Interest Rate Derivatives

M.Sc. Course in Quantitative Finance
School of Economics, Management and Statistics
University of Bologna, Sept. 2017
Septermber 2018

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Acknowledgments

M.B. gratefully acknowledges fruitful interactions with many colleagues at international conferences, Intesa Sanpaolo and Banca IMI.A particular mention goes to M. Carlicchi as co-author of some market data elaborations, G. Abbate and C. Maestri for the bond pricing slides, F. Ametrano and the QuantLib community for the open source software used for yield curve bootstrapping in sec. 6, N. Moreni for volatility bootstrapping in sec.10, M. Morini for the constant stimulating discussions, M. Scaringi for collaboration on sec. 6, M. Terraneo for collaboration on section 9, 10, 11.

How (not) to use these slides

These slides have been gradually developed over the years, update after update, following the author's personal experience, both on the job and in many courses with different audiences. Sometimes, something either new and interesting happens on the market, or is learnt by the author, or is required by students, and the slides are updated.

This is the reason why there are so many slides. Don' t panic. Typically only part of the slides will be used for a specific course. Why don't cut the presentation including only the slides to be used for each specific course? Because they could be useful during the lessons, according to questions from the students. Furthermore, the students could benefit later from slides dropped during the lessons, for consultation purposes.

Not all the slides are self-explaining (this would add more and more slides…), they need to be commented by the presenter. Thus the best way to use these slides is to follow the lessons and take notices directly on the slides. Using the slides without following the lessons is difficult, time consuming and inefficient.

Suggestions and corrections regarding these slides would be highly appreciated.

Summary

1: Introduction
Classical vs modern pricing framework [1]

The financial crisis begun in the second half of 2007 has triggered, among many consequences, a deep evolution phase of the classical framework adopted for trading derivatives.

Credit and liquidity issues, were found to have macroscopical impacts on financial instruments, both plain vanillas and exotics. In particular, the modern derivatives' market is highly sensible to the cost of funding on the central Bank and interbank market, and the effect of possible collateral agreements widely diffused among bank and central counterparties.

As a consequence, the classical theoretical framework adopted to price derivatives has become obsolete. Well-known relations described on standard textbooks and holding since decades had to be abandoned in one day.

The modern theoretical framework, still under active research and development, includes a larger set of market information and of relevant risk factors, credit and funding risk in particular, and requires to review "from scratch" the no-arbitrage models used on the market for derivatives' pricing and risk analysis.

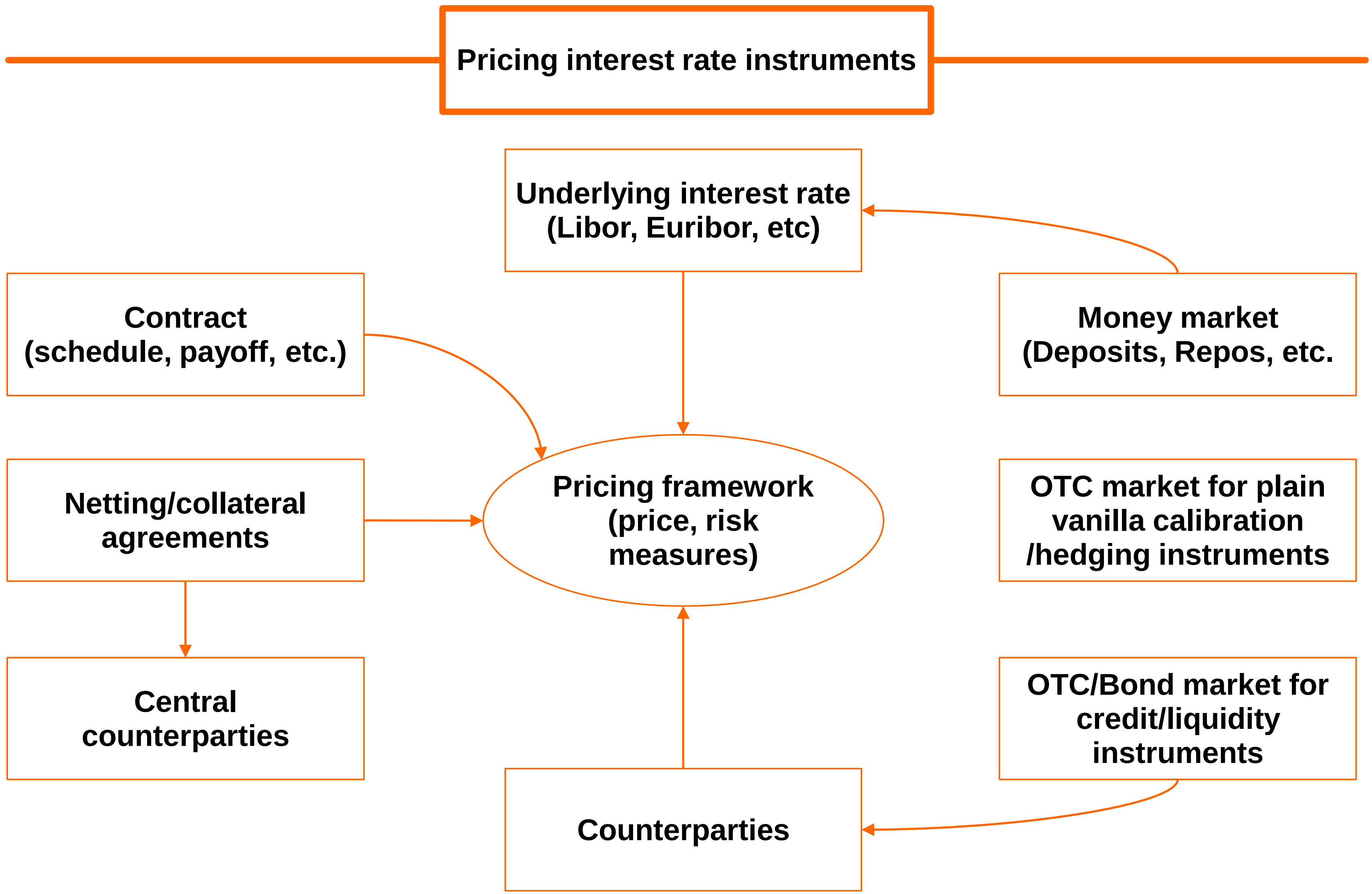
PS: notice the similarity with the transition from classical to modern (relativistic quantum) physics.

1: Introduction
Classical vs modern pricing framework [2]

In order to understand and to price interest rate linked instruments we must know their characteristics.

- The underlying: the interest rates, such as ECB rates, Libor, Euribor, Eonia, etc.
- The money market where such interest rates are traded and the basic lending/borrowing contracts, such as Deposits.
- The contract mechanics: the schedule with all the contract relevant dates, the payoff, and any other condition affecting the price.
- The counterparties: leading to credit/default issues, and to the corresponding credit/debt market (CDS and Bonds).
- The collateral: leading to liquidity/funding issues and to Central Counterparties
- The OTC market where the basic plain vanilla derivatives, are traded and used for yield curve and volatility construction, calibration and hedging purposes.
- The pricing model, to calculate prices and risk measures (sensitivities, VaR, etc.)

1: Introduction
Classical vs modern pricing framework [3]



1: Introduction
Where is the garbage ?

The gossip corner

In April 2011, during the opening plenary panel at an international conference, a (famous) quant compared the ongoing transition from classical to modern market - characterized by multiple yield curves including funding and counterparty risk - to a boring problem, such as "*carrying the garbage out of the door when it's raining*".

In the conference session 1 hour later, dedicated to "The Latest Practical Techniques In Funding, Discounting, Liquidity & CVA", another (less famous) quant commented that "*boring or not, we have to care about such issues, otherwise it's the garbage that enters the door*".

1: Introduction
Textbooks on interest rate modelling

- Brigo, D., Mercurio F. (2006), “*Interest Rate Models: Theory and Practice*”, Springer.
- Andersen, L.B.G, Piterbarg, V. V. (2010) “*Interest rate modelling*”, vol. 1,2,3, Atlantic Financial Press.
- Brigo, D., Morini, M., Pallavicini A. (2013), “*Counterparty Credit Risk, Collateral and Funding with Pricing Cases for All Asset Classes*”, Wiley.
- Henrard, M. (2014), “*Interest Rate Modelling in the Multi-Curve Framework*”, Palgrave Macmillan.
- Kienitz, J. (2014), “*Interest Rate Derivatives Explained: Volume 1: Products and Markets*”, Palgrave Macmillan.
- Björk, T. (2009), “*Arbitrage Theory in Continuous Time*”, Oxford University Press.

Summary

2. **Interest rate basic concepts**
   - Dimensions and units
   - Interest rate definitions
   - Interest rate conventions
   - Types of interest rates

2: Interest rate basic concepts
Dimensions and units [1]

In finance, as in any other scientific discipline dealing with measurable quantities, financial variables may have a dimension and can be measured using different units of measurement.

Also in finance there are:

- base and derived quantities
- different units and conversions
- calculations with units of measurement
- dimensional analysis

Dimensional analysis ("a sophisticated name for not mixing apples and oranges", C. G. Johnson, "*Dimensional Analysis and the Interpretation of Regression Coefficient*", Journal of Financial and Quantitative Analysis, pp. 75-99, Jan. 1972), is a powerful tool normally used in science to interpret mathematical expression regardless of the units used to measure them and to help the mathematical modelling of real systems.

In particular, dimensional analysis is useful when one

- defines new quantities,
- checks the formal correctness of mathematical expressions,
- **develops a piece of software to compute something**.

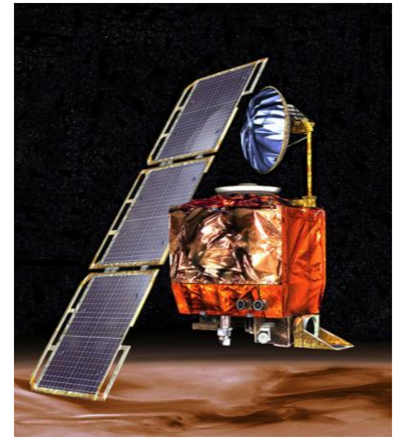
2: Interest rate basic concepts
Dimensions and units [2]

System of quantities and units in physics			
Base quantity	**Base quantity symbol**	**Base unit**	**Base unit symbol**
Length	L	meter	m
Mass	M	kilogram	kg
Time	T	second	s
Electric current	I	ampere	A
Temperature	Θ	kelvin	K
Amount of substance	N	mole	mol
Luminous intensity	J	candela	cd

System of quantities and units in finance			
Base quantity	**Base quantity symbol**	**Base unit**	**Unit symbol**
Time	T	year	y
Currency	C	Euro (in EU)	EUR

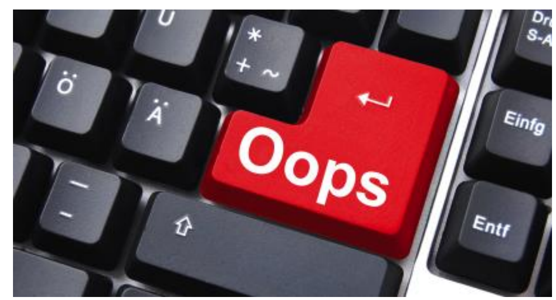
2: Interest rate basic concepts
Dimensions and units [3]

Examples from real life



- **Mars Climate Orbiter**: on September 23, 1999, communication with the spacecraft was lost as the spacecraft went into orbital insertion, due to ground-based computer software which produced output in units of pound x seconds instead of the SI units of newton x seconds. The spacecraft encountered Mars on a trajectory that brought it too close to the planet, causing it to pass through the upper atmosphere and disintegrate. The cost of the mission was $327.6 million (link).

- "**Gimli Glider**": on July 23, 1983, Air Canada Flight 143 ran out of fuel and landed at an emergency auto racing track in Gimli, Manitoba. The amount of fuel that had been loaded was miscalculated because of a confusion as to the calculation of the weight of fuel using the metric system, which had recently replaced the imperial system for use with the Boeing 767. (link).



- "**Fat fingers**": in June 2015, a junior employee at Deutsche Bank confused gross and net amounts while processing a trade, causing a payment to a US hedge fund of $6bn, orders of magnitude higher than the correct amount. The bank reported the error to the British Financial Conduct Authority, the European Central Bank and the US Federal Reserve Bank, and retrieved the money on the following day (link).

2: Interest rate basic concepts
Dimensions and units [4]

- **Example 1: interest rate**
  the interest accrued on an initial nominal $N(T_1)$ (dimension = c, units = *EUR*) over a time interval $[T_1,T_2]$ (dimension = t, units = y) is given by

$$N(T_2) = N(T_1)\left[1 + R(T_1,T_2)\tau(T_1,T_2)\right]$$

  where $R(T_1,T_2)\tau(T_1,T_2)$ is adimensional and both sides have the same dimension. Thus, the interest rate has dimension = t^{-1}, units = y^{-1}.

- **Example 2: brownian motion**
  $W(t)$ = sequence of random variables for $t>0$ is a brownian motion if

$$\begin{aligned}
&W(0) = 0,\\
&W(t_2) - W(t_1) \sim \mathcal{N}(0, t_2 - t_1) = \mathcal{N}(0,1)\sqrt{t_2 - t_1},\ \forall\, 0 < t_1 < t_2,\\
&W(t_2) - W(t_1) \text{ independent of } W(t),\ \forall\, 0 < t < t_1.
\end{aligned}$$

  where t_2-t_1 is the variance and the last requirement is the Markov property.
  The standard normal distribution $N(0,1)$ is an adimensional quantity, thus the Brownian step has dimension = $t^{1/2}$, units = $y^{1/2}$.
  The same applies to the infinitesimal Brownian step

$$dW(t) := W(t + dt) - W(t) \sim \mathcal{N}(0,1)\sqrt{dt}.$$

2: Interest rate basic concepts
Dimensions and units [5]

- **Example 3: lognormal dynamics**: for interest rate *R(t)* we assume the dynamics

Ratio, adimensional

$$\frac{dR(t)}{R(t)} = \sigma dW(t)$$

Brownian motion dimension = $T^{1/2}$

Lognormal volatility dimension = $T^{-1/2}$

$$1 = T^{-\frac{1}{2}} \times T^{\frac{1}{2}}$$

$$\sigma dW(t) \sim \mathcal{N}(0, \sigma^2 dt) = \mathcal{N}(0,1)\sigma\sqrt{dt}$$

- **Example 4: normal dynamics**: for interest rate *R(t)* we assume the dynamics

Rate, dimension = T^{-1}

$$dR(t) = \sigma dW(t)$$

Brownian motion dimension = $T^{1/2}$

Normal volatility dimension = $t^{-3/2}$

$$T^{-1} = T^{-\frac{3}{2}} \times T^{\frac{1}{2}}$$

- **Example 5: exchange rate**
  given an amount $N_f(t)$ of foreign currency at time t, we obtain the equivalent amount of domestic currency $N_d(t)$ at the same time t using the foreign exchange rate $x_{fd}(t)$ at the same time t as

Domestic currency units = EUR

$$N_d(t) = x_{fd}(t) N_f(t)$$

Foreign currency units = *USD*

Exchange rate dimension = *1*, units = *EUR/USD*

Same dimensions, different units.

2: Interest rate basic concepts
Dimensions and units [7]

- **Example 5: sensitivities**
  Given the present value $V(t, r, \sigma, \ldots)$ of some financial contract depending e.g. on time *t*, interest rate *r* and volatility σ at time *t*, we obtain the following sensitivities

Delta sensitivity	$\dfrac{\partial V}{\partial r}(t, r, \sigma)$	Currency/rate Dimension = *CT*
Lognormal vega sensitivity	$\dfrac{\partial V}{\partial \sigma}(t, r, \sigma)$	Currency/vola Dimension = $CT^{1/2}$

- **Example 6: Taylor expansion**
  Given the present value $V(t, r, \sigma, \ldots)$ of some financial contract depending e.g. on time *t*, interest rate *r* and volatility σ at time *t*, we obtain the following Taylor expansion

$$V(t, r, \sigma) = V(t, r_0, \sigma_0) + \frac{\partial V}{\partial r}(t, r_0, \sigma_0)(r - r_0) + \frac{\partial V}{\partial \sigma}(t, r_0, \sigma_0)(\sigma - \sigma_0) + \cdots$$

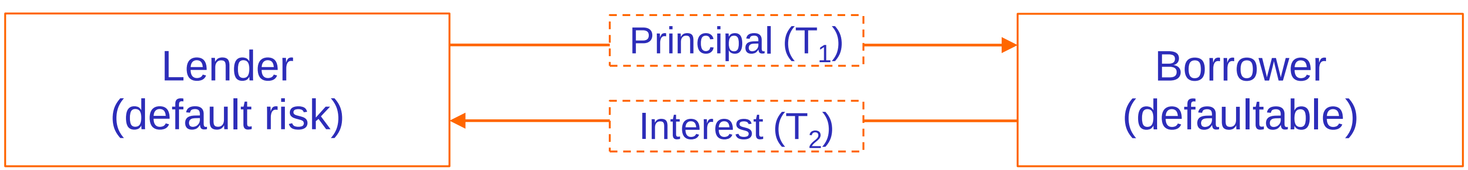
2: Interest rate basic concepts
Interest rate: definitions [1]

- Interest is a fee paid on borrowed money or asset. When a borrower B and a lender L agree to exchange at time T_1 a given amount of money N, called the principal, or, equivalently, an asset of value $N(T_1)$, for a given time interval $[T_1,T_2]$, with $T_1 < T_2$, they also agree for the compensation to be payed by the borrower to the lender for the loan at time T_2 (in the most natural case).

- Such compensation corresponds to the value of other equivalent investments that could have been made by the lender, in the same time interval $[T_1,T_2]$, with the loaned money or asset.

- On one side, the borrower enjoys the benefit of using the money or the asset avoiding the effort required to earn them, while on the other side the lender enjoys the benefit of the fee paid by the borrower for the loan.

- The compensation also ensures the lender for the risk of losing the money or the asset, in case that the borrower defaults before time T_2, called credit risk.



2: Interest rate basic concepts
Interest rate: definitions [2]

- The compensation is normally established in terms of an interest rate $R(T_1,T_2)$ to be applied to the initial invested amount $N(T_1)$ over the loan time length, such that the borrower will return to the lender at time T_2 the amount

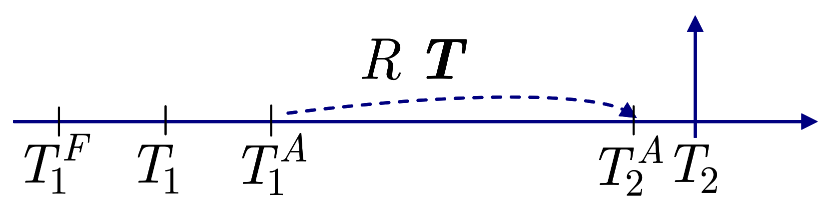
$$N(T_2) = N(T_1)\left[1 + R(T_1,T_2)\tau(T_1,T_2)\right],$$

where we have used the simple compounding rule for the rate, and $\tau(T_1,T_2)$ is the year fraction corresponding to the loan time length $[T_1,T_2]$.

- In general, the rate is observed at the beginning of the loan at some fixing date $T^F_1 \leq T_1$. The time interval $[T^F_1,T_1]$ is called settlement lag. Furthermore, the rate accrual period $[T^A_1,T^A_2]$ used to compute the interest may be different from the loan time length $[T_1,T_2]$

$$N(T_2) = N(T_1)\left[1 + R(\boldsymbol{T})\tau(T_1^A,T_2^A)\right],$$
$$\boldsymbol{T} := \left[T_1^F, T_1, T_1^A, T_2^A, T_2\right],$$

where we have indicated with a bold vector **T** the schedule of the loan contract, gathering all the relevant contract dates.



2: Interest rate basic concepts

Interest rate conventions

Interest rates practical usage implies the knowledge and careful application of a remarkable set of conventions and rules, in particular:

- date and year fraction,
- business calendar,
- date rolling,
- day count,
- rate compounding,
- rate fixing.

Despite the boring matter, these conventions are very important for the correct calculation of cash flows of interest rate derivatives, as one can read in a transaction termsheet. The devil is in the details.

See e.g. "*Interest Rate Instruments and Market Conventions Guide*", OpenGamma, http://docs.opengamma.com/display/DOC/Quantitative+Research

2: Interest rate basic concepts

Interest rate conventions: date and year fraction

- A date, or time instant, is commonly expressed in year/month/day format. In some particular cases, for example when vanishing time distances must be computed as limit cases, a date may be expressed in year/month/day/hour/minute/second format.

- The year fraction is the conventional time distance between two dates, $T_1=\{y_1,m_1,d_1\}$, and $T_2=\{y_1,m_1,d_1\}$ and is denoted with

$$\tau(T_1, T_2, \text{date rolling}, \text{day count}, \text{calendar}).$$

  The numerical value assumed by the year fraction above depends on the particular date rolling, day count and calendar conventions adopted (see the corresponding paragraphs below). In the following we will omit these additional dependencies in order to keep the notation lighter.

  Sometimes the special case of the year fraction between a maturity and a preceding date is called time to maturity.

  The dimension of year fraction is time (t) and the standard units are years (y).

Interest rate conventions: business calendar

The calendar convention refers to the choice of the calendar defining market business days and holidays. In general, we may have the following cases:

- Local exchanges adopt the local business calendars.

- OTC transactions may adopt a customised combination of calendars suited for market data settlements for counterparties possibly located in different nations.

- A standard in the Euro area is the TARGET calendar, defined as the business calendar of the Trans-European Automated Real-time Gross settlement Express Transfer system, used by EU countries and ECB (European Central Bank). It includes, other than Saturdays and Sundays, six additional holidays :
  - New Year's Day: 1 January
  - Good Friday: e.g. Friday 18 April 2014
  - Easter Monday: e.g. Monday 21 April 2014
  - Labour Day: 1 May
  - Christmas Day: 25 December
  - Christmas Holiday: 26 December

  Notice that some holidays are fixed and some are floating.

  See *http://www.ecb.europa.eu/home/html/holidays.en.html* for other info.

2: Interest rate basic concepts
Interest rate conventions: date rolling

The date rolling convention is the rule adopted to accommodate holidays. If a date T falls on a holiday, it is rolled forward or backward in time such that it falls in a business day, according with a given business calendar. The most common date rolling conventions used in financial markets are:

1. Actual: the date is not rolled, even if it is a non-business day.
2. Following: the date is rolled forward to the next business day.
3. Modified Following: as before, except if the new date falls in the following month, the date is rolled backward to the previous business day. This convention is designed for end of month accounting.
4. Preceding: the date is rolled backward to the previous business day. This convention is designed e.g. for loans to be settled before a specific date.
5. Modified Preceding: as before, except if the new date falls in the previous month, the date is rolled forward to the next business day..
6. End of Month: if the start date is the last working day in a given month, also the end date must be the last working day of the ending month too. This convention is sometimes applied to deposits with maturity larger than or equal to one month.

2: Interest rate basic concepts
Interest rate conventions: day count [1]

The day count convention is the rule adopted to compute the number of days between two dates T_1 and T_2. This value depends, in facts, on the conventional time lengths adopted for months and years. There are many different day count conventions commonly used on the market, developed over time to address different requirements, such as ease of calculation, constancy of time period, etc. Their number has been reduced after the introduction of the Euro. There is no central authority defining day count conventions. ISDA (International Swaps and Derivatives Association, `www.isda.org`) and ICMA (International Capital Markets Association, `www.icmagroup.org`) are the main reference sources.
There are two main classes of day count conventions.

1. 30/360 methods: all the conventions in this class calculate the year fraction between two dates T_1 and T_2 considering 30-days months and 360-days years, such that

$$\tau(T_1, T_2) = \frac{360(y_2 - y_1) + 30(m_2 - m_1) + (d_2 - d_1)}{360}$$

and are distinguished by each other by the rules adopted to adjust dates T_1 and/or T_2 for the end of the month. For instance, the "30E/360", or "Eurobond basis", day count convention is defined by the rule: "*if d_1 or d_2 is equal to 31, then set d_1 or d_2 equal to 30”*, while the "30/360 US", or "bond basis" is more complicated.

2: Interest rate basic concepts
Interest rate conventions: day count [2]

2. Actual methods: all the conventions in this class calculate the year fraction between two dates T_1 and T_2 considering the simple date difference at the numerator, according to the Julian calendar, and are distinguished by the choice of the conventional year's length at the denominator.
   For instance:
   - Actual/365 Fixed: $\tau(T_1, T_2) = \dfrac{\text{Days}(T_1, T_2)}{365}$
   - Actual/360: $\tau(T_1, T_2) = \dfrac{\text{Days}(T_1, T_2)}{360}$

   Notice that actual methods respect the additivity of time intervals, such that

$$\tau(T_1, T_2) + \tau(T_2, T_3) = \tau(T_1, T_3),$$

$$\int_{T_1}^{T_3} f(x)dx = \int_{T_1}^{T_2} f(x)dx + \int_{T_2}^{T_3} f(x)dx, \ \forall\, T_1 \leq T_2 \leq T_3.$$

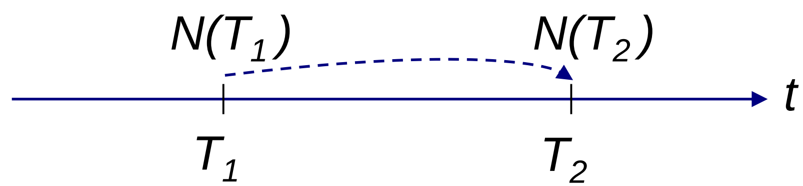
3. For time distances smaller than 1 day we can be compute the year fraction using the simplified expression $\tau(T_1,T_2) = T_2\text{-}T_1$, where T_1 and T_2 are real numbers associated to the two time instants.

See also Wikipedia https://en.wikipedia.org/wiki/Day_count_convention

2: Interest rate basic concepts

Interest rate conventions: compounding rules [1]

The compounding convention is the rule adopted to compute the interest and to add it to the principal, such that, from that moment on, the principal is increased and the next interest will be calculated onto the new principal. In finance there are different interest rate compounding rules that differs in the compounding frequency, that is, the frequency at which the interest is calculated and the principal increased. We focus here on spot interest rates observed at time T_1, spanning the time interval $[T_1;T_2]$, often called rate tenor, and applied to an initial nominal amount (principal) $N(T_1)$.



1. Discrete compounding: the discretely compounded interest rate with compounding frequency k (= number of compositions per year) is the constant annual rate R_k such that

$$\boxed{N(T_2) = N(T_1)\left[1 + \frac{R_k(T_1, T_2)}{k}\right]^{k\tau(T_1,T_2)}}$$

Notice that dimensions are correct because k is a frequency with dimension t^{-1}. This rule implies calculation and reinvestment of the interest amount k times per year. The day count convention often associated to these rates is Actual/365 Fixed.

2: Interest rate basic concepts
Interest rate conventions: compounding rules [2]

2. Annual compounding: the annually compounded interest rate is the constant annual rate with discrete compounding frequency *k=1*,

$$\boxed{N(T_2) = N(T_1)\left[1 + R_1(T_1, T_2)\right]^{\tau(T_1,T_2)}}$$

   Notice that dimensions are correct if one remembers that there is an hidden frequency factor *k=1* with dimension t^{-1}. Hence, this compounding rule implies calculation and reinvestment of the interest amount one single time per year. Day count as above.

3. Continuous compounding: the continuously compounded interest rate is the constant annual rate obtained from discrete compounding frequency k-> ∞

$$\boxed{N(T_2) = N(T_1)\left[1 + \frac{R_k(T_1, T_2)}{k}\right]^{k\tau(T_1,T_2)} \underset{k\to+\infty}{\longrightarrow} N(T_1)e^{R_\infty(T_1,T_2)\tau(T_1,T_2)}.}$$

   Hence, this compounding rule allows continuous calculation and reinvestment of the interest. In practice, the highest frequency for interest amount calculation is daily, but the difference is negligible. Day count as above.

2: Interest rate basic concepts
Interest rate conventions: compounding rules [3]

4. Simple compounding: the simply compounded interest rate is the constant annual rate defined such that

$$\boxed{N(T_2) := N(T_1)\left[1 + L(T_1,T_2)\tau(T_1,T_2)\right],}$$

where "L" stands for "Libor", the most famous simple compounded rate. It can be obtained as first order expansion of all the previous rules for small interest rates,

$$\begin{aligned} N(T_2) =& N(T_1)\left[1 + \frac{R_k(T_1,T_2)}{k}\right]^{k\tau(T_1,T_2)} \\ \underset{R_k\to 0^+}{\simeq}& N(T_1)\left[1 + R_1(T_1,T_2)\right]^{\tau(T_1,T_2)} \\ \underset{R_1\to 0^+}{\simeq}& N(T_1)e^{R_\infty(T_1,T_2)\tau(T_1,T_2)} \\ \underset{R_\infty\to 0^+}{\simeq}& N(T_1)\left[1 + L(T_1,T_2)\tau(T_1,T_2)\right]. \end{aligned}$$

The day count convention typically associated to these rates is Actual/360.

2: Interest rate basic concepts
Interest rate conventions: rate fixing rules

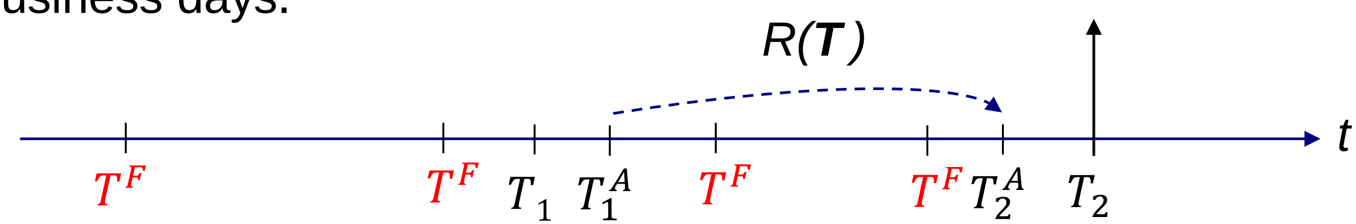
The rate fixing (or reset) convention is the rule adopted to compute the date T^F at which it is observed (fixed, or reset) the rate $R\left(T^F, T_1^A, T_2^A\right)$ used to compute the interest accrued over an accrual period $[T_1^A, T_2^A]$. There are essentially four rate fixing conventions, depending on the position of the rate fixing date T^F with respect to the interest accrual dates T_1^A, T_2^A.

1. Anticipated: the rate is fixed before the beginning of the accrual period, $T^F < T_1^A$.

2. In advance: the rate is fixed close to the beginning of the accrual period, $T^F \lesssim T_1^A$. The most diffused choices are $T^F = T_1^A$ or $T_1^A - T^F = 2$ business days. This latter minor time distance is also called settlement lag.

3. Retarded: the rate is fixed during the accrual period, $T_1^A < T^F < T_2^A$.

4. In arrears: the rate is fixed at the end of the accrual period, $T^F \lesssim T_2^A$, typically such that $T_2^A - T^F = 2$ business days.

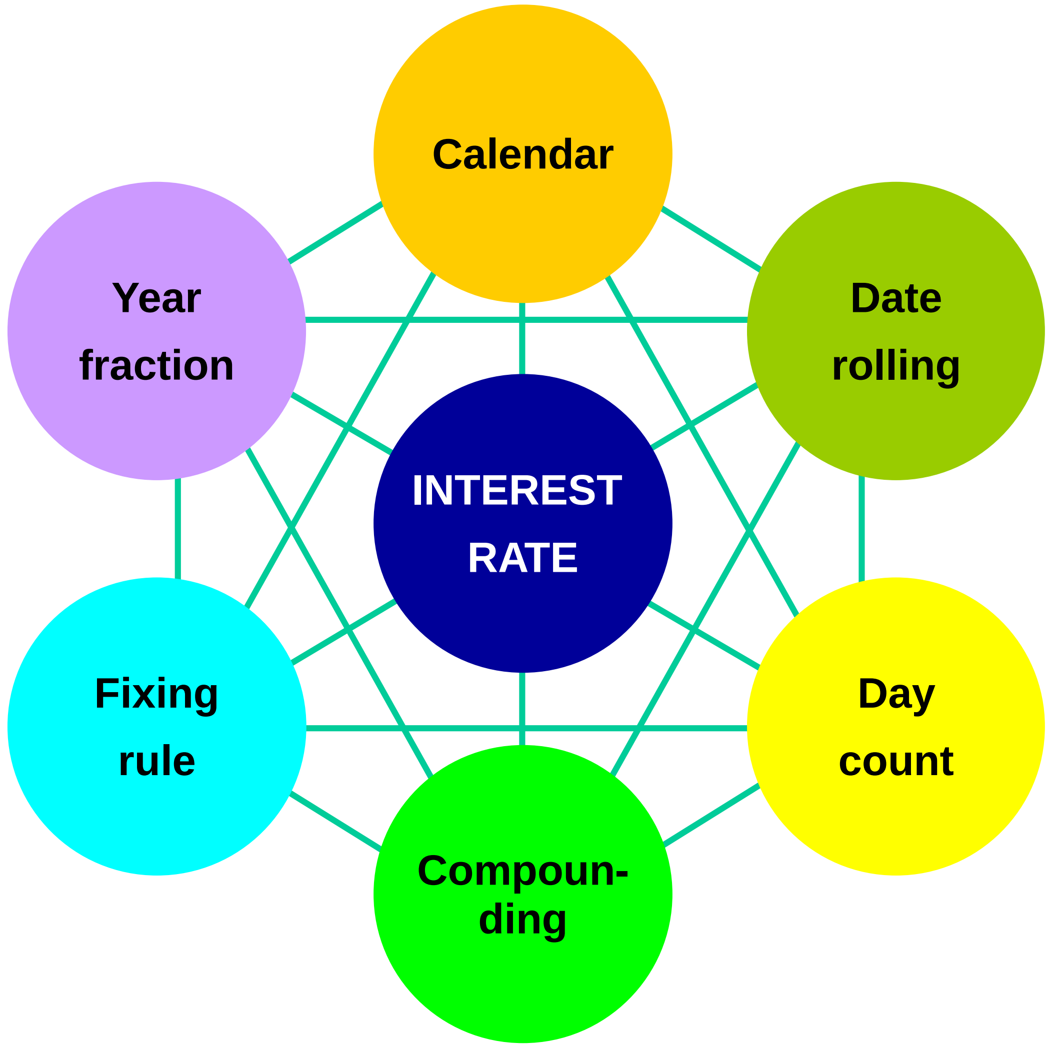


2: Interest rate basic concepts
Interest rate conventions: resumè

Interest rate conventions		
Convention	Definition	Examples
Calendar	Calendar adopted to define market business days and holidays	o Target Calendar o Local calendar o Combination of calendars
Date rolling	Rule adopted to accommodate holidays	o Actual o Preceding o Following o Modified Following o End of Month
Day count	Rule adopted to compute the year fraction between two dates	o 30/360 methods o Actual methods o T_2-T_1 (less than 1 day)
Compounding	Rule adopted to compute the interest amount	o Discrete (annual) o Continuous o Simple
Fixing	Rule adopted to compute the fixing date w.r.t. the accrual period	o Anticipated o In advance o Retarded o In arrears

2: Interest rate basic concepts
Interest rate conventions: resumè [2]

Interest rate: not a very simple quantity…



2: Interest rate basic concepts
Types of interest rates

There are different types of interest rates. A first main distinction can be made between fixed and floating rates. A second main distinction follows from the particular financial instrument linked to the interest rate, e.g. Deposits, Bonds, Swaps, etc.

- Fixed and floating interest rates
- Spot and forward interest rates
- Lending and borrowing rates
  - Deposit rates
  - Repo rates
  - Bank rates
  - Prime rates
- Bond rates
- Future/FRA/Swap rates
- Funding rates
- Risk free rates

2: Interest rate basic concepts
Problems

- Compounding: fix an interest rate and plot the accrued interest from 1d to 50Y for different compounding rules. Try different fixed rates. Deliverable: spreadsheet with data, results, charts and comments.

- Schedule: compute the schedule for a strip of floating interest rate payments with spot starting date, maturity 30Y and frequency 6M. Separate fixing, accrual and payment dates. Assume TARGET calendar, 2 business days settlement lag, modified following date rolling convention. Compute the relevant year fractions using different day count conventions. Deliverable: spreadsheet with data, results and comments.

Summary

3. **The market across the credit crunch**
   - Deposits
   - The money market
   - Libor/Euribor/Eonia/Repo interest rates
   - How the market changed: stylized facts and overview of market data
   - Symmetry breaking and market segmentation after the credit crunch
   - Credit and liquidity components
   - Counterparty risk and collateral
   - From Libor to OIS discounting

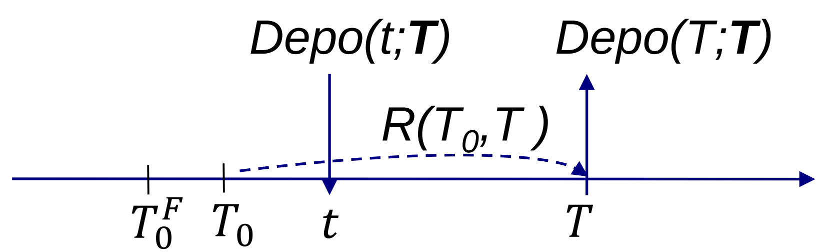
3: The market across the credit crunch
The basic lending/borrowing contracts: Deposits [1]

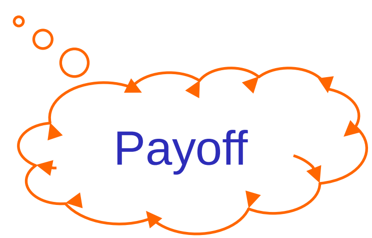
Interest Rate Certificates of Deposits are standard OTC zero coupon contracts such that:

- at start date T_0, counterparty A, called the Lender, pays a nominal amount N to counterparty B, called the Borrower,
- at maturity date T the Borrower pays back to the Lender the nominal amount N plus the interest accrued over the period $[T_0, T]$ (called rate tenor) at the annual simply compounded interest rate $R(T_0, T)$, fixed at time $T_0^F < T_0$, where $[T_0^F, T_0]$ is the settlement period, usually equal to two working days in the EUR market.

The payoff at maturity, from the point of view of the Lender (the receiver of the nominal amount plus interests), is given by

$$\mathbf{Depo}(T;\mathbf{T}) = N\left[1 + R(T_0, T)\tau(T_0, T)\right],$$





3: The market across the credit crunch
The basic lending/borrowing contracts: Deposits [2]

The price at time t, such that $T_0 \le t \le T$, when the deposit rate $R(T_0, T)$ is already fixed, is given by:

- the future cash flows (the nominal amount and the interest amount), which, in this case, are deterministic and happen at the same cash flow date T,
- each discounted at the pricing date $t \le T$

In formulas:

$$\boxed{\mathbf{Depo}(t;\mathbf{T}) = NP(t;T)\left[1 + R(T_0,T)\tau(T_0,T)\right],}$$

where $P(t;T)$ is a discount factor taking into account the time value of money. In case of Deposits, the discount factor is typically consistent with the deposit rate $R(t,T)$ quoted on the market at time t, such that

$$P(t;T) = \frac{1}{1 + R(t,T)\tau(t,T)}, \quad T_0 < t < T$$

Money market quotation at time t

This price is the same for all counterparties…

3: The market across the credit crunch
The basic lending/borrowing contracts: Deposits [3]

The previous formula does not take into account a very important fact: interbank deposits are unsecured borrowing/lending transactions between bank counterparties subject to default risk. In particular the lending counterparty will receive the payoff at maturity if the borrowing counterparty has not defaulted during the Deposit lifetime.

Hence, the actual Deposit payoff is, more precisely,

$$\mathbf{Depo}(T;\mathbf{T}) = N\left[1 + R(T_0,T)\tau(T_0,T)\right] 1_{[\tau_B > T]},$$

$$1_{[a>b]} := \begin{cases} 1, \, if \; a > b \\ 0, \, if \; a \leq b \end{cases}$$

Survival indicator

where τ_B is the (unknown) default time of the borrower. A more general formulation, including recovery, would be

$$\mathbf{Depo}(T;\mathbf{T}) = N\left[1 + R(T_0,T)\tau(T_0,T)\right]\left[1_{[\tau_B > T]} + R_B 1_{[\tau_B \leq T]}\right].$$

The recovery represents the percentage of notional *N* recovered by the lender in case of default of the borrower.

3: The market across the credit crunch
The basic lending/borrowing contracts: Deposits [4]

The default indicator has the following property

$$1_{[\tau_B > T]} + 1_{[\tau_B \leq T]} = 1$$

such that the default indicator with recovery can be manipulated as follows

$$\begin{aligned} 1_{[\tau_B > T]} + R_B 1_{[\tau_B \leq T]} &= 1 - 1_{[\tau_B \leq T]} + R_B 1_{[\tau_B \leq T]} \\ &= 1 - LGD_B 1_{[\tau_B \leq T]}, \\ LGD := 1 - R_B, \end{aligned}$$

where LGD = Loss Given Default is the percentage of notional *N* loss suffered by the lender in case of default of the borrower.

Actually the price of an interbank deposit contract depends also on other parameters, beyond the pure credit risk of the borrower, such as the nominal amount with respect to the credit line of the borrower on the lender side.

Typically, banks set up internal policies, called credit lines, to monitor and limit the lending amount to their counterparties.

3: The market across the credit crunch
The basic lending/borrowing contracts: Deposits [5]

Market practitioners usually include all such information in a synthetic borrower-specific spread $\Delta_B(t,T)$ over the interbank deposit rate $R(t,T)$ quoted on the market, such that

Deposit price at valuation date *t*

Deposit rate at start date T_0

Borrower's spread at start date T_0

$$\mathbf{Depo}(t;\mathbf{T}) = N P_B(t;T)\left\{1 + \left[R(T_0,T) + \Delta_B(T_0,T)\right]\tau(T_0,T)\right\},$$

$$P_B(t;T) = \frac{1}{1 + \left[R(t,T) + \Delta_B(t,T)\right]\tau(t,T)}.$$

Discount factor at valuation date *t*

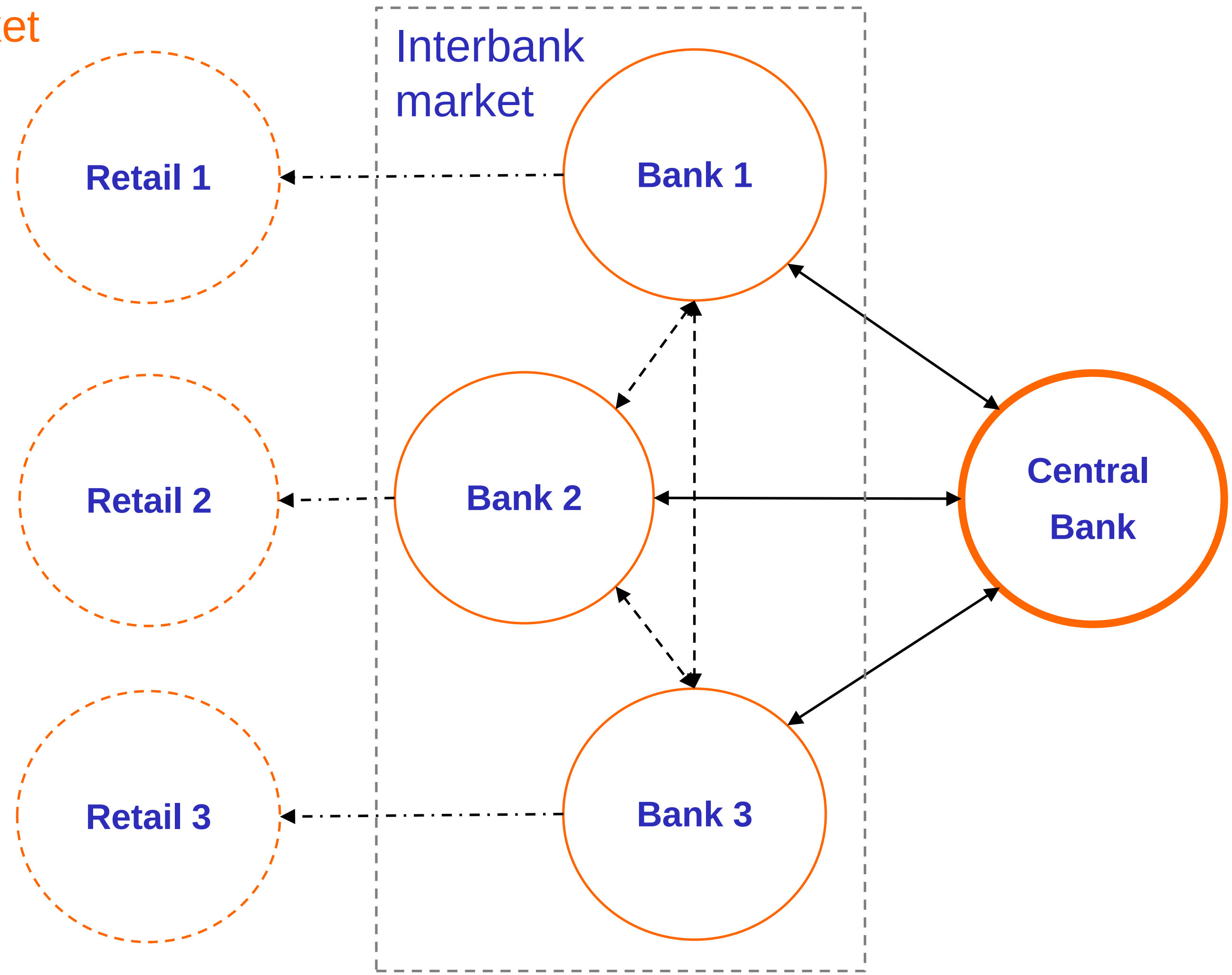
Money market quotation at valuation date *t*

Borrower's spread at valuation date *t*

3: The market across the credit crunch
The money market

The lending/borrowing market includes at least three areas:

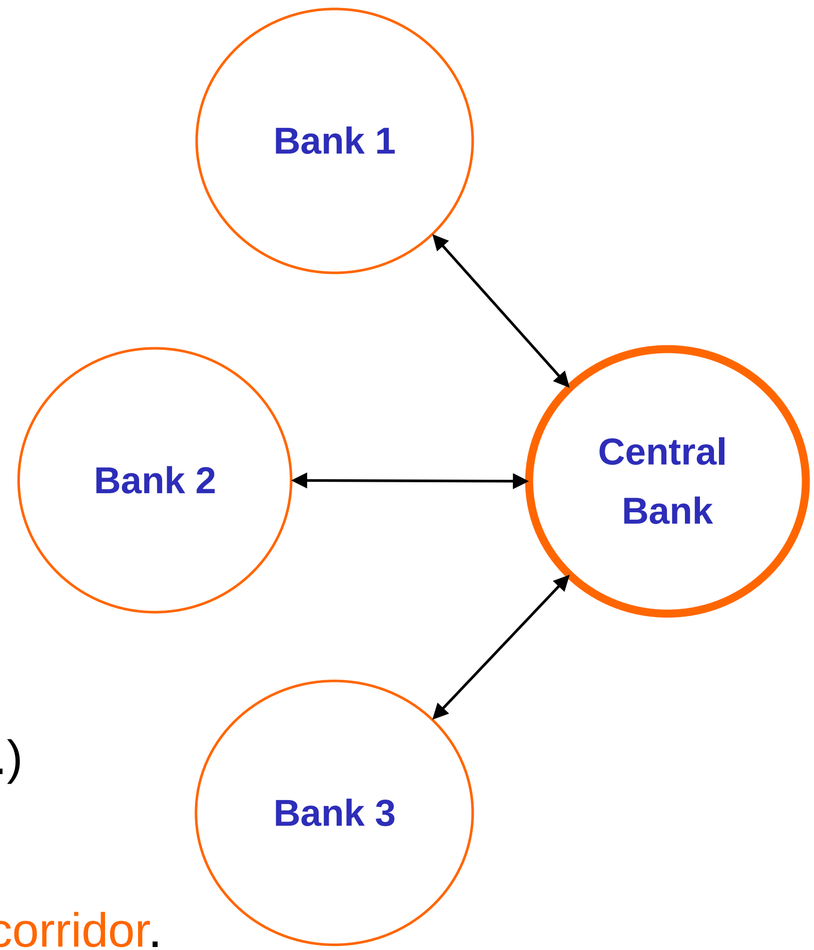
- Central Bank money market
- Interbank money market
- Retail money market



3: The market across the credit crunch
The money market: Central Bank market [1]

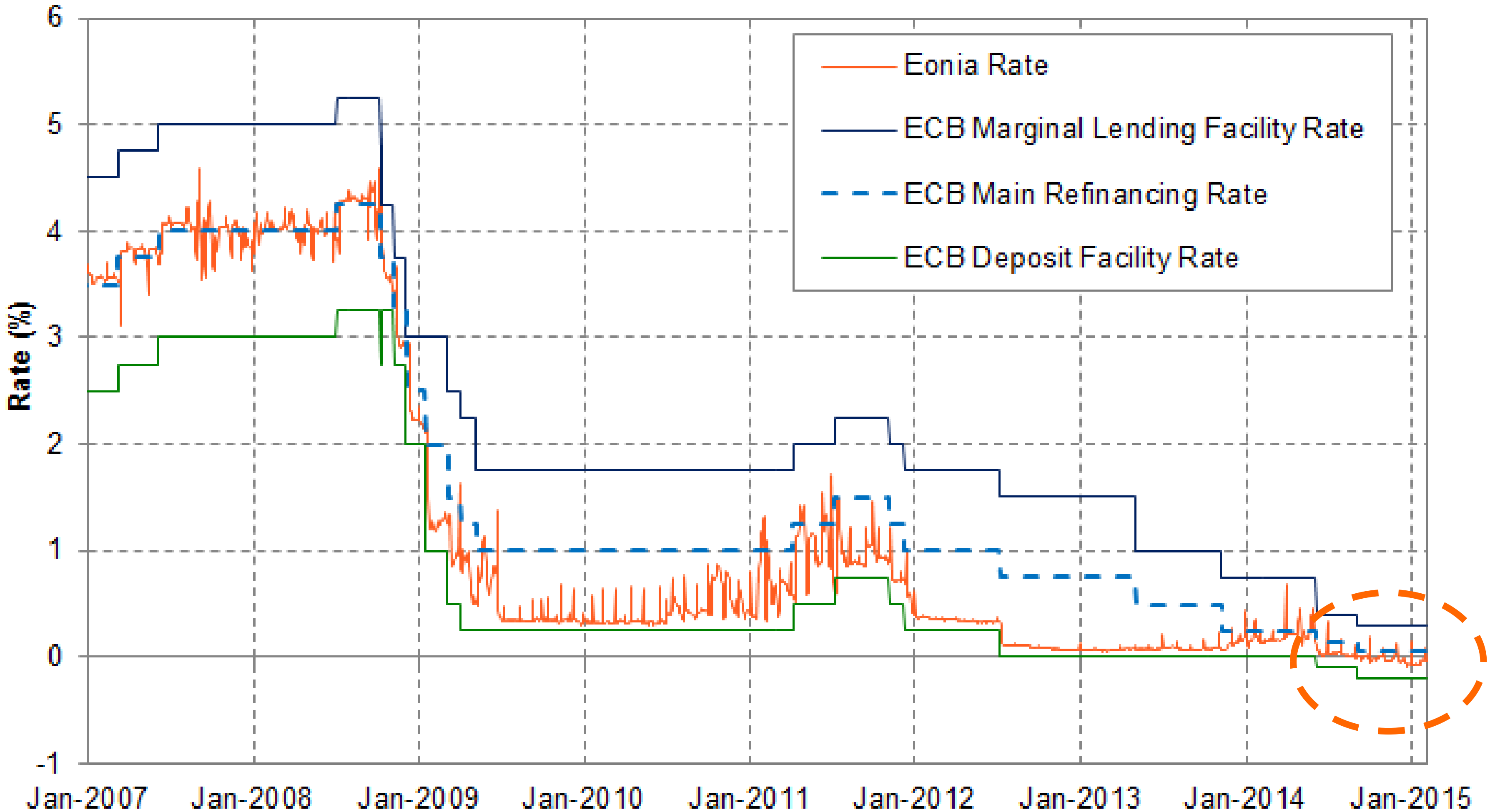
The Central Bank lending/borrowing market regards lending/borrowing transactions between the Central Bank (CB) and commercial/investment Banks.

- CB marginal lending facility: for Banks borrowing liquidity from the Central Bank, against the presentation of sufficient eligible asset (bonds), at the marginal lending facility rate.
- CB deposit facility: for Banks lending excess liquidity to the Central Bank, at the deposit facility rate.
- Typical time horizon: overnight.
- Liquidity: cash or eligible assets.
- Currency: national currency (e.g. EUR, USD, etc.)
- The deposit facility rate and the marginal lending facility rate define the Central Bank interest rate corridor.



3: The market across the credit crunch

The money market: Central Bank market [2]



ECB EUR interest rate corridor, *quotations Jan. 2007 – Dec. 2011*
(sources: ECB and Bloomberg)

3: The market across the credit crunch
The money market: Central Bank market [3]

Central Banks decide and transmit the monetary policy to the interbank money market by controlling the lending and borrowing interest rates.

During the financial crisis the ECB progressively lowered the main refinancing rate, to facilitate the funding of Banks:

- 8 Oct. 2008: the main refinancing rate is cut by 50 basis points;
- Nov. 2008 – May 2009: the main refinancing rate is cut further by a total of 325 basis points.

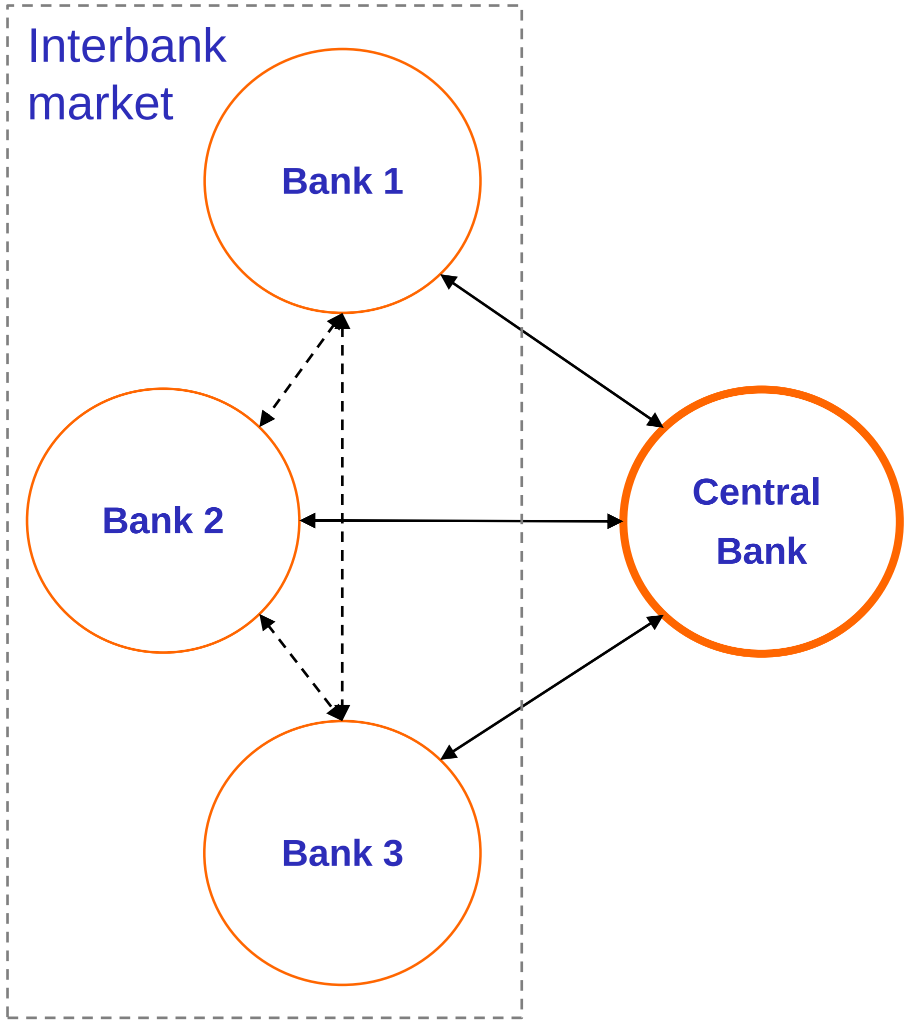
Furthermore, the ECB adopted a number of non-standard measures to support financing conditions and credit flows to the Euro area economy:

- unlimited secured access of Euro banks to ECB liquidity at the main refinancing rate;
- supplementary liquidity provision at longer maturities;
- currency swap agreements.

3: The market across the credit crunch
The money market: interbank market [1]

The interbank lending/borrowing market regards lending/borrowing transactions among Banks, it reallocates the liquidity originally supplied by the central bank as secured lending.

- Unsecured transactions based on interbank deposits (Depos) and interbank rates (Libor/Euribor/Eonia/Sonia/etc.)
- Secured transactions based on Repurchased Agreements (Repos) and repo rates.
- Maturity: ranging between overnight (1d) and one year (1Y), with different liquidity.
- Currency: the currency of the two banks, typically located in the same currency area (e.g. EUR, USD, etc.).



3: The market across the credit crunch
The money market: interbank market [2]

- Liquidity: cash or eligible assets.

- The CB deposit facility rate and marginal lending facility rate typically define a floor and a cap, respectively, for the corresponding interbank overnight rate.

- The interbank money market reallocates the liquidity originally supplied by the Central Bank.

- National banks are required to hold reserves on accounts with their national Central Bank. The holding requirement is averaged over a so-called reserve maintenance period. On the last day of the maintenance period banks with average reserves possibly under threshold borrow funds from other banks with excess of liquidity, thus causing spikes in volumes and lending interest rates, overnight in particular (since most of these funds are borrowed overnight).

3: The market across the credit crunch
The money market: interbank market [3]

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Alle Währungen werden gegen EUR kalkuliert
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	EUR	USD	GBP	JPY	CHF	
	03/01.	03/01.	04/01.	05/01.	04/01.	
ON	0.25 / 0.75	0.25 / 1.25	0.25 / 1.25		0.05 / 0.55	ON
TN	0.27 / 0.37	NYK. /HOLI.	0.55 / 0.75	0.01 / 0.21	0.03 / 0.23	TN
SN	0.30 / 0.40	0.25 / 0.35	0.55 / 0.75	0.05 / 0.25	0.03 / 0.23	SN
SW	0.55 / 0.65	0.60 / 0.80	0.90 / 1.10	0.35 / 0.55	0.25 / 0.45	SW
2W	0.65 / 0.75	0.70 / 0.90	1.05 / 1.25	0.35 / 0.55	0.28 / 0.48	2W
3W	0.70 / 0.80	0.80 / 1.00	1.15 / 1.35	0.35 / 0.55	0.32 / 0.52	3W
1M	0.93 / 1.03	1.00 / 1.20	1.40 / 1.60	0.48 / 0.68	0.47 / 0.77	1M
2M	1.10 / 1.20	1.30 / 1.50	1.67 / 1.87	0.67 / 0.87	0.59 / 0.89	2M
3M	1.26 / 1.36	1.54 / 1.74	1.90 / 2.10	0.80 / 1.00	0.70 / 1.00	3M
4M	1.33 / 1.43	1.66 / 1.86	2.01 / 2.21	0.88 / 1.08	0.77 / 1.07	4M
5M	1.41 / 1.51	1.77 / 1.97	2.15 / 2.35	0.97 / 1.17	0.86 / 1.16	5M
6M	1.51 / 1.61	1.89 / 2.09	2.28 / 2.48	1.07 / 1.27	0.95 / 1.25	6M
7M	1.57 / 1.67	1.98 / 2.18	2.36 / 2.56	1.13 / 1.33	1.00 / 1.30	7M
8M	1.64 / 1.74	2.08 / 2.28	2.45 / 2.65	1.19 / 1.39	1.04 / 1.34	8M
9M	1.69 / 1.79	2.15 / 2.35	2.53 / 2.73	1.26 / 1.46	1.09 / 1.39	9M
10M	1.74 / 1.84	2.23 / 2.43	2.60 / 2.80	1.32 / 1.52	1.12 / 1.42	10M
11M	1.79 / 1.89	2.28 / 2.48	2.67 / 2.87	1.38 / 1.58	1.19 / 1.49	11M
1Y	1.85 / 1.95	2.36 / 2.56	2.76 / 2.96	1.43 / 1.63	1.24 / 1.54	1Y
15M	1.73 / 2.13	2.38 / 2.78	2.76 / 3.16	1.46 / 1.86	1.35 / 1.75	15M
18M	1.66 / 2.06	2.40 / 2.80	2.75 / 3.15	1.46 / 1.86	1.26 / 1.66	18M
21M	1.62 / 2.02	2.41 / 2.81	2.78 / 3.18	1.46 / 1.86	1.18 / 1.58	21M
2Y	1.62 / 2.12	2.43 / 2.93	2.94 / 3.44	1.46 / 1.96	1.14 / 1.64	2Y
3Y	1.70 / 2.20	2.54 / 3.04	3.00 / 3.50	1.49 / 1.99	1.21 / 1.71	3Y
4Y	1.86 / 2.36	2.75 / 3.25	3.10 / 3.60	1.52 / 2.02	1.39 / 1.89	4Y
5Y	2.06 / 2.56	2.98 / 3.48	3.24 / 3.74	1.58 / 2.08	1.60 / 2.10	5Y
7Y	2.42 / 2.92	3.41 / 3.91	3.56 / 4.06	1.77 / 2.27	1.94 / 2.44	7Y
10Y	2.77 / 3.27	3.82 / 4.32	4.04 / 4.54	2.12 / 2.62	2.26 / 2.76	10Y
12Y	2.94 / 3.44	4.03 / 4.53	4.23 / 4.73	2.32 / 2.82	2.41 / 2.91	12Y
15Y	3.08 / 3.58	4.22 / 4.72	4.45 / 4.95	2.59 / 3.09	2.52 / 3.02	15Y
20Y	3.08 / 3.58	4.34 / 4.84	4.68 / 5.18	2.85 / 3.35	2.51 / 3.01	20Y
25Y	2.99 / 3.49	4.40 / 4.90	4.82 / 5.32	2.94 / 3.44	2.49 / 2.99	25Y
30Y	2.89 / 3.39	4.44 / 4.94	4.89 / 5.39	2.97 / 3.47	2.49 / 2.99	30Y
40Y	2.87 / 3.37		4.94 / 5.44			40Y
50Y	2.91 / 3.41		5.00 / 5.50			50Y

KLIEMM page with Interbank Deposits quotations (source: Reuters, 30 Dec. 2011)

3: The market across the credit crunch

The money market: interbank market [4]

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12:12 **DEPOS EUR** PAGE 1 / 1

Object	Ask	Bid	Time	Object	Ask	Bid	Time
1) O/N	0.3000	0.2000	7:41	16) 11M	1.1800	0.9800	7:00
2) T/N	0.3000	0.2000	7:00	17) 1 Y	1.2300	1.0300	7:00
3) 1 W	0.3300	0.2300	7:00				
4) 2 W	0.3500	0.2500	7:00				
5) 3 W	0.3900	0.2900	7:00				
6) 1 M	0.4600	0.2600	7:00				
7) 2 M	0.5500	0.3500	7:00				
8) 3 M	0.5900	0.3900	7:00				
9) 4 M	0.7400	0.5400	7:00				
10) 5 M	0.8600	0.6600	7:00				
11) 6 M	0.9800	0.7800	7:00				
12) 7 M	1.0400	0.8400	7:00				
13) 8 M	1.0600	0.8600	7:00				
14) 9 M	1.1100	0.9100	7:00				
15) 10M	1.1500	0.9500	7:00				

ICAP's new FRA Fixing platform launches Monday 2nd November

Manage your short term Euribor fixing risk through two 15 minute trading sessions of 0x3 and 0x6 EUR Fras for today & tomorrow's fix. Contact ICAP I-SWAP on BBerg for more info.

ICAP page with interbank Deposits quotations (source: Bloomberg, 20 Jan. 2010)

3: The market across the credit crunch
The age of negative rates [1]

- **Negative rates**

  - In September 2014 the ECB fixed the main refinancing rate at 5bp, the marginal lending rate at 30bp, the deposit facility rate at −20bp.
  - In the following weeks interbank short-term money market rates became negative. For instance, in December 2014 OIS rates were negative for maturities of up to three years.
  - In March 2016 the deposit facility rate was lowered to −40bp.
  - Also Japan, Sweden, Switzerland and Denmark adopted negative rates.
  - Banks may be reluctant to charge retail and corporate clients for negative rates, since they would lose customers in favor of other financial institutions (or the mattress).

See e.g. M. Bech and A. Malkhozov, "*How have central banks implemented negative policy rates?*", BIS Quarterly review, BIS Quarterly Review, March 2016.

3: The market across the credit crunch
The age of negative rates [2]

- **Why negative rates**

  - Central banks apply negative interest rates to their deposit facilities as a monetary policy tool to discourage banks from depositing their excess cash with the central banks.
  - The expectation is that banks will be discouraged from negative investing and instead will lend their excess funds to the market.
  - Since the central bank is (almost) default risk free, negative interest rates can be seen as the cost of the safe.
  - The narrowing of the ECB corridor aims to contrast the increasing volatility of overnight interest rates.

See e.g. M. Bech and A. Malkhozov, "*How have central banks implemented negative policy rates?*", BIS Quarterly review, BIS Quarterly Review, March 2016.

3: The market across the credit crunch
The age of negative rates [3]

- **Strange consequences of negative rates**
  - Pay for lending money !
  - Do not pay for a mortgage ! (being payed is too much optimistic...)
  - Discount factors > 1: one Euro today is worth less than one Euro tomorrow !
  - In a payer swap, pay fixed coupons and also pay floating coupons !
  - In a collateralized transaction, post collateral and also pay the collateral interest !
  - Stochastic interest rate models with a log-normal distribution (e.g. Black) are ruled out since they do not allow for negative rate distributions.
  - Stochastic interest rate models with a normal distribution (e.g. Hull-White) are no longer criticized since they allow for negative rate distributions.

3: The market across the credit crunch

The age of negative rates [4]

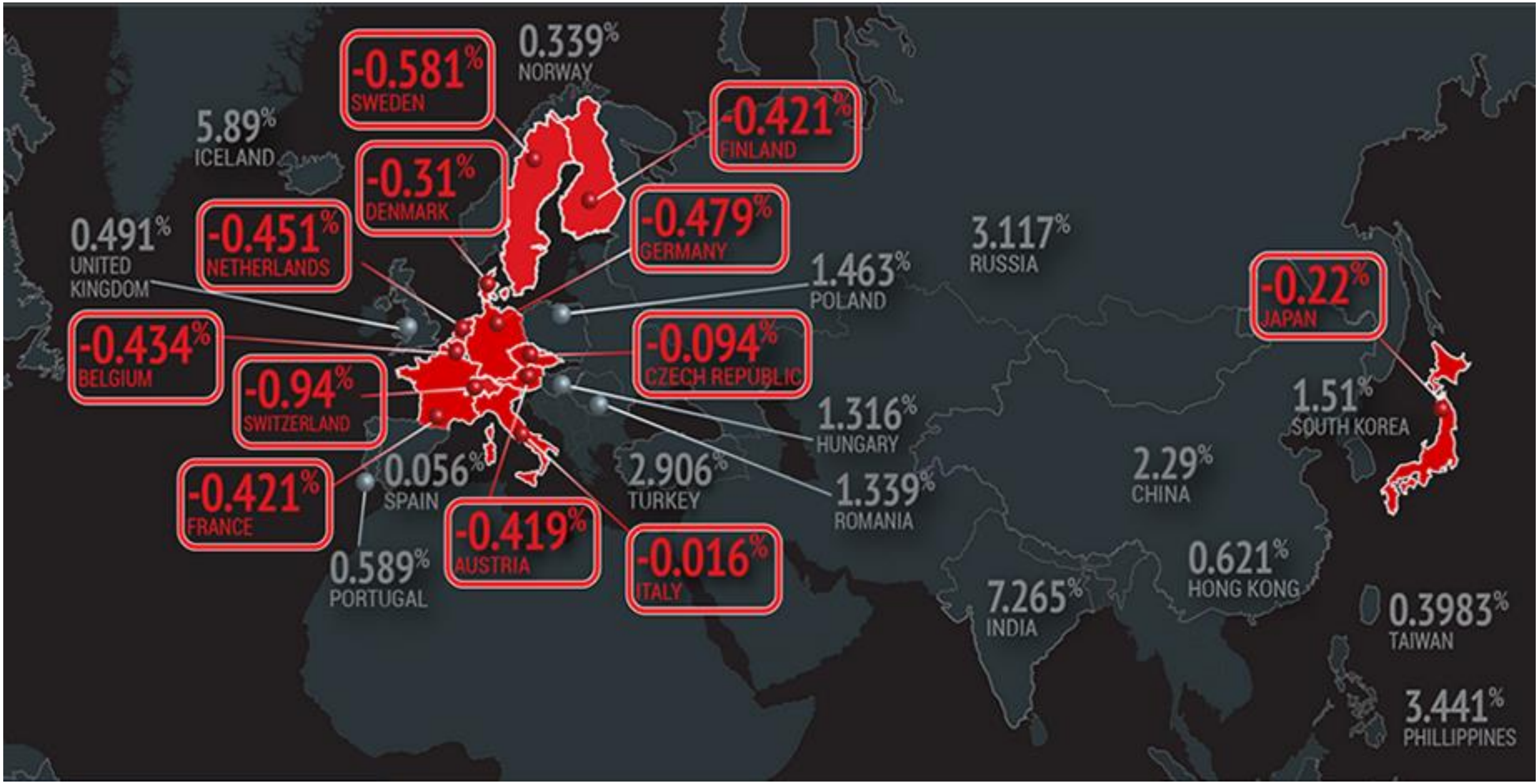
EUR003M -0.002 -.001 0.000 /0.000
At 4/22 Op -0.002 Hi -0.002 Lo -0.002 Prev -0.001 Vol 0
EUR003M Index | 96) Invio a Excel | Pag 1/6 Tabella prezzi storici
Euribor 3 Month ACT/360
Intrv 04/23/2014 - 04/22/2015 Period Giornalier
Merc Prz ask Valuta EUR
View Tabella dei prezzi

Mass .347 al 04/29/14
Min -.002 al 04/22/15
Media .129
Vzn net -.334 -100.60%

	Data	Prz ask		Data	Prz ask		Data	Prz ask
F	04/24/15		F	04/03/15		F	03/13/15	.025
T	04/23/15		T	04/02/15	.018	T	03/12/15	.027
W	04/22/15	L -.002	W	04/01/15	.018	W	03/11/15	.029
T	04/21/15	-.001	T	03/31/15	.019	T	03/10/15	.032
M	04/20/15	.001	M	03/30/15	.018	M	03/09/15	.035
F	04/17/15	.001	F	03/27/15	.021	F	03/06/15	.036
T	04/16/15	.002	T	03/26/15	.021	T	03/05/15	.036
W	04/15/15	.004	W	03/25/15	.021	W	03/04/15	.038
T	04/14/15	.008	T	03/24/15	.021	T	03/03/15	.038
M	04/13/15	.011	M	03/23/15	.022	M	03/02/15	.039
F	04/10/15	.012	F	03/20/15	.021	F	02/27/15	.039
T	04/09/15	.012	T	03/19/15	.024	T	02/26/15	.040
W	04/08/15	.014	W	03/18/15	.025	W	02/25/15	.042
T	04/07/15	.016	T	03/17/15	.025	T	02/24/15	.044
M	04/06/15		M	03/16/15	.025	M	02/23/15	.045

Euribor3M fixing historical series
(source: Bloomberg, 22 Apr. 2015)

3: The market across the credit crunch
The age of negative rates [5]



Source: Numerix + Bloomberg, snapshot of 2Y Governement bond yields on March 21, 2016
(`http://www.numerix.com/info-graphic/negative-rates-trends-continue-2016`)

3: The market across the credit crunch
Libor interest rate [1]

Libor definition and mechanics *(source: www.bbalibor.com, September 2010)*

- Libor = London Interbank Offered rate,
  - first published in 1986,
  - sponsored by British Banker's Association (BBA, see *http://www.bbalibor.com*),
  - reference rate mentioned in ISDA standards for OTC transactions.
- Fixing mechanics:
  - each TARGET business day no later than 11:00 GMT each panel Bank submits to the calculation agent "*at what rate could you borrow funds, were you to do so by asking for and then accepting inter-bank offers in a reasonable market size just prior to 11 am (GMT)?*" for a few maturities (1d-12M) in a given currency; between 11:00-11:45 each bank can adjust its contribution.
  - At 11:45 GMT the calculation agent computes the rate fixings, for each maturity, as the average of rates submissions after discarding highest and lowest quartiles (25%) and publishes the results (Reuters page "Libor=").
  - Rate conventions: annualised rate, act/360, three decimal places, modified following, end of month.
  - Calculation agent: Reuters.
- Currencies: GBP, USD, JPY, CHF, CAD, AUD, EUR, DKK, SEK, NZD (under review).

3: The market across the credit crunch
Libor interest rate [2]

Libor definition amplified

- the rate at which each bank submits must be formed from that bank's perception of its cost of funds in the interbank market;
- contributions must represent rates at which a bank would be offered funds in the London Money Market;
- contributions must be for the currency concerned, not the cost of producing one currency by borrowing in another currency and accessing the required currency via the foreign exchange markets;
- the rates must be submitted by members of staff at a bank with primary responsibility for management of a bank's cash, rather than a bank's derivative book;
- the definition of "funds" is: unsecured interbank cash or cash raised through primary issuance of interbank Certificates of Deposit.
- The rates are not necessarily based on actual transaction, because not all banks require funds each day, in size, in each currencies and maturities they quote. However, a bank is expected to know what its credit and liquidity risk profile is from rates at which it has dealt, and can construct a funding curve to predict accurately the correct rate for currencies or maturities in which it has not been active.

3: The market across the credit crunch
Libor interest rate [3]

Libor panels

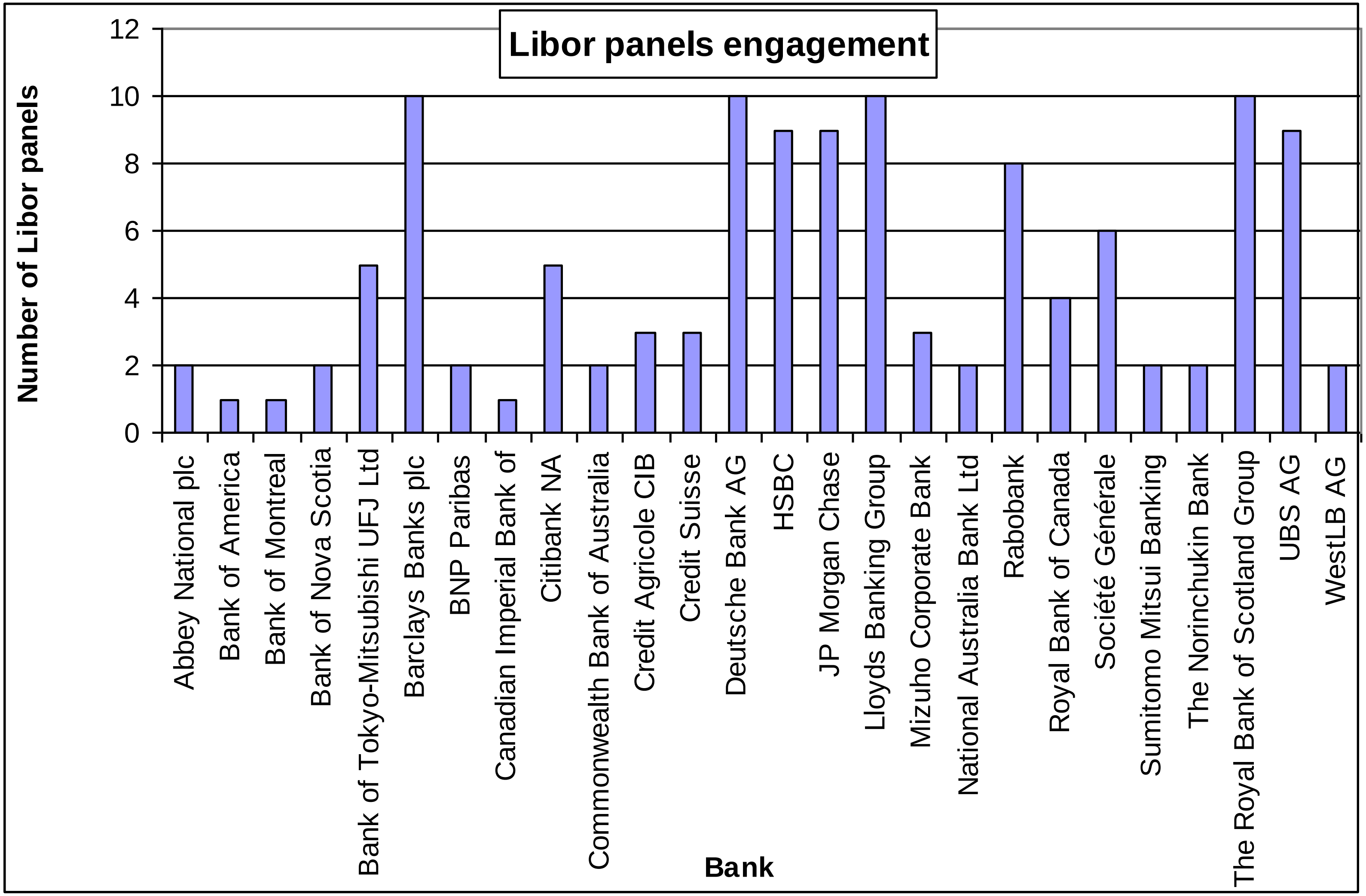
- Composition 8-12-16 contributors per currency (a multiple of 4 because of the average calculation rule above);
- Selection criteria:
  - Guiding principle: *"Banks chosen by the independent Foreign Exchange and Money Markets Committee to give the best representation of activity within the London money market for a particular currency*";
  - Criteria:
    - ✓ Scale of market activity
    - ✓ Reputation
    - ✓ Perceived expertise in the currency concerned
- Review: semi-annual review by BBA with FX & MM Committee; all panels and proposed banks are ranked according to their total money market and swaps activity over the previous year and selected according to the largest scale of activity with due concern given to the other 2 criteria.
- Sanctions: warning and successively exclusion from the panel.

3: The market across the credit crunch
Libor interest rate [4]

Libor panels per currency											
Banks	**AUD**	**CAD**	**CHF**	**EUR**	**GBP**	**JPY**	**USD**	**DKK**	**NZD**	**SEK**	**Panels**
Abbey National plc				X	X						**2**
Bank of America							X				**1**
Bank of Montreal		X									**1**
Bank of Nova Scotia		X					X				**2**
Bank of Tokyo-Mitsubishi UFJ Ltd			X	X	X	X	X				**5**
Barclays Banks plc	X	X	X	X	X	X	X	X	X	X	**10**
BNP Paribas					X		X				**2**
Canadian Imperial Bank of Commerce		X									**1**
Citibank NA			X	X	X	X	X				**5**
Commonwealth Bank of Australia	X								X		**2**
Credit Agricole CIB					X	X	X				**3**
Credit Suisse			X	X			X				**3**
Deutsche Bank AG	X	X	X	X	X	X	X	X	X	X	**10**
HSBC		X	X	X	X	X	X	X	X	X	**9**
JP Morgan Chase	X		X	X	X	X	X	X	X	X	**9**
Lloyds Banking Group	X	X	X	X	X	X	X	X	X	X	**10**
Mizuho Corporate Bank				X	X	X					**3**
National Australia Bank Ltd	X								X		**2**
Rabobank		X	X	X	X	X	X	X		X	**8**
Royal Bank of Canada		X		X	X		X				**4**
Société Générale		X	X	X	X	X	X				**6**
Sumitomo Mitsui Banking Corporation						X	X				**2**
The Norinchukin Bank						X	X				**2**
The Royal Bank of Scotland Group	X	X	X	X	X	X	X	X	X	X	**10**
UBS AG	X	X	X	X	X	X	X	X		X	**9**
WestLB AG				X		X					**2**
Totals	**8**	**12**	**12**	**16**	**16**	**16**	**19**	**8**	**8**	**8**	
Last review	May 2011										

Source: www.bbalibor.org, September 2011

3: The market across the credit crunch
Libor interest rate [5]



Source: www.bbalibor.org, September 2011

3: The market across the credit crunch
Libor interest rate [6]

Libor questioned during the crisis [1]

- Mar. 2008: the Bank for International Settlements reports that "*available data do not support the hypothesis that contributor banks manipulated their quotes to profit from positions based on fixings*“ (see J. Gyntelberg, P. Wooldridge, “*Interbank rate fixings during the recent turmoil*”, BIS Quarterly Review, Mar. 2008).
- Apr. 2008: Peng et al. from Citigroup (one of the largest Libor contributors) argue that “*...any Bank posting an high Libor level runs the risk of being perceived as needing funding*” (see Peng et al. “*Is Libor Broken*?”, Citi Fixed Income Strategies, Citigroup, Apr. 2008).
- Apr. 2008: the British Banker’s Association comments that Libor continues to be reliable, and that other proxies are not necessarily more sound than Libor at times of financial crisis.
- May 2008: the Wall Street Journal reports that some banks “*have been reporting significantly lower borrowing costs for the Libor, than what another market measure suggests they should be*” (see C. Mollenkamp, M. Whitehouse, The Wall Street Journal, 29 May 2008).
- Jun. 2008: Risk Magazine reports rumors that “*Libor rates are still not reflective of the true levels at which banks can borrow*” (see P. Madigan, “*Libor under attack*”, Risk, Jun. 2008).
- Oct. 2008: the International Monetary Fund reports that "*it appears that U.S. dollar Libor remains an accurate measure of a typical creditworthy bank’s marginal cost of unsecured U.S. dollar term funding*“ (see Global Financial Stability Report, Oct. 2008, ch. 2).
- Apr. 2010: an academic research paper reports evidences that Libor does not reflect the true bank’s borrowing costs (see C. Snider, T. Youle , “*Does the Libor reflect banks’ borrowing costs ?*”, SSRN working paper, 2 Apr. 2010, *http://ssrn.com/abstract=1569603*).

3: The market across the credit crunch
Libor interest rate [7]

Libor questioned during the crisis [2]

- Jun. 2012: Risk Magazine comments Barclays fined of 450 million USD by Commodity Futures Trading Commission (CFTC), Department of Justice (DOJ) and UK Financial Services Authority (FSA) for "*false, misleading or knowingly inaccurate submissions*" concerning Libor and Euribor in the period 2005-2009 (see P. Madigan, D. Wood, "*Libor manipulation lawsuits could cost banks tens of billions*", Risk, 28 Jun. 2012).
- Jun. 2012: FSA's publish "*The Wheatley review of LIBOR: final report*" (*http://cdn.hm-treasury.gov.uk/wheatley_review_libor_finalreport_280912.pdf*), stating that "*[…] retaining Libor unchanged in its current state is not a viable option, given the scale of identified weaknesses and the loss of credibility that it has suffered*" and designing a reform for strengthening the current Libor benchmark around four main themes:
    - the reform of the Libor fixing mechanism
    - the introduction of new rules and guidance for Libor contributions
    - the strengthening of the governance
    - changes to the regulatory framework.

3: The market across the credit crunch
Euribor interest rate [1]

Euribor definition and mechanics *(source: www.euribor.org, visited Sept. 2013)*

- Euribor = Euro Interbank Offered Rate
  - first published on 30 Dec. 1998;
  - sponsored by the European Banking Federation (EBF) and by the Financial Markets Association (ACI).

- Fixing mechanics:
  - each TARGET business day no later than 10:45 CET each panel Bank submits to the calculation agent "*what rate do you believe one prime bank is quoting to another prime bank for interbank term deposits within the euro zone?*" for a few maturities (T/N, 1w, 2w, 1M, 2M, 3M, 6M, 9M, 12M); between 10:45-11:00 each bank can adjust its contribution.
  - At 11:00 CET the calculation agent computes the rate fixings, for each maturity, as the average of rates submissions after discarding highest and lowest 15% and publishes the results (Reuters page "Euribor=").
  - Rate conventions: spot (T+2) value, annualised rate, act/360, three decimal places, modified following, end of month.
  - Calculation agent: Reuters.

- Currencies: EUR

3: The market across the credit crunch
Euribor interest rate [2]

Euribor panel

- Composition on Mar. 2010: 42 banks from 15 EU countries + 4 international banks;
- Selection criteria:
  - "…*active players in the euro money markets in the euro-zone or worldwide and if they are able to handle good volumes in euro-interest rate related instruments, especially in the money market, even in turbulent market condition*";
  - "*first class credit standing, high ethical standards and enjoying an excellent reputation*";
  - "…*ensure that the diversity of the euro money market is adequately reflected, thereby making Euribor an efficient and representative benchmark*."
- Review: "*periodically reviewed by the Steering Committee to ensure that the selected panel always truly reflects money market activities within the euro zone*".
- Banks obligations:
  - must quote "*the best price between the best banks*", "*for the complete range of maturities*", "*on time*", "*daily*", "*accurately*";
  - must make "*the necessary organisational arrangements to ensure that delivery of the rates is possible on a permanent basis without interruption due to human or technical failure*".
- Sanctions: warning and successively exclusion from the panel.
- Review in progress by EBF

3: The market across the credit crunch
Euribor interest rate [3]

Euribor panel (Sept. 2011)			
Country	**Bank**	**Country**	**Bank**
Austria (2)	Erste Group Bank AG RZB Raiffeisen Zentralbank Österreich AG	**Belgium (2)**	Dexia Bank KBC
Finland (2)	Nordea Pohjola	**Ireland (2)**	AIB Group Bank of Ireland
France (7)	Banque Postale BNP - Paribas HSBC France Société Générale Natixis Crédit Agricole s.a. Crédit Industriel et Commercial CIC	**Italy (3)**	Intesa Sanpaolo Unicredit Monte dei Paschi di Siena
		Luxembourg (1)	Banque et Caisse d'Épargne de l'État
		Netherlands (3)	RBS N.V. Rabobank ING Bank
Germany (11)	Landesbank Berlin Bayerische Landesbank Girozentrale Deutsche Bank WestLB AG Commerzbank DZ Bank Deutsche Genossenschaftsbank Norddeutsche Landesbank Girozentrale Landesbank Baden-Württemberg Girozentrale Landesbank Hessen Thüringen Girozentrale	**Portugal (1)**	Caixa Geral De Depósitos (CGD)
		Spain (4)	Banco Bilbao Vizcaya Argentaria Confederacion Española de Cajas de Ahorros Banco Santander Central Hispano La Caixa Barcelona
		Other EU Banks (3)	Barclays Capital Den Danske Bank Svenska Handelsbanken
		International Banks (4)	Bank of Tokyo - Mitsubishi J.P. Morgan Chase & Co. Citibank UBS (Luxembourg) S.A.
Greece (1)	National Bank of Greece		
Source: www.euribor.org			**Total: 46**

3: The market across the credit crunch

Euribor interest rate [4]

Euribor panel (Nov. 2016)			
Country	**Bank**	**Country**	**Bank**
Austria (0)		**Belgium (1)**	Belfius
Finland (0)	*(Nordea)* *(Pohjola)*	**Ireland (0)**	
France (5)	BNP - Paribas HSBC France Société Générale Natixis Crédit Agricole s.a.	**Italy (3)**	Intesa Sanpaolo Unicredit Monte dei Paschi di Siena
		Luxembourg (1)	Banque et Caisse d'Épargne de l'État
		Netherlands (1)	ING Bank
Germany (2)	Deutsche Bank DZ Bank Deutsche	**Portugal (1)**	Caixa Geral De Depósitos (CGD)
		Spain (4)	Banco Bilbao Vizcaya Argentaria Banco Santander CECABANK Caixa Bank S.A.
		Other EU Banks (1)	Barclays Capital *(Danske Bank)*
		International Banks (0)	*(Bank of Tokyo - Mitsubishi)* *(J.P. Morgan Chase Bank London Branch)*
Greece (1)	National Bank of Greece		
Source: www.emmi-benchmarks.eu/euribor-org/panel-banks.html			**Total: 20**

3: The market across the credit crunch
Euribor interest rate [5]

Euribor Reform

- Based on reccomendations from:
  - the IOSCO Principles on Financial Benchmarks (2013),
  - the ESMA-EBA Principles for Benchmark-Setting Processes in the EU (2013),
  - the draft EU Regulation on Indices Used as Benchmarks in Financial Instruments and Financial Contracts, expected to come into force in 2016.
- The European Money Markets Institute (EMMI) has started a Euribor reform process aiming to get benchmark rates which should:
  - minimize the opportunities for market manipulation;
  - be anchored in observable transactions wherever feasible;
  - be robust in the face of market dislocation and should command confidence that they remain resilient in times of stress.
- The new Euribor rate (transition in progress) will be
  - the rate at which banks of sound financial standings borrow funds on wholesale market
  - based on real transactions
  - determined by a proper methodology to ensure stability and liquidity

Go-live Q2 2017 after a 6-months Pre-Live verification program started September 2016

3: The market across the credit crunch
Other Ibor rates

- DKK CIBOR: Copenhagen Interbank Offered Rate, published by the Danish Bankers Association (www.finansraadet.dk).
- JPY TIBOR: Tokyo Interbank Offered Rate, published by the Japanese Bankers Association (www.zenginkyo.or.jp/en/tibor/the_jba_tibor) in two versions: the Japanese Yen TIBOR for the local market and the Euroyen TIBOR for the Japan offshore market.
- CAD CDOR: Canadian Dealer Offered Rate, published by the Montreal Exchange (www.m-x.ca/marc_terme_bax_cdor_en.php).
- AUD BBSW: Australian Bank Bill Rate, published by the Australian Financial Markets Association (www.afma.com.au/data/bbsw.html).
- Others:
  - STIBOR: Sweden
  - WIBOR: Poland
  - BUBOR: Hungary
  - PRIBOR: Czech Republic
  - JIBAR: South Africa

3: The market across the credit crunch
Eonia interest rate

Eonia definition and mechanics *(source: www.euribor.org, visited Sept. 2013)*

- Eonia = Euro Over Night Index Average
  - first published and sponsored as Euribor;
  - reference rate for overnight unsecured transactions in the Euro Market.

- Panel banks: same as Euribor.

- Fixing mechanics:
  - each TARGET business day no later than 18:30 CET each panel bank submits the total volume of overnight unsecured lending transactions of that day before 18:00 and the weighted average lending rate for these transactions for a single maturity (ON).
  - Between 18:30-18:45 (CET) the calculation agent computes the rate fixing as the average of all rates submissions (with no cuts) weighted with the corresponding transaction volumes and transmits the result to Reuters for publication within 18:45-19:00 (Reuters page "Eonia=").
  - Rate conventions: today value (T+0), annualised rate, act/360, three decimal places.
  - Calculation agent: European Central Bank.

Fed Fund definition and mechanics *(source: www.newyorkfed.org, visited Sept. 2013)*

- Fed Fund = Federal Fund Effective Rate
  - It is the rate of overnight unsecured loans of reserve balances at Federal Reserve Bank of New York that depository institutions make to one another.
  - reference rate for overnight unsecured transactions in the US market.
- Panel: there is no formal panel, the participants are all the institutions holding reserve balances at the Federal Reserve Bank of NY.
- Fixing mechanics:
  - each NY business day the Fed Fund fixing is calculated as the weighted average rate (with no cuts) of all unsecured USD transactions (mainly overnight) at the Domestic Trading Desk of the Federal Reserve Bank of NY.
- Comments:
  - Fed funds transactions do not change total bank reserves, they redistribute bank reserves and enable otherwise idle funds to yield a return.
  - The Federal Open Market Committee (FOMC) periodically sets a Federal fund target rate. The monetary policy is implemented by the Domestic Trading Desk of Fed NY, that “creates the conditions in reserve markets that will encourage fed funds to trade at a particular level”. In this way the Fed can create upward or downward pressure on the fed funds rate.

