Association*, 88: 1034-1043]
$\Rightarrow$ Read paper for more information.

- Assume $(X_1^{(1)}, X_2^{(1)}), \ldots, (X_1^{(M)}, X_2^{(M)}) \equiv$ random sample of bivariate observations
- Consider rvs $Z^{(i)} = F_X(X_1^{(i)}, X_2^{(i)})$ with cdf $K(z) = P(Z^{(i)} \leq z)$
- $K(z)$: linked to the generator of an Archimedean copula by:

$$K(z) = z - \frac{\phi(z)}{\phi'(z)} \tag{12}$$

Archimedean copula identification II

Identification of ϕ:

1. Estimate sample Kendall's tau:

$$\hat{\rho}_\tau(X_1, X_2) = \begin{pmatrix} M \\ 2 \end{pmatrix}^{-1} \sum_{1 \leq i \leq j \leq M} \text{sign}\left((X_1^{(i)} - X_1^{(j)})(X_2^{(i)} - X_2^{(j)})\right)$$

2. Non-parametric (sample) estimate of K is obtained as follows
   a) $\forall i = 1, \ldots, M$:
   $\hat{Z}^{(i)} = \frac{1}{M-1} \sum_{j=1}^{M} \mathbb{1}_{\{X_1^{(j)} < X_1^{(i)} \text{ and } X_2^{(j)} < X_2^{(i)}\}}$
   b) $\hat{K}(z) = \frac{1}{M} \sum_{j=1}^{M} \mathbb{1}_{\hat{Z}^{(i)} \leq z}$
3. Parametric (sample) parametric estimate of K:
   Choose ϕ and use $\hat{\rho}_\tau$ to estimate parameter θ of ϕ.

$$\hat{\rho}_\tau(X_1, X_2) \Rightarrow \hat{\theta} \text{ (from Table 1) } \Rightarrow \hat{\phi}(t) \Rightarrow \hat{K}_\phi(z) \text{ (from (12))}$$

   Repeat this step for several choices of ϕ.

Archimedean copula identification III

4. Select ϕ such that parametric estimate $\hat{K}_\phi(z)$ *closest* to non-parametric estimate $\hat{K}(z)$.
   - $\min_\phi \int \left(\hat{K}_\phi(z) - \hat{K}(z)\right)^2 d\hat{K}(z)$ OR
   - graphically: QQ plot can be used to see which generator ϕ best fits the data.

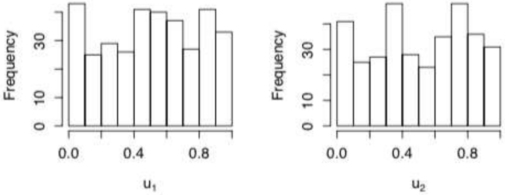
Table 1 again

Family	Kendall's tau	Spearman's rho
Independence	0	0
Clayton	$\frac{\theta}{\theta+2}$	Complicated
Gumbel	$1-\frac{1}{\theta}$	No closed-form
Frank	$1-\frac{4}{\theta}(D_1(-\theta)-1)$	$1-\frac{12}{\theta}(D_2(-\theta)-D_1(-\theta))$

Example: flows in pipeline I

We consider again the analysis of (first 2) pipeline flows [Ruppert].
Fully parametric pseudo-likelihood analysis

- univariate skewed t-model for flows 1 and 2 (use package sn of Azzalini).
- $U_{1,j}, \ldots, U_{n,j}$ flows in pipeline $j = 1, 2$ uniform-transformed by their estimated skewed-t CDF.



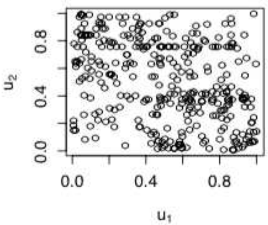
Histograms of both samples of uniform-transformed flows show some deviations from uniform distribution.
$\Rightarrow$ Skewed t may not be excellent fit (or deviations due to random variation).

Example: flows in pipeline II

We consider again the analysis of (first 2) pipeline flows.

Fully parametric pseudo-likelihood analysis

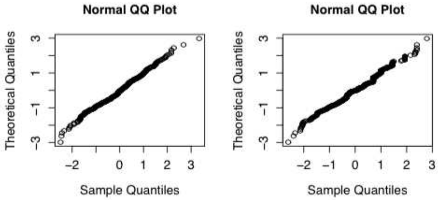
- univariate skewed t-model for flows 1 and 2 (use package sn of Azzalini).
- $U_{1,j}, \ldots, U_{n,j}$ flows in pipeline $j = 1, 2$ uniform-transformed by their estimated skewed-t CDF.