3: The market across the credit crunch
Secured Overnight Financing Rate (USD)

SOFR definition and mechanics *(source: xxx, visited xxx)*

- SOFR = Secured Overnight Financing Rate
  - TBD
  - TBD.
- Panel: TBD.
- Fixing mechanics:
  - TBD.
- Comments:
  - TBD

3: The market across the credit crunch
SONIA rate (GBP)

SONIA definition and mechanics *(source: www.bba.org.uk, visited Sept. 2013)*

- SONIA = Sterling Over Night Index Average
  - first published in Mar. 1997, sponsored by the Wholesale Markets Brokers' Association (WMBA, www.wmba.org.uk) and endorsed by the BBA (www.bba.org.uk);
  - reference rate for overnight unsecured transactions in the Sterling market.
- Panel: WMBA members (main Brokers in London).
- Fixing mechanics:
  - each London business day the SONIA fixing is calculated as the weighted average rate of all unsecured overnight sterling transactions brokered in London by WMBA members between 00:00 and 15.15 GMT in a minimum deal size of 25 million GBP with all counterparties listed under Section 43 of the Financial Services Act 1986.
  - Between 16:15 – 17:00 GMT the calculation agent computes the rate fixing and publishes the result (Reuters “SONIA 1“, Bloomberg “WMBA <GO>”).
  - Rate conventions: today value (T+0), annualised rate, act/360, four decimal places.
  - Calculation agent: Thomson Reuters.

3: The market across the credit crunch
SARON rate (CHF)

SARON definition and mechanics *(source: www.snb.ch, visited Sept. 2013)*

- SARON = Swiss Average Rate OverNight
  - first published on 25 Aug. 2009, sponsored by the Swiss National Bank (www.snb.ch);
  - reference rate for overnight repo transactions in the Swiss market.
- Panel: WMBA members (main Brokers in London).
- Fixing mechanics:
  - the SARON is calculated as the volume-weighted average rate of all repo overnight CHF transactions concluded and reference prices posted on the given trading day in the order book of Eurex Zurich Ltd electronic trading platform.
  - The reference price is calculated on the basis of tradable quotes provided they lie within the parameters of the quote filter. The quote filter is parameterised such that to minimize the possibilities of manipulation.
  - The SARON is calculated in real time and published every ten minutes. In addition, a fixing is conducted three times a day at 12.00, 16.00 and at the close of the trading day (18:00 at the earliest) (CET).

3: The market across the credit crunch
TONAR or Mutan rate (JPY)

Mutan definition and mechanics

(source: www.boj.or.jp/en/statistics/market/short/mutan, visited Sept. 2013)

- TONAR = Tokyo Over Night Average Rate
  - Also called Mutan = Uncollateralised Overnight Call Rate, is the Bank of Japan's uncollateralised overnight call rate
  - It's the reference rate for interbank unsecured (mainly) overnight transactions in the Japan market
  - It's the main tool for the transmission of Bank of Japan's monetary policy.
  - First published 11 April 1996.
  - See Reuters page "TONAR".

3: The market across the credit crunch
Other overnight rates

- DKKOIS: Danish National Bank Tomorrow/Next interest rate
- CORRA: Canadian Overnight Repo Rate Average
- AONIA: Australian OverNight Index Average
- NZIONA: New Zealand Index OverNight Average
- HONIX: Hong Kong OverNight Index
- SONAR: Singapore OverNight Average Rate.

3: The market across the credit crunch

Repo interest rate [1]

Eurepo definition and mechanics *(source: www.eurepo.org, March 2011)*

- Eurepo = Euro Repo (Repurchase Agreement Rate)
  - first published on 4 Mar. 2002;
  - sponsored by the European Banking Federation (EBF);
  - reference rate for Repo transaction in the Euro market
- Panel banks: 34 banks.
- Fixing mechanics:
  - each TARGET business day no later than 10:45 CET each panel Bank submits to the calculation agent "*the rate at which one prime bank offers funds in euro to another prime bank if in exchange the former receives from the latter the best collateral in terms of rating and liquidity within the Eurepo GC basket*" for 10 maturities (T/N, 1w, 2w, 3w, 1M, 2M, 3M, 6M, 9M, 12M); between 10:45-11:00 each bank can adjust its contribution.
  - At 11:00 CET the calculation agent computes the rate fixings, for each maturity, as the average of rates submissions after discarding highest and lowest 15% and publishes the results (Reuters page "EUREPO=").
  - Rate conventions: spot (T+2) value, annualised rate, act/360, three decimal places, modified following, end of month.
  - Calculation agent: Reuters.

3: The market across the credit crunch
Repo interest rate [2]

- Repurchase Agreement:
  allows a borrower to use a financial security as collateral for a cash loan at a fixed rate of interest. At Repo start date t the borrower sells the agreed security S to the lender and receives the cash amount corresponding to the security t-spot price $P_S(t)$. They also agree at time t that, at Repo maturity date $T>t$, the lender will sell back the security to the borrower at T-spot price $P_S(T)$ and the borrower will pay back to the lender the corresponding cash amount (here is the repurchase agreement). At time t, the difference between the forward price $F_S(t)$ and the spot price $P_S(t)$ of the security is thus proportional to the Repo (interest) rate.
  Thus a Repo is different from an (unsecured) Deposit contract, and is analogous to a secured loan. The difference with the latter is that there is a legal transfer of property of the security at times t and T (hence the different terminology). If the borrower defaults between t and T, the lender keep the security. If the lender does not resell back the security at time T, the borrower keeps the cash.
- The Eurepo GC (General Collateral) Basket:
  - consists of the most liquid government securities of EU-15 countries.
  - GC repos are regular repo transactions where the lender (buyer of the security collateral) does not specify the particular security to buy, and the borrower (seller of the security collateral) has the right to nominate the security that will be delivered as collateral of the loan.

3: The market across the credit crunch
Repo interest rate [3]

- Eurepo panel
  - Composition on Mar. 2011: 34 banks selected among the EU banks with the plus some large international bank from non-EU countries with important euro zone operations.
  - Selection criteria:
    - *“Active players in the euro repo markets in the euro-zone or worldwide”.*
    - *“Able to handle good volumes in euro repo rate related instruments, even in turbulent market conditions”.*
    - *“First class credit standing, high ethical standards and excellent reputation”*;
  - Review: “*periodically reviewed by the Steering Committee to ensure that the selected panel always truly reflects euro repo market activities within the euro zone and worldwide in accordance with criteria above*”.
  - Banks obligations:
    - Code of Conduct similar to those of Euribor and Eonia
  - Sanctions: warning and successively exclusion from the panel.

3: The market across the credit crunch
Other repo rates

- USD DTCC GCF:
  DTCC General Collateral Finance repo rate on Treasuries, Agencies or MBS, sponsored and published since 2009 by DTCC (Depository Trust and Clearing Corporation), it is based on actual transactions (www.dtcc.com/products/fi/gcfindex).

- GBP RONIA:
  Repurchase OverNight Index Average, sponsored and published since 2011 by the British Wholesale Market Brokers' Association (WMBA), it is measured as the weighted average of all sterling overnight secured transactions (www.wmba.org.uk/pages/index.cfm?page_id=33&title=ronia).

3: The market across the credit crunch
Libor/Euribor/Eonia/Eurepo interest rates discussion [1]

Libor discussion

- Libor is based on:
  - offered rates on unsecured funding;
  - expectations, views and beliefs of the panel banks about borrowing rates in the currency money market (see e.g. P. Madigan, "*Libor under attack*", Risk, Jun. 2008).
- As any interest rate expectation, Libor includes informations on:
  - the counterparty credit risk/premium,
  - the liquidity risk/premium

  and thus its not a risk free rate, as already well known before the crisis (see e.g. B. Tuckman, P. Porfirio, Jun. 2003).
- Lending/borrowing Libor rates is tenor dependent: *"The age of innocence – when banks lent to each other unsecured for three months or longer at only a small premium to expected policy rates – will not quickly, if ever, return"* (M. King, Bank of England Governor, 21 Oct. 2008).
- The Libor panels may change over time, panel banks may be replaced by other banks with higher credit standing. Borrowers and lenders will not be Libor forever.

3: The market across the credit crunch
Libor/Euribor/Eonia/Eurepo interest rates discussion [2]

Eonia discussion

- Eonia is based on:
    - lending (offer side) rates on unsecured funding;
    - actual transaction executed by the panel banks in the Euro money market
- Eonia is used by ECB as a method of effecting and observing the transmission of the monetary policy actions in the unsecured Euro money market and thus it includes informations on:
    - the monetary policy effects,
    - the short term cost of liquidity expectations of the panel banks in the unsecured Euro money market;
- Eonia holds the shortest rate tenor available (one day) and carries low counterparty credit and liquidity risk, thus it is a good market proxy to a risk free rate.

See also Goldman Sachs, "*Overview of Eonia and Update on Eonia Swap Market*", Mar. 2010.

3: The market across the credit crunch
Libor/Euribor/Eonia/Eurepo interest rates discussion [3]

Eurepo discussion

- Same points as Eonia apply, but for the secured Euro money market
- Eurepo carries the lowest counterparty credit and liquidity risk: thus it is the best market proxy to a risk free rate.
- Eonia and Eurepo are bracketed inside the interest rate corridor defined by the standing facilities provided by the european national banks to manage liquidity in the banking sector:
  - the marginal lending facility lets banks borrow liquidity from their national central bank against eligible assets: the marginal lending rate normallly defines a cap for the overnight market rates;
  - the deposit facility lets banks lend liquidity to their national central bank: the corresponding deposit rate normally definies a floor for the overnight market rates;
  - see graph later on.

3: The market across the credit crunch

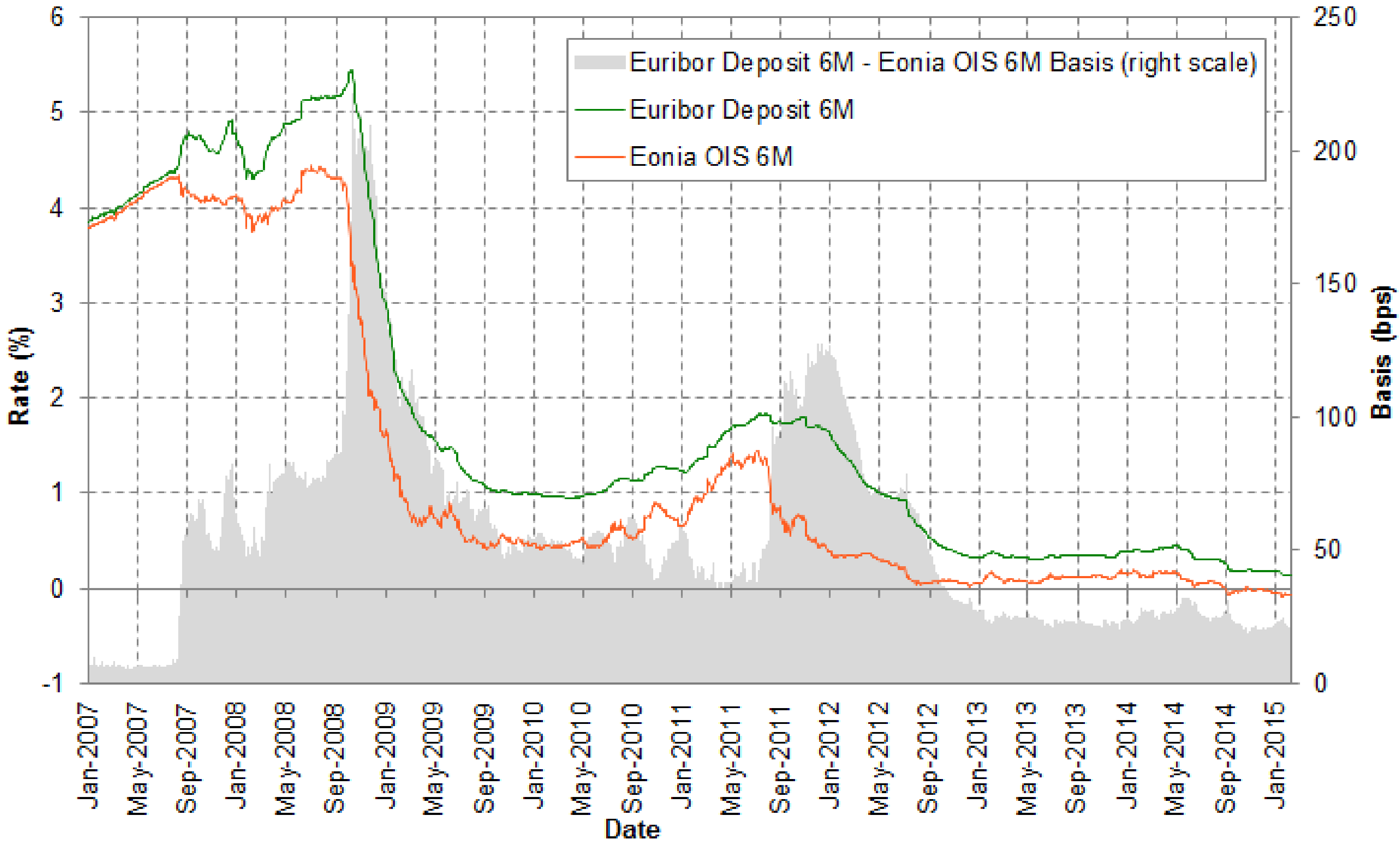
Libor/Euribor/Eonia/Eurepo interest rates discussion [4]

	Libor	Euribor	Eonia	Eurepo
Definition	**L**ondon **I**nter**B**ank **O**ffered **R**ate	**Euro** **I**nter**B**ank **O**ffered **R**ate	**E**uro **O**ver**N**ight **I**ndex **A**verage	**Euro** **Rep**urchase Agreement rate
Market	London Interbank	Euro Interbank	Euro Interbank	Euro Interbank
Side	Offer	Offer	Offer	Offer
Rate quotation specs	EURLibor = Euribor, Other currencies: minor differences (e.g. act/365, T+0, London calendar for GBPLibor).	TARGET calendar, settlement *T+2*, act/360, three decimal places, modified following, end of month, tenor variable.	TARGET calendar, settlement *T+1*, act/360, three decimal places, tenor 1d.	As Euribor
Maturities	1d-12m	1w-12m	1d	T/N-12m
Publication time	12.30 CET	11:00 am CET	6:45-7:00 pm CET	As Euribor
Panel banks	8-20 banks (London based) per currency	Most important EU-15 banks + a few international bank from non-EU countries	Same as Euribor	Most important EU-15 banks + a few international bank from non-EU countries
Calculation agent	Reuters	Reuters	European Central Bank	Reuters
Transactions based	No	No	Yes	No
Collateral	No (unsecured)	No (unsecured)	No (unsecured)	Yes (secured)
Counterparty risk	Yes	Yes	Low	Negligible
Liquidity risk	Yes	Yes	Low	Negligible
Tenor basis	Yes	Yes	No	No

3: The market across the credit crunch
How the market has changed: stylized facts

1. Banks are not credit risk free and are not too big to fail, credit and liquidity risk in market benchmark interest rates (Ibor), tenor dependency.
2. Explosion of spot/forward market Ibor/OIS and Ibor/Ibor tenor basis.
3. Explosion of single and cross currency basis swap rates.
4. Break of the classic-no-arbitrage relationships between market FRA rates and forward rates implicit in market Deposits.
5. Shift from unsecured to secured money market funding, diffusion of collateralization, CSA chaos, new ISDA Standard CSA.
6. Trades migration to Central Counterparties (CCPs).
7. Shift towards CSA discounting for collateralized trades, changes in market quotations for OTC derivatives, multiplication of interest rate yield curves used for pricing interest rate derivatives.
8. Reactions of regulators.

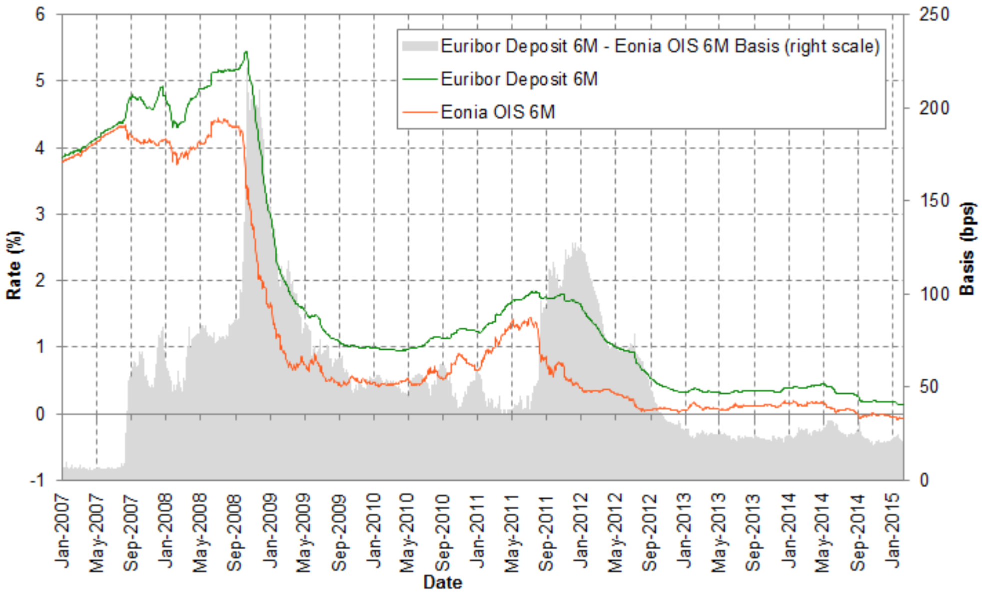
3: The market across the credit crunch
Spot rates [1]



Euribor6M Depo vs Eur OIS 6M (spot) rates

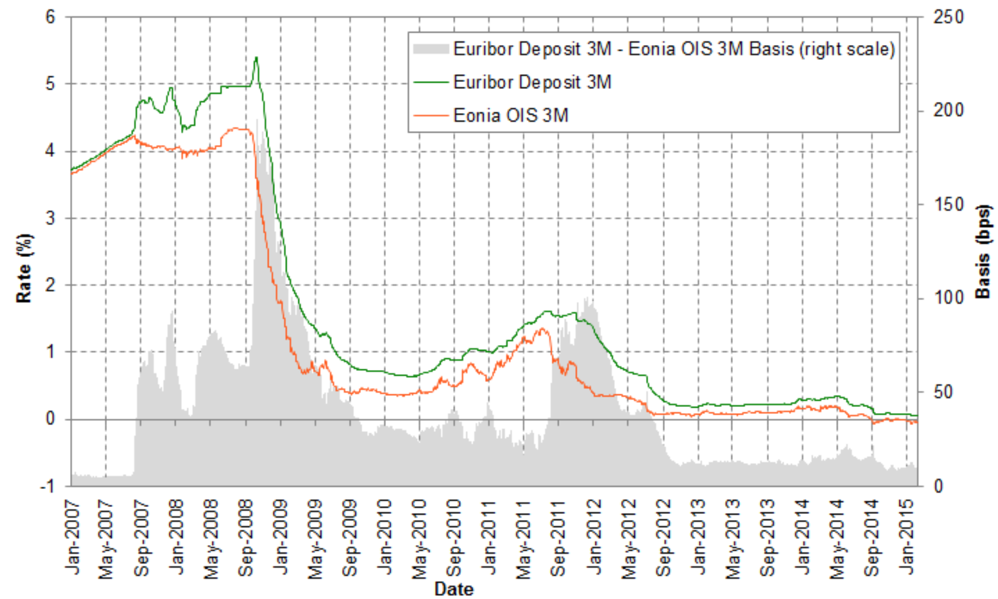
Quotations Jan. 2007 – Jan. 2015 (source: Bloomberg)

3: The market across the credit crunch
Spot rates [2]

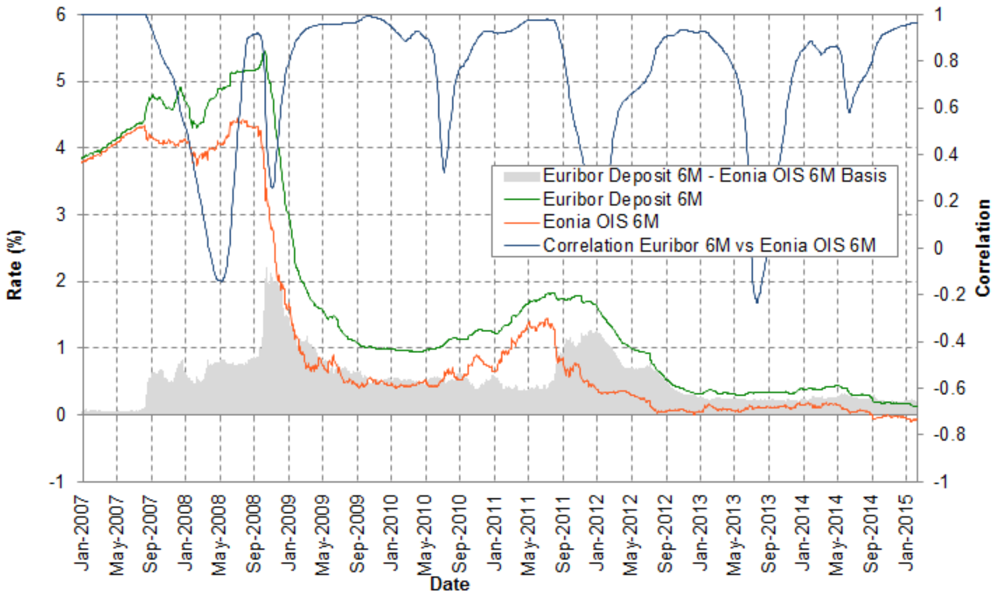


Euribor6M Depo vs Eur OIS 6M (spot) rates, *quotations Jan. 2007 – Jan. 2015 (source: Bloomberg)*

Euribor3M Depo vs Eur OIS 3M (spot) rates, *quotations Jan. 2007 – Jan. 2015 (source: Bloomberg)*

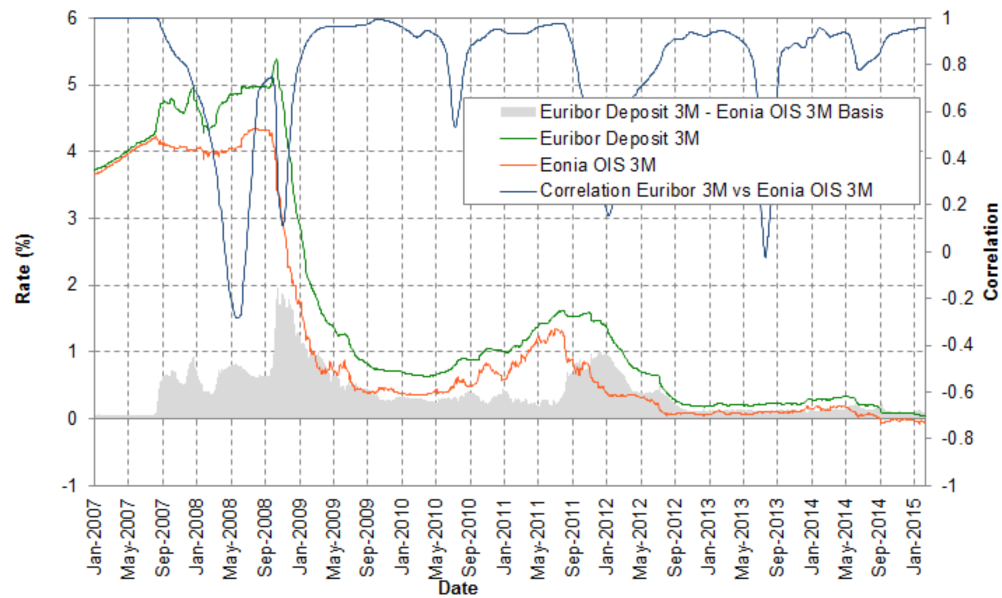


3: The market across the credit crunch
Spot rates [3]



Euribor6M Depo vs Eur OIS 6M (spot) rates, *quotations Jan. 2007 – Jan. 2015 (source: Bloomberg), historical correlations (annual window)*

Euribor3M Depo vs Eur OIS 3M (spot) rates, *quotations Jan. 2007 – Jan. 2015 (source: Bloomberg), historical correlations (annual window)*

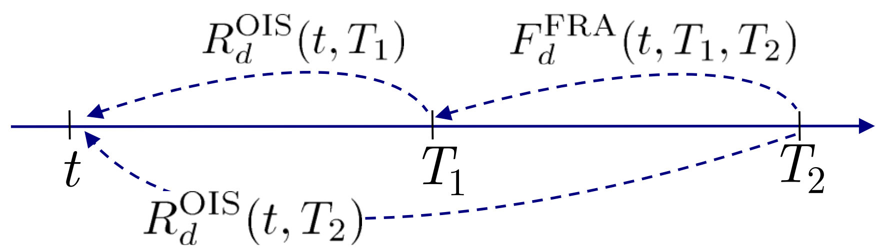


3: The market across the credit crunch
FRA rates [1]

```
16:16 30DEC10                        ICAP LONDON
Contact Reuters EXEU                        EURO
      Eonia                  Fwd EONIA
 1w   0.462-0.362      1X2   0.644-0.594
 2w   0.456-0.356      2X3   0.686-0.636
 3w   0.511-0.411      1x4   0.685-0.635
 1m   0.527-0.477      2x5   0.724-0.674
 2m   0.582-0.532      3x6   0.758-0.708
 3m   0.619-0.569     6x12   0.892-0.842
 4m   0.644-0.594
 5m   0.669-0.619      IMM Fra/Eonia
 6m   0.689-0.639
 7m   0.707-0.657      MAR   35.100-30.100
 8m   0.726-0.676      JUN   34.200-29.200
 9m   0.743-0.693      SEP   34.600-29.600
10m   0.761-0.711      DEC   35.300-30.300
11m   0.776-0.726
12m   0.792-0.742
Two Payments
15m   0.849-0.799
18m   0.914-0.864        All ICAP Euro pag
21m   0.989-0.939
2y    1.071-1.021
3y    1.419-1.369
ICAP Global Index <ICAP>
```

Check Eonia FRA replication (30 Dec. 2010)			
Tenor	FRA replica (%)	FRA market (%)	Difference (bp)
1x2	0.618	0.619	-0.1
2x3	0.664	0.661	0.3
1x4	0.659	0.66	-0.1
2x5	0.698	0.699	-0.1
3x6	0.732	0.733	-0.1
6x12	0.865	0.867	-0.2

Market Eonia FRA vs OIS implicit forward rates.
Quotations 30 Dec. 2010 (source: Reuters)

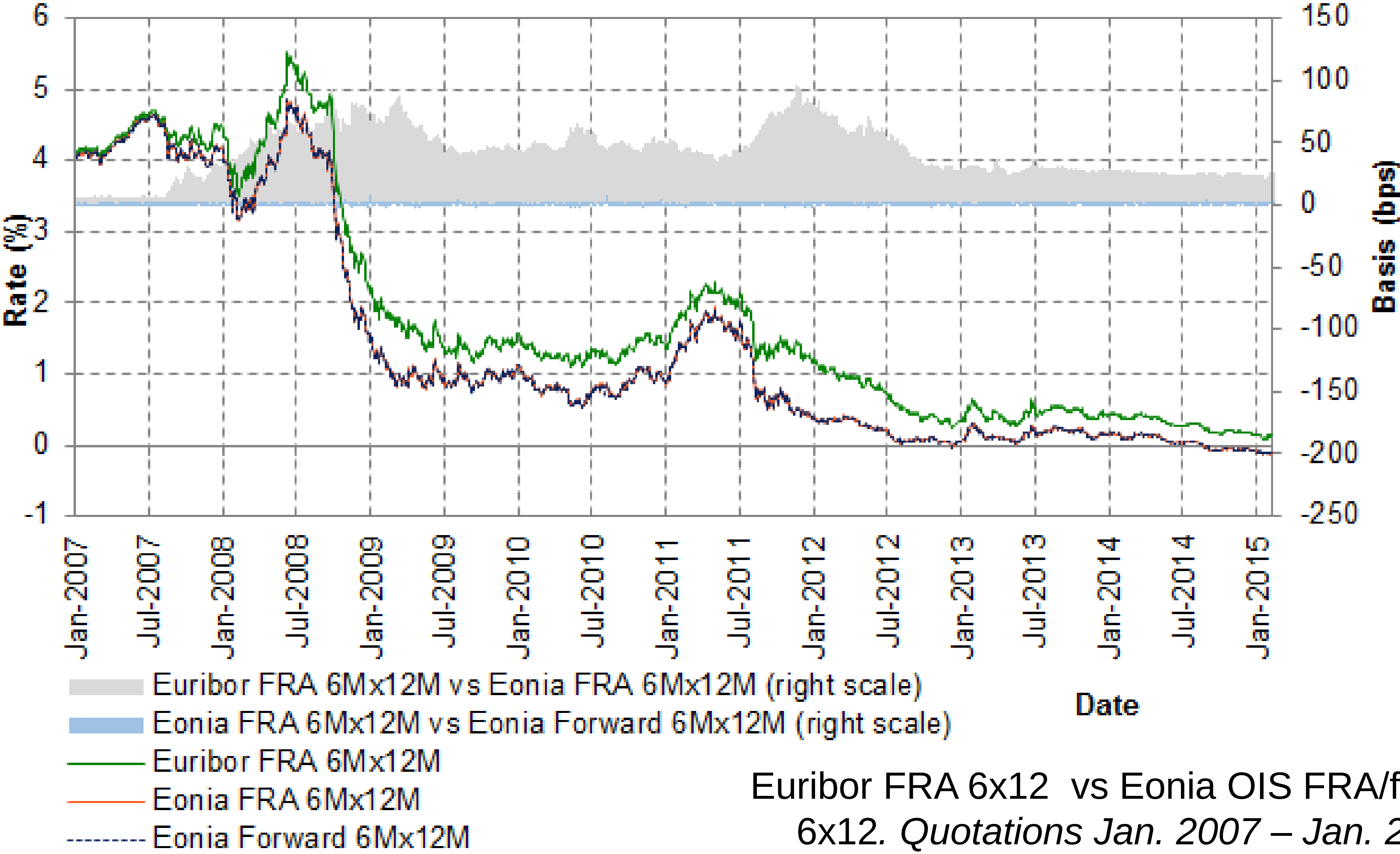


$$\frac{1}{1+R_{\mathrm{OIS}}(t,T_1)\tau(t,T_1)}\times\frac{1}{1+F_{\mathrm{OIS}}(t;T_1,T_2)\tau(T_1,T_2)}=\frac{1}{1+R_{\mathrm{OIS}}(t,T_2)\tau(t,T_2)},$$

$$\Rightarrow F_{\mathrm{OIS}}(t;T_1,T_2)=\frac{1}{\tau(T_1,T_2)}\left[\frac{1+R_{\mathrm{OIS}}(t,T_2)\tau(t,T_2)}{1+R_{\mathrm{OIS}}(t,T_1)\tau(t,T_1)}-1\right].$$

Mkt OIS and FRA rates are simple compounded

3: The market across the credit crunch
FRA rates [2]



Euribor FRA 6x12 vs Eonia OIS FRA/forward 6x12*. Quotations Jan. 2007 – Jan. 2015 (source: Bloomberg)*

3: The market across the credit crunch
FRA rates [3]

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KLIEMM
IRP        Tel. +49
CapMkt     Tel. +49
Fwds       Tel. +49
                Carl K
See <KLIEMMM2> f
16:52  30/12/10
        EUR
       CLOSED
 ON  0.28/0.38
 TN  0.50/1.00
 SN  0.40/0.50
 SW  0.53/0.63
 2W  0.56/0.66
 3W  0.62/0.72

 1M  0.70/0.80
 2M  0.82/0.92
 3M  0.92/1.02
 4M  0.97/1.07
 5M  1.05/1.15
 6M  1.14/1.24

 7M  1.19/1.29
 8M  1.23/1.33
 9M  1.28/1.38
10M  1.33/1.43
11M  1.37/1.47
 1Y  1.42/1.52

15M  1.60/1.85
18M  1.64/1.89
21M  1.70/1.95
 2Y  1.86/2.11
```

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16:16 30DEC10             ICAP LONDON                  UK69580        ICAPSHORT2
Contact Reuters EXEU        EURO Short Swaps / FRAs          +44 (0)20 7532 3530
          1M Swaps              IMM Dated                           3m FRAs
  2x1     0.827-0.777      1y  MAR/MAR 1.213-1.163          1x4   1.037-0.987
  3x1     0.839-0.789      1y  JUN/JUN 1.329-1.279          2x5   1.055-1.005
  4x1     0.850-0.800      1y  SEP/SEP 1.477-1.427          3x6   1.080-1.030
  5x1     0.863-0.813      1y  DEC/DEC 1.646-1.596          4x7   1.108-1.058
  6x1     0.875-0.825      2y  MAR/MAR 1.518-1.468          5x8   1.137-1.087
  7x1     0.888-0.838      2y  JUN/JUN 1.671-1.621          6x9   1.166-1.116
  8x1     0.907-0.857      3y  MAR/MAR 1.855-1.805
  9x1     0.924-0.874                                              6m FRAs
 10x1     0.941-0.891                  IMM Fras             1x7   1.264-1.214
 11x1     0.954-0.904                                       2x8   1.284-1.234
 12x1     0.969-0.919      1x7         1.273-1.223          3x9   1.307-1.257
                           2x8         1.292-1.242          4x10  1.332-1.282
 1y /3    1.158-1.108      3x9         1.320-1.270          5x11  1.357-1.307
 15m/3    1.209-1.159      4x10        1.344-1.294          6x12  1.391-1.341
 18m/3    1.270-1.220                                       12x18 1.649-1.599
 21m/3    1.345-1.295      0x3 Today   1.035-0.985          18x24 2.025-1.975
                           0x6 Today   1.259-1.209
 1y /6    1.347-1.297      0x3 Tom     1.033-0.983
 15m/6    1.352-1.302      0x6 Tom     1.260-1.210                12m FRA
 18m/6    1.454-1.404                                       12x24 2.001-1.951
 21m/6    1.497-1.447                        ICAP OIS Fix Menu <ICAPOISFIX01>
ICAP Global Index <ICAP>                      Forthcoming changes <ICAPCHANG
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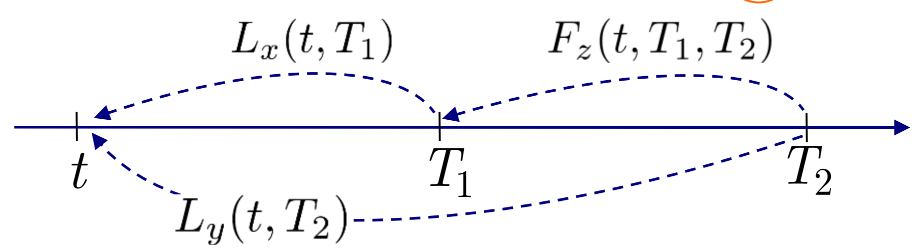
Market Euribor FRA vs Depo implicit forward rates. *Quotations 30 Dec. 2010 (source: Reuters)*

Check Euribor FRA replication (31 Dec 2010)			
Tenor	**FRA replica (%)**	**FRA market (%)**	**Difference (bp)**
1x4	1.101	1.012	8.9
2x5	1.273	1.03	24.3
3x6	1.420	1.055	36.5
4x7	1.556	1.083	47.3
5x8	1.603	1.112	49.1
6x9	1.631	1.141	49.0
1x7	1.323	1.239	8.4
2x8	1.433	1.259	17.4
3x9	1.520	1.282	23.8
4x10	1.636	1.307	32.9
5x11	1.695	1.332	36.3
6x12	1.764	1.366	39.8
12x18	2.335	1.624	71.1
18x24	2.613	2	61.3
12x24	2.469	1.976	49.3



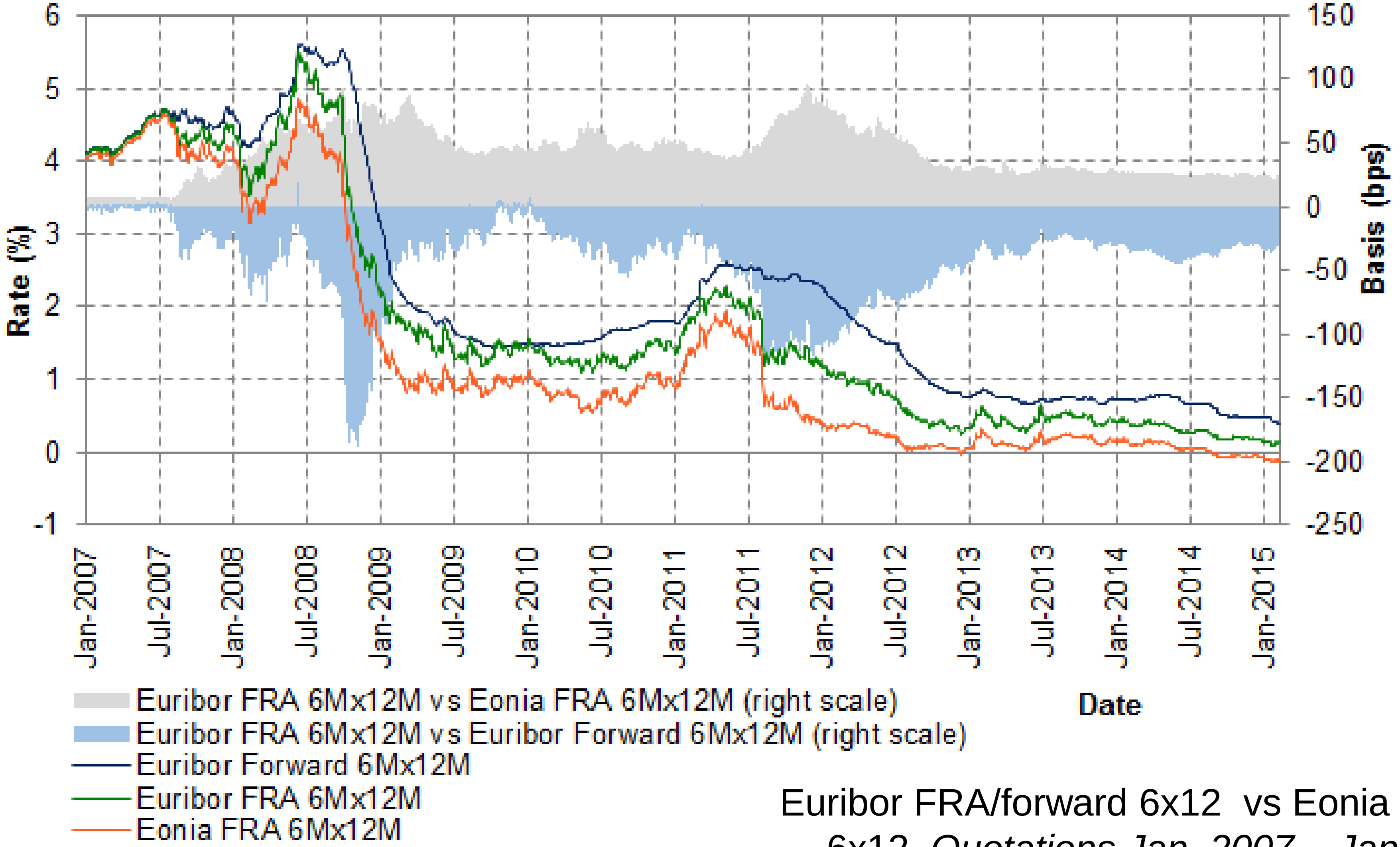
$$\frac{1}{1+L_x(t,T_1)\tau(t,T_1)}\times\frac{1}{1+F_z(t;T_1,T_2)\tau(T_1,T_2)}=\frac{1}{1+L_y(t,T_2)\tau(t,T_2)},$$

$$\Rightarrow F_z(t;T_1,T_2)=\frac{1}{\tau(T_1,T_2)}\left[\frac{1+L_y(t,T_2)\tau(t,T_2)}{1+L_x(t,T_1)\tau(t,T_1)}-1\right].$$

Simple compounded, x,y,z = rate tenors

3: The market across the credit crunch

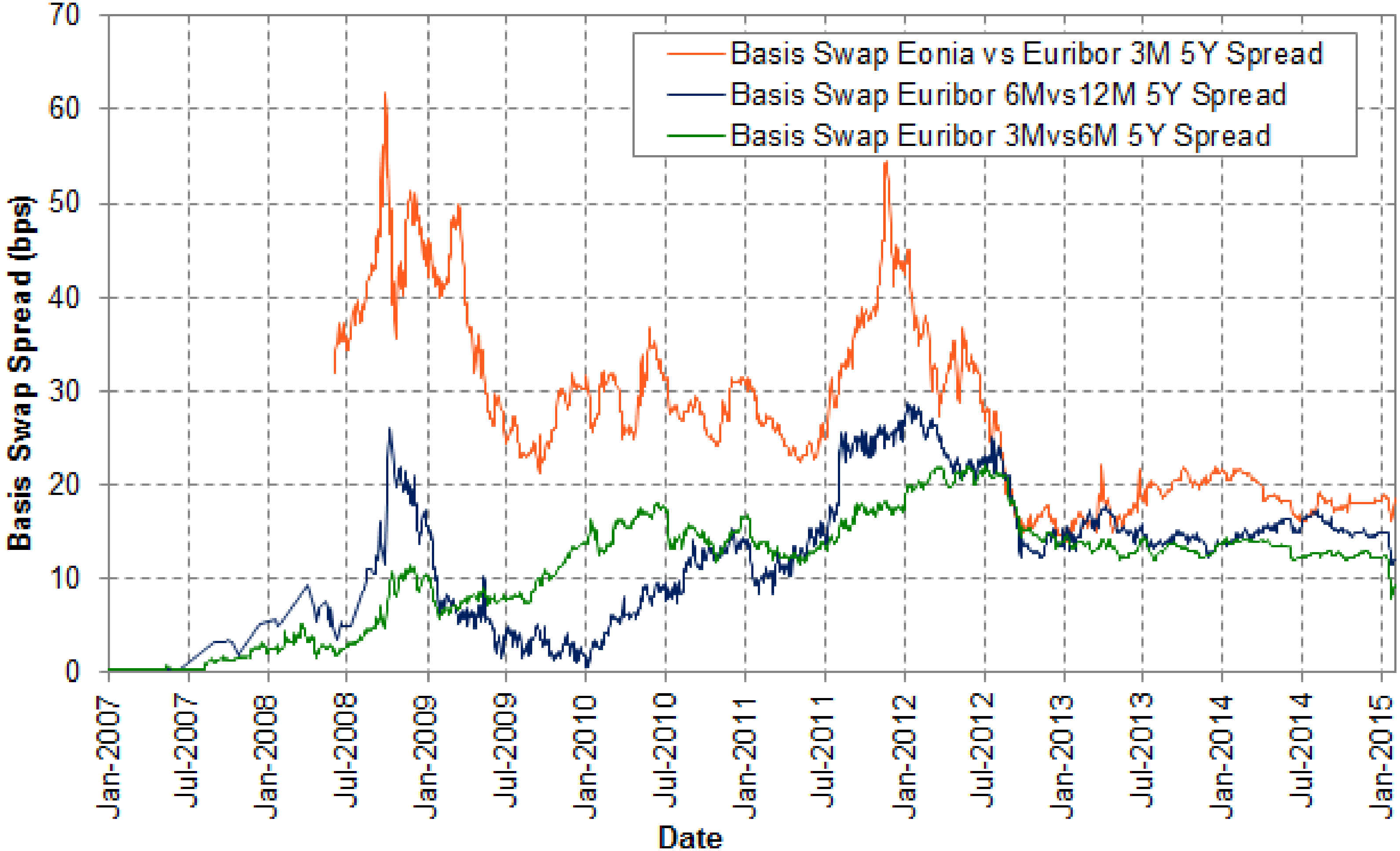
FRA rates [4]



Euribor FRA/forward 6x12 vs Eonia OIS FRA 6x12*. Quotations Jan. 2007 – Jan. 2015 (source: Bloomberg)*

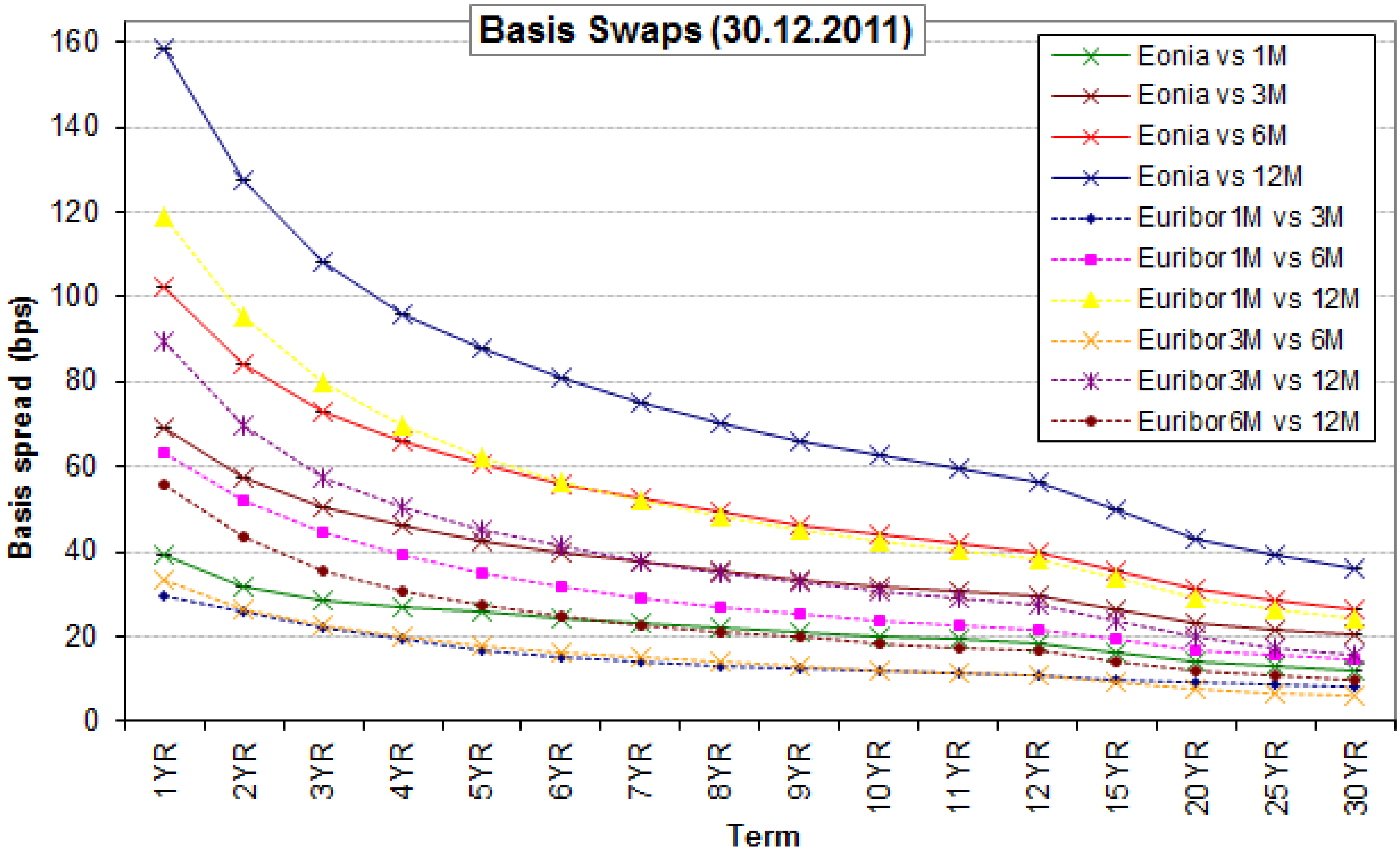
3: The market across the credit crunch

Basis Swap rates [1]



EUR Basis Swap 5Y, Euribor 3M vs 6M vs 12M vs Eonia,
Quotations Jan. 2007 – Dec. 2011 (source: Bloomberg)

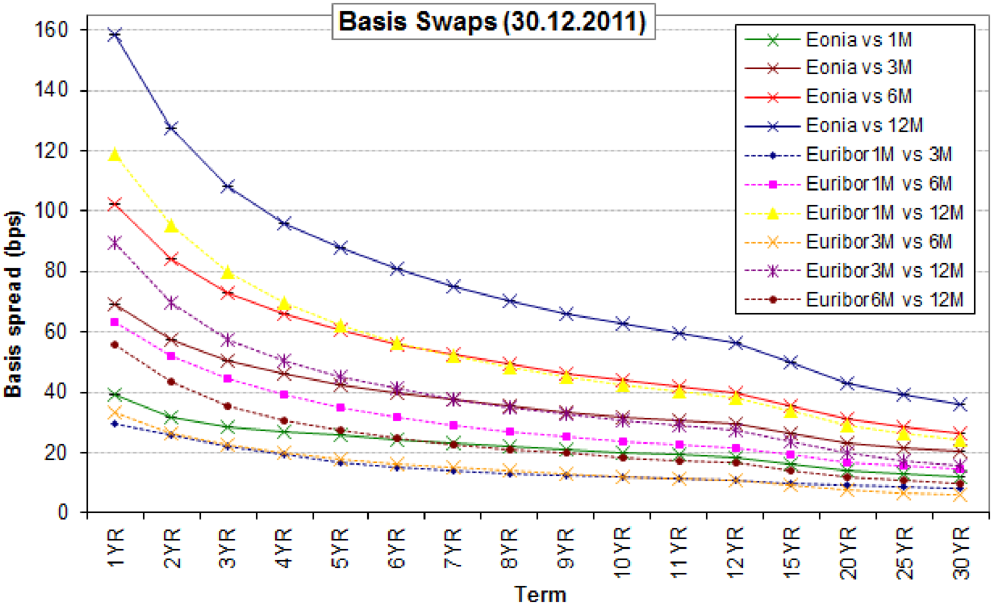
3: The market across the credit crunch
Basis Swap rates [2]



EUR Basis Swaps term structure

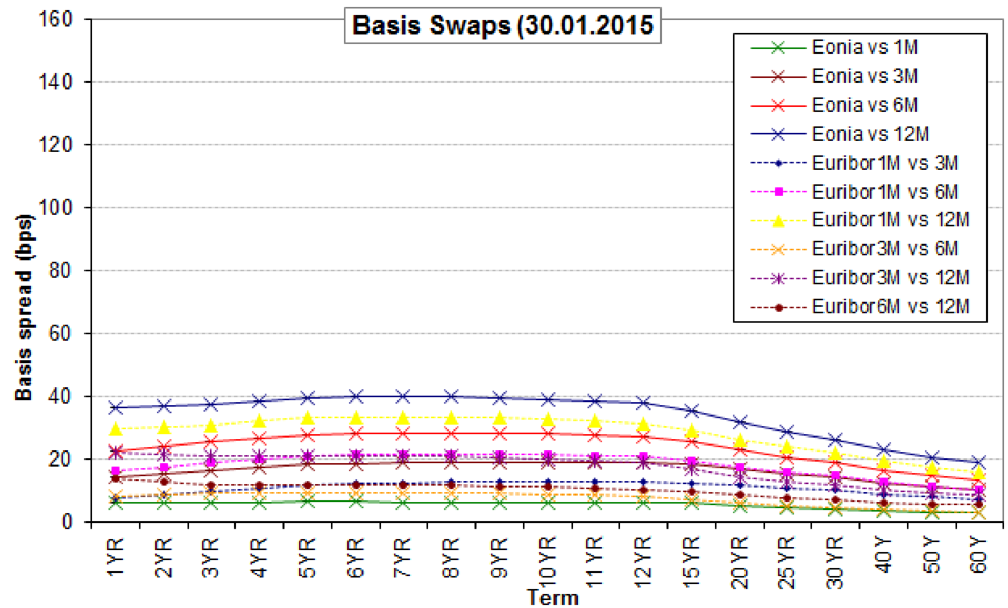
Quotations as of 30 Dec 2011 (source: Reuters, ICAP)

3: The market across the credit crunch
Basis Swap rates [3]



EUR Basis Swaps term structure
Quotations as of 30 Dec 2011
(source: Reuters, ICAP)

EUR Basis Swaps term structure
Quotations as of 30 Jan. 2015
(source: Reuters, ICAP)



3: The market across the credit crunch
Libor and default/liquidity risk [1]

Suppose a Bank B (lender) with excess liquidity, interested to lend today at Libor rate for 6 month. There are (at least) two different strategies:

1) Today Bank B checks its liquidity exposure over the next 6 months, looks for a borrower counterparty C_1, loans the excess liquidity for 6M (e.g. entering into a 6M Deposit), and recovers liquidity plus Libor6M interest back in 6 months, if the borrowing counterparty C_1 has not defaulted.

2) Today Bank B checks its liquidity exposure over the next 3 months, looks for a borrower counterparty C_1, loans the excess liquidity for 3M (e.g. entering into a 3M Deposit), and recovers liquidity plus Libor3M interest back in 3 months, if the borrowing counterparty C_1 has not defaulted.
   Three months later, Bank B checks again its liquidity exposure over the following 3 months, looks for a borrower counterparty C_2 (the same or another one), loans the excess liquidity for the following 3M (e.g. entering into a 3M Deposit), and recovers liquidity plus Libor3M interest back in 3 months, if the borrowing counterparty C_2 has not defaulted.

3: The market across the credit crunch
Libor and default/liquidity risk [2]

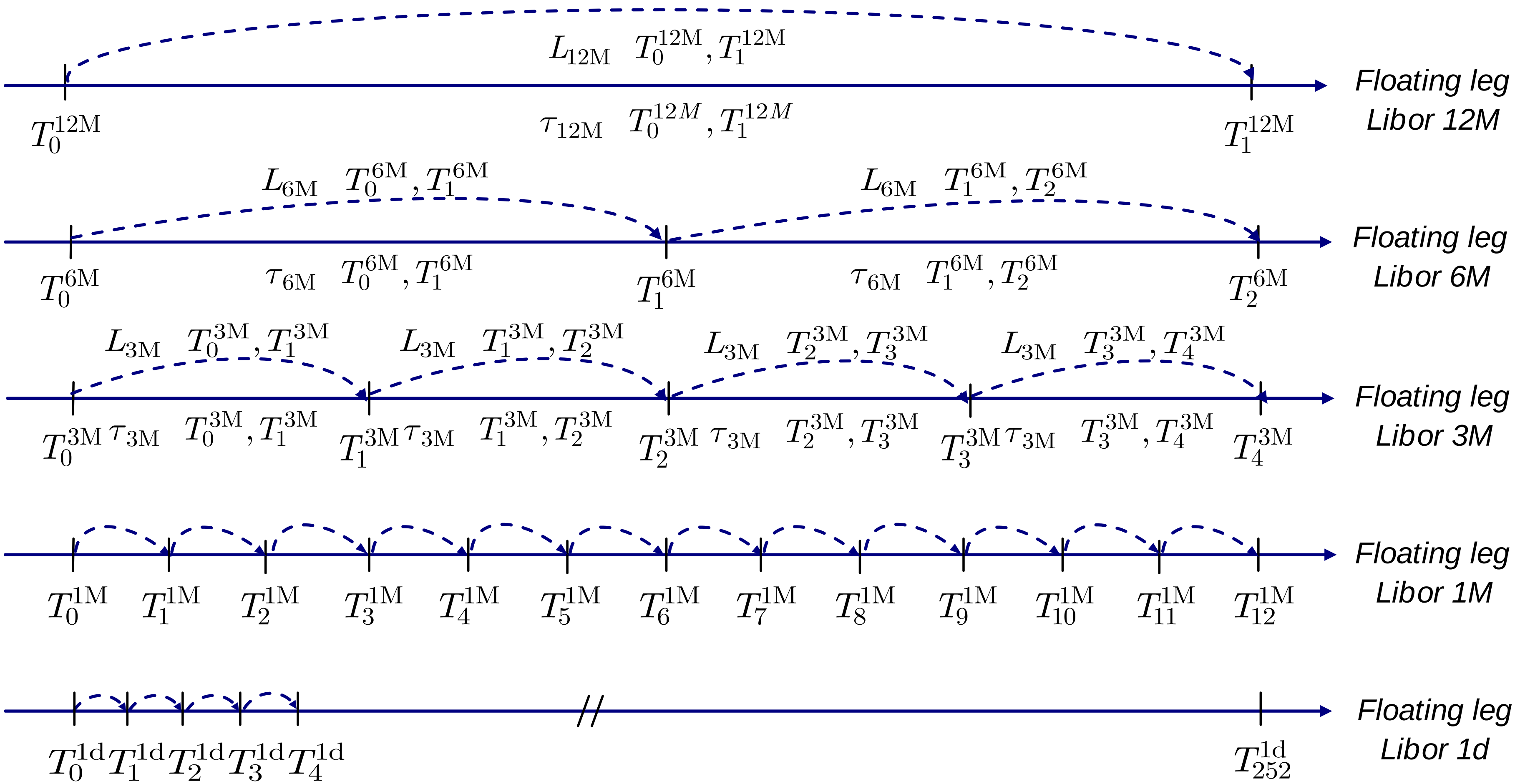
Bank B suffers a lower credit/liquidity risk adopting the second strategy, because the expected default probability of counterparty C_1 over [0-6M] is higher than the expected default probability composed for C_1 over [0,3M] and C_2 over [3M-6M].

This is reflected into lower Libor 3M+3M rate with respect to Libor 6M rate.

The two strategies above may be packed into a Basis Swap Libor3M vs Libor6M, with maturity 6M. The floating leg tied to Libor6M represents the first strategy, and the floating leg tied to Libor3M represent the second strategy. In this case only interests are exchanged, not the notional amounts.

Since the Basis Swap is a collateralized instrument, there is a negligible counterparty risk. Thus a (positive) spread is required onto the 3M leg to reach an equilibrium.

3: The market across the credit crunch
Basis Swap rates [3]



Picture of floating Swap legs with equal maturities ($T_1^{12M} = T_2^{6M} = T_4^{3M} = T_{12}^{1M} = T_{252}^{1d} = 1Y$) and different Libor tenors (12M, 6M, 3M, 1M, 1d from top to bottom).

3: The market across the credit crunch
Basis Swap rates [4]

$$\begin{aligned}
\Pi(T_0) &\to P(T_0;T_1^{12M})L_{12M}(T_0^{12M},T_1^{12M})\tau_{12M}(T_0^{12M},T_1^{12M}) \\
&\simeq \sum_{j=1}^{2} P(T_0;T_j^{6M})F_{6M}(T_{j-1}^{6M},T_j^{6M})\tau_{6M}(T_{j-1}^{6M},T_j^{6M}) \\
&\simeq \sum_{j=1}^{4} P(T_0;T_j^{3M})F_{3M}(T_{j-1}^{3M},T_j^{3M})\tau_{3M}(T_{j-1}^{3M},T_j^{3M}) \\
&\simeq \sum_{j=1}^{12} P(T_0;T_j^{1M})F_{1M}(T_{j-1}^{1M},T_j^{1M})\tau_{1M}(T_{j-1}^{1M},T_j^{1M}) \\
&\simeq \sum_{j=1}^{252} P(T_0;T_j^{1d})F_{1d}(T_{j-1}^{1d},T_j^{1d})\tau_{1d}(T_{j-1}^{1d},T_j^{1d}) \\
&\simeq 1 - P(T_0;T_1^{12M}),
\end{aligned}$$

$$F_{x,i}(t) \to \frac{1}{\tau(T_{i-1},T_i)}\left[\frac{P(t;T_{i-1})}{P(t;T_i)} - 1\right] \ \forall x.$$

Classical pricing of floating Swap legs with equal maturities and different Libor tenors (12M, 6M, 3M, 1M, 1d). All floating legs with any tenor may be replicated with a single Zero Coupon Bond with the corresponding maturity, and thus they have the same value. In geometrical terms: replication invariance under Libor tenor transformations.

3: The market across the credit crunch
Basis Swap rates [5]

$$
\begin{aligned}
\Pi(T_0) &\rightarrow P(T_0;T_1^{12M})L_{12M}(T_0^{12M},T_1^{12M})\tau_{12M}(T_0^{12M},T_1^{12M}) \\
&\neq \sum_{j=1}^{2} P(T_0;T_j^{6M})F_{6M}(T_{j-1}^{6M},T_j^{6M})\tau_{6M}(T_{j-1}^{6M},T_j^{6M}) \\
&\neq \sum_{j=1}^{4} P(T_0;T_j^{3M})F_{3M}(T_{j-1}^{3M},T_j^{3M})\tau_{3M}(T_{j-1}^{3M},T_j^{3M}) \\
&\neq \sum_{j=1}^{12} P(T_0;T_j^{1M})F_{1M}(T_{j-1}^{1M},T_j^{1M})\tau_{1M}(T_{j-1}^{1M},T_j^{1M}) \\
&\neq \sum_{j=1}^{252} P(T_0;T_j^{1d})F_{1d}(T_{j-1}^{1d},T_j^{1d})\tau_{1d}(T_{j-1}^{1d},T_j^{1d}) \\
&\neq 1 - P(T_0;T_1^{12M}),
\end{aligned}
$$

$$
F_{x,i}(t) \neq \frac{1}{\tau(T_{i-1},T_i)}\left[\frac{P(t;T_{i-1})}{P(t;T_i)} - 1\right] \; \forall x.
$$

Modern pricing of floating Swap legs with equal maturities and different Libor tenors. All floating legs with any tenor may not be replicated with a single Zero Coupon Bond with the corresponding maturity, and thus they have different values. In geometrical terms: broken replication invariance under Libor tenor transformations.

3: The market across the credit crunch
Credit risk component: Synthetic CDS Euribor Index [1]

Synthetic CDS Euribor Index

Includes daily quotations of CDS spread for a given maturity, referred to financial institutions that belong to the Euribor panel in December 2011.

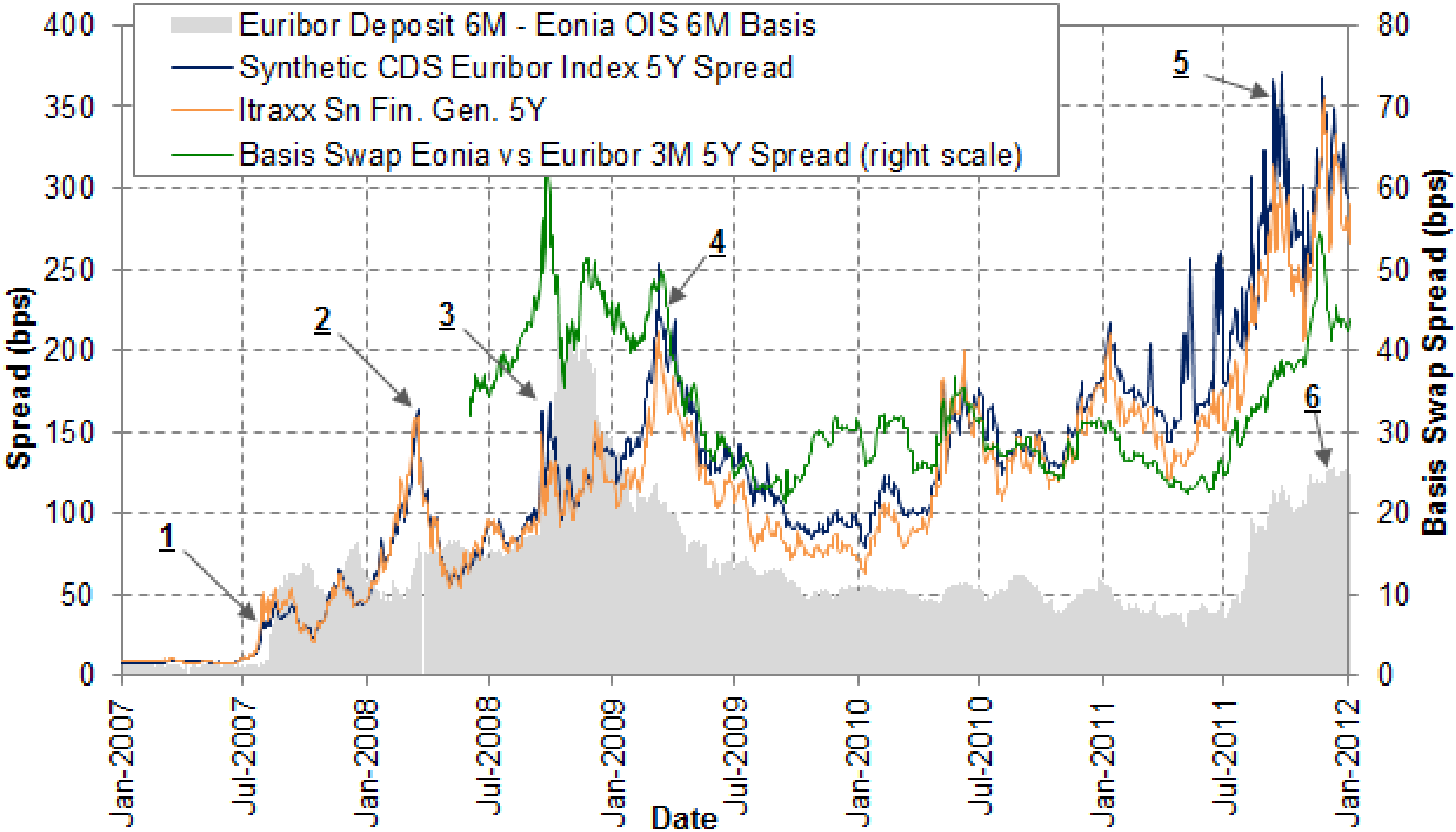
Its computation replicates the fixing mechanism of Euribor rates: for each reference date, we exclude the highest and the lowest 15% CDS spread quotations and compute the average of the remaining 70% quotes.

It represents the average cost for protection against the default of an Euribor panel bank within the European financial market.

A better construction would consider the actual composition of the Euribor panel at each date.

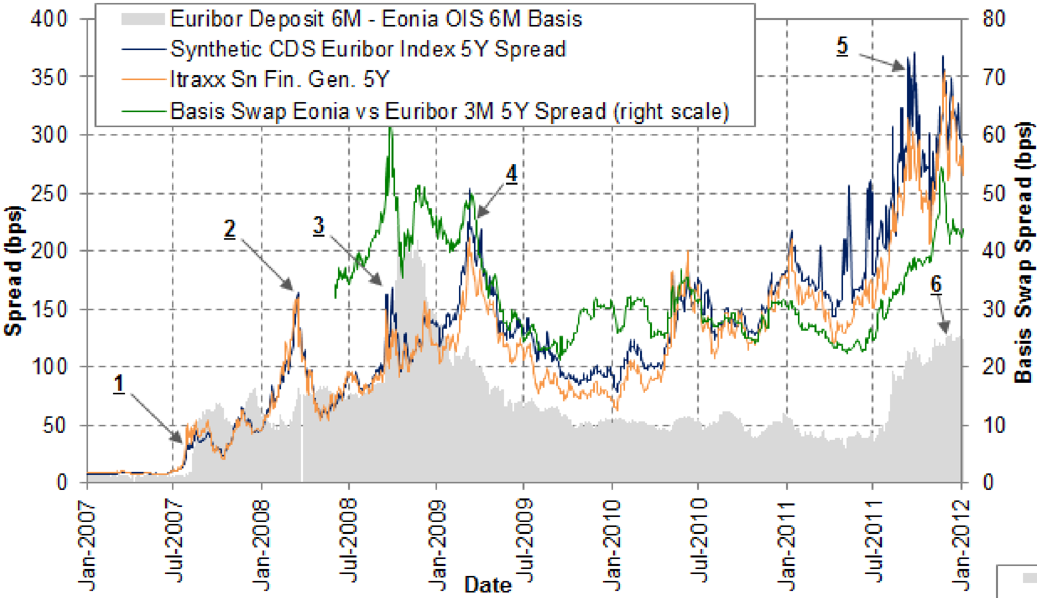
3: The market across the credit crunch

Credit risk component: Synthetic CDS Euribor Index [2]



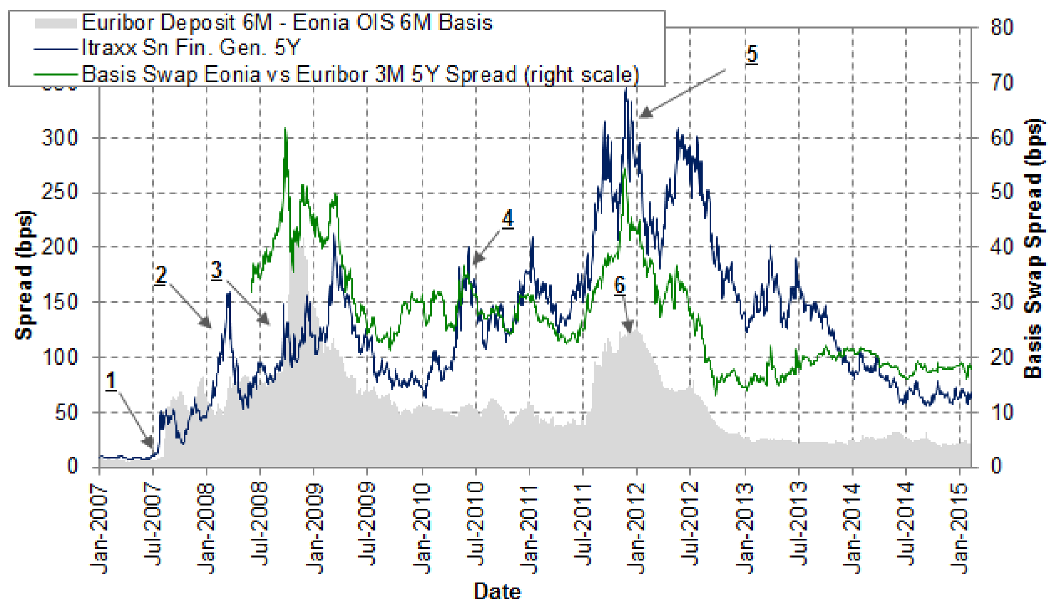
Synthetic CDS Euribor Index 5Y, vs Itraxx Senior Financial Generic 5Y, vs Euribor-OIS 6M basis vs Basis Swap Euribor6M/Eonia (right scale). (Quotations Jan. 2007 – Dec. 2011, source: Bloomberg).

3: The market across the credit crunch
Credit risk component: Synthetic CDS Euribor Index [3]

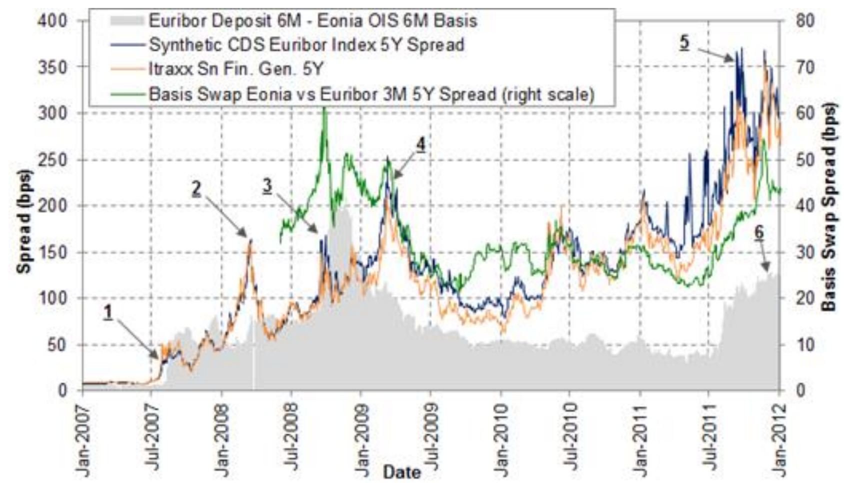


Itraxx Senior Financial Generic 5Y, vs Euribor-OIS 6M basis vs Basis Swap Euribor6M/Eonia (right scale). (Quotations Jan. 2007 – Jan. 2015, source: Bloomberg).

Synthetic CDS Euribor Index 5Y, vs Itraxx Senior Financial Generic 5Y, vs Euribor-OIS 6M basis vs Basis Swap Euribor6M/Eonia (right scale). (Quotations Jan. 2007 – Dec 2011, source: Bloomberg).



3: The market across the credit crunch
Credit risk component: Synthetic CDS Euribor Index [3]



- Peak 1 (Aug. 2007): rise of concerns over banks' exposure to credit structured products (i.e. CDO, ABS etc.), triggering the explosion of the Euribor – Eonia basis and an increase of Euribor rates.
- Peak 2 (14 Mar. 2008) : bail-out of Bear Stearns.
- Peak 3 (15 Sep. 2008): bankruptcy of Lehman Brothers.
- Peak 4 (Mar. 2009): deterioration of financial markets unlashed by the failure of Lehman. End of the "Banks are too big to fail" paradigm. Stress not reflected in the Euribor – Eonia basis, mainly driven by the loosening monetary policy decisions of central banks.
- Peak 5 (Sep. 2011): Euro sovereign debts crisis and Italy's credit rating cut, End of the "States are too big to fail" paradigm.
- Peak 6: rise of the Euribor – Eonia basis (127 bps on 1st Dec. 2011, following concerns about EU possibility to face the crisis.

- The Basis Swap Eonia Vs Euribor 3M generally tracks the Synthetic CDS Euribor Index, especially in correspondence of high perceived credit risk, thus revealing a stronger relevance of the credit risk component, at least over longer maturities (5Y).

3: The market across the credit crunch
Liquidity risk component: Liquidity Surplus Index [1]

Liquidity Risk

The liquidity risk component in Euribor and Eonia interbank rates is distinct but strongly related to the credit risk component.
According to Acerbi and Scandolo (2007), liquidity risk may appear in at least three circumstances:

- funding liquidity risk: lack of liquidity to cover short term debt obligations,
- market liquidity risk: difficulty to liquidate assets on the market due excessive bid-offer spreads,
- systemic liquidity risk: difficulty to borrow funds on the market due to excessive funding cost.

Following Morini (2009), these three elements are, in principle, not a problem until they do not appear together, because a bank with, for instance, problem 1 and 2 (or 3) will be able to finance itself by borrowing funds (or liquidating assets) on the market.
During the crisis these three scenarios manifested themselves jointly at the same time, thus generating a systemic lack of liquidity (see e.g. Michaud and Upper 2008).
Clearly, it is difficult to disentangle liquidity and credit risk components in the Euribor and Eonia rates, because, in particular, they do not refer to the default risk of one counterparty in a single derivative deal but to a money market with bilateral credit risk (see the discussion in Morini (2009) and references therein).

Liquidity risk component: Liquidity Surplus Index [2]

Liquidity Surplus Index

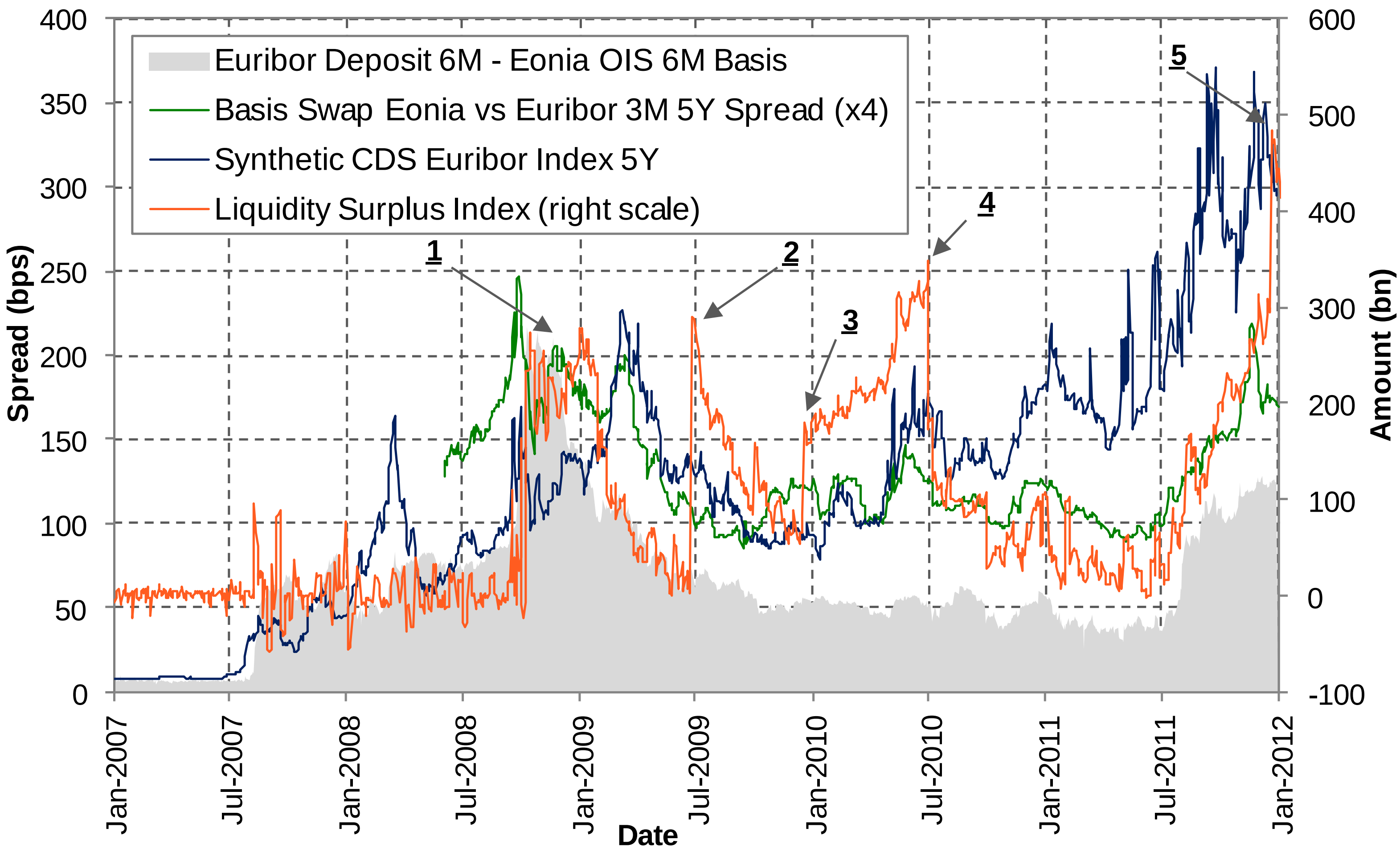
Includes official ECB's data regarding the ECB Deposit Facility. Is given by the sum of:

- the total amount of the deposits posted by the EU financial institutions at the ECB's Deposit Facility
- the current account holdings exceeding the EUR market-wide level minimum reserve requirement that are held by EU financial institutions at the ECB.

The higher the Liquidity Surplus Index, the stronger the preference of EU financial institutions to deposit cash at the ECB instead of lending in the interbank market or investing in more profitable (and risky) activities. Hence the Index reflects the liquidity stress in the Euro zone interbank market.

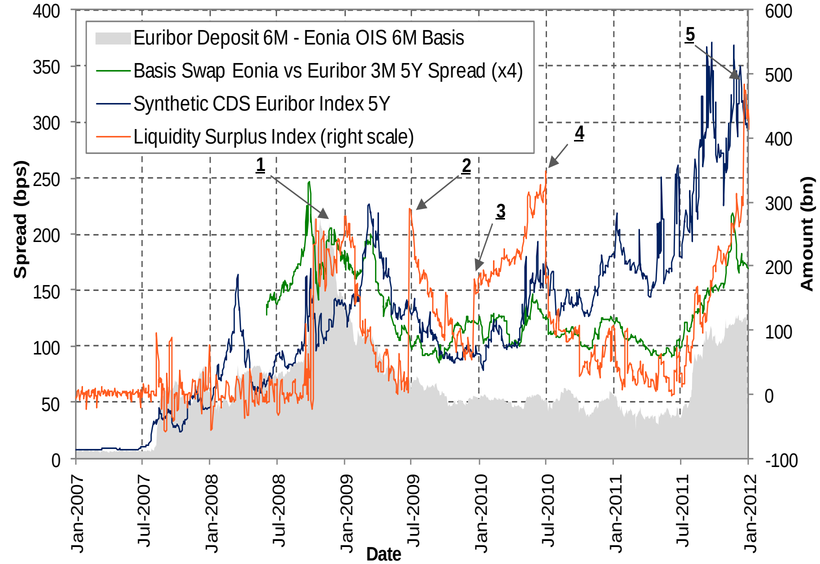
The ECB requires EU financial institutions to hold minimum reserves amounts on accounts managed by National Central Banks, in order to stabilize the market interest rates and to facilitate the role of the ECB as liquidity supplier for the interbank market. The amount of minimum reserves is fixed, on a monthly basis, according to each financial institution's reserve base. The requirement is verified considering the average, during a certain maintenance period, of the amounts posted by the financial institution at the reserve accounts, thus allowing some flexibility to face minimum reserves provisions without compromising their business or investing opportunities. Holdings of required reserves are remunerated at the Main Refinancing Operation (MRO) rate, while holdings that exceeded the reserve requirement are not rewarded.

3: The market across the credit crunch
Liquidity risk component: Liquidity Surplus Index [3]



Synthetic CDS Euribor index 5Y vs Liquidity Surplus Index vs Basis Swap Eonia Vs Euribor 3M 5Y (x4) vs Euribor Deposit 6M – Eonia OIS 6M basis (area). (Jan. 2007 – Dec. 2011, sources: Bloomberg and ECB).

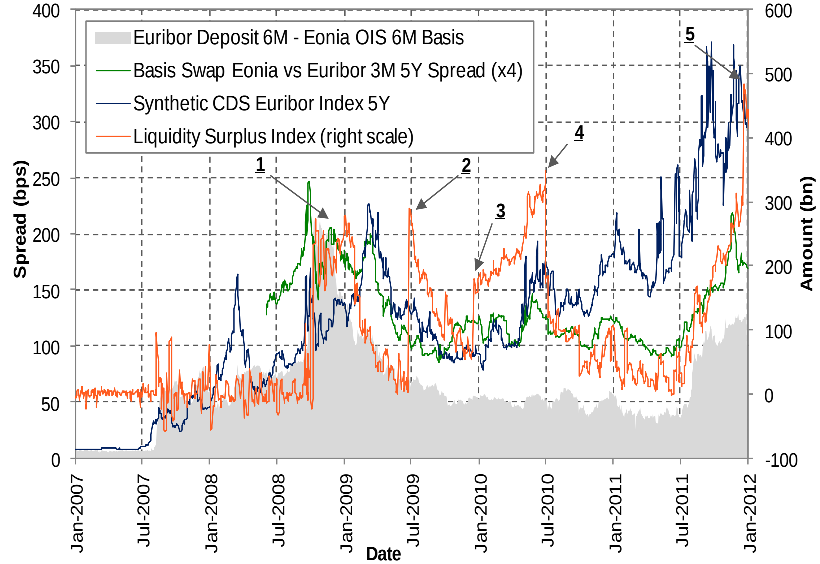
3: The market across the credit crunch
Liquidity risk component: Liquidity Surplus Index [4]



- Peak 1 (Oct. 2008): sudden explosion of the Liquidity Surplus Index, due to ECB's adoption of several monetary policy measures (cut of the official interests rates in conjunction with others central banks , introduction of a fixed-rate refinancing operation with full-allotment, extension of the securities accepted as collateral, increase of the number of financial institutions that can access to the ECB monetary policy channels).
- Peak 2 (25 Jun. 2009): ECB introduces the Long Term Refinancing Operations (LTRO) with 12M term, reducing the liquidity shortage the EU financial market. Contextual decrease of the Euribor – Eonia basis and of the Synthetic CDS Euribor Index.
- Peak 3 (Dec. 2009): extension of the fixed-rate refinancing operations with full allotment.
- Peak 4 (May 2010): worsening of the sovereign debt crisis related to market concerns on Greece's capability to maintain its debt obligations, reflected also by the increase of Synthetic CDS Euribor index (200 bps in June 2010). LTRO expiration and sudden decrease of the Liquidity Surplus Index, until July 2011.

3: The market across the credit crunch
Liquidity risk component: Liquidity Surplus Index [5]

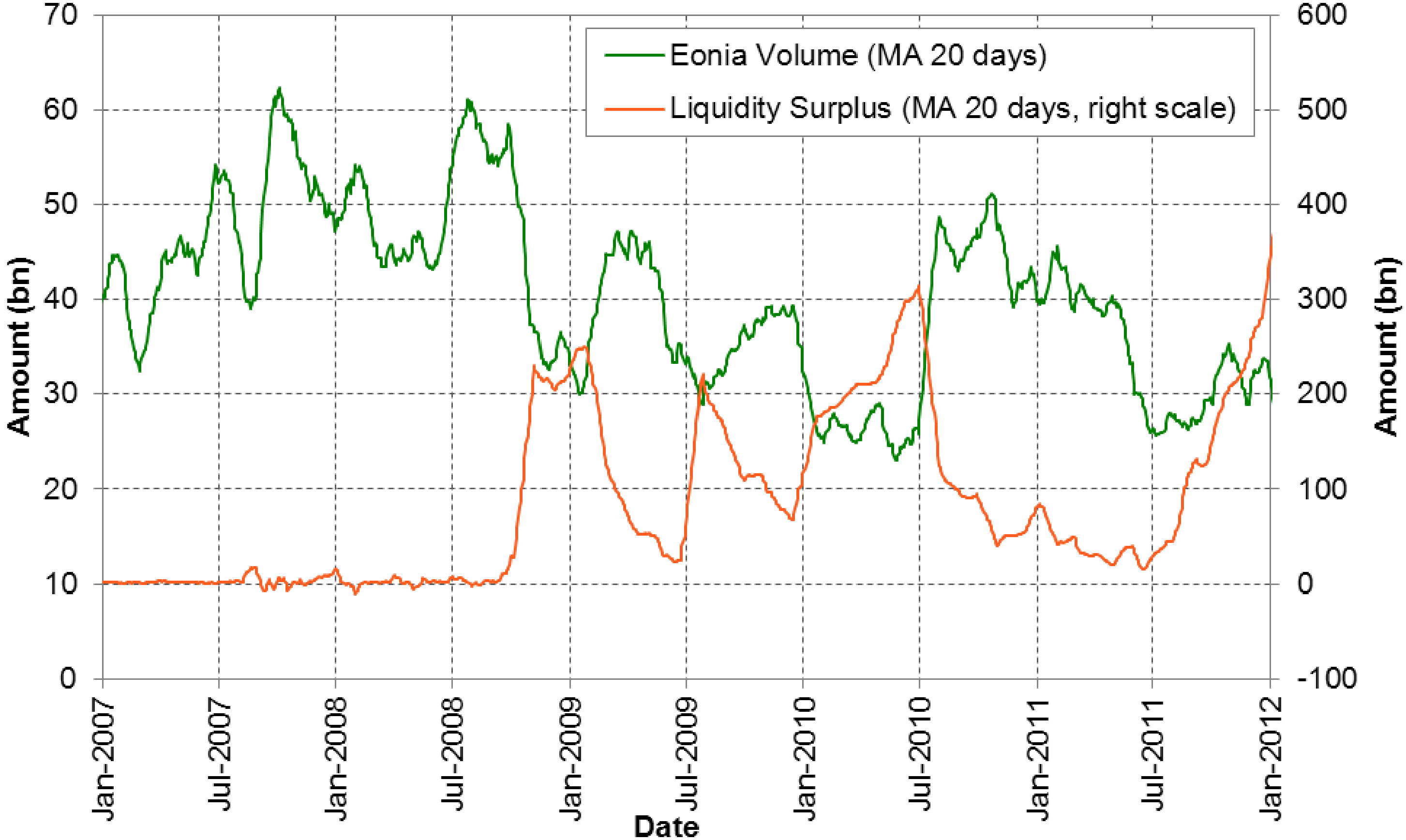
- Jul. 2009 – Jul. 2011: the Euribor – Eonia basis maintained a relatively stable and low level compared to Aug. 2007 – Jun. 2009, showing contained peaks in correspondence of a simultaneous increase of both the Liquidity Surplus Index and the Synthetic Surplus Index.



- Second half of 2011: second phase of the sovereign debt crisis affecting Italy and Spain. Increased credit and liquidity risk reflected by the rise of Synthetic CDS Euribor Index and Liquidity Surplus Index, respectively. Rise of the Euribor-Eonia basis due to stable Euribor and decreasing Eonia.
- Peak 5 (22 Dec. 2011): ECB's allows multi-tranche LTRO with maturity 3Y. The first LTRO tranche provided €489.2 billion to 523 financial institutions on 21 Dec. 2011, pushing the Liquidity Surplus Index at €483 billion. Almost all the liquidity offered to the market by the ECB was posted at the ECB's accounts. Significant increase in the Euribor – Eonia basis.

3: The market across the credit crunch

Liquidity risk component: Liquidity Surplus Index [6]



Liquidity Surplus Index vs Eonia Volumes (Jan. 2007 – Dec. 2011, sources: Bloomberg, ECB). An increase in the liquidity amount posted at the ECB is always accompanied by a reduction of the total amount traded in the European money market. The drain of liquidity on the money market is the main reason of the closeness of the Eonia rate to the Deposit Facility Rate during the financial crisis.

3: The market across the credit crunch
Liquidity vs credit risk: CDS vs Bond rates

Bond prices vs CDS spreads vs Euribor6M-Eonia OIS 6M basis

Quotations Jun. 2008 – Oct. 2010 (source: Bloomberg)

3: The market across the credit crunch
Liquidity vs credit risk: conclusions

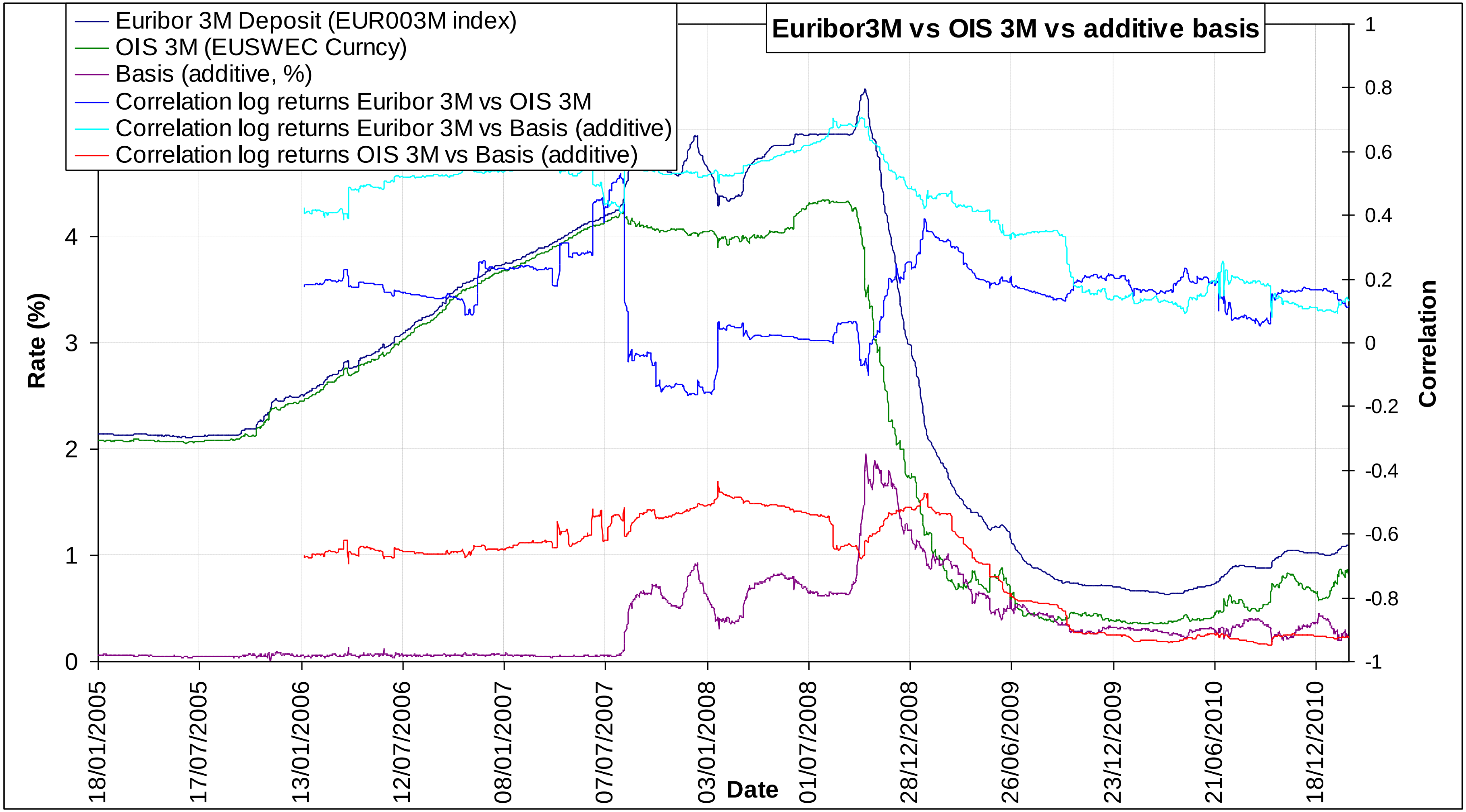
By considering the trend of the Liquidity Surplus Index we argue that, from the Lehman Brothers' bankruptcy up to the end of the 2011, the liquidity risk factor has played a key role, in conjunction with the credit risk, in explaining the trend of the Euribor – Eonia basis.

In general, we observe an increase of the Euribor - Eonia basis when both the Synthetic CDS Euribor Index and the Liquidity Surplus Index start to rise. This combined upward movement is caused by an increase of the overall risk perceived within the interbank market.

Looking at the historical series, we claim that the Euribor – Eonia basis' peak of October 2008 was caused, initially, by the increase in the average default risk of the market in correspondence of the Lehman crash and, subsequently, by the liquidity risk in the interbank market and the drastic official interest rate cut operated by the ECB during that period.

Also, the upward trend of the Euribor – Eonia basis in the second half of 2011 was driven by a simultaneous rise of both the credit and liquidity risk in the interbank market, reflected in the market through a decrease of the Eonia OIS rates and almost stable Euribor rates.

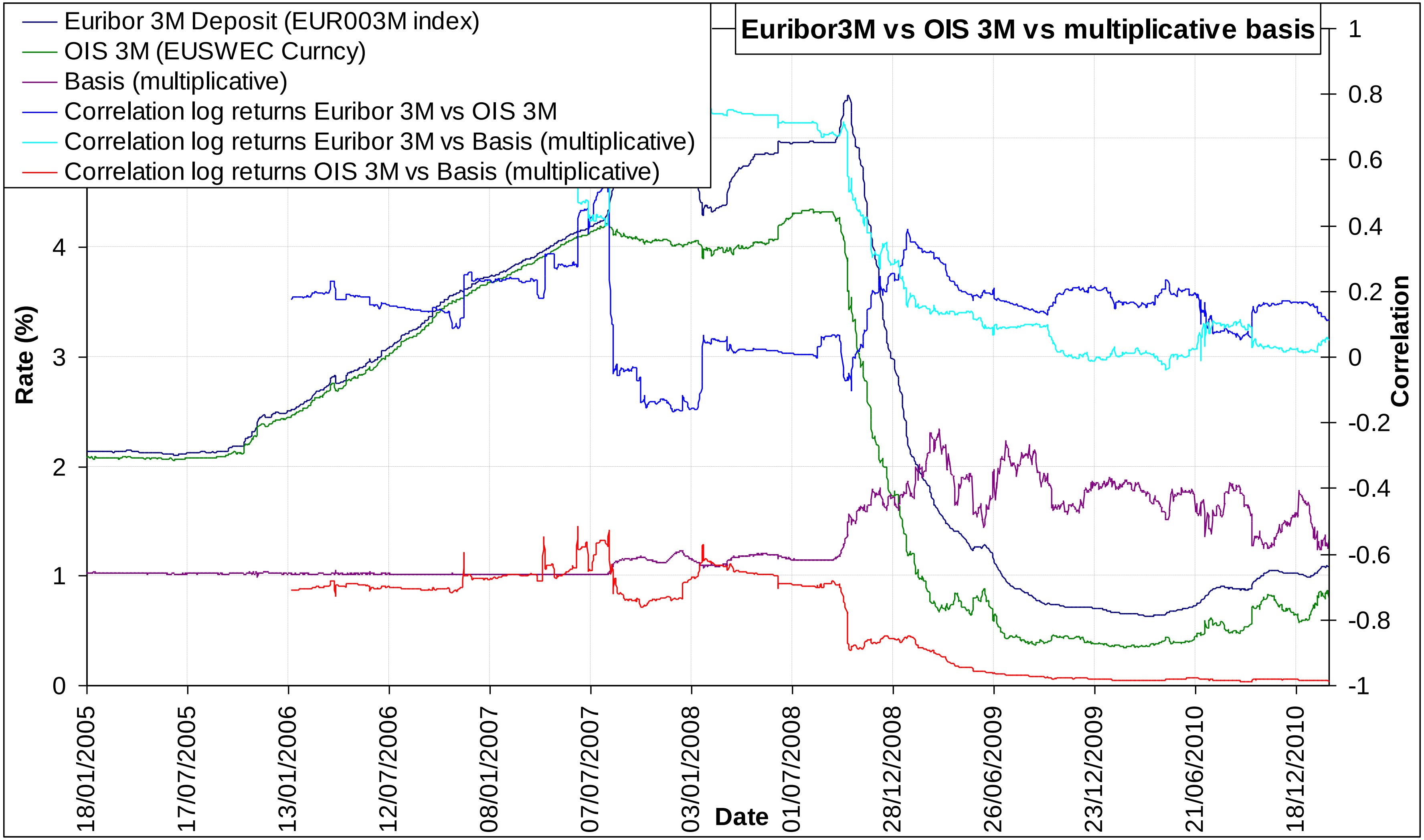
3: The market across the credit crunch
Stochastic basis [1]



EUR 3M OIS rates vs 3M Depo (spot) rates

Additive basis, 1 year correlations on log returns (source: Bloomberg)

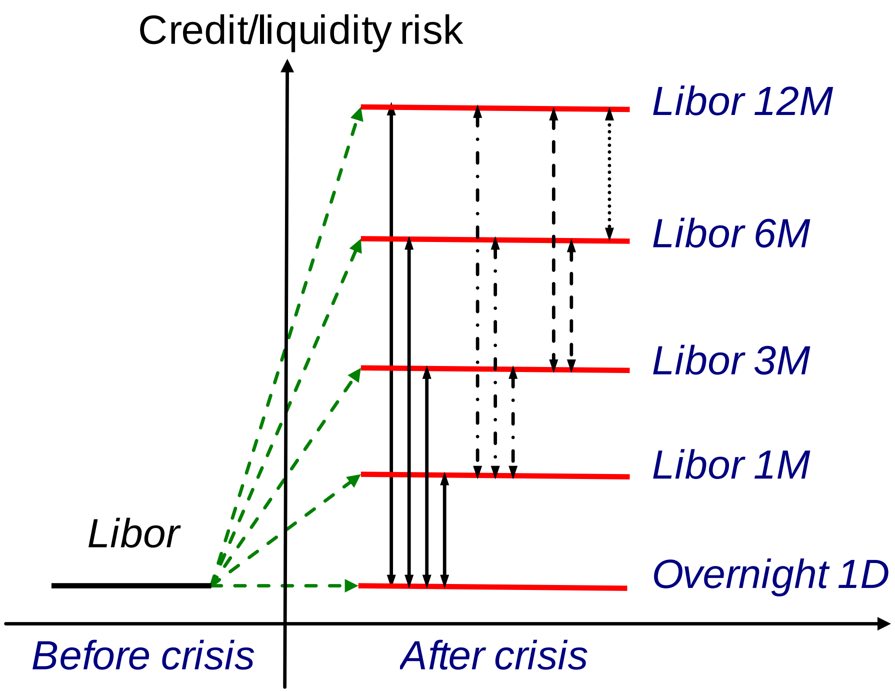
3: The market across the credit crunch
Stochastic basis [2]

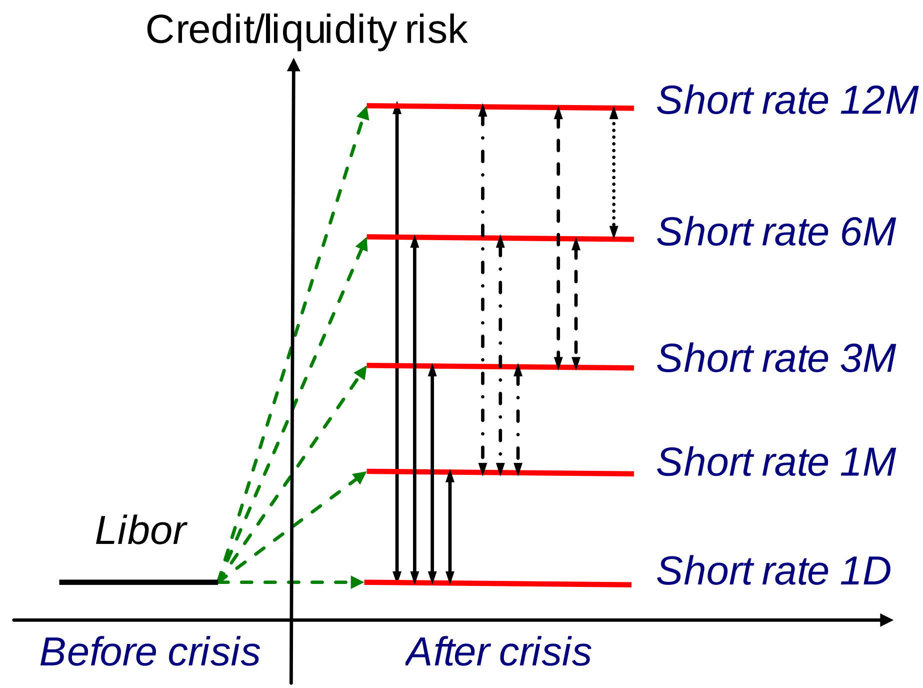


As before, multiplicative basis

3: The market across the credit crunch
Interest rate market segmentation: rationale

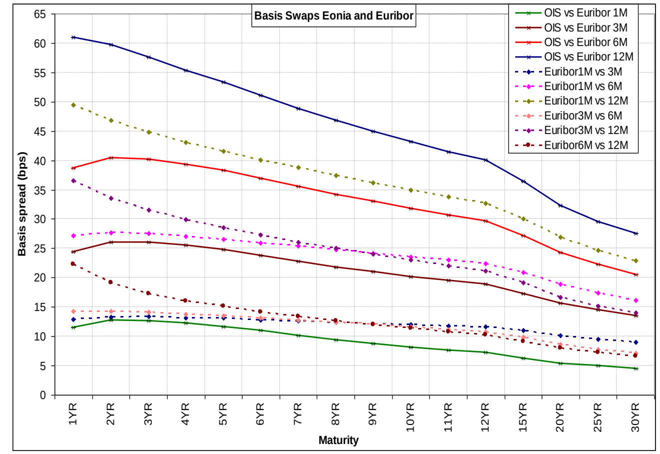
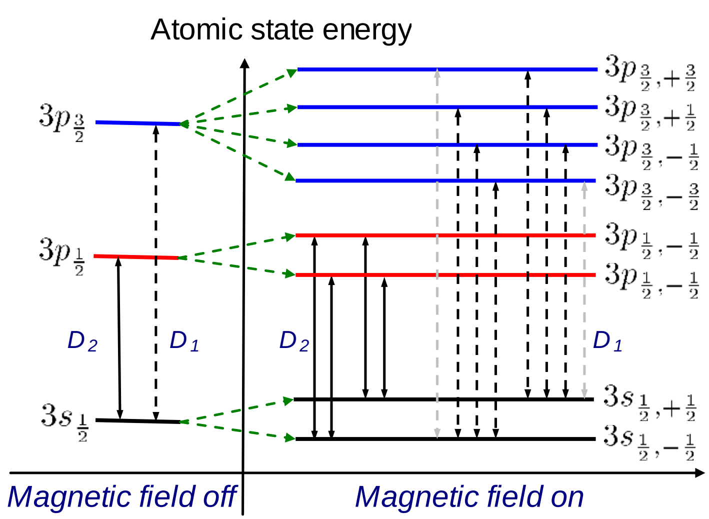
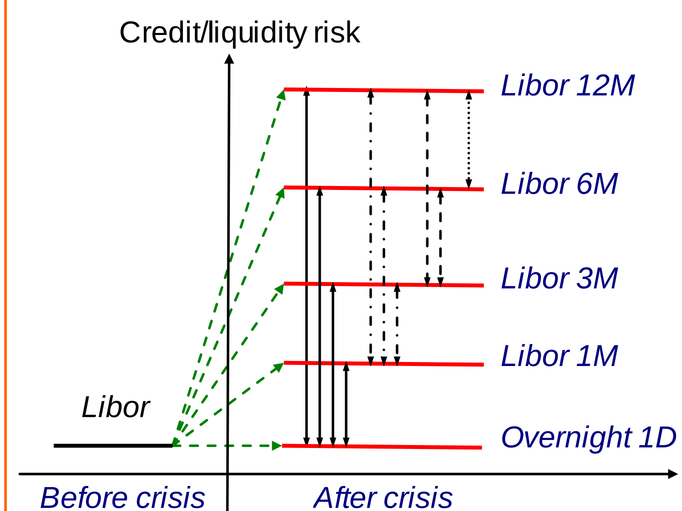
Thinking in terms of more fundamental dynamical variables, e.g. a short rate, the credit crunch has acted as a sort of symmetry breaking mechanism: from a (unstable) situation in which an unique short rate process was able to model and explain the whole term structure of interest rates of all tenors, towards a sort of market segmentation into sub-areas corresponding to instruments with different underlying rate tenors, characterised, in principle, by distinct dynamics, e.g. distinct short rate processes (the Zeeman effect in finance…) (see also B. Tuckman, P. Porfirio, "*Interest Rate Parity, Money Market Basis Swaps, and Cross-Currency Basis Swaps*", Lehman Brothers, Jun. 2003).





3: The market across the credit crunch

The Zeeman effect in physics and finance [1]

	The Zeeman effect in Physics	The Zeeman effect in finance
Experiment / observation		
Theory / explanation		

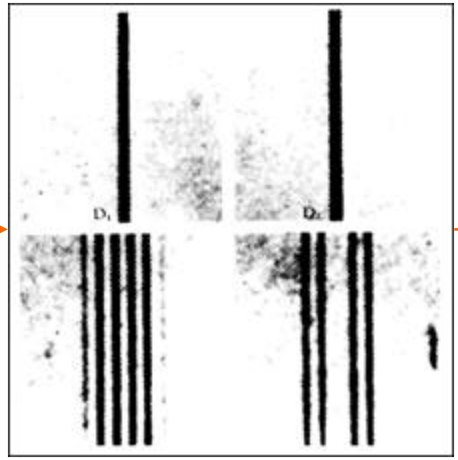
3: The market across the credit crunch
The Zeeman effect in physics and finance [2]

	The Zeeman effect in Physics	The Zeeman effect in finance
Discovery	Pieter Zeeman, Leyden, Aug. 1896.	Global markets, Aug. 2007.
Observation	Atomic spectral lines in a magnetic field split into groups, called Zeeman multiplets (Zeeman, 1896).	Basis swap with different tenors show, after the credit crunch, large gaps, called basis spreads.
Technology	Light spectroscopy, spectrometer.	Libor spectroscopy, market data providers.
Interpretation	In normal conditions (i.e. zero magnetic field): o distinct atomic states displaying symmetry properties (e.g. rotational invariance) have the same energy, o the corresponding spectral lines overlaps.	In pre-credit crunch conditions (i.e. negligible credit/liquidity risk), o distinct Libor floating legs with equal maturity and different tenors have the same value (replication invariance); o the corresponding Basis Swaps display negligible basis spread.
	The external magnetic field breaks the atomic rotational symmetry and splits the atomic states at different energies.	The credit and liquidity risk breaks the Libor tenor symmetry and splits equivalent Libor floating legs at different values.
	The corresponding spectral lines split into observable multiplets (H. Lorentz, Nov. 1896).	The corresponding basis swaps split into non-negligible basis spreads.
Consequences	Electron discovery, atomic structure, quantum mechanics.	Risky interest rates, multiple yield curves, CSA discounting, multi-curve pricing models.

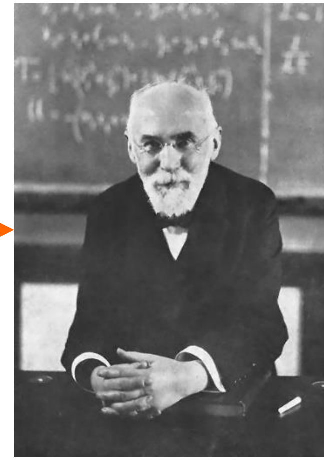
(see M. Bianchetti, "*The Zeeman Effect in Finance*", *www.ssrn.com*, 2011)

3: The market across the credit crunch
The Zeeman effect in physics and finance [3]

Pieter Zeeman (1865-1943)

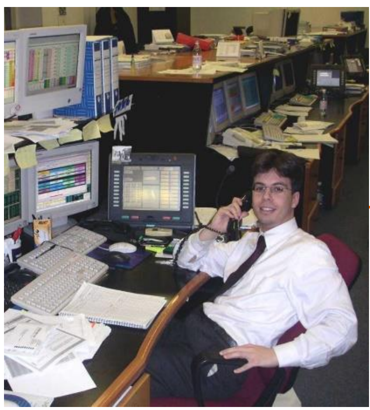


Zeeman's observation of the Zeeman effect (1896)

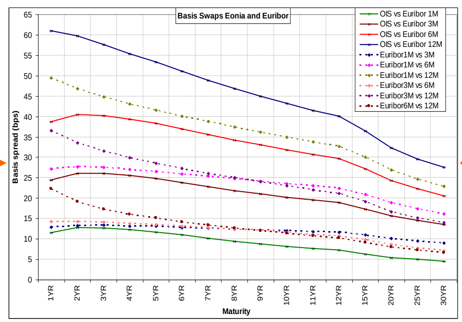


Lorentz's theoretical interpretation (1896)

Zeeman + Lorentz Nobel Prize in physics (1902)



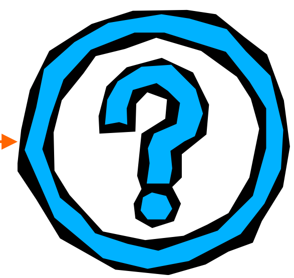
A typical trader (living)



Libor/OIS basis market observation (2007)

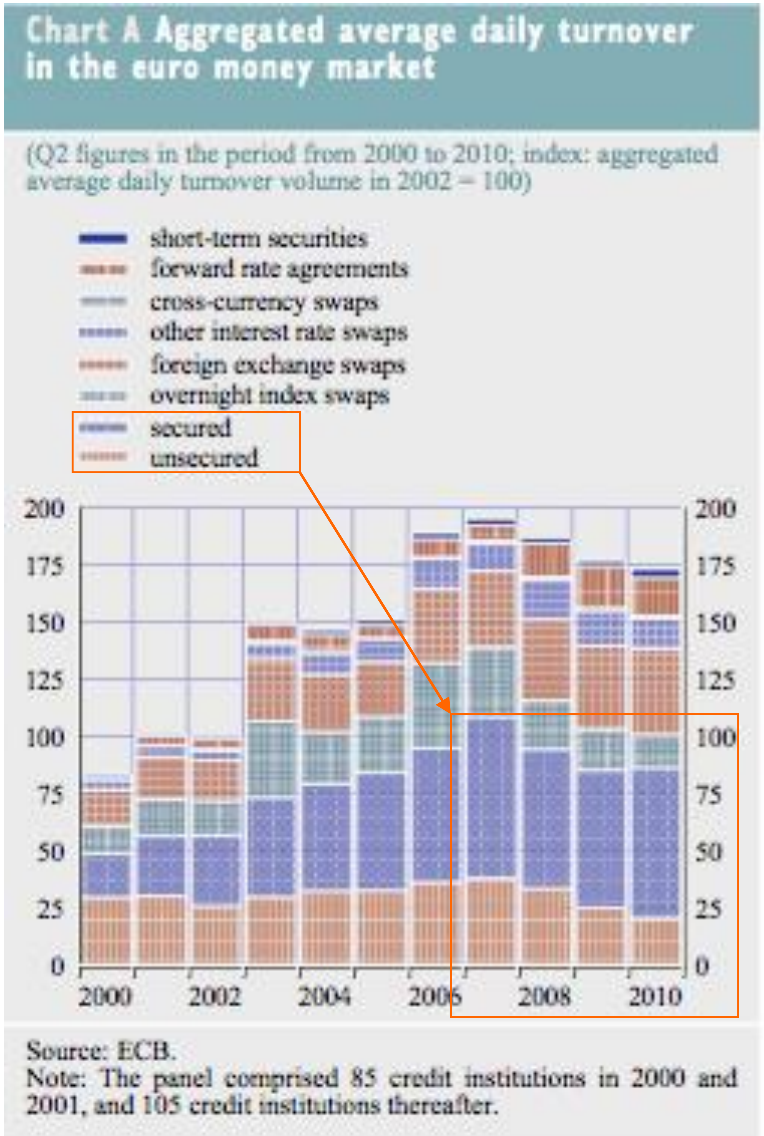
(M. Morini, 2009, et al.)

Theoretical interpretation



Never say never…

3: The market across the credit crunch
Unsecured vs secured transactions



EUR money market average daily turnover

The crisis has:

1. inverted the overall trend, from increasing to decreasing turnover;
2. Shift of transaction volumes:
   - from the unsecured money market (Eonia, bottom sections, -18% in Q2-2010 vs Q2-2009)
   - to the secured money market (Eurepo, second bottom sections, +8% in Q2-2010 vs Q2-2009)

(source: European Central Bank Financial Stability Review, Dec. 2010, p. 65)

3: The market across the credit crunch
Counterparty risk and collateral [1]

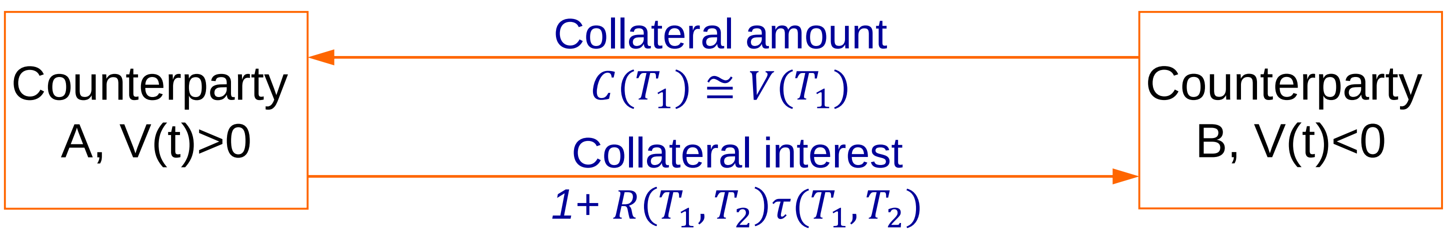
- Typical financial transactions generate streams of future cashflows, whose total net present value (NPV = algebraic sum of all discounted expected cashflows) implies a credit exposure between the two counterparties.
  If, for counterparty A at time t, $NPV_A(t)>0$ => counterparty A expects to (globally) receive future cashflows from counterparty B (A has a credit with B), and, on the other side, counterparty B has $NPV_B(t)<0$ and expects to (globally) pay future cashflows to counterparty A (B has a debt with A). The reverse holds if $NPV_A(t)<0$ and $NPV_B(t)>0$.
  Such mutual credit exposure can be mitigated through a guarantee, called collateral.

- In banking, collateral has two meanings:
  - asset-based lending: the traditional secured lending, with unilateral obligations, secured in the form of property, surety, guarantee or other;
  - capital market collateralization: used to secure trade transactions, with bilateral obligations, secured by more liquid assets such as cash or securities, also known as margin.

- Capital market collateralization: physical delivery from the debtor to the creditor (with or without transfer of property) of liquid financial instruments or cash with NPV corresponding to the NPV of the trade, as a guarantee for mutual obligations.

3: The market across the credit crunch
Counterparty risk and collateral [2]

Trade btw two counterparties under collateral	Counterparty A	Counterparty B
Trade value V(t)	Positive (receiver)	Negative (payer)
Collateral account C(t)	Receives collateral	Posts collateral
Collateral interest R(t)	Pays interest	Receives interest
Trade value V(t)	Negative (payer)	Positive (receiver)
Collateral account C(t)	Posts collateral	Receives collateral
Collateral interest R(t)	Receives interest	Pays interest
Collateral account as Cash Deposit	Collateral account grows at CSA rate $C(T_2) = C(T_1)[1 + R(T_1, T_2)\tau(T_1, T_2)]$	



3: The market across the credit crunch

Counterparty risk and collateral [2]

Collateral mechanics: regulated vs OTC markets		
	Regulated markets	Over the counter markets
Collateralisation	All trades are collateralised	Not all trades are collateralised, it depends on the agreements between the counterparties
Financial instruments	highly standardised	highly customised
Clearing House	There is a Clearing House that acts as counterparty for any trade and establish settlement and margination rules	There is no Clearing House, direct interaction between the counterparties, ad hoc contracts are used
Settlement and margination execution	Daily settlement and margination, collateral in cash of main currencies or highly rated bonds (govies)	Most used contracts are: ✓ISDA Master Agreement ✓Credit Support Annex (CSA)
Collateral interest	Overnight rate	Depend on the agreements

3: The market across the credit crunch
Counterparty risk and collateral [3]

OTC: Bilateral CSA

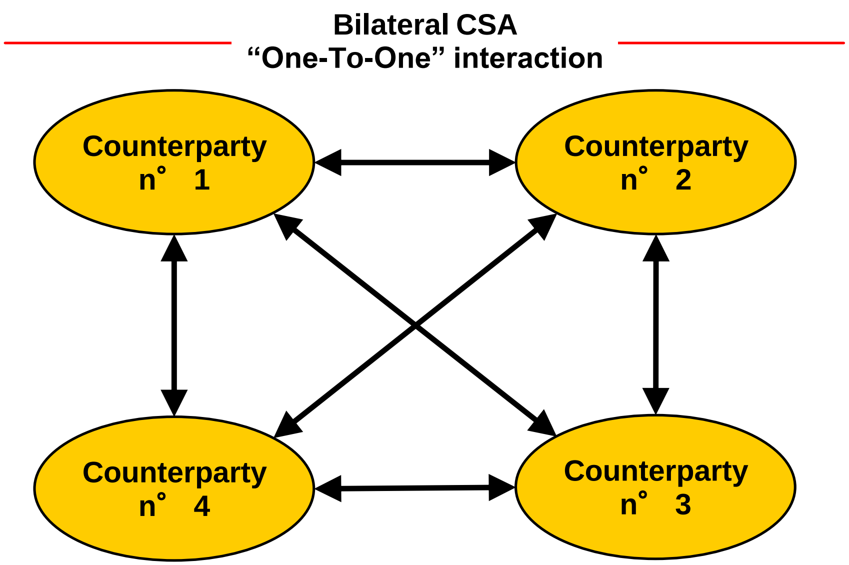
OTC counterparties subscribe bilateral collateral agreements, typically under ISDA Master Agreement with CSA.

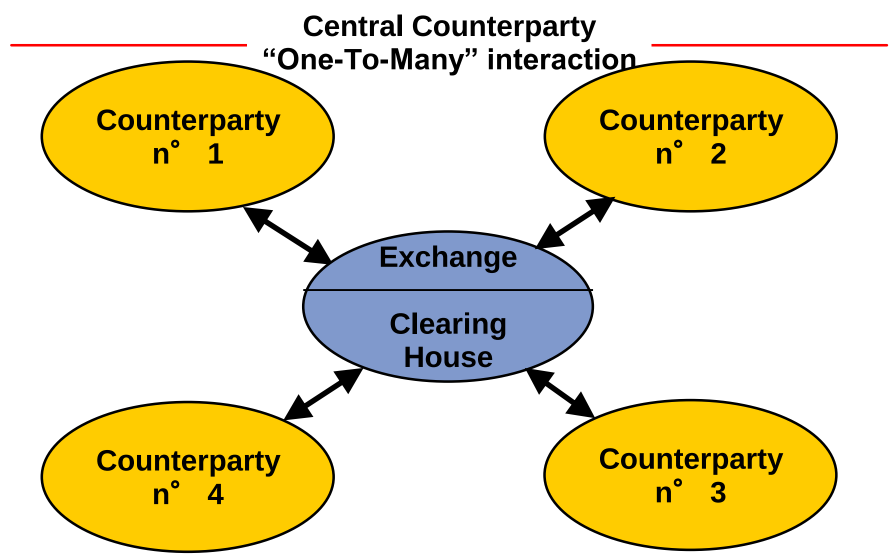
Netting is allowed at counterparty level.

OTC: Central Counterparty

The CCPs centralises OTC transactions and reduces the counterparty risks by:

- netting between multiple counterparties,
- requiring collateral deposits,
- providing independent valuation of trades and collateral,
- monitoring the credit worthiness of the clearing firms,
- providing a guarantee fund.





3: The market across the credit crunch
Counterparty risk and collateral [4]

Collateral in OTC markets

- **ISDA Master Agreement**
  - Master contracts, proposed and maintained by the International Swaps and Derivatives Association (ISDA), that set out standard terms that apply to all the transactions of a given type. It is completed by the Schedule, the Confirmation, and the Credit Support Annex (four documents in total).
  - Widely used by most financial operators for OTC transactions.
  - Netting clause: in case of early termination event, the counterparties are allowed to calculate the total net reciprocal credit exposure (total NPV = algebraic sum of the NPVs of all mutual transactions)
- **Credit Support Annex**:
  optional accessory document to the ISDA Master Agreement that establish the collateral and margination rules between the counterparties.
  There are two main CSA versions:
  - UK CSA: most used in Europe, with property transfer of the collateral (cash or assets) from the debtor to the creditor, that can freely use it;
  - US CSA: most used in the US, the collateral (cash or assets) is deposited by the debtor in a locked bank account of the creditor (there is no property transfer of the collateral).

3: The market across the credit crunch
Counterparty risk and collateral [6]

CSA characteristics:

- Exposure: potential loss that the creditor would suffer in case of default of the counterparty before trade maturity. It is measured in terms of cost of replacement the cost for the creditor to enter in the same deal with another counterparty.
- Base currency: reference currency of the contract and of the collateral.
- Eligible currency: one or more currencies alternative to the base currency.
- Eligible credit support: the collateral assets agreed by the counterparties, generally cash or AAA bonds (mainly govies).
- Haircut: valuation percentage applied to the Eligible Credit Support to reduce the collateral asset volatility, proportional to the asset residual life.
- Independent amount: the amount transferred at CSA inception, independent on the NPV dynamics.
- Threshold: the maximum exposure allowed between two counterparties without CSA; it depends on the credit worthiness of the counterparties.
- Minimum transfer amount (MTA): the threshold for margination; it depends on the counterparties' ratings.
- Rounding: the rounding to be applied to the MTA.

CSA characteristics (cont'd):

- Valuation agent: the counterparty that calculates the exposure and the collateral for margination; if not specified, the burden lies with the counterparty that calls the Collateral.
- Valuation date: exposure calculation and margination frequency; it may be daily, weekly or monthly; daily margination allows for the best guarantee against credit risk.
- Notification time: when the Valuation Agent communicates to the other counterparty the exposure and the collateral to be exchanged.
- Interest rate: the rate of remuneration of the collateral; normally it is the flat overnight rate in the base currency. Sometimes, a spread on overnight may be present.
- Dispute resolution: how to redeem any disagreements on the exposure and collateral valuation.
- Side: bilateral (or two-ways), unilateral (or one-way).

- The collateral agreements have had a strong diffusion after the credit crunch, such that, presently, virtually all the interbank market is collateralised.

3: The market across the credit crunch
Counterparty risk and collateral [8]

CSA diffusion (ISDA Margin Survey, 2012)

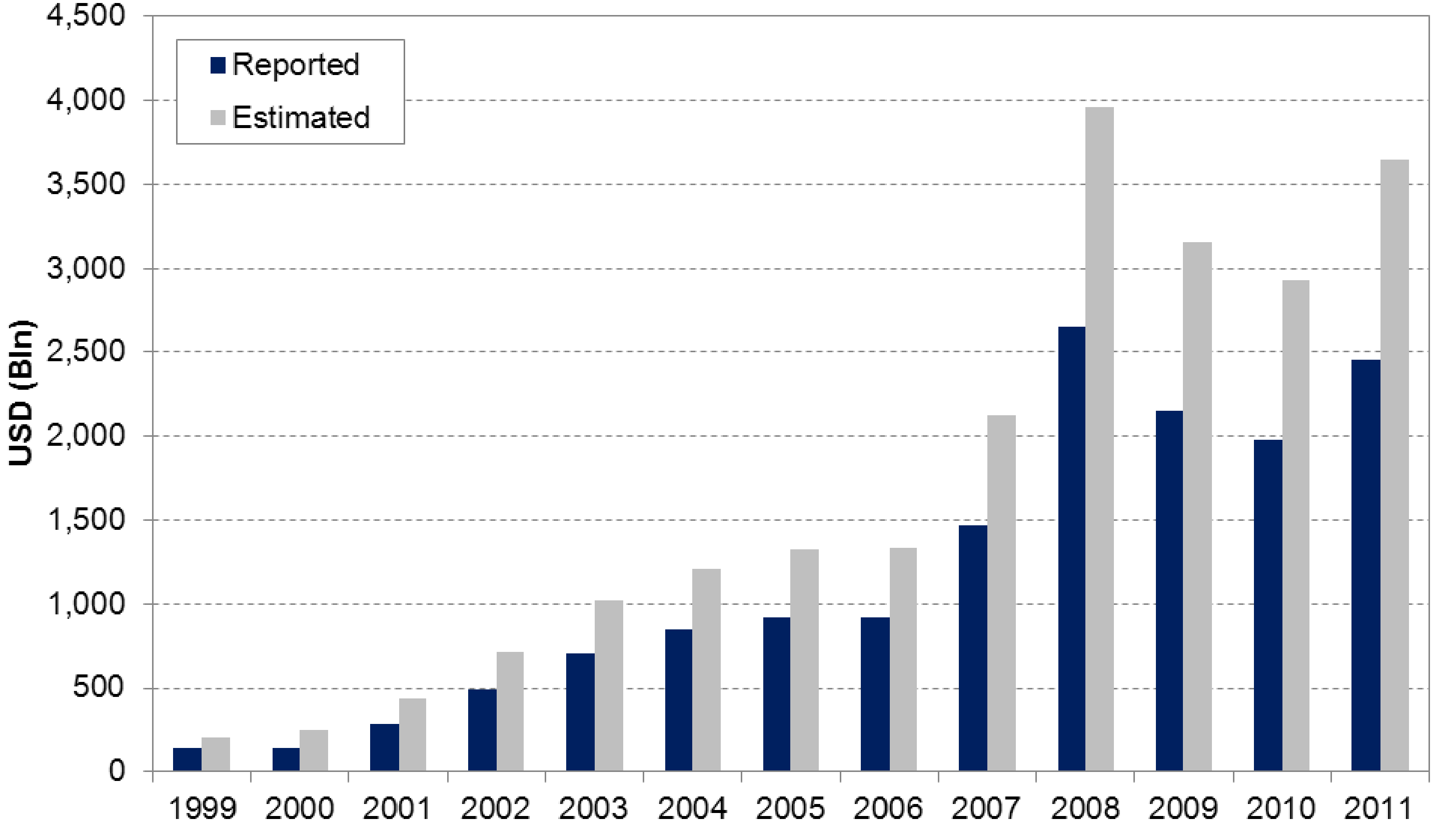
- Survey on ISDA members, 51 respondents, 14 large dealers (with more than 3.000 active collateral agreements).

	All	Large Dealers
All OTC Derivatives	71.4%	83.7%
Fixed Income Derivatives	78.1%	89.9%
Credit Derivatives	93.4%	96.1%
FX Derivatives	55.6%	70.6%
Equity Derivatives	72.7%	85.3%
Commodities Derivatives	56.3%	63.9%

- FX derivatives show the lowest degree of collateralization, because of generally short maturities that help to mitigate the counterparty risk.
- The percentages of collateralization related to large dealers are always higher than those of other classes of market participants, regardless the type of OTC Derivatives.

3: The market across the credit crunch
Counterparty risk and collateral [9]

CSA diffusion (ISDA Margin Survey, 2012) (cont'd)



Reported and estimated collateral in circulation within the OTC derivatives market. We observe an upward trend over all the past 12 years, except between 2008 and 2010 when the crisis has triggered a reduction in the market activity.

CSA chaos: the problem (see [1])

- Most of the CSAs do not fix the currency of the collateral. Counterparties are allowed, in principle, to post collateral in different currencies ("multiple currency CSA").

- Changing collateral currency $\Rightarrow$ changing the discounting curve $\Rightarrow$ changing the NPV. Hence, multi-currency CSA implies a cheapest-to-deliver collateral currency option on the currency of the underlying portfolio of instruments under CSA.

- Banks resist to unwinding and back-loading existing trades on to Central Counterparties (CCP) because changes in the discounting curves and collateral currency options may be very expensive.

- Pricing cheapest to deliver collateral currency option: see later and [2].

[1] Nick Sawyer, "*Multi-currency CSA chaos behind push to standardised CSA*", Risk Magazine, 1 Mar 2011
[2] M. Fujii and A. Takahashi, *"Choice of Collateral Currency",* Risk, Jan. 2011.

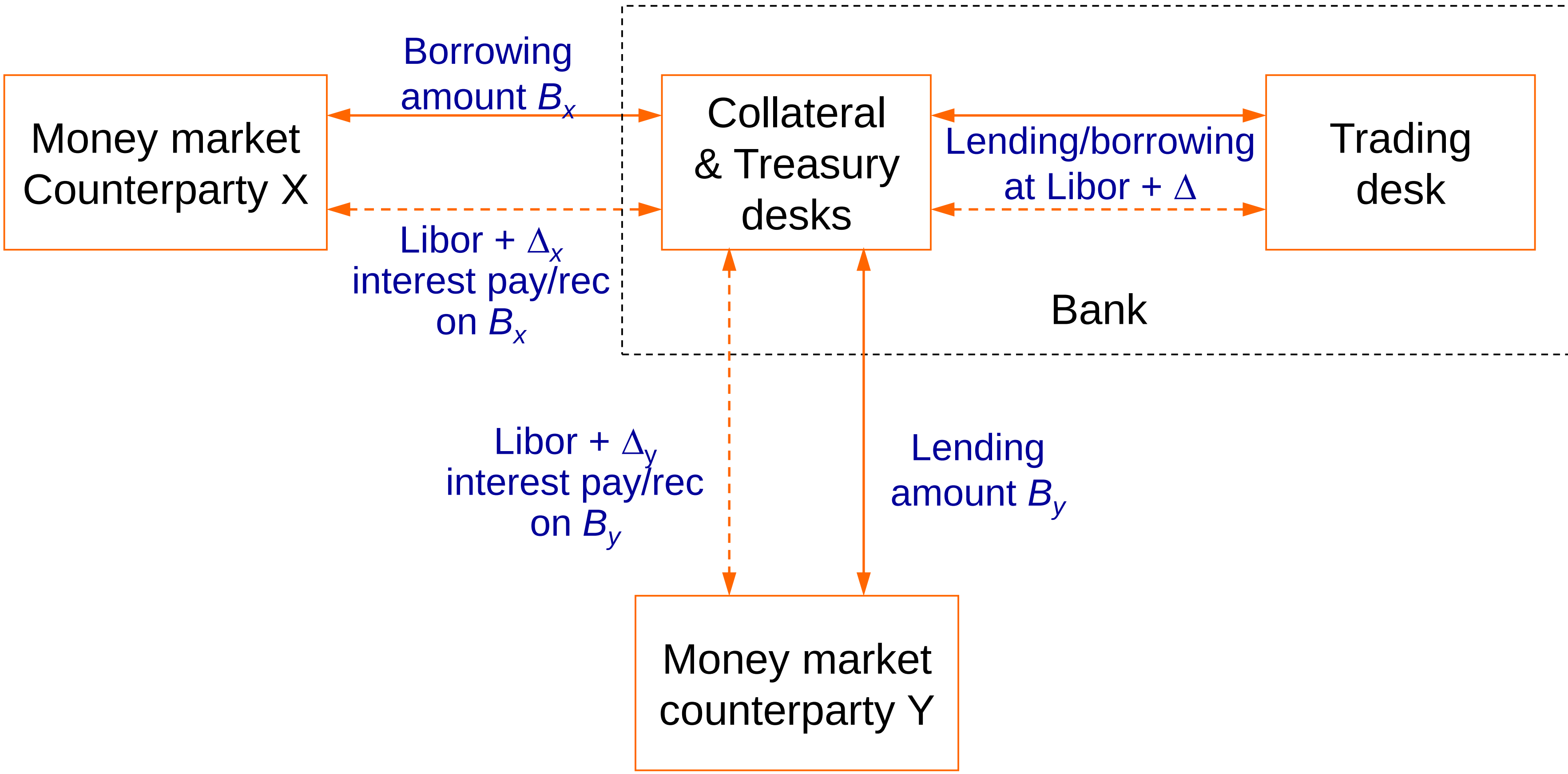
3: The market across the credit crunch
Counterparty risk and collateral [11]

CSA chaos: the new ISDA Standard CSA (see [1,2]) **:**

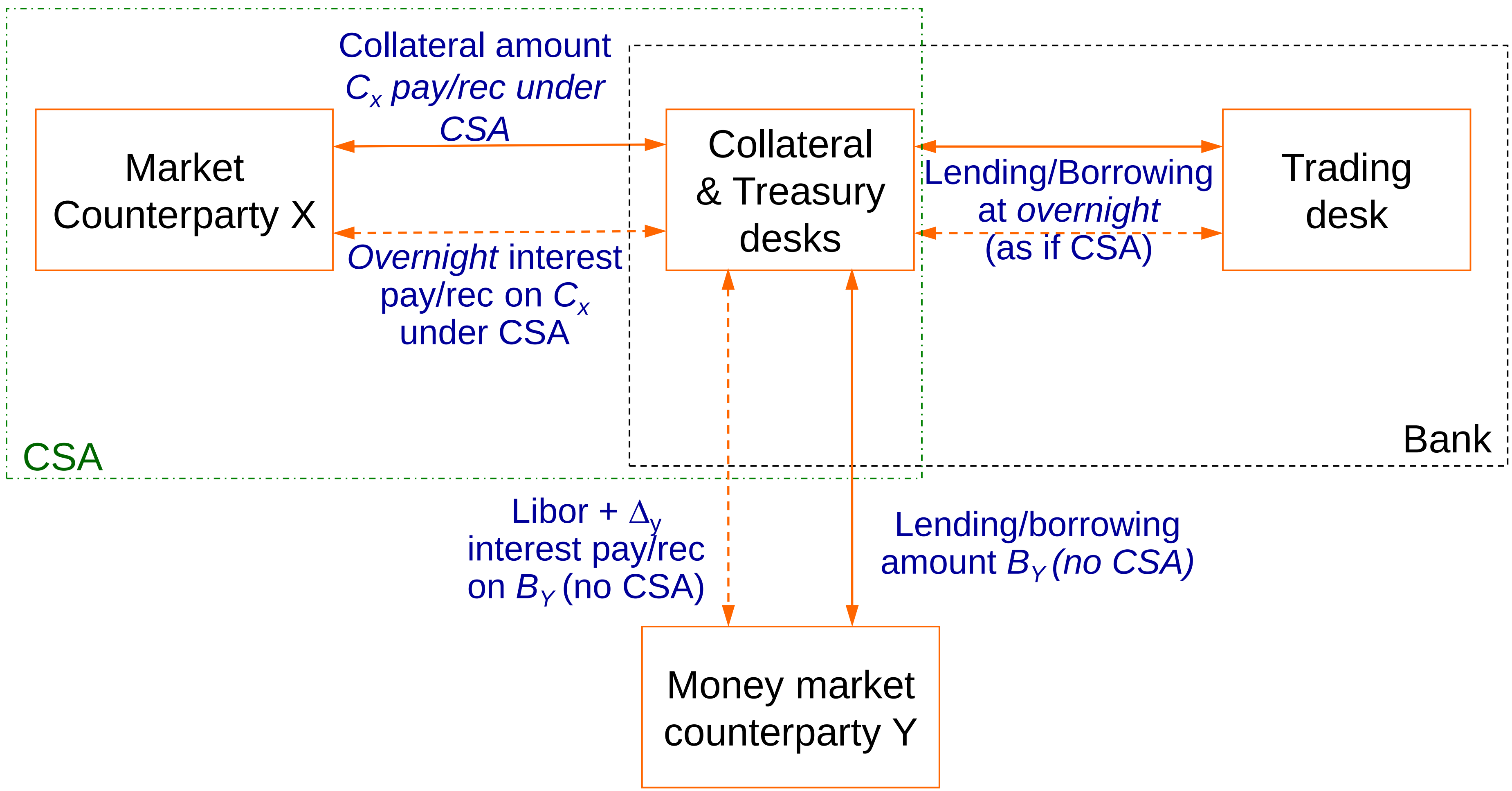
- ISDA has set up a working group to study a new standard CSA in order to avoid the CSA chaos. Final draft under development, expected delivery within in 2011, expected roll out 2nd Quarter 2011.
- LCH-SwapClear methodology is the driver. There will be five "silos" corresponding to the five currencies with the most liquid OIS curves (EUR, GBP, CHF, USD, JPY).
- Each transaction will be allocated to one silo, according with the currency of the underlying, with the relevant OIS rate used to discount all the trade's cash flows.
- Problem: multiple flows of collateral in different currencies appear, thus creating an huge systemic cross currency settlement risk (example: two counterparties, two swaps, one in EUR, one in USD).
- Solution: counterparties are allowed by the CSA to net the collateral flows in multiple currencies into a single payment in a single currency using the overnight currency swap market to exchange the flows.
- Problem: assigning multiple currency trades (e.g. CCS) to USD silo will imply a pricing impact and possible funding squeeze for non-US-dollar-funded banks.

[1] N. Sawyer, "*Standard CSA: industry's solution to novation gets bottleneck nearer*", Risk, 5 Sept. 2011
[2] V. Vaghela, N. Sawyer, "*Standard CSA: the dollar dominance dispute*", Risk,10 Jan. 2012

Typical unsecured funding mechanics in a Bank (no CSA)

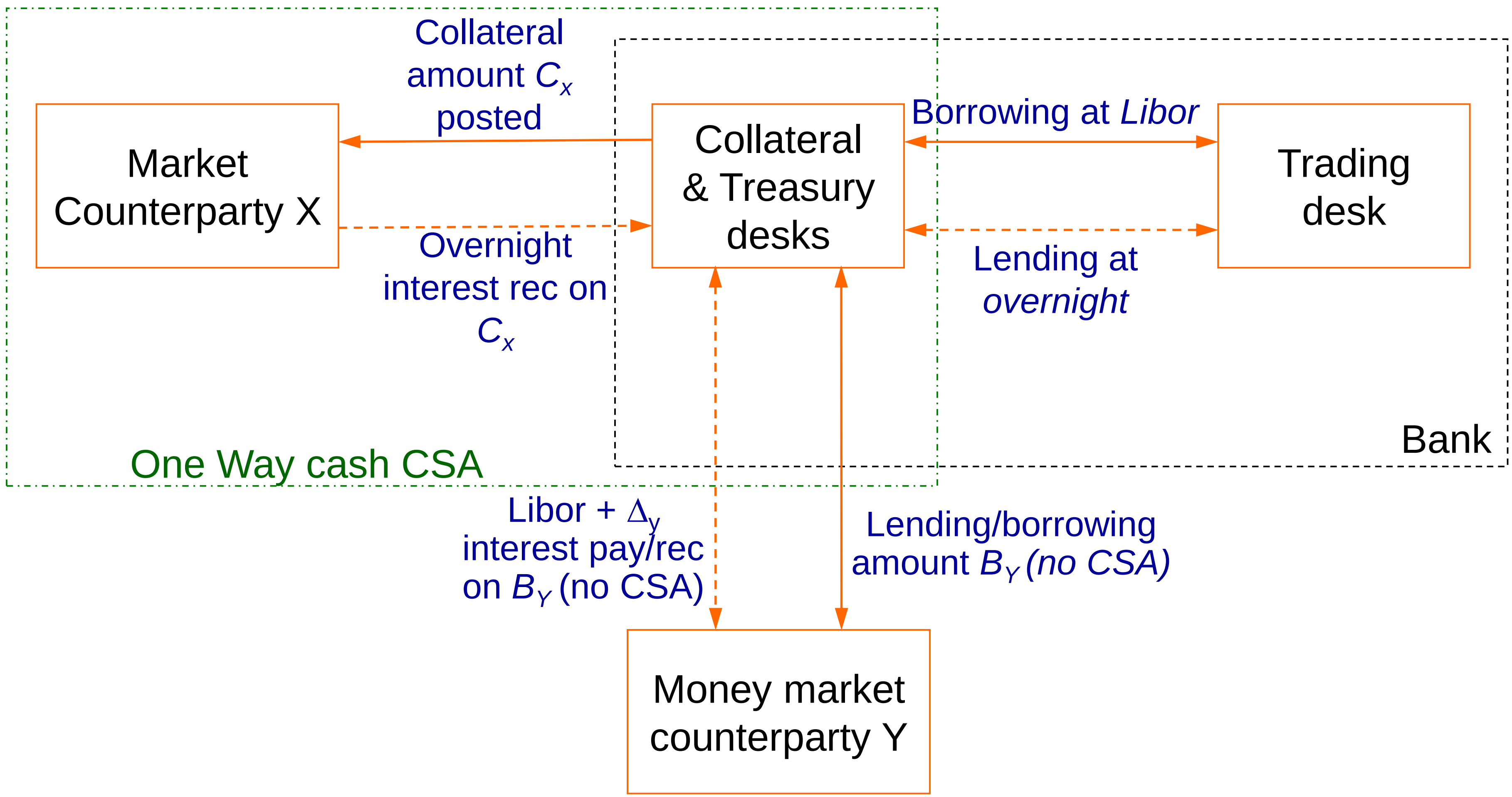


Typical secured funding mechanics in a Bank
(Two Ways cash CSA)



3: The market across the credit crunch
Counterparty risk and collateral [14]

Typical secured funding mechanics in a Bank
(One Way cash CSA)



3: The market across the credit crunch
Counterparty risk and collateral [15]

Collateral Pros

- Counterparty risk reduction
- Credit management optimization
- Capital ratios reduction (Basilea II)
- Increased business opportunities
- Funding at overnight rate
- Periodic check of credit exposure and portfolio NPV

Collateral Cons

- Funding volatility sensitivity
- Possible liquidity squeeze
- Structural and running costs
- Operational risks: settlement and MTM mismatch
- Legal risk
- Pricing impact
- CSA chaos, new ISDA CSA

3: The market across the credit crunch
From Libor to CSA discounting

Collateralised trades (CSA)	**Uncollateralised trades (no CSA)**
Assuming perfect collateralisation: ■ Fully symmetric ■ Zero threshold ■ Cash collateral only ■ Collateral currency = deal currency ■ Daily margination ■ Flat overnight margination rate ■ Immediate settlement $\Rightarrow$ By no arbitrage **discounting rate** **= funding rate** **= collateral rate**	Assuming no collateralisation: ■ Funding on the money market (Deposits, Repos) or securities market (Bonds, etc.) $\Rightarrow$ By no arbitrage **discounting rate** **= funding rate**
Overnight or OIS discounting	**Libor^{+} discounting**

More on this picture in the following

3: The market across the credit crunch
From Libor to CSA discounting: Clearing Houses

1. **June 17th 2010**: LCH press release: "*LCH.Clearnet Ltd […] which operates […] SwapClear, is to begin using the overnight index swap (OIS) rate curves to discount its $218 trillion IRS portfolio. Previously, in line with market practice, the portfolio was discounted using Libor. However, an increasing proportion of trades are now priced using OIS discounting. After extensive consultation with market participants, LCH.Clearnet has decided to move to OIS to ensure the most accurate valuation of its portfolio for risk management purposes. LCH.Clearnet already uses OIS rates to price the rate of return on cash collateral. From 29 June 2010, USD, Euro and GBP trades in SwapClear will be revalued using OIS*. […]" *[1]*
2. **15 July 2011**: International Derivatives Clearing Group (IDCG) moves Swaps to OIS discounting *[2]*
3. **August 2011**: LCH moves JPY Swaps to OIS discounting using Mutan rate
4. **August 2011**: Chicago Mercantile Exchange (CME) moves Swaps to OIS discounting. *"We are moving to discount swaps at OIS as a result of what has been going on in the swaps market for the past two years. All major swaps dealers have started discounting their collateralised swaps portfolios at the relevant OIS rate. Our service is designed to eliminate basis risk between current over-the-counter and cleared products, so we strive to reflect current market practice," (Kim Taylor, president of CME Clearing in Chicago, cited in [2]).*

[1] C. Whittall, "*LCH.Clearnet re-values $218 trillion swap portfolio using OIS*", Risk, 17 Jun. 2010.
[2] M. Cameron, "*CME and IDCG revalue swaps using OIS discounting*", Risk, 4 Oct. 2011.

3: The market across the credit crunch
From Libor to CSA discounting: Broker Market [1]

The market of OTC derivatives transacted through interdealer brokers (e.g. ICAP) has moved to OIS discounting:

1. August 4th 2010: Goldman Sachs communicates that
"*From the 1st of September, the vanilla vol market in EUR/GBP/CHF/Scandies will be moving to forward premia in the interdealer broker market. The reason for this change is to mitigate any impact on pricing from the effects of different collateral agreements. As a first step, only cash settled swaptions will move to this convention. It is important to stress that this change will not impact clients in terms of pricing or liquidity. If anything, liquidity should be improved as dealers will be able to trade with any name when hedging their risk in the broker market. Clients will still have the option to settle their premium spot or forward, or at any point in between, just as they do currently*."

3: The market across the credit crunch

From Libor to CSA discounting: Broker Market [2]

2. August 11th 2010: ICAP communicates that
   "*We are planning to us the following methodology for pricing and for the publication of our information. We plan to gross up our spot premium to fwd premiums and intend to run two pages concurrently. This process will be driven by the spot premia until the 1st of Sept and vice-versa from then on. We will be using a eonia curve to discount from the moment we publish both numbers, thus immediately affecting our bp vol page VCAP6 (the vols will be lower). All pages will remain the same with the addition of VCAP2A for fwd premia. It will be made clear that VCAP2 is for indicative purposes only, discounted using our eonia curve and therefore CSA dependant.*"

3. August 2010:
   also Tradition begins to quote forward premium swaptions.

3: The market across the credit crunch
From Libor to CSA discounting: Broker Market [3]

4. September 15th, 2010: ICAP explains that
   "*Until very recently all prices quoted in the Euro IR Swaption market were quoted as spot premium. [...] Around 6 weeks ago, xxx began showing prices in the market specifying that these were only for names with whom they did not have a $$ CSA. Those counterparties responded in kind. The result was a two-tier market with very little prospect of any long-term liquidity. The obvious solution was to move to forward premium. A discussion was started [...] with most of the main dealers who all agreed to migrate to forward premium from Sept 1st.*
   We have moved all of our pricing to this methodology and currently still give the market an indication of where we expect the spot premium price to be, using our discount curve. Clearly this price is entirely CSA dependant.
   At the moment 95% of our prices and trades are forward premium."

5. December 15th, 2011: ICAP communicates that since Monday 3rd, Jan. 2012, ICAP will no longer publish Libor volatility pages

3: The market across the credit crunch

From Libor to CSA discounting: Broker Market [4]

15:10 30JUN10 ICAP UK69580 VCAP2

EUR ATM Swaption Straddles - Premium Mids

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	11.0	24.5	38.5	53.0	68.0	83.5	99.0	115	129	144	205	261	319	389
2M Opt	16.5	37.0	58.0	79.5	101	122	143	163	183	203	290	373	456	548
3M Opt	22.5	50.5	77.5	104	129	153	177	201	225	250	355	458	564	666
6M Opt	37.5	81.0	121	158	195	231	265	298	330	362	499	624	763	919
9M Opt	50.0	107	157	205	252	295	337	378	417	455	622	767	931	1104
1Y Opt	61.5	126	186	245	300	352	402	449	495	536	734	900	1081	1256
18M Opt	78.5	158	233	305	373	437	498	557	612	666	902	1116	1318	1512
2Y Opt	94.0	183	268	349	427	503	574	641	705	767	1028	1259	1492	1708
3Y Opt	111.0	214	312	405	494	581	663	740	815	888	1182	1447	1726	1991
4Y Opt	121.5	235	341	441	537	630	719	803	885	965	1282	1571	1874	2153
5Y Opt	126.5	244	357	464	565	661	754	842	929	1011	1351	1656	1963	2274
7Y Opt	129.0	250	367	480	587	686	783	878	971	1062	1411	1730	2049	2375
10Y Opt	128.5	247	364	477	586	689	791	891	988	1082	1444	1763	2092	2404
15Y Opt	123.0	235	348	461	569	673	775	874	968	1057	1407	1722	2043	2352
20Y Opt	113.5	220	326	434	540	640	736	829	915	994	1337	1627	1910	2164
25Y Opt	104.5	201	300	401	500	591	676	764	843	918	1220	1483	1726	1946
30Y Opt	95.0	180	267	355	442	522	598	676	746	812	1092	1337	1569	1791

15:10 30JUN10 ICAP UK69580 VCAP1

EUR ATM Swaption Straddles - Implied Volatilities

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	46.4	39.2	35.9	32.3	29.9	28.1	27.0	26.3	25.6	25.1	23.3	23.5	24.9	27.6
2M Opt	48.8	40.6	37.0	33.4	30.8	28.6	27.1	25.9	25.2	24.7	23.1	23.5	25.0	27.3
3M Opt	51.7	44.1	39.5	34.8	31.2	28.7	27.0	25.8	25.0	24.6	22.9	23.4	25.1	27.0
6M Opt	57.9	47.4	41.2	35.9	32.2	29.7	27.8	26.5	25.5	24.8	22.7	22.6	24.1	26.4
9M Opt	59.9	48.4	41.4	36.2	32.7	30.1	28.2	26.9	25.9	25.1	22.9	22.6	24.0	26.0
1Y Opt	59.5	46.6	40.2	35.7	32.4	30.0	28.3	27.1	26.1	25.1	23.2	22.9	24.1	25.6
18M Opt	53.9	42.1	36.9	33.2	30.6	28.7	27.4	26.3	25.4	24.8	23.0	23.1	24.0	25.2
2Y Opt	48.2	37.6	33.5	30.6	28.6	27.3	26.3	25.4	24.6	24.1	22.5	22.5	23.6	24.7
3Y Opt	35.5	29.9	27.4	25.8	24.7	24.0	23.3	22.8	22.3	22.0	20.9	21.2	22.5	23.8
4Y Opt	28.3	25.2	23.8	22.9	22.2	21.7	21.2	20.9	20.6	20.5	19.8	20.3	21.6	22.8
5Y Opt	24.0	22.1	21.5	21.0	20.5	20.1	19.8	19.6	19.4	19.4	19.1	19.7	21.0	22.3
7Y Opt	19.7	19.0	18.8	18.5	18.3	18.1	18.0	18.0	18.0	18.2	18.2	19.0	20.2	21.4
10Y Opt	16.9	16.5	16.5	16.6	16.8	16.9	17.0	17.2	17.5	17.7	18.2	19.1	20.2	21.2
15Y Opt	17.0	16.9	17.2	17.6	18.1	18.5	18.8	19.3	19.7	20.0	20.7	21.3	22.0	22.6
20Y Opt	19.9	20.2	20.8	21.8	22.6	23.2	23.8	24.3	24.6	24.7	24.4	23.8	23.6	23.3
25Y Opt	26.0	26.1	26.8	27.8	28.5	28.6	28.4	28.4	28.0	27.7	25.3	23.7	22.7	22.4
30Y Opt	27.6	26.2	26.0	26.0	26.1	25.8	25.5	25.4	25.0	24.6	22.4	21.1	20.8	21.2

EUR ATM Swaption market quotes on 30 June 2010.

Spot premia (left panel) and Black implied volatilities (right panel).

The ATM implied volatilities surface were obtained using the classical single-curve approach.
(Sources: Reuters, ICAP).

3: The market across the credit crunch

From Libor to CSA discounting: Broker Market [5]

15:12 30SEP10 ICAP UK69580 VCAP2A

EUR ATM Swaption Straddles - Fwd Premium Mids

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	10.5	25.5	41.0	57.5	76.5	96.5	117	138	161	185	270	346	421	495
2M Opt	15.5	37.5	60.0	86.0	115	141	169	198	228	261	379	486	592	701
3M Opt	20.0	49.5	78.5	110	142	176	209	243	278	315	459	584	712	842
6M Opt	32.0	76.5	116	160	206	250	294	341	386	431	622	799	964	1135
9M Opt	44.5	99.0	150	203	257	310	363	417	471	526	756	959	1155	1366
1Y Opt	55.0	118	178	237	299	363	423	483	545	605	859	1083	1311	1539
18M Opt	76.0	151	225	300	374	447	519	590	664	739	1035	1303	1564	1817
2Y Opt	94.5	183	268	351	437	520	601	681	762	842	1179	1479	1767	2042
3Y Opt	116.5	225	327	426	528	625	722	814	906	1002	1376	1726	2064	2388
4Y Opt	131.5	254	369	480	588	695	798	900	999	1108	1507	1878	2244	2585
5Y Opt	141.5	273	396	514	630	744	856	967	1076	1189	1618	2001	2374	2754
7Y Opt	152.0	295	431	564	691	819	942	1059	1177	1302	1757	2168	2567	2974
10Y Opt	170.5	329	487	638	785	927	1065	1195	1327	1463	1960	2419	2848	3255
15Y Opt	193.0	374	552	725	890	1055	1217	1373	1532	1688	2256	2810	3309	3780
20Y Opt	209.0	410	608	804	994	1181	1363	1539	1714	1866	2523	3110	3645	4115
25Y Opt	218.0	424	633	843	1047	1244	1430	1607	1791	1961	2622	3209	3755	4241
30Y Opt	220.5	425	629	833	1035	1227	1414	1590	1771	1939	2613	3226	3803	4369

15:12 30SEP10 ICAP UK69580 VCAP1A

EUR ATM Swaption Straddles - Implied Volatilities

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	40.6	38.4	37.9	36.8	36.3	35.8	35.5	35.4	35.8	36.4	34.4	34.4	36.1	38.6
2M Opt	40.4	38.6	37.6	37.1	37.1	35.8	34.9	34.7	34.6	35.1	33.2	33.3	35.0	37.7
3M Opt	41.1	40.8	39.5	38.3	37.1	36.1	35.1	34.6	34.4	34.5	32.8	32.7	34.4	37.0
6M Opt	44.1	42.7	39.7	38.0	36.7	35.3	34.1	33.6	33.1	32.8	31.2	31.5	33.0	35.3
9M Opt	46.9	43.0	40.2	37.8	36.0	34.5	33.4	32.7	32.3	32.1	30.6	30.6	32.1	34.5
1Y Opt	47.5	42.5	39.6	36.8	35.1	34.0	32.9	32.2	31.7	31.3	29.8	29.8	31.4	33.6
18M Opt	48.7	41.0	37.9	35.6	33.8	32.5	31.6	30.9	30.5	30.3	28.8	29.0	30.5	32.3
2Y Opt	47.6	39.8	36.4	33.8	32.4	31.3	30.5	29.8	29.4	29.1	28.0	28.3	29.7	31.4
3Y Opt	40.5	34.6	31.9	30.0	29.2	28.4	27.9	27.4	27.0	27.0	26.0	26.6	28.2	29.8
4Y Opt	34.0	29.9	28.1	27.0	26.2	25.7	25.3	25.0	24.8	24.9	24.2	24.9	26.5	27.9
5Y Opt	28.9	26.3	25.2	24.4	23.9	23.6	23.3	23.2	23.2	23.4	23.0	23.7	25.1	26.7
7Y Opt	23.2	22.0	21.5	21.1	20.8	20.8	20.8	20.8	20.9	21.2	21.1	22.0	23.3	24.7
10Y Opt	19.6	18.9	19.0	19.1	19.2	19.3	19.4	19.5	19.7	20.0	20.4	21.6	22.6	23.5
15Y Opt	18.8	18.4	18.6	18.8	19.0	19.4	19.8	20.3	20.8	21.4	22.2	23.3	23.9	24.2
20Y Opt	20.6	20.9	21.6	22.4	23.1	23.9	24.5	25.2	25.8	26.0	26.0	25.8	25.2	24.7
25Y Opt	27.2	27.0	28.1	29.0	29.4	29.6	29.4	29.2	29.3	29.6	27.2	25.3	24.1	23.7
30Y Opt	29.2	26.9	26.7	26.9	28.0	27.1	27.2	27.0	27.1	26.8	23.9	22.3	22.0	22.3

15:12 30SEP10 ICAP UK69580 VCAP2

EUR ATM Swaption Straddles - Implied Spot Premium Mids

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	10.5	25.5	41.0	57.5	76.5	96.5	117	138	161	185	270	346	420	494
2M Opt	15.5	37.5	60.0	85.5	115	141	168	198	227	261	379	486	592	700
3M Opt	20.0	49.5	78.5	110	142	176	209	243	278	315	458	583	711	841
6M Opt	32.0	76.5	116	159	205	250	293	340	385	430	620	796	961	1131
9M Opt	44.5	98.5	149	202	255	308	361	414	468	523	751	953	1148	1358
1Y Opt	54.5	117	177	235	296	360	419	479	540	599	852	1074	1300	1526
18M Opt	75.0	149	222	296	369	441	512	582	655	729	1020	1286	1542	1792
2Y Opt	92.5	179	262	344	428	510	589	667	747	825	1156	1449	1731	2002
3Y Opt	112.5	217	315	411	509	604	697	785	874	967	1328	1665	1992	2305
4Y Opt	124.5	240	349	455	556	658	756	852	946	1049	1427	1778	2125	2447
5Y Opt	131.0	253	367	476	583	689	792	895	996	1100	1497	1852	2197	2549
7Y Opt	133.5	259	377	494	606	717	825	928	1031	1141	1539	1900	2249	2605
10Y Opt	136.0	263	388	508	626	739	849	953	1058	1166	1563	1928	2270	2594
15Y Opt	130.0	251	371	487	598	709	817	922	1029	1134	1515	1888	2223	2539
20Y Opt	119.5	235	348	460	569	676	780	881	981	1068	1444	1780	2087	2356
25Y Opt	110.0	214	320	426	528	628	722	811	904	990	1324	1621	1896	2142
30Y Opt	101.5	196	289	384	477	565	651	732	816	893	1203	1485	1751	2012

15:12 30SEP10 ICAP UK69580 VCAP1

EUR ATM Swaption Straddles - Implied Volatilities (Euribor disc)

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	40.5	38.4	37.9	36.8	36.4	35.9	35.5	35.4	35.8	36.4	34.4	34.4	36.1	38.6
2M Opt	40.4	38.6	37.6	37.2	37.1	35.8	34.9	34.7	34.6	35.1	33.2	33.3	35.0	37.7
3M Opt	41.1	40.8	39.5	38.4	37.2	36.1	35.1	34.6	34.4	34.5	32.8	32.7	34.5	37.1
6M Opt	44.1	42.8	39.8	38.1	36.8	35.3	34.2	33.7	33.2	32.9	31.2	31.6	33.0	35.4
9M Opt	47.0	43.1	40.3	38.0	36.1	34.6	33.5	32.8	32.4	32.1	30.6	30.7	32.1	34.6
1Y Opt	47.6	42.7	39.7	37.0	35.2	34.1	33.0	32.3	31.8	31.4	29.9	29.9	31.5	33.7
18M Opt	48.8	41.2	38.1	35.8	34.0	32.7	31.8	31.1	30.7	30.5	29.0	29.2	30.6	32.5
2Y Opt	47.8	40.1	36.7	34.1	32.6	31.5	30.7	30.0	29.6	29.3	28.2	28.5	29.9	31.6
3Y Opt	40.8	35.0	32.2	30.4	29.5	28.7	28.2	27.6	27.3	27.3	26.3	26.9	28.5	30.2
4Y Opt	34.3	30.3	28.5	27.4	26.6	26.1	25.6	25.3	25.1	25.3	24.5	25.3	26.9	28.4
5Y Opt	29.2	26.7	25.6	24.8	24.3	24.0	23.7	23.6	23.6	23.8	23.4	24.1	25.6	27.2
7Y Opt	23.6	22.5	22.0	21.6	21.3	21.2	21.3	21.3	21.4	21.7	21.6	22.6	23.9	25.4
10Y Opt	20.0	19.5	19.6	19.7	19.8	19.9	20.0	20.1	20.3	20.6	21.1	22.3	23.4	24.3
15Y Opt	19.3	19.2	19.4	19.6	19.8	20.2	20.7	21.2	21.7	22.3	23.3	24.4	25.1	25.4
20Y Opt	21.4	22.1	22.9	23.7	24.5	25.3	26.1	26.8	27.4	27.7	27.8	27.5	27.0	26.3
25Y Opt	28.2	29.3	30.3	31.4	32.1	32.6	32.6	32.5	32.6	32.3	29.6	27.5	26.1	25.5
30Y Opt	32.5	31.5	31.1	31.1	31.1	30.9	30.7	30.3	30.1	29.6	26.2	24.4	23.9	24.4

EUR ATM Swaption market quotes on 30 September 2010. Forward premia (left upper panel, page VCAP2A) and spot premia (left lower panel, page VCAP2), obtained from forward premia using Eonia discounting.

Black implied volatilities are now differentiated between Eonia (right upper panel, page VCAP1A) and Euribor discounting (right lower panel, page VCAP1). (Sources: Reuters, ICAP).

3: The market across the credit crunch

From Libor to CSA discounting: Broker Market [6]

15:16 31MAY12 ICAP UK69580 VCAP2A

EUR ATM Swaption Straddles - Fwd Premium Mids

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	12.0	24.5	39.0	56.0	73.5	92.0	112	133	154	177	271	362	445	523
2M Opt	16.5	34.5	56.0	78.5	106	132	161	189	218	249	380	510	628	734
3M Opt	19.5	43.0	69.0	100	131	164	198	234	271	310	461	613	753	877
6M Opt	27.5	60.0	98.5	143	192	240	288	336	386	435	640	834	1027	1206
9M Opt	33.0	75.0	121	175	239	297	356	416	477	534	772	999	1221	1434
1Y Opt	39.0	86.5	143	211	283	350	419	487	557	622	894	1159	1412	1647
18M Opt	52.5	115	188	274	360	442	525	606	687	765	1088	1398	1693	1981
2Y Opt	71.0	151	241	337	435	526	615	704	792	882	1254	1602	1935	2258
3Y Opt	107.5	217	329	443	560	666	770	873	973	1074	1509	1921	2319	2700
4Y Opt	135.0	265	394	525	652	772	890	1004	1115	1224	1701	2157	2601	3019
5Y Opt	154.5	301	443	583	721	854	983	1109	1231	1349	1851	2334	2806	3250
7Y Opt	180.0	348	507	665	821	971	1119	1263	1405	1544	2086	2595	3094	3569
10Y Opt	200.0	392	573	748	922	1094	1264	1431	1594	1753	2343	2889	3404	3880
15Y Opt	221.0	437	641	844	1041	1234	1424	1611	1794	1983	2620	3203	3695	4182
20Y Opt	234.0	463	684	900	1111	1318	1521	1720	1915	2102	2778	3348	3885	4362
25Y Opt	241.0	478	710	932	1149	1363	1574	1783	1988	2197	2875	3440	3997	4467
30Y Opt	251.5	494	727	951	1169	1386	1602	1816	2033	2262	2950	3534	4091	4612

15:16 31MAY12 ICAP UK69580 VCAP1

EUR ATM Swaption Straddles - Implied Volatilities

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	92.6	65.3	63.9	61.8	57.9	54.7	53.0	51.9	51.4	51.2	48.6	50.7	52.9	54.9
2M Opt	89.9	63.3	63.2	59.2	56.6	53.6	52.0	50.5	49.7	49.4	46.9	49.3	51.4	53.2
3M Opt	89.8	64.4	62.8	60.7	56.5	53.4	51.5	50.5	49.9	49.7	46.3	48.1	50.2	51.7
6M Opt	88.4	62.3	61.1	58.7	56.4	53.7	51.7	50.2	49.3	48.6	45.1	46.3	48.5	50.4
9M Opt	85.9	61.8	58.9	56.2	55.1	52.6	50.9	49.8	48.9	47.9	44.2	45.2	47.1	48.9
1Y Opt	84.4	59.4	57.7	55.9	54.2	51.9	50.4	49.2	48.4	47.4	44.0	45.2	47.0	48.6
18M Opt	80.8	58.2	55.2	53.9	52.1	50.2	48.8	47.6	46.7	45.9	43.0	44.2	45.9	47.5
2Y Opt	81.5	58.9	55.1	52.4	50.7	48.6	47.0	45.8	44.8	44.2	42.3	43.5	45.2	46.8
3Y Opt	72.6	54.8	50.6	48.3	47.0	45.2	43.9	42.8	41.9	41.5	40.6	42.1	44.0	45.4
4Y Opt	58.5	48.1	45.7	44.4	43.2	41.9	41.0	40.1	39.5	39.3	39.1	40.7	42.6	43.8
5Y Opt	50.0	44.0	42.3	41.1	40.2	39.4	38.7	38.2	38.0	37.9	38.0	39.5	41.3	42.3
7Y Opt	42.5	38.8	37.4	36.6	36.1	35.8	35.8	35.9	36.2	36.6	36.9	38.0	39.2	39.6
10Y Opt	35.1	33.6	33.2	33.2	33.6	34.1	34.9	35.7	36.5	37.1	37.1	37.6	37.6	37.1
15Y Opt	37.0	37.2	37.9	38.8	39.6	40.2	40.8	41.3	41.8	42.6	40.3	38.2	35.7	34.6
20Y Opt	44.9	43.8	43.9	44.1	44.4	44.9	45.4	45.7	45.9	45.7	39.6	35.1	33.1	31.7
25Y Opt	49.1	48.1	48.3	47.6	46.7	45.7	44.6	43.6	42.7	43.0	35.0	31.4	29.8	28.6
30Y Opt	45.7	40.2	38.6	37.2	37.6	35.5	35.1	34.8	34.8	35.1	30.5	28.0	26.8	26.3

15:16 31MAY12 ICAP UK69580 VCAP2

EUR ATM Swaption Straddles - Implied Spot Premium Mids

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	12.0	24.5	39.0	56.0	73.5	92.0	112	133	154	177	271	362	445	523
2M Opt	16.5	34.5	56.0	78.5	106	132	161	189	218	249	379	510	627	734
3M Opt	19.5	43.0	69.0	100	131	163	198	233	271	310	461	613	753	876
6M Opt	27.5	60.0	98.0	143	192	240	287	335	385	435	639	833	1026	1205
9M Opt	33.0	75.0	120	174	239	297	355	415	476	533	771	997	1219	1431
1Y Opt	39.0	86.5	143	210	282	349	418	486	555	620	892	1157	1409	1643
18M Opt	52.0	115	187	273	359	441	523	604	684	762	1084	1393	1687	1974
2Y Opt	71.0	150	240	335	433	523	612	701	788	877	1247	1594	1925	2247
3Y Opt	106.5	215	326	439	554	659	762	864	963	1064	1494	1902	2297	2674
4Y Opt	132.5	260	387	515	640	758	874	986	1095	1203	1670	2119	2554	2965
5Y Opt	150.0	292	429	566	700	829	954	1076	1195	1308	1796	2264	2723	3153
7Y Opt	169.0	327	476	624	771	912	1051	1186	1320	1450	1958	2437	2905	3351
10Y Opt	176.5	347	506	661	815	967	1117	1265	1409	1549	2071	2554	3008	3429
15Y Opt	175.0	346	508	669	825	977	1127	1276	1421	1571	2075	2537	2926	3312
20Y Opt	169.5	335	496	652	805	955	1102	1246	1387	1523	2013	2426	2816	3161
25Y Opt	162.0	321	477	625	771	915	1057	1197	1335	1475	1931	2310	2683	2999
30Y Opt	157.5	309	455	595	732	868	1003	1137	1273	1416	1847	2213	2562	2888

15:16 31MAY12 ICAP UK69580 VCAP2B

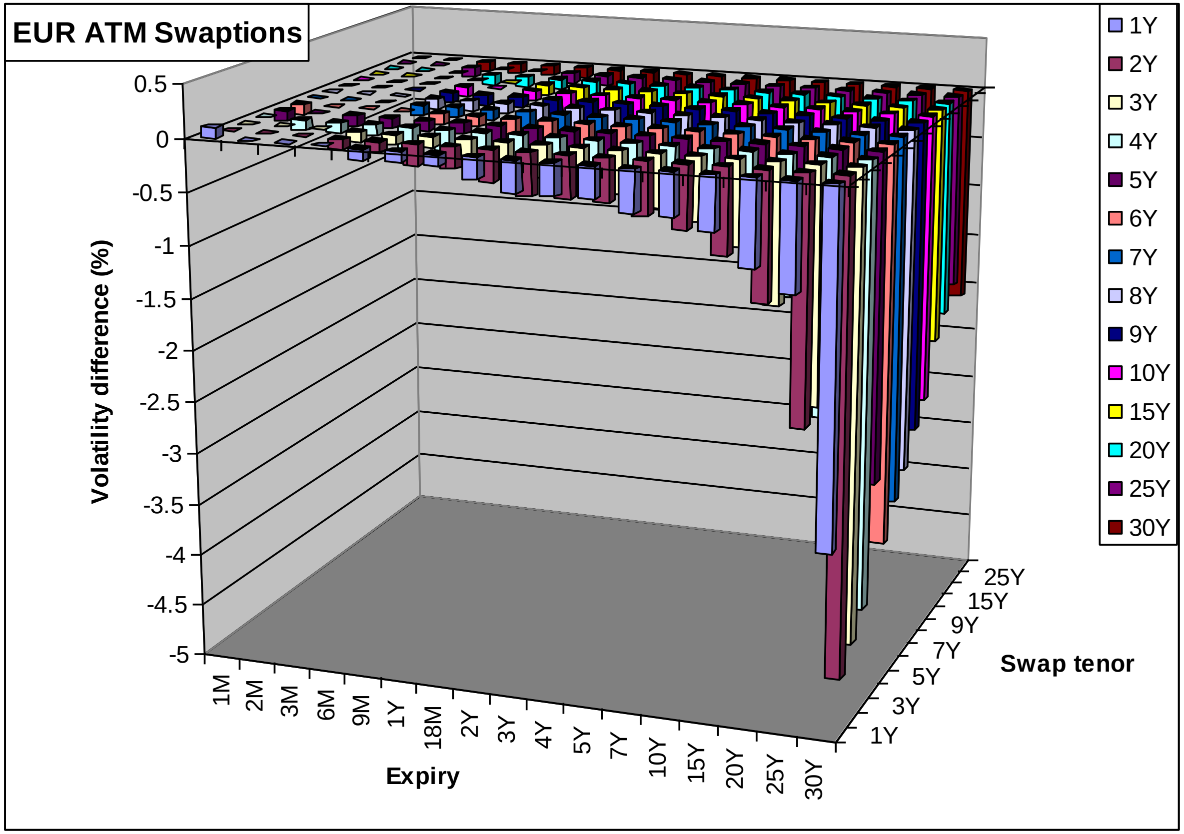
EUR ATM Swaption Straddles - Implied Spot Premium Mids (Euribor disc)

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	12	25	39	56	74	92	112	132	154	177	270	361	444	523
2M Opt	16	34	56	78	105	132	161	188	217	248	378	509	626	733
3M Opt	20	43	69	100	131	163	197	233	270	308	460	611	751	874
6M Opt	27	59	97	142	192	239	286	334	384	433	637	830	1023	1201
9M Opt	33	74	120	173	238	295	354	414	473	529	766	991	1214	1424
1Y Opt	39	86	142	209	281	347	416	484	553	618	888	1150	1403	1635
18M Opt	52	114	186	272	357	438	520	600	679	757	1077	1383	1676	1959
2Y Opt	70	149	238	333	430	519	607	695	781	869	1236	1578	1908	2224
3Y Opt	106	213	323	434	547	651	753	853	950	1049	1474	1875	2266	2633
4Y Opt	132	257	381	507	630	745	859	969	1075	1181	1639	2078	2507	2903
5Y Opt	148	286	420	553	684	810	932	1051	1168	1279	1753	2212	2659	3079
7Y Opt	166	317	462	606	747	883	1019	1151	1280	1406	1896	2361	2811	3238
10Y Opt	172	333	485	635	783	930	1074	1215	1352	1486	1987	2450	2885	3288
15Y Opt	171	328	481	631	778	922	1064	1205	1342	1486	1963	2399	2768	3132
20Y Opt	162	313	464	611	756	899	1036	1172	1303	1429	1887	2275	2639	2963
25Y Opt	156	300	445	581	716	847	973	1099	1222	1369	1793	2144	2492	2786
30Y Opt	151	277	406	531	673	781	910	1038	1168	1304	1701	2038	2360	2659

EUR ATM Swaption market quotes on 31 May 2012. Forward premia (left upper panel, page VCAP2A) and spot premia (left lower panel, page VCAP2) obtained from forward premia using Eonia discounting, and spot premia (right lower panel, page VCAP2B) obtained from forward premia using Euribor discounting.

There is an unique ATM implied volatility surface (right upper panel, page VCAP1), obtained from forward premia using Eonia discounting. (Sources: Reuters, ICAP).

3: The market across the credit crunch
From Libor to CSA discounting: Broker Market [7]



Implicit market volatility differences (Eonia-Euribor) on 30 Sep. 2010.

The Eonia Black implied volatilities (VCAP1A) are smaller than the corresponding Euribor volatilities (VCAP1) because, lowering the discounting rate from Euribor to Eonia, a larger discount factor is obtained, thus leading, at constant spot premium, to a smaller implied volatility.

The larger differences correspond to the longer option maturities (30Y) with the shorter underlying swap tenors (1Y).

3: The market across the credit crunch
From Libor to CSA discounting: balance sheets (May 2011)

At 2010 year end some Banks have given disclosure the switch to OIS discounting [1]:

- BNP: -108 MM EUR (IRS)
- Credit Agricole: -120 MM EUR (Fixed Income)
- Morgan Stanley: +176 MM USD (IRD)
- RBS: +127 MM GBP (???)
- UBS: +76 MM CHF (???)
- HSBC: not significant

"In the fourth quarter of 2010, the Company began using the overnight indexed swap ("OIS") curve as an input to value substantially all of its collateralized interest rate derivative contracts. The Company believes using the OIS curve, which reflects the interest rate typically paid on cash collateral, more accurately reflects the fair value of collateralized interest rate derivative contracts. Previously, the Company discounted these collateralized interest rate derivative contracts based on London Interbank Offered Rates ("LIBOR")." [2]

[1] M. Cameron, ""BNP Paribas takes €108 million on swaps after switch to OIS discounting", Risk, 6 May 2011.
[2] Morgan Stanley & Co. Inc., Consolidated Statement of Financial Condition as of Dec. 31, 2010 and Independent Auditors' report.

3: The market across the credit crunch
From Libor to CSA discounting: the gold rush

Risk.net
Financial Risk Management News and Analysis

Goldman and the OIS gold rush: How fortunes were made from a discounting change

/risk-magazine/feature/2270178/goldman-and-the-ois-gold-rush-how-fortunes-were-made-from-a-discounting-change

29 May 2013, Matt Cameron, Risk magazine

"They rang everyone on the Street with this one," says one head of swaps trading at an Asian bank, recalling how his firm was repeatedly asked by Goldman Sachs to step into a package of swap trades in 2008. The trades in question were two cross-currency swaps in the same currency pair – one would be out of the money for the new counterparty, while the other was flat. Goldman was offering to pay around $200 million to the bank to step in – money that would immediately be posted back to the US bank as collateral.

3: The market across the credit crunch

From Libor to CSA discounting: KPMG survey (Oct. 2011) [1]

KPMG, "*New valuation and pricing approaches for derivatives in the wake of the financial crisis - Moving towards a new market standard ?*", October 2011, www.kpmg.com

1. A new valuation and pricing framework
   - CSA discounting has become the market standard for pricing collateralized deals, and will become the market standard for valuation at trade level.
   - The treatment of uncollateralized deals is still open to discussion
2. Collateralized deals:
   - major banks are driving the transformation of the market towards CSA discounting, medium sized banks lag behind.
   - the adoption of CSA discounting by central clearing services [make] this market transformation […] irreversible.
   - […] the most critically affected asset class [are] interest rate derivatives, long-running FX or inflation products.
   - valuation inconsistencies [are managed] by running a parallel infrastructure capable of both CSA and Libor discounting.
   - the consideration of eligible collateral currencies in their valuation appears to be an important detail

3: The market across the credit crunch
From Libor to CSA discounting: KPMG survey (Oct. 2011) [2]

3. Uncollateralized deals
   - the majority of the banks plans to consider their own funding costs in the valuation of uncollateralized deals in some form
   - Institutions planning to adopt a funding cost discounting approach mostly tend to determine the spread based on an analysis of their current short-term and long-term (unsecured) funding costs. Others use the average of their current funding mix or short-term funding costs only.
   - Many of them have [instead] adopted a symmetric CVA/DVA approach. They are, as a result, forced to handle the double counting problem when they wish to implement funding cost discounting. […they] stick to Libor discounting and might adjust the DVA by the bond-CDS basis.
4. Implications for bank management
   - the new valuation schemes deeply affect transfer pricing between derivative desks, treasury/funding functions and loan units. […] internal deals are, or will be, valued as if they are collateralized
   - Analogously to a CVA, the funding needs and benefits from collateralized and uncollateralized trades could also be transferred to a central desk via upfront fees or internal trades. This central desk would manage the funding needs of the derivatives business and the second order effects of the underlying market movements on the funding of the collateral pool. A central desk ensures that netting effects over all desks can be exploited and that the funding needs of the capital market business can be pooled.

3: The market across the credit crunch

From Libor to CSA discounting: KPMG survey (Oct. 2011) [3]

5. What institutions need to do now

Key actions

- In order not to be arbitraged out, each institution must adopt CSA discounting for collateralized deals, at least for asset classes where the market indicates that most of the banks have switched their pricing to the new method.
- In times of rising funding costs, no bank can afford to ignore its funding costs. The economic valuation of uncollateralized deals should be linked to the actual funding spread of each bank. A business model which does not allow to fully price in its actual cost of funding is not feasible in the long-term.
- CSA or funding related valuation is not a pure playground for quants, but rather a topic that evokes questions about transfer pricing, steering of risk and, most importantly, the business model of each bank.

3: The market across the credit crunch
Conclusions Part 1

What we have understood up to now:

- How the market has changed.
- Credit and liquidity risk are important.
- Collateral is important, new standard CSA.
- Funding is important.

We need a generalised theoretical framework able to include these new elements.

3: The market across the credit crunch
Problems

- Deposits: using the KLIEMM quotations, price, for each of the five currencies, a strip of Deposits with spot starting date and weekly-spaced maturities until 1Y (i.e. 52 deposits). Use real schedules. Plot the resulting yield curves, accrued interest amounts and discount factors until maximum quoted maturities. Repeat the calculations assuming a borrower with a flat spread of 100 bps. Deliverable: spreadsheet with data, results and charts.

- Historical series 1: build the chart of the historical series of OIS6M, Euribor6M, and of the corresponding basis and correlation. Use spreadsheet and Reuters/Bloomberg. Deliverable: spreadsheet with charts and comments.

- Historical series 2: extend the previous problem to other tenors (1M,3M,12M) and relevant interest rates (e.g. Repo). Deliverable: spreadsheet with charts and comments.

Summary

4. Modern interest rate modelling

- Theoretical framework
  - Short rate, bank account and risk neutral measure
  - Feynman-Kac theorem
  - Zero Coupon Bond and forward measure
  - Change of measure, Girsanov theorem
  - Replication
- Funding, collateral, funding value adjustment (FVA)
  - Black-Scholes-Merton from a modern perspective
  - Multiple funding sources
  - Collateral: discrete margination
  - Perfect collateral
  - Perfect collateral for derivative and hedge
  - Perfect collateral, dividends, repo
  - Partial collateral
  - Perfect collateral, stochastic rates
  - Multiple currency

4: Modern interest rate modelling
Motivations

- In order to understand the modern interest rate market after the credit crunch, we must set up a framework with solid theoretical basis able to explain the observed market data, or, in other words, to price plain vanilla derivatives according to the available market quotations.

- Hence, we must go back to basics and restart from scratch the interest rate theory, with the aim to refresh the foundations and to (re)discover hidden assumptions, in particular concerning the single versus the multi-curve approach.

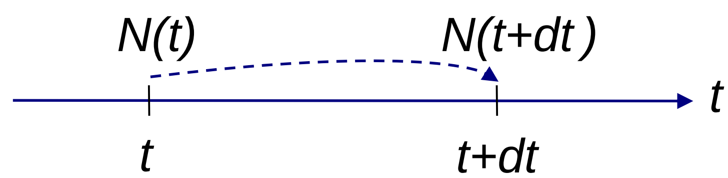
4: Modern interest rate modelling
Short rate

The (spot) instantaneous rate, abbreviated into short rate, is an abstract rate spanning an infinitesimal time interval (with infinitesimal rate tenor). In fact, setting $T_1 = t$, $T_2 = T$ and taking the limit $t^+ \leftarrow T$ we obtain

$$R_k(t,T) \underset{t^+ \leftarrow T}{\longrightarrow} R_1(t,T) \underset{t^+ \leftarrow T}{\longrightarrow} R_\infty(t,T) \underset{t^+ \leftarrow T}{\longrightarrow} L_k(t,T) \underset{t^+ \leftarrow T}{\longrightarrow} r(t).$$

From any eq. above, we may obtain, setting $T = t + dt$,

$$N(t+dt) = N(t)\left[1 + r(t)dt\right],$$



where *dt* is an infinitesimal time interval. Thus the dynamics of *N(t)* is given by

$$dN(t) := N(t+dt) - N(t) = N(t)r(t)dt$$

and can be easily integrated over a finite time interval $[T_1,T_2]$ to obtain

$$N(T_2) = N(T_1)\exp\int_{T_1}^{T_2} r(t)\,dt.$$

Thus the short rate is a continuously compounded annual rate.

4: Modern interest rate modelling
Bank account and risk neutral measure [1]

The bank account, or money market account, is, in financial mathematics, an ideal financial instrument representing the behaviour of an abstract loan that rewards its holder with the risk free rate. Denoting with *B(t)* the bank account value at time t, it must evolve in time according to the dynamics

$$\begin{cases} dB(t) = r(t)B(t)dt, \\ B(0) = 1. \end{cases}$$

This simple differential problem can be integrated to obtain the solution

$$\boxed{B(T) = B(0) \exp \int_0^T r(t)\, dt,}$$

Bank account

The factor *B(0)=1* on the r.h.s. is typically omitted, in this case one should remember that the dimensionality is correct (currency on both sides).

Being the short rate a stochastic process, also the bank account is a stochastic process. From a financial point of view, the bank account is such that one unit of currency invested at time *t = 0* accrues, over an infinitesimal time interval *dt*, at the (stochastic) short rate *r(t)*.

4: Modern interest rate modelling
Bank account and risk neutral measure [2]

Thus the bank account is particularly suitable as reference asset, or numeraire, since it allows to put into relation amounts of currencies observed at different times.
The value (price) of any contract Π can be expressed in units of *B(t)* and *B(T)* through the risk neutral pricing formula

$$\frac{\Pi(t)}{B(t)} = \mathbb{E}_t^Q\left[\frac{\Pi(T)}{B(T)}\right],$$

$$\Pi(t) = \mathbb{E}_t^Q\left[D(t,T)\Pi(T)\right],$$

$$D(t,T) := \frac{B(t)}{B(T)} = \exp\left[-\int_t^T r(u)\,du\right],$$

Stochastic discount factor

where *Q* denotes the risk neutral measure associated to the numeraire *B(t)* and $\mathbb{E}_t^Q\left[.\right]$ denotes the expectation at time *t<T* under measure *Q*.

We remark that the stochastic discount factor is adimensional, being the ratio between two bank account values, and it depends on the short rate over the time interval *[t,T]*.

4: Modern interest rate modelling
Bank account and risk neutral measure [3]

In case of a contract paying multiple coupons $\{\Pi(T_1), \dots, \Pi(T_N)\}$ at multiple cash flow dates $\{T_1, \dots, T_N\}$ we have

$$\Pi(t) = \sum_{i=1}^{N} \mathbb{E}_t^Q \left[D(t, T_i) \Pi(T_i) \right].$$

Notice that the expectation is a linear operator. The expression above can be written in integral form

$$\Pi(t) = \int_t^T \mathbb{E}_t^Q \left[D(t, u) d\pi(u) \right].$$

where $T = T_N$, by introducing the cumulative coupon process

$$\pi(t) := \sum_{i=1}^{N} 1_{[t > T_i]} \Pi(T_i).$$

In fact

Dirac's delta

$$\Pi(t) = \int_t^T \mathbb{E}_t^Q \left[D(t, u) d\pi(u) \right] = \int_t^T \mathbb{E}_t^Q \left[D(t, u) \sum_{i=1}^{N} \delta\left(T_i - u\right) \Pi(u)\, du \right] = \sum_{i=1}^{N} \mathbb{E}_t^Q \left[D(t, T_i) \Pi(T_i) \right]$$

4: Modern interest rate modelling
Feynman-Kac Theorem [1]

The Feynman-Kac theorem allows to bridge between the PDE representation and the SDE representation of the derivatives' price Π.

Feynman-Kac theorem:
let *A* be a generic asset with price process *A(t)* solution of the SDE

$$dA(t) = \boxed{\mu(t, A)} dt + \sigma(t, A) dW^Q(t), \quad 0 \le t \le T,$$
$$\mu, \sigma \in \mathcal{L}^2[0, T],$$
$$A(t = 0) = A_0 \in \mathbb{R}^+,$$

under some probability measure *Q*.
Let also Π be a derivative on *A* with price $\Pi[t,A(t)]= \Pi(t)=$ at time *t*, solution of the parabolic PDE

$$\hat{\mathcal{L}}_\mu \Pi(t) = \boxed{r(t)} \Pi(t),$$
$$\hat{\mathcal{L}}_\mu := \frac{\partial}{\partial t} + \boxed{\mu(t, A)} \frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2}{\partial A^2},$$
$$\Pi \in \mathcal{C}^{1,2}\big[[0, T] \times \mathbb{R}\big], \quad \sigma(t, A)\Pi(t) \in \mathcal{L}^2[0, T],$$
$$\Pi(T) \in \mathbb{R}^+, \; t \in [0, T] \subset \mathbb{R}^+$$

4: Modern interest rate modelling
Feynman-Kac Theorem [2]

…then the derivatives' price $\Pi(t)$ admits the representation

$$\Pi(t) = \mathbb{E}_t^Q \left[D(t,T)\Pi(T) \right],$$

$$D(t,T) = \exp\left[-\int_t^T \boxed{r(u)} du \right],$$

where Q is the measure such that

$$dA(t) = \boxed{\mu(t,A)} dt + \sigma(t,A) dW^Q(t),$$

The Feynman-Kac theorem allows to switch from the PDE representation to the SDE representation of the derivatives' price Π.

Proof: homework
Hint: apply Ito's formula to $D(t,T)\Pi(t)$

Generalised Feynman-Kac theorem:

In case of the more general parabolic PDE

$$\mathcal{L}_\mu \Pi(t) = \boxed{r(t, A(t))}\Pi(t) + \boxed{\phi(t, A(t))},$$

we have the more general expectation

$$\Pi(t) = \mathbb{E}_t^Q \left[D(t,T,A)\Pi(T) + \int_t^T D(t,u,A(u))\boxed{\phi(u, A(u))}du \right],$$

$$D(t,T,A) = \exp\left[-\int_t^T \boxed{r(u, A(u))}du \right].$$

where Q is the measure such that

$$dA(t) = \mu(t,A)dt + \sigma(t,A)dW^Q(t),$$

See e.g. Darrel Duffie (2001), Tomas Bjork (2009).

Zero Coupon Bond

The Zero Coupon Bond is the simplest interest rate derivative. It is a contract in which one party guarantees to the other party the payment of one unit of currency at maturity date T, with no other payments. The contract payoff at time T is thus denoted by $P(T;T)=1$ and the contract value at time $t<T$ by $P(t;T)$. The dimension is currency (c) and the units are, e.g., Euro.
Using the risk neutral pricing formula we have the pricing expression

$$\boxed{P(t;T) = \mathbb{E}_t^Q \left[D(t,T) P(T;T) \right] = \mathbb{E}_t^Q \left[D(t,T) \right].}$$

Zero coupon bond

As for the bank account the dimensionality of the eq. above is correct when one remembers that there is an hidden nominal amount $N=1$ units of currency on the r.h.s.

Notice that the Zero Coupon Bond value, being the price of a contract between two counterparties, has to be exactly known at any time $t<T$, and thus it is a deterministic (not stochastic) quantity (being an expectation). This is the main difference with respect to the stochastic discount factor.

4: Modern interest rate modelling
Zero Coupon Bond vs discount factor

Both the stochastic discount factor $D(t,T)$ and the Zero Coupon Bond $P(t;T)$ "move" an amount of money backward in time.
In financial terms we say that the amount of money is discounted from time T to time $t<T$, thus $D(t,T), P(t;T)$ are both called discount factors. The reciprocals $1/D(t;T)$ and $1/P(t;T)$ "move" an amount of money forward in time from t to $T>t$ and are called capitalization factors.

The main difference between the two types of discount/capitalization factors is that, in general, the Zero Coupon Bond is deterministic, while the stochastic discount factor is not. Thus, given a deterministic amount of money $N(T_2)$ at time T_2, we have

$$N(T_1) = P(T_1;T_2)N(T_2),$$

$$N'(T_1) = D(T_1,T_2)N(T_2) \neq N(T_1),$$

$$\mathbb{E}^Q_{T_1}[N'(T_1)] = P(T_1;T_2)N(T_2) = N(T_1),$$

where $N(T_1)$ is deterministic and $N'(T_1)$ is, in general, stochastic.

In case of deterministic interest rates, we have $D(t,T) = P(t;T)$ and $N'(T_1) = N(T_1)$ are all deterministic quantities.

4: Modern interest rate modelling
Zero Coupon Bond and interest rates [1]

There exist a relationship between interest rates and Zero Coupon Bonds.

Using the general expression

$$N(T_1) = P(T_1;T_2)N(T_2)$$

and the definitions of simple/discrete/continuous compounded rates give before, we obtain the following expressions

Interest rate	Expression in terms of Zero Coupon Bond
Simple compounding (Libor)	$L(T_1,T_2) = \frac{1}{\tau(T_1,T_2)}\left[\frac{1}{P(T_1;T_2)} - 1\right],$
Discrete compounding	$R_k(T_1,T_2) = \frac{k}{P(T_1;T_2)^{\frac{1}{k\tau(T_1,T_2)}}} - k,$
Continuous compounding	$R_\infty(T_1,T_2) = -\frac{1}{\tau(T_1,T_2)}\ln P(T_1;T_2).$

4: Modern interest rate modelling
Zero Coupon Bond and interest rates [2]

We may invert the preceding relations to express the Zero Coupon Bond in terms of the different types of interest rates

Interest rate	Expression in terms of Zero Coupon Bond
Simple compounding (Libor)	$P(T_1;T_2) = \dfrac{1}{1 + L(T_1,T_2)\tau(T_1,T_2)},$
Discrete compounding	$P(T_1;T_2) = \dfrac{1}{\left[1 + \frac{R_k(T_1,T_2)}{k}\right]^{k\tau(T_1,T_2)}},$
Continuous compounding	$P(T_1;T_2) = e^{-R_\infty(T_1,T_2)\tau(T_1,T_2)}.$

4: Modern interest rate modelling
Zero Coupon Bond and forward measure

The Zero Coupon Bond is particularly important in interest rate modeling because, similarly to the bank account, it can be used as reference asset (numeraire) to put into relation amounts of currencies observed at different times.

The value (price) of any asset π at any times *t* and *T>t* can be expressed in units of *P(t,T)* and *P(T,T)*, respectively, through the T-forward (Libor) pricing formula

$$\boxed{\begin{gathered}\frac{\Pi(t)}{P(t;T)} = \mathbb{E}_t^{Q^T}\left[\frac{\Pi(T)}{P(T;T)}\right] = \mathbb{E}_t^{Q^T}\left[\Pi(T)\right], \\ \Pi(t) = P(t;T)\mathbb{E}_t^{Q^T}\left[\Pi(T)\right],\end{gathered}}$$

where Q_T denotes the T-forward (Libor) measure associated to the numeraire *P(t;T)*.

4: Modern interest rate modelling
Change of measure [1]

Definition

Q_{N_1}, Q_{N_2} are equivalent martingale measures associated to the two numeraires N_1 and N_2 if

$$\frac{\Pi(t)}{N_1(t)} = \mathbb{E}_t^{Q_{N_1}}\left[\frac{\Pi(T)}{N_1(T)}\right],$$

$$\frac{\Pi(t)}{N_2(t)} = \mathbb{E}_t^{Q_{N_2}}\left[\frac{\Pi(T)}{N_2(T)}\right]$$

Comparing the two equations above we obtain

$$\Pi(t) = \mathbb{E}_t^{Q_{N_1}}\left[\frac{N_1(t)}{N_1(T)}\Pi(T)\right] = \mathbb{E}_t^{Q_{N_2}}\left[\frac{N_2(t)}{N_2(T)}\Pi(T)\right],$$

thus the change from measure Q_{N_1} to measure Q_{N_2} is given by

$$\boxed{\mathbb{E}_t^{Q_{N_2}}\left[\frac{\Pi(T)}{N_2(T)}\right] = \frac{N_1(t)}{N_2(t)}\mathbb{E}_t^{Q_{N_1}}\left[\frac{\Pi(T)}{N_1(T)}\right].}$$

4: Modern interest rate modelling
Change of measure [2]

Why changing the numeraire ?

Let *A(t)* the stochastic process underlying the payoff of the derivative Π:

- Suppose that $N_2(t)$ is a numeraire, a strictly positive tradable asset and $A(t)N_2(t)$ is the price of a tradable asset;
- In this case $[A(t)N_2(t)]/N_2(t)$ is a martingale under Q_2, such that we may assume simple stochastic dynamics for it with simple distributions under Q_2, e.g. lognormal martingale dynamics

$$\frac{dA(t)}{A(t)} = \sigma(t)dW^{Q_{N_2}}(t),$$

$$\ln A(T) \simeq \mathcal{N}\left[\ln A(t) - \frac{1}{2}\int_t^T \sigma^2(u)du;\ \int_t^T \sigma^2(u)du\right].$$

Now, if $\Pi(t)/N_2(t)$ is simple enough w.r.t. $\Pi(t)/N_1(t)$, we are able to compute its expectation under Q_2

$$\mathbb{E}_t^{Q_{N_2}}\left[\frac{\Pi(T)}{N_2(T)}\right].$$

- Example 1: change between risk neutral and T-forward measures

If we choose

$$
\begin{aligned}
&Q_{N_1} = Q,\; Q_{N_2} = Q_T, \\
&N_1(t) = B(t),\; N_2(t) = P(t;T),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\Pi(t) = \mathbb{E}_t^{Q}\left[D(t,T)\Pi(T)\right] &= B(t)\mathbb{E}_t^{Q}\left[\frac{\Pi(T)}{B(T)}\right] \\
&= B(t)\frac{P(t;T)}{B(t)}\mathbb{E}_t^{Q_T}\left[\frac{\Pi(T)}{P(T;T)}\right] \\
&= P(t;T)\mathbb{E}_t^{Q_T}\left[\Pi(T)\right].
\end{aligned}
$$

- Example 2: change between T_1 and T_2 forward measures

If we choose

$$Q_{N_1} = Q_{T_1} =, Q_{N_2} = Q_{T_2}, T_1 < T_2$$
$$N_1(t) = P(t; T_1), \; N_2(t) = P(t; T_2),$$

we obtain

$$\Pi(t) = \mathbb{E}_t^{Q_{T_1}} \left[\frac{P(t; T_1)}{P(T'; T_1)} \Pi(T') \right] = \mathbb{E}_t^{Q_{T_2}} \left[\frac{P(t; T_2)}{P(T'; T_2)} \Pi(T') \right],$$
$$t \leq T_1 < T_2 \leq T'$$

- Example 3: change between foreign and domestic risk neutral measures

If we choose

$$
\begin{gathered}
Q_{N_1} = Q^f =, \; Q_{N_2} = Q^d, \\
N_1(t) = B_f(t), \; N_2(t) = B_d(t), \\
\Pi_d(t) = x_{fd}(t)\Pi_f(t),
\end{gathered}
$$

where $x_{fd}(t)$ is the spot exchange rate from currency f to currency d, we obtain

$$
\begin{aligned}
\Pi_f(t) &= \mathbb{E}_t^{Q^f}\left[D_f(t,T)\Pi_f(T)\right], \\
\Pi_d(t) &= \mathbb{E}_t^{Q^d}\left[D_d(t,T)\Pi_d(T)\right] \\
&= x_{fd}(t)\mathbb{E}_t^{Q^f}\left[D_f(t,T)\Pi_f(T)\right] \\
&= \mathbb{E}_t^{Q^d}\left[D_d(t,T)x_{fd}(T)\Pi_f(T)\right],
\end{aligned}
$$

hence

$$
\mathbb{E}_t^{Q^f}\left[D_f(t,T)\Pi_f(T)\right] = \mathbb{E}_t^{Q^d}\left[D_d(t,T)\frac{x_{fd}(T)}{x_{fd}(t)}\Pi_f(T)\right], \; \forall t \leq T
$$

- Example 4: change between risk neutral and forward swap measures

If we choose, in the general change of measure formula,

$$\begin{aligned} &Q_{N_1} = Q,\ Q_{N_2} = Q_{\mathbf{S}}, \\ &N_1(t) = B(t),\ N_2(t) = A(t; \mathbf{S}), \end{aligned}$$

where $A(t; \boldsymbol{S})$ is the swap annuity,

$$A(t, \mathbf{S}) = \sum_{i=1}^{n} P(t, S_i)\tau_K(S_{i-1}, S_i),$$

we obtain

$$\begin{aligned} \Pi(t) = \mathbb{E}_t^Q \left[D(t,T)\Pi(T) \right] &= \mathbb{E}_t^Q \left[\frac{B(t)}{B(T)} \Pi(T) \right] \\ &= A(t; \mathbf{S}) \mathbb{E}_t^{Q_S} \left[\frac{\Pi(T)}{A(T, \mathbf{S})} \right], \ \forall\, t \le T \le S_0. \end{aligned}$$

Theorem: change of numeraire (Girsanov)

Given two numeraires $N_1(t), N_2(t)$, and a generic asset $A(t)$ following the stochastic diffusion processes under the martingale measure Q_1 associated to N_1,

$$dA(t) = \boxed{\mu_A^{Q_1}[t, A(t)]} + \boldsymbol{\sigma}_A[t, A(t)]' \cdot d\boldsymbol{W}^{Q_1}(t),$$

$$dN_1(t) = \mu_{N_1}^{Q_1}[t, N_1(t)] + \boldsymbol{\sigma}_{N_1}[t, N_1(t)]' \cdot d\boldsymbol{W}^{Q_1}(t),$$

$$dN_2(t) = \mu_{N_2}^{Q_1}[t, N_2(t)] + \boldsymbol{\sigma}_{N_2}[t, N_2(t)]' \cdot d\boldsymbol{W}^{Q_1}(t),$$

$$dW_i^{Q_1}(t)dW_j^{Q_1}(t) = \rho_{ij}(t)dt, \; i, j = 1, ..., F,$$

where $\boldsymbol{W}^{Q_1}(t)$ is a F-dimensional vector of correlated brownian motions under Q_1 and the volatilities are F-dimensional vectors, with *$1 \leq F \leq 3$*. The dynamics of $A(t)$ under Q_2 is

$$dA(t) = \boxed{\mu_A^{Q_2}[t, A(t)]} + \boldsymbol{\sigma}_A[t, A(t)] \cdot d\boldsymbol{W}^{Q_2}(t),$$

$$\mu_A^{Q_2}[t, A(t)] = \mu_A^{Q_1}[t, A(t)] + \boldsymbol{\sigma}_A[t, A(t)] \cdot \boldsymbol{\rho}(t) \cdot \left[\frac{\boldsymbol{\sigma}_{N_2}[t, A(t)]}{N_2(t)} - \frac{\boldsymbol{\sigma}_{N_1}[t, A(t)]}{N_1(t)} \right]',$$

$$d\boldsymbol{W}^{Q_2}(t) = d\boldsymbol{W}^{Q_1}(t) - \boldsymbol{\rho}(t) \cdot \left[\frac{\boldsymbol{\sigma}_{N_2}[t, A(t)]}{N_2(t)} - \frac{\boldsymbol{\sigma}_{N_1}[t, A(t)]}{N_1(t)} \right]' dt,$$

Problems

- Check the following no-arbitrage relationship

$$D(t,T) = D(t,u)D(u,T), \quad \forall\, u \in [t,T]\,.$$

- Check the following no-arbitrage relationship

$$P(t,T) = P(t,u)\mathbb{E}_t^{Q_u}\left[P(u,T)\right], \quad \forall\, u \in [t,T]\,.$$

4: Modern interest rate modelling
Replication [1]

- **Market**
We assume a market *M* trading *n* assets ***A*** with price process $\boldsymbol{A}(t)$, dividend process $\boldsymbol{D}(t)$ and cumulative gain $\boldsymbol{G}(t)$ processes given by

$$\boldsymbol{G}(t) = \boldsymbol{A}(t) + \boldsymbol{D}(t) = \begin{bmatrix} G_1(t) \\ \vdots \\ G_n(t) \end{bmatrix} = \begin{bmatrix} A_1(t) \\ \vdots \\ A_n(t) \end{bmatrix} + \begin{bmatrix} D_1(t) \\ \vdots \\ D_n(t) \end{bmatrix}.$$

Price, dividend and cumulative gain processes

- **Asset price dynamics**
We assume that the assets ***A*** follows a Ito dynamics under measure *P*

$$d\boldsymbol{A}(t) = \boldsymbol{\mu}(t, \boldsymbol{A})dt + \boldsymbol{\sigma}(t, \boldsymbol{A}) \cdot d\boldsymbol{W}^P(t), \quad \boldsymbol{A}(0) = \boldsymbol{A}_0,$$
$$dW_i^P(t)dW_j^P(t) = \delta_{ij}dt, \ \forall i, j = 1, \cdots, d,$$

Asset price dynamics

where the drift $\boldsymbol{\mu}$ is a *n*-dimensional vector, the volatility $\boldsymbol{\sigma}$ is a $d \times n$ matrix, and ***W*** is a *d*-dimensional standard Brownian motion vector with $d \leq n$. Here we also have a time horizon $T > 0$, a probability space $(\Omega, \mathcal{F}, Q)$ and a right-continuous filtration $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$.

Replication [2]

Dividend dynamics

We assume that the assets $\boldsymbol{A}$ generate continuous dividends proportional to $\boldsymbol{A}$ with instantaneous dividend rate $r_D(t)$ following the dynamics

$$d\boldsymbol{D}(t) = \boldsymbol{A}(t)\boldsymbol{r}_D(t)dt, \quad \boldsymbol{D}(0) = \boldsymbol{0},$$

$$\boldsymbol{D}(t) = \int_0^t \boldsymbol{A}(u)\boldsymbol{r}_D(u)du,$$

Dividend dynamics

(notice the component by component product).

Cumulative gain dynamics

As a consequence, the cumulative gain dynamics is given by

$$\begin{aligned} d\boldsymbol{G}(t) &= d\boldsymbol{A}(t) + d\boldsymbol{D}(t) \\ &= [\boldsymbol{\mu}(t, \boldsymbol{A}) + \boldsymbol{r}_D(t)\boldsymbol{A}(t)]\, dt + \boldsymbol{\sigma}(t, \boldsymbol{A}) \cdot d\boldsymbol{W}^P(t), \\ \boldsymbol{G}(0) &= \boldsymbol{A}(0) + \boldsymbol{D}(0) = \boldsymbol{G}_0, \end{aligned}$$

Cumulative gain dynamics

Derivative dynamics

We also assume that the market trades also a derivative Π written on the assets $\boldsymbol{A}$. Denoting with $\Pi(t, \boldsymbol{A})$ the derivative price at time t, it's dynamics is obtained from Ito's lemma,

$$
\begin{aligned}
d\Pi(t, \boldsymbol{A}) &= \frac{\partial \Pi}{\partial t} dt + \frac{\partial \Pi}{\partial \boldsymbol{A}}' \cdot d\boldsymbol{A}(t) + \frac{1}{2} d\boldsymbol{A}(t)' \cdot \frac{\partial^2 \Pi}{\partial \boldsymbol{A}^2} \cdot d\boldsymbol{A}(t) \\
&= \hat{\mathcal{L}}_\mu(t, \boldsymbol{A}) \Pi(t) dt + \frac{\partial \Pi}{\partial \boldsymbol{A}}' \cdot \boldsymbol{\sigma}(t, \boldsymbol{A}) \cdot d\boldsymbol{W}^P(t),
\end{aligned}
$$

Derivative price dynamics

$$
\hat{\mathcal{L}}_\mu(t, \boldsymbol{A}) := \frac{\partial}{\partial t} + \boldsymbol{\mu}(t, \boldsymbol{A})' \cdot \frac{\partial}{\partial \boldsymbol{A}} + \frac{1}{2} Tr \left[\boldsymbol{\sigma}(t, \boldsymbol{A})' \cdot \frac{\partial^2}{\partial \boldsymbol{A}^2} \cdot \boldsymbol{\sigma}(t, \boldsymbol{A}) \right],
$$

where

$$
Tr \left[\boldsymbol{\sigma}(t, \boldsymbol{A})' \frac{\partial^2}{\partial \boldsymbol{A}^2} \boldsymbol{\sigma}(t, \boldsymbol{A}) \right] = \sum_{i,j=1}^{n} \sum_{f=1}^{d} \sigma_{i,f}(t, \boldsymbol{A}) \sigma_{j,f}(t, \boldsymbol{A}) \frac{\partial^2}{\partial A_i \partial A_j}.
$$

We denote with $G_\Pi(t, \boldsymbol{A})$ the cumulative gain process of the derivative Π.

4: Modern interest rate modelling
Replication [4]

- **Exchange rate**
  In order to deal with the multiple currency funding case, we introduce in the market *M* also a spot exchange rate at time *t*,

$$N^\alpha(t) = x^{\alpha,\beta}(t) N^\beta(t).$$

- **Trading strategy and portfolio**
  A trading strategy is a process $\boldsymbol{\Theta}(t)$ of weights of assets $\boldsymbol{A}$ such that the associated trading portfolio $\boldsymbol{\Theta}' \cdot \boldsymbol{A}$ has price, dividend and cumulative gain processes given by

$$V_{\boldsymbol{\Theta}}(t, \boldsymbol{A}) := \boldsymbol{\Theta}(t)' \cdot \boldsymbol{A}(t), \qquad V_{\boldsymbol{\Theta}}(0, \boldsymbol{A}) = \boldsymbol{\Theta}(0)' \cdot \boldsymbol{A}(0) := V_{\boldsymbol{\Theta},0}(\boldsymbol{A}),$$

$$G_{\boldsymbol{\Theta}}(t, \boldsymbol{A}) := \int_0^t \boldsymbol{\Theta}(u)' \cdot d\boldsymbol{G}(u), \qquad G_{\boldsymbol{\Theta}}(0, \boldsymbol{A}) = 0,$$

$$D_{\boldsymbol{\Theta}}(t, \boldsymbol{A}) := G_{\boldsymbol{\Theta}}(t, \boldsymbol{A}) - \left[V_{\boldsymbol{\Theta}}(t, \boldsymbol{A}) - V_{\boldsymbol{\Theta},0}(\boldsymbol{A})\right], \qquad D_{\boldsymbol{\Theta}}(0, \boldsymbol{A}) = 0.$$

Price, dividend and cumulative gain processes of trading portfolio

The components of the trading strategy $\boldsymbol{\Theta}$ are interpreted as the number of units (or nominal) of the assets $\boldsymbol{A}$ held in the trading portfolio at time *t*.

See e.g. Darrel Duffie (2001), Tomas Bjork (2009).

4: Modern interest rate modelling
Replication [5]

Self financing strategy and portfolio

A trading strategy $\boldsymbol{\Theta}$ is self-financing if the dividend process of the associated trading portfolio is always null

$$D_{\boldsymbol{\Theta}}(t, \boldsymbol{A}) = 0, \quad \forall\, t \in [0, T]\,,$$
$$\Rightarrow dG_{\boldsymbol{\Theta}}(t, \boldsymbol{A}) = dV_{\boldsymbol{\Theta}}(t, \boldsymbol{A})$$

Self financing condition

Intuitively, a self-financing trading strategy is such that cumulative gains of the trading portfolio are generated only by the changes in the asset prices A, and no additional cash in/out flows occur during it's life.

Replication strategy and portfolio

A derivative Π is replicated through a self-financing trading strategy Θ if

$$\Pi(t, \boldsymbol{A}) = V_{\boldsymbol{\Theta}}(t, \boldsymbol{A}), \quad \forall\, t \in [0, T]\,,$$
$$G_{\Pi}(t, \boldsymbol{A}) = G_{\boldsymbol{\Theta}}(t, \boldsymbol{A}), \quad \forall\, t \in [0, T]\,,$$

Replication condition

that is, if both the price and cumulative gain processes of the derivative match the corresponding price and cumulative gain processes of the replication portfolio.

Classical Black-Scholes-Merton, modern perspective [1]

We consider a generic derivative Π depending on a single generic underlying asset *A*, with payoff $\Pi(T)$ at time *T* and price $\Pi(t)$ at time $t < T$.

We assume a market M that trades three financial instruments:

- the asset *A*, with no dividends
- the derivative Π
- the funding account B_f (cash units) for funding unsecured at rate r_f

We stress the following assumptions:

- Single asset *A*
- No collateral
- No counterparty risk
- No dividends
- Generic funding for asset A, no repo
- Deterministic interest rates (see the derivation)
- All the classical Black-Scholes-Merton assumptions

We will derive the classical Black-Scholes-Merton pricing formulas using replication arguments, PDE and Feynman-Kac. In particular, we will be able to understand when and where funding enters into the derivation and into the final result.

Using the replication approach, we have the following equations.

- Assets price, dividend and cumulated gain processes

$$\boldsymbol{X}(t) = \begin{bmatrix} A(t) \\ B_f(t) \end{bmatrix}, \quad \boldsymbol{D}(t) = \mathbf{0}, \quad \boldsymbol{G}(t) = \boldsymbol{X}(t),$$

- Dynamics under measure P

$$\begin{aligned}
dA(t) &= \mu(t, A)dt + \sigma(t, A)dW^P(t), \\
dB_f(t) &= r_f(t)B_f(t)dt, \\
d\Pi(t) &= \frac{\partial \Pi}{\partial t}dt + \frac{\partial \Pi}{\partial A}dA(t) + \frac{1}{2}\frac{\partial^2 \Pi}{\partial A^2}dA^2(t) \\
&= \hat{\mathcal{L}}_\mu \Pi(t)dt + \sigma(t, A)\frac{\partial \Pi}{\partial A}dW^P(t), \\
\hat{\mathcal{L}}_\mu &= \frac{\partial \Pi}{\partial t} + \mu(t, A)\frac{\partial \Pi}{\partial A} + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2 \Pi}{\partial A^2},
\end{aligned}$$

4: Modern interest rate modelling
Classical Black-Scholes-Merton, modern perspective [3]

The gain processes of the assets, in SDE form, are given directly by the dynamics chosen before, as

$$d\boldsymbol{G}(t) = d\boldsymbol{X}(t) = \begin{bmatrix} dA(t) \\ dB_f(t) \end{bmatrix} = \begin{bmatrix} \mu(t,A)dt + \sigma(t,A)dW^P(t) \\ r_f(t)B_f(t)dt \end{bmatrix}.$$

We now construct a replication strategy $\boldsymbol{\Theta}$ of the derivative Π by setting up a replication portfolio as follows

$$\begin{aligned}
\boldsymbol{\Theta}(t) &:= \begin{bmatrix} \Delta(t) \\ \psi_f(t) \end{bmatrix}, \\
V_\Theta(t) &= \boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t) = \Delta(t)A(t) + \psi_f(t)B_f(t), \\
dG_\Theta(t) &:= \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) = \Delta(t)dA(t) + \psi_f(t)dB_f(t) \\
&= \left[\mu(t,A)\Delta(t) + r_f(t)\psi_f(t)B_f(t)\right]dt + \Delta(t)\sigma(t,A)dW^P(t).
\end{aligned}$$

where:

- $\boldsymbol{\Theta}$ is the (column) vector of the portfolio positions, or number of units, in each asset, and $\boldsymbol{\Theta}'$ denotes vector transposition,
- $V_\Theta(t)$ and $G_\Theta(t)$ are the (scalar) value and gain processes of the replication portfolio.

4: Modern interest rate modelling
Classical Black-Scholes-Merton, modern perspective [4]

We now impose replication condition, and we obtain

$$\Pi(t) = V(t) = \Delta(t)A(t) + \psi_f(t)B_f(t), \quad \forall\, t \leq T$$
$$\Rightarrow \psi_f(t)B_f(t) = \Pi(t) - \Delta(t)A(t),$$

consistently with the fact that the funding account B_f is used to finance the borrowing of $\Delta(t)$ units of the underlying $A(t)$ at the funding rate $r_f(t)$.

The gain process of the replication portfolio becomes

$$dG_\Theta(t) = \mu(t,A)\Delta(t)dt + \boxed{d\Gamma(t,A)} + \Delta(t)\sigma(t,A)dW^P(t),$$
$$\boxed{d\Gamma(t,A)} = \left[-r_f(t)\Delta(t)A(t) + r_f(t)\Pi(t)\right] dt$$

the cash amount $\Gamma(t)$ contained in the replication portfolio is split between:

- the derivative $\Pi(t)$, growing at the funding rate $r_f(t)$,
- the amount $\Delta(t)A(t)$, borrowed at the funding rate $r_f(t)$ to finance the purchase of $\Delta(t)$ units of the underlying asset $A(t)$.

We now impose the self-financing condition. The replication strategy is said self-financing if its dividend process (in/out cash flows generated by the strategy) is null,