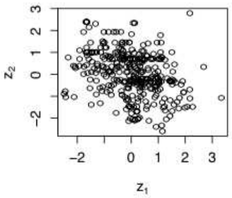
Scatterplot of uniform-transformed flows shows negative correlation
$\Rightarrow$ Gumbel will not fit well.
Scatterplot shows data in region where u_1 and u_2 have small values
$\Rightarrow$ this region excluded by Clayton with negative dependence.

Example: flows in pipeline III

- $Z_{i,j} = \Phi^{-1}(U_{i,j})$ should be approximately $N(0,1)$



Approximately linear normal plots.



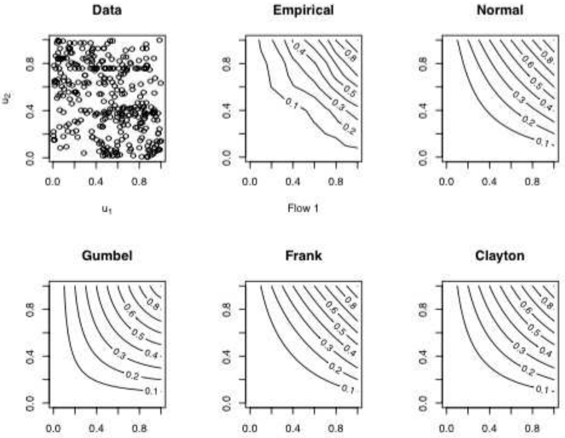
Scatterplot shows negative correlation.

Example: flows in pipeline IV

- Assume that flows have meta-Gaussian distribution
  - Spearman's rho is -0.357
  - $sin(\pi\hat{\tau}/2) = -0.372$ where $\hat{\tau}$ is sample Kendall's tau.
  - Pearson's correlation of normal-transformed flows is -0.335.

  Reasonably close agreement among 3 values.
- Five parametric copulas were fit to uniform-transformed flows
  - Student
  - Gaussian
  - Gumbel
  - Frank
  - Clayton
- Transforming flows by their empirical CDFs yields semiparametric pseudo maximum-likelihood estimates.

Example: flows in pipeline V



t copula similar to Gaussian since $\hat{\nu}$ is large.

Example: flows in pipeline VI

Copula family	Estimates	Maximized log-likelihood	AIC
t	$\widehat{\rho} = -0.34$ $\widehat{\nu} = 22.3$	21.0	−38.0
Gaussian	$\widehat{\rho} = -0.331$	20.4	−38.8
Gumbel	$\widehat{\theta} = 0.988$	1.06	−0.06
Frank	$\widehat{\theta} = -2.25$	23.1	−44.1
Clayton	$\widehat{\theta} = -0.167$	9.87	−17.7

Frank copula fits best

- minimizes AIC
- contours are closest to those of empirical copula

Exercise I

Exercise 6

Install and load the R packages `copula`, `fCopulae`, `Ecdat`, `fGarch` and `MASS`. Run and study the following R code.

```
cop_t_dim3 = tCopula(c(-.6,.75,0), dim = 3, dispstr = "un", df = 1)
set.seed(5640)
rand_t_cop = rCopula(500,cop_t_dim3)
```

1. What type of copula has been sampled? (Give the copula family, the correlation matrix, and any other parameters that specify the copula.)
2. What is the sample size?
3. Create a scatterplot matrix of the sample and print its sample correlation.
4. Var 2 and Var 3 are uncorrelated. Do they seem independent? Why or why not?
5. Do you see signs of tail dependence? If so, where?

Exercise II

6. What are the effects of correlation upon the plots?
7. The nonzero correlations in the copula do not have the same values as the corresponding sample correlations. Do you think this is just due to random variation or is something else going on?

Run and study the following R code

```
cop_normal_dim3 = normalCopula(c(-.6,.75,0), dim = 3, dispstr = "un'
mvdc_normal <- mvdc(cop_normal_dim3, c("exp", "exp","exp"),
list(list(rate=2), list(rate = 3), list(rate=4)) )
set.seed(5640)
rand_mvdc = rMvdc(1000,mvdc_normal)
pairs(rand_mvdc)
par(mfrow=c(2,2))
plot(density(rand_mvdc[,1]))
plot(density(rand_mvdc[,2]))
plot(density(rand_mvdc[,3]))
```

Exercise III

8. What are the marginal distributions of the three variables in `rand_mvdc`? What are their expected values?
9. Are the second and third variables independent? Why or why not?