$$D_\Theta(t) = G_\Theta(t) - V_\Theta(t) = 0.$$

We have just seen that this latter condition is already satisfied. Combining the conditions above, we have

$$dG_\Theta(t) = dV_\Theta(t) = d\Pi(t).$$

Introducing in this latter equation the expressions of *dG(t)* and *dΠ(t)* obtained before, and rearranging terms we obtain the SDE

$$\left[\frac{\partial \Pi}{\partial t} + \mu(t,A)\left(\frac{\partial \Pi}{\partial A} - \Delta(t)\right) + \frac{1}{2}\sigma^2(t,A)\frac{\partial^2 \Pi}{\partial A^2}\right] dt \\ + \sigma(t,A)\left(\frac{\partial \Pi}{\partial A} - \Delta(t)\right) dW^P(t) = d\Gamma(t,A).$$

We finally impose the risk neutral condition $\Delta(t) = \frac{\partial \Pi}{\partial A}$ such that the stochastic (risky) term with $dW^P(t)$ disappears, and we obtain…

…a Black-Scholes PDE equation for the derivative's price $\Pi(t)$

$$\hat{\mathcal{L}}_{r_f}\Pi(t) = \boxed{r_f(t)}\Pi(t),$$
$$\hat{\mathcal{L}}_{r_f} := \frac{\partial}{\partial t} + \boxed{r_f(t)}A(t)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t,A)\frac{\partial^2}{\partial A^2}$$

Using the Feynman-Kac theorem we may switch from the PDE representation to the SDE representation given by

$$\Pi(t) = \mathbb{E}_t^{\boxed{Q}}\left[\boxed{D_f(t;T)}\Pi(T)\right],$$
$$D_f(t;T) := \exp\left[-\int_t^T \boxed{r_f(u)}du\right],$$
$$dA(t) = \boxed{r_f(t)}A(t)dt + \sigma(t,A)dW^{\boxed{Q}}(t),$$

under the risk neutral funding probability measure Q associated, in this case, to the funding account $B_f(t)$. We conclude that we discount at the funding rate.

Remarks:

1. **Replication at work**:
This proof makes clear how the replication and funding mechanism works in practice. The market risk generated by the derivatives' position is hedged using the risky asset A. The replication strategy is constructed with a combination of the instruments available on the market: the (single) asset A and the funding account B_f. The former allows to include into the replication strategy the appropriate amount of risk to hedge the risk generated by the derivative. The latter describe the amount of cash that we must borrow or lend on the market at the funding rate r_f to finance the hedging. The cash is split between the amount $\Delta(t)A(t)$, borrowed to finance the purchase of $\Delta(t)$ units of the risky asset $A(t)$, and the amount Π.

2. **Probability measure**: the probability measure Q introduced via Feynman-Kac is associated to the risk neutral drift r_f appearing in the SDE dynamics of the asset A. In the classical financial world Q was traditionally associated to a Libor Bank account, reflecting the average funding rate on the interbank money market, considered a good proxy of a risk free rate. Nowadays, in the modern financial world, there are no risk free rates, and Q must be interpreted simply as the risk neutral measure associated to the funding account B_f. We call it the funding measure.

3. **Static hedge**: in particular, self-financing implies the absence of strategy dividends and

$$
\begin{aligned}
& dV_\Theta(t) = dG_\Theta(t), \\
\Rightarrow\ & d\left[\boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t)\right] = \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) = \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{X}(t), \\
\Rightarrow\ & d\left[\Delta(t)A(t) + \psi_f(t)B_f(t)\right] = \Delta(t)dA(t) + \psi_f(t)dB_f(t), \\
\Rightarrow\ & d\boldsymbol{\Theta}(t) = \mathbf{0}, \\
\Rightarrow\ & \boldsymbol{\Theta}(t) = \ constant.
\end{aligned}
$$

This is the well known feature of the classical Black-Scholes derivation:

- the position $\Delta(t)A(t)$ in the risky asset S is self-financing in its own, because its variation $d[\Delta(t)A(t)]$ is funded by the risky asset variation alone, $\Delta(t)dA(t)$.
- The position is static, $\Delta(t)= constant$.

We stress that this is a consequence of the absence of dividends. In general this equality does not hold,

$$
d\left[\boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t)\right] \neq \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{X}(t).
$$

4. **Zero Coupon Bonds**:
   We can define unsecured Zero Coupon Bonds, such that

$$P_f(T;T) = 1,$$
$$P_f(t;T) = \mathbb{E}_t^Q\left[D_f(t;T)\right],$$

5. **T-forward measure**:
   we can switch from risk neutral funding measure Q, associated to numeraire $B_f(t)$, to T-forward measure Q^T, associated to numeraire $P_f(t,T)$, using the Radon-Nicodym derivative

$$M_f(t;T) := \frac{P_f(t;T)}{B_f(t)},$$

Radon-Nicodym derivative

$$\begin{aligned}
\Pi(t;T) &= \mathbb{E}_t^Q\left[D_f(t,T)\Pi(T)\right] \\
&= M_f(t;T)\mathbb{E}_t^{Q^T}\left[\frac{D_f(t,T)}{M_f(T;T)}\Pi(T)\right] \\
&= \frac{P_f(t;T)}{B_f(t)}\mathbb{E}_t^{Q^T}\left[\frac{B_f(t}{P_f(T;T)}\Pi(T)\right] \\
&= P_f(t;T)\mathbb{E}_t^{\boxed{Q^T}}\left[\Pi(T)\right].
\end{aligned}$$

T-forward measure

Multiple funding sources

we assume that derivatives' counterparties may finance their derivatives' acivity by borrowing and lending funds on the market through a variety of market operations, such as trading Deposits, Repos (Repurchase Agreements), Bonds, etc. at their corresponding funding rates.

We also assume that derivatives' counterparties eventually reduce the counterparty risk through the adoption of bilateral collateral agreements (CSA) or trade migration to Central Counterparties (CCPs).

In particular, we will identify three sources of funding associated with derivatives.

- **Money market**
  Money market funding is the traditional unsecured funding source for banks and financial institutions. Borrowing and lending is based on the trading of Certificates of Deposit (Depo). A Depo is an unsecured cash zero coupon loan. In case of default of the borrower during the Depo life, the lender suffers a loss.
  In banks and financial institutions, a derivative trading desk may borrow and lend unsecured funds through a treasury desk.

- **Repo market**

Another common funding source is repo funding. In this case, borrowing and lending is based on trading Repurchase Agreement contracts (Repo). A Repo is a secured cash zero coupon loan such that, at time t, counterparty B, the borrower, sales an asset A to counterparty L, the lender, receiving upfront the corresponding asset value $A(t)$, under the agreement to buy it back and pay an interest $R_R(t,T)$, called repo rate, at maturity $T>t$. Thus, the borrower pays, at maturity T the amount

$$\Pi_R(T,A) = A(t)\left[1 + R_R(t,T)\tau(t,T)\right]$$

The repo is secured by the asset A itself, used as collateral. In case of default of the borrower during the repo life, the lender keeps the asset A. Thus, a repo is equivalent to a combination between a spot sale (the initial legal transfer of the asset to the lender in exchange for transfer of money to the borrower), and a forward contract (repayment of the loan to the lender and return of the collateral of the borrower at maturity). Possible coupons and or dividends generated by the asset A during the repo life are transferred by the lender to the borrower.

Looking at the forward contract component of the repo, the Repo price $\Pi_R(t)$ is such that $\Pi_R(t)=0$ if the contract is traded at par at time t.

In banks and financial institutions, a derivative trading desk may borrow and lend secured repo funds through a repo desk.

Multiple funding sources [3]

- **Collateral**

  Real collateral agreements are regulated mostly under the Credit Support Annex (CSA) of the ISDA Standard Master Agreement. For pricing purposes it is useful to introduce an abstract "perfect" collateral, with the following properties.
  - Zero initial margin or initial deposit
  - Zero threshold
  - Zero minimum transfer amount
  - Fully symmetric
  - Cash collateral only
  - Continuous margination
  - Instantaneous settlement
  - Instantaneous margination rate $r^{\beta}_{c}(t)$ in currency β
  - In case of default of one counterparty: neither close out amounts nor legal risk to the closing of the deal or availability of the collateral

  As a consequence we have that, in general, the collateral value perfectly matches the derivative's value,

$$\Pi^{\alpha}(t, A) = x_{\alpha\beta}(t) C^{\beta}(t), \quad \forall\, t \leq T.$$

  In banks and financial institutions, the collateral associated with a derivative trading desk is operated by a collateral desk.

- **Multiple funding accounts**

  Following the previous discussion, we assume that the amount of cash borrowed or lent by a counterparty in the market *M* from multiple funding sources is associated to multiple funding accounts B_x, where index x denotes the specific source of funding, with value $B^\alpha_x(t)$ and (symmetric) funding rate $r^\alpha_x(t)$ in currency α at time t, such that

$$dB^\alpha_x(t) = r^\alpha_x(t)B^\alpha_x(t)dt, \quad B^\alpha_x(0) = 1,$$

$$B^\alpha_x(t) = \exp\left[\int_0^t r^\alpha_x(u)du\right],$$

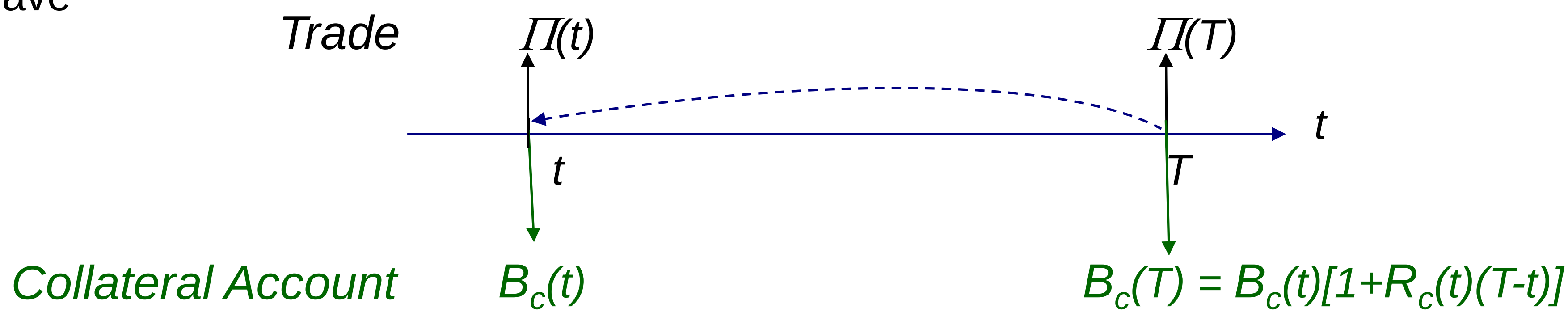
$$D^\alpha_x(t,T) := \frac{B^\alpha_x(t)}{B^\alpha_x(T)} = \exp\left[-\int_t^T r^\alpha_x(u)du\right].$$

Multiple funding accounts

- **Remarks**:
  1. the collection of funding accounts $B_x(t)$ is assumed locally market risk free, since their dynamics do not contain stochastic terms, and thus the value of the account at time $t+dt$ depends only on the value of the account and of the rate at previous time t.
  2. The collection of funding accounts $B_x(t)$ is assumed credit risk free, since the default of the borrowing counterparty is not included into their dynamics
  3. The funding rates $r_x(t)$ may be, in general, stochastic.

4: Modern interest rate modelling
Collateral: a trivial (but intuitive) example

Let's suppose that the trade consists of single cash flow, such that we receive/pay an amount $\Pi(T)$ at cash flow date T, corresponding to a present value $\Pi(t)$ at time $t<T$. Let's suppose also that the trade is under perfect collateral, with two margination dates, at t and T: at time t we post the amount $B_c(t)$ into the collateral account, where it grows at the collateral rate $R_c(t)$ up to maturity T. By no arbitrage and self-financing, we must have



$$B_c(T) = B_c(t)\left[1 + R_c(t)(T-t)\right] = \Pi(T),$$

$$\Pi(t) = P_d(t,T)\Pi(T) = B_c(t),$$

$$\Rightarrow P_d(t,T) = \frac{1}{\left[1 + R_c(t)(T-t)\right]}.$$

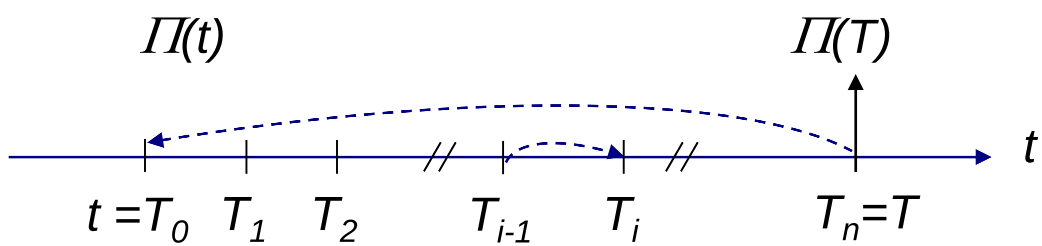
Collateral discounting

Thus no arbitrage requires discounting at the collateral rate.

Collateral: discrete margination case [1]

In case of multiple discrete marginations, at each margination date T_i the counterparties must regulate the margin over the last time interval $\Delta T_i = [T_{i-1}, T_i]$ by exchanging the amount

$$\begin{aligned}\mathcal{M}(T_i) &= \Pi(T_i) - \Pi(T_{i-1}) - B_c(T_{i-1}) R_c(T_{i-1}) \Delta T_i \\ &= \Pi(T_i) - \Pi(T_{i-1}) \left[1 + R_c(T_{i-1}) \Delta T_i\right],\end{aligned}$$



The counterparty whose NPV has increased/decreased must receive/post the amount $M(T_i)$. At the same time interval the bank account $B(t)$ and discount factor $D(t,T)$ evolve with the (simple compounded) rate R as

$$B(T_i) = B(T_{i-1}) \left[1 + R(T_{i-1}) \Delta T_i\right],$$

$$D(t;T_i) = \frac{D(t, T_{i-1})}{1 + R(T_{i-1}) \Delta T_i}, \quad D(t;t) = 1,$$

Collateral: discrete margination case [2]

The value at time *t* of all the future margination amounts is given by the discounted sum

$$
\begin{aligned}
\mathcal{M}(t;\boldsymbol{T}) &= \sum_{i=1}^{n} \mathbb{E}_t^Q \left[D(t;T_i)\mathcal{M}(T_i) \right] \\
&= \sum_{i=1}^{n} \mathbb{E}_t^Q \left\{ D(t;T_i) \left[\Pi(T_i) - \Pi(T_{i-1}) \left(1 + R_c(T_{i-1})\Delta T_i\right) \right] \right\} \\
&= \sum_{i=1}^{n} \mathbb{E}_t^Q \left[D(t;T_i)\Pi(T_i) - D(t;T_i) \frac{D_c(t;T_{i-1})}{D_c(t;T_i)} \Pi(T_{i-1}) \right],
\end{aligned}
$$

where $D_c(t;T_i)$ is the discount factor associated to the collateral rate such that

$$
\begin{aligned}
B_c(T_i) &= B_c(T_{i-1}) \left[1 + R_c(T_{i-1})\Delta T_i \right], \\
D_c(t;T_i) &= \frac{D_c(t;T_{i-1})}{1 + R_c(T_{i-1})\Delta T_i}, \\
D_c(t;t) &= 1.
\end{aligned}
$$

4: Modern interest rate modelling
Collateral: discrete margination case [3]

By no-arbitrage, the total value of the margination amount must be null,

$$\mathcal{M}(t;\boldsymbol{T}) = \sum_{i=1}^{n} \mathbb{E}_t^Q \left[D(t;T_i)\Pi(T_i) - D(t;T_i)\frac{D_c(t;T_{i-1})}{D_c(t;T_i)}\Pi(T_{i-1}) \right] = 0.$$

This may be true if and only if $D_d(t;T_i) = D_c(t;T_i)$. Hence

$$\begin{aligned}
\mathcal{M}(t;\boldsymbol{T}) &= \sum_{i=1}^{n} \mathbb{E}_t^Q \left[D_c(t;T_i)\Pi(T_i) - D_c(t;T_{i-1})\Pi(T_{i-1}) \right] \\
&= \mathbb{E}_t^Q \left[D_c(t;T_n)\Pi(T_n) - D_c(t,T_0)\Pi(T_0) \right] \\
&= \mathbb{E}_t^Q \left[D_c(t;T)\Pi(T) \right] - \Pi(t) = 0,
\end{aligned}$$

We conclude that no arbitrage implies discounting at the collateral rate

$$\begin{aligned}
\Pi(t) &= \mathbb{E}_t^Q \left[D_c(t;T)\Pi(T) \right], \\
D(t;T_i) &= D_c(t;T_i), \\
R(t;T_i) &= R_c(t;T_i).
\end{aligned}$$

CVD

4: Modern interest rate modelling
Collateral: discrete margination case [4]

In the limit of continuous margination $\Delta T_i \rightarrow dt$ we have

$$D(t;T_i) \longrightarrow D(t;T) = \exp\left[-\int_t^T r(u)\,du\right],$$

$$D_c(t;T_i) \longrightarrow D_c(t;T) = \exp\left[-\int_t^T r_c(u)\,du\right],$$

$$\mathcal{M}(t;\boldsymbol{T}) \longrightarrow \mathcal{M}(t;T) = \mathbb{E}_t^Q\left[D_c(t;T)\Pi(T)\right] - \Pi(t) = 0,$$

$$\Pi(t) = \mathbb{E}_t^Q\left[D_c(t;T)\Pi(T)\right],$$

where $r(t)$ and $r_c(t)$ are the short rate and the collateral short rate, respectively.

4: Modern interest rate modelling
Collateral: discrete margination case [5]

In the limit of no margination $\Delta T_i \rightarrow T\text{-}t$ we have the same equations as above, but we make funding not at the collateral rate but at a generic funding spread $s_f(t)$ over the risk free rate

$$
\begin{aligned}
D_c(t;T_i) &\longrightarrow \exp\left\{-\int_t^T \left[r(u) + s_f(u)\right] du\right\} := D(t;T)D_f(t;T), \\
\mathcal{M}(t;\boldsymbol{T}) &\longrightarrow \mathcal{M}_f(t;\boldsymbol{T}) = \mathbb{E}_t^Q\left[D(t;T)D_f(t;T)\Pi(T)\right] - \Pi(t) = 0, \\
\Pi(t) &\longrightarrow \Pi_f(t) = \mathbb{E}_t^Q\left[D(t;T)D_f(t;T)\Pi(T)\right], \\
|\Pi_f(t)| &\leq |\Pi(t)|.
\end{aligned}
$$

Hence we discount at the funding rate.

Collateral: discrete margination case [6]

If $s_f(t)$ is deterministic we obtain

$$\Pi_f(t) = P_f(t;T)\mathbb{E}_t^Q\left[D(t;T)\Pi(T)\right] = P_f(t;T)\Pi(t),$$

$$P_f(t;T) = D_f(t;T) = \exp\left[-\int_t^T s_f(u)\,du\right],$$

$$|\Pi_f(t)| \leq |\Pi(t)|.$$

4: Modern interest rate modelling
Funding: perfect collateral [1]

We now consider the case of derivative under perfect collateral.

Our economy admits, in this case, four financial instruments :

- the asset A, with no dividends
- the derivative Π under perfect collateral C
- the funding account B_f for funding unsecured at rate r_f
- the collateral account B_c for funding secured by collateral at rate r_c

We hold all the assumptions of the previous case

See e.g. V. Piterbarg (Feb 2010), V. Piterbarg (Aug. 2012), C. Burgard, M. Kjaer (Oct. 2011), A. Castagna (Dec. 2011), D. Brigo et al (Jul. 2012), A. Pallavicini, D. Brigo (2013), A. Antonov, M. Bianchetti (2013).

4: Modern interest rate modelling
Funding: perfect collateral [2]

Using the replication approach, we have the following equations.

- Assets price, dividend and cumulated gain processes

$$\boldsymbol{X}(t) = \begin{bmatrix} A(t) \\ B_f(t) \\ B_c(t) \end{bmatrix}, \quad \boldsymbol{D}(t) = \mathbf{0}, \quad \boldsymbol{G}(t) = \boldsymbol{X}(t),$$

- Dynamics under measure P

$$\begin{aligned}
dA(t) &= \mu(t, A)dt + \sigma(t, A)dW^P(t), \\
dB_f(t) &= r_f(t)B_f(t)dt, \\
dB_c(t) &= r_c(t)B_c(t)dt, \\
d\Pi(t) &= \hat{\mathcal{L}}_\mu \Pi(t)dt + \sigma(t, A)\frac{\partial \Pi}{\partial A}dW^P(t), \\
\hat{\mathcal{L}}_\mu &= \frac{\partial}{\partial t} + \mu(t, A)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2}{\partial A^2}.
\end{aligned}$$

4: Modern interest rate modelling

Funding: perfect collateral [3]

The gain processes of the assets, in SDE form, are given directly by the dynamics chosen before, as

$$d\boldsymbol{G}(t) = d\boldsymbol{X}(t) = \begin{bmatrix} dA(t) \\ dB_f(t) \\ dB_c(t) \end{bmatrix} = \begin{bmatrix} \mu(t,A)dt + \sigma(t,A)dW^P(t) \\ r_f(t)B_f(t)dt \\ r_c(t)B_c(t)dt \end{bmatrix}.$$

We now construct a replication strategy $\boldsymbol{\Theta}$ of the derivative Π by setting up a replication portfolio as follows

$$\boldsymbol{\Theta}(t) := \begin{bmatrix} \Delta(t) \\ \psi_f(t) \\ \psi_c(t) \end{bmatrix},$$

$$V_\Theta(t) = \boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t) = \Delta(t)A(t) + \psi_f(t)B_f(t) + \psi_c(t)B_c(t),$$

$$\begin{aligned} dG_\Theta(t) &:= \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) = \Delta(t)dA(t) + \psi_f(t)dB_f(t) + \psi_c(t)dB_c(t) \\ &= \left[\mu(t,A)\Delta(t) + r_f(t)\psi_f(t)B_f(t) + r_c(t)\psi_c(t)B_c(t)\right]dt + \Delta(t)\sigma(t,A)dW^P(t). \end{aligned}$$

4: Modern interest rate modelling

Funding: perfect collateral [4]

We now impose the perfect collateral and replication conditions,

$$\Pi(t) = C(t) = \psi_c(t) B_c(t),$$
$$V_\Theta(t) = \Pi(t), \quad \forall\, t \leq T$$

respectively, and we obtain

$$\begin{aligned} V_\Theta(t) &= \Delta(t)A(t) + \psi_f(t)B_f(t) + \boxed{\psi_c(t)B_c(t)} \\ &= \Delta(t)A(t) + \psi_f(t)B_f(t) + \boxed{\Pi(t)}, \\ &\Rightarrow \;\; \psi_f(t)B_f(t) = -\Delta(t)A(t). \end{aligned}$$

Notice that:

- the funding account B_f is used to finance, at the funding rate r_f, the borrowing of Δ units of the asset *A*,
- the collateral account is used to finance, at the collateral rate r_c, the derivative Π,
- no cash is left out the replication because of the perfect collateral (see the partial collateral case later).

Funding: perfect collateral [5]

The gain process of the replication portfolio becomes

$$dG_{\Theta}(t) = \mu(t, A)\Delta(t)dt + \boxed{d\Gamma(t, A)} + \Delta(t)\sigma(t, A)dW^{P}(t),$$
$$\boxed{d\Gamma(t, A)} := \left[-r_f(t)\Delta(t)A(t) + r_c(t)\Pi(t)\right] dt.$$

Notice that the cash amount $\Gamma(t, A)$ contained in the replication portfolio is split between:

- the derivative amount, equal to the cash in the collateral account B_c, growing at the collateral rate r_c,
- the amount $\Delta(t)A(t)$, borrowed at the funding rate $r_f(t)$ to finance the purchase of $\Delta(t)$ units of the asset $A(t)$.

4: Modern interest rate modelling
Funding: perfect collateral [6]

We now impose the self-financing condition on the replication strategy

$$D_\Theta(t) = G_\Theta(t) - V_\Theta(t) = 0.$$

We have seen that this latter condition is already satisfied. Combining the conditions above, we have

$$dG_\Theta(t) = dV_\Theta(t) = d\Pi(t).$$

Introducing in this latter equation the expressions of $dG_\Theta(t)$ and $d\Pi(t)$ obtained before, and rearranging terms, we obtain the SDE

$$\begin{aligned}\left[\frac{\partial \Pi}{\partial t} + \mu(t,A)\left[\frac{\partial \Pi}{\partial A} - \Delta(t)\right] + \frac{1}{2}\sigma^2(t,A)\frac{\partial^2 \Pi}{\partial A^2}\right] dt \\ + \sigma(t,A)\left[\frac{\partial \Pi}{\partial A} - \Delta(t)\right] dW^P(t) = d\Gamma(t,A).\end{aligned}$$

We finally impose the risk neutral condition $\Delta(t) = \frac{\partial \Pi}{\partial A}$ such that the stochastic (risky) term with $dW^P(t)$ disappears, and we obtain…

4: Modern interest rate modelling
Funding: perfect collateral [7]

…a generalised Black-Scholes PDE equation for the derivative's price *Π(t,A)*

$$\hat{\mathcal{L}}_{r_f}\Pi(t) = \boxed{r_c(t)}\Pi(t),$$
$$\hat{\mathcal{L}}_{r_f} = \frac{\partial}{\partial t} + \boxed{r_f(t)}A(t)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t,A)\frac{\partial^2}{\partial A^2}.$$

Using the Feynman-Kac theorem we may switch from the PDE representation to the SDE representation given by

$$\Pi(t) = \mathbb{E}_t^{\boxed{Q}}\left[\boxed{D_c(t;T)}\Pi(T)\right],$$
$$D_c(t;T) := \exp\left[-\int_t^T \boxed{r_c(u)}du\right],$$
$$dA(t) = \boxed{r_f(t)}A(t)dt + \sigma(t,A)dW^{\boxed{Q}}(t),$$

Classical Black-Scholes –Merton with collateral

We conclude that we discount at the collateral rate.

4: Modern interest rate modelling
Funding: perfect collateral [8]

Remarks

1. **Funding measure**:
   the probability measure Q introduced via Feynman-Kac is associated with the risk neutral drift r_f appearing in the SDE dynamics of the asset A. It is the same measure of the uncollateralised case, but now the numeraire is the collateral account B_c.

2. **Borrowing/lending return**:
   we notice that, over the time interval $[t, t+dt)$, the derivative $\Pi(t)$ and the cash flow $r_c(t)\Pi(t)dt$ generated by the (perfect) collateral margination, are equivalent to a derivative $\Pi(t)$ without collateral but with a continuous dividend yield $r_c(t)\Pi(t)dt$, such that $r_f - rc$ the rate is the actual (instantaneous) borrowing/lending cost/return including the collateral.

3. **Zero Coupon Bonds**:
   We can define secured (collateralized) Zero Coupon Bonds, such that

$$P_c(T;T) = 1,$$

$$P_c(t;T) = \mathbb{E}_t^Q \left[D_c(t;T) \right].$$

3. T-Forward measure:

we can switch from the funding measure associated with numeraire $B_c(t)$ to T-forward measure $Q_c{}^T$ associated with secured (collateralised) numeraire $P_c(t;T)$ using the corresponding Radon-Nicodym derivative as follows

$$M_c(t,T) = \frac{P_c(t;T)}{B_c(t)},$$

$$\begin{aligned}\Pi_c(t) &= \mathbb{E}_t^Q\left[D_c(t;T)\Pi(T)\right] \\ &= M(t,T)\mathbb{E}_t^{Q_c^T}\left[\frac{D_c(t;T)}{M(T,T)}\Pi(T)\right] \\ &= P_c(t;T)\mathbb{E}_t^{\boxed{Q_c^T}}\left[\frac{1}{P_c(T,T)}\Pi(T)\right], \\ &= P_c(t;T)\mathbb{E}_t^{Q_c^T}\left[\Pi(T)\right].\end{aligned}$$

Collateralised T-forward measure

We remember that any positive martingale process (not necessary the ratio between two numeraires) can be used as Radon-Nicodym derivative in a measure change.

4: Modern interest rate modelling

Funding: perfect collateral [10]

4. **(Im)perfect collateral**
   Since real CSAs are far from being perfect, a more general proof is required to take into account imperfect collateral, such that $B_c(t) \quad \Pi(t)$, also in terms of different currencies.

5. **Equity asset**
   The lending/borrowing of some assets is often realised through repo contracts and funded at the repo rate. Furthermore, assets may pay, in general, dividends.
   Typical example is equity asset.
   Thus a more general proof is necessary to deal with these special case.

4: Modern interest rate modelling
Funding: perfect collateral [11]

6. Funding Valuation Adjustment (FVA)

Comparing the collateralised vs uncollateralised prices we can define a Funding Value Adjustment (FVA) such that, in additive form,

$$\begin{aligned}
\Pi_f(t) &= \mathbb{E}_t^Q\left[D_f(t;T)\Pi(T)\right] \\
&= \mathbb{E}_t^Q\left[\frac{D_f(t;T)}{D_c(t;T)}D_c(t;T)\Pi(T)\right] \\
&= \mathbb{E}_t^Q\left[D_c(t;T)\Pi(T)\right] + FVA_{f,c}(t) \\
&= \Pi_c(t) + FVA_{f,c}(t),
\end{aligned}$$

$$FVA_{f,c}(t) := \Pi_f(t) - \Pi_c(t).$$

Funding Value Adjustment

The calculation of the FVA above depends on the correlated dynamics of the funding and collateral rates, and is thus model dependent. If the variance of the ratio between the funding and the collateral (stochastic) discounts is negligible with respect to the variance of the remaining discounted payoff [*],

$$Var\left[\frac{D_f(t;T)}{D_c(t;T)}\right] \ll Var\left[D_c(t;T)\Pi(T)\right],$$

[*] this condition is less restrictive than the deterministic basis limit used below.

4: Modern interest rate modelling
Funding: perfect collateral [12]

we may write

$$\begin{aligned}\Pi_f(t) &= \mathbb{E}_t^Q\left[\frac{D_f(t;T)}{D_c(t;T)}D_c(t;T)\Pi(T)\right]\\ &\simeq \mathbb{E}_t^Q\left[\frac{D_f(t;T)}{D_c(t;T)}\right]\mathbb{E}_t^Q\left[D_c(t;T)\Pi(T)\right]\\ &= \mathbb{E}_t^Q\left[\frac{D_f(t;T)}{D_c(t;T)}\right]\Pi_c(t),\\ FVA_{f,c}(t) &\simeq \left\{\mathbb{E}_t^Q\left[\frac{D_f(t;T)}{D_c(t;T)}\right]-1\right\}\Pi_c(t).\end{aligned}$$

Funding
Value
Adjustment

In the limit of deterministic basis we obtain the simple expression

$$\mathbb{E}_t^Q\left[\frac{D_f(t;T)}{D_c(t;T)}\right] \simeq \frac{\mathbb{E}_t^Q\left[D_f(t;T)\right]}{\mathbb{E}_t^Q\left[D_c(t;T)\right]} = \frac{P_f(t;T)}{P_c(t;T)} = e^{-\int_t^T s_{f,c}(u)\,du},$$

$$FVA_{f,c}(t) \simeq \left[e^{-\int_t^T s_{f,c}(u)\,du}-1\right]\Pi_c(t) \simeq -\left[\int_t^T s_{f,c}(u)\,du\right]\Pi_c(t),$$

$$s_{f,c}(t) := r_f(t) - r_c(t).$$

Funding: perfect collateral for derivative and hedge [1]

We now consider the special case of perfect collateral, in which also the asset A is perfectly collateralised, with its distinct collateral rate.

In this case our economy admits, four financial instruments:

- the asset A under collateral C_A
- the derivative Π under collateral C_Π
- the derivatives' collateral account $B_{c\Pi}$ for funding Π secured by collateral at rate $r_{c\Pi}$
- the asset's collateral account B_{cA} for funding A secured by collateral at rate r_{cA}

The proof follows the previous case of perfect collateral, with the substitutions

$$
\begin{aligned}
B_f &\to B_{c_A}, \quad r_f \to r_{c_A}, \\
B_c &\to B_{c_\Pi}, \quad r_c \to r_{c_\Pi}, \\
C &\to C_\Pi.
\end{aligned}
$$

We obtain the PDE

$$
\begin{aligned}
\hat{\mathcal{L}}_{r_c^A} \Pi(t) &= r_{c_\Pi}(t)\Pi(t), \\
\hat{\mathcal{L}}_{r_c^A} &= \frac{\partial}{\partial t} + r_{c_A}(t)A(t)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t,A)\frac{\partial^2}{\partial A^2}.
\end{aligned}
$$

4: Modern interest rate modelling
Funding: perfect collateral for derivative and hedge [2]

Using Feynman-Kac

$$\Pi(t) = \mathbb{E}_t^Q \left[D_{c_\Pi}(t;T)\Pi(T) \right],$$

$$D_{c_\Pi}(t;T) := \exp\left[-\int_t^T r_{c_\Pi}(u)du \right],$$

$$dA(t) = r_{c_A}(t)A(t)dt + \sigma(t,A)dW^Q(t).$$

We conclude that, in case of perfect collateral for both the derivative and the hedge, the stochastic process for the asset *A* is driven by the collateral rate r_{cA} associated with the asset *A*.

Further, if the asset and the derivatives have identical CSAs, $B_{c_A} = B_{c_\Pi} = B_c$, $r_{c_A} = r_{c_\Pi} = r_c$, we obtain the cleanest situation with the same collateral rate everywhere,

$$\Pi(t) = \mathbb{E}_t^Q \left[D_c(t;T)\Pi(T) \right],$$

$$D_c(t;T) := \exp\left[-\int_t^T r_c(u)du \right],$$

$$dA(t) = r_c(t)A(t)dt + \sigma(t,A)dW^Q(t).$$

4: Modern interest rate modelling
Funding: perfect collateral + dividends + repo [1]

We now consider the case of derivatives on a dividend paying asset A subject to repo funding.

In practice, the asset A, instead of being traded directly, funded through an unsecured funding account B_f at the funding rate r_f, is traded indirectly through repo contracts, secured by the asset A and funded at the repo rate r_R.

Hence, our economy admits, in this case, four financial instruments:

- the asset A, and its dividends at rate r_D
- the derivative Π under collateral C
- the collateral account B_c for funding secured by collateral at rate r_c
- the repo contract $\Pi_R(A)$, for funding secured by asset A at repo rate r_R

See D. Brigo et al, "*Illustrating a problem in the self-financing condition in two 2010-2011 papers on funding, collateral and discounting*", 10 Jul. 2012, SSRN, *http://ssrn.com/abstract=2103121*.

4: Modern interest rate modelling
Funding: perfect collateral + dividends + repo [2]

Dynamics under the real measure *P*,

$$
\begin{aligned}
dA(t) &= \mu(t,A)dt + \sigma(t,A)dW^P(t), \\
dB_c(t) &= r_c(t)B_c(t)dt, \\
d\Pi(t) &= \hat{\mathcal{L}}_\mu \Pi(t)dt + \sigma(t,A)\frac{\partial \Pi}{\partial A}dW^P(t), \\
\hat{\mathcal{L}}_\mu &= \frac{\partial}{\partial t} + \mu(t,A)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t,A)\frac{\partial^2}{\partial A^2}.
\end{aligned}
$$

Regarding the repo contract dynamics, we must remember that, for the repo holder:

- o there is a continuous positive cash flow of dividends $+r_D(t)A(t)dt$,
- o there is a continuous negative cash flow of repo interests $-r_R(t)A(t)dt$,
- o the price is linearly dependent on the asset *A*,

Thus, using Ito's Lemma, we obtain

$$
\begin{aligned}
d\Pi_R(t) &= \frac{\partial \Pi_R}{\partial t}dt + \frac{\partial \Pi_R}{\partial A}dA(t) + \frac{1}{2}\frac{\partial^2 \Pi_R}{\partial A^2}dA^2(t) + r_D(t)A(t)dt - r_R(t)A(t)dt \\
&= dA(t) + [r_D(t) - r_R(t)]\,A(t)dt
\end{aligned}
$$

Repo dynamics

(we have considered unit nominal).

4: Modern interest rate modelling
Funding: perfect collateral + dividends + repo [3]

The replication strategy of the derivative Π is obtained by combining appropriate amounts $[\Delta, \psi]$ of the available assets $[\Pi_R, B_c]$,

$$\boldsymbol{X}(t) := \begin{bmatrix} 0 \\ B_c(t) \end{bmatrix},$$

$$\boldsymbol{\Theta}(t) := \begin{bmatrix} \Delta(t) \\ \psi_c(t) \end{bmatrix},$$

where we have taken into account that the repo contract is always traded at par on the market, such that $\Pi_R(t) = 0$ for time t trading.

The value of the replication portfolio V is thus simply given by

$$V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = \boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t) = \psi_c(t) B_c(t)$$

(remember there is a zero repo value on the r.h.s.)

4: Modern interest rate modelling

Funding: perfect collateral + dividends + repo [4]

Gain processes of the assets

$$
\begin{aligned}
d\boldsymbol{G}(t) :&= \begin{bmatrix} d\Pi_R(t) \\ dB_c(t) \end{bmatrix} \\
&= \begin{bmatrix} \{\mu(t,A) + [r_D(t) - r_R(t)]\, A(t)\}\, dt + \sigma(t,A) dW^P(t) \\ r_c(t) B_c(t) dt \end{bmatrix}.
\end{aligned}
$$

Gain process of the replication portfolio

$$
\begin{aligned}
dG(t, \boldsymbol{\Theta}, \boldsymbol{X}) :&= \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) \\
&= \{[\mu(t,A) + (r_D(t) - r_R(t)) A(t)]\, \Delta(t) + r_c(t)\psi_c(t) B_c(t)\}\, dt \\
&+ \Delta(t)\sigma(t,A) dW^P(t)
\end{aligned}
$$

Funding: perfect collateral + dividends + repo [5]

Dividend processes of the assets may be obtained by difference

$$
\begin{aligned}
d\boldsymbol{D}(t) &= d\boldsymbol{G}(t) - d\boldsymbol{X}(t) \\
&= \begin{bmatrix} dA(t) + \left[r_D(t) - r_R(t)\right] A(t)dt \\ dB_c(t) \end{bmatrix} - \begin{bmatrix} 0 \\ dB_c(t) \end{bmatrix} \\
&= \begin{bmatrix} dA(t) + \left[r_D(t) - r_R(t)\right] A(t)dt \\ 0 \end{bmatrix} \\
\boldsymbol{D}(0) &= \boldsymbol{0}.
\end{aligned}
$$

This is consistent with the presence of dividends assumed at the beginning.
Notice that in this case we have

$$
d\left[\boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t)\right] \neq \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{X}(t).
$$

4: Modern interest rate modelling
Funding: perfect collateral + dividends + repo [6]

We now impose the perfect collateral and replication conditions,

$$\psi_c(t)B_c(t) = C(t) = \Pi(t) = V(t, \boldsymbol{\Theta}, \boldsymbol{X}), \;\; \forall\, t \leq T.$$

The gain process of the replication portfolio becomes

$$\begin{aligned} dG(t, \boldsymbol{\Theta}, \boldsymbol{X}) &= \Delta(t)\left\{dA(t) + \left[r_D(t) - r_R(t)\right]A(t)dt\right\} + dB_c(t) \\ &= \mu(t, A)\Delta(t)dt + d\Gamma(t, A) + \Delta(t)\sigma(t, A)dW^P(t), \\ d\Gamma(t, A) &= \Big\{\left[r_D(t) - r_R(t)\right]\Delta(t)A(t) + r_c(t)\Pi(t)\Big\}dt. \end{aligned}$$

We observe at this stage that the cash amount $\Gamma(t,A)$ contained in the replication portfolio is split between:

- the cash in the collateral account B_c, growing at the collateral rate r_c,
- the cash generated by the dividends paid by the asset A at the dividend rate r_D,
- the cash $\Delta(t)A(t)$ borrowed at the repo rate $r_R(t)$ to finance the purchase of $\Delta(t)$ units of the asset $A(t)$, secured by the asset itself.

We now impose the self-financing condition: the dividend process of the replication strategy must be null,

$$
\begin{aligned}
D(t, \boldsymbol{\Theta}, \boldsymbol{X}) = G(t, \boldsymbol{\Theta}, \boldsymbol{X}) - V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = 0, \\
\Rightarrow dG(t, \boldsymbol{\Theta}, \boldsymbol{X}) = dV(t, \boldsymbol{\Theta}, \boldsymbol{X}) = d\Pi(t).
\end{aligned}
$$

Introducing the gain process of the strategy on the l.h.s., the derivative process on the r.h.s of the previous equation, and rearranging terms, we have

$$
\begin{aligned}
\left\{ \frac{\partial \Pi}{\partial t} + \mu(t, A) \left[\frac{\partial \Pi}{\partial A} - \Delta(t) \right] + \frac{1}{2} \sigma^2(t, A) \frac{\partial^2 \Pi}{\partial A^2} \right\} dt \\
+ \sigma(t, A) \left[\frac{\partial \Pi}{\partial A} - \Delta(t) \right] dW^P(t) = d\Gamma(t, A).
\end{aligned}
$$

Finally, imposing the risk neutral condition $\Delta(t) = \dfrac{\partial \Pi}{\partial A}$, we obtain…

4: Modern interest rate modelling
Funding: perfect collateral + dividends + repo [8]

… a generalised Black-Scholes equation for the derivative's price *Π(t)*

$$\hat{\mathcal{L}}_{r_R - r_D} = \frac{\partial}{\partial t} + \boxed{[r_R(t) - r_D(t)]}\, A(t) \frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t, A) \frac{\partial^2}{\partial A^2},$$

$$\hat{\mathcal{L}}_{r_R - r_D} \Pi(t) = \boxed{r_c(t)}\,\Pi(t).$$

Using the Feynman-Kac theorem we obtain the SDE representation

$$\Pi(t) = \mathbb{E}_t^{\boxed{Q}}\left[\boxed{D_c(t;T)}\,\Pi(T)\right],$$

$$D_c(t;T) := \exp\left[-\int_t^T \boxed{r_c(u)}\, du\right],$$

$$dA(t) = \boxed{[r_R(t) - r_D(t)]}\, A(t)dt + \sigma(t, A)dW^{\boxed{Q}}(t).$$

We conclude that we discount at the collateral rate.

Funding: perfect collateral + dividends + repo [9]

Remarks:

1. **Funding measure**:
   the probability measure Q introduced via Feynman-Kac is associated with the risk neutral drift r_R appearing in the SDE dynamics of the asset A. It is the same measure of the collateralised case, with numeraire the collateral account B_c.

2. **Risk neutral drift**:
   The repo rate r_R is the correct rate to be used in the risk neutral dynamics of assets subject to repo. We may think that the repo (short) rate $r_R(t)$ is associated with a repo account B_R, such that
   $$dB_R(t) = r_R(t)B_R(t)dt,$$
   The repo rate, being associated with a secured transaction, may be considered a good proxy of a risk free rate. In practice, overnight repos are close to unsecured overnight rates.

3. **Funding Valuation Adjustment (FVA)**
   The price $\Pi(t)$ is different from the no repo case, because of the different risk neutral drift in the SDE dynamics of the asset A. According to the consideration above, the difference is small. We conclude that FVA is negligible.

4: Modern interest rate modelling
Funding: partial collateral [1]

We now relax the hypothesis of perfect collateral and consider the more general case of partial collateral $C(t) \neq \Pi(t)$, in the same currency of the derivative.

Our economy admits, in this case, four financial instruments:

- the asset A
- the derivative Π under partial collateral C
- the funding account B_f for funding unsecured at rate r_f
- the collateral account B_c for funding secured by collateral at rate r_c

Funding: partial collateral [2]

In general, we may assume the following dynamics under the real measure *P*,

$$
\begin{aligned}
dA(t) &= \mu(t, A)dt + \sigma(t, A)dW^P(t), \\
dB_f(t) &= r_f(t)B_f(t)dt, \\
dB_c(t) &= r_c(t)B_c(t)dt, \\
d\Pi(t) &= \hat{\mathcal{L}}_\mu \Pi(t)dt + \sigma(t, A)\frac{\partial \Pi}{\partial A}dW^P(t), \\
\hat{\mathcal{L}}_\mu &= \frac{\partial}{\partial t} + \mu(t, A)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2}{\partial A^2}.
\end{aligned}
$$

Funding: partial collateral [3]

The replication strategy of the derivative Π is obtained by combining appropriate amounts Θ of the available assets,

$$\boldsymbol{X}(t) := \begin{bmatrix} A(t) \\ B_f(t) \\ B_c(t) \end{bmatrix}, \quad \boldsymbol{\Theta}(t) := \begin{bmatrix} \Delta(t) \\ \psi_f(t) \\ \psi_c(t) \end{bmatrix},$$

The value of the replication portfolio V is given by

$$V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = \boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t) = \Delta(t)A(t) + \psi_f(t)B_f(t) + \psi_c(t)B_c(t).$$

The gain and dividend processes of the assets, in SDE form, are given directly by the dynamics discussed before, as

$$
d\boldsymbol{G}(t) := \begin{bmatrix} dA(t) \\ dB_f(t) \\ dB_c(t) \end{bmatrix} = \begin{bmatrix} \mu(t,A)dt + \sigma(t,A)dW^P(t) \\ r_f(t)B_f(t)dt \\ r_c(t)B_c(t)dt \end{bmatrix},
$$

$$
d\boldsymbol{D}(t) = \boldsymbol{0}
$$

The gain process of the replication portfolio is given, in SDE form, by

$$
\begin{aligned}
& dG(t, \boldsymbol{\Theta}, \boldsymbol{X}) := \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) \\
= & \left\{\mu(t,A)\Delta(t) + r_f(t)\psi_f(t)B_f(t) + r_c(t)\psi_c(t)B_c(t)\right\} dt + \Delta(t)\sigma(t,A)dW^P(t)
\end{aligned}
$$

4: Modern interest rate modelling

Funding: partial collateral [6]

We now impose the perfect replication condition,

$$
\begin{aligned}
&V(t,\boldsymbol{\Theta},\boldsymbol{X}) = \Pi(t) = \Delta(t)A(t) + \psi_f(t)B_f(t) + \psi_c(t)B_c(t), \quad \forall\, t \le T, \\
&\Rightarrow \psi_f(t)B_f(t) = \Pi(t) - \psi_c(t)B_c(t) = \Pi(t) - \Delta(t)A(t) - C(t).
\end{aligned}
$$

consistently with the fact that the funding account B_f is used to finance the borrowing of $\Delta(t)$ units of the asset $A(t)$ at the funding rate $r_f(t)$.

The gain process of the replication portfolio becomes

$$
\begin{aligned}
dG(t,\boldsymbol{\Theta},\boldsymbol{X}) &= \mu(t,A)\Delta(t)dt + d\Gamma(t,A) + \Delta(t)\sigma(t,A)dW^P(t), \\
d\Gamma(t,A) &= \left\{-r_f(t)\Delta(t)A(t) + r_f(t)\Pi(t) - \left[r_f(t) - r_c(t)\right]C(t)\right\}dt, \\
&= \left\{-r_f(t)\Delta(t)A(t) + r_c(t)C(t) - r_f(t)\left[\Pi(t) - C(t)\right]\right\}dt
\end{aligned}
$$

We observe that the cash amount $\Gamma(t,A)$ in the replication portfolio is split between:

- o the collateral C, growing at the collateral rate r_c,
- o the amount $\Delta(t)A(t)$, borrowed at the funding rate $r_f(t)$ to finance the purchase of $\Delta(t)$ units of the asset $A(t)$,
- o the off-collateral amount $\Pi(t)-C(t)$, borrowed/lent at the funding rate $r_f(t)$.

We now impose the self-financing condition: the dividend process of the replication strategy must be null,

$$D(t, \boldsymbol{\Theta}, \boldsymbol{X}) = G(t, \boldsymbol{\Theta}, \boldsymbol{X}) - V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = 0,$$
$$\Rightarrow dG(t, \boldsymbol{\Theta}, \boldsymbol{X}) = dV(t, \boldsymbol{\theta}, \boldsymbol{X}) = d\Pi(t).$$

Introducing, in the previous equation, the gain process of the replication portfolio on the l.h.s., the derivative process on the r.h.s, and rearranging terms, we have

$$\left\{ \frac{\partial \Pi}{\partial t} + \mu(t, A) \left[\frac{\partial \Pi}{\partial A} - \Delta(t) \right] + \frac{1}{2} \sigma^2(t, A) \frac{\partial^2 \Pi}{\partial A^2} \right\} dt$$
$$+ \sigma(t, A) \left[\frac{\partial \Pi}{\partial A} - \Delta(t) \right] dW^P(t) = d\Gamma(t, A).$$

Finally, imposing the risk neutral condition $\Delta(t) = \dfrac{\partial \Pi}{\partial A}$, we obtain…

Funding: partial collateral [8]

… a generalised Black-Scholes equation for the derivative's price *Π(t)*

$$\hat{\mathcal{L}}_{r_f}\Pi(t) = \boxed{r_f(t)\Pi(t) - \left[r_f(t) - \boxed{r_c(t)}\right] C(t),}$$

$$\hat{\mathcal{L}}_{r_f} = \frac{\partial}{\partial t} + \boxed{r_f(t)}A(t)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t, A)\frac{\partial^2}{\partial A^2}.$$

Using the Feynman-Kac theorem we obtain the SDE representation

$$\Pi(t) = \mathbb{E}_t^{\boxed{Q}}\left[\boxed{D_c(t;T)}\Pi(T) + \int_t^T \boxed{D_c(t;u)}\,\boxed{\left[r_f(u) - r_c(u)\right]\left[\Pi(u) - C(u)\right] du}\right],$$

$$D_c(t;T) := \exp\left[-\int_t^T \boxed{r_c(u)}du\right],$$

$$dA(t) = \boxed{r_f(t)}A(t)dt + \sigma(t, A)dW^{\boxed{Q}}(t),$$

Remarks:

- **Funding Value Adjustment (FVA)**:
  we may rewrite the previous formula as follows

$$
\begin{aligned}
\Pi_{f,c}(t) &= \mathbb{E}_t^Q \left[D_c(t;T)\Pi(T) + \int_t^T D_c(t;u) \left[r_f(u) - r_c(u) \right] \left[\Pi_{f,c}(u) - C(u) \right] du \right], \\
&= \Pi_c(t) + FVA_{f,c}(t), \\
FVA_{f,c}(t) &:= \Pi_{f,c}(t) - \Pi_c(t) \\
&= \mathbb{E}_t^Q \left[\int_t^T D_c(t;u) \left[r_f(u) - r_c(u) \right] \left[\Pi_{f,c}(u) - C(u) \right] du \right].
\end{aligned}
$$

  In this case, the FVA amounts to the expected difference between the trade and collateral values, weighted with the difference between funding and collateral rates, integrated over the residual life of the trade.

We consider now the case of stochastic funding rates with perfect collateral. Stochastic funding means that funding rates are stochastic, there are more risk factors to hedge.

Our economy admits, in this case, six financial instruments:

- the asset A,
- the uncollateralized asset A_f, e.g. a zero coupon bond $P_f(t,s)$
- the collateralized asset A_c, e.g. a zero coupon bond $P_c(t,s)$
- the derivative Π under perfect collateral C
- the funding account B_f for funding unsecured at rate r_f
- the collateral account B_c for funding secured by collateral at rate r_c

Notice that:

- Non tradable assets, such as interest rates, inflation, fx, etc. can't appear directly as hedging instruments, but through some corresponding tradable financial instruments. In particular, funding rates r_x may enter in the form of Zero Coupon Bonds, $A_x(t) = P_x(t,s)$.
- Additional financial instruments associated with (stochastic) funding rates are required to hedge the additional risk factors.

4: Modern interest rate modelling
Funding: perfect collateral, stochastic rates [2]

In general, we may assume the following dynamics under the real measure *P*

$$
\begin{aligned}
&dA(t) = \mu_A(t,A)dt + \boldsymbol{\sigma}_A(t,A)\cdot d\boldsymbol{W}^P(t),\\
&dA_x(t) = \mu_x(t,A_x)dt + \boldsymbol{\sigma}_x(t,A_x)\cdot d\boldsymbol{W}^P(t),\\
&dB_x(t) = r_x(t)B_x(t)dt,\\
&d\Pi(t) = \hat{\mathcal{L}}_\mu\Pi(t)dt + \left[\frac{\partial\Pi}{\partial A}\boldsymbol{\sigma}_A(t,A) + \frac{\partial\Pi}{\partial A_f}\boldsymbol{\sigma}_f(t,A_f) + \frac{\partial\Pi}{\partial A_c}\boldsymbol{\sigma}_c(t,A_c)\right]\cdot d\boldsymbol{W}^P(t),\\
&\hat{\mathcal{L}}_\mu := \frac{\partial}{\partial t} + \mu_A(t,A)\frac{\partial}{\partial A} + \mu_f(t,A_f)\frac{\partial}{\partial A_f} + \mu_c(t,A_c)\frac{\partial}{\partial A_c} + \frac{1}{2}\boldsymbol{\Sigma^2(t,A,A_f,A_c)}(t)\cdot\frac{\boldsymbol{\partial^2}}{\boldsymbol{\partial A^2}},\\
&\boldsymbol{\Sigma}^2(t,A,A_f,A_c)\cdot\frac{\boldsymbol{\partial^2}}{\boldsymbol{\partial A^2}} := \boldsymbol{\sigma}_A^2(t,A)\frac{\partial^2}{\partial A^2} + \boldsymbol{\sigma}_f^2(t,A_f)\frac{\partial^2}{\partial A_f^2} + \boldsymbol{\sigma}_c^2(t,A_c)\frac{\partial^2}{\partial A_c^2}\\
&\qquad + \boldsymbol{\sigma}_A(t,A)\cdot\left[\boldsymbol{\sigma}_f(t,A_f)\frac{\partial^2}{\partial A\partial A_f} + \boldsymbol{\sigma}_c(t,A_c)\frac{\partial^2}{\partial A\partial A_c}\right] + \boldsymbol{\sigma}_f(t,f)\cdot\boldsymbol{\sigma}_c(t,A_c)\frac{\partial^2}{\partial A_f\partial A_c},\\
&dW_i^P(t)dW_j^P(t) = \delta_{i,j}dt,\\
&x := \{f,c\}.
\end{aligned}
$$

4: Modern interest rate modelling
Funding: perfect collateral, stochastic rates [3]

The replication strategy of the derivative Π is obtained by combining appropriate amounts Θ of the available assets,

$$\boldsymbol{X}(t) := \begin{bmatrix} A(t) \\ A_f(t) \\ A_c(t) \\ B_f(t) \\ B_c(t) \end{bmatrix}, \quad \boldsymbol{\Theta}(t) := \begin{bmatrix} \Delta_A(t) \\ \Delta_f(t) \\ \Delta_c(t) \\ \psi_f(t) \\ \psi_c(t) \end{bmatrix},$$

The value of the replication portfolio V is given by

$$\begin{aligned} V(t, \boldsymbol{\Theta}, \boldsymbol{X}) &= \boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t) \\ &= \Delta_A(t)A(t) + \Delta_f(t)A_f(t) + \Delta_c(t)A_c(t) + \psi_f(t)B_f(t) + \psi_c(t)B_c(t). \end{aligned}$$

4: Modern interest rate modelling
Funding: perfect collateral, stochastic rates [4]

The gain and dividend processes of the assets, in SDE form, are given directly by the dynamics discussed before, as

$$
d\boldsymbol{G}(t) := \begin{bmatrix} dA(t) \\ dA_f(t) \\ dA_c(t) \\ dB_f(t) \\ dB_c(t) \end{bmatrix} = \begin{bmatrix} \mu_A(t,A)dt + \boldsymbol{\sigma}_A(t,A)\cdot d\boldsymbol{W}^P(t) \\ \mu_f(t,A_f)dt + \boldsymbol{\sigma}_f(t,A_f)\cdot d\boldsymbol{W}^P(t) \\ \mu_c(t,A_c)dt + \boldsymbol{\sigma}_c(t,A_c)\cdot d\boldsymbol{W}^P(t) \\ r_f(t)B_f(t)dt \\ r_c(t)B_c(t)dt \end{bmatrix},
$$

$$
d\boldsymbol{D}(t) = \boldsymbol{0}
$$

The gain process of the replication portfolio is given, in SDE form, by

$$
\begin{aligned}
dG(t,\boldsymbol{\Theta},\boldsymbol{X}) &:= \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) \\
&= \{\mu_A(t,A)\Delta_A(t) + \mu_f(t,A_f)\Delta_f(t) + \mu_c(t,A_c)\Delta_c(t) \\
&\qquad + r_f(t)\psi_f(t)B_f(t) + r_c(t)\psi_c(t)B_c(t)\}\, dt \\
&\quad + [\Delta_A(t)\boldsymbol{\sigma}_A(t,A) + \Delta_f(t)\boldsymbol{\sigma}_f(t,A_f) + \Delta_c(t)\boldsymbol{\sigma}_c(t,A_c)]\cdot d\boldsymbol{W}^P(t)
\end{aligned}
$$

We now impose the perfect replication condition,

$$
\begin{aligned}
&V(t, \boldsymbol{\Theta}, \boldsymbol{X}) = \Pi(t) \\
&= \Delta_A(t)A(t) + \Delta_f(t)A_f(t) + \Delta_c(t)A_c(t) + \psi_f(t)B_f(t) + \psi_c(t)B_c(t) \\
&= \Delta_A(t)A(t) + \Delta_f(t)A_f(t) + \psi_f(t)B_f(t) + \Pi(t),
\end{aligned}
$$

where we have used the generalised perfect collateral condition

$$\Pi(t) = C(t) = \psi_c(t)B_c(t) + \Delta_c(t)A_c(t).$$

Thus we obtain the generalised unsecured funding condition

$$\psi_f(t)B_f(t) = -\Delta_A(t)A(t) - \Delta_f(t)A_f(t),$$

consistently with the fact that the funding account B_f is used to finance the borrowing of $\Delta_A(t)$ units of the asset $A(t)$ and $\Delta_f(t)$ units of the asset $A_f(t)$ at the funding rate $r_f(t)$.

4: Modern interest rate modelling
Funding: perfect collateral, stochastic rates [6]

The gain process of the replication portfolio becomes

$$
\begin{aligned}
dG(t, \boldsymbol{\Theta}, \boldsymbol{X}) &= d\Pi(t) = \hat{\mathcal{L}}_\mu \Pi(t) dt \\
&= \left[\mu_A(t, A)\Delta_A(t) + \mu_f(t, A_f)\Delta_f(t) + \mu_c(t, A_c)\Delta_c(t)\right] dt \\
&+ d\Gamma(t, A, A_f, A_c) \\
&+ \left[\Delta_A(t)\boldsymbol{\sigma}_A(t, A) + \Delta_f(t)\boldsymbol{\sigma}_f(t, A_f) + \Delta_c(t)\boldsymbol{\sigma}_c(t, A_c)\right] \cdot d\boldsymbol{W}^P(t), \\
d\Gamma(t, A) &= -r_f(t)\left[\Delta_A(t)A(t) + \Delta_f(t)A_f(t)\right] + r_c(t)\left[\Pi(t) - \Delta_c(t)A_c(t)\right] dt.
\end{aligned}
$$

We observe that the cash amount Γ in the replication portfolio is split between:

- the collateral amount $C(t) = \Pi - \Delta_c(t)A_c(t)$ growing at the collateral rate r_c,
- the amount $\Delta(t)A(t) + \Delta_f(t)A_f(t)$, borrowed at the funding rate $r_f(t)$ to finance the purchase of $\Delta(t)$ units of the asset $A(t)$, and $\Delta_f(t)$ units of the asset $A_f(t)$.

4: Modern interest rate modelling
Funding: perfect collateral, stochastic rates [7]

We now impose the self-financing condition: the dividend process of the replication strategy must be null,

$$D(t,\mathbf{\Theta},\boldsymbol{X}) = G(t,\mathbf{\Theta},\boldsymbol{X}) - V(t,\mathbf{\Theta},\boldsymbol{X}) = 0,$$
$$\Rightarrow dG(t,\mathbf{\Theta},\boldsymbol{X}) = dV(t,\boldsymbol{\theta},\boldsymbol{X}) = d\Pi(t).$$

Introducing, in the previous equation, the gain process of the replication portfolio on the l.h.s., the derivative process on the r.h.s, and rearranging terms, we have

$$\begin{aligned}\left\{\frac{\partial\Pi}{\partial t}+\mu_A(t,A)\left[\frac{\partial\Pi}{\partial A}-\Delta_A(t)\right]+\mu_f(t,A_f)\left[\frac{\partial\Pi}{\partial A_f}-\Delta_f(t)\right]+\mu_c(t,A_c)\left[\frac{\partial\Pi}{\partial A_c}-\Delta_c(t)\right]\right.\\ \left.+\frac{1}{2}\mathbf{\Sigma}^2(t,A,A_f,A_c)\cdot\frac{\boldsymbol{\partial}^2}{\boldsymbol{\partial A}^2}\right\}dt+\left\{\left[\frac{\partial\Pi}{\partial A}-\Delta_A(t)\right]\boldsymbol{\sigma}_A(t,A)\right.\\ \left.+\left[\frac{\partial\Pi}{\partial A_f}-\Delta_f(t)\right]\boldsymbol{\sigma}_f(t,A_f)+\left[\frac{\partial\Pi}{\partial A_c}-\Delta_c(t)\right]\boldsymbol{\sigma}_c(t,A_c)\right\}\cdot d\boldsymbol{W}^P(t) = d\Gamma(t,A,A_f,A_c).\end{aligned}$$

Finally, imposing the generalised risk neutral condition

$$\Delta_A(t) = \frac{\partial\Pi}{\partial A},\ \Delta_f(t) = \frac{\partial\Pi}{\partial A_f},\ \Delta_c(t) = \frac{\partial\Pi}{\partial A_c},$$

we obtain…

4: Modern interest rate modelling
Funding: perfect collateral, stochastic rates [8]

…a generalised Black-Scholes PDE equation for the derivative's price *Π(t)*

$$\hat{\mathcal{L}}_{f,c}\Pi(t) = \boxed{r_c(t)}\Pi(t),$$

$$\hat{\mathcal{L}}_{f,c} = \frac{\partial}{\partial t} + \boxed{r_f(t)}\left[A(t)\frac{\partial}{\partial A} + A_f(t)\frac{\partial}{\partial A_f}\right] + r_c(t)A_c(t)\frac{\partial}{\partial A_c}\frac{1}{2}\boldsymbol{\Sigma}^2(t, A, A_f, A_c)\cdot\frac{\boldsymbol{\partial}^2}{\boldsymbol{\partial A}^2}.$$

Using the Feynman-Kac theorem we may switch from the PDE representation to the SDE representation given by

$$\Pi(t) = \mathbb{E}_t^{\boxed{Q}}\left[\boxed{D_c(t;T)}\Pi(T)\right],$$

$$D_c(t;T) := \exp\left[-\int_t^T \boxed{r_c(u)}du\right],$$

$$dA(t) = \boxed{r_f(t)}A(t)dt + \boldsymbol{\sigma}_A(t, A)\cdot d\boldsymbol{W}^{\boxed{Q}}(t),$$

$$dA_x(t) = r_x(t)A_x(t)dt + \boldsymbol{\sigma}_x(t, A_x)\cdot d\boldsymbol{W}^{Q}(t),$$

We conclude that we discount at the collateral rate.

Remarks

- IR dynamics

  In the risk neutral measure the dynamics of the stochastic rates r_x is expressed via the functional dependence on the instruments A_x. We could proceed the other way around, postulating from the very beginning, stochastic process of the rates in the risk-neutral measure.

  This freedom allows to select their dynamics with appropriate characteristics that can be calibrated to the corresponding yield curves. For example, we could choose the Hull-White dynamics such that, under the risk neutral measure Q,

$$dr_x(t) = [k_x(t) - a_x(t) r_x(t)]\, dt + \boldsymbol{\sigma}_{r_x}(t) \cdot d\boldsymbol{W}^Q(t),$$

4: Modern interest rate modelling
Funding: perfect collateral, stochastic rates [10]

- IR instruments

Next, we can specify the interest rate instruments A_x used in the replication. One possible choice are unsecured and (perfectly) collateralised zero coupon bonds, $A_x(t) = P_x(t,T')$, with some maturity $T>T$.

The parameters $k_x(t)$ in the Hull-White dynamics above can be calibrated to the market zero coupon bond curves.

Using Ito's lemma we obtain the price dynamics of the zero coupon bonds as

$$dP_x(t,T') = \boxed{r_x(t)P_x(t,T')}dt - P_x(t,T')\boldsymbol{\sigma}_{P_x}(t,T') \cdot d\boldsymbol{W}^Q(t),$$

$$\boldsymbol{\sigma}_{P_x}(t,T') = \boldsymbol{\sigma}_{r_x}(t) \int_t^{T'} e^{-\int_t^u a_x(v)dv} du,$$

$$x = f, c.$$

Notice that the zero coupon bonds dynamics inherit appropriate $\boxed{\text{drifts}}$ corresponding to their underlying (short) rates r_x.

4: Modern interest rate modelling
Funding: general case [1]

We now consider the more general case of multiple assets $\boldsymbol{A}$, multiple stochastic funding rates r_x, dividends and partial collateral.

Our economy admits, in this case, multiple financial instruments:

- the vector of assets $\boldsymbol{A_x} = [A_f, A_c, A_R]$ according to the funding of their associated hedging strategies (unsecured, collateral, repo)
- the derivative Π on assets $\boldsymbol{A_x}$ under partial CSA
- the vector of funding accounts $\boldsymbol{B_x} = [B_f, B_c, B_R]$ with funding rates $\boldsymbol{r_x} = [r_f, r_c, r_R]$

The generalised funding conditions become

$$\begin{aligned}
\Delta_f(t)A_f(t) + \psi_f(t)B_f(t) &= \Pi(t) - C_\Pi(t), \\
\Delta_c(t)A_c(t) + \psi_c(t)B_c(t) &= C_\Pi(t), \\
\Delta_R(t)A_R(t) + \psi_R(t)B_R(t) &= 0.
\end{aligned}$$

4: Modern interest rate modelling
Funding: general case [2]

We obtain a generalised Black-Scholes PDE equation for the derivative's price $\Pi(t)$

$$\hat{\mathcal{L}}_r \Pi(t) = r_f(t)\Pi(t) - \left[r_f(t) - r_c(t)\right] C_\Pi(t),$$

$$\hat{\mathcal{L}}_r = \frac{\partial}{\partial t} + \sum_x \left[r_x(t) - r_D(t)\right] \boldsymbol{A}_x(t) \cdot \frac{\partial}{\partial \boldsymbol{A}_x} + \frac{1}{2}\sum_{x,y} \boldsymbol{\sigma}_x(t, \boldsymbol{A}_x) \cdot \boldsymbol{\sigma}_y(t, \boldsymbol{A}_y) \cdot \frac{\partial^2}{\partial \boldsymbol{A}_x \partial \boldsymbol{A}_y},$$

and, using Feynman-Kac, the SDE representation

$$\Pi(t) = \mathbb{E}_t^Q \left[D_c(t;T)\Pi(T) - \int_t^T D_c(u,T)\left[r_f(u) - r_c(u)\right]\left[\Pi(u) - C_\Pi(u)\right] du \right],$$

$$D_x(t;T) := \exp\left[-\int_t^T r_x(u) du \right],$$

$$dA_x(t) = \left[r_x(t) - r_D(t)\right] A_x(t) dt + \boldsymbol{\sigma}_x(t, \boldsymbol{A}_x) \cdot d\boldsymbol{W}^Q(t),$$

see Antonov-Bianchetti, in preparation.

Funding: multiple currency [1]

We consider now the case of different currency for the derivative and its funding.

Our economy admits, in this case, five financial instruments:

- the asset A^α, in currency α, with no dividends
- the derivative Π^α in currency α under collateral C^β in currency β
- the funding account B^α_f for funding unsecured in currency α at rate r^α_f
- the funding account B^β_f for funding unsecured in currency β at rate r^β_f
- the collateral account $B^\beta_c(t)$ for funding secured by collateral C^β at rate r^β_f

We also assume that the derivative Π is under perfect collateral, such that

$$\Pi^\alpha(t) = x^{\alpha\beta}(t) C^\beta(t), \quad \forall\, t \leq T,$$

where $x_{\alpha\beta}$ is the spot exchange rate expressing the value in currency α of one unit of currency β.

4: Modern interest rate modelling
Funding: multiple currency [2]

We have the following dynamics under the real measure P^α

$$
\begin{aligned}
dA^\alpha(t) &= \mu(t, A^\alpha)dt + \sigma(t, A^\alpha)dW^{P,\alpha}(t),\\
dB_f^\alpha(t) &= r_f^\alpha(t)B_f^\alpha(t)dt,\\
dB_f^\beta(t) &= r_f^\beta(t)B_f^\beta(t)dt,\\
dB_c^\beta(t) &= r_c^\beta(t)B_c^\beta(t)dt,\\
d\Pi^\alpha(t) &= \left[\frac{\partial \Pi^\alpha}{\partial t} + \mu(t, A^\alpha)\frac{\partial \Pi}{\partial A} + \frac{1}{2}\sigma^2(t, A^\alpha)\frac{\partial^2 \Pi^\alpha}{\partial A^2}\right]dt + \sigma^2(t, A^\alpha)\frac{\partial \Pi}{\partial A}dW^{P,\alpha}(t).
\end{aligned}
$$

4: Modern interest rate modelling
Funding: multiple currency [3]

We now construct a replication strategy of the derivative Π^α, by setting up a replication portfolio V^α such that

$$V^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) = \Pi^\alpha(t), \quad \forall\, t \leq T,$$

by combining appropriate amounts $\boldsymbol{\Theta}$ of the available assets $\boldsymbol{X}$,

$$\boldsymbol{X}(t) := \begin{bmatrix} A^\alpha(t) \\ B_f^\alpha(t) \\ B_f^\beta(t) \\ B_c^\beta(t) \end{bmatrix}, \quad \boldsymbol{\Theta}(t) := \begin{bmatrix} \Delta(t) \\ \psi_f^\alpha(t) \\ \psi_f^\beta(t) x^{\alpha\beta}(t) \\ \psi_c^\beta(t) x^{\alpha\beta}(t) \end{bmatrix},$$

$$\begin{aligned} V^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) &= \boldsymbol{\Theta}(t)' \cdot \boldsymbol{X}(t) \\ &= \Delta(t) A^\alpha(t) + \psi_f^\alpha(t) B_f^\alpha(t) + \psi_f^\beta(t) x^{\alpha\beta}(t) B_f^\beta(t) + \psi_c^\beta(t) x^{\alpha\beta}(t) B_c^\beta(t) \end{aligned}$$

4: Modern interest rate modelling
Funding: multiple currency [4]

The gain processes of the assets, in SDE form, are given directly by the dynamics discussed before, as

$$
\begin{aligned}
d\boldsymbol{G}(t) :&= \begin{bmatrix} dA^{\alpha}(t) \\ dB_f^{\alpha}(t) \\ dB_f^{\beta}(t) \\ dB_c^{\beta}(t) \end{bmatrix} = \begin{bmatrix} \mu(t, A^{\alpha})dt + \sigma(t, A^{\alpha})dW^{P,\alpha}(t) \\ r_f^{\alpha}(t)B_f^{\alpha}(t)dt \\ r_f^{\beta}(t)B_f^{\beta}(t)dt \\ r_c^{\beta}(t)B_c^{\beta}(t)dt \end{bmatrix}, \\
d\boldsymbol{D}(t) &= \boldsymbol{0}, \\
d\boldsymbol{G}(t) &= d\boldsymbol{X}(t).
\end{aligned}
$$

4: Modern interest rate modelling
Funding: multiple currency [5]

We now impose the perfect collateral and replication conditions, and we obtain

$$
\begin{aligned}
V^\alpha(t,\boldsymbol{\Theta},\boldsymbol{X}) &= \Delta(t)A^\alpha(t) + \psi_f^\alpha(t)B_f^\alpha(t) + \psi_f^\beta(t)x^{\alpha\beta}(t)B_f^\beta(t) + \boxed{\psi_c^\beta(t)x^{\alpha\beta}(t)B_c^\beta(t)},\\
&= \Delta(t)A^\alpha(t) + \psi_f^\alpha(t)B_f^\alpha(t) + \psi_f^\beta(t)x^{\alpha\beta}(t)B_f^\beta(t) + \boxed{\Pi(t)},\\
V^\alpha(t,\boldsymbol{\Theta},\boldsymbol{X}) &= \Pi^\alpha(t), \quad \forall\, t \leq T\\
&\Rightarrow\ \psi_f^\alpha(t)B_f^\alpha(t) = -\Delta(t)A^\alpha(t) - \psi_f^\beta(t)x^{\alpha\beta}(t)B_f^\beta(t)
\end{aligned}
$$

consistently with the fact that the funding account B^α_f is used to finance, in currency α, the borrowing of $\Delta(t)$ units of the asset $A^\alpha(t)$ and $\psi^\beta_f(t)x^{\alpha\beta}(t)$ units of cash $B^\beta_f(t)$ in currency β, at the funding rate $r^\alpha_f(t)$.

4: Modern interest rate modelling

Funding: multiple currency [6]

The gain process of the replication portfolio, in SDE form, is given by

$$
\begin{aligned}
dG^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) &= dX^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) = \boldsymbol{\Theta}(t)' \cdot d\boldsymbol{G}(t) \\
&= \mu(t, A^\alpha)\Delta(t)dt + d\Gamma^\alpha(t, A^\alpha) + \sigma(t, A^\alpha)\Delta(t)dW^{P,\alpha}(t), \\
d\Gamma^\alpha(t, A^\alpha) &:= \left\{-r_f^\alpha(t)\Delta(t)A^\alpha(t) + \left[r_f^\beta(t) - r_f^\alpha(t)\right]\psi_f^\beta(t)x^{\alpha,\beta}(t)B_f^\beta(t) + r_c^\beta(t)\Pi^\alpha(t)\right\}dt \\
&= \left\{-r_f^\alpha(t)\Delta(t)A^\alpha(t) + \left[r_c^\beta(t) + r_f^\beta(t) - r_f^\alpha(t)\right]\Pi^\alpha(t)\right\}dt,
\end{aligned}
$$

where we have chosen

$$
\psi_2(t)x^{\alpha,\beta}(t)B_f^\beta(t) = \Pi^\alpha(t),
$$

such that the $\psi^\beta_f(t)x^{\alpha\beta}(t)$ units of cash $B^\beta_f(t)$ are used to fund the derivative $\Pi^\alpha(t)$.

We observe that the cash amount $\Gamma^\alpha(t)$ in he replication portfolio is split between:

- the amount $\Delta(t)A^\alpha(t)$ (currency α), borrowed at the funding rate $r^\alpha_f(t)$ to finance the purchase of $\Delta(t)$ units of the asset $A^\alpha(t)$,
- the amount $B^\beta_c(t)$ in the collateral account (currency β), growing at the cross currency collateral rate $r^\beta_c(t)+r^\beta_f(t)-r^\alpha_f(t)$,

4: Modern interest rate modelling
Funding: multiple currency [7]

We now impose the self-financing condition: the dividend process of the replication portfolio must be null,

$$D^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) = G^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) - V^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) = 0,$$
$$\Rightarrow dG^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) = dV^\alpha(t, \boldsymbol{\Theta}, \boldsymbol{X}) = d\Pi^\alpha(t).$$

Introducing, in the previous equation, the gain process of the replication portfolio on the l.h.s., the derivative process on the r.h.s, and rearranging terms, we have

$$\left\{\frac{\partial \Pi^\alpha}{\partial t} + \mu(t, A^\alpha)\left[\frac{\partial \Pi^\alpha}{\partial A} - \Delta(t)\right] + \frac{1}{2}\sigma^2(t, A^\alpha)\frac{\partial^2 \Pi^\alpha}{\partial A^2}\right\}dt$$
$$+ \sigma(t, A^\alpha)\left[\frac{\partial \Pi^\alpha}{\partial A} - \Delta(t)\right] dW^{P,\alpha}(t) = d\Gamma^\alpha(t, A^\alpha).$$

Finally, imposing the risk neutral condition $\Delta(t) = \dfrac{\partial \Pi^\alpha}{\partial A^\alpha}$, we obtain…

… a generalised Black-Scholes equation for the derivative's price $\Pi^\alpha(t,A)$

$$\hat{\mathcal{L}}_{r_f^\alpha}\Pi^\alpha(t) = \left[\boxed{r_c^\beta(t) + r_f^\beta(t) - r_f^\alpha(t)}\right]\Pi^\alpha(t),$$

$$\hat{\mathcal{L}}_{r_f^\alpha} = \frac{\partial}{\partial t} + \boxed{r_f^\alpha(t)}A^\alpha(t)\frac{\partial}{\partial A} + \frac{1}{2}\sigma^2(t,A^\alpha)\frac{\partial^2}{\partial A^2}.$$

Using the Feynman-Kac theorem we obtain the SDE representation

$$\Pi^\alpha(t) = \mathbb{E}_t^{\boxed{Q^\alpha}}\left[\boxed{D_{c,f}^{\alpha,\beta}(t;T)}\Pi^\alpha(T)\right],$$

$$D_{c,f}^{\alpha,\beta}(t;T) := \exp\left[-\int_t^T \left[\boxed{r_c^\beta(u) + r_f^\beta(u) - r_f^\alpha(u)}\right]du\right]$$

$$= D_c^\beta(t;T)\frac{D_f^\beta(t;T)}{D_f^\alpha(t;T)},$$

$$dA^\alpha(t) = \boxed{r_f^\alpha(t)}A^\alpha(t)dt + \sigma_A(t,A^\alpha)dW^{\boxed{Q^\alpha}}(t),$$

Remarks:

1. **Discounting**:
   the discount factor

$$D_{c,f}^{\alpha,\beta}(t;T) := \exp\left[-\int_t^T \left[r_c^\beta(u) + r_f^\beta(u) - r_f^\alpha(u)\right] du\right]$$

   contains the cross currency basis.

2. Funding Valuation Adjustment (FVA):
comparing the single vs double currency collateralised prices we can define a Funding Value Adjustment (FVA) such that, in additive form,

$$
\begin{aligned}
\Pi^\alpha(t) &= \mathbb{E}_t^{Q^\alpha}\left[D_c^\beta(t;T)\frac{D_f^\beta(t;T)}{D_f^\alpha(t;T)}\Pi^\alpha(T)\right] \\
&\simeq \mathbb{E}_t^{Q^\alpha}\left[\frac{D_c^\beta(t;T)D_f^\beta(t;T)}{D_c^\alpha(t;T)D_f^\alpha(t;T)}\right]\mathbb{E}_t^{Q^\alpha}\left[D_c^\alpha(t;T)\Pi^\alpha(T)\right] \\
&= \mathbb{E}_t^{Q^\alpha}\left[\frac{D_c^\beta(t;T)D_f^\beta(t;T)}{D_c^\alpha(t;T)D_f^\alpha(t;T)}\right]\Pi_c^\alpha(t), \\
FVA_{f,c}^{\alpha,\beta}(t) &\simeq \left\{\mathbb{E}_t^{Q^\alpha}\left[\frac{D_c^\beta(t;T)D_f^\beta(t;T)}{D_c^\alpha(t;T)D_f^\alpha(t;T)}\right]-1\right\}\Pi_c^\alpha(t).
\end{aligned}
$$

4: Modern interest rate modelling
Funding: multiple currency [11]

In the limit of deterministic basis we obtain the simple expression

$$\mathbb{E}_t^{Q^\alpha}\left[\frac{D_c^\beta(t;T)D_f^\beta(t;T)}{D_c^\alpha(t;T)D_f^\alpha(t;T)}\right] \simeq e^{-\int_t^T s_{c,f}^{\alpha,\beta}(u)\,du},$$

$$\begin{aligned} FVA_{c,f}^{\alpha,\beta}(t) &\simeq \left[e^{-\int_t^T s_{c,f}^{\alpha,\beta}(u)\,du} - 1\right]\Pi_c^\alpha(t) \\ &\simeq -\left[\int_t^T s(u)\,du\right]\Pi_c^\alpha(t), \\ s_{c,f}^{\alpha,\beta}(t) &:= r_c^\beta(t) + r_f^\beta(t) - r_c^\alpha(t) - r_f^\alpha(t). \end{aligned}$$

3. **Special cases**:
   All the cases analysed before may be recovered as special cases of the last formula.
   - perfect collateral: set $x_{\alpha\beta}(t)B^\beta{}_c(t) \rightarrow \Pi^\alpha(t,A)\ \ \forall t$
   - single currency: set $\alpha \rightarrow \beta$, $x_{\alpha\beta}(t) \rightarrow 1\ \ \forall t$
   - no collateral: set $B_c(t) \rightarrow 0\ \forall t$

4: Modern interest rate modelling
Funding: interest rate derivatives [1]

Interest rate derivatives are a significative special case, since they combine together multiple funding rates plus the underlying interest rate.
The underlying rate is typically a risky Ibor rate, associated with unsecured (uncollateralised) lending/borrowing transactions, e.g. deposits traded on the interbank money market, with a given tenor.

A general interest rate derivative Π pays a stream of coupons $\Pi(T_i, L_{f,i})$, $i=1,\dots,n$, at cash flow dates T_i linked to a corresponding stream of risky Ibor rates $L_{f,i} := L_f(T_{i-1}, T_i)$ fixing at time $T_{i-1} < T_i$ associated with tenor $T_i - T_{i-1}$.

The Ibor rates are often associated with risky (unsecured) Zero Coupon Bonds $P_f(T_{i-1}, T_i)$, such that

$$L_f(T_{i-1}, T_i) = \frac{1}{\tau(T_{i-1}, T_i)}\left[\frac{1}{P_f(T_{i-1}, T_i)} - 1\right],$$

$$P_f(T_{i-1}, T_i) = \mathbb{E}^Q_{T_{i-1}}\left[\exp\left(-\int_{T_{i-1}}^{T_i} r_f(u)du\right)\right].$$

Funding: interest rate derivatives [2]

In the general case of partial collateral the price of the interest rate derivative is set to $\Pi(T_n) = \Pi(T_n, L_{f,n})$ and computed iteratively for intervals $[T_{i-1}, T_i]$

$$\Pi(T_{i-1}) = \mathbb{E}^Q_{T_{i-1}} \left[D_c(t, T_i)\Pi(T_i) - \int_{T_{i-1}}^{T_i} D_c(u, T_i) \left[r_f(u) - r_c(u) \right] \left[\Pi(u) - C(u) \right] du \right] \\ + \Pi(T_{i-1}, L_{f,i-1}).$$

Note that the expectation between cash flow dates is given by the first expectation, and and the payment is added explicitly in the last term.

4: Modern interest rate modelling
Funding: the discounting and FVA Table

Type of contract	Discounting	Asset drift	FVA
Perfectly collateralised trades (perfect CSA)	Collateral rate	Funding/repo rate	NO
Perfectly collateralised trades and asset (perfect CSAs)	Collateral rate	Asset collateral rate	NO
Partially collateralised trades (real CSA)	Collateral rate	Funding/repo rate	YES
Double currency collateralised trades (real CSA)	Collateral rate (collateral ccy) + CCS basis	Funding/repo rate (asset ccy)	YES
Uncollateralised trades (no CSA)	Funding rate	Funding/repo rate	YES

4: Modern interest rate modelling
Funding: summary and open issues

- **Summary**

  Using consistently standard pricing techniques (no-arbitrage, replication, self financing, Feynman-Kac) and multiple sources of funding, we have been able to derive generalised pricing formulas with small effort (with respect to Black-Scholes-Merton), and to define the Funding Value Adjustments corresponding to each case.

- **Modelling Issues**
  - Generalisation to exotics
  - FVA computation
  - What is the right funding curve ?
  - Add credit risk: CVA/DVA terms + corrections to FVA (free of double counting)

5: Linear interest rate products
Summary

5. **Interest rate products**
   - Interest rate derivatives: modeling approaches
   - A simple credit model
   - Forward rate
   - Forward Rate Agreement
   - Futures
   - Instantaneous forward rate
   - Swap
   - Forward swap measure
   - Overnight Indexed Swap
   - Basis Swap
   - Summary of rate conventions

Interest rate derivatives: modeling approaches [1]

Pricing interest rate derivatives is a non-trivial problem, for which we may identify two distinct approaches:

1. Modeling the joint evolution of a default-free rate, plus counterparty's default times.
   - Interest rate risk: model one single risk free stochastic rate.
   - Credit risk: model the default of the interbank sector, not of a precise counterparty, taking into account that the Libor panels themselves are not static but their composition changes over time, depending on the relative default probability of candidate Libor banks (the panels are themselves stochastic !). E.g. model the default of all candidate Libor banks, then select the subset of the less risky banks composing the panel, and finally compute the risky Libor as risk free rate + trimmed average of risky components.
   - Correlations: we need a complex correlation matrix with rate/credit and credit/credit correlations.

2. Modeling the joint evolution of multiple distinct rates: this implies taking the approach of multi-curves constructions to its logical consequences, and to introduce a generalised interest rate model where such distinct curves are modeled jointly.

Interest rate derivatives: modeling approaches [2]

We will:

- Exploit the first approach showing how a simple rate + credit model is able to describe a multi-curve world.

- Use systematically the second approach, as described in the recent financial literature. In particular we will borrow mainly from a few authors: Kijima et al. (2008), F. Mercurio (2009-2010), Fujii et al. (2009-2010).

5: Linear interest rate products
A simple credit model [1]

The market basis and the multi-curve approach can be justified using the simple credit model from Mercurio (2009), adapted to the present context and notation (see also Bianchetti and Carlicchi, 2012).

Our economy admits, in this case, two counterparties and two financial instruments:

- One riskless counterparty, characterised by zero default risk
- One risky (Libor) counterparty, characterised by non zero default risk and:
  - $LGD_x = 1 - R_x$ = loss given default and recovery rate
  - $\tau_x(t) > t$ = default time observed at time *t*
- $P(t,T)$ = riskless Zero Coupon Bond, emitted by the riskless counterparty for maturity $T > t$, equivalent to a perfectly collateralised Zero Coupon Bond
- $P_x(t,T)$ = risky Zero Coupon Bond, emitted by the risky (Libor) counterparty for maturity *T*

The payoffs of the riskless and risky Zero Coupon Bonds, respectively, are

$$
\begin{aligned}
P(T,T) &= 1, \\
P_x(T,T) &= 1_{[\tau_x(t)>T]} + R_x \, 1_{[\tau_x(t)\leq T]} \boxed{\leq P(T,T).}
\end{aligned}
$$

Notice that the default time $\tau_x(t)$ is a stochastic process.
In the limit of vanishing counterparty risk $\tau_x(t) \to \infty$ or $R_x \to 1$ we obtain $P_x(t,T) \to P(t,T)$.

5: Linear interest rate products
A simple credit model [2]

Using the relation

$$1_{[\tau_x(t)>T]} + R_x 1_{[\tau_x(t)\le T]} = 1 - 1_{[\tau_x(t)\le T]} + R_x 1_{[\tau_x(t)\le T]} = 1 - LGD_x 1_{[\tau_x(t)\le T]}$$

the prices of the riskless and risky Zero Coupon Bonds are given, respectively, by

$$\begin{aligned}
P(t,T) &= E_t^Q\{D(t,T)P(T,T)\},\\
P_x(t,T) &= E_t^Q\{D(t,T)P_x(T,T)\}\\
&= E_t^Q\{D(t,T)[1_{[\tau_x(t)>T]} + R_x 1_{[\tau_x(t)\le T]}]\}\\
&= E_t^Q\{D(t,T)\} - LGD_x E_t^Q\{D(t,T)1_{[\tau_x(t)\le T]}]\}\\
&= P(t,T) - LGD_x P(t,T) Q_x(t,T)\\
&= P(t,T)R(t,t,T,R_x)\\
&\le P(t,T),
\end{aligned}$$

In principle they are correlated.
For simplicity we assume independence.

where:

$$\begin{aligned}
R(t,T_1,T_2,R_x) &:= 1 - LGD_x\, Q_x(t,T_1,T_2) \le 1,\\
Q_x(t,T_1,T_2) &:= E_t^Q[Q_x(T_1,T_2)],\ \ T_1,T_2 \ \textit{forward default probability}\\
Q_x(T_1,T_2) &:= E_{T_1}^Q[1_{[\tau_x(T_1)\le T_2]}],\ \ T_1\textit{-spot default probability.}
\end{aligned}$$

5: Linear interest rate products
A simple credit model [3]

We may define corresponding riskless and risky zero coupon/Libor interest rates as

$$P(T_1,T_2) = e^{-R(T_1,T_2)\tau(T_1,T_2)} = \frac{1}{\tau(T_1,T_2)}\left[\frac{1}{L(T_1,T_2)} - 1\right],$$

$$P_x(T_1,T_2) = e^{-R_x(T_1,T_2)\tau(T_1,T_2)}, = \frac{1}{\tau(T_1,T_2)}\left[\frac{1}{L_x(T_1,T_2)} - 1\right],$$

using the appropriate compounding and day count rules, respectively. Using the previous relation between risky and riskless Zero Coupon Bonds we obtain

$$P_x(T_1,T_2) = P(T_1,T_2)R(T_1,T_1,T_2,R_x) \le P(T_1,T_2),$$

$$R(T_1,T_2) = -\frac{1}{T}\ln\left[P(T_1,T_2)\right],$$

$$R_x(T_1,T_2) = -\frac{1}{T}\ln\left[P_x(T_1,T_2)\right] = -\frac{1}{T}\ln\left[P(T_1,T_2)R(T_1,T_1,T_2,R_x)\right] \boxed{\ge R(T_1,T_2),}$$

$$L(T_1,T_2) = \frac{1}{\tau(T_1,T_2)}\left[\frac{1}{P(T_1,T_2)} - 1\right]$$

$$L_x(T_1,T_2) = \frac{1}{\tau(T_1,T_2)}\left[\frac{1}{P(T_1,T_2)R(T_1,T_1,T_2,R_x)} - 1\right] \boxed{\ge L(T_1,T_2).}$$

We conclude that the risky zero coupon/Libor interest rates are always higher than the corresponding riskless rates. In other words, the default risk of the average Libor counterparty, induces a positive basis spread between the risky zero coupon/Libor rates and the corresponding riskless rates

$$
\begin{aligned}
\Delta_x^R(T_1,T_2) :&= R_x(T_1,T_2) - R(T_1,T_2),\\
&= -\frac{1}{T}\ln\left[P_x(T_1,T_2)\right] - \frac{1}{T}\ln\left[P(T_1,T_2)\right],\\
&= -\frac{1}{T}\ln\left[R(T_1,T_1,T_2,R_x)\right] > 0,\\
\Delta_x^L(T_1,T_2) :&= L_x(T_1,T_2) - L(T_1,T_2)\\
&= \frac{1}{\tau_f(T_1,T_2)P(T_1,T_2)}\left[\frac{1}{R(T_1,T_1,T_2,R_x)} - 1\right] > 0.
\end{aligned}
$$

5: Linear interest rate products
A simple credit model [5]

The price at time t of a risky standard FRA contract exchanging in T_2 the fixed rate K versus the risky Libor rate $L_x(T_1,T_2)$ is given by

$$\begin{aligned}
\mathrm{FRA}_x^{\mathrm{Std}}(t,T_1,T_2,K) &= N\omega\mathbb{E}_t^Q\left\{D(t,T_2)\left[L_x(T_1,T_2)-K\right]\tau(T_1,T_2)\right\} \\
&= N\omega\mathbb{E}_t^Q\left\{D(t,T_1)D(T_1,T_2)\left[\frac{1}{P_x(T_1,T_2)}-1-K\tau(T_1,T_2)\right]\right\} \\
&= N\omega\mathbb{E}_t^Q\left\{D(t,T_1)\mathbb{E}_{T_1}^Q\left\{P(T_1,T_2)\left[\frac{1}{P_x(T_1,T_2)}-1-K\tau(T_1,T_2)\right]\right\}\right\} \\
&= N\omega\mathbb{E}_t^Q\left\{D(t,T_1)\mathbb{E}_{T_1}^Q\left\{\frac{1}{R(T_1,T_1,T_2,R_x)}-P(T_1,T_2)\left[1+K\tau(T_1,T_2)\right]\right\}\right\} \\
&= N\omega\mathbb{E}_t^Q\left\{\frac{D(t,T_1)}{R(T_1,T_1,T_2,R_x)}-D(t,T_2)\left[1+K\tau(T_1,T_2)\right]\right\} \\
&= N\omega\left\{\frac{P(t,T_1)}{R(t,T_1,T_2,R_x)}-P(t,T_2)\left[1+K\tau(T_1,T_2)\right]\right\} \\
&\geq N\omega\left\{P(t,T_1)-P(t,T_2)\left[1+K\tau(T_1,T_2)\right]\right\} \\
&= \mathrm{FRA}^{\mathrm{Std}}(t,T_1,T_2,K).
\end{aligned}$$

See subseqent slides for the discussion of the FRA.

5: Linear interest rate products

A simple credit model [6]

The price at time *t* of a risky market FRA contract is obtained through analogous calculations as

$$
\begin{aligned}
\mathrm{FRA}_x^{\mathrm{Mkt}}(t, T_1, T_2, K) &= N\omega \mathbb{E}_t^Q \left\{ D(t, T_1) \left[\frac{L_x(T_1, T_2) - K}{1 + L_x(T_1, T_2)\tau(T_1, T_2)} \right] \tau(T_1, T_2) \right\} \\
&= N\omega \left\{ P(t, T_1) - P(t, T_2) \left[1 + K\tau(T_1, T_2) \right] R(t, T_1, T_2, R_x) \right\} \\
&= \mathrm{FRA}_x^{\mathrm{Std}}(t, T_1, T_2, K) R(t, T_1, T_2, R_x) \\
&\leq \mathrm{FRA}^{\mathrm{Std}}(t, T_1, T_2, K).
\end{aligned}
$$

Notice that the FRA prices above have been derived under the assumption that the FRA contract (not the underlying Libor rate) is counterparty risk free. Otherwise the payoff would contain the default indicators of the two counterparties involved in the contract (not that of the average Libor counterparty as in the risky Zero Coupon Bond).

5: Linear interest rate products
A simple credit model [7]

The equilibrium FRA rates at time *t* are given by

$$
\begin{aligned}
R_x^{\mathrm{FRA,Std}}(t,T_1,T_2) &= R_x^{\mathrm{FRA,Mkt}}(t,T_1,T_2) \\
&= \frac{1}{\tau(T_1,T_2)}\left[\frac{P(t,T_1)}{P(t,T_2)}\frac{1}{R(t,T_1,T_2,R_x)}-1\right] \\
&\geq \frac{1}{\tau(T_1,T_2)}\left[\frac{P(t,T_1)}{P(t,T_2)}-1\right] \\
&= F(t,T_1,T_2) \\
&= R^{\mathrm{FRA,Std}}(t,T_1,T_2) = R^{\mathrm{FRA,Mkt}}(t,T_1,T_2)
\end{aligned}
$$

We conclude that the risky FRA rate is always higher than the corresponding riskless FRA rate. In other words, the default risk of the average Libor counterparty, included inside the risky Libor rate underlying a counterparty risk free FRA contract, induces a positive basis spread between the risky FRA rate and the corresponding riskless FRA rate.

5: Linear interest rate products
Interest rate derivatives: description approach [1]

We will discuss now the most important interest rate financial instruments. In particular, we will focus on the most important plain vanilla instruments, also quoted on the market: Deposit, FRA, Futures, Swap, OIS, Basis Swap, Cap, Floor, Swaption, etc.

We will adopt a systematic description approach based on the following scheme.

- Instrument description: payoff, pictures, discussion, etc.
- Instrument pricing: derivation of the relevant pricing formulas.
- Market data: possible quotations of the financial instrument available on the market.
- Discussion: classical vs modern pricing, etc.

Remind: that interest rate instruments depend, in general, on two distinct interest rates.

- The underlying rate of the instrument, and related quantities, indexed with *"x"*, such as $L_x(T_1, T_2)$.
- The discount rate associated to the instrument, and related quantities, indexed with *"d"*, such as $P_d(T_1, T_2)$. Notice that the discount rate depends on the funding sources.

Interest rate derivatives: description approach [2]

In general, we describe the time structure of financial contracts using time grids, or schedules, collect all the relevant contract dates (period start/end dates, fixing dates, accrual dates, cash flow dates, etc.) and year fractions known at the beginning of the contract, written in or derived from the termsheet. For example, the schedule of a Swap is described by

$$\begin{aligned}
&\boldsymbol{T}_x = \{T_{x,0}, ..., T_{x,m}\}\,, \text{ floating leg schedule,}\\
&\boldsymbol{S} = \{S_0, ..., S_n\}\,, \text{ fixed leg schedule,}\\
&S_0 = T_{x,0}, S_n = T_{x,m},
\end{aligned}$$

In particular, floating leg year fractions $\tau_x(T_{i-1}, T_i)$ are both regular and consistent with the corresponding floating Libor rate tenor *x*, e.g. if the rate is Libor6M, with tenor *x=6M*, then the floating leg frequency is $2y^{-1}$ and the dates T_i are six-month spaced.

5: Linear interest rate products
Forward rates [1]

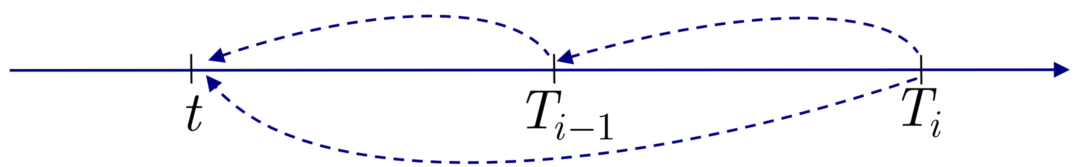
Forward rates $F(t;T_{i-1},T_i)$ are interest rates observed at a generic time instant *t*, resetting at future time T_{i-1} and spanning the future time interval $[T_{i-1},T_i]$ (called rate tenor), with $t < T_{i-1} < T_i$. Forward rates can be expressed in terms of Zero Coupon Bonds by recurring to a simple no-arbitrage argument. If we define the forward Zero Coupon Bond observed at time *t* as

$$P(t;T_{i-1},T_i) := \mathbb{E}_t^Q\left[P(T_{i-1};T_i)\right]$$

we may write the following no arbitrage relation for deterministic $N(T_i)$

$$N(t) = P(t;T_i)N(T_i) = P(t;T_{i-1})P(t;T_{i-1},T_i)N(T_i).$$

The financial meaning of expression above is that, given a deterministic amount of money $N(T_i)$ at time T_i, its equivalent amount of money, or value, at time $t < T_i$ must be unique, both if we discount directly in one single step from T_i to *t*, using the discount factor $P(t,T_i)$, and if we discount in two steps, first from T_i to T_{i-1}, using the forward discount factor $P(t;T_{i-1},T_i)$ and then from T_{i-1} to *t*, using $P(t;T_{i-1})$.



5: Linear interest rate products
Forward rates [2]

At this point we may define the simple compounded forward rate $F(t;T_{i-1},T_i)$ associated to $P(t;T_{i-1},T_i)$ as

$$P(t;T_{i-1},T_i) = \frac{P(t;T_i)}{P(t;T_{i-1})} := \frac{1}{1+F(t;T_{i-1},T_i)\tau(T_{i-1},T_i)},$$

By inverting we obtain the familiar no arbitrage expression

$$\boxed{\begin{aligned} F_i(t) :&= F(t;T_{i-1},T_i) = \frac{1}{\tau(T_{i-1},T_i)}\left[\frac{1}{P(t;T_{i-1},T_i)} - 1\right] \\ &= \frac{P(t;T_{i-1}) - P(t;T_i)}{\tau(T_{i-1},T_i)P(t;T_i)}. \end{aligned}}$$

We notice that, for $t \to T_{i-1}^{-}$ forward rates converge to spot Libor rates.

5: Linear interest rate products
Forward rates [3]

Theorem: forward rates are martingales under their "natural" T_i - forward measure Q^{T_i}.

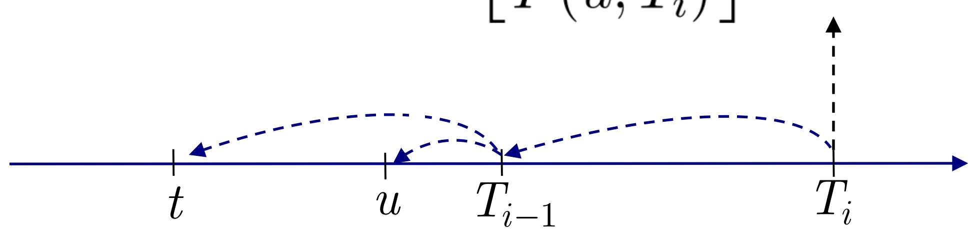
$$\boxed{F_i(t) = \mathbb{E}_t^{Q^{T_i}}\left[F_i(u)\right], \; \forall \, t < u < T_{i-1} < T_i.}$$

Proof: the quantity

$$\Pi(t) := P(t;T_i)F_i(t)\tau(T_{i-1},T_i) = P(t;T_{i-1}) - P(t;T_i),$$

is the time-t price of a tradable asset, since it is a combination of two (tradable) Zero Coupon Bonds. Hence, under the Q^{T_i} - forward measure,

$$\frac{\Pi(t)}{P(t;T_i)} = F_i(t)\tau(T_{i-1},T_i) = \mathbb{E}_t^{Q^{T_i}}\left[\frac{\Pi(u)}{P(u;T_i)}\right] = \mathbb{E}_t^{Q^{T_i}}\left[F_i(u)\right]\tau(T_{i-1},T_i).$$



Qed

In particular, setting $u = T_{i-1}$ we obtain

$$\boxed{F_i(t) = \mathbb{E}_t^{Q^{T_i}}\left[L(T_{i-1},T_i)\right].}$$

5: Linear interest rate products
Forward rates [4]

Notice that (risky) Libor rates $L_x(T_{i-1}, T_i)$ are not martingales under the T_i - forward measure Q^{T_i}

$$F_i(t) = \mathbb{E}_t^{Q^{T_i}}\left[L(T_{i-1}, T_i)\right] \neq \mathbb{E}_t^{Q^{T_i}}\left[L_x(T_{i-1}, T_i)\right] := F_{x,i}(t),$$

since the underlying Libor rate is, in general, different from the funding rate associated to the probability measure.

We define

$$\boxed{F_{x,i}(t) := \mathbb{E}_t^{Q^{T_i}}\left[L_x(T_{i-1}, T_i)\right],}$$

Risky forward rate

the risky forward rate.

When the funding rate and the underlying rate are the same, or in case of vanishing interest rate basis, we obtain the classical (pre-credit crunch) single curve limit

$$F_{x,i}(t) = \mathbb{E}_t^{Q^{T_i}}\left[L_x(T_{i-1}, T_i)\right], \longrightarrow \mathbb{E}_t^{Q^{T_i}}\left[L(T_{i-1}, T_i)\right] = F_i(t).$$

5: Linear interest rate products
Forward rates [5]

Properties of the risky forward rate:

1. at fixing date T_{i-1} it coincides with the Libor rate

$$F_{x,i}(T_{i-1}) = L_x(T_{i-1}, T_i).$$

2. It is a martingale under the T_i - forward discounting measure Q^{T_i} associated to the numeraire $P(t;T_i)$:

$$\boxed{F_{x,i}(t) = \mathbb{E}_t^{Q^{T_i}}\left[L_x(T_{i-1}, T_i)\right] = \mathbb{E}_t^{Q^{T_i}}\left[F_{x,i}(T_{i-1})\right]},$$

3. FRA contracts are quoted on the market in terms of their forward rates, thus it is "what you read on the screen". A forward rate term structure can be stripped from market FRA quotations.

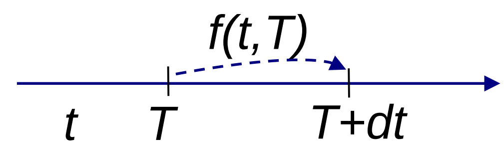
4. The risky forward rate is the basic building block of the new theoretical interest rate framework.

5: Linear interest rate products
Instantaneous forward rate

Instantaneous forward rates $f(t,T)$ are abstract forward rates observed at time t and spanning an infinitesimal future time interval $[T, T+dt]$ (with infinitesimal rate tenor). They are thus obtained through the limit

$$f(t;T) := \lim_{T'\to T^+} F(t;T,T') = -\lim_{T'\to T^+} \frac{1}{P(t;T')} \frac{P(t;T') - P(t;T)}{\tau(T,T')}$$

$$= -\frac{1}{P(t;T)} \frac{\partial P(t;T)}{\partial T} = -\frac{\partial \ln P(t;T)}{\partial T}.$$



Integrating the eq. above we can express the Zero Coupon Bond as an integral of instantaneous forward rates as

$$\boxed{P(t;T) = \exp\left[-\int_t^T f(t;u)\,du\right].}$$

Instantaneous forward rates can also be calculated as expectations of future short rates under the T-forward measure Q_T. In fact, setting $u = T$ in the martingality relation for the forward rate we obtain

$$\boxed{f(t;T) = \mathbb{E}_t^{Q_T}\left[r(T)\right].}$$

The eq. above can also be verified directly.

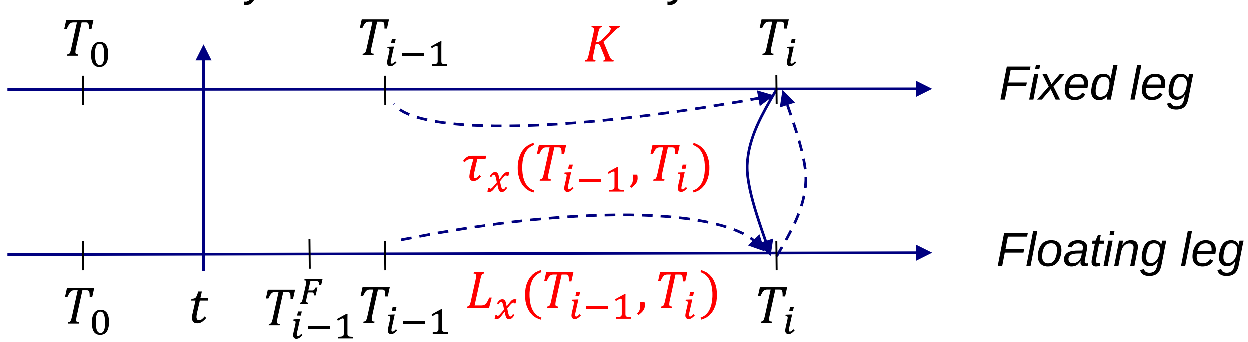
5: Linear interest rate products
Forward Rate Agreement [1]

Forward Rate Agreements (FRA) are standard OTC contracts with two legs starting at time T_0. The floating leg pays the interest accrued with a (risky) Libor $L_x(T_{i-1}, T_i)$ reset at time T_{i-1}^F, and spanning the time interval $[T_{i-1}, T_i]$. The fixed leg pays the interest accrued with a fixed rate K over the same time interval $[T_{i-1}, T_i]$.
There are two types of FRA:

1. Standard (or textbook) FRA:
   the payoff of the standard FRA at payment date T_i is given by

$$\mathbf{FRA}_{\mathrm{Std}}(T_i; \mathbf{T}, K, \omega) = N\omega \left[L_x(T_{i-1}, T_i) - K\right] \tau_x(T_{i-1}, T_i),$$

where $\omega = \pm 1$ for a payer/receiver FRA (referred to the fixed leg), respectively, and for simplicity we have assumed that both rates are annual, simply compounded, and share the same year fraction and day count convention.



5: Linear interest rate products
Forward Rate Agreement [2]

The price of the standard FRA at time $t<T_{i-1}$ is given, under the payment T_i -forward measure, by

$$\begin{aligned}
\mathbf{FRA}_{\mathrm{Std}}(t;\mathbf{T},K,\omega) &= P_d(t;T_i)\mathbb{E}_t^{Q^{T_i}}\left[\mathbf{FRA}_{\mathrm{Std}}(T_i;\mathbf{T},K,\omega)\right] \\
&= N\omega P_d(t;T_i)\left\{\mathbb{E}_t^{Q^{T_i}}\left[L_x(T_{i-1},T_i)\right]-K\right\}\tau_x(T_{i-1},T_i) \\
&= N\omega P_d(t;T_i)\left[F_{x,i}(t)-\boxed{K}\right]\tau_x(T_{i-1},T_i).
\end{aligned}$$

The FRA rate at time t is defined as the fixed rate K that makes null the FRA present value,

$$R_{x,\mathrm{Std}}^{\mathrm{FRA}}(t;\mathbf{T}) = \boxed{F_{x,i}(t)},$$

FRA rate

Obviously, the FRA rate collapses on the Deposit rate for $T_1 \to T_0^+$

$$\lim_{T_1\to T_0^+} R_{\mathrm{Std}}^{\mathrm{FRA}}(t,\mathbf{T}) = R^{\mathrm{Depo}}(t;\mathbf{T})).$$

5: Linear interest rate products
Forward Rate Agreement [3]

What do we need to use the FRA pricing formula above in practice?

$$\boxed{\mathbf{FRA}_{\mathrm{Std}}(t;\mathbf{T},K,\omega) = N\omega P_d(t;T_i)\left[F_{x,i}(t) - K\right]\tau_x(T_{i-1},T_i).}$$

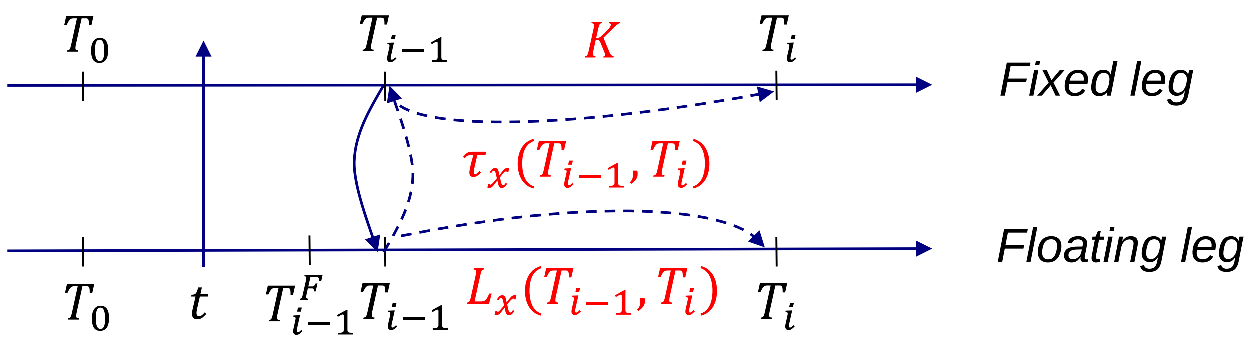
- Static trade data: $N, \omega, K, T_0, \dots, T_n$ and the corresponding trade schedule **T**.
- Discount factors $P_d(t;T_0), \dots, P_d(t;T_n)$: calculated on the discounting curve $\mathcal{C}_d$.
- Forward rates $F_{x,0}(t), \dots, F_{x,n}(t)$: calculated on the forwarding curve $\mathcal{C}_x$.

Important:
to price interest rate derivatives we need the appropriate yield curves to compute the relevant forward rates and discount factors.

5: Linear interest rate products
Forward Rate Agreement [4]

2. Market FRA:
   the payoff of the market FRA at payment date T_{i-1} (not T_i) is given by

$$
\begin{aligned}
\mathbf{FRA}_{\mathrm{Mkt}}(T_{i-1};\mathbf{T},K,\omega) &= N\,\frac{\omega\left[L_x(T_{i-1},T_i)-K\right]\tau_x(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)} \\
&= \frac{\mathbf{FRA}_{\mathrm{Std}}(T_{i-1};\mathbf{T},K,\omega)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}.
\end{aligned}
$$



Notice that the payment is anticipated at date T_{i-1}, discounted from T_i to T_{i-1} using the Libor rate itself.

5: Linear interest rate products
Forward Rate Agreement [5]

The price of the market FRA at time $t < T_i$ is given, under the payment T_{i-1} forward measure, by

$$\begin{aligned}
\mathbf{FRA}_{\text{Mkt}}(t;\mathbf{T},K,\omega) &= P_d(t;T_{i-1})\mathbb{E}_t^{Q^{T_{i-1}}}\left[\mathbf{FRA}_{\text{Mkt}}(T_{i-1};\mathbf{T},K,\omega)\right] \\
&= N\omega P_d(t;T_{i-1})\mathbb{E}_t^{Q^{T_{i-1}}}\left\{\frac{\left[L_x(T_{i-1},T_i)-K\right]\tau_x(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right\} \\
&= N\omega P_d(t;T_{i-1})\left\{1-\left[1+K\tau_x(T_{i-1},T_i)\right]\mathbb{E}_t^{Q^{T_{i-1}}}\left[\frac{1}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right]\right\}.
\end{aligned}$$

Notice that, in this case, the price depends on the expectation of the forward discount factor

$$P_x(T_{i-1},T_i) := \frac{1}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}$$

under the payment T_{i-1} forward measure.

5: Linear interest rate products
Forward Rate Agreement [6]

Switching from T_{i-1} to T_i forward measure we obtain

$$
\begin{aligned}
&\mathbb{E}_t^{Q^{T_{i-1}}}\left[\frac{1}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right] \\
&\qquad = \frac{P_d(t,T_{i-1})}{P_d(t,T_i)}\mathbb{E}_t^{Q^{T_i}}\left[\frac{1}{P_d(T_{i-1},T_i)}\frac{1}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right] \\
&\qquad = \frac{1}{1+\tau_d(T_{i-1},T_i)F_{d,i}(t)}\mathbb{E}_t^{Q^{T_i}}\left[\frac{1+L_d(T_{i-1},T_i)\tau_d(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right],
\end{aligned}
$$

where we have defined the spot and forward Libor associated to discount zero bonds as follows

$$
\begin{aligned}
L_d(T_{i-1},T_i) &:= \frac{1}{\tau_d(T_{i-1},T_i)}\left[\frac{1}{P_d(T_{i-1},T_i)}-1\right], \\
F_d(t;T_{i-1},T_i) &:= \frac{1}{\tau_d(T_{i-1},T_i)}\left[\frac{P_d(t,T_{i-1})}{P_d(t,T_i)}-1\right].
\end{aligned}
$$

Forward Rate Agreement [7]

The price and the FRA rate of the market FRA becomes

$$\mathbf{FRA}_{\mathrm{Mkt}}(t;\mathbf{T},K,\omega) = N\omega P_d(t;T_{i-1}) \times \left\{1 - \frac{1+K\tau_x(T_{i-1},T_i)}{1+F_{d,i}(t)\tau_d(T_{i-1},T_i)}\mathbb{E}_t^{Q^{T_i}}\left[\frac{1+L_d(T_{i-1},T_i)\tau_d(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right]\right\},$$

$$R^{\mathrm{FRA}}_{x,\mathrm{Mkt}}(t;\mathbf{T}) = \frac{1}{\tau_x(T_{i-1},T_i)}\left\{\frac{1+F_{d,i}(t)\tau_d(T_{i-1},T_i)}{\mathbb{E}_t^{Q^{T_i}}\left[\frac{1+L_d(T_{i-1},T_i)\tau_d(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right]} - 1\right\}.$$

Thus the price of the market FRA depends on the model chosen for the joint distribution of the two Libor rates $L_d(T_{i-1},T_i)$ and $L_x(T_{i-1},T_i)$ under the forward measure Q^{T_i}.

5: Linear interest rate products
Forward Rate Agreement [8]

Assuming some model for the dynamics of $L_d(T_{i-1}, T_i)$ and $L_x(T_{i-1}, T_i)$ under the forward measure Q^{T_i} we may write, without loss of generality,

$$\mathbb{E}_t^{Q^{T_i}}\left[\frac{1+L_d(T_{i-1},T_i)\tau_d(T_{i-1},T_i)}{1+L_x(T_{i-1},T_i)\tau_x(T_{i-1},T_i)}\right]=\frac{1+F_{d,i}(t)\tau_d(T_{i-1},T_i)}{1+F_{x,i}(t)\tau_x(T_{i-1},T_i)}e^{C_x^{\mathrm{FRA}}(t;T_{i-1})},$$

$$\mathbf{FRA}_{\mathrm{Mkt}}(t;\mathbf{T},K,\omega)=N\omega P_d(t;T_{i-1})\left[1-\frac{1+K\tau_x(T_{i-1},T_i)}{1+F_{x,i}(t)\tau_x(T_{i-1},T_i)}e^{C_x^{\mathrm{FRA}}(t;T_{i-1})}\right],$$

$$R_{x,\mathrm{Mkt}}^{\mathrm{FRA}}(t;\mathbf{T})=\frac{1}{\tau_x(T_{i-1},T_i)}\left\{\left[1+F_{x,i}(t)\tau_x(T_{i-1},T_i)\right]e^{C_x^{\mathrm{FRA}}(t;T_{i-1})}-1\right\},$$

where $C_x(t;T_{i-1})$ is a convexity adjustment, whose detailed expression depends on the chosen model.

5: Linear interest rate products
Forward Rate Agreement [9]

A possible choice is that of Mercurio (2010), in which the two FRA rates are modeled as shifted lognormal martingales under the forward measure Q^{T_i},

$$
\begin{aligned}
\frac{dF_{d,i}(t)}{F_{d,i}(t) + \frac{1}{\tau_d(T_{i-1},T_i)}} &= \sigma_{d,i} dW_d^{Q^{T_i}}(t),\\
\frac{dF_{x,i}(t)}{F_{x,i}(t) + \frac{1}{\tau_x(T_{i-1},T_i)}} &= \sigma_{x,i} dW_x^{Q^{T_i}}(t),\\
dW_d^{Q^{T_i}}(t) dW_x^{Q^{T_i}}(t) &= \rho_{d,x,i} dt,\\
C(t;T_{i-1}) &= \left[\sigma_{x,i}^2 - \sigma_{x,i}\sigma_{d,i}\rho_{d,x,i}\right]\tau(t,T_{i-1}).
\end{aligned}
$$

The size of the convexity adjustment results to be below 1 bp, even for long maturities, for typical post credit crunch market situations (see Mercurio 2010)).

5: Linear interest rate products
Forward Rate Agreement [10]

$$
\begin{aligned}
\mathbf{FRA}_{\text{Mkt}}(t;\mathbf{T},K,\omega) &= N\omega P_d(t;T_{i-1})\left[1-\frac{1+K\tau_x(T_{i-1},T_i)}{1+F_{x,i}(t)\tau_x(T_{i-1},T_i)}e^{C_x^{\text{FRA}}(t;T_{i-1})}\right] \\
&\simeq \mathbf{FRA}_{\text{Std}}(t;\mathbf{T},K,\omega),
\end{aligned}
$$

$$
\begin{aligned}
R_{x,\text{Mkt}}^{\text{FRA}}(t;\mathbf{T}) &= \frac{1}{\tau_x(T_{i-1},T_i)}\left\{\left[1+F_{x,i}(t)\tau_x(T_{i-1},T_i)\right]e^{C_x^{\text{FRA}}(t;T_{i-1})}-1\right\} \\
&\simeq R_{x,\text{Std}}^{\text{FRA}}(t;\mathbf{T}) = F_{x,i}(t).
\end{aligned}
$$

- Prices and FRA rates of the Standard and Market FRAs are different because of the interest rate basis, but such difference is negligible.

- We are still allowed to say that the market quotes FRA rates.

5: Linear interest rate products

Forward Rate Agreement: market quotes [11]

```
12:57 30DEC11              ICAP LONDON                    UK69580       ICAPSHORT2
Contact Reuters EXEU        EURO Short Swaps / FRAs             +44 (0)20 7532 3530
         1M Swaps                 IMM Dated                              3m FRAs
  2x1    1.008-0.958        1y  MAR/MAR 0.986-0.936             1x4   1.248-1.198
  3x1    0.978-0.928        1y  JUN/JUN 0.948-0.898             2x5   1.155-1.105
  4x1    0.942-0.892        1y  SEP/SEP 0.962-0.912             3x6   1.092-1.042
  5x1    0.905-0.855        1y  DEC/DEC 1.004-0.954             4x7   1.041-0.991
  6x1    0.879-0.829        2y  MAR/MAR 1.025-0.975             5x8   0.989-0.939
  7x1    0.858-0.808        2y  JUN/JUN 1.046-0.996             6x9   0.956-0.906
  8x1    0.839-0.789        3y  MAR/MAR 1.177-1.127
  9x1    0.825-0.775                                                  6m FRAs
 10x1    0.813-0.763                       IMM Fras            1x7    1.496-1.446
 11x1    0.806-0.756                                           2x8    1.390-1.340
 12x1    0.801-0.751        1x7         1.454-1.404            3x9    1.317-1.267
                            2x8         1.336-1.286            4x10   1.271-1.221
 1y /3   1.112-1.062        3x9         1.295-1.245            5x11   1.225-1.175
 15m/3   1.075-1.025        4x10        1.253-1.203            6x12   1.197-1.147
 18m/3   1.060-1.010                                           12x18  1.150-1.100
 21m/3   1.059-1.009        0x3 Today   1.381-1.331            18x24  1.249-1.199
                            0x6 Today   1.642-1.592
 1y /6   1.447-1.397        0x3 Tom     1.371-1.321
 15m/6   1.285-1.235        0x6 Tom     1.632-1.582                   12m FRA
 18m/6   1.350-1.300                                           12x24  1.506-1.456
 21m/6   1.265-1.215                          ICAP OIS Fix Menu <ICAPOISFIX01>
ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls
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5: Linear interest rate products

Forward Rate Agreement: market quotes [12]

16:14 30JAN15 ICAP LONDON UK69580 ICAPSHORT2

Contact Reuters EXEU

EURO Short Swaps / FRAs +44 (0)20 7532 3530

1M Swaps

2x1	0.029/-0.021
3x1	0.023/-0.027
4x1	0.018/-0.032
5x1	0.013/-0.037
6x1	0.010/-0.040
7x1	0.008/-0.042
8x1	0.005/-0.045
9x1	0.004/-0.046
10x1	0.002/-0.048
11x1	0.001/-0.049
12x1	-0.001/-0.051

1y /3	0.078/ 0.028
15m/3	0.077/ 0.027
18m/3	0.078/ 0.028
21m/3	0.082/ 0.032

1y /6	0.162/ 0.112
15m/6	0.150/ 0.100
18m/6	0.165/ 0.115
21m/6	0.159/ 0.109

IMM Dated

1y	MAR/MAR	0.078-0.028
1y	JUN/JUN	0.075-0.025
1y	SEP/SEP	0.078-0.028
1y	DEC/DEC	0.088-0.038
2y	MAR/MAR	0.090-0.040
2y	JUN/JUN	0.099-0.049
3y	MAR/MAR	0.128-0.078

IMM Fras

1x7	0.174-0.124
2x8	0.170-0.120
3x9	0.167-0.117
4x10	0.165-0.115

3m FRAs

1x4	0.087-0.037
2x5	0.086-0.036
3x6	0.082-0.032
4x7	0.081-0.031
5x8	0.078-0.028
6x9	0.075-0.025

6m FRAs

1x7	0.173-0.123
2x8	0.171-0.121
3x9	0.168-0.118
4x10	0.166-0.116
5x11	0.164-0.114
6x12	0.163-0.113
12x18	0.169-0.119
18x24	0.199-0.149

12m FRA

12x24	0.298-0.248

ICAP OIS Fix Menu <ICAPOISFIX01>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANG

Forward Rate Agreement: single curve limit [13]

In the classical, single-curve limit, with vanishing interest rate basis, we have

$$
\begin{aligned}
\mathbf{FRA}_{\mathrm{Std}}(t;\mathbf{T},K,\omega) &\longrightarrow N\omega P(t;T_i)\left[F_i(t)-K\right]\tau(T_{i-1},T_i),\\
\mathbf{FRA}_{\mathrm{Mkt}}(t;\mathbf{T},K,\omega) &\longrightarrow N\omega P(t;T_{i-1})\left[1-\frac{1+\tau(T_{i-1},T_i)K}{1+\tau(T_{i-1},T_i)F_i(t)}\right]\\
&= N\omega P(t;T_i)\left[F_i(t)-K\right]\tau(T_{i-1},T_i)\\
&= \mathbf{FRA}_{\mathrm{Std}}(t;\mathbf{T},K,\omega),\\
R^{\mathrm{FRA}}_{\mathrm{Mkt}}(t;\mathbf{T}) &\longrightarrow R^{\mathrm{FRA}}_{\mathrm{Std}}(t;\mathbf{T}) = F_i(t) = \frac{1}{\tau(T_{i-1},T_i)}\left[\frac{P(t;T_{i-1})}{P(t;T_i)}-1\right]
\end{aligned}
$$

Forward Rate Agreement: pricing table [14]

	FRA pricing formulas
Classical (single-curve)	$\mathbf{FRA}_{\text{Std}}(t;T_{i-1},T_i,K,\omega) = N\omega P(t;T_i)\left[F_i(t)-K\right]\tau(T_{i-1},T_i),$ $R^{\text{FRA}}_{\text{Std}}(t;\mathbf{T}) = F_i(t) = \mathbb{E}_t^{Q^{T_i}}\left[L(T_{i-1},T_i)\right],$ $\mathbf{FRA}_{\text{Mkt}}(t;T_{i-1},T_i,K,\omega) = \mathbf{FRA}_{\text{Std}}(t;T_{i-1},T_i,K,\omega),$ $R^{\text{FRA}}_{\text{Mkt}}(t,\mathbf{T}) = R^{\text{FRA}}_{\text{Std}}(t,\mathbf{T}).$
Modern (multi-curve)	$\mathbf{FRA}_{\text{Std}}(t;T_{i-1},T_i,K,\omega) = N\omega P_d(t;T_i)\left[F_{x,i}(t)-K\right]\tau_x(T_{i-1},T_i),$ $R^{\text{FRA}}_{x,\text{Std}}(t;\mathbf{T}) = F_{x,i}(t) := \mathbb{E}_t^{Q^{T_i}}\left[L_x(T_{i-1},T_i)\right],$ $\mathbf{FRA}_{\text{Mkt}}(t;\mathbf{T},K,\omega) = N\omega P_d(t;T_{i-1})\left[1-\frac{1+\tau_x(T_{i-1},T_i)K}{1+\tau_x(T_{i-1},T_i)F_{x,i}(t)}e^{C_x^{\text{FRA}}(t;T_{i-1})}\right],$ $R^{\text{FRA}}_{x,\text{Mkt}}(t;\mathbf{T}) = \frac{1}{\tau_x(T_{i-1},T_i)}\left\{\left[1+\tau_x(T_{i-1},T_i)F_{x,i}(t)\right]e^{C_x^{\text{FRA}}(t;T_{i-1})}-1\right\}.$

5: Linear interest rate products
Futures [1]

Interest rate Futures are the exchange-traded contracts equivalent to the over-the-counter (market) FRAs. The Futures' payoff at the last settlement date T_{i-1} (as for the market FRA, not T_i), is given by

$$\mathbf{Futures}(T_{i-1};\mathbf{T}) = N\left[1 - L_x(T_{i-1},T_i)\right].$$

This payoff is a classical example of "mixing apples and oranges" because, clearly, on the r.h.s. 1 is adimensional while the Libor rate $L_x(T_{i-1},T_i)$ has dimension t^{-1} and they cannot be directly summed together without an year fraction $\tau(T_{i-1},T_i)$. Thus we must look at it as a mere rule for computing the amount of currency to be margined everyday.

The Futures' price at time $t < T_{i-1}$ is given by

$$\begin{aligned}\mathbf{Futures}(t;\mathbf{T}) &= \mathbb{E}_t^Q\left[D_d(t;t)\mathbf{Futures}(T_{i-1};\mathbf{T})\right] \\ &= N\left\{1 - \mathbb{E}_t^Q\left[L_x(T_{i-1},T_i)\right]\right\} := N\left[1 - R_x^{\mathrm{Fut}}(t;\mathbf{T})\right],\end{aligned}$$

under the risk neutral measure Q associated to the funding bank account $B(t)$.
Notice that the Futures' daily margination mechanism implies that the payoff is regulated everyday, thus generating the unitary discount factor $D(t;t)=1$ appearing in the first line above.
The daily margination amount is calculated as Δ = 1.000.000€ x (P_{today} - $P_{yesterday}$)/4 .

5: Linear interest rate products
Futures [2]

Hence the pricing of Futures requires the computation of the Futures' rate

$$R^{\text{Fut}}(t;T_{i-1},T_i) := \mathbb{E}_t^Q\left[L_x(T_{i-1},T_i)\right] = \mathbb{E}_t^Q\left[F_{x,i}(T_{i-1})\right].$$

Since the forward rate $F_{x,i}(t)$ is not a martingale under the risk neutral measure Q, such computation requires the adoption of a model for the dynamics of $F_{x,i}(t)$. In general, we obtain that the Futures' rate is given by the corresponding (risky) forward rate corrected with a convexity adjustment

$$\boxed{\begin{aligned} R_x^{\text{Fut}}(t;\mathbf{T}) := \mathbb{E}_t^Q\left[L_x(T_{i-1},T_i)\right] &= \mathbb{E}_t^{Q^{T_i}}\left[L_x(T_{i-1},T_i)\right] + C_x^{\text{Fut}}(t;T_{i-1}) \\ &= F_{x,i}(t) + C_x^{\text{Fut}}(t;T_{i-1}). \end{aligned}}$$

From a financial point of view, an investor long a Futures contract will have a loss when the Futures' price increases (and the Futures' rate decreases) but he will finance such loss at a lower spot rate; viceversa when the Futures' price decreases (rate increases) the profit will be reinvested at an higher (spot) rate.
Hence the volatility of the forward rates and their correlation to the spot rates have to be accounted for.

5: Linear interest rate products

Futures [3]

The expression of the convexity adjustment will depend on the particular model adopted and will contain, in general, the model's volatilities and correlations.

Different approaches are available in the literature, see for example:

- Kirikos and Novak 1997 for the derivation within the Hull&White model,
- Jackel and Kawai 2005, and Brigo&Mercurio 2006 for the Libor Market Model,
- Brigo and Mercurio 2006 for the Two Factor Short Rate Gaussian G2++ Model,
- Piterbarg and Renedo 2006 for stochastic volatility models,
- Henrard 2009 for the One Factor HJM Model.
- Mercurio 2009 for the multi-curve Libor Market Model

5: Linear interest rate products
Futures [4]

For instance, under the multi-curve Libor Market Model of Mercurio (2009), the convexity adjustment takes the form

$$C_x^{\mathrm{Fut}}(t;T_{i-1}) \simeq F_{x,i}(t)\exp\left[\int_t^{T_{i-1}} \mu_{x,i}(u)\,du - 1\right],$$

where

$$\int_t^{T_{i-1}} \mu_{x,i}(u)\,du \simeq \sigma_{x,i}\sum_{j=1}^{i}\frac{\tau_{d,j}\sigma_{d,j}\rho_{j,i}^{x,d}F_{d,j}(t)}{1+\tau_{d,j}F_{d,j}(t)}\left(T_{j-1}-t\right),$$

$$\frac{dF_{x,i}(t)}{F_{x,i}(t)} = \mu_{x,i}(t)dt + \sigma_{x,i}dW_x^{Q^{T_i}}(t),$$

$$\frac{dF_{d,i}(t)}{F_{d,i}(t)} = \mu_{d,i}(t)dt + \sigma_{d,i}dW_d^{Q^{T_i}}(t),$$

$$F_{d,j}(t) := \mathbb{E}_t^{Q^{T_j}}\left[L_d(T_{j-1},T_j)\right] = \frac{1}{\tau_{d,j}}\left[\frac{P_d(t;T_{j-1})}{P_d(t;T_j)} - 1\right],$$

$\sigma_{x,i},\ \sigma_{d,j},\ \rho_{i,j}^{x,d}\ =$ instantaneous (deterministic) volatilities and correlation of $F_{x,i}(t), F_{d,j}(t)$, respectively.

5: Linear interest rate products
Futures: pricing table [5]

	Futures pricing formulas
Classical (single-curve)	$\mathbf{Futures}(t;\mathbf{T}) = N\left[1 - R^{\text{Fut}}(t;\mathbf{T})\right],$ $R^{\text{Fut}}(t;\mathbf{T}) := \mathbb{E}_t^Q\left[L(T_{i-1},T_i)\right] = F_i(t) + C^{\text{Fut}}(t,T_{i-1}).$
Modern (multi-curve)	$\mathbf{Futures}(t;\mathbf{T}) = N\left[1 - R_x^{\text{Fut}}(t;\mathbf{T})\right],$ $R_x^{\text{Fut}}(t;\mathbf{T}) := \mathbb{E}_t^Q\left[L_x(T_{i-1},T_i)\right] = F_{x,i}(t) + C_x^{\text{Fut}}(t,T_{i-1}).$

Futures: market quotes [6]

0#FEI: LIFFE EURIBOR LIF/LIF EUR

Mth	Last	Net.Ch	Bid	Ask	Bid/Asksize	Settle	Open	High	Low	Volume	Op.Int	Time
JAN2	↓98.745	-0.005	98.740	98.745	62x85	98.750	98.750	98.750	98.730	9877	48513	14:01
FEB2	↓98.835	-0.025	I98.775	99.040	62x50	98.860	98.835	98.835	98.835	25		11:04
MAR2	S↓98.910	-0.005	c98.910	c98.915	1603x1910	98.915	98.915	98.925	98.900	25561	493029	14:03
APR2					x	98.905						:
MAY2					x	98.905						:
JUN2	↓99.060	+0.005	99.060	c99.065	21x6276	99.055	99.050	99.070	99.045	25517	405066	14:04
SEP2	S↓99.100		c99.100	c99.105	857x2134	99.100	99.090	99.115	99.090	30125	406357	14:04
DEC2	S↓99.080		c99.080	c99.085	1278x2837	99.080	99.070	99.095	99.070	11328	333524	14:04
MAR3	↓99.055	+0.005	c99.050	99.055	2658x369	99.050	99.045	99.065	99.035	9120	323774	14:04
JUN3	↓98.995	+0.010	c98.990	98.995	1330x351	98.985	98.985	99.005	98.975	8852	247782	14:05
SEP3	↓98.925	+0.010	c98.920	98.925	964x297	98.915	98.915	98.940	98.905	5163	222578	14:05
DEC3	↓98.825	+0.010	98.825	c98.830	131x2264	98.815	98.815	98.840	98.805	3467	167857	14:05
MAR4	S↓98.725	+0.015	c98.720	c98.725	310x400	98.710	98.710	98.735	98.700	1971	121954	14:02
JUN4	↓98.600	+0.025	c98.590	c98.595	243x111	98.575	98.595	98.605	98.570	2107	73221	13:42
SEP4	↓98.455	+0.025	c98.445	c98.455	579x391	98.430	98.420	98.460	98.420	1729	64464	13:46
DEC4	S↑98.300	+0.030	c98.285	I98.290	100x4	98.270	98.260	98.300	98.260	3161	33348	13:42
MAR5	S↑98.160	+0.030	c98.145	c98.160	219x335	98.130	98.135	98.160	98.135	81	17550	11:29
JUN5	S↑97.980	+0.010	97.990	c98.005	131x55	97.970	97.980	97.980	97.980	50	7349	08:49
SEP5	S↓97.835	+0.015	c97.835	c97.855	130x138	97.820	97.825	97.850	97.825	437	7287	13:17
DEC5	S↓97.680	+0.010	c97.685	c97.710	170x178	97.670	97.700	97.700	97.680	70	4531	13:17
MAR6	S↑97.560		97.570	I97.590	25x10	97.560	97.560	97.560	97.560	30	2013	13:17
JUN6			97.445	c97.480	25x50	97.450					4851	17:14
SEP6			97.325	c97.365	25x45	97.335					1953	10:30
DEC6			97.215	97.255	25x25	97.215					343	18:33
MAR7					x	97.435						:
JUN7					x	97.435						:
SEP7					x	97.435						:
DEC7					x	97.435						:

5: Linear interest rate products

Futures: market quotes [7]

0#FEI: LIFFE EURIBOR LIF/IEU EUR

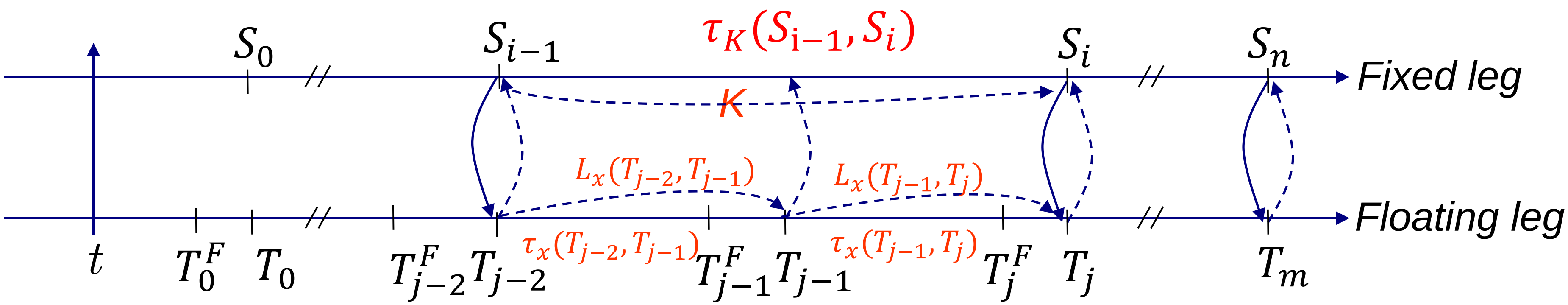
Mth		Last	Net.Ch	Bid	Ask	Bid/Asksize	Settle	Open	High	Low	Volume	Op.Int	Time
FEB5	b	99.940	-0.005	99.940	c99.945	100x668	99.945	99.945	99.950	99.940	3302	21425	17:08
MAR5		99.930	-0.015	c99.930	c99.935	11186x5606	99.945	99.935	99.945	99.930	74491	423669	17:17
APR5	s	99.945	-0.010	i99.935	99.945	3x356	99.955	99.950	99.950	99.945	500	757	17:09
MAY5	s	99.950	-0.005	99.935	i99.950	200x152	99.955	99.955	99.960	99.950	501	0	17:14
JUN5	s	↓99.940	-0.025	c99.940	99.945	32256x939	99.965	99.960	99.965	99.940	67266	325564	17:17
JUL5						0x0	99.965				0	0	:
SEP5		99.950	-0.020	c99.950	c99.955	8894x5456	99.970	99.970	99.975	99.950	75434	355202	17:17
DEC5	s	99.950	-0.020	c99.950	c99.955	5332x11524	99.970	99.970	99.980	99.950	36325	267475	17:17
MAR6		99.945	-0.020	99.945	c99.950	828x30480	99.965	99.965	99.970	99.945	38881	228914	17:17
JUN6		↓99.930	-0.020	c99.930	c99.935	2215x2606	99.950	99.950	99.960	99.930	36979	207720	17:17
SEP6		99.915	-0.015	c99.910	c99.915	6333x980	99.930	99.930	99.940	99.905	29657	182033	17:17
DEC6	s	99.890	-0.015	c99.885	c99.890	8126x359	99.905	99.905	99.910	99.885	20033	189236	17:17
MAR7	s	99.865	-0.010	c99.860	99.865	5542x33	99.875	99.875	99.885	99.860	28156	143624	17:17
JUN7	s	99.835	-0.005	c99.830	c99.835	705x109	99.840	99.840	99.850	99.825	21880	150198	17:17
SEP7	s	↓99.800	-0.005	99.800	c99.805	4x1167	99.805	99.800	99.810	99.790	18388	131362	17:17
DEC7	s	99.765	0	c99.760	c99.765	255x69	99.765	99.765	99.770	99.750	16558	136632	17:17
MAR8	b	99.730	+0.010	c99.725	c99.730	10x138	99.720	99.710	99.735	99.705	9886	62439	17:17
JUN8		99.690	+0.020	c99.685	c99.690	44x14	99.670	99.660	99.695	99.655	8515	31941	17:17
SEP8	s	99.650	+0.025	c99.645	c99.655	151x203	99.625	99.615	99.655	99.610	6342	17807	17:16
DEC8	b	99.610	+0.030	99.605	i99.610	18x22	99.580	99.580	99.610	99.565	5728	17878	17:16
MAR9		99.560	+0.030	i99.555	c99.565	18x7	99.530	99.530	99.560	99.515	406	6531	17:15
JUN9	b	99.510	+0.035	i99.505	i99.520	18x54	99.475	99.470	99.510	99.470	677	2990	17:15
SEP9	b	↑99.455	+0.035	i99.445	i99.465	18x37	99.420	99.455	99.455	99.455	96	716	17:15
DEC9		99.385	+0.025	i99.385	i99.400	18x30	99.360	99.385	99.385	99.380	56	1013	17:15
MAR0		99.330	+0.020	i99.325	99.340	18x30	99.310	99.330	99.330	99.330	5	293	17:15
JUN0	s	99.275	+0.015	i99.270	99.290	18x30	99.260	99.275	99.275	99.275	5	252	17:15
SEP0				i99.215	i99.245	18x30	99.210				0	50	:
DEC0				i99.150	99.195	18x20	99.155				0	85	:

5: Linear interest rate products
Swap [1]

Interest rate swaps are OTC contracts in which two counterparties agree to exchange two streams of cash flows, typically tied to a fixed rate *K* against floating rate. These payment streams are called fixed and floating leg of the swap, respectively, and they are characterized by two schedules ***S***, ***T*** and coupon payoffs

$$\mathbf{S} = \{S_0, \ldots, S_n\}, \text{ fixed leg schedule,}$$
$$\mathbf{T} = \{T_0, \ldots, T_m\}, \text{ floating leg schedule,}$$
$$S_0 = T_0, S_n = T_m,$$
$$\mathbf{Swaplet}_{\text{fix}}(S_i; S_{i-1}, S_i, K) = NK\tau_K(S_{i-1}, S_i),$$
$$\mathbf{Swaplet}_{\text{float}}(T_j; T_{j-1}, T_j) = NL_x(T_{j-1}, T_j)\tau_x(T_{j-1}, T_j),$$

Where τ_K and τ_x are the year fractions with the fixed and floating rate conventions.



Swap [2]

The fixed vs floating interest rate swap coupon payoffs are

$$\begin{aligned}\mathbf{Swaplet}_{\text{fix}}(S_i;S_{i-1},S_i,K) &= NK\tau_K(S_{i-1},S_i),\ i=1,...,n,\\ \mathbf{Swaplet}_{\text{float}}(T_j;T_{j-1},T_j) &= NL_x(T_{j-1},T_j)\tau_x(T_{j-1},T_j),\ j=1,...,m.\end{aligned}$$

The coupon prices at time $t<Max(S_i,T_j)$ are given by

$$\begin{aligned}\mathbf{Swaplet}_{\text{fix}}(t;S_{i-1},S_i,K) &= P_d(t;S_i)\mathbb{E}_t^{Q^{S_i}}\left[\mathbf{Swaplet}_{\text{fix}}(S_i;S_{i-1},S_i,K)\right]\\ &= NP_d(t;S_i)K\tau_K(S_{i-1},S_i),\\ \mathbf{Swaplet}_{\text{float}}(t;T_{j-1},T_j) &= P_d(t;T_j)\mathbb{E}_t^{Q^{T_j}}\left[\mathbf{Swaplet}_{\text{float}}(T_j;T_{j-1},T_j)\right]\\ &= NP_d(t;T_j)\mathbb{E}_t^{Q^{T_j}}\left[L_x(T_{j-1},T_j)\right]\tau_x(T_{j-1},T_j)\\ &= NP_d(t;T_j)F_{x,j}(t)\tau_x(T_{j-1},T_j).\end{aligned}$$

5: Linear interest rate products
Swap [3]

The price of the fixed and floating swap legs is given, at time $t<T_0$, by

$$\mathbf{Swap}_{\text{fix}}(t;\mathbf{S},K) = \sum_{i=1}^{n} \mathbf{Swaplet}_{\text{fix}}(t;S_{i-1},S_i,K) = K\,A_d(t;\mathbf{S}),$$

$$\mathbf{Swap}_{\text{float}}(t;\mathbf{T}) = \sum_{j=1}^{m} \mathbf{Swaplet}_{\text{float}}(t;T_{j-1},T_j)$$

$$= \sum_{j=1}^{m} P_d(t;T_j)F_{x,j}(t)\tau_x(T_{j-1},T_j),$$

where we have defined the swap annuity

$$A_d(t;\mathbf{S}) = \sum_{i=1}^{n} P_d(t,S_i)\tau_K(S_{i-1},S_i).$$

The index “d” reminds that the annuity is linked to the discount rate.

5: Linear interest rate products
Swap [4]

The total swap price is given, at time $t<T_0$, by

$$\mathbf{Swap}(t;\mathbf{T},\mathbf{S},K,\omega) = \omega\left[\mathbf{Swap}_{\text{float}}(t;\mathbf{T}) - \mathbf{Swap}_{\text{fix}}(t;\mathbf{S},K)\right]$$

$$= N\omega\left[\sum_{j=1}^{m} P_d(t;T_j)F_{x,j}(t)\tau_x(T_{j-1},T_j) - K A_d(t;\mathbf{S})\right],$$

where ω = +/-1 for a payer/receiver swap (referred to the fixed leg).
The swap rate at time t is

$$R_x^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) = \frac{\mathbf{Swap}_{\text{float}}(t;\mathbf{T})}{N\omega A_d(t,\mathbf{S})}$$

$$= \frac{\sum_{j=1}^{m} P_d(t;T_j)F_{x,j}(t)\tau_x(T_{j-1},T_j)}{A_d(t;\mathbf{S})}.$$

Hence the swap price can be written in terms of the swap rate as

$$\boxed{\mathbf{Swap}(t;\mathbf{T},\mathbf{S},K,\omega) = N\omega\left[R_x^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) - K\right]A_d(t;\mathbf{S}).}$$

General IRS

IRS traded on the market may have coupon-dependent characteristics, for example:

- variable nominal, e.g. amortized (i.e. decreasing), accreting (i.e. increasing), roller coaster (i.e. decreasing and increasing), etc.,
- variable fixed rate,
- floating rate with margin (spread).

A straightforward generalization of the previous formulas leads to the following expression,

$$\mathbf{Swap}(t;\mathbf{T},\mathbf{S},\mathbf{K},\mathbf{N},\omega) = \omega\left\{\sum_{j=1}^{m} N_j^{Float} P_d(t;T_j)\left[F_{x,j}(t)+\Delta_j\right]\tau_x(T_{j-1},T_j)\right.$$
$$\left. -\sum_{i=1}^{n} N_i^{Fix} P_d(t;S_i) K_i \tau_K(S_{i-1},S_i)\right\}.$$

The most general IRS may be characterized by:

- more than two legs,
- different phases with different floating/fixed rates and margins.

IRS and FRA

- IRS as FRA portfolio

  An IRS can be described as a portfolio of standard FRAs, but with some attention to the different floating vs fixed frequencies

$$\mathbf{Swap}(t;\mathbf{T},\mathbf{S},\mathbf{K},N,\omega) = \omega N \left\{ \sum_{j=1}^{m} P_d(t;T_j) F_{x,j}(t) \tau_x(T_{j-1},T_j) - \sum_{i=1}^{n} P_d(t;S_i) K_i \tau_K(S_{i-1},S_i) \right\} = \sum_{j=1}^{m} \mathbf{FRA}_{\mathrm{Std}}(t;T_{j-1},T_j,K_j,N,\omega)$$

  where

$$\mathbf{FRA}_{\mathrm{Std}}(t;T_{j-1},T_j,K_j,N,\omega) = \omega N \left[P_d(t;T_j) F_{x,j}(t) \tau_x(T_{j-1},T_j) - P_d(t;T_j) K_j \tau_K(T_{j-1},T_j) \delta_{K_j,K_i} \delta_{T_{j-1},S_{i-1}} \delta_{T_j,S_i} \right],$$

- FRA as special IRS

  From the formula above we find that a FRA is a special IRS with $m = 1$.

Swap: single curve limit [6]

In the classical, single-curve limit, with vanishing interest rate basis, we have

$$\mathbf{Swap}_{\text{fix}}(t;\mathbf{S},K) \longrightarrow NKA(t,\mathbf{S}),$$

$$\begin{aligned}\mathbf{Swap}_{\text{float}}(t;\mathbf{T}) \longrightarrow\ & N\sum_{j=1}^{m} P(t,T_j)F(t;T_{j-1},T_j)\tau_L(T_{j-1},T_j)\\ &\simeq N\sum_{j=1}^{m}\left[P(t,T_{j-1})-P(t,T_j)\right] = N\left[P(t,T_0)-P(t,T_m)\right],\end{aligned}$$

where we have used, in the last line, the single-curve expression of the forward rate and the telescopic property of the summation. The latter does hold exactly only if the floating leg schedule is regular (the periods do concatenate exactly with no gaps or overlappings). In practice the error is very small (of the order of 0.1 basis points).

The swap price and swap rate are given by

$$\mathbf{Swap}(t,\mathbf{T},\mathbf{S},K,\omega) \simeq N\omega\left[P(t,T_0)-P(t,T_m)-KA(t,\mathbf{S})\right],$$

$$R^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) \simeq \frac{P(t,T_0)-P(t,T_m)}{A(t,\mathbf{S})}.$$

5: Linear interest rate products

Swap: pricing table [7]

	Swap pricing formulas
Classical (single-curve)	$\mathbf{Swap}(t;\mathbf{T},\mathbf{S},K,\omega) = N\omega\left[R^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) - K\right] A(t,\mathbf{S}),$ $R^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) \simeq \dfrac{P(t,T_0) - P(t,T_m)}{A(t,\mathbf{S})}.$
Modern (multi-curve)	$\mathbf{Swap}(t;\mathbf{T},\mathbf{S},K,\omega) = N\omega\left[R_x^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) - K\right] A_d(t,\mathbf{S}),$ $R_x^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) = \dfrac{\sum_{j=1}^{m} P_d(t,T_j) F_{x,j}(t) \tau_x(T_{j-1},T_j)}{A_d(t,\mathbf{S})}.$

We stress that market spot-starting swaps are worth zero. The two swap legs thus must have the same value. They are not required to be, singularly, worth par (when a fictitious exchange of notionals is introduced at maturity).

5: Linear interest rate products

Swap: market quotes [8]

12:59 30DEC11 ICAP UK69580 ICAPEURO

Euribor vs 6 mth — 3/6 basis — Spot Starting Date

1 Yr	1.442-1.402	16Yrs	2.717-2.677	1 Yr	+33.5		
2 Yrs	1.330-1.290	17Yrs	2.727-2.687	2 Yrs	+26.4		
3 Yrs	1.400-1.360	18Yrs	2.729-2.689	3 Yrs	+22.4		
4 Yrs	1.565-1.525	19Yrs	2.725-2.685	4 Yrs	+19.8		
5 Yrs	1.756-1.716	20Yrs	2.717-2.677	5 Yrs	+17.9		
6 Yrs	1.941-1.901			6 Yrs	+16.3		
7 Yrs	2.096-2.056	21Yrs	2.707-2.667	7 Yrs	+15.0		
8 Yrs	2.220-2.180	22Yrs	2.696-2.656	8 Yrs	+13.9		
9 Yrs	2.324-2.284	23Yrs	2.683-2.643	9 Yrs	+12.9		
10Yrs	2.414-2.374	24Yrs	2.669-2.629	10Yrs	+12.1		
		25Yrs	2.654-2.614				
11Yrs	2.495-2.455						
12Yrs	2.566-2.526	26Yrs	2.639-2.599			10X12	0.172/0.132
13Yrs	2.624-2.584	27Yrs	2.624-2.584			10X15	0.304/0.264
14Yrs	2.667-2.627	28Yrs	2.610-2.570			10X20	0.323/0.283
15Yrs	2.698-2.658	29Yrs	2.597-2.557			10X25	0.260/0.220
		30Yrs	2.587-2.547			10X30	0.193/0.153
		35Yrs	2.571-2.531			10X35	0.177/0.137
		40Yrs	2.578-2.538			10X40	0.184/0.144
		50Yrs	2.590-2.550			10X50	0.196/0.156
		60Yrs	2.596-2.556			10X60	0.202/0.162

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

12:57 30DEC11 ICAP UK69580 ICAPEURO2

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	Eonia IRS ACT/360	Euro Swap vs 3M Euribor 30/360	Euro Swap vs 1M Euribor 30/360
1YR	0.412-0.342	1.112-1.062	0.824-0.754
2YR	0.487-0.417	1.071-1.021	0.824-0.754
3YR	0.668-0.598	1.181-1.131	0.969-0.899
4YR	0.902-0.832	1.372-1.322	1.189-1.119
5YR	1.143-1.073	1.582-1.532	1.422-1.352
6YR	1.370-1.300	1.783-1.733	1.640-1.570
7YR	1.560-1.490	1.951-1.901	1.821-1.751
8YR	1.713-1.643	2.086-2.036	1.965-1.895
9YR	1.843-1.773	2.200-2.150	2.086-2.016
10YR	1.956-1.886	2.298-2.248	2.190-2.120
11YR	2.057-1.987	2.386-2.336	2.283-2.213
12YR	2.146-2.076	2.465-2.415	2.366-2.296
15YR	2.318-2.248	2.610-2.560	2.520-2.450
20YR	2.384-2.314	2.646-2.596	2.564-2.494
25YR	2.349-2.279	2.592-2.542	2.514-2.444
30YR	2.302-2.232	2.531-2.481	2.458-2.388

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

12:57 30DEC11 ICAP UK69580 ICAP14

Euro Forwards

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Maturity

		1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
	1Y	1.198	1.359	1.587	1.817	2.025	2.190	2.318	2.423	2.512	2.591
	2Y	1.521	1.785	2.029	2.239	2.399	2.517	2.611	2.691	2.763	2.825
F	3Y	2.055	2.292	2.490	2.632	2.731	2.809	2.876	2.937	2.989	3.029
o	4Y	2.534	2.715	2.833	2.910	2.971	3.025	3.076	3.120	3.152	3.168
r	5Y	2.901	2.988	3.042	3.087	3.131	3.175	3.214	3.240	3.249	3.246
w	6Y	3.079	3.116	3.153	3.193	3.235	3.272	3.294	3.298	3.291	3.271
a	7Y	3.153	3.192	3.233	3.277	3.314	3.334	3.334	3.321	3.296	3.263
r	8Y	3.232	3.275	3.321	3.357	3.374	3.367	3.348	3.316	3.277	3.231
d	9Y	3.320	3.368	3.401	3.412	3.397	3.370	3.330	3.283	3.231	3.176
	10Y	3.417	3.444	3.444	3.417	3.381	3.332	3.277	3.218	3.158	3.099
	11Y	3.472	3.459	3.418	3.371	3.313	3.251	3.186	3.121	3.058	3.001
	12Y	3.445	3.389	3.336	3.270	3.203	3.133	3.064	2.999	2.941	2.889
	13Y	3.331	3.278	3.208	3.137	3.065	2.994	2.928	2.869	2.818	2.769
	14Y	3.223	3.143	3.068	2.993	2.920	2.853	2.795	2.746	2.698	2.653
	15Y	3.060	2.988	2.912	2.839	2.773	2.717	2.670	2.624	2.581	2.541
	20Y	2.416	2.389	2.350	2.314	2.280	2.249	2.223	2.202	2.187	2.183
	25Y	2.089	2.069	2.063	2.060	2.076	2.103	2.134	2.167	2.200	2.231
	30Y	2.245	2.290	2.331	2.369	2.403	2.433	2.458	2.480	2.499	2.514

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

12:57 30DEC11 ICAP UK69580 ICAP15

Euro Forwards

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Maturity

		11Y	12Y	13Y	14Y	15Y	20Y	25Y	30Y
	1Y	2.660	2.716	2.755	2.783	2.798	2.767	2.683	2.613
	2Y	2.873	2.905	2.926	2.933	2.932	2.852	2.746	2.675
F	3Y	3.052	3.064	3.064	3.055	3.039	2.918	2.793	2.725
O	4Y	3.172	3.164	3.148	3.125	3.097	2.947	2.813	2.753
R	5Y	3.232	3.210	3.180	3.147	3.111	2.943	2.808	2.759
W	6Y	3.243	3.209	3.170	3.130	3.091	2.912	2.786	2.748
A	7Y	3.223	3.179	3.135	3.092	3.052	2.867	2.756	2.729
R	8Y	3.182	3.133	3.087	3.043	3.000	2.815	2.724	2.706
D	9Y	3.122	3.072	3.025	2.980	2.936	2.757	2.688	2.679
	10Y	3.045	2.996	2.948	2.903	2.859	2.695	2.649	2.648
	11Y	2.950	2.901	2.855	2.810	2.770	2.630	2.605	2.611
	12Y	2.839	2.793	2.749	2.708	2.671	2.563	2.558	2.571
	13Y	2.723	2.680	2.640	2.604	2.572	2.501	2.513	2.531
	14Y	2.611	2.572	2.537	2.507	2.481	2.447	2.474	2.496
	15Y	2.504	2.470	2.442	2.418	2.402	2.402	2.440	2.465
	20Y	2.188	2.199	2.214	2.231	2.249	2.331	2.381	2.416
	25Y	2.260	2.287	2.311	2.333	2.352	2.413	2.454	2.478
	30Y	2.525	2.534	2.542	2.548	2.554	2.578	2.591	2.594

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

5: Linear interest rate products

Swap: market quotes [9]

16:15 30JAN15 ICAP UK69580 ICAPEURO

Euribor vs 6 mth — 3/6 basis — Spot Starting Date

1 Yr	0.157-0.117	16Yrs	1.001-0.961	1 Yr	8.45
2 Yrs	0.170-0.130	17Yrs	1.034-0.994	2 Yrs	8.70
3 Yrs	0.208-0.168	18Yrs	1.062-1.022	3 Yrs	9.05
4 Yrs	0.256-0.216	19Yrs	1.087-1.047	4 Yrs	9.15
5 Yrs	0.316-0.276	20Yrs	1.109-1.069	5 Yrs	9.30
6 Yrs	0.389-0.349			6 Yrs	9.35
7 Yrs	0.468-0.428	21Yrs	1.128-1.088	7 Yrs	9.25
8 Yrs	0.548-0.508	22Yrs	1.144-1.104	8 Yrs	9.15
9 Yrs	0.627-0.587	23Yrs	1.158-1.118	9 Yrs	9.05
10Yrs	0.700-0.660	24Yrs	1.170-1.130	10Yrs	8.90
		25Yrs	1.181-1.141		
11Yrs	0.766-0.726				
12Yrs	0.825-0.785	26Yrs	1.191-1.151	10X12	0.145/0.105
13Yrs	0.879-0.839	27Yrs	1.199-1.159	10X15	0.285/0.245
14Yrs	0.926-0.886	28Yrs	1.207-1.167	10X20	0.429/0.389
15Yrs	0.965-0.925	29Yrs	1.214-1.174	10X25	0.501/0.461
		30Yrs	1.220-1.180	10X30	0.540/0.500
		35Yrs	1.243-1.203	10X35	0.563/0.523
		40Yrs	1.252-1.212	10X40	0.572/0.532
		50Yrs	1.212-1.172	10X50	0.532/0.492
		60Yrs	1.197-1.157	10X60	0.517/0.477

Disclaimer <IDIS> Page live in London hours ONLY (between 0700 - 1800)

16:17 30JAN15 ICAP UK69580 ICAPEURO2

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	Eonia IRS ACT/360	Euro Swap vs 3M Euribor 30/360	Euro Swap vs 1M Euribor 30/360
1YR	-0.054/-0.124	0.0780/0.0280	0.0090/-0.061
2YR	-0.057/-0.127	0.0880/0.0380	0.0080/-0.062
3YR	-0.034/-0.104	0.1210/0.0710	0.0320/-0.038
4YR	0.0040/-0.066	0.1710/0.1210	0.0700/
5YR	0.0540/-0.016	0.2310/0.1810	0.1230/0.0530
6YR	0.1220/0.0520	0.3040/0.2540	0.1910/0.1210
7YR	0.1990/0.1290	0.3840/0.3340	0.2690/0.1990
8YR	0.2780/0.2080	0.4650/0.4150	0.3480/0.2780
9YR	0.3560/0.2860	0.5450/0.4950	0.4280/0.3580
10YR	0.4290/0.3590	0.6190/0.5690	0.5020/0.4320
11YR	0.4980/0.4280	0.6880/0.6380	0.5710/0.5010
12YR	0.5600/0.4900	0.7510/0.7010	0.6330/0.5630
15YR	0.7110/0.6410	0.9000/0.8500	0.7850/0.7150
20YR	0.8800/0.8100	1.0560/1.0060	0.9490/0.8790
25YR	0.9750/0.9050	1.1360/1.0860	1.0370/0.9670
30YR	1.0300/0.9600	1.1800/1.1300	1.0880/1.0180
40YR	1.0860/1.0160	1.2190/1.1690	1.1390/1.0690
50YR	1.0630/0.9930	1.1830/1.1330	1.1130/1.0430
60YR	1.0610/0.9910	1.1710/1.1210	1.1090/1.0390

16:17 30JAN15 ICAP UK69580 ICAP14

Euro Forwards
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Maturity

Forward	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
1Y	0.163	0.210	0.269	0.339	0.419	0.503	0.588	0.670	0.744	0.811
2Y	0.258	0.322	0.398	0.483	0.572	0.659	0.743	0.818	0.885	0.943
3Y	0.387	0.470	0.559	0.652	0.740	0.825	0.900	0.965	1.021	1.072
4Y	0.553	0.646	0.740	0.830	0.914	0.987	1.050	1.103	1.150	1.188
5Y	0.739	0.835	0.923	1.005	1.076	1.134	1.184	1.228	1.261	1.288
6Y	0.931	1.016	1.095	1.161	1.215	1.260	1.300	1.330	1.352	1.373
7Y	1.101	1.178	1.240	1.288	1.328	1.364	1.389	1.407	1.424	1.439
8Y	1.255	1.310	1.352	1.386	1.418	1.439	1.453	1.467	1.479	1.489
9Y	1.367	1.402	1.431	1.461	1.477	1.487	1.499	1.508	1.517	1.522
10Y	1.437	1.464	1.492	1.505	1.512	1.522	1.530	1.537	1.541	1.543
11Y	1.491	1.520	1.529	1.531	1.539	1.546	1.552	1.554	1.555	1.554
12Y	1.550	1.548	1.545	1.552	1.557	1.563	1.564	1.564	1.562	1.557
13Y	1.545	1.542	1.552	1.559	1.565	1.566	1.566	1.563	1.558	1.552
14Y	1.539	1.556	1.564	1.570	1.571	1.569	1.566	1.560	1.553	1.546
15Y	1.574	1.576	1.581	1.579	1.576	1.571	1.563	1.555	1.546	1.540
20Y	1.544	1.529	1.518	1.507	1.501	1.496	1.487	1.478	1.473	1.468
25Y	1.469	1.449	1.438	1.437	1.432	1.429	1.426	1.423	1.420	1.416
30Y	1.412	1.410	1.407	1.404	1.398	1.392	1.384	1.376	1.367	1.358

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

16:17 30JAN15 ICAP UK69580 ICAP15

Euro Forwards
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Maturity

FORWARD	11Y	12Y	13Y	14Y	15Y	20Y	25Y	30Y
1Y	0.870	0.924	0.969	1.007	1.042	1.162	1.219	1.247
2Y	0.995	1.039	1.074	1.107	1.136	1.232	1.273	1.293
3Y	1.113	1.146	1.176	1.203	1.227	1.297	1.324	1.337
4Y	1.218	1.246	1.269	1.290	1.308	1.355	1.371	1.376
5Y	1.312	1.333	1.351	1.366	1.377	1.405	1.410	1.408
6Y	1.390	1.406	1.417	1.427	1.434	1.446	1.441	1.434
7Y	1.452	1.461	1.468	1.473	1.476	1.474	1.464	1.451
8Y	1.496	1.501	1.504	1.505	1.504	1.493	1.479	1.462
9Y	1.526	1.527	1.526	1.524	1.521	1.504	1.487	1.466
10Y	1.543	1.540	1.537	1.533	1.530	1.508	1.489	1.464
11Y	1.551	1.546	1.541	1.537	1.533	1.508	1.487	1.459
12Y	1.552	1.546	1.541	1.536	1.530	1.504	1.481	1.451
13Y	1.546	1.540	1.535	1.528	1.521	1.497	1.473	1.438
14Y	1.539	1.534	1.527	1.519	1.514	1.490	1.463	1.424
15Y	1.533	1.526	1.518	1.512	1.506	1.482	1.453	1.407
20Y	1.463	1.459	1.455	1.451	1.446	1.416	1.366	1.286
25Y	1.411	1.405	1.399	1.392	1.384	1.326	1.234	1.202
30Y	1.348	1.336	1.322	1.306	1.286	1.177	1.147	1.145

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

5: Linear interest rate products
Swap: real-life example [10]

Interest Rate Swap Term

BLACKROCK
DBS
Trade No. 17955, Vs 1

Sep 9, 2009 15:38:22

TID 001695200942100

NEW TRADE

BUY

5 YR SWAP

Termination Date	9/16/2014
Contract Type	Interest Rate Swap
Notional Amount (see Face Amount)	4,500,000.000
Business Days	NY_Bank and LN
Business Day Convention	Mod Following
Accrual Dates Adjusted?	Yes

FLOATING

Counterparty A (Iowa Public Employees Retirement Services) Pays	
Coupon	3M LIBOR + 0 BPS
Index Definition	USD-LIBOR-BBA-3MO
Initial Coupon	TBD
Accrual Date	9/16/2009
First Swap Payment Date	12/16/2009
Daycount Basis	QUARTERLY, ACT 360

FIXED

Counterparty B (Deutsche Bank AG Germany) Pays	
Coupon	5.231% FIXED
Accrual Date	9/16/2009
First Swap Paymen Date	3/16/2014
Daycount Basis	SEMI-ANNUALLY, D30/360

Trade purpose	MDF-CHG-RSK-TRG

Example of a typical IRS term sheet.

Forward swap measure

The swap annuity

$$A(t;\mathbf{S}) = \sum_{i=1}^{n} P(t;S_i)\tau_K(S_{i-1},S_i),$$

is such that the quantity

$$R^{\text{Swap}}(t;\mathbf{T},\mathbf{S})A(t;\mathbf{S}) \simeq P(t;T_0) - P(t;T_m),$$

is the price of a tradable asset (portfolio of Zero Coupon Bonds). Thus the value (price) of any other asset Π at any time $t<T\leq S_0$ can be expressed in units of $A(t,\mathbf{S})$ and $A(T,\mathbf{S})$, respectively, through the forward swap pricing formula

$$\boxed{\frac{\Pi(t)}{A(t;\mathbf{S})} = \mathbb{E}_t^{Q_\mathbf{S}}\left[\frac{\Pi(T)}{A(T;\mathbf{S})}\right], \ \forall\, t < T \leq S_0,}$$

where $Q_\mathbf{S}$ denotes the forward swap measure associated to the numeraire $A(t,\mathbf{S})$.

5: Linear interest rate products
Overnight Indexed Swap (OIS) [1]

Overnight interest rate swaps (OIS) are swap contracts with a fixed leg with rate *K* against a floating leg with an overnight floating rate with daily composition. Assuming overlapping fixed and floating leg schedules ***S*** =***T***,

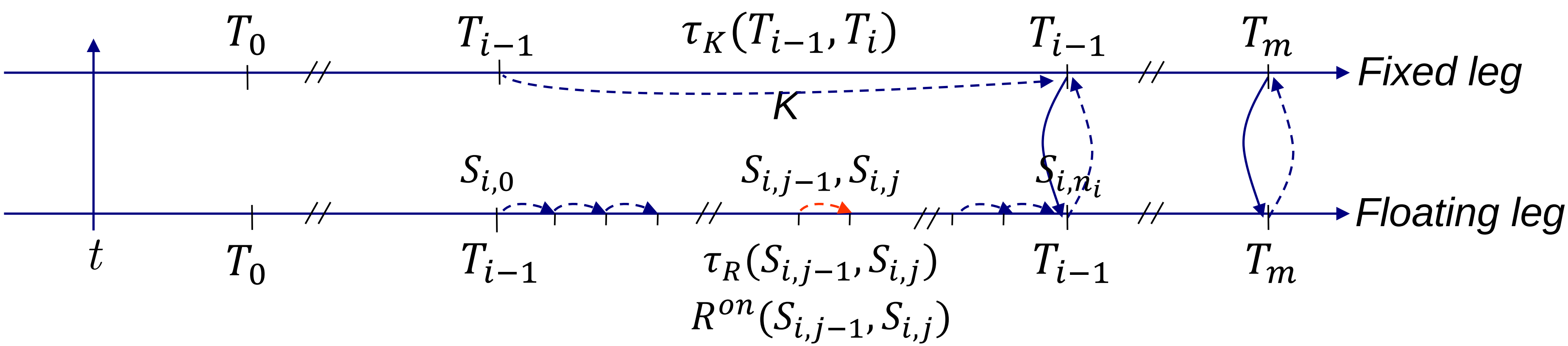
$$\mathbf{T} = \{T_0, ..., T_i, ..., T_m\}\,, \text{ fixed and floating leg payment schedule,}$$

$$\mathbf{S}_i = \{S_{i,0}, ..., S_{i,n_i}\}\,, \text{ floating leg fixing sub-schedules,}$$

$$S_{i,0} = T_{i-1},\ S_{i,n_i} = T_i = S_{i+1,0},$$

$$(T_{i-1}, T_i] = \bigcup_{j=1}^{n_i} (S_{i,j-1}, S_{i,j}]\,,$$

where $\mathbf{S}_i$ is the i-th sub-schedule of the daily compounded overnight rate, including n_i-1 fixing dates for each coupon period $(T_{i-1}; T_i\,]$. Notice the complex nested schedule.



5: Linear interest rate products
Overnight Indexed Swap (OIS) [2]

The OISlet payoffs are given by

$$\mathbf{OISlet}_{\text{fix}}(T_i; T_{i-1}, T_i, K) = NK\tau_K(T_{i-1}, T_i),$$
$$\mathbf{OISlet}_{\text{float}}(T_i; \mathbf{T}_i) = NR^{\text{on}}(T_i; \mathbf{T}_i)\tau_R(T_{i-1}, T_i),$$

where τ_K and τ_R are the year fractions with the fixed and floating rate conventions, and the total overnight rate $R^{on}(T_i;\mathbf{T}_i)$ compounded on the coupon period $(T_{i-1};T_i]$ and observed at time T_i is given by

$$R^{\text{on}}(T_i; \mathbf{T}_i) := \frac{1}{\tau_R(T_{i-1}, T_i)} \left\{ \prod_{j=1}^{n_i} [1 + R^{\text{on}}(S_{i,j-1}, S_{i,j})\tau_R(S_{i,j-1}, S_{i,j})] - 1 \right\}$$

where each single overnight rate $R^{\text{on}}(S_{i,j-1}, S_{i,j})$ covers the overnight time interval $[S_{i,j-1}, S_{i,j}]$.

Overnight Indexed Swap (OIS) [3]

The price of the single overnight indexed swaplet at time $t<T_i$ is given by

$$\mathbf{OISlet}_{\mathrm{float}}(t;\mathbf{T}_i) = NP_d(t;T_i)R^{\mathrm{on}}(t;\mathbf{T}_i)\tau_R(T_{i-1},T_i).$$

where, under the T_i-forward measure,

$$\begin{aligned}
R^{\mathrm{on}}(t;\mathbf{T}_i) &:= \mathbb{E}_t^{Q^{T_i}}\left[R^{\mathrm{on}}(T_i;\mathbf{T}_i)\right] \\
&= \frac{1}{\tau_R(T_{i-1},T_i)}\mathbb{E}_t^{Q^{T_i}}\left\{\prod_{j=1}^{n_i}\left[1+R^{\mathrm{on}}(S_{i,j-1},S_{i,j})\tau_R(S_{i,j-1},S_{i,j})\right]-1\right\} \\
&= \frac{1}{\tau_R(T_{i-1},T_i)}\left\{\prod_{j=1}^{n_i}\left[1+\mathbb{E}_t^{Q^{S_{i,j}}}\left[R^{\mathrm{on}}(S_{i,j-1},S_{i,j})\right]\tau_R(S_{i,j-1},S_{i,j})\right]-1\right\} \\
&= \frac{1}{\tau_R(T_{i-1},T_i)}\left\{\prod_{j=1}^{n_i}\left[1+R^{\mathrm{on}}(t;S_{i,j-1},S_{i,j})\tau_R(S_{i,j-1},S_{i,j})\right]-1\right\},
\end{aligned}$$

$R^{on}(t;S_{i,j-1},S_{i,j})$ is the forward overnight rate observed at time t for the time interval $(S_{i,j-1},S_{i,j}]$, and we have used the tower rule for nested conditioned expectations.

Using the martingale property of forward overnight rates $R^{on}(t;S_{i,j-1},S_{i,j})$ over each time interval $(S_{i,j-1},S_{i,j}]$

$$R^{\text{on}}(t;S_{i,j-1},S_{i,j}) = \mathbb{E}_t^{Q^{S_{i,j}}}\left[R^{\text{on}}(S_{i,j-1},S_{i,j})\right] = \frac{1}{\tau_R(S_{i,j-1},S_{i,j})}\left[\frac{P_d(t;S_{i,j-1})}{P_d(t;S_{i,j})}-1\right]$$

we obtain

$$\begin{aligned}
R^{\text{on}}(t;\mathbf{T}_i) &= \frac{1}{\tau_R(T_{i-1},T_i)}\left\{\prod_{j=1}^{n_i}\left[1+R^{\text{on}}(t;S_{i,j-1},S_{i,j})\tau_R(S_{i,j-1},S_{i,j})\right]-1\right\}\\
&= \frac{1}{\tau_R(T_{i-1},T_i)}\left\{\prod_{j=1}^{n_i}\left[\frac{P_d(t;S_{i,j-1})}{P_d(t;S_{i,j})}\right]-1\right\}\\
&= \frac{1}{\tau_R(T_{i-1},T_i)}\left[\frac{P_d(t;S_{i,0})}{P_d(t;S_{i,1})}\frac{P_d(t;S_{i,1})}{P_d(t;S_{i,2})}\cdots\frac{P_d(t;S_{i,n_i-1})}{P_d(t;S_{i,n_i})}-1\right]\\
&= \frac{1}{\tau_R(T_{i-1},T_i)}\left[\frac{P_d(t;T_{i-1})}{P_d(t;T_i)}-1\right]\\
&= F_d(t;T_{i-1},T_i).
\end{aligned}$$

Thus the price of the single overnight indexed swaplet at time $t<T_i$ is given by

$$
\begin{aligned}
\mathbf{OISlet}_{\text{float}}(t;\mathbf{T}_i) &= N P_d(t;T_i) R^{\text{on}}(t;\mathbf{T}_i)\tau_R(T_{i-1},T_i) \\
&= N P_d(t;T_i) F_d(t;T_{i-1},T_i)\tau_R(T_{i-1},T_i) \\
&= N\left[P_d(t;T_{i-1}) - P_d(t;T_i)\right],
\end{aligned}
$$

and the price of the total OIS is given by

$$
\begin{aligned}
\mathbf{OIS}(t;\mathbf{T},K,\omega) &= \omega\left[\mathbf{OIS}_{\text{float}}(t;\mathbf{T}) - \mathbf{OIS}_{\text{fix}}(t;\mathbf{S},K)\right] \\
&= \omega\sum_{i=1}^{n}\left[\mathbf{OISlet}_{\text{float}}(t;\mathbf{T}_i) - \mathbf{OISlet}_{\text{fix}}(t;T_{i-1},T_i,K)\right] \\
&= N\omega\left[\sum_{i=1}^{n}\left[P_d(t;T_{i-1}) - P_d(t;T_i)\right] - K A_d(t;\mathbf{T})\right] \\
&= N\omega\left[P_d(t;T_0) - P_d(t;T_n) - K A_d(t;\mathbf{T})\right],
\end{aligned}
$$

where $\omega = {}^{+}/_{-}\,1$ for a payer/receiver swap (referred to the fixed leg).

5: Linear interest rate products
Overnight Indexed Swap (OIS) [6]

Another proof of the OISlet pricing formula

$$\mathbf{OISlet}_{\text{float}}(t;\mathbf{T}_i) = N\left[P_d(t;T_{i-1}) - P_d(t;T_i)\right],$$

is the following:

$$\mathbf{OISlet}_{\text{float}}(t;\mathbf{T}_i) = N\mathbb{E}_t^Q\left\{D(t;T_i)\prod_{j=1}^{n_i}\left[1+R^{\text{on}}(S_{i,j-1},S_{i,j})\tau_R(S_{i,j-1},S_{i,j})\right]-1\right\}$$

$$= N\mathbb{E}_t^Q\left\{D(t;S_{i,n_i-1})D(S_{i,n_i-1};S_{i,n_i})\prod_{j=1}^{n_i}\left[1+R^{\text{on}}(S_{i,j-1},S_{i,j})\tau_R(S_{i,j-1},S_{i,j})\right]\right\}-NP_d(t;T_i)$$

$$= N\mathbb{E}_t^Q\left\{D(t;S_{i,n_i-1})\prod_{j=1}^{n_i}\left[1+R^{\text{on}}(S_{i,j-1},S_{i,j})\tau_R(S_{i,j-1},S_{i,j})\right]\mathbb{E}_{S_{i,n_i-1}}^Q\left[D(S_{i,n_i-1},S_{i,n_i})\right]\right\}-NP_d(t;T_i)$$

$$= N\mathbb{E}_t^Q\left\{D(t;S_{i,n_i-1})\prod_{j=1}^{n_i}\left[1+R^{\text{on}}(S_{i,j-1},S_{i,j})\tau_R(S_{i,j-1},S_{i,j})\right]P(S_{i,n_i-1},S_{i,n_i})\right\}-NP_d(t;T_i)$$

$$= N\mathbb{E}_t^Q\left\{D(t;S_{i,n_i-1})\prod_{j=1}^{n_i-1}\left[1+R^{\text{on}}(S_{i,j-1},S_{i,j})\tau_R(S_{i,j-1},S_{i,j})\right]\right\}-NP_d(t;T_i).$$

Definition of discounting bond

Overnight Indexed Swap (OIS) [7]

By recursive application of the previous result, we obtain

$$
\begin{aligned}
&\mathbf{OISlet}_{\text{float}}(t;\mathbf{T}_i)\\
&= N\mathbb{E}_t^Q\left\{D(t;S_{i,n_i-1})\prod_{j=1}^{n_i-1}\left[1+R^{\text{on}}(S_{i,j-1},S_{i,j})\tau_R(S_{i,j-1},S_{i,j})\right]\right\} - NP_d(t;T_i)\\
&= N\mathbb{E}_t^Q\left\{D(t;S_{i,n_i-2})\prod_{j=1}^{n_i-2}\left[1+R^{\text{on}}(S_{i,j-1},S_{i,j})\tau_R(S_{i,j-1},S_{i,j})\right]\right\} - NP_d(t;T_i)\\
&= \cdots\\
&= N\left[P_d(t;T_{i-1}) - P_d(t;T_i)\right].
\end{aligned}
$$

5: Linear interest rate products
Overnight Indexed Swap (OIS) [8]

The OIS rate at time t, denoted by $R^{OIS}(t; \boldsymbol{T})$, is defined as the OIS contract equilibrium rate, that makes null the swap present value,

$$\begin{aligned}
\mathbf{OIS}(t; \mathbf{T}, K, \omega) &= N\omega \left[\sum_{i=1}^{n} P_d(t; T_i) F_d(t; T_{i-1}, T_i) \tau_R(T_{i-1}, T_i) - R_d^{\mathrm{OIS}}(t; \mathbf{T}) A_d(t; \mathbf{T}) \right] \\
&= 0, \\
R_d^{\mathrm{OIS}}(t; \mathbf{T}) &= \frac{\sum_{i=1}^{n} P_d(t; T_i) F_d(t; T_{i-1}, T_i) \tau_R(T_{i-1}, T_i)}{A_d(t; \mathbf{T})} \\
&\simeq \frac{P_d(t; T_0) - P_d(t; T_n)}{A_d(t; \mathbf{T})}.
\end{aligned}$$

Thus the price of the OIS can also be written as

$$\boxed{\mathbf{OIS}(t; \mathbf{T}, K, \omega) = N\omega \left[R_d^{\mathrm{OIS}}(t; \mathbf{T}) - K \right] A_d(t; \mathbf{T}).}$$

5: Linear interest rate products

Overnight Indexed Swap (OIS): pricing table [9]

	OIS pricing formulas
Classical (single-curve)	$$\mathbf{OIS}(t;\mathbf{T},K,\omega) = N\omega\left[R^{\mathrm{OIS}}(t;\mathbf{T}) - K\right]A(t;\mathbf{T}),$$ $$R^{\mathrm{OIS}}(t;\mathbf{T}) = \frac{P(t;T_0) - P(t;T_n)}{A(t;\mathbf{T})}$$
Modern (multi-curve)	$$\mathbf{OIS}(t;\mathbf{T},K,\omega) = N\omega\left[R_d^{\mathrm{OIS}}(t;\mathbf{T}) - K\right]A_d(t;\mathbf{T}),$$ $$R_d^{\mathrm{OIS}}(t;\mathbf{T}) = \frac{P_d(t;T_0) - P_d(t;T_n)}{A_d(t;\mathbf{T})}.$$

5: Linear interest rate products

Overnight Indexed Swap (OIS): market quotes [10]

12:53 30DEC11 ICAP LONDON UK69580 ICAPSHORT1
Contact Reuters EXEU EURO FRAs / OIS +44 (0)20 7532 3530

	Eonia		Fwd EONIA		ECB Dates		Eonia v 3m E'bor A/360
1w	0.434-0.334	1X2	0.417-0.367	JAN	0.430-0.380	1Yr	+071.6/+066.6
2w	0.428-0.328	2X3	0.411-0.361	FEB	0.403-0.353	18M	+064.7/+059.7
3w	0.439-0.339	1x4	0.408-0.358	MAR	0.417-0.367	2Yr	+060.1/+055.1
1m	0.421-0.371	2x5	0.396-0.346	APR	0.388-0.338	3Yr	+053.1/+048.1
2m	0.419-0.369	3x6	0.395-0.345	MAY	0.385-0.335	4Yr	+048.5/+043.5
3m	0.416-0.366	6x12	0.397-0.347	JUN	0.403-0.353	5Yr	+045.1/+040.1
4m	0.411-0.361			JUL	0.379-0.329	6Yr	+042.2/+037.2
5m	0.405-0.355	IMM Fra/Eonia		AUG	0.379-0.329	7Yr	+039.9/+034.9
6m	0.406-0.356			SEP	0.403-0.353	8Yr	+037.8/+032.8
7m	0.401-0.351	MAR	74.200-69.200			9Yr	+036.0/+031.0
8m	0.398-0.348	JUN	60.300-55.300			10Y	+034.4/+029.4
9m	0.399-0.349	SEP	54.400-49.400			11Y	+033.1/+028.1
10m	0.398-0.348	DEC	52.600-47.600			12Y	+031.9/+026.9
11m	0.398-0.348					15Y	+028.9/+023.9
12m	0.402-0.352					20Y	+025.9/+020.9
Two Payments						25Y	+024.1/+019.1
15m	0.408-0.358					30Y	+022.8/+017.8
18m	0.424-0.374						
21m	0.448-0.398						(2 Swaps)
2y	0.477-0.427						
3y	0.658-0.608				ICAP OIS Fix Menu <ICAPOISFIX01>		

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

ICAP UK69580 ICAPEURO2
ils Please Call David Shepherd on +44 (0)207 532 3530

	Eonia IRS ACT/360	Euro Swap vs 3M Euribor 30/360	Euro Swap vs 1M Euribor 30/360
1YR	0.412-0.342	1.112-1.062	0.824-0.754
2YR	0.487-0.417	1.071-1.021	0.824-0.754
3YR	0.668-0.598	1.181-1.131	0.969-0.899
4YR	0.902-0.832	1.372-1.322	1.189-1.119
5YR	1.143-1.073	1.582-1.532	1.422-1.352
6YR	1.370-1.300	1.783-1.733	1.640-1.570
7YR	1.560-1.490	1.951-1.901	1.821-1.751
8YR	1.713-1.643	2.086-2.036	1.965-1.895
9YR	1.843-1.773	2.200-2.150	2.086-2.016
10YR	1.956-1.886	2.298-2.248	2.190-2.120
11YR	2.057-1.987	2.386-2.336	2.283-2.213
12YR	2.146-2.076	2.465-2.415	2.366-2.296
15YR	2.318-2.248	2.610-2.560	2.520-2.450
20YR	2.384-2.314	2.646-2.596	2.564-2.494
25YR	2.349-2.279	2.592-2.542	2.514-2.444
30YR	2.302-2.232	2.531-2.481	2.458-2.388

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

USD OISs pay floating coupons indexed to the average of the over night Fed Funds rates fixed during the coupon period. Thus they are not properly plain vanilla instruments.

See K. Takada (2011)

5: Linear interest rate products

Overnight Indexed Swap (OIS): market quotes [11]

16:15 30JAN15 ICAP LONDON UK69580 ICAPSHORT1
Contact Reuters EXEU EURO FRAs / OIS +44 (0)20 7532 3530

	Eonia		Fwd EONIA		ECB Dates		Eonia v 3m E'bor A/360
1w	0.009/-0.091	1X2	0.0000/-0.050	MAR	-0.0190/-0.0690	1Yr	+016.7/+011.7
2w	0.007/-0.093	2X3	-0.046/-0.096	APR	-0.0580/-0.1080	18M	+017.3/+012.3
3w	0.006/-0.094	1x4	-0.035/-0.085	JUN	-0.0780/-0.1280	2Yr	+017.8/+012.8
1m	-0.007/-0.057	2x5	-0.061/-0.111	JUL	-0.0810/-0.1310	3Yr	+018.9/+013.9
2m	-0.003/-0.053	3x6	-0.071/-0.121	SEP	-0.0860/-0.1360	4Yr	+019.9/+014.9
3m	-0.016/-0.066	6x12	-0.086/-0.136	OCT	-0.0880/-0.1380	5Yr	+020.9/+015.9
4m	-0.028/-0.078			DEC	-0.0900/-0.1400	6Yr	+021.2/+016.2
5m	-0.037/-0.087	IMM Fra/Eonia		JAN	-0.0930/-0.1430	7Yr	+021.4/+016.4
6m	-0.044/-0.094			MAR	-0.0790/-0.1290	8Yr	+021.5/+016.5
7m	-0.050/-0.100	MAR	16.400-11.400			9Yr	+021.6/+016.6
8m	-0.054/-0.104	JUN	18.400-13.400			10Y	+021.6/+016.6
9m	-0.057/-0.107	SEP	18.500-13.500			11Y	+021.5/+016.5
10m	-0.061/-0.111	DEC	18.500-13.500			12Y	+021.5/+016.5
11m	-0.062/-0.112					15Y	+021.1/+016.1
12m	-0.065/-0.115					20Y	+019.6/+014.6
Two Payments						25Y	+018.0/+013.0
15m	-0.069/-0.119					30Y	+016.8/+011.8
18m	-0.071/-0.121					40Y	
21m	-0.071/-0.121					50Y	
2y	-0.068/-0.118					60Y	
3y	-0.043/-0.093						

ICAP OIS Fix Menu <
ICAP Global Index <ICAP> Forthcoming chang

16:17 30JAN15 ICAP UK69580 ICAPEUR02

For Further Details Please Call David Shepherd on +44 (0)207 532 3530

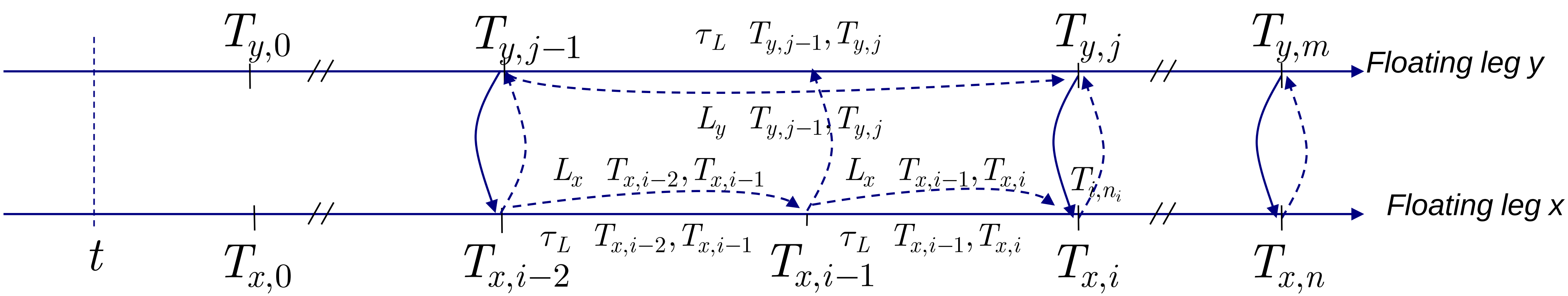
	Eonia IRS ACT/360	Euro Swap vs 3M Euribor 30/360	Euro Swap vs 1M Euribor 30/360
1YR	-0.054/-0.124	0.0780/0.0280	0.0090/-0.061
2YR	-0.057/-0.127	0.0880/0.0380	0.0080/-0.062
3YR	-0.034/-0.104	0.1210/0.0710	0.0320/-0.038
4YR	0.0040/-0.066	0.1710/0.1210	0.0700/
5YR	0.0540/-0.016	0.2310/0.1810	0.1230/0.0530
6YR	0.1220/0.0520	0.3040/0.2540	0.1910/0.1210
7YR	0.1990/0.1290	0.3840/0.3340	0.2690/0.1990
8YR	0.2780/0.2080	0.4650/0.4150	0.3480/0.2780
9YR	0.3560/0.2860	0.5450/0.4950	0.4280/0.3580
10YR	0.4290/0.3590	0.6190/0.5690	0.5020/0.4320
11YR	0.4980/0.4280	0.6880/0.6380	0.5710/0.5010
12YR	0.5600/0.4900	0.7510/0.7010	0.6330/0.5630
15YR	0.7110/0.6410	0.9000/0.8500	0.7850/0.7150
20YR	0.8800/0.8100	1.0560/1.0060	0.9490/0.8790
25YR	0.9750/0.9050	1.1360/1.0860	1.0370/0.9670
30YR	1.0300/0.9600	1.1800/1.1300	1.0880/1.0180
40YR	1.0860/1.0160	1.2190/1.1690	1.1390/1.0690
50YR	1.0630/0.9930	1.1830/1.1330	1.1130/1.0430
60YR	1.0610/0.9910	1.1710/1.1210	1.1090/1.0390

5: Linear interest rate products

Basis Swap [1]

Interest rate basis swaps are OTC contracts in which two counterparties agree to exchange two streams of cash flows tied to two floating rates with different tenor, characterized by two schedules $\boldsymbol{T}_x$, $\boldsymbol{T}_y$ and coupon payoffs

$$
\begin{aligned}
&\mathbf{T}_x = \{T_{x,0}, ..., T_{x,n_x}\}, \text{ floating leg } x \text{ schedule,} \\
&\mathbf{T}_y = \{T_{y,0}, ..., T_{y,n_y}\}, \text{ floating leg } y \text{ schedule,} \\
&T_{x,0} = T_{y,0}, \; T_{x,n_x} = T_{y,n_y}, \\
&\mathbf{Swaplet}_x(T_{x,i}; T_{x,i-1}, T_{x,i}) = N L_x(T_{x,i-1}, T_{x,i}) \tau_x(T_{x,i-1}, T_{x,i}), \\
&\mathbf{Swaplet}_y(T_{y,j}; T_{y,j-1}, T_{y,j}) = N L_y(T_{y,j-1}, T_{y,j}) \tau_y(T_{y,j-1}, T_{y,j}).
\end{aligned}
$$



Basis swaps are defined in two ways:

1. As a floating vs floating swap plus spread (called basis swap spread).

$$
\begin{aligned}
\mathbf{BasisSwap}(t;\boldsymbol{T}_x,\boldsymbol{T}_y,\Delta) &:= \mathbf{Swap}_{\text{Float}}(t;\boldsymbol{T}_x) - \mathbf{Swap}_{\text{Float}}(t;\boldsymbol{T}_y,\Delta) \\
&= N\left[\sum_{i=1}^{n_x} P_d(t;T_{x,i})F_{x,i}(t)\tau_x(T_{x,i-1},T_{x,i})\right. \\
&\quad \left. - \sum_{j=1}^{n_y} P_d(t;T_{y,j})\left[F_{y,j}(t)+\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y)\right]\tau_y(T_{y,j-1},T_{y,j})\right] \\
&= \mathbf{Swap}_{\text{Float}}(t;\boldsymbol{T}_x) - \mathbf{Swap}_{\text{Float}}(t;\boldsymbol{T}_y) - N\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y)A_{d,y}(t;\boldsymbol{T}_y), \\
A_{d,y}(t;\boldsymbol{T}_y) &= \sum_{j=1}^{n_y} P_d(t;T_{y,j})\tau_y(T_{y,j-1},T_{y,j}).
\end{aligned}
$$

Imposing the fair value condition $\mathbf{BasisSwap}(t;\boldsymbol{T}_x,\boldsymbol{T}_y,\Delta)=0,$ we obtain the basis swap spread expression

$$
\Delta := \Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y) = \frac{\mathbf{Swap}_{\text{Float}}(t;\boldsymbol{T}_x,\omega) - \mathbf{Swap}_{\text{Float}}(t;\boldsymbol{T}_y,\omega)}{NA_{d,y}(t;\boldsymbol{T}_y)}.
$$

5: Linear interest rate products
Basis Swap [3]

2. As portfolios of two standard floating vs fixed swaps with two different floating rate tenors and coincident fixed legs. The basis swap spread is defined as the difference between the two equilibrium swap rates:

$$\mathbf{Swap}_x(t;\boldsymbol{T}_x,\boldsymbol{S},K_x,\omega) = N\omega\left[R_x^{\text{Swap}}(t;\boldsymbol{T}_x,\boldsymbol{S}) - K_x\right]A_d(t,\boldsymbol{S}) = 0,$$
$$\mathbf{Swap}_y(t;\boldsymbol{T}_y,\boldsymbol{S},K_y,\omega) = N\omega\left[R_y^{\text{Swap}}(t;\boldsymbol{T}_x,\boldsymbol{S}) - K_y\right]A_d(t,\boldsymbol{S}) = 0,$$

$$\begin{aligned}\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y,\boldsymbol{S}) &:= R_x^{\text{Swap}}(t;\boldsymbol{T}_x,\boldsymbol{S}) - R_y^{\text{Swap}}(t;\boldsymbol{T}_y,\boldsymbol{S}) \\ &= \frac{\sum_{i=1}^{n_x} P_d(t;T_{x,i})F_{x,i}(t)\tau_x(T_{x,i-1},T_{x,i}) - \sum_{j=1}^{n_y} P_d(t;T_{y,j})F_{y,j}(t)\tau_y(T_{y,j-1},T_{y,j})}{A_d(t;\boldsymbol{S})} \\ &= \frac{\mathbf{Swap}_{\text{float}}(t;\boldsymbol{T}_x) - \mathbf{Swap}_{\text{float}}(t;\boldsymbol{T}_y)}{NA_d(t;\boldsymbol{S})}\end{aligned}$$

5: Linear interest rate products
Basis Swap [4]

We notice that:

1. In both definitions the basis swap spread is positive when the fixings of the first floating rate are higher, on average, than the fixings of the second floating rate, such that the basis spread is needed on the second leg to keep the swap at equilibrium. This is precisely what happens in the modern interest rate market with Libor rates with different tenors, e.g. Libor6M vs Libor3M.

2. The two definitions are equivalent but the two basis spreads are slightly different because of the different annuities involved:

$$\Delta(t; \boldsymbol{T}_x, \boldsymbol{T}_y, \boldsymbol{S}) = \frac{A_{d,y}(t; \boldsymbol{T}_y)}{A_d(t; \boldsymbol{S})} \Delta(t; \boldsymbol{T}_x, \boldsymbol{T}_y).$$

3. The second definition is usually preferred because the basis swap rate shares the same rate conventions of the swap rates, while in the first definition it shares the same conventions of the FRA rate.

Basis Swap: pricing table [5]

	Basis Swap pricing formulas
Classical (single-curve)	$\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y,\boldsymbol{S},\omega)=\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y)=0$
Modern (multi-curve)	$\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y,\boldsymbol{S},\omega)=\dfrac{\mathbf{Swap}_{\text{float}}(t;\boldsymbol{T}_x)-\mathbf{Swap}_{\text{float}}(t;\boldsymbol{T}_y)}{N\omega A_d(t;\boldsymbol{S})},$ $\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y)=\dfrac{\mathbf{Swap}_{\text{Float}}(t;\boldsymbol{T}_x,\omega)-\mathbf{Swap}_{\text{Float}}(t;\boldsymbol{T}_y,\omega)}{N\omega A_d(t;\boldsymbol{T}_y)}.$

5: Linear interest rate products

Basis Swap: market quotes [6]

12:57 30DEC11 ICAP UK69580 ICAPEUROBASIS

EUR Basis Swaps (as 2 Swaps)

For Further Details Please Call Jamie Mockett on +44 (0)207 532 3937

These are indicative mids priced out of a spot starting date

All prices are Euribor vs Euribor quoted Bond Basis

	3M vs 6M	1M vs 3M	1M vs 6M	6M vs 12M	3M vs 12M
1YR	33.5	29.7	63.2	55.8	89.4
2YR	26.4	25.7	52.2	43.4	69.8
3YR	22.4	22.2	44.6	35.2	57.5
4YR	19.8	19.2	39.1	30.4	50.2
5YR	17.9	17.0	34.9	27.2	45.1
6YR	16.3	15.3	31.6	24.8	41.1
7YR	15.0	14.0	29.0	22.8	37.8
8YR	13.9	13.1	27.0	21.1	35.0
9YR	12.9	12.4	25.3	19.7	32.6
10YR	12.1	11.8	23.9	18.5	30.6
11YR	11.4	11.3	22.7	17.4	28.8
12YR	10.6	10.9	21.6	16.5	27.2
15YR	9.3	10.0	19.3	14.3	23.6
20YR	7.6	9.2	16.9	12.1	19.8
25YR	6.7	8.8	15.5	10.7	17.5
30YR	6.1	8.4	14.4	9.7	15.8
40YR	5.2				
50YR	4.7				

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

12:53 30DEC11 ICAP LONDON UK69580 ICAPSHORT1

Contact Reuters EXEU EURO FRAs / OIS +44 (0)20 7532 3530

	Eonia		Fwd EONIA		ECB Dates		Eonia v 3m E'bor A/360
1w	0.434-0.334	1X2	0.417-0.367	JAN	0.430-0.380	1Yr	+071.6/+066.6
2w	0.428-0.328	2X3	0.411-0.361	FEB	0.403-0.353	18M	+064.7/+059.7
3w	0.439-0.339	1x4	0.408-0.358	MAR	0.417-0.367	2Yr	+060.1/+055.1
1m	0.421-0.371	2x5	0.396-0.346	APR	0.388-0.338	3Yr	+053.1/+048.1
2m	0.419-0.369	3x6	0.395-0.345	MAY	0.385-0.335	4Yr	+048.5/+043.5
3m	0.416-0.366	6x12	0.397-0.347	JUN	0.403-0.353	5Yr	+045.1/+040.1
4m	0.411-0.361			JUL	0.379-0.329	6Yr	+042.2/+037.2
5m	0.405-0.355	IMM Fra/Eonia		AUG	0.379-0.329	7Yr	+039.9/+034.9
6m	0.406-0.356			SEP	0.403-0.353	8Yr	+037.8/+032.8
7m	0.401-0.351	MAR	74.200-69.200			9Yr	+036.0/+031.0
8m	0.398-0.348	JUN	60.300-55.300			10Y	+034.4/+029.4
9m	0.399-0.349	SEP	54.400-49.400			11Y	+033.1/+028.1
10m	0.398-0.348	DEC	52.600-47.600			12Y	+031.9/+026.9
11m	0.398-0.348					15Y	+028.9/+023.9
12m	0.402-0.352					20Y	+025.9/+020.9
Two Payments						25Y	+024.1/+019.1
15m	0.408-0.358					30Y	+022.8/+017.8
18m	0.424-0.374						
21m	0.448-0.398						(2 Swaps)
2y	0.477-0.427						
3y	0.658-0.608						

ICAP OIS Fix Menu <ICAPOISFIX01>

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

12:58 30DEC11 ICAP UK69580 ICAPEUROBASIS2

EUR Basis Swaps (as 2 Swaps)

3mth Euribor vs 6mth Euribor Basis Swaps with various start dates

For Further Details Please Call Jamie Mockett on +44 (0)207 532 3937

All prices are quoted Bond Basis

	Spot	IMM	1Wk Fwd	2Wk Fwd	3Wk Fwd	1 Mth Fwd	3 Mth Fwd	6 Mth Fwd
1YR	33.5	26.8	32.5	31.5	31.0	30.2	26.3	23.0
2YR	26.4	22.3	25.8	25.3	24.9	24.5	22.0	19.9
3YR	22.4	19.4	22.0	21.6	21.3	21.0	19.1	17.6
4YR	19.8	17.5	19.5	19.2	19.0	18.8	17.3	16.0
5YR	17.9	15.9	17.6	17.4	17.2	17.0	15.7	14.5
6YR	16.3	14.6	16.0	15.8	15.7	15.5	14.4	13.4
7YR	15.0	13.5	14.8	14.6	14.5	14.3	13.3	12.4
8YR	13.9	12.5	13.7	13.5	13.4	13.3	12.4	11.5
9YR	12.9	11.7	12.7	12.6	12.5	12.3	11.5	10.8
10YR	12.1	11.0	11.9	11.8	11.7	11.6	10.8	10.1
11YR	11.4	10.3	11.2	11.1	11.0	10.9	10.1	9.4
12YR	10.6	9.6	10.5	10.3	10.3	10.1	9.5	8.9
15YR	9.3	8.5	9.2	9.1	9.0	8.9	8.4	7.9
20YR	7.6	6.9	7.5	7.4	7.4	7.3	6.9	6.4
25YR	6.7	6.1	6.6	6.6	6.5	6.4	6.1	5.7
30YR	6.1	5.6	6.0	5.9	5.9	5.8	5.5	5.2
40YR	5.2	4.8	5.2	5.1	5.1	5.0	4.8	4.5
50YR	4.7	4.3	4.7	4.6	4.6	4.5	4.3	4.1

ICAP Euro IRS pages close @ 13:00 GMT. Pls refer to LIFFE/EUREX for futures lvls

5: Linear interest rate products

Basis Swap: market quotes [7]

16:15 30JAN15 ICAP UK69580 ICAPEUROBASIS

EUR Basis Swaps (as 2 Swaps)

These are indicative mids priced out of a spot starting date

All prices are Euribor vs Euribor quoted Bond Basis

	3M vs 6M	1M vs 3M	1M vs 6M	6M vs 12M	3M vs 12M
1YR	8.45	7.9	16.3	13.7	22.1
2YR	8.70	9.0	17.7	12.6	21.4
3YR	9.05	10.0	19.0	12.0	21.0
4YR	9.15	11.0	20.2	11.9	21.1
5YR	9.30	11.8	21.1	12.0	21.3
6YR	9.35	12.2	21.6	11.9	21.3
7YR	9.25	12.5	21.7	11.8	21.0
8YR	9.15	12.6	21.8	11.6	20.8
9YR	9.05	12.7	21.8	11.4	20.5
10YR	8.90	12.7	21.6	11.2	20.1
11YR	8.60	12.7	21.3	10.9	19.5
12YR	8.25	12.7	21.0	10.5	18.8
15YR	7.20	12.5	19.7	9.6	16.8
20YR	6.00	11.7	17.7	8.6	14.5
25YR	5.20	10.9	16.1	7.8	13.0
30YR	4.65	10.2	14.8	7.2	11.9
40YR	3.95	9.0	12.9	6.4	10.4
50YR	3.50	8.0	11.5	5.9	9.4
60YR	3.20	7.2	10.4	5.6	8.7

16:15 30JAN15 ICAP LONDON UK69580 ICAPSHORT1

Contact Reuters EXEU EURO FRAs / OIS +44 (0)20 7532 3530

Eonia		Fwd EONIA		ECB Dates		Eonia v 3m E'bor A/360	
1w	0.009/-0.091	1X2	0.0000/-0.050	MAR	-0.0190/-0.0690	1Yr	+016.7/+011.7
2w	0.007/-0.093	2X3	-0.046/-0.096	APR	-0.0580/-0.1080	18M	+017.3/+012.3
3w	0.006/-0.094	1x4	-0.035/-0.085	JUN	-0.0780/-0.1280	2Yr	+017.8/+012.8
1m	-0.007/-0.057	2x5	-0.061/-0.111	JUL	-0.0810/-0.1310	3Yr	+018.9/+013.9
2m	-0.003/-0.053	3x6	-0.071/-0.121	SEP	-0.0860/-0.1360	4Yr	+019.9/+014.9
3m	-0.016/-0.066	6x12	-0.086/-0.136	OCT	-0.0880/-0.1380	5Yr	+020.9/+015.9
4m	-0.028/-0.078			DEC	-0.0900/-0.1400	6Yr	+021.2/+016.2
5m	-0.037/-0.087	IMM Fra/Eonia		JAN	-0.0930/-0.1430	7Yr	+021.4/+016.4
6m	-0.044/-0.094			MAR	-0.0790/-0.1290	8Yr	+021.5/+016.5
7m	-0.050/-0.100	MAR	16.400-11.400			9Yr	+021.6/+016.6
8m	-0.054/-0.104	JUN	18.400-13.400			10Y	+021.6/+016.6
9m	-0.057/-0.107	SEP	18.500-13.500			11Y	+021.5/+016.5
10m	-0.061/-0.111	DEC	18.500-13.500			12Y	+021.5/+016.5
11m	-0.062/-0.112					15Y	+021.1/+016.1
12m	-0.065/-0.115					20Y	+019.6/+014.6
Two Payments						25Y	+018.0/+013.0
15m	-0.069/-0.119					30Y	+016.8/+011.8
18m	-0.071/-0.121					40Y	+015.0/+010.0
21m	-0.071/-0.121					50Y	+013.8/+008.8
2y	-0.068/-0.118					60Y	+012.9/+007.9
3y	-0.043/-0.093						

ICAP OIS Fix Menu <ICAPOISFIX01>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANG

16:17 30JAN15 ICAP UK69580 ICAPEUROBASIS2

EUR Basis Swaps (as 2 Swaps)

3mth Euribor vs 6mth Euribor Basis Swaps with various start dates

For Further Details Please Call Jamie Mockett on +44 (0)207 532 3937

All prices are quoted Bond Basis

	Spot	IMM	1Wk Fwd	2Wk Fwd	3Wk Fwd	1 Mth Fwd	3 Mth Fwd	6 Mth Fwd
1YR	8.40	9.10	8.70	9.00	9.00	9.10	9.10	9.10
2YR	8.70	9.10	8.90	9.00	9.00	9.10	9.10	9.20
3YR	9.05	9.30	9.20	9.20	9.30	9.30	9.30	9.30
4YR	9.15	9.30	9.20	9.30	9.30	9.30	9.40	9.40
5YR	9.30	9.50	9.40	9.40	9.40	9.50	9.50	9.50
6YR	9.35	9.50	9.40	9.40	9.50	9.50	9.50	9.40
7YR	9.25	9.30	9.30	9.30	9.30	9.30	9.30	9.30
8YR	9.15	9.20	9.20	9.20	9.20	9.20	9.20	9.20
9YR	9.05	9.10	9.10	9.10	9.10	9.10	9.10	9.10
10YR	8.90	8.90	8.90	9.00	9.00	9.00	8.90	8.80
11YR	8.60	8.60	8.60	8.60	8.60	8.60	8.60	8.50
12YR	8.25	8.30	8.30	8.30	8.30	8.30	8.20	8.10
15YR	7.20	7.20	7.20	7.20	7.20	7.20	7.20	7.10
20YR	6.00	6.00	6.00	6.00	6.00	6.00	6.00	5.90
25YR	5.20	5.20	5.20	5.20	5.20	5.20	5.20	5.10
30YR	4.65	4.60	4.70	4.70	4.70	4.70	4.60	4.60
40YR	3.95	3.90	4.00	4.00	4.00	4.00	3.90	3.90
50YR	3.50	3.50	3.50	3.50	3.50	3.50	3.50	3.40

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

5: Linear interest rate products
Summary of rate conventions

Rate	Associated day count
Discount rate	ACT/365
Forward rate	ACT/360
IRS fixed rate	EUR: 30/360 (see e.g. Reuters page ICAPEURO1)
OIS fixed rate	EUR: ACT/360 (see e.g. Reuters page ICAPEURO1)
Basis swap spread	EUR: 30/360 (see e.g. Reuters page ICAPEUROBASIS)

- The discount rate day count is the convention to calculate the year fractions associated to discount factors. Notice that ACT/365 method respects the additivity of time intervals: $\tau\left(T_1, T_3, \frac{act}{365}\right) = \tau\left(T_1, T_2, \frac{act}{365}\right) + \tau\left(T_2, T_3, \frac{act}{365}\right), \forall\, T_1 < T_2 < T_3$.
- The IRS/OIS fixed rate day count is the convention to calculate the year fractions associated to IRS/OIS fixed leg.
- The forward rate day count is the convention to calculate the year fractions associated to forward rates and to IRS/OIS floating leg.
- The Basis Swap day count is the convention to calculate the year fractions associated to basis swap spreads.
- Other currencies or particular cases may require different conventions.

5: Linear interest rate products
Problems

- Basis swap: starting from the market quotes for basis swaps, build the all the possible basis swaps. Plot the corresponding chart. Explain the reason why the basis Eonia vs Euribor3M quoted in page ICAPSHORT1 is different from the same basis implicit in Eonia and Euribor3M quotations in page ICAPEURO2. Use static data or possibly Reuters/Bloomberg real time. Deliverable: spreadsheet with charts and comments.

- Pricing: price an OIS, a Swap, and a Basis Swap. Hint: assume same maturities, and three different yield curves: discounting and forwarding with tenors x and y (e.g. OIS, Euribor3M and Euribor6M). Try simple curve shapes, e.g. flat, linear ascending/descending, convex. Deliverable: spreadsheet with charts and comments.

- Pricing: pricing a IRS requires interpolations on the discounting and forwarding curves C_d and C_x, respectively. Is this price fully arbitrage free ? In what sense ?

Summary

6. Multi-curve framework

- Introduction
- Modern multi-curve market practice
- Multi-curves construction
- Bootstrapping instruments
- Market data
- Bootstrapping formulas
- Interpolation
- Negative rates
- Exogenous bootstrapping
- Turn of year effect
- Multiple curves, multiple deltas, multiple hedging
- Performance
- Sanity checks

6: Multi-curve framework
Introduction

The OTC market quotes linear interest rate instruments with specific:

- schedules (e.g. semi-annual floating vs annual fixed),
- underlying rates (e.g. Euribor6M),
- maturities (e.g. 5Y, 10Y).

However the market participants need to manage past or new interest rate trades with customized characteristics, such as:

- broken periods and residual maturities,
- different schedules.

Example: a 10Y swap annual fixed vs semiannual floating traded 2Y5M ago, is now a 7Y7M swap with an ongoing period…

Interest rate curves allows us to link the price non-quoted instruments in terms of quoted market instruments, with the aim of

- getting arbitrage-free relative prices,
- determine how variations in market quotes affect the price of non-quoted instruments (delta sensitivity),
- determine how to neutralize the delta risk a portfolio of non-quoted instruments (hedging).

6: Multi-curve framework
Modern multi-curves market practice

In case of vanilla linear derivatives the modern approach proceeds is as follows:

1. Build a single discounting curve $\mathcal{C}_d$ using the preferred bootstrapping procedure.
2. Build multiple distinct forwarding curves $\mathcal{C}_{f1}\ldots\ \mathcal{C}_{fn}$ using distinct selections of vanilla interest rate instruments, each homogeneous in the underlying rate tenor (typically 1M, 3M, 6M, 12M).
3. Compute the FRA/Swap rates with tenor *f* on the corresponding forwarding curve $\mathcal{C}_f$ and calculate the corresponding expected cash flows.
4. Compute the corresponding discount factors using the discounting curve $\mathcal{C}_d$.
5. Compute prices as sum of discounted cash flows.
6. Compute delta sensitivities and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the corresponding set of vanillas.

6: Multi-curve framework
Yield curve construction [1]

A yield curve is a two-column matrix of pairs {date, rate}) corresponding to a set of homogeneous interest rate financial instruments with different maturities. Homogeneity relates to the market rate the instruments are based upon.

Yield curve bootstrapping is an iterative optimization procedure to build an yield curve from market quotations of interest rate instruments.

The bootstrapping procedure produces an yield curve such that either each input instrument must be repriced exactly (in exact fit approaches) or within a predetermined precision (in best fit approaches).

In a very natural way, an yield curve is associated to an interpolation rule to determine discounts/pseudo discounts for any maturity.

A very common bootstrapping approach is to perform a sequence of single-dimensional optimizations each of which targets an input instrument with increasing maturity. The procedure may be based either on analytical formulas or on iterative numerical algorithms (e.g. Newton-Raphson).

6: Multi-curve framework
Yield curve construction [2]

Yield curve bootstrapping is a mathematically simple but practically very delicate exercise, more an art than a science, since it is the result of the interplay of many details, such that:

- Interpretation of market quotes
- Selection of bootstrapping instruments
- Market data available
- Bootstrapping algorithm
- Bootstrapping formulas
- Choice of interpolation
- Dealing with possible negative rates
- Endogenous vs exogenous bootstrapping
- Turn of year effect
- Multiple sensitivities
- Performance
- Sanity checks of bootstrapping results

"When dealing with curves, nothing ever goes straight"
(A. Bugin, Financial Engineering, Banca IMI)

Main references (see bibliography)

- F. Ametrano and M. Bianchetti (2013)
- F. Ametrano, L. Ballabio and P. Mazzocchi (2015)
- And many others

6: Multi-curve framework
Interpretation of market quotes

We assume that:

- the OTC interbank market is fully collateralized under standard CSAs (daily margination, overnight collateral rate, no frictions);
- the market uses coherently an OIS discounting curve;
- market quotes observable on the OTC interbank market, e.g. Broker quotes, reflect the characteristics above.

```
16:16 30DEC10     ICAP                                           UK69580          ICAPEURO
                             Euribor vs 6 mth             3/6 basis
                                                     Spot Starting Date
1 Yr     1.347-1.297   16Yrs    3.704-3.654    1 Yr    +19.0
2 Yrs    1.610-1.560   17Yrs    3.724-3.674    2 Yrs   +18.2
3 Yrs    1.946-1.896   18Yrs    3.735-3.685    3 Yrs   +17.7
4 Yrs    2.256-2.206   19Yrs    3.737-3.687    4 Yrs   +17.1
5 Yrs    2.529-2.479   20Yrs    3.733-3.683    5 Yrs   +16.6
6 Yrs    2.749-2.699                           6 Yrs   +15.9
7 Yrs    2.934-2.884   21Yrs    3.724-3.674    7 Yrs   +15.2
8 Yrs    3.089-3.039   22Yrs    3.710-3.660    8 Yrs   +14.5
9 Yrs    3.215-3.165   23Yrs    3.693-3.643    9 Yrs   +13.8
10Yrs    3.323-3.273   24Yrs    3.673-3.623    10Yrs   +13.1
                       25Yrs    3.651-3.601
11Yrs    3.419-3.369  All ICAP Euro pages to close at 5.30pm on 30th Dec
12Yrs    3.505-3.455   26Yrs    3.627-3.577               10X12    0.202/0.162
13Yrs    3.575-3.525   27Yrs    3.603-3.553               10X15    0.371/0.331
14Yrs    3.631-3.581   28Yrs    3.578-3.528               10X20    0.430/0.390
15Yrs    3.674-3.624   29Yrs    3.554-3.504               10X25    0.348/0.308
                       30Yrs    3.531-3.481               10X30    0.228/0.188
                       35Yrs    3.438-3.388               10X35    0.135/0.095
                       40Yrs    3.378-3.328               10X40    0.075/0.035
                       50Yrs    3.335-3.285               10X50    0.032/-0.008
                       60Yrs    3.285-3.235               10X60   -0.018/-0.058
Disclaimer <IDIS>          Page live in London hours ONLY (between 0700 - 1800)
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Quotations in a fully collateralized interbank market

6: Multi-curve framework
Bootstrapping instruments

EUR yield curves market bootstrapping instruments				
Discount curve	1M tenor curve	3M tenor curve	6M tenor curve	12M tenor curve
▪ OIS ▪ ECB OIS	▪ Synthetic Depo <1M ▪ Depo 1M ▪ Swap 1M ▪ Basis swap 6M-1M	▪ Synthetic Depo <3M ▪ Depo 3M ▪ FRA 3M tod/tom. ▪ IMM/serial Futures 3M ▪ Basis swap 6M-3M	▪ Synthetic Depo <6M ▪ Depo 6M ▪ FRA 6M tod/tom. ▪ FRA 6M 1x7-18x24 ▪ Swaps 6M	▪ Synthetic Depo <12M ▪ Depo 12M ▪ FRA 12x24 ▪ Basis swap 6M-12M

- We select multiple distinct sets of bootstrapping instruments homogeneous in the underlying rate tenor, no “mixing apples and oranges”.

- The actual choice of the bootstrapping instruments is subject to many practical trading and risk management considerations: liquidity, bid-ask spreads, transaction costs, information to be included or not in the curve, etc.

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See <KLIEMM2> for Scandinavian/Eastern deposits like DKK,SEK,NOK,CZK,PLN,HUF
16:52 30/12/10

	EUR CLOSED	USD CLOSED	FWDS CLOSED	GBP CLOSED	JPY CLOSED	CHF CLOSED	
ON	0.28/0.38	0.35/0.55	-0.040/-0.010	0.50/0.65		0.00/0.25	ON
TN	0.50/1.00	0.50/1.00	1.150/ 1.350	0.50/1.00	-0.01/0.14	0.10/0.35	TN
SN	0.40/0.50	0.30/0.50	0.030/ 0.060	0.50/0.75	-0.01/0.14	0.01/0.26	SN
SW	0.53/0.63	0.40/0.65	0.10/ 0.25	0.65/0.90	0.05/0.20	0.02/0.22	SW
2W	0.56/0.66	0.40/0.65	0.05/ 0.25	0.70/0.95	0.05/0.20	0.03/0.23	2W
3W	0.62/0.72	0.40/0.65	-0.35/ -0.10	0.75/1.00	0.05/0.20	0.04/0.24	3W
1M	0.70/0.80	0.55/0.75	-0.70/ -0.40	0.80/1.05	0.15/0.35	0.08/0.28	1M
2M	0.82/0.92	0.60/0.80	-2.30/ -1.90	0.85/1.10	0.20/0.40	0.18/0.38	2M
3M	0.92/1.02	0.65/0.85	-4.20/ -3.70	0.95/1.20	0.30/0.50	0.25/0.45	3M
4M	0.97/1.07	0.70/0.90	-6.30/ -5.55	1.00/1.25	0.30/0.55	0.31/0.51	4M
5M	1.05/1.15	0.74/0.94	-8.60/ -7.85	1.10/1.35	0.35/0.60	0.36/0.56	5M
6M	1.14/1.24	0.80/1.00	-10.80/ -9.80	1.20/1.45	0.40/0.65	0.40/0.60	6M
7M	1.19/1.29	0.84/1.04	-13.60/-12.10	1.25/1.50	0.45/0.70	0.45/0.65	7M
8M	1.23/1.33	0.88/1.08	-16.20/-14.70	1.30/1.55	0.45/0.70	0.49/0.69	8M
9M	1.28/1.38	0.94/1.14	-17.10/-15.10	1.35/1.60	0.45/0.70	0.53/0.73	9M
10M	1.33/1.43	0.98/1.18	-18.90/-16.40	1.40/1.65	0.45/0.70	0.57/0.77	10M
11M	1.37/1.47	1.01/1.21	-20.90/-18.40	1.45/1.70	0.45/0.70	0.60/0.80	11M
1Y	1.42/1.52	1.05/1.25	-22.50/-19.50	1.55/1.80	0.50/0.75	0.63/0.83	1Y
15M	1.60/1.85	1.14/1.44	-29.00/-23.00	1.64/2.14	0.49/0.74	0.66/1.16	15M
18M	1.64/1.89	1.23/1.53	-30.00/-23.00	1.74/2.24	0.50/1.00	0.70/1.20	18M
21M	1.70/1.95	1.34/1.64	-32.00/-24.00	1.87/2.37	0.49/0.99	0.76/1.26	21M
2Y	1.86/2.11	1.46/1.76	-34.00/-24.00	2.13/2.63	0.49/0.99	0.89/1.39	2Y
3Y	2.20/2.50	1.96/2.36	-10.00/ 10.00	2.62/3.12	0.53/1.03	1.21/1.71	3Y
4Y	2.52/2.82	2.46/2.86	55.00/ 85.00	3.03/3.53	0.58/1.08	1.51/2.01	4Y
5Y	2.80/3.10	2.91/3.31	150.00/190.00	3.38/3.88	0.67/1.17	1.78/2.28	5Y
7Y	3.30/3.70	3.67/4.17	CENTRALBANKNEWS	3.96/4.46	0.96/1.46	2.23/2.73	7Y
10Y	3.74/4.14	4.31/4.81	EZBnext13/01/11	4.47/4.97	1.34/1.84	2.61/3.11	10Y
12Y	3.95/4.35	4.58/5.08	1.00%v.07/05/09	4.70/5.20	1.56/2.06	2.77/3.27	12Y
15Y	4.14/4.54	4.85/5.35	FOMCnxt26/01/11	4.87/5.37	1.82/2.32	2.87/3.37	15Y
20Y	4.19/4.59	5.06/5.56	0-.25%v16/12/08	4.93/5.43	2.07/2.57	2.79/3.29	20Y
25Y	4.04/4.44	5.09/5.59	BoEnext13/01/11	4.89/5.39	2.14/2.64	2.72/3.22	25Y
30Y	3.82/4.22	5.14/5.64	0.50%v.05/03/09	4.81/5.31	2.15/2.65	2.62/3.12	30Y

Source: Reuters

6: Multi-curve framework
Market data [2]

OIS curve (Eonia):

16:16 30DEC10 ICAP LONDON UK69580 ICAPSHORT1
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	Eonia
1w	0.462-0.362
2w	0.456-0.356
3w	0.511-0.411
1m	0.527-0.477
2m	0.582-0.532
3m	0.619-0.569
4m	0.644-0.594
5m	0.669-0.619
6m	0.689-0.639
7m	0.707-0.657
8m	0.726-0.676
9m	0.743-0.693
10m	0.761-0.711
11m	0.776-0.726
12m	0.792-0.742
Two Payments	
15m	0.849-0.799
18m	0.914-0.864
21m	0.989-0.939
2y	1.071-1.021
3y	1.419-1.369

	Fwd EONIA		ECB Dates
1X2	0.644-0.594	JAN	0.607-0.557
2X3	0.686-0.636	FEB	0.656-0.606
1x4	0.685-0.635	MAR	0.697-0.647
2x5	0.724-0.674	APR	0.735-0.685
3x6	0.758-0.708	MAY	0.771-0.721
6x12	0.892-0.842	JUN	0.795-0.745

	IMM Fra/Eonia
MAR	35.100-30.100
JUN	34.200-29.200
SEP	34.600-29.600
DEC	35.300-30.300

	Eonia v 3m E'bor A/360
1Yr	+037.5/+032.5
18M	+036.5/+031.5
2Yr	+036.1/+031.1
3Yr	+035.0/+030.0
4Yr	+034.1/+029.1
5Yr	+033.2/+028.2
6Yr	+032.2/+027.2
7Yr	+031.2/+026.2
8Yr	+030.2/+025.2
9Yr	+029.3/+024.3
10Y	+028.2/+023.2
11Y	+027.3/+022.3
12Y	+026.5/+021.5
15Y	+024.4/+019.4
20Y	+022.4/+017.4
25Y	+021.4/+016.4
30Y	+020.7/+015.7

All ICAP Euro pages to close at 5.30pm on 30th Dec
(2 Swaps)

ICAP OIS Fix Menu <ICAPOISFIX01>
ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANG

16:21 30DEC10 ICAP UK69580 ICAPEURO2

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All ICAP Euro pages to close at 5.30pm on 30th Dec

	Eonia IRS ACT/360	Euro Swap vs 3M Euribor 30/360	Euro Swap vs 1M Euribor 30/360
1YR	0.804-0.734	1.159-1.109	0.994-0.924
2YR	1.081-1.011	1.428-1.378	1.265-1.195
3YR	1.429-1.359	1.769-1.719	1.609-1.539
4YR	1.745-1.675	2.081-2.031	1.922-1.852
5YR	2.026-1.956	2.356-2.306	2.201-2.131
6YR	2.260-2.190	2.584-2.534	2.432-2.362
7YR	2.459-2.389	2.776-2.726	2.627-2.557
8YR	2.631-2.561	2.939-2.889	2.795-2.725
9YR	2.770-2.700	3.072-3.022	2.930-2.860
10YR	2.894-2.824	3.187-3.137	3.048-2.978
11YR	3.006-2.936	3.290-3.240	3.156-3.086
12YR	3.106-3.036	3.383-3.333	3.253-3.183
15YR	3.310-3.240	3.569-3.519	3.449-3.379
20YR	3.405-3.335	3.645-3.595	3.541-3.471
25YR	3.344-3.274	3.572-3.522	3.481-3.411
30YR	3.239-3.169	3.459-3.409	3.376-3.306

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

Source: Reuters

6: Multi-curve framework
Market data [3]

IRS curves (Euribor):

16:16 30DEC10 ICAP UK69580 ICAPEURO

Tenor	Euribor vs 6 mth	Tenor	Euribor vs 6 mth	3/6 basis Spot Starting Date	
1 Yr	1.347-1.297	16Yrs	3.704-3.654	1 Yr	+19.0
2 Yrs	1.610-1.560	17Yrs	3.724-3.674	2 Yrs	+18.2
3 Yrs	1.946-1.896	18Yrs	3.735-3.685	3 Yrs	+17.7
4 Yrs	2.256-2.206	19Yrs	3.737-3.687	4 Yrs	+17.1
5 Yrs	2.529-2.479	20Yrs	3.733-3.683	5 Yrs	+16.6
6 Yrs	2.749-2.699			6 Yrs	+15.9
7 Yrs	2.934-2.884	21Yrs	3.724-3.674	7 Yrs	+15.2
8 Yrs	3.089-3.039	22Yrs	3.710-3.660	8 Yrs	+14.5
9 Yrs	3.215-3.165	23Yrs	3.693-3.643	9 Yrs	+13.8
10Yrs	3.323-3.273	24Yrs	3.673-3.623	10Yrs	+13.1
		25Yrs	3.651-3.601		
11Yrs	3.419-3.369	All ICAP Euro pages to close at 5.30pm on 30th Dec			
12Yrs	3.505-3.455	26Yrs	3.627-3.577	10X12	0.202/0.162
13Yrs	3.575-3.525	27Yrs	3.603-3.553	10X15	0.371/0.331
14Yrs	3.631-3.581	28Yrs	3.578-3.528	10X20	0.430/0.390
15Yrs	3.674-3.624	29Yrs	3.554-3.504	10X25	0.348/0.308
		30Yrs	3.531-3.481	10X30	0.228/0.188
		35Yrs	3.438-3.388	10X35	0.135/0.095
		40Yrs	3.378-3.328	10X40	0.075/0.035
		50Yrs	3.335-3.285	10X50	0.032/-0.008
		60Yrs	3.285-3.235	10X60	-0.018/-0.058

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16:16 30DEC10 ICAP LONDON UK69580 ICAPSHORT2

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1M Swaps		IMM Dated		3m FRAs	
2x1	0.827-0.777	1y MAR/MAR	1.213-1.163	1x4	1.037-0.987
3x1	0.839-0.789	1y JUN/JUN	1.329-1.279	2x5	1.055-1.005
4x1	0.850-0.800	1y SEP/SEP	1.477-1.427	3x6	1.080-1.030
5x1	0.863-0.813	1y DEC/DEC	1.646-1.596	4x7	1.108-1.058
6x1	0.875-0.825	2y MAR/MAR	1.518-1.468	5x8	1.137-1.087
7x1	0.888-0.838	2y JUN/JUN	1.671-1.621	6x9	1.166-1.116
8x1	0.907-0.857	3y MAR/MAR	1.855-1.805		
9x1	0.924-0.874			6m FRAs	
10x1	0.941-0.891	IMM Fras		1x7	1.264-1.214
11x1	0.954-0.904			2x8	1.284-1.234
12x1	0.969-0.919	1x7	1.273-1.223	3x9	1.307-1.257
		2x8	1.292-1.242	4x10	1.332-1.282
1y /3	1.158-1.108	3x9	1.320-1.270	5x11	1.357-1.307
15m/3	1.209-1.159	4x10	1.344-1.294	6x12	1.391-1.341
18m/3	1.270-1.220			12x18	1.649-1.599
21m/3	1.345-1.295	0x3 Today	1.035-0.985	18x24	2.025-1.975
		0x6 Today	1.259-1.209		
1y /6	1.347-1.297	0x3 Tom	1.033-0.983		
15m/6	1.352-1.302	0x6 Tom	1.260-1.210	12m FRA	
18m/6	1.454-1.404			12x24	2.001-1.951
21m/6	1.497-1.447				

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EUR Basis Swaps (as 2 Swaps)

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These are indicative mids priced out of a spot starting date

All prices are Euribor vs Euribor quoted Bond Basis

	3M vs 6M	1M vs 3M	1M vs 6M	6M vs 12M	3M vs 12M
1YR	19.0	17.5	36.5	21.3	40.2
2YR	18.2	17.3	35.5	18.5	36.7
3YR	17.7	17.1	34.7	17.1	34.8
4YR	17.1	16.8	34.0	15.5	32.6
5YR	16.6	16.5	33.1	14.0	30.6
6YR	15.9	16.2	32.1	12.9	28.9
7YR	15.2	15.9	31.1	12.1	27.3
8YR	14.5	15.5	30.0	11.4	25.9
9YR	13.8	15.2	29.0	10.9	24.7
10YR	13.1	14.9	28.1	10.4	23.6
11YR	12.4	14.5	26.9	10.1	22.5
12YR	11.7	14.1	25.8	9.7	21.4
15YR	10.0	13.1	23.1	9.0	19.0
20YR	8.3	11.5	19.8	8.4	16.7
25YR	7.3	10.2	17.5	8.0	15.4
30YR	6.7	9.4	16.0	7.8	14.4
40YR	5.7	All ICAP Euro pages to close at 5.30pm on 30th Dec			
50YR	5.2				

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

Source: Reuters

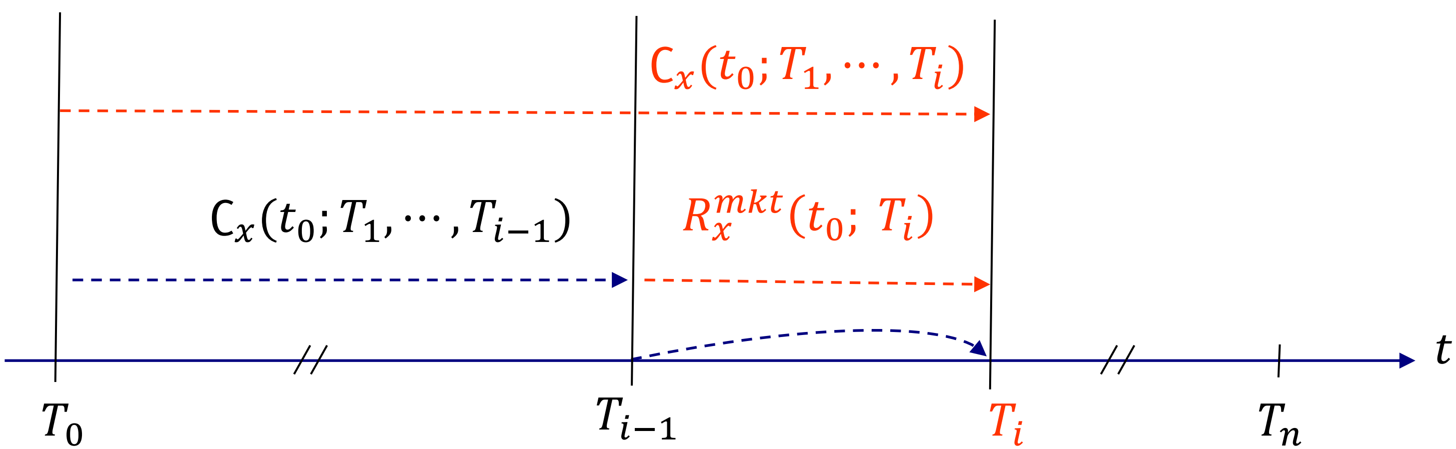
6: Multi-curve framework
Bootstrapping algorithm [1]

For a given yield curve $\mathcal{C}_x(t_0)$ at time t_0 let

- $\boldsymbol{T} = \{t_0, T_1, \cdots, T_i, \cdots, T_n\}$ be the discrete time grid of the market data selected as bootstrapping instruments. The terms $T_1, \cdots, T_n$ are also called pillars;
- $R_x^{mkt}(t_0;\ T_i)$ be the market rate quoted at time t_0 for the instrument with maturity T_i.

The bootstrapping algorithm proceeds recursively as follows:

- given the yield curve $\mathcal{C}_x(t_0; T_1, \cdots, T_{i-1})$ already built for pillars $T_1, \cdots, T_{i-1}$,
- obtain the yield curve $\mathcal{C}_x(t_0; T_1, \cdots, T_i)$ at pillar T_i using the yield curve $\mathcal{C}_x(t_0; T_1, \cdots, T_{i-1})$ for previous pillar T_{i-1}, the market quote $R_x^{mkt}(t_0; T_0, T_i)$ for pillar T_i, and the pricing formula associated to the market instrument, using appropriate analytical formulas or numerical algorithms,
- proceed until the last pillar T_n.



6: Multi-curve framework
Bootstrapping algorithm [2]

The bootstrapping target is the target quantity in the bootstrapping procedure of a given yield curve, that is, the unknown quantity that is recursively extracted from the market quotations using the bootstrapping formulas.
There are at least four alternatives bootstrapping targets:

- spot/forward rates: the quantities directly entering the pricing formulas
- Zero Coupon Bonds,
- log Zero Coupon Bonds,
- zero coupon rates.

The quantities above are related To spot/forward rates according to the usual formulas,

$$\begin{aligned} P_x(t;T_i) &:= \frac{1}{1 + L_{x,i}(t)\tau_x(t,T_i)}, \\ P_x(t;T_i) &:= \frac{P_x(t;T_{i-1})}{1 + F_{x,i}(t)\tau_x(T_{i-1},T_i)}, \\ Z_x(t,T_i) &:= -\frac{lnP_x(t;T_i)}{\tau_x(t,T_i)}. \end{aligned}$$

We stress that the risky zero coupon bond $P_x(t,T_i)$ above is just a recursive definition based on the FRA rates observed on the market, not a consequence of no-arbitrage between consecutive spot/forward rates in the classical sense.

6: Multi-curve framework
Bootstrapping algorithm [3]

Thus, the bootstrapping terminates with the construction of one of the target yield curves below,

$$
\begin{aligned}
\mathcal{C}_x^F(t_0) &:= \{T_i \longrightarrow F_x(t_0;T_{i-1},T_i),\ T_i \geq t_0\}\,,\ \text{FRA rate curve,}\\
\mathcal{C}_x^P(t_0) &:= \{T_i \longrightarrow P_x(t_0;T_i),\ T_i \geq t_0\}\,,\ \text{zero coupon bond curve,}\\
\mathcal{C}_x^{lnP}(t_0) &:= \{T_i \longrightarrow -lnP_x(t_0;T_i),\ T_i \geq t_0\}\,,\ \text{log zero coupon bond curve,}\\
\mathcal{C}_x^z(t_0) &:= \{T_i \longrightarrow Z_x(t_0,T_i),\ T_i \geq t_0\}\,,\ \text{zero rate curve,}
\end{aligned}
$$

for the relevant tenors, e.g. $x = \{d, 1M, 3M, 6M, 12M\}$ for the EUR market.

Once an yield curve $\mathcal{C}_x(t_0)$ has been bootstrapped on its discrete time grid $\boldsymbol{T} = \{t_0, T_1, \cdots, T_i, \cdots, T_n\}$, any value of target quantity at a non-pillar date T can be calculated from the yield curve using appropriate interpolation schemes.

6: Multi-curve framework
Bootstrapping formulas [1]

The yield curve bootstrapping formulas are just the pricing formulas of the plain vanilla instruments selected as bootstrapping instruments. The index *i* runs on the yield curve pillars. We shorten some notation.

Instrument	Quotation	Pricing formula
Deposits	Spot rate	$L_x(t, T_i)$
FRAs	(Market) FRA rate	$R^{\text{FRA}}_{x,\text{Mkt}}(t; \mathbf{T}_i) = \dfrac{\left[1 + \tau_x(T_{i-1}, T_i) F_{x,i}(t)\right] e^{C^{\text{FRA}}_x(t; T_{i-1})} - 1}{\tau_x(T_{i-1}, T_i)}$
Futures	Futures price	$\mathbf{Futures}(t; \mathbf{T}_i) = N\left\{1 - \left[F_{x,i}(t) + C^{\text{Fut}}_x(t, T_{i-1})\right]\right\}$
OIS	OIS rate	$R^{\text{OIS}}_d(t; \mathbf{T}_i) = \dfrac{P_d(t; T_0) - P_d(t; T_{n_i})}{A_d(t; \mathbf{T}_i)}$
Swaps	Swap rate	$R^{\text{Swap}}_x(t; \mathbf{T}_i, \mathbf{S}_i) = \dfrac{\sum_{j=1}^{m_i} P_d(t, T_j) F_{x,j}(t) \tau_{x,j}}{A_d(t, \mathbf{S}_i)}$
Basis Swaps	Basis swap rate	$\Delta(t; \boldsymbol{T}_{x,i}, \boldsymbol{T}_{y,i}, \boldsymbol{S}_i, \omega) = R^{\text{Swap}}_x(t; \boldsymbol{T}_{x,i}, \boldsymbol{S}_i) - R^{\text{Swap}}_y(t; \boldsymbol{T}_{y,i}, \boldsymbol{S}_i)$

6: Multi-curve framework
Bootstrapping formulas [2]

Then, the bootstrapping algorithm of the forward and ZC bond curves, for each typology of bootstrapping instruments, proceeds as follows. Market data in red.

- **Deposits**

$$L_{x,i}^{\text{Depo}}(t_0),$$

$$P_x(t_0, T_i) = \frac{1}{1 + L_{x,i}^{\text{Depo}}(t_0)\tau_x(t_0, T_i)}$$

- **FRA**

$$F_{x,i}(t_0) = \frac{\left[1 + R_{x,i}^{\text{FRA}}(t_0)\tau_{x,i}\right] e^{-C_{x,i-1}^{\text{FRA}}(t_0)} - 1}{\tau_{x,i}} \simeq R_{x,i}^{\text{FRA}}(t_0),$$

$$P_x(t_0, T_i) = \frac{P_x(t_0, T_{i-1}) e^{-C_{x,i-1}^{\text{FRA}}(t_0)}}{1 + R_{x,i}^{\text{FRA}}(t_0)\tau_{x,i}} \simeq \frac{P_x(t_0, T_{i-1})}{1 + R_{x,i}^{\text{FRA}}(t_0)\tau_{x,i}}$$

6: Multi-curve framework
Bootstrapping formulas [3]

- **Futures**:

$$F_{x,i}(t_0) = 1 - \frac{\mathbf{Futures}(t_0;T_i)}{100} - C^{\text{Fut}}_{x,i-1}(t_0),$$

$$P_x(t_0,T_i) = \frac{P_x(t_0,T_{i-1})}{1+\left[1-\frac{\mathbf{Futures}(t_0;T_i)}{100} - C^{\text{Fut}}_{x,i-1}(t_0)\right]\tau_{x,i}}.$$

- **OIS**:

$$F_{d,i}(t_0) = \frac{1}{\tau_{R,i}}\left\{\frac{P_d(t_0;T_{i-1})\left[1+R^{\text{OIS}}_i(t_0)\tau_{R,i}\right]}{\left[R^{\text{OIS}}_{i-1}(t_0)-R^{\text{OIS}}_i(t_0)\right]A_{d,i-1}(t_0)+P_d(t_0;T_{i-1})} - 1\right\},$$

$$P_d(t_0;T_i) = \frac{\left[R^{\text{OIS}}_{i-1}(t_0)-R^{\text{OIS}}_i(t_0)\right]A_{d,i-1}(t_0)+P_d(t_0;T_{i-1})}{1+R^{\text{OIS}}_i(t_0)\tau_{R,i}}.$$

- **Swap:**

$$F_{x,i}(t_0) = \frac{R^{\text{Swap}}_{x,i}(t_0)A_{d,i}(t_0) - R^{\text{Swap}}_{x,i-1}(t_0)A_{d,i-1}(t_0)}{P_d(t_0,T_i)\tau_{x,i}},$$

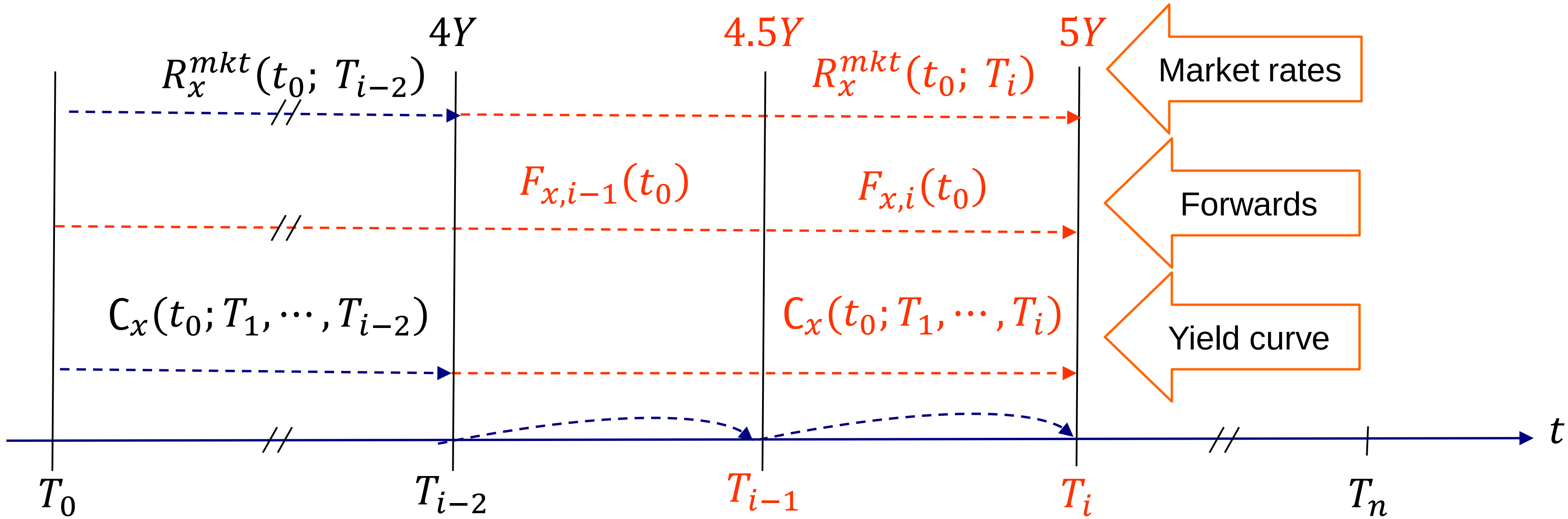
$$P_x(t_0,T_i) = \frac{P_d(t_0,T_i)P_x(t_0,T_{i-1})}{R^{\text{Swap}}_{x,i}(t_0)A_{d,i}(t_0) - R^{\text{Swap}}_{x,i-1}(t_0)A_{d,i-1}(t_0) + P_d(t_0,T_i)}$$

These formulas hold if the fixed and floating schedules coincide ($\boldsymbol{T} = \boldsymbol{S}$)

6: Multi-curve framework
Bootstrapping formulas [4]

When the fixed and floating schedules are different ($\boldsymbol{T} \neq \boldsymbol{S}$), e.g. in the case of a market IRS on Euribor6M, the bootstrapping formulas must take into account that bridge rate fixing dates not quoted on the market.

- The yield curve $\mathsf{C}_x(t_0; T_1, \cdots, T_{i-2})$ up to T_{i-2} is available from the previous bootstrapping step.
- The next market data available is $R_x^{mkt}(t_0;\ T_i)$.
- The yield curve $\mathsf{C}_x(t_0; T_1, \cdots, T_i)$ up to step T_i depends on TWO unknown forwards, $F_{x,i-1}(t_0)$ and $F_{x,i}(t_0)$.
- Example: the 4.5Y coupon in a 5Y swap on Euribor6M.



6: Multi-curve framework
Exogenous bootstrapping [1]

We stress that, looking at discounting:

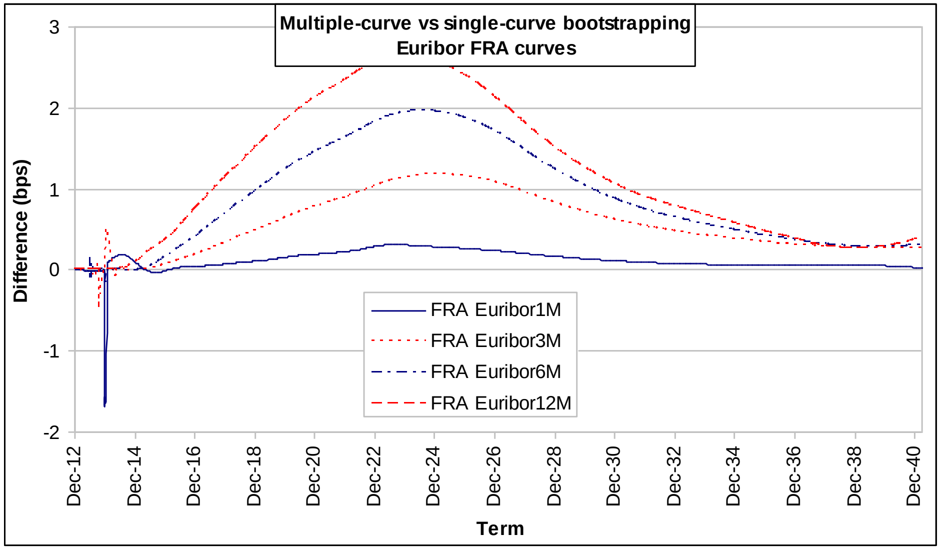
- the OIS curve uses itself to discount cash flows. In this case the bootstrapping is called “single-curve” or “endogenous”,

- the Swap curves use the OIS curve to discount cash flows. In this case the bootstrapping is called “multi-curve” or “exogenous”.

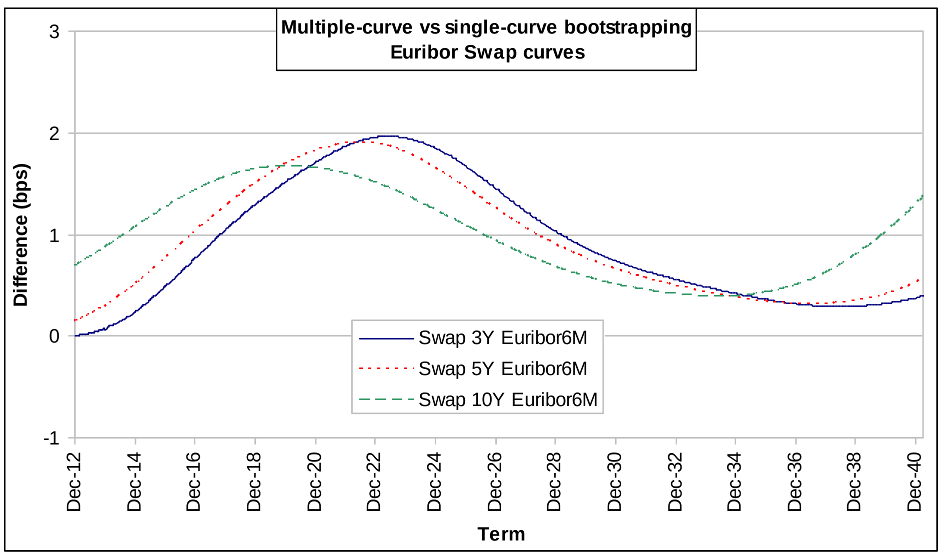
Hence, there exist a bootstrapping hierarchy:

- first, bootstrap the OIS curve (single-curve bootstrapping),
- next, bootstrap the Euribor curves for all tenors x using the previous OIS curve (multi-curve bootstrapping).

6: Multi-curve framework
Exogenous bootstrapping [2]



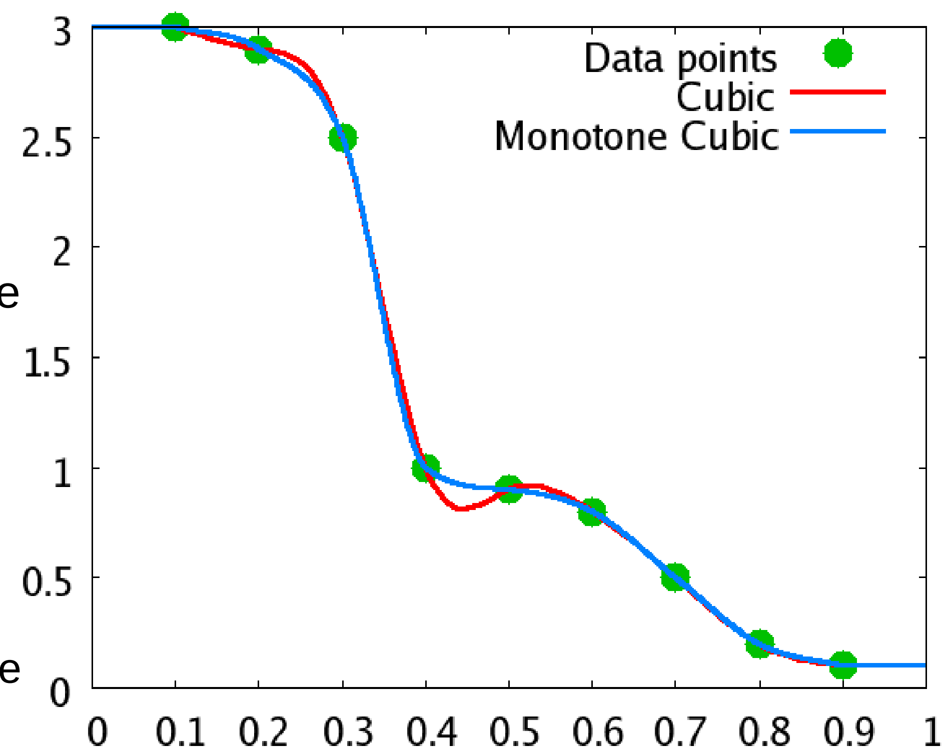
Difference (bps) between multi-curve vs single-curve bootstrapping for Euribor FRA curves. Each FRA rate shown in the chart has a tenor consistent with its underlying Euribor tenor. The spikes in the 1M and 3M curves are due to turn of year jumps included in the exogenous bootstrapping and excluded in the endogenous bootstrapping.



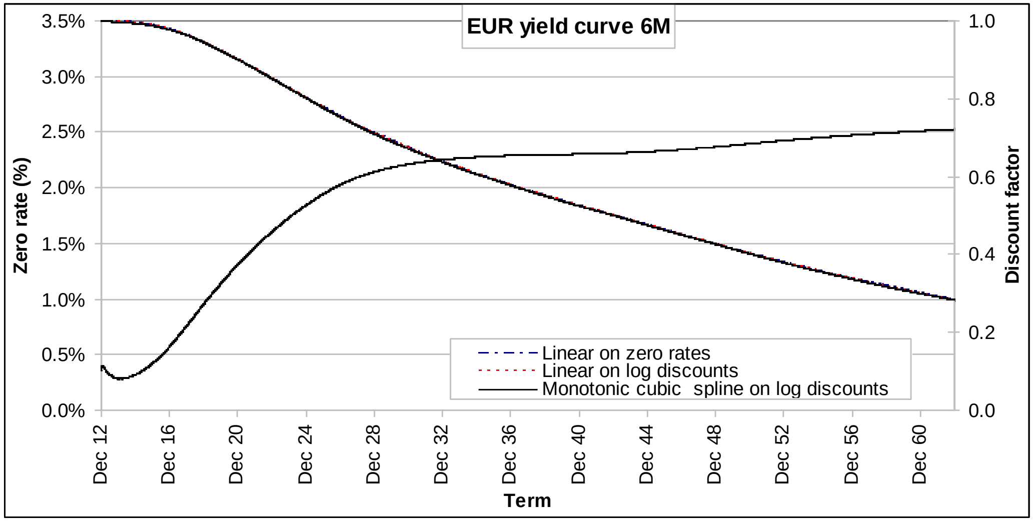
Effect on Swap curves.

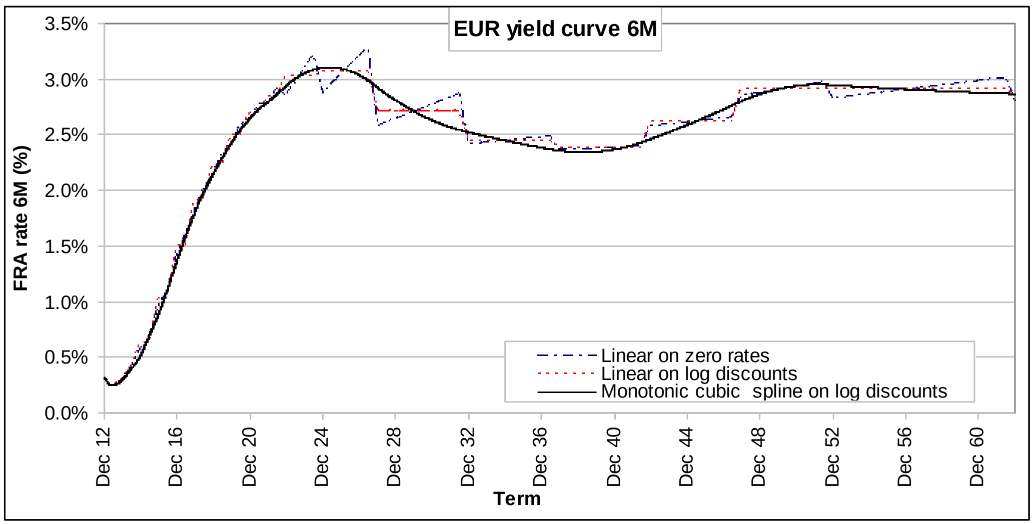
6: Multi-curve framework
Interpolation [2]

- Since interpolation is used both during the bootstrapping and in pricing, a consistent interpolation scheme must be adopted in both phases.
- The choice of cubic interpolations is a very delicate issue. Simple splines suffer from well-documented problems such as spurious inflection points, excessive convexity and lack of locality after input price perturbations (delocalised sensitivities).
- Constrained cubic splines (e.g. Kruger) guarantee monotonicity, with very small curvature.
- Andersen (2007) addressed these issues through the use of shape preserving splines from the class of generalised tension splines.
- Hagan and West (2006, 2008) developed a new scheme based on positive-preserving interpolation on forward rates.
- Ametrano and Bianchetti (2013) found that the classic Hyman monotonic cubic filter (Hyman1983) applied to natural splines of log discounts is the easiest approach ensuring non-negative forward curves and removes most of the unpleasant waviness.
- Le Floc'h (2013) reviews many choices and also analyses the delta breakdown suggesting the use of Hermite splines.



6: Multi-curve framework
Interpolation [3]





Examples of “bad” (but very popular…) interpolation schemes.

Upper panel: zero rate/discount curves (Euribor6M) bootstrapped using different interpolation schemes display similar smooth behaviours. Closer inspection reveals non differentiable points (discontinuous first derivative) at the interpolation sites (the larger the gap between two consecutive quotes, the larger may be the discontinuity)

Lower panel: FRA rate curves (daily FRA rates with 6M tenor) reveal different non-smooth behaviours, with ugly oscillations larger than 50bp. The monotonic cubic spline interpolation on log discounts (continuous black line) is clearly the smoothest choice.

Data as of 11 Dec. 2012

6: Multi-curve framework
Interpolation [4]

Drawbacks of non-local interpolation

1. **Non-local sensitivity in pricing**

   - Analytical formulas and their corresponding implementations are more complex.
   - Delta sensitivity is non-local, yielding more complex hedges.
   - Delta sensitivity may be unstable: small perturbations of single pillars may alter the shape of the yield curve, yielding jumping delta sensitivities.
     - Constrained cubic splines have a strong “tension” and waveness is kept under control.
     - Tension splines require calibration of a parameter which may lead to spurius effects.
     - Hyman filters have digitality → delta instability when curves are almost flat.
     - Hermite splines seem to be quite well behaved.

6: Multi-curve framework
Interpolation [5]

Drawbacks of non-local interpolation (cont'd)

2. Non-local sensitivity in bootstrapping

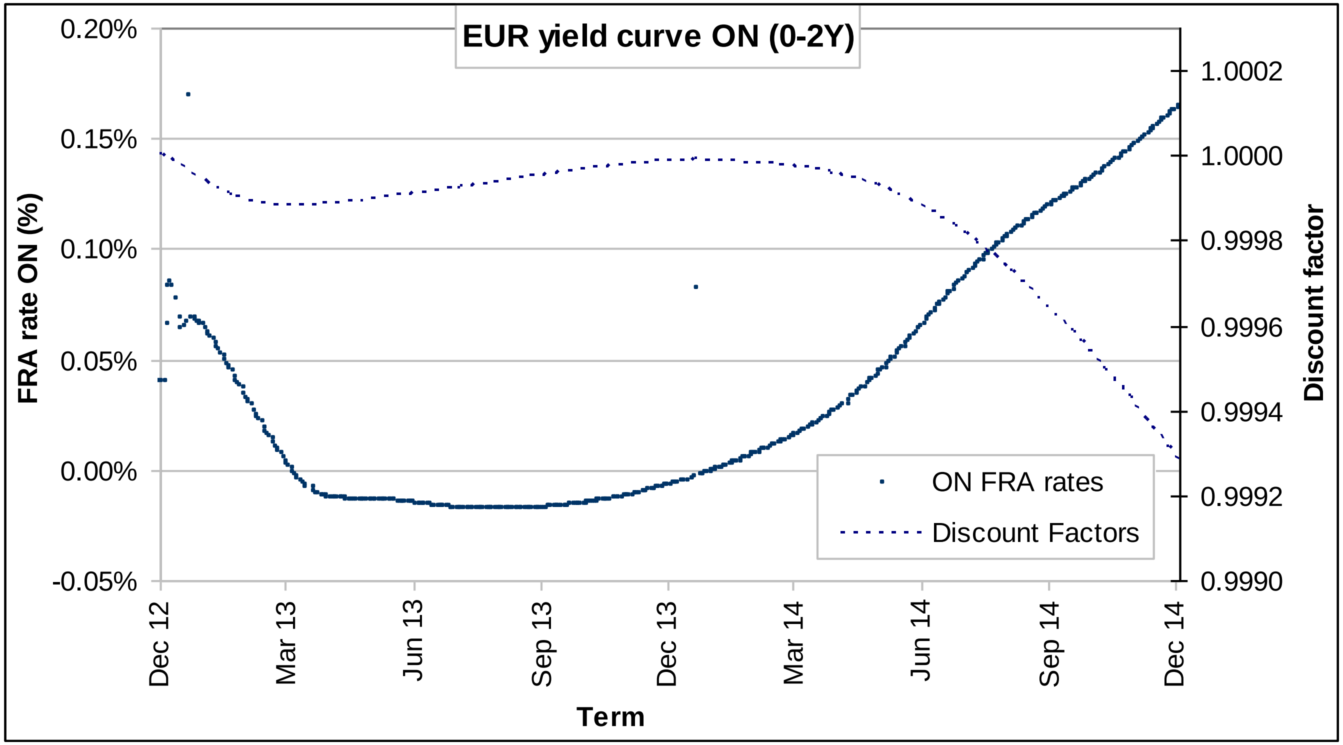
- Using non-local interpolations inside the bootstrapping procedure makes the yield curve already bootstrapped until pillar T_{i-1} dependent on the addition of further pillar T_i.

- This feature requires an iterative bootstrapping procedure where, at each step, the bootstrapping procedure is applied to the yield curve calculated at the previous step, until convergence is achieved.

- The first step in the iterative bootstrap might use a simple and faster local interpolation scheme, or even be replaced by a good curve guess. In a real time dynamical environment, the most natural guess is the yield curve available after the last bootstrapping.

- Furthermore, iterative bootstrapping creates a performance bottleneck.

6: Multi-curve framework
Negative rates



Short term structure of the EUR OIS curve as of 11 Dec. 2012. Left scale: over night FRA rates daily sampled. The two outlier dots are the turn of the year jumps. Right scale: discount factors.

Negative rates up to -2 basis points and increasing discount factors appear in the 3M-12M window.

6: Multi-curve framework
Yield curve jumps: turn of year effect [1]

For even the best interpolation schemes to be effective, the forward rate curve must be smooth, i.e. any jump must be removed, and added back only at the end of the bootstrapping procedure.

Because of periodic liquidity constraints imposed by regulators and liquidity injection by central banks, each end of month/quarter/year, we may observe a jump in the overnight rate, which is transmitted, with lower sizes, to Libor rates with larger tenors.

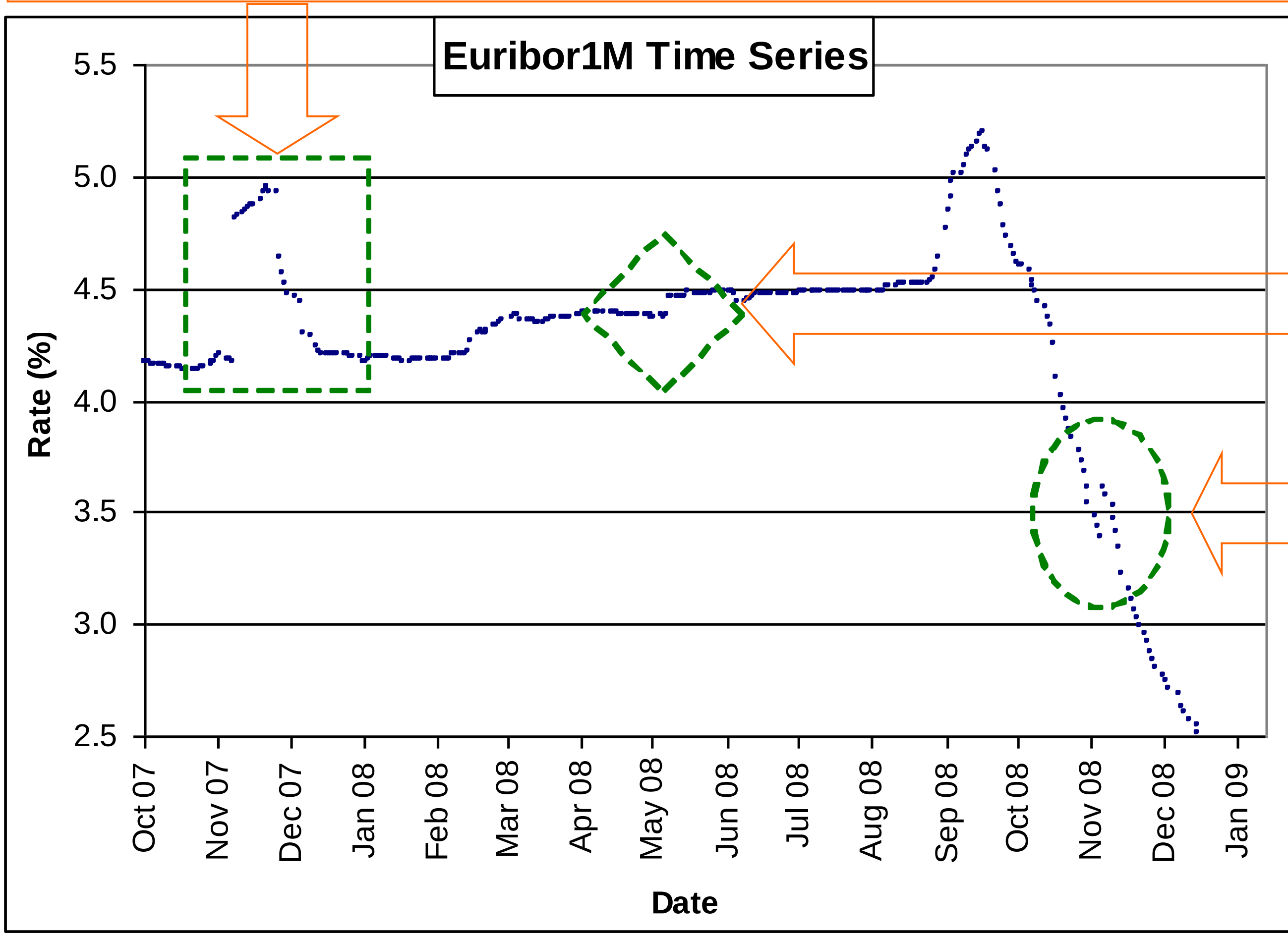
In the EUR market:

- the larger jump is observed on the last working day of the year (e.g., December 31, 2008) for the overnight deposit maturing the first working day of the next year (e.g. January 2, 2009). The same happens for the tomorrow-next and spot-next deposits one and two business days before, respectively (eg, December 30 and 29, 2008).
- Other instruments with longer underlying rate tenors display smaller jumps when their maturity crosses the same border: for instance, the 1M deposit quotation jumps two business days before the first business day of December; the 12M deposit always include a jump except two business days before the end of the year (due to the end-of-month rule); the December IMM futures always include a jump, as well as the October and November serial futures; 2Y swaps always include two jumps; etc.
- The effect is generally observable in market quotes up to the first two ends of year and becomes negligible at the following crosses.

6: Multi-curve framework

Yield curve jumps: turn of year effect [2]

The 2007 turn-of-year jump (64 bp) is clearly visible on November 29, 2007 just when the spot starting 1M tenor rate spans the end of 2007, with rates reverting towards the previous levels one month later.



In lower volatility regimes even the much smaller 2008 turn of semester jump may be observable on May 29, 2008 (9 bp).

The 2008 turn-of-year jump on November 27, 2008 (22 bp) is partially hidden by the high market volatility realised in that period.

6: Multi-curve framework
Yield curve jumps: turn of year effect [3]

- The decreasing jump with increasing underlying rate tenor can be easily understood once we distinguish between jumping rates and non-jumping rates. For instance, we may think of the 1M deposit as a weighted average of 22 (business days in one month) overnight rates (plus a basis). If such depo spans an end of year, there must be a single overnight rate, weighting 1/22, which crosses that end of year and displays the jump, while the others do not. Considering rates with longer tenors, there are still single jumping overnight rates, but these have smaller weights. Hence, longer deposits/FRAs display smaller jumps. The same holds for swaps, as portfolios of depos/FRAs.

- From a financial point of view, the turn-of-year effect is due to the increased search for liquidity by financial institutions to fulfill periodic regulatory requirements.

- The yield curve discontinuities induced by the turn-of-year effect may appear, to a non-market-driven reader, to be a fuzzy effect breaking the desired yield curve smoothness. On the contrary, they are neither a strangeness of the market quotations nor an accident of the bootstrapping, but correspond to true and detectable financial effects that should be included in any yield curve used for marking-to-market interest rate derivatives.

6: Multi-curve framework

Yield curve jumps: turn of year effect [4]

- The turn of the year effect (as well as the turn of the month/quarter) has the nice advantage that its location is known *ex ante*.

- By means of some heuristics applied to OIS curve or to Libor 1M, 3M, 6M, 12M curves we can roughly estimate its size.

- We can “clean up” quotes whose rates cross the end of the year and bootstrap forward curves with no jumps.

- The end of month/quarter/year jump effect can be easily incorporated after the bootstrapping, by means of a multiplicative jump coefficient applied to discount factors (or, equivalently, an additive coefficient applied to zero or forward rates), affecting the relevant set of dates following a given end of month/quarter/year.

- In this way we may estimate the jump coefficient using instruments with a given underlying rate tenor (e.g. Euribor3M), and to apply it to other curves with different tenors taking into account the proper weights.

6: Multi-curve framework
Yield curve jumps: turn of year effect [5]

The jump coefficient (i.e. the jump size) can be estimated from market quotations using different approaches:

1. The jump in the 3M futures strip: the (no-jump) end-of-year crossing forward is obtained through interpolation of non-crossing forwards; the jump coefficient is given by the difference between the latter and the quoted value. The first turn of year can be obtained only up to the third Wednesday of September, when the corresponding futures expires. In the period October–December there are no non-crossing futures to interpolate, we can infer the first turn of the year from the second.

2. The jump in the 6M FRA strip: this is equivalent to the approach above but it allows the estimation of the first turn of year up to June (included).

3. The jump in the 1M swaps strip: this is equivalent to the approaches above and it allows the estimation of the first turn of year up to November (included).

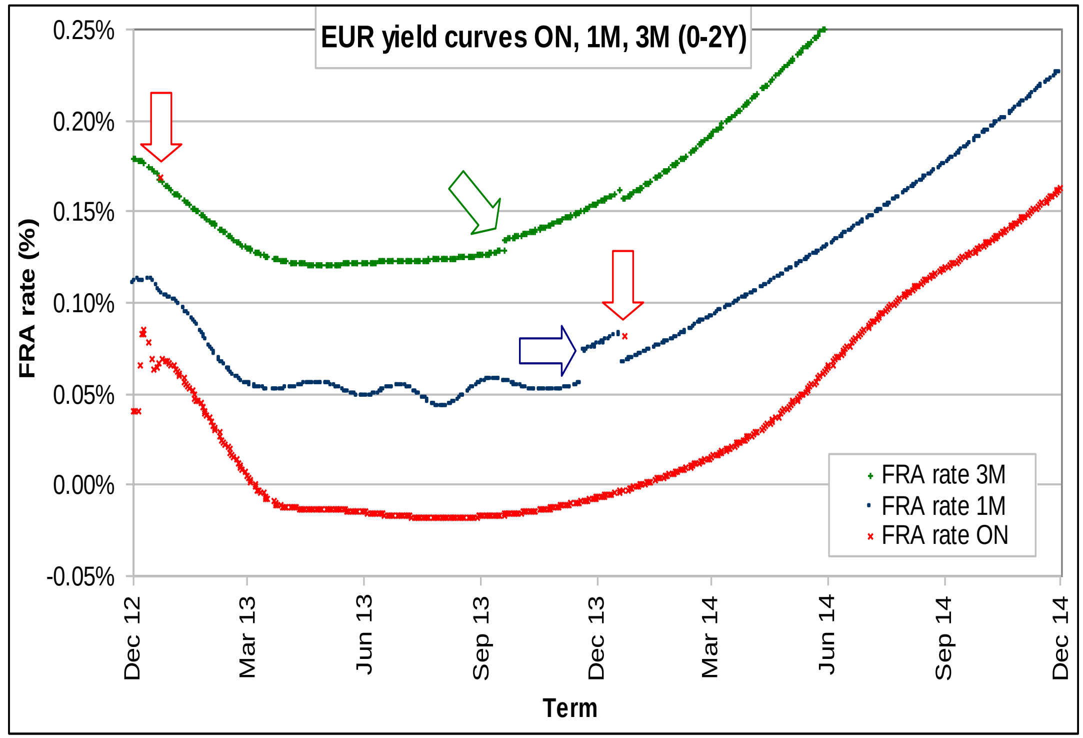
4. The jump in the FRA strip quoted by brokers each Monday: this approach is valid all year long, but it allows only a discontinuous weekly update.

5. On the OIS curve , assuming piecewise constant interest rates, it is also possible to estimate turn of month, quarter effects and apply them to other tenors.

6: Multi-curve framework
Yield curve jumps: turn of year effect [6]

Example of turn of year effect included in the FRA curves OIS, Euribor1M, Euribor3M.

- The OIS FRA curve displays both the 2013 (10.2 bps) and 2014 (8.5 bps) turn of year jumps.
- The Euribor 1M FRA curve displays the 2014 turn of year jump between 1° Dec. 2013 (+1.8 bps) and 2° Jan. 2014 (-1.6 bps) with size roughly equal to 1/20 of the OIS jumps.
- The Euribor 3M FRA curve displays the 2014 turn of year jump between 1° Oct. 2013 (+0.6 bps) and 2° Jan. 2014 (-0.5 bps), with size roughly equal to 1/3 of the 3M jumps.
- We stress that a single turn of the year induces one discontinuity in the zero rate and discount curves, and two discontinuities in the forward rate curve (remember that the forward rate is given by the ratio of two discounts).



6: Multi-curve framework
Yield curve jumps: OIS [7]

- The central bank periodically fixes the rate for Main Refinancing Operations as well as the rates for lending/borrowing facilities. Such central bank rates are constant between fixing dates. Fixing dates are calendarized and thus known in advance up to some time horizon.

- The market may quote IR instruments on such fixing dates, reflecting the market expectations of the central bank decisions.

- For example, the EUR market quotes a strip of OIS FRAs on ECB dates.

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Contact Reuters EXEU                       EURO FRAs / OIS                +44 (0)20 7532 3530
     Eonia                  Fwd EONIA                  ECB Dates    Eonia v 3m E'bor A/360
 1w  0.009/-0.091  1X2 0.0000/-0.050  MAR -0.0190/-0.0690   1Yr  +016.7/+011.7
 2w  0.007/-0.093  2X3 -0.046/-0.096  APR -0.0580/-0.1080   18M  +017.3/+012.3
 3w  0.006/-0.094  1x4 -0.035/-0.085  JUN -0.0780/-0.1280   2Yr  +017.8/+012.8
 1m -0.007/-0.057  2x5 -0.061/-0.111  JUL -0.0810/-0.1310   3Yr  +018.9/+013.9
 2m -0.003/-0.053  3x6 -0.071/-0.121  SEP -0.0860/-0.1360   4Yr  +019.9/+014.9
 3m -0.016/-0.066 6x12 -0.086/-0.136  OCT -0.0880/-0.1380   5Yr  +020.9/+015.9
 4m -0.028/-0.078                     DEC -0.0900/-0.1400   6Yr  +021.2/+016.2
 5m -0.037/-0.087 IMM Fra/Eonia       JAN -0.0930/-0.1430   7Yr  +021.4/+016.4
 6m -0.044/-0.094                     MAR -0.0790/-0.1290   8Yr  +021.5/+016.5
 7m -0.050/-0.100  MAR  16.400-11.400                       9Yr  +021.6/+016.6
 8m -0.054/-0.104  JUN  18.400-13.400                       10Y  +021.6/+016.6
 9m -0.057/-0.107  SEP  18.500-13.500                       11Y  +021.5/+016.5
10m -0.061/-0.111  DEC  18.500-13.500                       12Y  +021.5/+016.5
11m -0.062/-0.112                                           15Y  +021.1/+016.1
12m -0.065/-0.115                                           20Y  +019.6/+014.6
Two Payments                                                25Y  +018.0/+013.0
15m -0.069/-0.119                                           30Y  +016.8/+011.8
18m -0.071/-0.121                                           40Y  +015.0/+010.0
21m -0.071/-0.121                                           50Y  +013.8/+008.8
2y  -0.068/-0.118                                           60Y  +012.9/+007.9
3y  -0.043/-0.093                             ICAP OIS Fix Menu <ICAPOISFIX01>
ICAP Global Index <ICAP>                       Forthcoming changes <ICAPCHANG
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6: Multi-curve framework
Yield curve jumps: OIS [8]

- Naturally, the OIS curve, on its short end (up to 1Y) is better described by piece-wise constant instantaneous forward rates.

- The mid-long portions may be described by smoother functions.

- Thus, the OIS yield curve may be constructed with a mixed interpolation technique:
  - the short part with piece-wise constant instantaneous forward rates,
  - the medium/long part with smooth and stable interpolation methods as discussed before.

- The two types of OIS yield curve have different usages:
  - the mixed interpolation curve is typically preferred by treasurers, more sensitive to short term funding costs,
  - The fully smooth curve is typically preferred by swap traders, more sensitive to pricing and hedging long term instruments with stable delta sensitivities.

6: Multi-curve framework
Multiple curves, multiple deltas, multiple hedging [1]

We consider a generic portfolio of interest rate derivatives with price $\Pi(t, \boldsymbol{R}^{mkt})$, where:

- $\boldsymbol{R}^{mkt} = \{\boldsymbol{R}_k^{mkt}\}_{k=1}^{N_C}$ is the collection of N_C market interest rates vectors underlying the portfolio $\Pi\big(t, \boldsymbol{R}^{mkt}\big)$ (also called "risk factors"),

- $\mathrm{C} = \{\mathrm{C}_k\}_{k=1}^{N_C}$ is the collection of the related N_C yield curves,

- $\boldsymbol{R}_k^{mkt} = \{R_{k,j}^{mkt}\}_{j=1}^{N_k}$ is the vector of the N_k market interest rates associated to the corresponding bootstrapping instruments of the k-th yield curve C_k .

Example: EUR market rates

In the case of EUR market we have $N_C = 5$ yield curves, → the collection of market rates is

d	1M	3M	6M	12M
$R_{d,1}^{mkt}$	$R_{1M,1}^{mkt}$	$R_{3M,1}^{mkt}$	$R_{6M,1}^{mkt}$	$R_{12M,1}^{mkt}$
$R_{d,2}^{mkt}$	$R_{1M,2}^{mkt}$	$R_{3M,2}^{mkt}$	$R_{6M,2}^{mkt}$	$R_{12M,2}^{mkt}$
…	…	…	…	…
R_{d,N_d}^{mkt}	$R_{1M,N_{1M}}^{mkt}$	$R_{3M,N_{3M}}^{mkt}$	$R_{6M,N_{6M}}^{mkt}$	$R_{12M,N_{12M}}^{mkt}$

where

$$\sum_{k=1}^{5} N_k = N_d + N_{1M} + N_{3M} + N_{6M} + N_{12M} = N.$$

Notice that, in general, columns may have different lengths (i.e. $N_d \neq N_{1M} \neq N_{3M} \neq N_{6M} \neq N_{12M}$).

6: Multi-curve framework
Multiple curves, multiple deltas, multiple hedging [3]

We want to compute the multi-curve delta sensitivity of the portfolio, that is, the first derivative of the portfolio price Π w.r.t all the market rates $\boldsymbol{R}^{mkt}$.

This is actually a complex task, that we develop into six steps.

1. First of all, the market rate delta sensitivity is, by definition, given by

$$\Delta^{\pi}(t, \boldsymbol{R}^{mkt}) = \sum_{k=1}^{N_C} \sum_{j=1}^{N_k} \frac{\partial \Pi(t, \boldsymbol{R}^{mkt})}{\partial R_{k,j}^{mkt}}$$

   where k indexes the N_C yield curves and j the N_k pillars of yield curve C_k.

6: Multi-curve framework
Multiple curves, multiple deltas, multiple hedging [4]

2. Typically, yield curves $\mathsf{C} = \{\mathsf{C}_k\}_{k=1}^{N_C}$ are represented as zero rate curves $\mathsf{C}^z = \{\mathsf{C}_k^z\}_{k=1}^{N_C}$. Thus we have to consider the zero rate delta sensitivity and the corresponding cross derivatives

$$\frac{\partial \Pi(t, \boldsymbol{R}^{mkt})}{\partial R_{k,j}^{mkt}} = \sum_{\alpha=1}^{N_k} \frac{\partial \Pi(t, \boldsymbol{R}^{mkt})}{\partial Z_{k,\alpha}} J_{\alpha,j}^{k,k},$$

$$J_{\alpha,j}^{k,h} := \frac{\partial Z_{k,\alpha}}{\partial R_{h,j}^{mkt}}$$

where $\{Z_{k,\alpha}\}_{\alpha=1}^{N_k}$ is the vector of N_k zero rates in the k-th zero rate curve C_k^z, and $J_{\alpha,j}^{k,h}$ is the yield curve Jacobian.

We may have two cases:

a) quasi-local interpolation → the Jacobian is a quasi-diagonal matrix, with small off-diagonal elements → the zero rate delta sensitivities are very close to the market rate delta sensitivities;

b) non-local interpolation → market rate delta is delocalized across different pillars.

6: Multi-curve framework

Multiple curves, multiple deltas, multiple hedging [5]

3. Take properly into account all the delta sensitivity components due to multi-curve bootstrapping:
   - the direct delta sensitivity to the discounting zero curve $\mathcal{C}_d^z$ and to its corresponding bootstrapping instruments $\boldsymbol{R}_d^{mkt}$;
   - the direct delta sensitivity to the forwarding zero curves $\left\{\mathcal{C}_{f_1}^z, \dots, \mathcal{C}_{f_n}^z\right\}$ and to its corresponding bootstrapping instruments $\left\{\boldsymbol{R}_{f_1}^{mkt}, \dots, \boldsymbol{R}_{f_n}^{mkt}\right\}$ with tenors $\{f_1, \dots, f_n\}$;
   - the indirect delta sensitivity of the the forwarding zero curves $\left\{\mathcal{C}_{f_1}^z, \dots, \mathcal{C}_{f_n}^z\right\}$ to the discounting zero curve $\mathcal{C}_d^z$.

Thus, we obtain the following three multi-curve deltas

$$\Delta^\pi(t, \boldsymbol{R}^{mkt}) = \sum_{j=1}^{N_d} \sum_{\alpha=1}^{N_d} \frac{\partial \Pi(t, \boldsymbol{R}^{mkt})}{\partial Z_{d,\alpha}} J_{\alpha,j}^{d,d},$$

$$= \sum_{j=1}^{N_f} \sum_{\alpha=1}^{N_f} \frac{\partial \Pi(t, \boldsymbol{R}^{mkt})}{\partial Z_{f,\alpha}} J_{\alpha,j}^{f,f} + \sum_{j=1}^{N_d} \sum_{\alpha=1}^{N_f} \frac{\partial \Pi(t, \boldsymbol{R}^{mkt})}{\partial Z_{f,\alpha}} J_{\alpha,j}^{f,d}.$$

See “One price, two curves, three deltas”, Henrard (2009).

6: Multi-curve framework
Multiple curves, multiple deltas, multiple hedging [6]

4. Aggregate market rate delta sensitivities on the selected subset *H* of the most liquid market instruments used for hedging $\boldsymbol{R}^H = \{R_1^H, \dots, R_{N_H}^H\}$ (hedging instruments),

$$\Delta^\pi(t, \boldsymbol{R}^H) \simeq \sum_{j=1}^{N^H} \frac{\partial \Pi(t, \boldsymbol{R}^H)}{\partial R_j^H}.$$

5. Calculate hedge ratios:

$$h_j(t, \boldsymbol{R}^H) := \frac{\frac{\partial \Pi(t, \boldsymbol{R}^H)}{\partial R_j^H}}{\delta_j^H(t, R_j^H)},$$

$$\delta_j^H(t, R_j^H) := \frac{\partial \pi_j^H(t, R_j^H)}{\partial R_j^H},$$

where

- π_j^H is the price (unit nominal) of the corresponding hedging instrument,
- δ_j^H is its delta sensitivity.

6. Build an hedged portfolio with zero delta adding the appropriate amount of hedging instruments

$$
\begin{aligned}
\Pi^{Tot}(t, \boldsymbol{R}^H) &:= \Pi(t, \boldsymbol{R}^H) - \sum_{j=1}^{N_H} h_j(t) \pi_j^H(t, R_j^H), \\
\Delta^{Tot}(t, \boldsymbol{R}^H) &\simeq \sum_{k=1}^{N_H} \left[\frac{\partial \Pi(t, \boldsymbol{R}^H)}{\partial R_k^H} - \sum_{j=1}^{N_H} h_j(t) \frac{\partial \pi_j^H(t, R_j^H)}{\partial R_k^H} \right] \\
&= \sum_{k=1}^{N_H} \left[\frac{\partial \Pi(t, \boldsymbol{R}^H)}{\partial R_k^H} - h_k(t) \delta_k^H(t) \right] = 0.
\end{aligned}
$$

Remark: such hedged portfolio is not only hegded globally (if all rates R^H moves), but also on a hedge by hedge basis (only one hedge rate moves). This way of visualizing sensivity is called delta breakdown.

Numerical calculation of sensitivities: finite differences method

- Delta sensitivity (central finite difference)

$$\frac{\partial \Pi(t, \boldsymbol{R}^{mkt})}{\partial R_{k,j}^{mkt}} \approx \frac{\Pi(t, \boldsymbol{R}^{mkt} + \epsilon_{k,j}) - \Pi(t, \boldsymbol{R}^{mkt} - \epsilon_{k,j})}{2\epsilon_{k,j}}$$

- Gamma sensitivity (central finite difference)

$$\frac{\partial^2 \Pi(t, \boldsymbol{R}^{mkt})}{\partial (R_{k,j}^{mkt})^2} \approx \frac{\Pi(t, \boldsymbol{R}^{mkt} + \epsilon_{k,j}) - 2\Pi(t, \boldsymbol{R}^{mkt}) + \Pi(t, \boldsymbol{R}^{mkt} - \epsilon_{k,j})}{(\epsilon_{k,j})^2}$$

- Cross-gamma sensitivity (central finite difference)

$$\begin{aligned}\frac{\partial^2 \Pi(t, \boldsymbol{R}^{mkt})}{\partial R_{k,j}^{mkt} \partial R_{h,i}^{mkt}} \approx & \frac{\Pi(t, \boldsymbol{R}^{mkt} + \epsilon_{k,j} + \epsilon_{h,i}) - \Pi(t, \boldsymbol{R}^{mkt} + \epsilon_{k,j}) - \Pi(t, \boldsymbol{R}^{mkt} + \epsilon_{h,i})}{\epsilon_{k,j}\epsilon_{h,i}} \\ & + \frac{2\Pi(t, \boldsymbol{R}^{mkt}) - \Pi(t, \boldsymbol{R}^{mkt} - \epsilon_{k,j}) - \Pi(t, \boldsymbol{R}^{mkt} - \epsilon_{h,i}) + \Pi(t, \boldsymbol{R}^{mkt} - \epsilon_{k,j} - \epsilon_{h,i})}{\epsilon_{k,j}\epsilon_{h,i}}\end{aligned}$$

where $\epsilon_{k,j}$ is the finite shock applied to $R_{k,j}^{mkt}$

6: Multi-curve framework
Multiple curves, multiple deltas, multiple hedging [9]

Parallel Delta vs Pillar Delta

- Parallel delta: for a each yield curve, bump the market rates all together, re-bootstrap, reprice, and compute finite difference (one single calculation).
- Pillar delta: for each yield curve, bump each single market rate, re-bootstrap reprice, and compute finite difference (as many calculation as the number of pillars).

Pros & cons:

- Parallel delta cannot help visualizing mismatching positions long-short end of a given curve (exposure to flattening/steepening),
- Hedging the parallell delta sensitivity with very few liquid instruments is cheap but exposes to unhedged risks (flattening/steepening, basis).
- Hedging the pillar delta sensitivity to each single curve pillar is much more precise but may be very expensive.

Sustainable hedging is somewhere in between:

- hedge the shortest part of the pool of curves with separate instruments (ON,1M,…) to better capture the inter-tenor basis,
- aggregate the longest part of the curve sensitivities and hedge in a “parallel” way with most liquid instruments.

Example: delta sensitivity calculation for IRS

We select three IRS with different rates and maturities as follows,

Trades			
Field	**IRS 1**	**IRS 2**	**IRS 3**
Start date	15-09-15	15-09-15	15-09-15
Spot date	17-09-15	17-09-15	17-09-15
Maturity date	17-09-25	17-09-25	17-03-26
Maturity (Y)	10y	10y	10y6M
Floating rate (rec)	Euribr6M	Euribr6M	Euribr6M
Floating rate freq.	6M	6M	6M
Floating rate conv.	ACT/360	ACT/360	ACT/360
Fixed rate (pay)	1.078%	2.000%	1.078%
Fixed rate freq.	1y	1y	1y
Fixed rate conv.	30/360	30/360	30/360
Nominal (€)	100,000,000	100,000,000	100,000,000
Floating leg NPV (€)	10,523,347	10,523,347	11,512,459
Fixed leg NPV (€)	-10,523,347	-19,523,835	-11,512,459
NPV (€)	0	-9,000,488	0

where 1.078% is the equilibrium swap rate, and we compute the corresponding OIS and Euribor6M delta sensitivities with 1 bps (see spreadsheet).

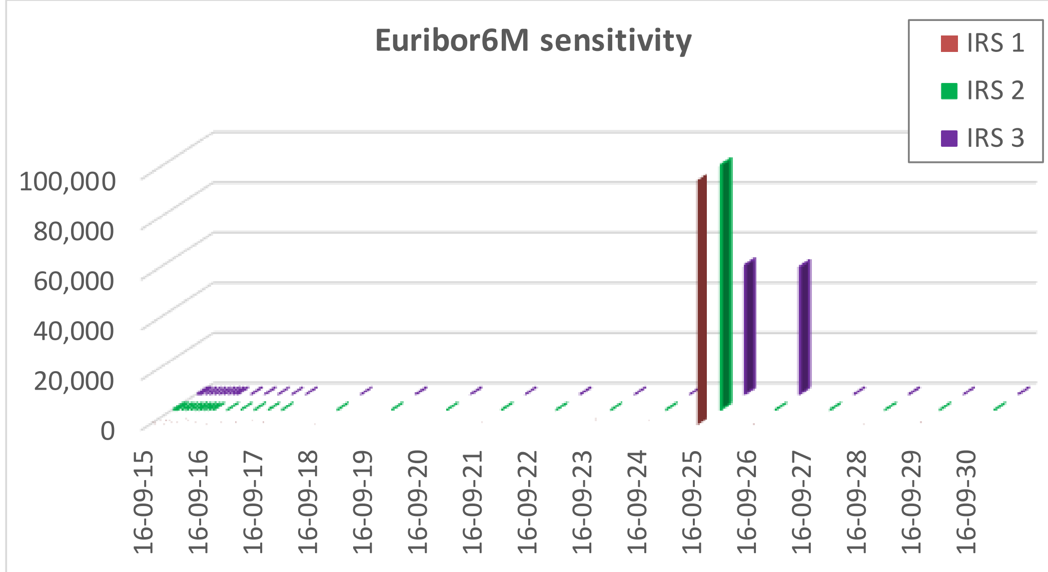
6: Multi-curve framework
Multiple curves, multiple deltas, multiple hedging [11]

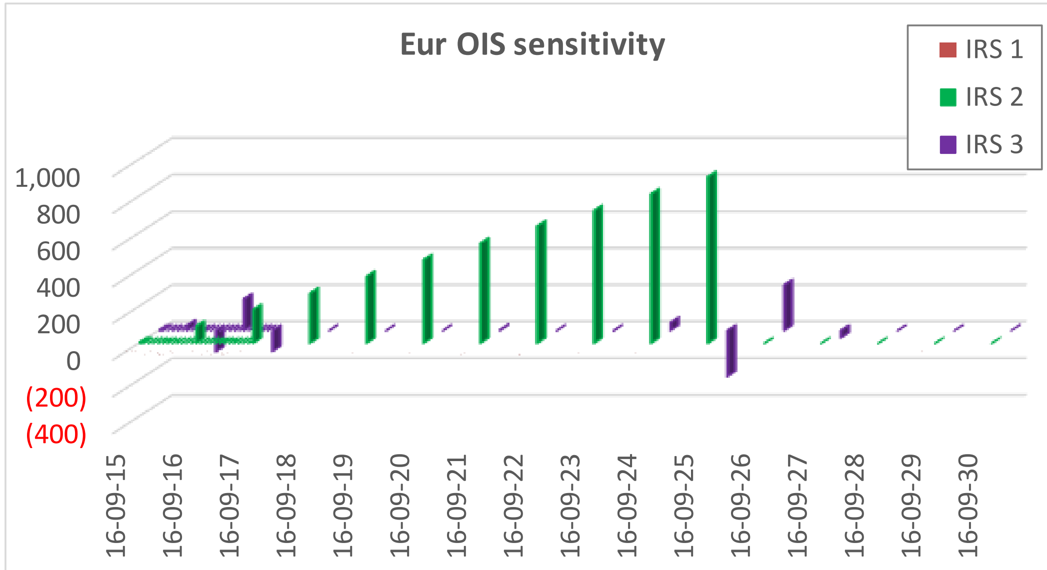
Example: delta sensitivity calculation for IRS (cont'd)

Results for delta sensitivity wrt Euribor6M (top) and OIS (bottom).

- Euribor6M delta is concentrated at maturity. For IRS 3 it is split between the two neighbour pillars.
- Eur OIS delta is either zero or much smaller.
- See spreadsheet for details.

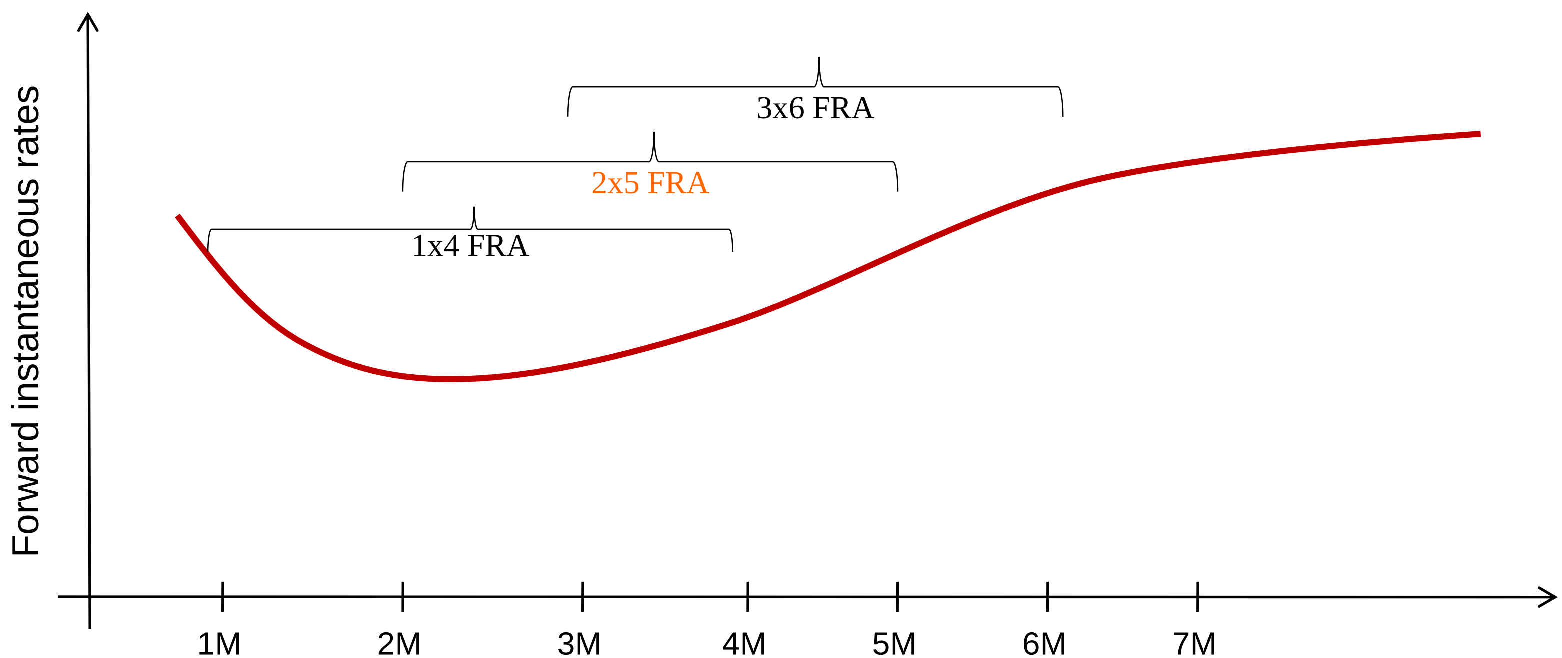
Trades			
Field	IRS 1	IRS 2	IRS 3
Euribor 6M delta (€/bps)	97,620	97,620	102,119
Eur OIS delta (€/bps)	0	4,993	-25





Example: delta sensitivity with forward starting overlapping instruments (1/4)

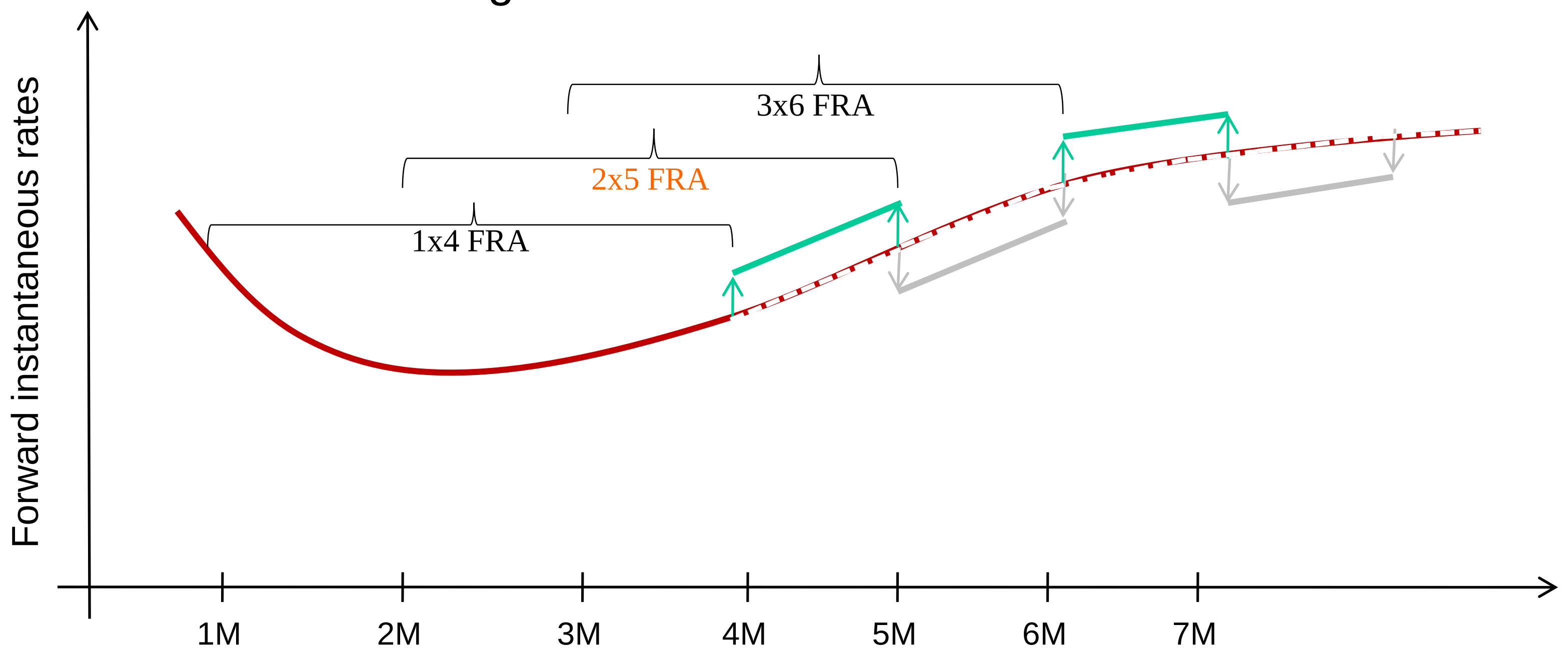
Even in presence of a local interpolator (linear on zero rates), the presence of many overlapping instruments, leads to smooth and rich curves but requires caution when valuing delta breakdowns. Bumping a single quote keeping the overlapping quotes unchanged may cause distorsions because it could be an unnatural market move. The size of the bump must be carefully chosen.



6: Multi-curve framework
Multiple curves, multiple deltas, multiple hedging [13]

Example: delta sensitivity with forward starting overlapping instruments (2/4)

FRA rates with tenor *T* lock the average value of instantaneous (or overnight) forward rates over a given period. For example, bumping the 2x5 FRA rate alone and keeping the neighbour 1x4 and 3x6 FRA rates unchanged means changing instantaneous forwards over [4M,5M], then rates over [5M,6M] will move such that the 3x6 remains constant, and so on. Such propagation effect may lead to non-local, dislocated sensitivities, which are more relevant for forward starting instruments with short tenors.



6: Multi-curve framework

Multiple curves, multiple deltas, multiple hedging [14]

Example: delta sensitivity with forward starting overlapping instruments (3/4)

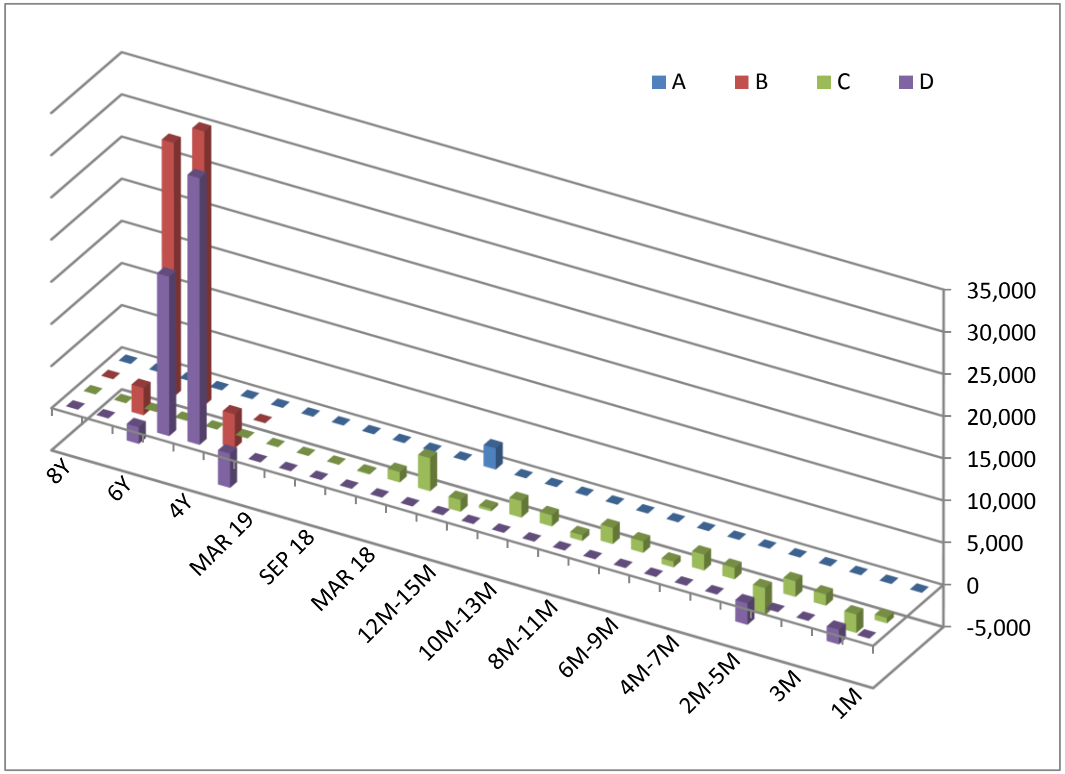
Instrument	Start date	Maturity	Market quote
Dp 1M	06/12/2016	06/01/2017	-0.310
Dp 2M	06/12/2016	06/02/2017	-0.311
Dp 3M	06/12/2016	06/03/2017	-0.310
FRA 1M-4M	06/01/2017	06/04/2017	-0.308
FRA 2M-5M	06/02/2017	08/05/2017	-0.301
FRA 3M-6M	06/03/2017	06/06/2017	-0.293
FRA 4M-7M	06/04/2017	06/07/2017	-0.286
FRA 5M-8M	08/05/2017	07/08/2017	-0.281
FRA 6M-9M	06/06/2017	06/09/2017	-0.277
FRA 7M-10M	06/07/2017	06/10/2017	-0.272
FRA 8M-11M	07/08/2017	06/11/2017	-0.269
FRA 9M-12M	06/09/2017	06/12/2017	-0.266
FRA 10M-13M	06/10/2017	08/01/2018	-0.261
FRA 11M-14M	06/11/2017	06/02/2018	-0.258
FRA 12M-15M	06/12/2017	06/03/2018	-0.256
FUT DEC 17	20/12/2017	20/03/2018	100.256
FUT MAR 18	21/03/2018	21/06/2018	100.240
FUT JUN 18	20/06/2018	20/09/2018	100.218
FUT SEP 18	19/09/2018	19/12/2018	100.188
FUT DEC 18	19/12/2018	19/03/2019	100.153
FUT MAR 19	20/03/2019	20/06/2019	100.108
SWP 3Y	06/12/2016	06/12/2019	-0.205
SWP 4Y	06/12/2016	07/12/2020	-0.122
SWP 5Y	06/12/2016	06/12/2021	-0.015
SWP 6Y	06/12/2016	06/12/2022	0.110

Euribor3M curve built with many FRAs (from 1x4 to 12x15), IMM Futures and swaps, interpolation constrained cubic spline on zero rates.

Let us check the delta sensitivity of

A. A fwd starting instrument which is included in curve calibration: the 12x15 FRA.
B. A spot starting ATM swap fixed annual vs Euribor3M, 2D→5Y6M (6/12/16→6/12/22).
C. A fwd starting ATM swap fixed annual vs Euribor3M, 5M→1Y5M (8/5/17→8/5/18).
D. A fwd starting ATM swap fixed annual vs Euribor3M, 5M→4Y5M (8/5/17→8/5/21). The length is greater than in (C), Propagation effects will eventually compensate.

Example: delta sensitivity with forward starting overlapping instruments (4/4)



	A	B	C	D
1M	0		638.4	0.8
2M	0		-2,184.8	-1,731.4
3M	0		1,324.0	15.6
1M-4M	0		1,854.2	2.4
2M-5M	0		-3,207.5	-2,541.8
3M-6M	0		1,353.3	13.6
4M-7M	0		1,874.7	2.5
5M-8M	0		-680.8	-7.9
6M-9M	0		1,353.3	11.3
7M-10M	0		1,907.8	2.6
8M-11M	0		-680.5	-5.6
9M-12M	0		1,328.0	8.8
10M-13M	0		1,963.1	2.7
11M-14M	0		-304.2	-2.7
12M-15M	2,511.1		-1,396.5	2.8
DEC 17	0		3,915.2	5.2
MAR 18	0		1,230.3	6.6
JUN 18	0		-111.2	6.6
SEP 18	0		0	6.6
DEC 18	0		0	6.6
MAR 19	0		0	6.8
3Y	0	-1.2	0	-4,058.6
4Y	0	-4,015.5	0	31,621.7
5Y	0	32,503.6	0	18,908.8
6Y	0	30,104.2	0	-1,970.8
7Y	0	-3,235.6	0	0
TOTAL	2,511.1	55,355.5	10,176.8	40,313.2

Notional 100 mln€, such that each 3M coupon has about 2,500€ sensitivity for a 1%* bump of mkt rates.

(*just for visualization: we have bumped rates of a smaller amount!)

6: Multi-curve framework
Performance

A good computational performance of yield curve bootstrapping algorithms is a key feature of real-time applications in liquid markets. Possible bottlenecks are:

- Analytical vs numerical interpolation: while simple interpolation schemes may use analytical formulas (e.g. linear interpolation), more general schemes require a solver or zero-finding numerical algorithm. Hence, efficient algorithms (e.g. Newton-Raphson) and careful choice of their parameters (e.g. initial guess and convergence criteria) are required for fast interpolation.
- Non-local interpolation: non-local schemes, such as splines, require an iterative algorithm to converge to the final solution. Again careful control of numerical parameters is required for fast convergence.
- Exogenous bootstrapping: this bootstrapping scheme implies a cross-dependence of the forwarding yield curves C_f on the discounting yield curve C_d. Hence, a market change in a single OIS quotation triggers the recalculation of all the dependent yield curves. Since this dependence may be often very small, the cross recalculation may be disabled.
- Delta sensitivity: this is clearly the most expensive computation, since each single pillar shock triggers the recalculation of the entire yield curve (or of the whole set of yield curves, in case of OIS shock and exogenous bootstrapping), and the revaluation of the corresponding portfolio of derivatives. This huge task may be made more efficient by pre-computing and storing the Jacobians.

We conclude that a good design and careful test of the basic yield curve bootstrapping component will affect the overall performance of any pricing system.

6: Multi-curve framework
Sanity checks

Once the desired set of yield curves has been constructed, one wish to check and monitor their correctness. Obviously the bootstrapping algorithm must be carefully debugged, but even a correct implementation requires a constant monitoring against possible market changes and failures of the basic yield curve construction hypothesis. A typical example of market change is the switch to (exogenous) OIS discounting in 2010.

There are essentially four methods to check and monitor yield curves.

- Market knowledge: clearly, any good yield curve assumes there is a good trader behind, with real-time knowledge of the corresponding market bootstrapping instruments. She is the best check that anyone can set up.
- Visual inspection: the bare shape of the yield curve is a good sentinel of possible bootstrapping problems. All the yield curve typologies (discount, zero/FRA/instantaneous FRA rates) must be monitored all together, since they convey different and complementary information.
- Repricing of bootstrapping instruments: by construction, each market instrument selected as input bootstrapping instrument must be repriced exactly (in exact fit approaches) or within a predetermined precision (in best fit approaches).
- Repricing of market instruments: finally, instruments quoted on the market but not included in the bootstrapping should be repriced within their bid-ask window. A typical example are quotes of long term Futures and forward starting IRS.

6: Multi-curve framework
Problems

- Yield curves: using the yield curves provided in the market data sheets, plot the corresponding forward curves and forward basis curves. Check different interpolation schemes. Deliverable: spreadsheet with chart and comments.

- Yield curves: using the market data provided in the market data sheets, bootstrap the EUR OIS, Euribor 1M, 3M, 6M and 12M yield curves.. Plot the corresponding discount, zero rate, forward rate and forward basis rate curves. Check different interpolation schemes both pre and post bootstrapping. Deliverable: spreadsheet/VBA or Matlab with charts and comments.

- Sensitivity: starting from the IRS pricing problem (see exercises for previous section 5), compute the delta sensitivity of an OIS, a swap and a basis swap. Proof analytically and numerically the reason why such sensitivities are concentrated on the maturity date. Check the analytical result by numerically computing the sensitivities using the finite difference method. Deliverable: analytical proof + spreadsheet with chart and comments.

Summary

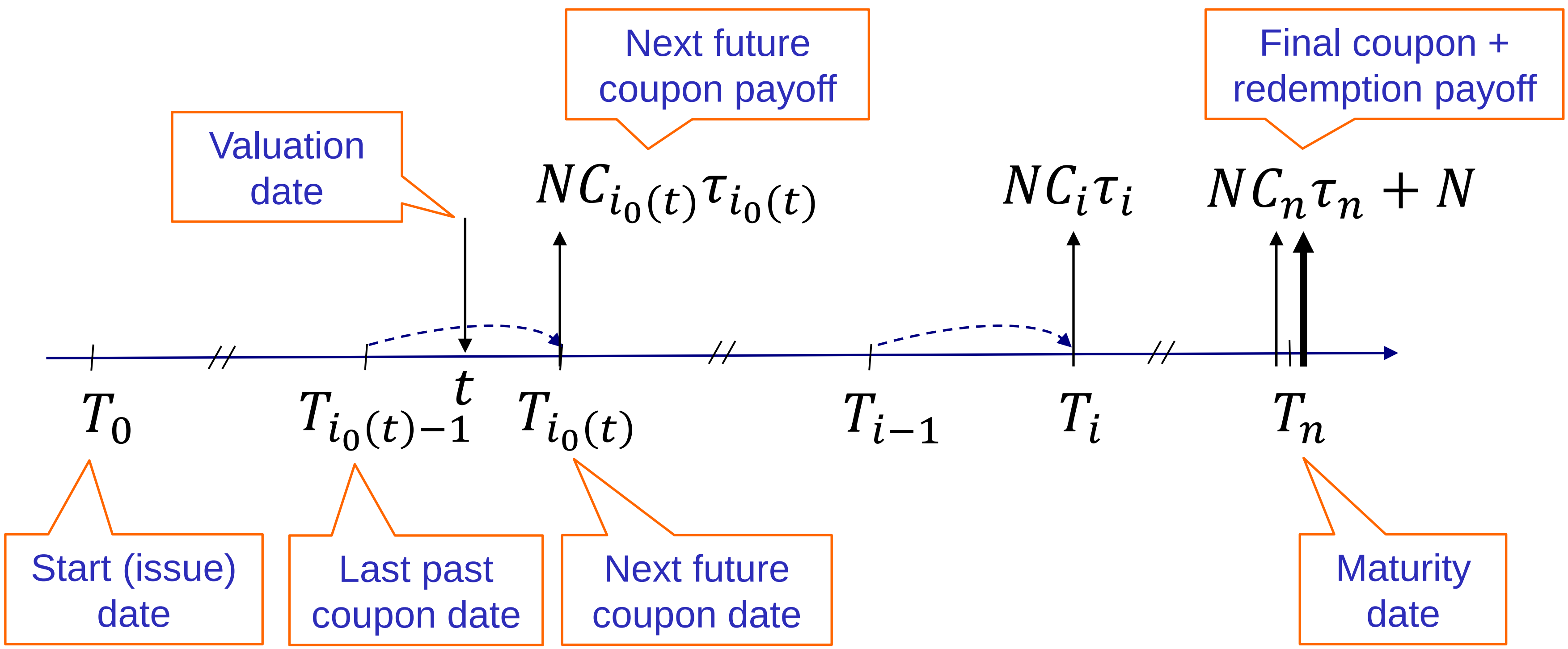
7. Bonds

- Payoff
- Pricing
- Risk management
- Spread curve construction
- Spread measures
- CDS vs cash spread
- Bloomberg example
- Negative rates

7: Bonds
Payoff [1]

Bonds are debt financial instruments issued by a Bond issuer and bought by a Bond buyer or holder. The bond buyer provides liquidity (the initial Bond value, typically 100 units of currency) to the bond issuer at the bond issue date, in exchange of a stream of future bond coupons and a final redemption (a.k.a. face value, or nominal value).

In particular, interest rate bonds provide coupons indexed to interest rates.



7: Bonds
Payoff [2]

There exist many different types of Bonds, depending on:

- Coupon structure
  - Fixed rate coupon
  - Floating rate coupon
  - Zero rate coupon
  - Structured coupon (Cap&Floor, CMS, CMS-Spread options, Ratchet, Reverse Floater, …)
  - Etc.

- Redemption structure
  - Bullet: entire face value is paid at once at the maturity date.
  - Amortizing: bond face value is paid through a schedule of payments.
  - Callable: the bond issuer can estinguish the bond by repaying the face value prior to maturity.
  - Putable: the bond holder can force the issuer to repay the face value prior to maturity.
  - Inflation-linked: face value is linked to fluctuations of inflation rates.
  - Etc.

7: Bonds
Pricing [1]

Bond pricing requires the usual discounted expectation of future cash flows, both coupons and redemption (face value). Since Bond issuers are defaultable, Bond cash flows, in particular the final redemption, are subject to the issuer's (or guarantors', if any) possibility to honour the bond holders with their credit rights as agreed in the Bond emission (creditworthiness).

The price of a defaultable bond is thus given by

$$V(t;\boldsymbol{T}_n) = E_t^Q\left[\sum_{i=i_0(t)}^{n} N D(t,T_i) C_i\, \tau_i 1\left[\tau_I(t) > T_i\right] + N\, D(t,T_n) 1\left[\tau_I(t) > T_n\right]\right]$$

Issuer default indicator

where:

$\boldsymbol{T}_n$	= bond schedule (vector of relevant date)
T_n	= maturity
τ_i	= year fraction for time interval $[T_{i-1}, T_i]$
$\tau_I(t)$	= default time of the bond issuer or guarantors at time t
$1[\tau_I(t) > T_i]$	= survival indicator of the issuer at time T_i
C_i	= coupon cash flow at date T_i (interest payments)
N	= nominal amount (commonly N=100)
$i_0(t)$	= index of the next coupon at time t, such that $t \in \left[T_{i_0(t)-1}, T_{i_0(t)}\right]$
$D(t,T_i)$	= discount factor

7: Bonds
Pricing [2]

Hence, Bond pricing requires the inclusion of the creditworthiness of the bond issuer at pricing time *t* (represented by the survival indicator). It is market practice to include the Issuer default component introducing an Issuer risky discount factor $P_d^I(t,T)$ such that the Bond price is given by

$$V(t;\boldsymbol{T}_n) = E_t^Q\left[\sum_{i=i_0(t)}^{n} N D(t,T_i) C_i\, \tau_i 1[\tau_I(t) > T_i] + N\, D(t,T_n) 1[\tau_I(t) > T_n]\right]$$

$$\cong N \sum_{i=i_0(t)}^{n} E_t^Q[C_i]\tau_i E_t^Q\big[D(t,T_i)1[\tau_I(t) > T_i]\big] + N E_t^Q\big[D(t,T_n)1[\tau_I(t) > T_n]\big]$$

$$= N \sum_{i=i_0(t)}^{n} E_t^Q[C_i]\tau_i P^I(t,T_i) + N P^I(t,T_n)$$

We assume coupons are independent of discount rates and default times

where we have defined the Issuer's risky discount factor

$$P^I(t,T_i) := E_t^Q\big[D(t,T_i)1[\tau_I(t) > T_i]\big].$$

Risky discount factor

The risky discount factor $P^I(t,T_i)$ is a defaultable zero coupon bond issued by *I* for maturity T_i.

7: Bonds
Pricing [3]

How to compute the risky discount factors?
We further assume that discount rates are independent of default times, such that

$$P^I(t,T_i) \cong E_t^Q[D(t,T_i)]E_t^Q\left[1[\tau_I(t) > T_i]\right] = P(t,T_i)Q^I(t,T_i),$$

where $Q^I(t,T_i)$ is the survival probability of the issuer in the time interval $[t,T_i]$. Thinking in terms of hazard rates [*] we can write

$$Q^I(t,T_i) = E_t^Q\left[1[\tau_I(t) > T_i]\right] = E_t^Q\left[exp\left(-\int_t^{T_i} \lambda_I(u)du\right)\right].$$

We further assume that hazard rates (or default times) are deterministic, and we finally obtain

$$Q^I(t,T_i) = exp[-z_I(t,T_i)\tau(t,T_i)],$$

$$z_I(t,T_i) := \frac{1}{\tau(t,T_i)}\int_t^{T_i} \lambda_I(u)du\,,$$

where $z_I(t,T_i)$ is a risky zero coupon rate for each coupon date T_i.

[*] See e.g. Brigo and Mercurio (2006), ch. 21.1.1.

7: Bonds
Pricing [4]

Hence, the risky discount factor is given by

$$P^I(t,T_i) \cong P(t,T_i)exp[-z_I(t,T_i)\tau(t,T_i)],$$

Finally, we rebase the risky zero coupon rate introducing a reference rate, usually Libor/Euribor (with rate tenor x) and a constant zero coupon spread, called Issuer z-spread, such that

$$P^I(t,T_i) = P_x^I(t,T_i) \coloneqq exp\{-[R_x(t,T_i) + z_I(t,T_n)]\tau(t,T_i)\}.$$

Reference rate

Issuer's z-spread

In conclusion, the pricing formula is

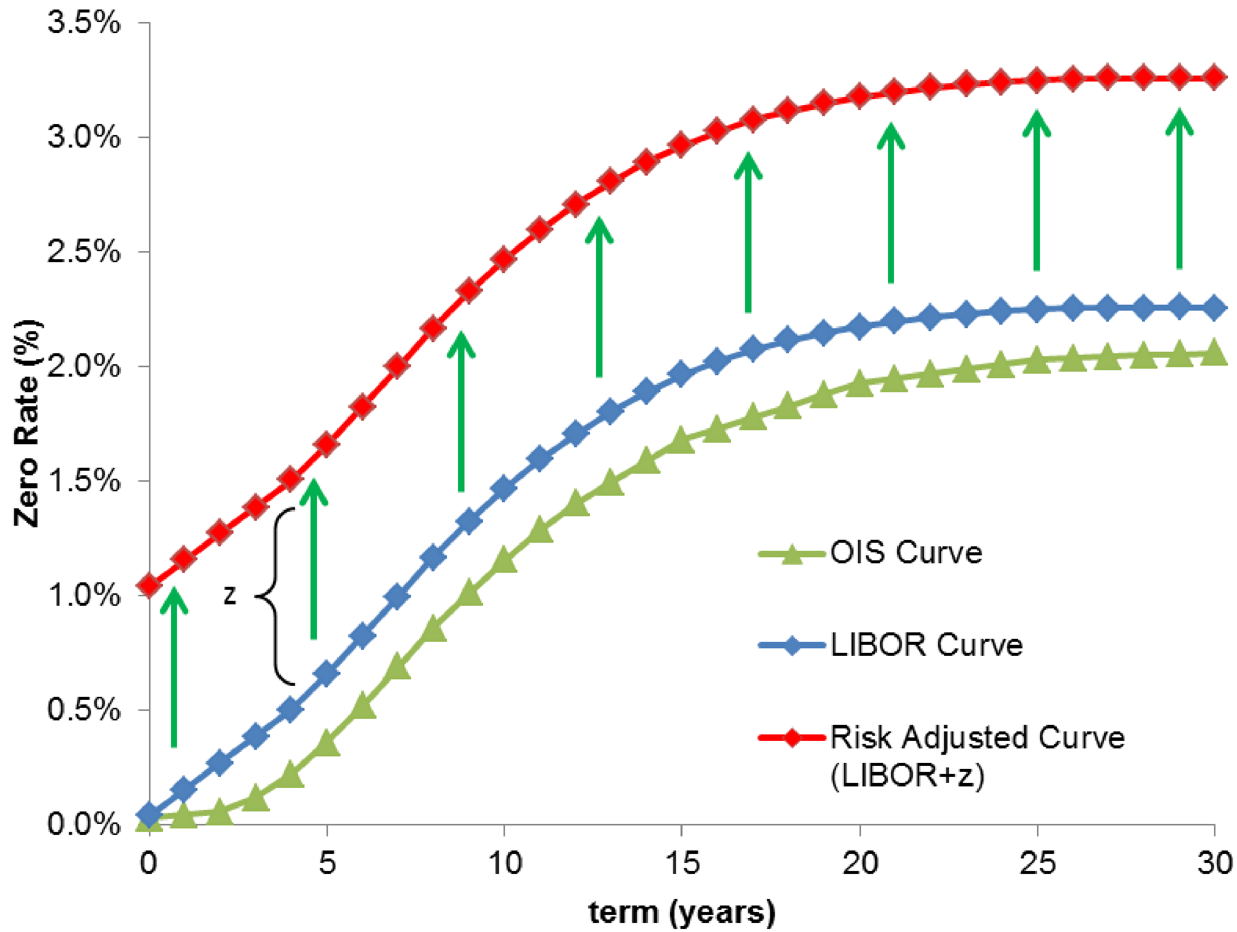
$$V(t;\boldsymbol{T}_n) = N\sum_{i=i_0(t)}^{n} P_x^I(t,T_i)E_t^Q[C_i]\tau_i + NP_x^I(t,T_n).$$

This formula is exact in the sense that all the information regarding Issuer's default and its correlations with the bond's underlyings is encoded in the z-spread.

Pricing [5]

We stress that

- both $R_x(t, T_i)$ and $z_I(t, T_n)$ are zero coupon rates,
- only the Libor rate $R_x(t, T_i)$ has a term structure,
- the z-spread is constant over the residual lifetime of the bond $[t, T_n]$.
- Hence, there exist a specific z-spread $z_I(t, T_n)$ for each bond issuer and each maturity.



7: Bonds
Pricing: Fixed Rate Notes

The coupons are given by

$$C_i = c \qquad \forall\, i\, \epsilon [1,2, \dots, n].$$

The payment stream of the FiRN is characterized by a schedule of fixed coupons

$$V(t;\boldsymbol{T_n}) = cNA^I_{x,n}(t\ ) + NP^I_x(t, T_n),$$

$$A^I_{x,n}(t) \coloneqq \sum_{i=i_0(t)}^{n} P^I_x(t, T_i)\tau_i\,.$$

Risky
annuity

7: Bonds
Pricing: Floating Rate Notes

The coupons are given by

$$C_i = L_y(t_{i-1}, t_i) + \Delta, \qquad \forall\, i = 2, \cdots n, C_1 = c_1$$

The payment stream of the FRN is characterized by a schedule of LIBOR coupons, plus a margin Δ. The current coupon is typically already fixed (also called last reset),

$$V(t; \boldsymbol{T_n}) = N P_x^I\left(t, T_{i_0(t)}\right)\left[L_y\left(T_{i_0(t)-1}, T_{i_0(t)}\right) + \Delta\right]\tau_{i_0(t)} + N \sum_{i=i_0(t)+1}^{n} P_x^I(t, T_i)\left[F_y(t; T_{i-1}, T_i) + \Delta\right]\tau_i + N\, P_x^I(t, T_n)$$

If $x = y$ and Δ is close to Z-spread we obtain a simplified formula

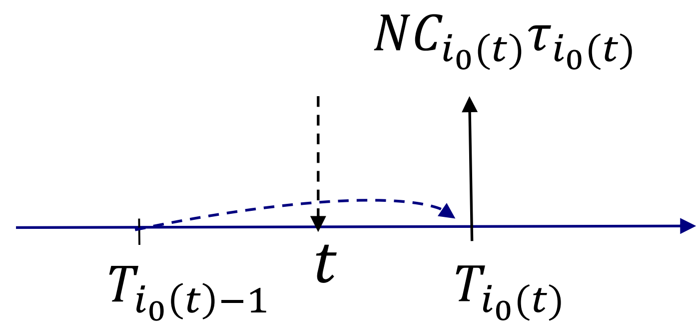
$$\left[F_y(t; T_{i-1}, T_i) + \Delta\right] \simeq \frac{1}{\tau_i}\left[\frac{P_x^I(t, T_{i-1})}{P_x^I(t, T_i)} - 1\right],$$

$$\begin{aligned} V(t; \boldsymbol{T_n}) &\simeq N P_x^I\left(t, T_{i_0(t)}\right)\left[L_y\left(T_{i_0(t)-1}, T_{i_0(t)}\right) + \Delta\right]\tau_{i_0(t)} \\ &+ N P_x^I\left(t, T_{i_0(t)}\right) - N P_x^I(t, T_n) + N P_x^I(t, T_n) = \\ &N P_x^I\left(t, T_{i_0(t)}\right)\left[L_x\left(T_{i_0(t)-1}, T_{i_0(t)}\right) + \Delta\right]\tau_{i_0(t)} + N P_x^I\left(t, T_{i_0(t)}\right) \end{aligned}$$

Pricing [6]

Clean and dirty prices

The Bond price discussed so far includes also the current coupon, such that $t \in \left[T_{i_0(t)-1}, T_{i_0(t)}\right]$, thus including the interest component accrued since last payment date, during the interval $[T_{i-1}, t\,]$, called accrued interest. This is called a dirty price.



In case of bond selling, the accrued interest is kept by the bond seller, not transferred to the bond buyer. Hence, Bonds are usually quoted in terms of clean prices, excluding the accrued interest. The clean price is given by

$$V(t;\boldsymbol{T}_n) = V_A\left(t, T_{i_0(t)}, I\right) + V_{Clean}(t;\boldsymbol{T}_n, I),$$

$$V_A\left(t, T_{i_0(t)}\right) \coloneqq NC_{i_0(t)}\tau\left(T_{i_0(t)-1}, t\right),$$

Accrued interest

$$V_{Clean}(t;\boldsymbol{T}_n) \coloneqq V(t;\boldsymbol{T}_n) - V_A\left(t, T_{i_0(t)}\right)$$

$$= \sum_{i=i_0(t)}^{n} NP_x^I(t, T_i)E_t^Q[C_i]\tau_i + NP_x^I(t, T_n) - NC_{i_0(t)}\tau\left(T_{i_0(t)-1}, t\right).$$

Bonds depends mainly on the level of spot interest rates and credit spreads.

1: Fixed Rate Notes (FiRN) sensitivity

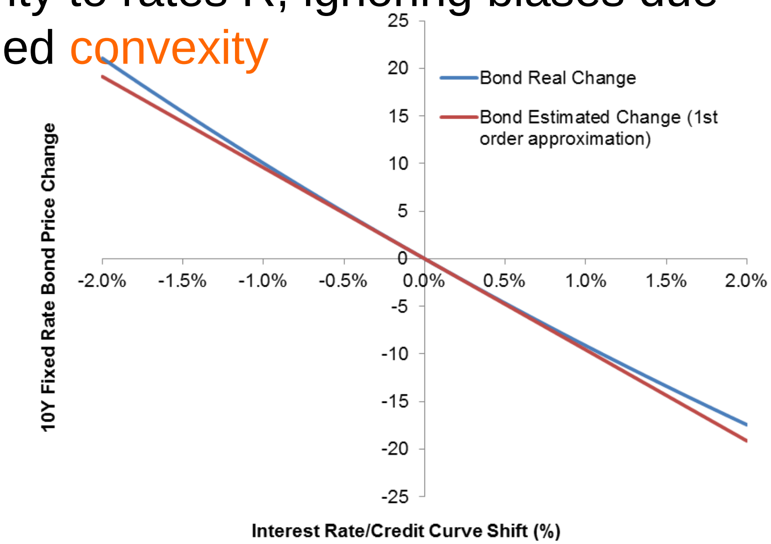
$$\frac{\partial V(t;\boldsymbol{T_n})}{\partial R} \approx -\frac{N\sum_{i=i_0(t)}^{n} cP_x^I(t,T_i)\tau_i\,(T_i-t)+NP_x^I(t,T_n)(T_n-t)}{V(t;\boldsymbol{T_n})}V(t;\boldsymbol{T_n}) = -Modified\ Duration\ V(t;\boldsymbol{T_n})$$

$$\frac{\partial V(t;T_n)}{\partial z}=\frac{\partial V(t;T_n)}{\partial R}$$

In case of fixed rate bonds, interest rate shift (R) affects only the discount factors. Then the sensitivity to credit spread (z) equals the sensitivity to rates R, ignoring biases due to different conventions. Second order effect is defined convexity

- DV01 is defined as the price change related to 1bp interest rate curve shift (R)
- CR01 is defined as the price change related to 1bp credit spread curve shift

$$\text{DV01}=\frac{\partial V(t;\boldsymbol{T_n})}{\partial R}1bp \qquad \text{CR01}=\frac{\partial V(t;\boldsymbol{T_n})}{\partial z}1bp$$



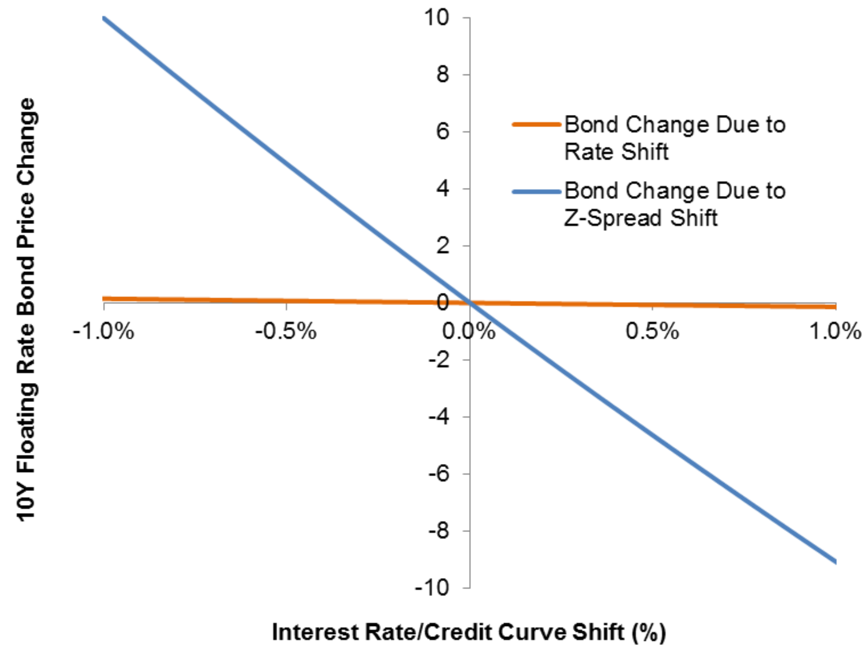
2: Floating Rate Notes (FRN) sensitivity:

$$\frac{\partial V(t;T_n)}{\partial z} = -Modified\ Duration\ V(t;\boldsymbol{T_n})$$

If $x = y$ and Δ is close to Z-spread

$$\frac{\partial V(t;T_n)}{\partial R} \approx \frac{\partial\left[NP_x^I\left(t,T_{i_0(t)}\right)\left[L_x\left(T_{i_0(t)-1},T_{i_0(t)}\right)+\Delta\right]\tau_{i_0(t)}+NP_x^I\left(t,T_{i_0(t)}\right)\right]}{\partial R} \gg \frac{\partial V(t;T_n)}{\partial z}$$

FRNs are fairly insensitive to interest rate curve shifts (DV01 $\approx$ 0) since the discounting sensitivity is balanced by floating rate sensitivity.
On fixing dates, the sensitivity is null.



Spread curve construction [1]

$$V(t;\boldsymbol{T_n}) = N \sum_{i=i_0(t)}^{n} E[C_i]\tau_i e^{-[R_x(t,T_i)+z_I(t,T_n)]\tau(t,T_i)} + N\, e^{-[R_x(t,T_n)+z_I(t,T_n)]\tau(t,T_n)}$$

R_x Rates

Interest rate curves (i.e. Libor/Euribor.

Z_I Z-Spread

Credit spread curves from selected bonds issued by ***I*** that are actively quoted on the market (contributed) called Benchmarks.

7: Bonds
Spread curve construction [2]

From **PRICE** of FiRNs contributed in the market (Benchmarks)

Reverse Engineering

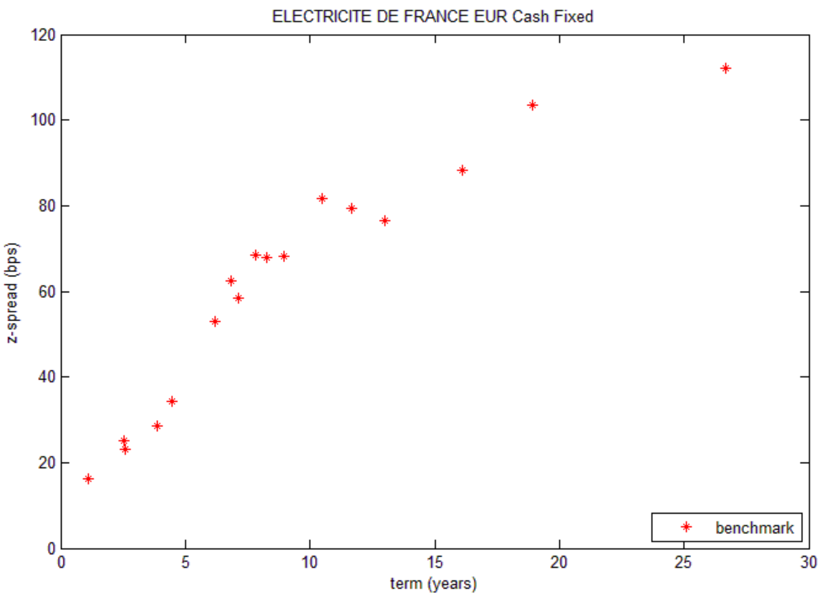
Z-SPREAD for each maturity of benchmarks

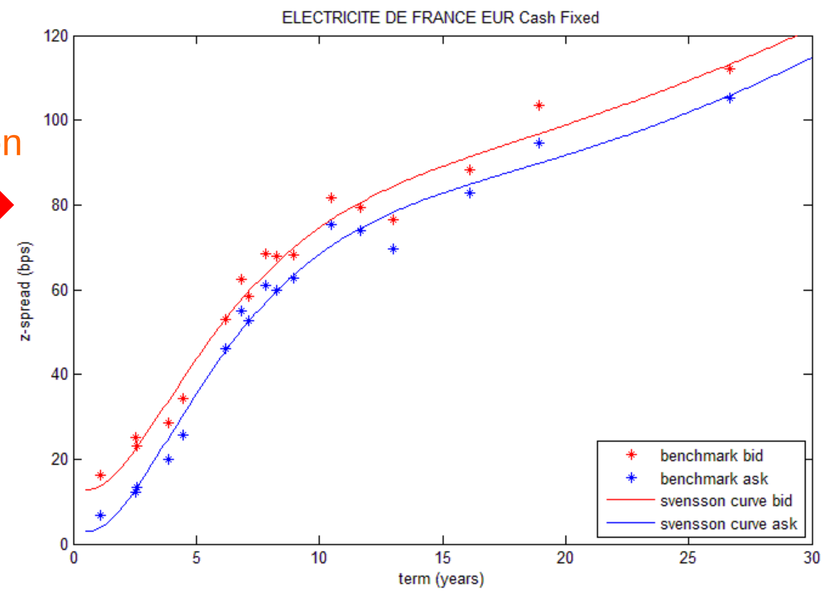
Interpolation

Z-SPREAD calculated for term of Non Contributed Bond + rates

Bond Pricing

PRICE of non-contributed Bond





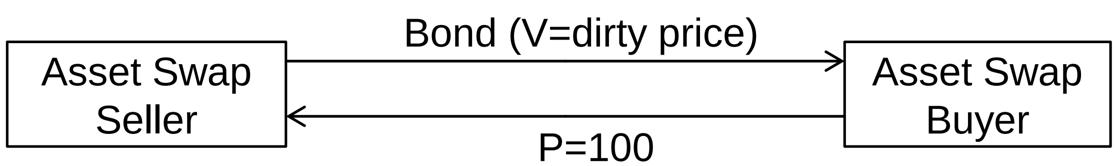
7: Bonds

Spread measures: asset swap spread [1]

Asset Swaps are combinations of interest rate Bonds and Swaps used to transform the cash flow stream of the Bond, e.g. fixed into floating or viceversa. In this way Bonds investors may hedge the currency and interest rate risks of the Bond using interest rate derivatives. The asset swap market developed with the swap market in the early 1990s.

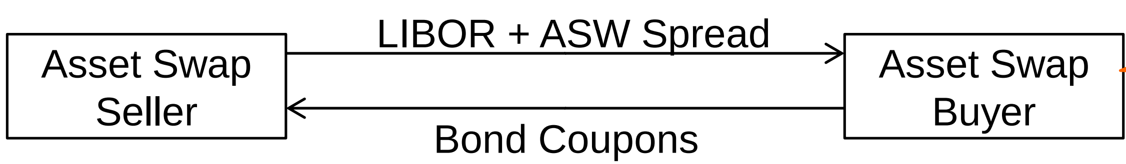
ASW combines:

- A Bond purchase at par (P=100)



The ASW buyer is exposed to market and default risk

- An interest rate swap, with the same Bond maturity, to exchange bond coupons with floating payments plus a margin, called Asset Swap Spread (ASW)



The ASW buyer transforms the market risk and is compensated for the credit/liquidity risk by the ASW spread

The ASW spread reflects the creditworthiness of the Bond issuer, and is another common credit risk measure in bond market.

7: Bonds
Spread measures: asset swap spread [2]

The ASW-Spread is defined as the equilibrium spread such that the total NPV of all the cash flows is null. From the point of view of the Asset Swap seller we have

$$0 = N(100 - V) + N\left\{\sum_{i=1}^{n} P_x(t,T_i)E_t^Q[C_i]\tau_i - \sum_{j=1}^{m} P_d\left(t,T_j\right)\left[F_{x,j}(t) + R_m^{asw}(t)\right]\tau_j\right\}$$

$$= N(100 - V) + N\left\{\sum_{i=1}^{n} P_x(t,T_i)E_t^Q[C_i]\tau_i - \sum_{j=1}^{m} P_d\left(t,T_j\right)F_{x,j}(t)\tau_j + R_m^{asw}(t)A_{d,m}(t)\right\}$$

Hence the asset swap spread is given by

$$R_m^{asw}(t) = \frac{\sum_{i=1}^{n} P_x(t,T_i)E_t^Q[C_i]\tau_i - \sum_{j=1}^{m} P_d\left(t,T_j\right)F_{x,j}(t)\tau_j + 100 - V}{A_{d,m}(t)}$$

Notice that discounting is consistent with the bond and IRS conventions: Bond's cashflow are discounted at Libor flat (without z-spread), while the IRS cash flows (collateralised) are discounted at collateral rate.

See e.g.

- D. O'Kane, "*Introduction to Asset Swaps*", Lehman Brothers Structured Credit Research, Jan. 2000,
- D. O'Kane and S. Sen, "*Credit Spreads Explained*", Lehman Brothers Structured Credit Research, Mar. 2004.

7: Bonds
Spread measures: YTM and co.

- The yield to maturity (YTM), $R_{YTM}(t, T_n)$, is the constant discount rate (annual compounding) such that

$$V(t; \boldsymbol{T}_n) = \sum_{i=i_0(t)}^{n} NP_x^I(t, T_i) E_t^Q[C_i]\tau_i + NP_x^I(t, T_n)$$

$$:= \sum_{i=i_0(t)}^{n} \frac{N E_t^Q[C_i]\tau_i}{(1 + R_{YTM}(t, T_n))^{T_i - t}} + \frac{N}{(1 + R_{YTM}(t, T_n))^{T_n - t}}$$

- The yield spread is the difference between a bond R_{YTM} and the R_{YTM} of a benchmark bond (ex. spread BTP-Bund 10y, “lo spread”)

> Italians Obsessed by ‘Lo Spread’ as Advance in Bond Yields Makes Headlines

- The I-spread is the difference between bond R_{YTM} and the linearly interpolated rate to the same maturity on an appropriate reference curve (government benchmark bonds, swap rates, …). The I-spread against Government reference curve is also called G-spread.

Spread measures: summary

Spread	Defnition
Z spread	Constant discount rate, continuous compounding
Asset swap spread	Equilibrium spread of the asset swap
Yield to maturity	Constant discount rate, discrete compounding
Yield spread	Difference between a bond YTM and the YTM of a benchmark bond
I-spread	Difference between bond R_{YTM} and the linearly interpolated rate to the same maturity on an appropriate reference curve

7: Bonds

CDS spread vs cash spread *[1]*

- The Credit Default Swap (CDS) is a credit derivative transaction in which two parties enter into an agreement, whereby one party (the Protection Buyer) pays the other party (the Protection Seller) periodic premiums (a.k.a. *spread*) for the specified life of the agreement. The Protection Seller makes no payment unless a credit event related to a predetermined reference asset occurs. If such an event occurs, it triggers the Protection Seller's settlement obligation, which can be either cash or physical.

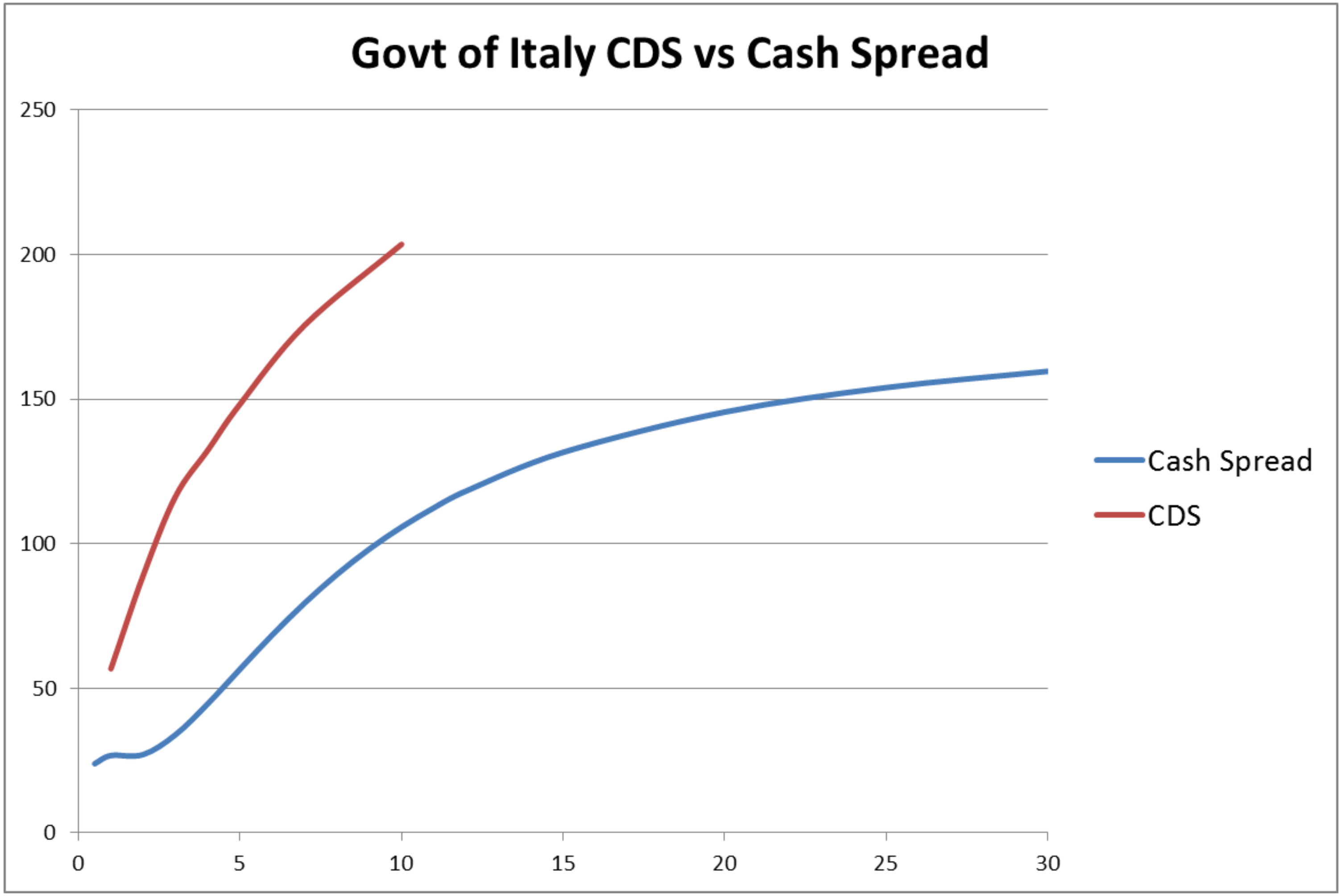
- CDS spreads are independent from funding concerns since they are an expression of derivative market. The difference between Credit Default Spread and Cash Spread is the CDS-Bond Basis.

- The main factors affecting the CDS-Bond basis are:
  - Fundamental factors, related to the different mechanics of the two instruments
  - Market factors, related to different liquidity supply and demand, CDS vs Bond markets segmentation, etc.

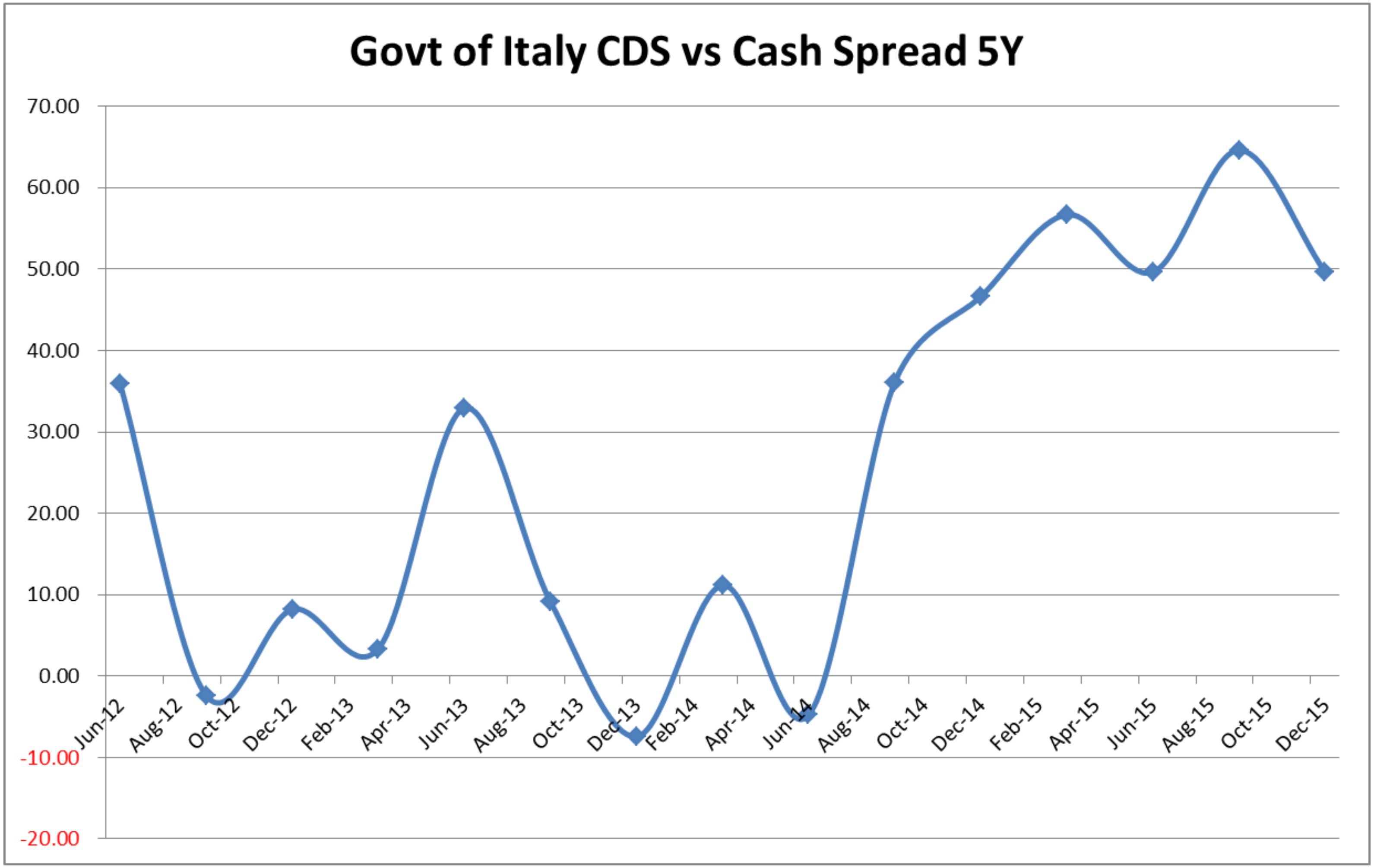
- The CDS-Bond basis is quite volatile, mostly in liquidity shortage periods when sell-off pressure on bonds drives the basis even in negative territory.

See e.g. D. O'Kane R. McAdie, "*Explaining the Basis: Cash versus Default Swaps*", Lehman Brothers Structured Credit Research, May 2001.

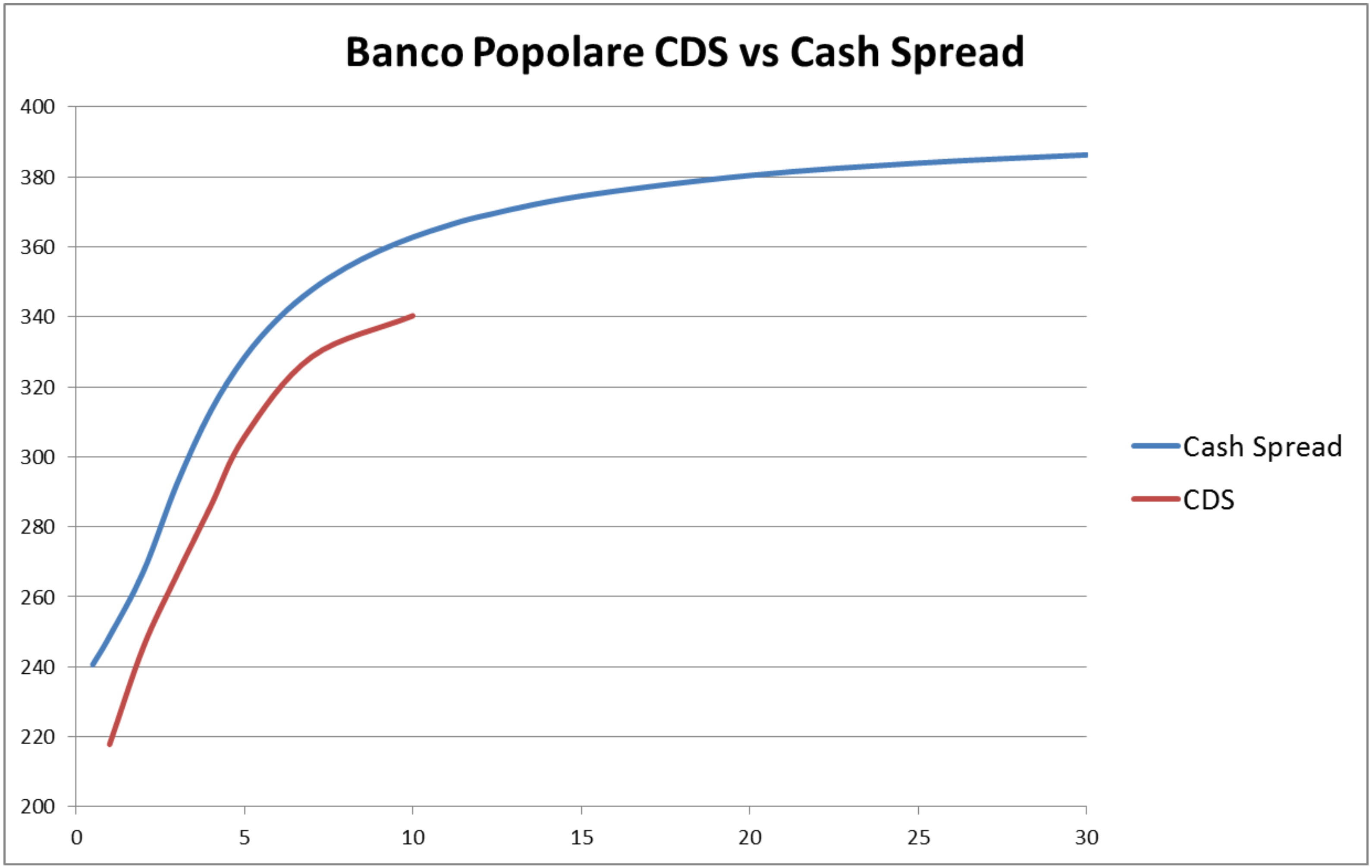
CDS spread vs cash spread *[2]*



7: Bonds
CDS spread vs cash spread *[3]*

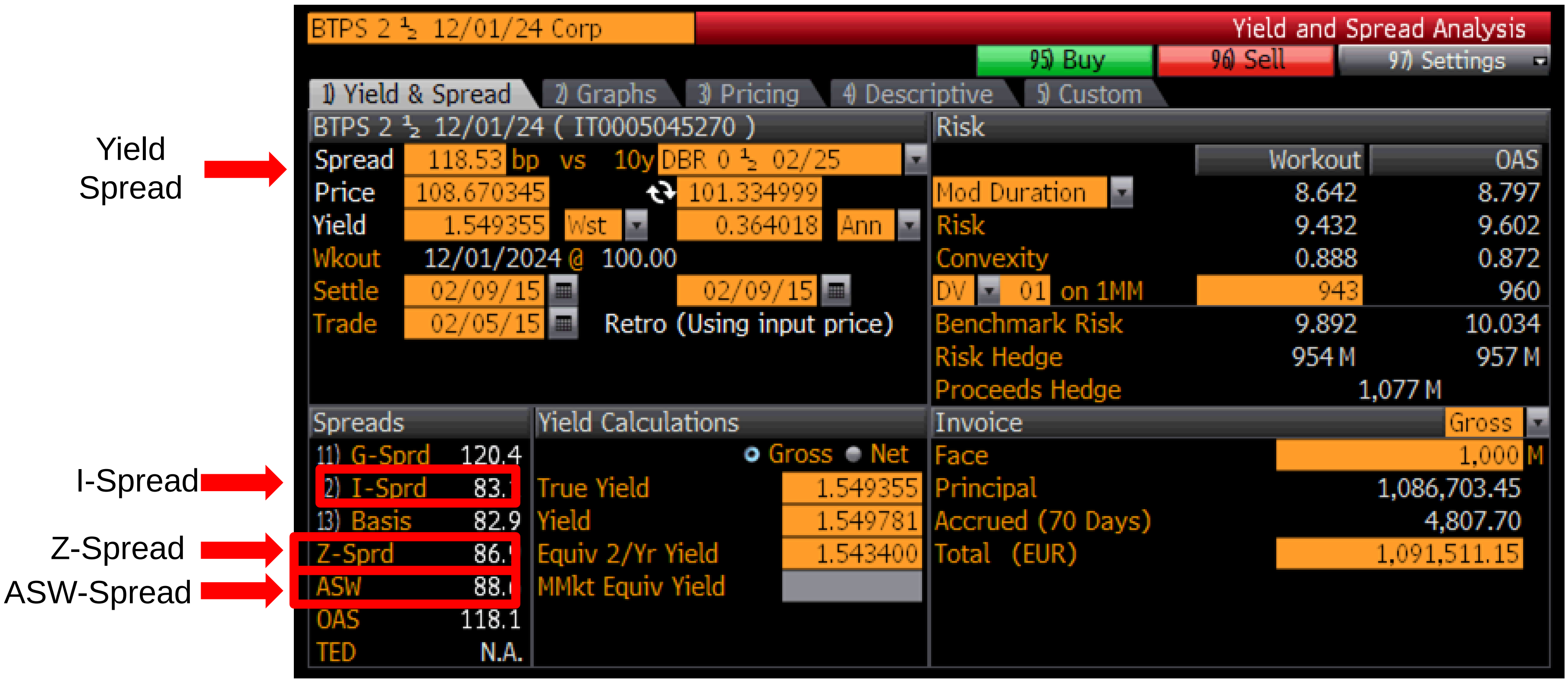


7: Bonds
CDS spread vs cash spread *[4]*



Bloomberg example [1]

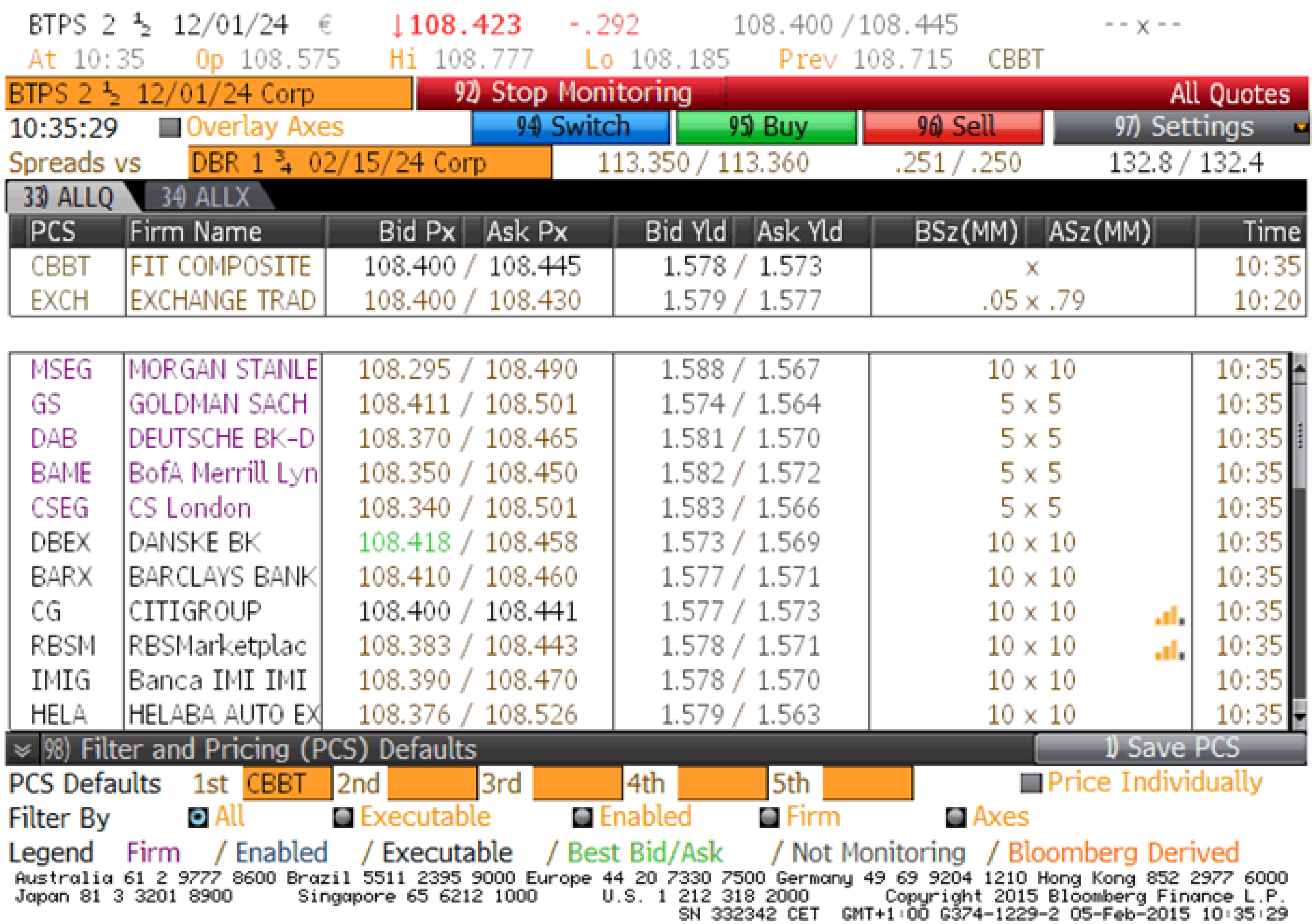
Bloomberg is the most widespread platform for bond pricing. YAS is the Bloomberg page for bond pricing and analytics (credit spreads, sensitivities, comparisons, …)



Source: Bloomberg Screenshot YAS

7: Bonds
Bloomberg example [2]

Market dealers share price information (quotes, spreads, executability, volumes) in ALLQ pages. Similar pages exist in Reuters.

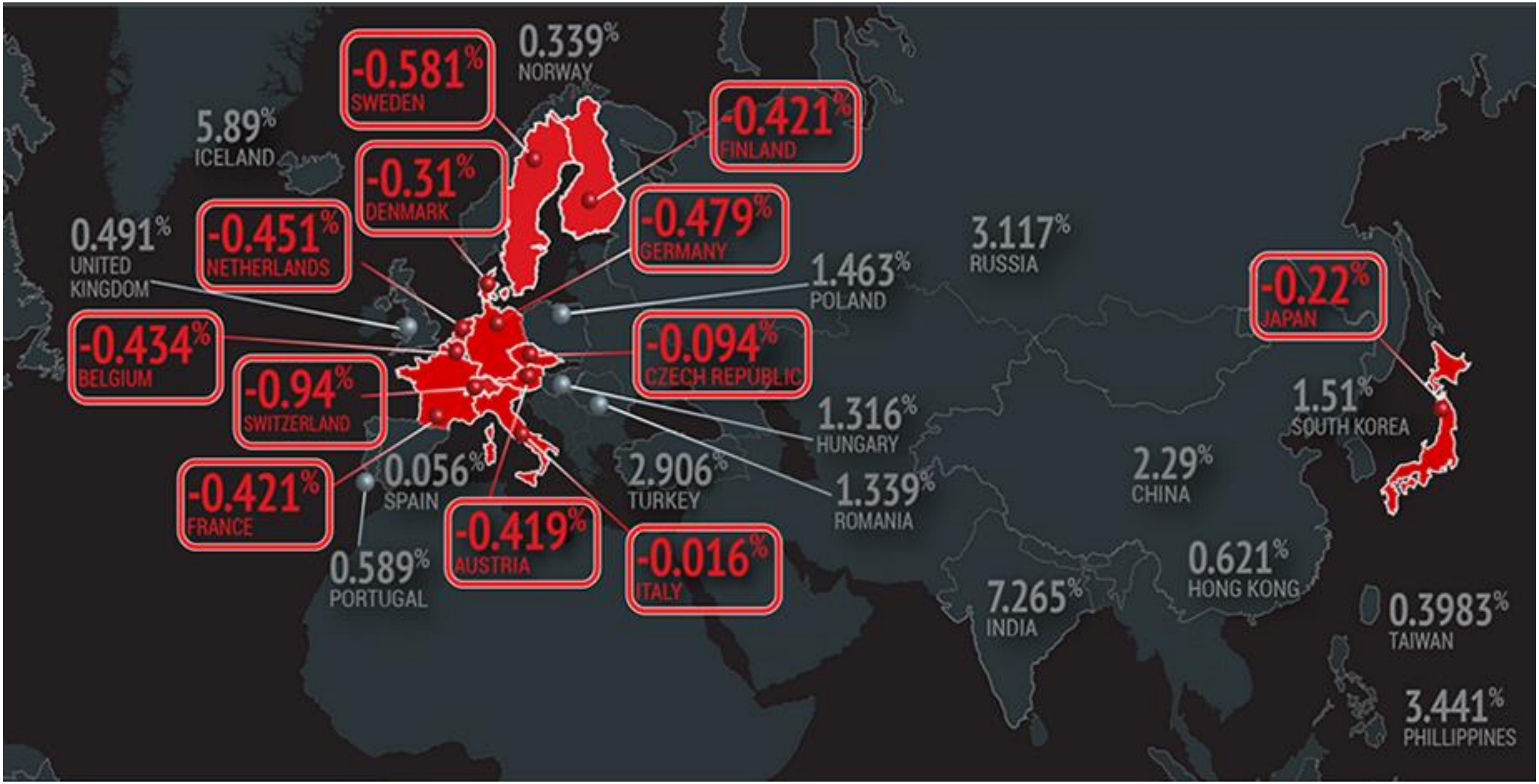


BTPS 2 ½ 12/01/24 € ↓108.423 -.292 108.400 / 108.445 --x--
At 10:35 Op 108.575 Hi 108.777 Lo 108.185 Prev 108.715 CBBT
BTPS 2 ½ 12/01/24 Corp | 92) Stop Monitoring | All Quotes
10:35:29 Overlay Axes | 94) Switch | 95) Buy | 96) Sell | 97) Settings
Spreads vs DBR 1 ¾ 02/15/24 Corp | 113.350 / 113.360 | .251 / .250 | 132.8 / 132.4
33) ALLQ | 34) ALLX

PCS	Firm Name	Bid Px / Ask Px	Bid Yld / Ask Yld	BSz(MM) x ASz(MM)	Time
CBBT	FIT COMPOSITE	108.400 / 108.445	1.578 / 1.573	x	10:35
EXCH	EXCHANGE TRAD	108.400 / 108.430	1.579 / 1.577	.05 x .79	10:20
MSEG	MORGAN STANLE	108.295 / 108.490	1.588 / 1.567	10 x 10	10:35
GS	GOLDMAN SACH	108.411 / 108.501	1.574 / 1.564	5 x 5	10:35
DAB	DEUTSCHE BK-D	108.370 / 108.465	1.581 / 1.570	5 x 5	10:35
BAME	BofA Merrill Lyn	108.350 / 108.450	1.582 / 1.572	5 x 5	10:35
CSEG	CS London	108.340 / 108.501	1.583 / 1.566	5 x 5	10:35
DBEX	DANSKE BK	108.418 / 108.458	1.573 / 1.569	10 x 10	10:35
BARX	BARCLAYS BANK	108.410 / 108.460	1.577 / 1.571	10 x 10	10:35
CG	CITIGROUP	108.400 / 108.441	1.577 / 1.573	10 x 10	10:35
RBSM	RBSMarketplac	108.383 / 108.443	1.578 / 1.571	10 x 10	10:35
IMIG	Banca IMI IMI	108.390 / 108.470	1.578 / 1.570	10 x 10	10:35
HELA	HELABA AUTO EX	108.376 / 108.526	1.579 / 1.563	10 x 10	10:35

98) Filter and Pricing (PCS) Defaults | 1) Save PCS
PCS Defaults 1st CBBT 2nd 3rd 4th 5th | Price Individually
Filter By All | Executable | Enabled | Firm | Axes
Legend Firm / Enabled / Executable / Best Bid/Ask / Not Monitoring / Bloomberg Derived
Australia 61 2 9777 8600 Brazil 5511 2395 9000 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2015 Bloomberg Finance L.P.
SN 332342 CET GMT+1:00 G374-1229-2 05-Feb-2015 10:35:29

7: Bonds
Negative yields



Source: Numerix + Bloomberg, snapshot of 2Y Governement bond yields on March 21, 2016
(*http://www.numerix.com/info-graphic/negative-rates-trends-continue-2016*)

Summary

8. Forward rate modelling

- Black model
- Volatility smile
- Beyond the Black model
- Shifted-Black model
- SABR model
- Shifted-SABR model

8: Forward rate modelling
Introduction

We have seen that the risky forward rate

$$F_{x,i}(t) := \mathbb{E}_t^{Q^{T_i}} \left[L_x(T_{i-1}, T_i) \right],$$

- is a martingale under the T_i-forward measure Q_{T_i} associated to the discounting zero coupon bond $P_d(t, T_i)$,
- is quoted on the market of linear interest rate instruments (FRAs, IRSs, Basis Swaps),
- can be used as the basic building block of the modern interest rate modelling framework.

We wish to extend our framework to non-linear financial instruments, such as options, characterised by the prototypical payoff

$$\mathrm{Max} \left\{ \omega \left[L_x(T_{i-1}, T_i) - K; 0 \right] \right\},$$

where K is a strike value for the Libor rate L_x.

Thus, we need to model the dynamics of the risky forward rate $F_{x,i}(t)$.

8: Forward rate modelling
Black model [1]

We remind that risky forward rates are martingales under their natural T_i-forward discounting measure. We thus assume, for each forward rate, a lognormal martingale (driftless) dynamics

$$\boxed{\begin{gathered} dF_{x,i}(t) = F_{x,i}(t)\sigma_{x,i}(t)dW_{x,i}^{Q^{T_i}}(t), \\ dF_{x,i}(t)dF_{y,j}(t) = \delta_{x,y}\delta_{i,j}dt, \quad \forall\, x, y, i, j, \\ F_{x,i}(t_0) = F_{x,i,0}, \end{gathered}}$$

Lognormal forward dynamics

where

- $F_{x,i}(t)$ is a generic risky forward rate with tenor *x* , related to the future time interval $[T_{i-1}, T_i]$, observed at time $t < T_{i-1}$,
- $\sigma_{x,i}(t)$ is lognormal instantaneous volatility of the process $dF_{x,i}(t)$,
- $dW_{x,i}^{Q^{T_i}}(t)$ is a standard brownian motion under Q^{T_i},
- Q^{T_i} is the T_i - forward measure associated to the discounting numeraire $P_d(t, T_i)$,
- $F_{x,i,0}$ is the initial condition of the SDE.

Notice that this model has one parameter: the instantaneous volatility $\sigma_{x,i}(t)$. This parameter is, in general, not set by *a priori considerations* (e.g. historical analysis), but chosen such that a given market price is recovered → see the following.

8: Forward rate modelling
Black model [2]

The lognormal dynamics allows a relevant closed-form solution, called Black formula, for the expectation

$$\mathbb{E}_t^{Q^{T_i}}\left\{\mathrm{Max}\left\{\omega\left[L_x(T_{i-1},T_i)-K;0\right]\right\}\right\}=\mathrm{Black}\left[F_{x,i}(t),K,v_x(t;T_{i-1}),\omega\right],$$

where

$$\mathrm{Black}\left[F_{x,i}(t),K,v_{x,i}(t;T_{i-1}),\omega\right]=\omega\left\{F_{x,i}(t)\Phi\left(\omega d_i^{+}\right)-K\Phi\left(\omega d_i^{-}\right)\right\},$$

$$d_i^{\pm}=\frac{\ln\frac{F_{x,i}(t)}{K}\pm\frac{1}{2}v_x(t;T_{i-1})}{\sqrt{v_x(t;T_{i-1})}},$$

$$\Phi(x)=\frac{1}{2\pi}\int_{-\infty}^{x}e^{-\frac{1}{2}y^2}\,dy,$$

LN implied forward variance

$$v_x(t;T_{i-1})=\int_t^{T_{i-1}}\sigma_{x,i}(u)^2\,du,$$

LN implied forward volatility

$$\sigma_x(t;T_{i-1}):=\sqrt{\frac{v_x(t;T_{i-1})}{\tau_x(t;T_{i-1})}}=\sqrt{\int_t^{T_{i-1}}\frac{\sigma_{x,i}(u)^2}{\tau_x(t;T_{i-1})}\,du}.$$

8: Forward rate modelling
Black model [3]

Remarks

- The LN Black formula is a monotonic increasing function of the LN implied volatility $\sigma_x(t;T_{i-1})$. Hence, there exist a unique implied volatility corresponding to a given price, often called LN (or Black) implied volatility.

- The LN dynamics is not assumed to be a good proxy for the true underlying dynamics. The Black formula is actually used as a model-based parametric form useful to map option prices into LN implied volatilities.

- There exist simple analytical formulas for Black greeks, i.e. delta, vega, etc.

8: Forward rate modelling
Black model [4]

Criticism

The basic assumption of the Black model is that forward rates $F_{x,i}(t)$ are independent lognormal martingales under the discounting T_i-forward measure associated to the discounting numeraire $P_d(t, T_i)$ has two main drawbacks.

1. The LN instantaneous volatility $\sigma_{x,i}(t)$ and the corresponding LN implied volatility $\sigma_x(t; T_{i-1})$ are an intrinsic property of the FRA rate $F_{x,i}(t)$ itself, irrespective of the specific contract payoff in which they enter. In particular, they do not depend on the option's strike. Actually, since the 1987 market crash the market option prices display Black implied volatilities with a strong strike dependence, called "skew" or "smile".

2. The lognormal distribution for the forward rates is strictly positive, such that the model can't describe negative rates. Actually, since Dec. 2012, negative interest rates have systematically appeared on the market.

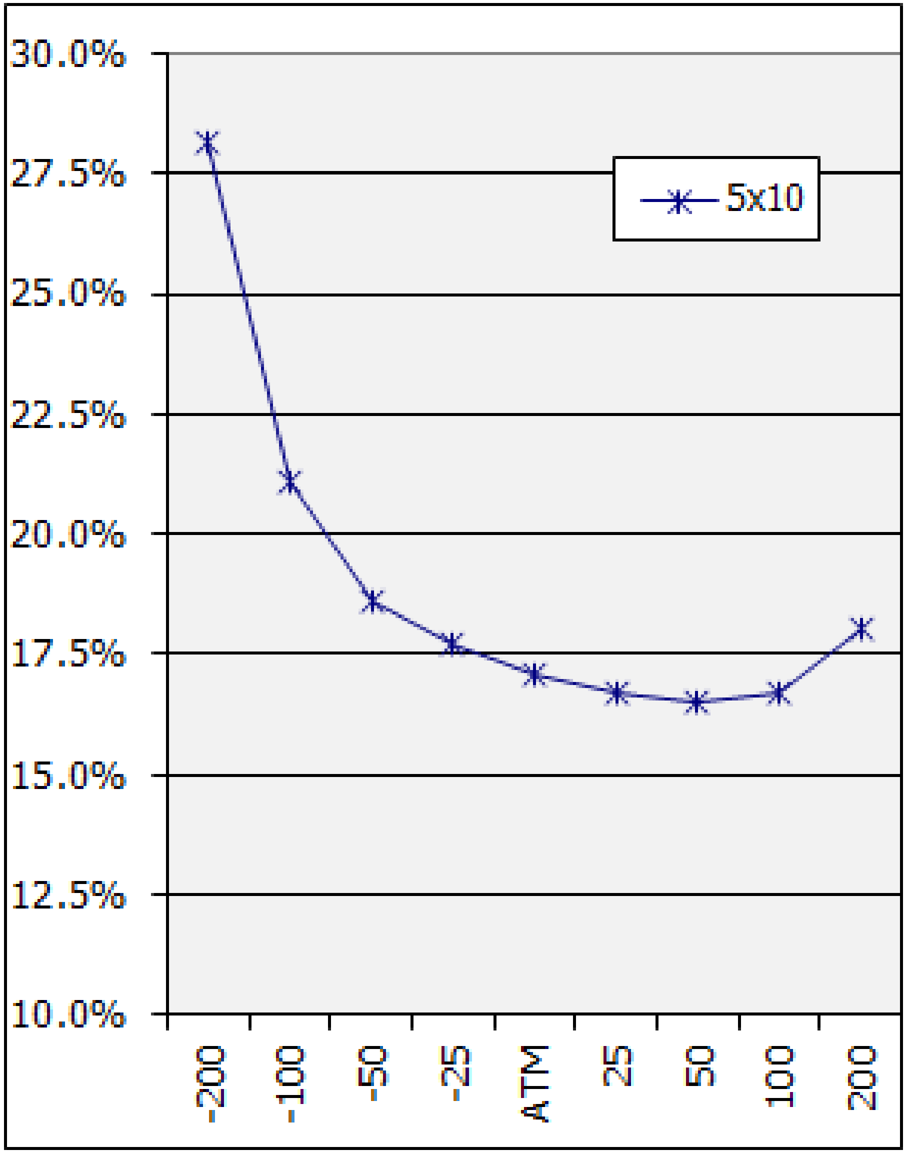
We conclude that the Black model is too simple to explain the observed market data (it is falsified in Popper's sense) and needs a generalisation.

8: Forward rate modelling
IR Volatility smile

Example of volatility smile.

The picture reports the lognormal Black implied volatility for an Eur Swaption with maturity 5Y and underlying swap tenor 10Y, for 9 strike values relative to the ATM forward swap rate.

Source: Reuters, 30 Sep. 2009.



8: Forward rate modelling
Beyond the Black model [1]

We may write, in a general single-FRA rate framework,

$$dF_{x,i}(t) = f\left[t, F_{x,i}(t)\right] dW_{x,i}^{Q^{T_i}}(t),$$
$$F_{x,i}(T_{i-1}) = L_x(T_{i-1}, T_i),$$
$$dW_{x,i}^{Q^{T_i}}(t) dW_{x,j}^{Q^{T_i}}(t) = \delta_{i,j} dt,$$

under the discounting T_i-forward measure associate to the discounting numeraire $P_d(t;T_i)$, where f is a function of the FRA rate.

We distinguish two main cases:

- when f is a deterministic function of the FRA rate, we have local volatility models,
- when f is a stochastic function of the FRA rate, we have stochastic volatility models.

Thus choosing a model M is equivalent to choose a particular form for the function f.

8: Forward rate modelling
Beyond the Black model [2]

Given the pricing model *M*, we can always obtain a Black implied volatility surface with the following procedure (see Brigo and Mercurio (2006), sec. 3.6).

1. Select a set of options with different maturities $\{T_1, \cdots, T_n\}$ and strikes $\{K_1, \cdots, K_m\}$.

2. Select a pricing model *M* for these options, and compute the *M*-model prices $V^M(t)$ for the set of options, i.e. compute the expectation

$$V^{\mathcal{M}}(t, T_i, K_j) = N P_d(t, T_i) \mathbb{E}_t^{Q^{T_i}} \left[V(T_i, T_i, K_j) \right].$$

3. Invert Black formula, i.e. solve the equation

$$V^{\mathcal{M}}(t, T_i, K_j) = N P_d(t, T_i) \text{Black} \left[F_{x,i}(t), K_j, v_x^{\text{Black}}(t; T_{i-1}, K_j), \omega \right]$$

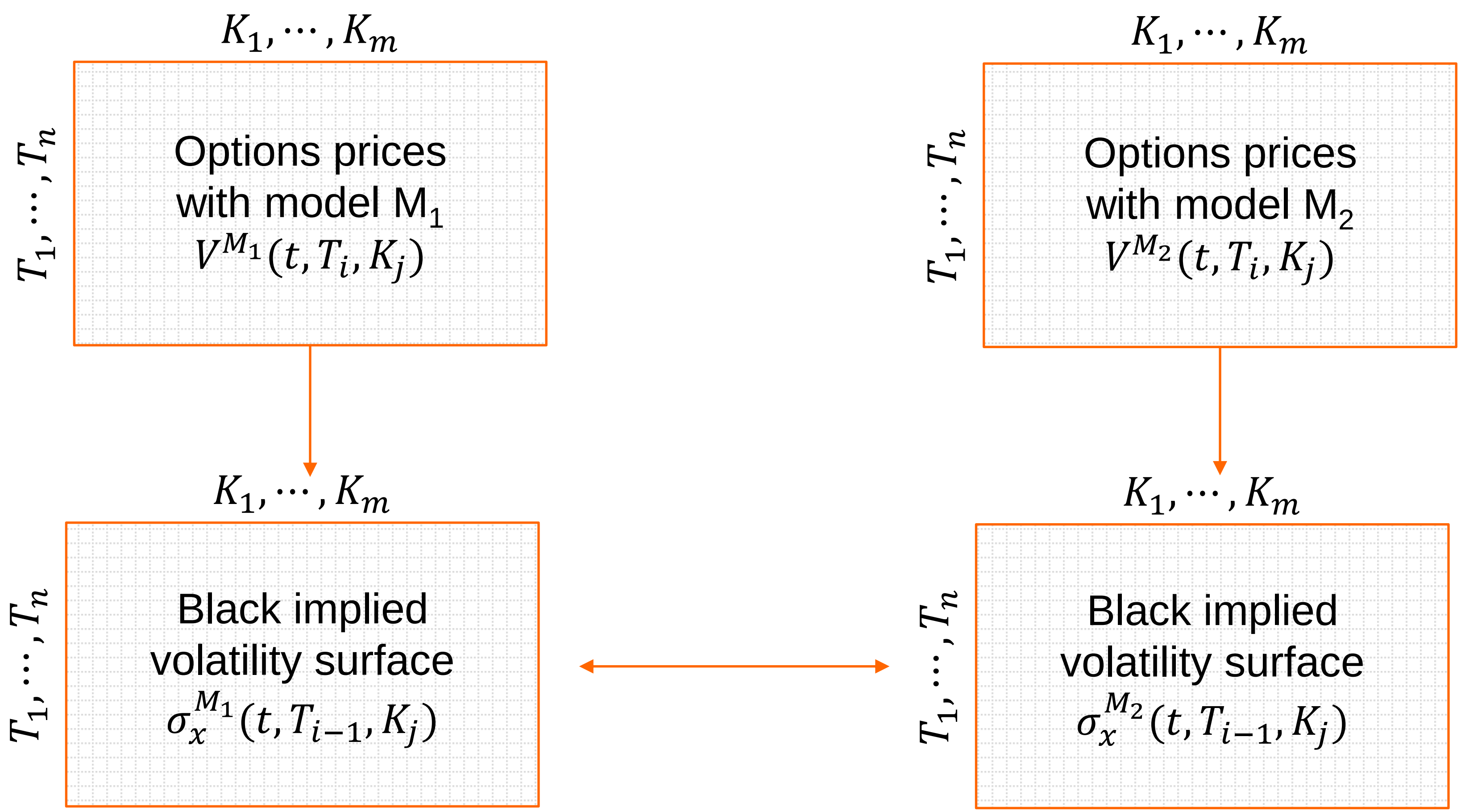
for the Black implied volatility

$$\sigma_x^{\text{Black}}(t; T_{i-1}, K_j) = \sqrt{\frac{v_x^{\text{Black}}(t; T_{i-1}, K_j)}{\tau_x(t; T_{i-1})}},$$

4. Repeat for all option strikes and maturities, gather results and obtain the *M*-model (Black) implied volatility smile.

8: Forward rate modelling
Beyond the Black model [3]

Black implied volatility surface



Different models calibrated to the same market prices will display equal implied volatilities at market points but different implied volatilities elsewhere.

8: Forward rate modelling
Beyond the Black model [4]

1. Normal model:

$$f\left[t, F_{x,i}(t)\right] := \sigma_{x,i}(t),$$
$$dF_{x,i}(t) = \sigma_{x,i}(t) dW_{x,i}^{Q^{T_i}}(t).$$

2. Shifted lognormal model:

$$f\left[t, F_{x,i}(t)\right] := \left[F_{x,i}(t) + \lambda_{x,i}\right] \sigma_{x,i}(t),$$
$$\frac{dF_{x,i}(t)}{F_{x,i}(t) + \lambda_{x,i}} = \sigma_{x,i}(t) dW_{x,i}^{Q^{T_i}}(t).$$

For large shift $\lambda_{x,i}$ we recover the normal model, for $\lambda_{x,i} = 0$ the lognormal model.

3. Constant Elasticity of Variance (CEV) model:

$$f\left[t, F_{x,i}(t)\right] := F_{x,i}^{\beta}(t) \sigma_{x,i}(t), \ 0 \leq \beta \leq 1,$$
$$\frac{dF_{x,i}(t)}{F_{x,i}^{\beta}(t)} = \sigma_{x,i}(t) dW_{x,i}^{Q^{T_i}}(t).$$

For $\beta = 0,1$ we recover the normal and lognormal models, respectively.

4. SABR model:

$$\frac{dF_{x,i}(t)}{F_{x,i}^{\beta}(t)} = V_x(t)dW_{x,i}^{Q^{T_i}}(t),$$

$$\frac{dV_x(t)}{V_x(t)} = \nu dZ_x^{Q^{T_i}}(t),$$

$$dW_{x,i}^{Q^{T_i}}(t)dZ_x^{Q^{T_i}}(t) = \rho dt, \ \forall\, i = 1, ..., n.$$

5. Shifted SABR model:

$$\frac{dF_{x,i}(t)}{F_{x,i}^{\beta}(t) + \lambda_{x,i}} = V_x(t)dW_{x,i}^{Q^{T_i}}(t),$$

$$\frac{dV_x(t)}{V_x(t)} = \nu dZ_x^{Q^{T_i}}(t),$$

$$dW_{x,i}^{Q^{T_i}}(t)dZ_x^{Q^{T_i}}(t) = \rho dt, \ \forall\, i = 1, ..., n.$$

Beyond the Black model [6]

Model	Negative rates	Volatility smile	Analytical
Lognormal (Black)	NO	NO	YES exact
Shifted Lognormal	YES, bounded below to $-\lambda_{x,i}$	NO	YES exact
Normal	YES but unbounded below	NO	YES exact
CEV	YES	NO	YES approximated (complex)
SABR	NO	YES	YES approximated (simple)
Shifted SABR	YES	YES	YES approximated (simple)

8: Forward rate modelling
Shifted-Black model [1]

Since risky forward rates are martingales under their natural T_i-forward discounting measure Q^{T_i} , we may assume shifted lognormal martingale (driftless) dynamics for each of them

$$\boxed{\begin{aligned} dF_{x,i}(t) &= [F_{x,i}(t) + \lambda_{x,i}]\,\sigma_{x,i}(t)dW_{x,i}^{Q^{T_i}}(t), \\ dF_{x,i}(t)dF_{y,j}(t) &= \delta_{x,y}\delta_{i,j}dt, \quad \forall\, x, y, i, j, \\ F_{x,i}(t_0) &= F_{x,i,0}, \end{aligned}}$$

Shifted lognormal forward dynamics

where

- $F_{x,i}(t)$ is a generic forward rate with tenor x , for future time interval $[T_{i-1}, T_i]$, at time $t < T_{i-1}$,
- $\lambda_{x,i}$ is the constant lognormal shift associated to $F_{x,i}(t)$,
- $\sigma_{x,i}(t)$ is shifted lognormal instantaneous volatility of the process $dF_{x,i}(t)$,
- $dW_{x,i}^{Q^{T_i}}(t)$ is a standard brownian motion under Q^{T_i},
- Q^{T_i} is the T_i - forward measure associated to the discounting numeraire $P(t, T_i)$,
- $F_{x,i,0}$ is the initial condition of the SDE.

Notice that this model has two parameters: the instantaneous volatility $\sigma_{x,i}(t)$ and the shift $\lambda_{x,i}$.

Shifted-Black model [2]

The shifted lognormal dynamics allows a relevant closed-form solution, called (shifted) Black formula, for the expectation

$$\mathbb{E}_t^{Q^{T_i}}\left\{\text{Max}\left\{\omega\left[L_x(T_{i-1},T_i)-K;0\right]\right\}\right\} = \text{Black}\left[F_{x,i}(t),K,\lambda_{x,i},v_x(t;T_{i-1}),\omega\right],$$

$$\text{Black}\left[F_{x,i}(t),K,\lambda_{x,i},v_{x,i}(t;T_{i-1}),\omega\right]$$

$$=\omega\left\{\left[F_{x,i}(t)+\lambda_{x,i}\right]\Phi\left(\omega d_i^+\right)-\left[K+\lambda_{x,i}\right]\Phi\left(\omega d_i^-\right)\right\},$$

$$d_i^{\pm}=\frac{\ln\frac{F_{x,i}(t)+\lambda_{x,i}}{K+\lambda_{x,i}}\pm\frac{1}{2}v_x(t;T_{i-1})}{\sqrt{v_x(t;T_{i-1})}},$$

$$\Phi(x)=\frac{1}{2\pi}\int_{-\infty}^{x}e^{-\frac{1}{2}y^2}\,dy,$$

$$v_x(t;T_{i-1})=\int_t^{T_{i-1}}\sigma_{x,i}(u)^2\,du,$$

SLN implied forward variance

$$\sigma_x(t;T_{i-1}):=\sqrt{\frac{v_x(t;T_{i-1})}{\tau_x(t;T_{i-1})}}=\sqrt{\int_t^{T_{i-1}}\frac{\sigma_{x,i}(u)^2}{\tau_x(t;T_{i-1})}\,du}.$$

SLN implied forward volatility

Remarks

- Notice the relation

$$\begin{aligned}\mathrm{Black}\left[F_{x,i}(t), K, \lambda_{x,i}, v_x(t;T_{i-1}), \omega\right] \\ = \mathrm{Black}\left[F_{x,i}(t) + \lambda_{x,i}, K + \lambda_{x,i}, v_x(t;T_{i-1}), \omega\right]\end{aligned}$$

  telling that the shifted Black formula is equivalent to the classic Black formula (with no shift) fed with shifted forward and strike.

- Again, the SLN Black formula is a monotonic increasing function of the SLN implied volatility $\sigma_x(t;T_{i-1})$. Hence, there exist a unique implied volatility corresponding to a given price, often called SLN (or shifted-Black) implied volatility.

- Again, the SLN dynamics is not assumed to be a good proxy for the true underlying dynamics. The Shifted-Black formula is actually used as a model-based parametric form useful to remap option prices into SLN implied volatilities.

8: Forward rate modelling
SABR model [1]

SABR (Stochastic Alpha Beta Rho) model: (Hagan et al. 2002) is one of the simplest possible generalisations of the single-FRA rate Black model with stochastic volatility, choosing

$$f\left[t, F_{x,i}(t)\right] := F_{x,i}^{\beta}(t) V_x(t),$$

thus the SABR dynamics are

$$\boxed{\begin{gathered}
\frac{dF_{x,i}(t)}{F_{x,i}^{\beta}(t)} = V_x(t)\, dW_{x,i}^{Q^{T_i}}(t), \\
\frac{dV_x(t)}{V_x(t)} = \nu\, dZ_x^{Q^{T_i}}(t), \\
V_x(t_0) = \alpha, \quad F_{x,i}(t_0) = F_{x,i,0}, \\
dW_{x,i}^{Q^{T_i}}(t)\, dZ_x^{Q^{T_i}}(t) = \rho\, dt,\ \forall\, i = 1, ..., n, \\
t < T_{i-1} < T_i,\ \nu > 0,\ \alpha > 0,\ 0 \le \beta \le 1,\ -1 \le \rho \le 1.
\end{gathered}}$$

8: Forward rate modelling
SABR model [2]

The classical SABR model derivation by Hagan et al. (2002) makes neither explicit nor hidden assumptions regarding the nature of the yield curves underlying the FRA rates. Hence, the extension of the classical model to the modern framework is trivial, just requiring the replacement of the classical forward rate with the modern FRA rate and of the T_i - forward measure associated to the classical single curve numeraire $P(t;T_i)$ with the modern T_i–forward measure associated to the discounting numeraire $P_d(t;T_i)$.

	Classical vs modern SABR
Classical SABR	$\frac{dF_i(t)}{F_i^\beta(t)} = V(t)dW_i^{Q^{T_i}}(t),$ $\frac{dV(t)}{V(t)} = \nu dZ^{Q^{T_i}}(t),\ V(t_0) = \alpha,$ $dW_i^{Q^{T_i}}(t)dZ^{Q^{T_i}}(t) = \rho dt,\ \forall\, i = 1, ..., n.$
Modern SABR	$\frac{dF_{x,i}(t)}{F_{x,i}^\beta(t)} = V_x(t)dW_{x,i}^{Q^{T_i}}(t),$ $\frac{dV_x(t)}{V_x(t)} = \nu dZ_x^{Q^{T_i}}(t),\ V_x(t_0) = \alpha,$ $dW_{x,i}^{Q^{T_i}}(t)dZ_x^{Q^{T_i}}(t) = \rho dt,\ \forall\, i = 1, ..., n.$

The SABR model admits a very nice analytical solution for plain vanilla options: the pricing expression for caplets/floorlets/swaptions is the well-known Black formula:

	SABR pricing formulas for plain vanillas
Caplet/ Floorlet	$$\mathbf{cf}_{\text{SABR}}(t;T_{i-1},T_i,K,\omega) = NP_d(t,T_i) \times \text{Black}\left[F_{x,i}(t),K,v_x^{\text{SABR}}(t;T_{i-1},F_{x,i}(t),K),\omega\right]\tau_x(T_{i-1},T_i),$$ $$\sigma_x^{\text{SABR}}(t;T_{i-1},F_{x,i}(t),K) := \sqrt{\frac{v_x^{\text{SABR}}(t;T_{i-1},F_{x,i}(t),K)}{\tau_x(t;T_{i-1})}},$$
Swaption	$$\mathbf{Swaption}_{\text{SABR}}(t;\mathbf{T},\mathbf{S},K,\omega) = NA_d(t,\mathbf{S}) \times \text{Black}\left[R_x^{\text{Swap}}(t,\mathbf{T},\mathbf{S}),K,v_x^{\text{SABR}}(t,\mathbf{T},\mathbf{S},K),\omega\right],$$ $$\sigma_x^{\text{SABR}}(t;R_x^{\text{Swap}}(t,\mathbf{T},\mathbf{S}),K) = \sqrt{\frac{v_x^{\text{SABR}}(t,\mathbf{T},\mathbf{S},K)}{\tau_x(t;T_{i-1})}}.$$

with a modified SABR implicit volatility given by the perturbative expansion

	SABR implicit volatility
Out of the money ($F \neq K$)	$\sigma^{\mathrm{SABR}}(t;T,F,K) = \dfrac{\nu \ln\frac{F}{K}}{x(z)} \dfrac{A(F,K)}{B(F,K)},$ $A(F,K) := 1 + \left[\dfrac{\alpha^2(1-\beta)^2}{24(FK)^{1-\beta}} + \dfrac{\alpha\beta\nu\rho}{4(FK)^{\frac{1-\beta}{2}}} + \nu^2\dfrac{2-3\rho^2}{24}\right]\tau(t;T)...,$ $B(F,K) := 1 + \dfrac{1}{24}\left[(1-\beta)\ln\dfrac{F}{K}\right]^2 + \dfrac{1}{1920}\left[(1-\beta)\ln\dfrac{F}{K}\right]^4 + ...,$ $x(z) := \ln\dfrac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho},\ \ z := \dfrac{\nu}{\alpha}(FK)^{\frac{1-\beta}{2}}\ln\dfrac{F}{K}.$
At The Money ($F=K$)	$\sigma^{\mathrm{SABR}}_{\mathrm{ATM}}(t;T,F,K) := \sigma^{\mathrm{SABR}}(t;T,F,F)$ $= \dfrac{\alpha}{F^{1-\beta}}\left\{1 + \left[\dfrac{\alpha^2(1-\beta)^2}{24F^{2-2\beta}} + \dfrac{\alpha\beta\nu\rho}{4F^{1-\beta}} + \nu^2\dfrac{2-3\rho^2}{24}\right]\tau(t;T) + ...\right\}$

8: Forward rate modelling
SABR model [5]

Remarks:

- The SABR model, like the Black model, describes the time evolution of each single, isolated FRA/Swap rate. There are no rate correlations (as e.g. in the Libor Market Model).
- The number of SABR parameters is intentionally kept low: there is just a single volatility process $V_x(t)$ common to all FRA/Swap rates (with a given fixing date) and a single correlation ρ between the volatility and the rate processes.
- The model admits a very nice analytical solution for caps/floors/swaptions that makes it:
  - easy to code even inside a spreadsheet,
  - extremely fast to calibrate even on real time data.
- The model is easy to interpret from a financial point of view (see next slide).
- The same model can be applied as well to swap rates, just changing the notation

$$F_{x,i}(t) \longrightarrow R_x^{\mathrm{Swap}}(t; \mathbf{T}, \mathbf{S}),$$
$$Q^{T_i} \longrightarrow Q^{\mathbf{S}}.$$

8: Forward rate modelling
SABR model: parameters [6]

The SABR implicit volatility expression admits a simple and transparent financial interpretation: let's consider the following expansion in powers of *ln(F/K)*:

$$\sigma^{\mathrm{SABR}}(t;T,F,K) = \underbrace{\boxed{\frac{\alpha}{F^{1-\beta}}}}_{A} \times$$

$$\left[1-\frac{1}{2}\Big(\underbrace{\boxed{1-\beta}}_{B}-\underbrace{\boxed{\rho\lambda}}_{C}\Big)\ln\frac{K}{F}+\frac{1}{12}\Big[\underbrace{\boxed{(1-\beta)^2}}_{D}+\underbrace{\boxed{(2-3\rho^2)\lambda^2}}_{E}\Big]\ln^2\frac{K}{F}+\ldots\right],$$

$$\lambda := \frac{\nu}{\alpha}F^{1-\beta}$$

- A = "backbone", leading term setting the SABR volatility level.
- B+C = first order linear correction to the backbone, proportional to the SABR volatility slope with respect to the strike K => skew term:
- B = "beta skew", always negative;
- C = "vanna skew", the sign depends on the correlation.
- D+E = second order quadratic correction to the backbone, proportional to the SABR volatility curvature with respect to the strike K => smile term:
- D = "beta skew" squared, always positive;
- E = "volga" (volatility gamma) term, the sign depends on the correlation.

8: Forward rate modelling

SABR model: parameters [7]

SABR parameters redundancy:

- The leading order term (backbone) depends on two parameters (α and β).
- Both first order (slope) and second order (curvature) corrections depend on two concurrent terms depending on β and $\rho\lambda$.
- Thus parameter β is redundant. In fact market volatilities can be fitted equally well with different values of β. Hagan et al. suggest to determine β by historical calibration or to fix it once and for all, e.g. at $\beta = 0.5$.

8: Forward rate modelling
Shifted-SABR model [1]

The shifted-SABR model is the simplest possible generalisation of the original SABR model to accommodate negative forward rates, such that

$$f\left[t, F_{x,i}(t)\right] := \left[F^{\beta}_{x,i}(t) + \lambda_{x,i}\right] V_x(t).$$

Thus the shifted-SABR dynamics are

$$\boxed{\begin{gathered}
\frac{dF_{x,i}(t)}{F^{\beta}_{x,i}(t) + \lambda_{x,i}} = V_x(t) dW^{Q^{T_i}}_{x,i}(t), \\
\frac{dV_x(t)}{V_x(t)} = \nu dZ^{Q^{T_i}}_{x}(t), \\
V_x(t_0) = \alpha, \quad F_{x,i}(0) = F_{x,i,0}, \\
dW^{Q^{T_i}}_{x,i}(t) dZ^{Q^{T_i}}_{x}(t) = \rho dt, \ \forall\, i = 1, ..., n, \\
t < T_{i-1} < T_i, \ \nu > 0, \ \alpha > 0, \ 0 \leq \beta \leq 1, \ -1 \leq \rho \leq 1.
\end{gathered}}$$

8: Forward rate modelling
Shifted-SABR model [2]

Comments

- The rate shift $\lambda_{x,i}$ allows forward rates to assume negative values, down to the lower bound $F_{x,i}(t) + \lambda_{x,i}$.

- This choice of the shifted-SABR dynamics allows the usage of all the previous formalism with the substitutions:

$$F_{x,i}(t) \longrightarrow \tilde{F}_{x,i}(t) := F_{x,i}(t) + \lambda_{x,i},$$

$$K \longrightarrow \tilde{K} := K + \lambda_{x,i}.$$

8: Forward rate modelling
SABR model: calibration [1]

The SABR model is calibrated to the market in the following way.

1. Assume to have discounting and forwarding yield curves $\mathcal{C}_d$ and $\mathcal{C}_x$ available from a previous yield curve bootstrapping.

2. Fix the shift parameter $\lambda_{x,i}$ a priori as the desired floor for negative rates, according either to market quotations or to a desired level.

3. For each market smile section, i.e. a set of market option prices $\{P_1^{mkt}, \dots, P_N^{mkt}\}$ corresponding to the same forward rate (or forward swap rate) and different strikes:
   a) compute their corresponding shifted-lognormal market-implied volatilities $\{\sigma_1^{mkt}, \dots, \sigma_N^{mkt}\}$.
   b) Choose an initial guess for the SABR parameters $p_0 \coloneqq \{\alpha_0, \nu_0, \rho_0, \beta_0\}$.
   c) Using the shifted-SABR formulas, compute the shifted-SABR volatilities $\{\sigma_1^{SABR}(p_0), \dots, \sigma_N^{SABR}(p_0)\}$ corresponding to the market smile section above.
   d) $\rightarrow$ to be continued

8: Forward rate modelling
SABR model: calibration [2]

d) Solve numerically the minimization problem

$$p = ArgMin_p \; D\left[\sigma_1^{mkt}, \cdots \sigma_N^{mkt}; \sigma_1^{SABR}(p), \cdots \sigma_N^{SABR}(p)\right],$$

where $\mathcal{D}$ is a distance function, e.g. a weighted root mean square error

$$D\left[\sigma_1^{mkt}, \cdots \sigma_N^{mkt}; \sigma_1^{SABR}(p), \cdots \sigma_N^{SABR}(p)\right] = \sqrt{\sum_{i=1}^{N} \left\{w_i \left[\sigma_i^{mkt} - \sigma_i^{SABR}(p)\right]\right\}^2},$$

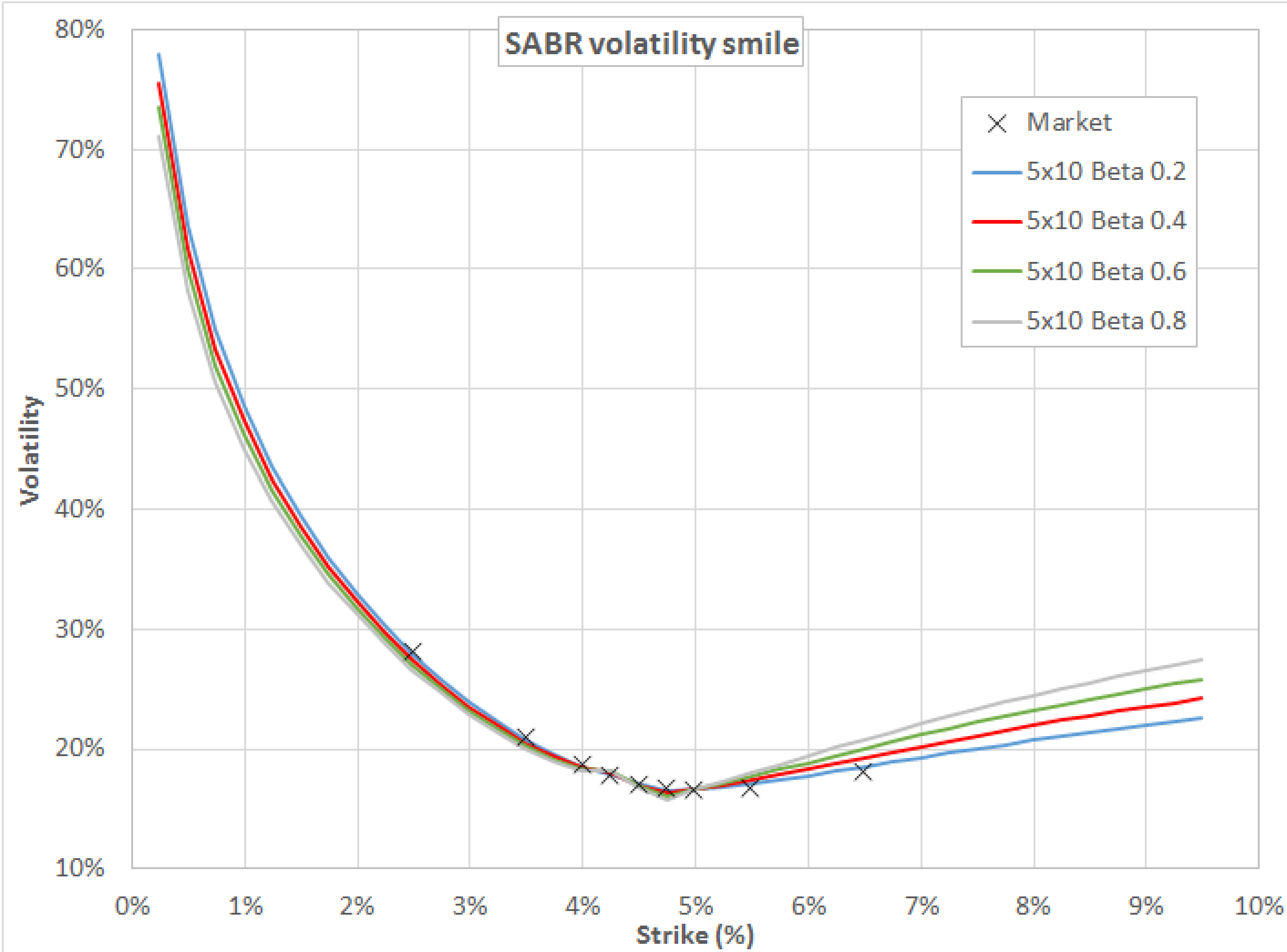
and $\{w_1, \dots, w_N\}$ are appropriate weights, e.g. the options vega sensitivities,

$$w_i = \frac{Vega_i}{\sum_{i=1}^{N} Vega_i}, \quad \sum_{i=1}^{N} w_i = 1.$$

Notice that, at first order,

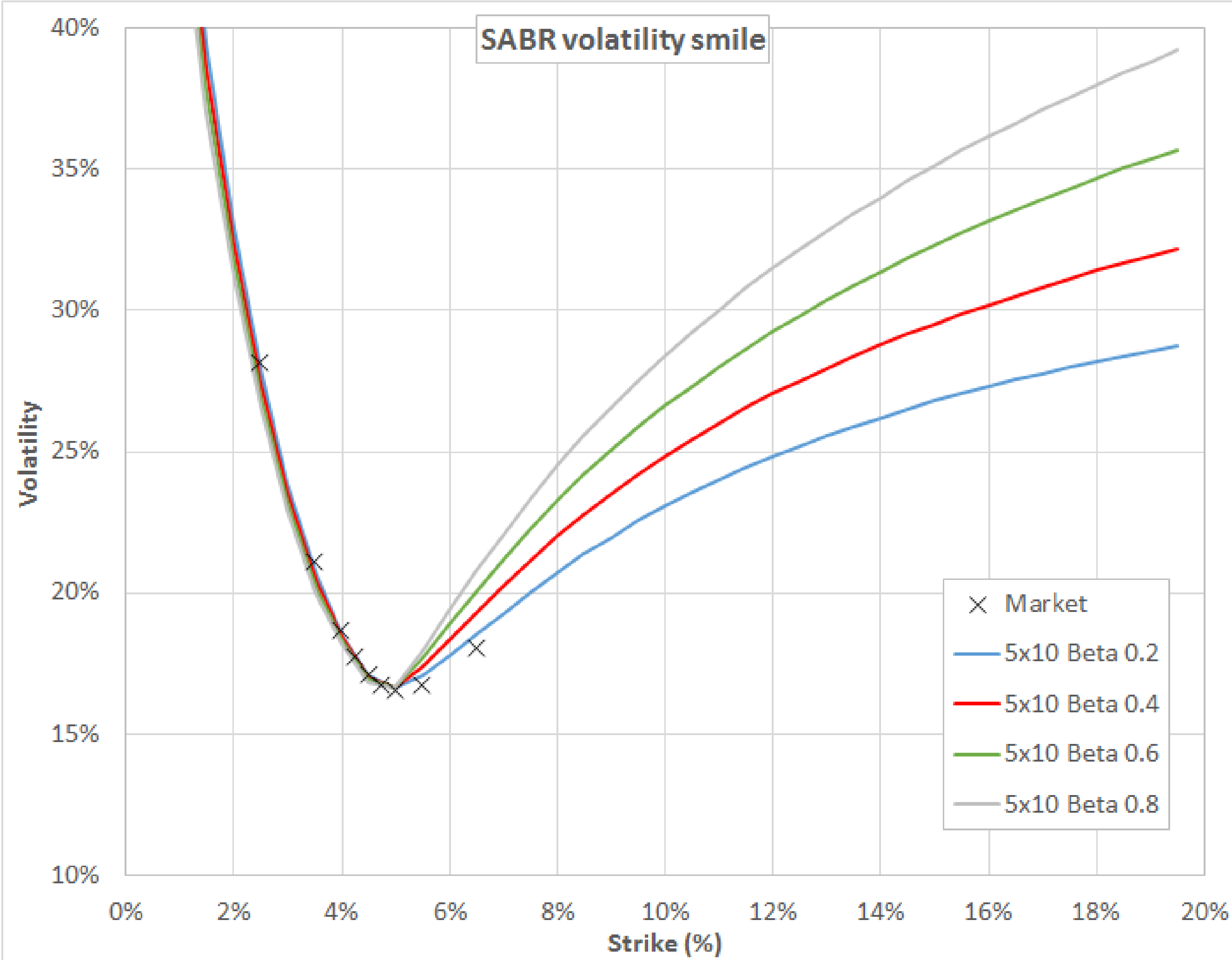
$$w_i \left[\sigma_i^{mkt} - \sigma_i^{SABR}(p)\right] \div \frac{\partial P}{\partial \sigma_i} \left[\sigma_i^{mkt} - \sigma_i^{SABR}(p)\right] \simeq P_i^{mkt} - P_i^{SABR}.$$

8: Forward rate modelling
SABR model: calibration [3]



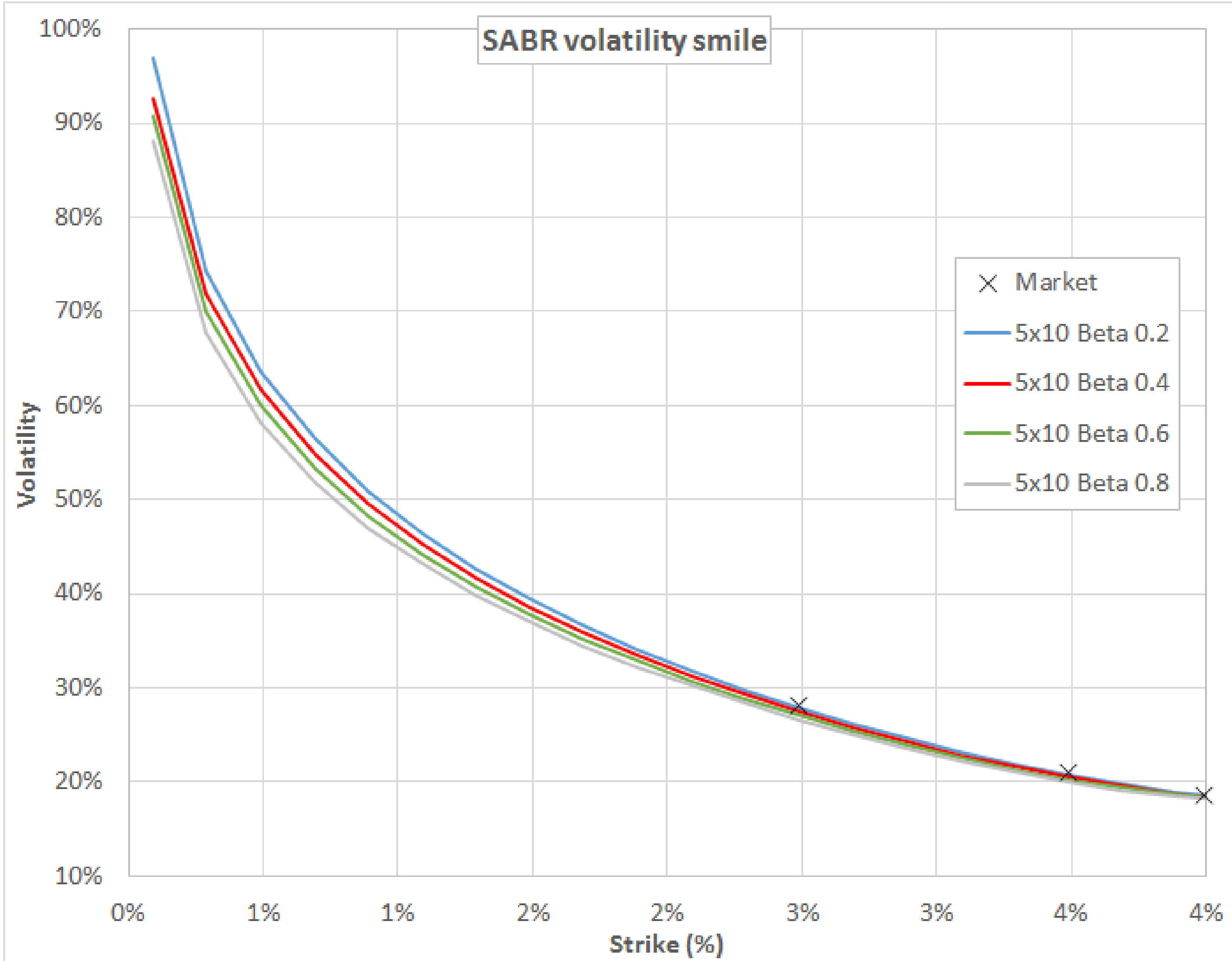
SABR volatility smiles for different beta values (alpha, nu, rho calibrated).

SABR model: calibration [4]



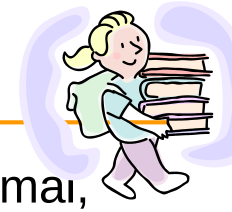
SABR volatility smiles for different beta values: focus on the right wing.

8: Forward rate modelling
SABR model calibration [5]



SABR volatility smiles for different beta values: focus on the left wing.

8: Forward rate modelling
Problems



- Simulation: plot scenarios of the stochastic dynamics described before: normal, shifted lognormal, CEV, SABR. Compare the paths and discuss. Hint: use the same random numbers for each model. Plot the corresponding distributions for short, medium and long time horizons. Use spreadsheet and possibly VBA or Matlab. Deliverable: spreadsheet with charts and comments.

- SABR: plot the SABR volatility as a function of strike, maturity and the four parameters α, β, ρ and λ . Find out the most significant dependencies on the parameters. Deliverable: spreadsheet with charts and comments.

8: Forward rate modelling
Proof of Black formula [1]

We prove the Black formula for a single caplet with payoff at time T_{i-1}

$$\mathbf{Cpl}(T_{i-1}; T_{i-1}, T_i, K) = N\mathrm{Max}\left[L(T_{i-1}, T_i) - K; 0\right]\tau(T_{i-1}, T_i).$$

We assume a martingale (driftless) lognormal dynamics for the underlying forward rate

$$dF(t; T_{i-1}, T_i) = F(t; T_{i-1}, T_i)\sigma_i(t)dW^{Q_{T_i}}(t),$$

where W is a standard brownian motion under T_i-forward measure Q_{Ti}, and $\sigma_i(t)$ is the instantaneous volatility of the forward rate. We move to the logarithm and use Ito's lemma

$$\begin{aligned} d\ln F_i(t) &= \frac{dF_i(t)}{F_i(t)} - \frac{1}{2}\frac{[dF_i(t)]^2}{[F_i(t)]^2} = \sigma_i(t)dW(t) - \frac{1}{2}[\sigma_i(t)dW(t)]^2 \\ &= \sigma_i(t)dW(t) - \frac{1}{2}\sigma_i^2(t)dt, \end{aligned}$$

we integrate both sides

$$\begin{aligned} \int_t^T d\ln F_i(u) &= \int_t^T \sigma_i(u)dW(u) - \frac{1}{2}\int_t^T \sigma_i^2(u)du \\ &= \int_t^T \sigma_i(u)dW(u) - \frac{1}{2}v_i(t,T) \end{aligned}$$

8: Forward rate modelling
Proof of Black formula [2]

$$\ln\frac{F_i(T)}{F_i(t)} = \int_t^T \sigma_i(u)dW(u) - \frac{1}{2}v_i(t,T),$$

$$F_i(T) = F_i(t)\exp\left[\int_t^T \sigma_i(u)dW(u) - \frac{1}{2}v_i(t,T)\right] := F_i(t)\exp\left[I_i(t,T)\right].$$

We compute mean and variance of the exponent $I_i(t,T)$

$$\mathbb{E}_t[I_i(t,T)] = -\frac{1}{2}v_i(t,T),$$

$$\mathrm{Var}[I_i(t,T)] = \mathrm{Var}\left[\int_t^T \sigma_i(u)dW(u)\right]$$

$$= \mathbb{E}\left\{\left[\int_t^T \sigma_i(u)dW(u)\right]^2\right\} - \mathbb{E}\left[\int_t^T \sigma_i(u)dW(u)\right]^2$$

$$= \int_t^T \sigma_i^2(u)du = v_i(t,T).$$

Thus $\quad I_i(t,T) \sim \mathcal{N}\left[-\frac{1}{2}v_i(t,T); v_i(t,T)\right] \sim -\frac{1}{2}v_i(t,T) + \sqrt{v_i(t,T)}\mathcal{N}(0,1).$

We compute now the expectation

$$\mathbb{E}_t^{Q_{T_i}}\{\mathrm{Max}\left[F_i(T)-K;0\right]\}=\mathbb{E}_t^{Q_{T_i}}\left\{\mathrm{Max}\left[F_i(t)e^{I_i(t,T)}-K;0\right]\right\}$$

$$=\int_{-\infty}^{+\infty}\mathrm{Max}\left[F_i(t)e^{-\frac{1}{2}v_i(t,T)+\sqrt{v_i(t,T)}u}-K;0\right]p_{\mathcal{N}(0,1)}(u)\,du$$

$$=\int_{u_0}^{+\infty}\left[F_i(t)e^{-\frac{1}{2}v_i(t,T)+\sqrt{v_i(t,T)}u}-K\right]p_{\mathcal{N}(0,1)}(u)\,du$$

$$=F_i(t)\int_{u_0}^{+\infty}e^{-\frac{1}{2}v_i(t,T)+\sqrt{v_i(t,T)}u}p_{\mathcal{N}(0,1)}(u)\,du-K\int_{u_0}^{+\infty}p_{\mathcal{N}(0,1)}(u)\,du$$

$$:=F_i(t)\mathcal{I}_{1,i}(t,T)-K\mathcal{I}_{2,i}(t,T)$$

Where we have set

$$u_0:=\frac{-\ln\frac{F_i(t)}{K}+\frac{1}{2}v_i(t,T)}{\sqrt{v_i(t,T)}}$$

8: Forward rate modelling
Proof of Black formula [4]

We compute now the two integrals

$$\mathcal{I}_{2,i}(t,T) = \int_{u_0}^{+\infty} p_{\mathcal{N}(0,1)}(u)\,du = 1 - \Phi(u_0) = \Phi(-u_0) = \Phi\left[\frac{\ln\frac{F_i(t)}{K} - \frac{1}{2}v_i(t,T)}{\sqrt{v_i(t,T)}}\right],$$

$$\begin{aligned}
\mathcal{I}_{1,i}(t,T) &= \int_{u_0}^{+\infty} e^{-\frac{1}{2}v_i(t,T)+\sqrt{v_i(t,T)}u} p_{\mathcal{N}(0,1)}(u)\,du \\
&= \frac{1}{\sqrt{2\pi}} \int_{u_0}^{+\infty} e^{-\frac{1}{2}v_i(t,T)+\sqrt{v_i(t,T)}u-\frac{1}{2}u^2}\,du \\
&= \frac{1}{\sqrt{2\pi}} e^{-v_i(t,T)} \int_{u_0}^{+\infty} e^{-\frac{1}{2}\left[u-\sqrt{v_i(t,T)}\right]^2}\,du \\
&= \frac{1}{\sqrt{2\pi}} e^{-v_i(t,T)} \int_{u_0-\sqrt{v_i(t,T)}}^{+\infty} e^{-\frac{1}{2}z^2}\,dz \\
&= e^{-v_i(t,T)}\left\{1 - \Phi\left[u_0 - \sqrt{v_i(t,T)}\right]\right\} \\
&= e^{-v_i(t,T)}\Phi\left[-u_0 + \sqrt{v_i(t,T)}\right] = \Phi\left[\frac{\ln\frac{F_i(t)}{K} + \frac{1}{2}v_i(t,T)}{\sqrt{v_i(t,T)}}\right].
\end{aligned}$$

8: Forward rate modelling
Proof of Black formula [5]

Thus, gathering pieces, we obtain

$$
\begin{aligned}
\mathbb{E}_t^{Q_{T_i}}\left\{\mathrm{Max}\left[F_i(T_i)-K;0\right]\right\} &= F_i(t)\Phi(d_+)-K\Phi(d_-)\\
&:= \mathrm{Black}\left[F_i(t),K,v_i(t,T_{i-1})\right],\\
d_\pm &:= \frac{\ln\frac{F_i(t)}{K}\pm\frac{1}{2}v_i(t,T_{i-1})}{\sqrt{v_i(t,T_{i-1})}},\\
v_i(t,T_{i-1}) &= \int_t^{T_{i-1}}\sigma_i(u)\,du,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Cpl}(t;T_{i-1},T_i,K) &= NP(t;T_i)\mathbb{E}_t^{Q_{T_i}}\left\{\mathrm{Max}\left[F_i(T_i)-K;0\right]\right\}\tau(T_{i-1},T_i)\\
&= NP(t;T_i)\tau(T_{i-1},T_i)\mathrm{Black}\left[F_i(t),K,v_i(t,T_{i-1})\right].
\end{aligned}
$$

Exercise: derive the analytical formulas for call and put greeks: delta, gamma, vega.

Summary

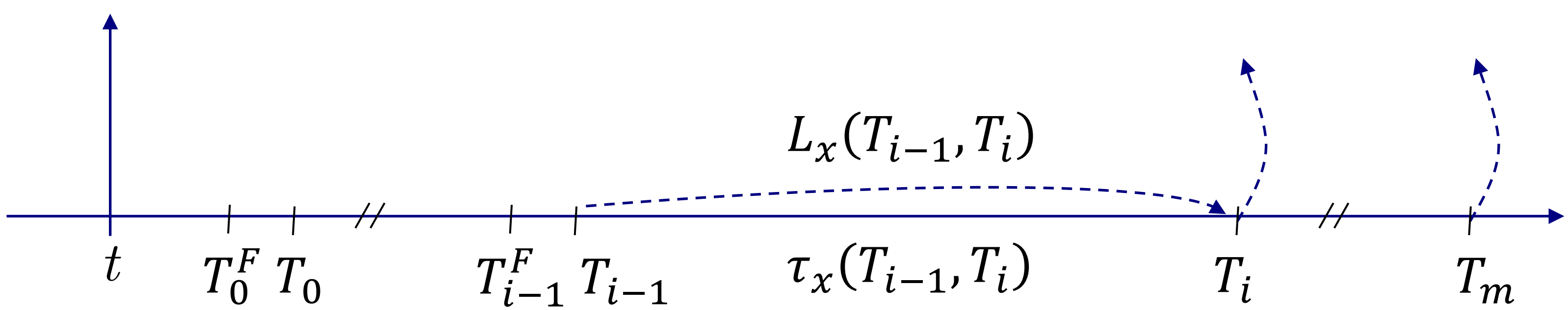
9. Interest rate volatility products

- Cap/Floor
- Swaption, cash vs physical settlement
- Constant Maturity Swap
- CMS Cap/Floor
- CMS Spread Option

Interest rate caps and floors are portfolios of caplet/floorlet options on a standard interest rate FRA. Each caplet/floorlet in the portfolio is characterized by an expiry date T_{i-1} at the FRA fixing date T_{i-1}, a cash flow date T_i at the FRA maturity date T_i, and a payoff at time T_i given by

$$\begin{aligned}\mathbf{cf}(T_i;T_{i-1},T_i,K,\omega) &= \mathrm{Max}\left[\mathbf{FRA}_{\mathrm{Std}}(T_i;T_{i-1},T_i,K,\omega);0\right]\\ &= N\tau_x(T_{i-1},T_i)\mathrm{Max}\left\{\omega\left[L_x(T_{i-1},T_i)-K\right];0\right\}.\end{aligned}$$

where $\omega = +/-1$ for caplet/floorlet, respectively.



Usages of Caps/Floors

Caps and floors can be used to protect against the risk of interest rate fluctuations. For example, a borrower locked into a floating rate mortgage, e.g. paying Euribor1M + spread each month for e.g. 10Y, is exposed to the risk of rising interest rates (she is in a short position on Euribor1M).

To hedge the short position, the borrower may buy a cap on Euribor1M with maturity 10Y and strike e.g. 2%. As a consequence, for each payment:

- if the Euribor1M fixes below or equal 2%, the borrower pays Euribor1M + spread on the mortgage and receives nothing from the cap;
- if the Euribor1M fixes above 2%, the borrower pays Euribor1M + spread on the mortgage (as before) but receives Euribor1M – 2% from the Cap. Thus the total (mortgage + cap) amount payed by the borrower is "capped" at 2% + spread.
  In formulas, the total borrower's cash flow is given by

$$
\begin{aligned}
C_i &= -\left[L_{1M}(T_{i-1},T_i)+\Delta\right]\tau(T_{i-1},T_i)+\left[L_{1M}(T_{i-1},T_i)-K\right]\tau(T_{i-1},T_i)\\
&= -\left(\Delta+K\right)\tau(T_{i-1},T_i).
\end{aligned}
$$

9: Interest rate volatility products
Cap/Floor [3]

The standard market-like caplet/floorlet pricing expression at time $t<T_i$ is given by the (shifted lognormal - SLN) Black formula

$$
\begin{aligned}
\mathbf{cf}(t;T_{i-1},T_i,K,\omega) &= NP_d(t,T_i)\tau_x(T_{i-1},T_i)\mathbb{E}_t^{Q^{T_i}}\left[\mathrm{Max}\left\{\omega\left[L_x(T_{i-1},T_i)-K\right];0\right\}\right] \\
&= NP_d(t,T_i)\tau_x(T_{i-1},T_i)\times \mathrm{Black}\left[F_{x,i}(t),K,\lambda_x,v_x(t;T_{i-1}),\omega\right]
\end{aligned}
$$

We stress that the Black formula is based on the assumption that each single forward rate $F(t;T_{i-1},T_i)$ follows an independent shifted-lognormal (driftless) martingale dynamics under its natural discounting T_i-forward measure Q_{T_i} over the time interval $[t;T_{i-1}]$ with instantaneous volatility $\sigma_i(t)$. The formula depends only on the single caplet/floorlet lognormal variance $v(t,T_{i-1})$. The shift regulates the minimum value of the rate distribution, and may be used to accomodate negative rates.

The corresponding volatilities

$$
\sigma_x(t;T_{i-1}) := \sqrt{\frac{v_x(t;T_{i-1})}{\tau_x(t;T_{i-1})}} = \sqrt{\int_t^{T_{i-1}} \frac{\sigma_{x,i}(u)^2}{\tau_x(t;T_{i-1})}\,du}.
$$

Shifted-lognormal implied forward volatility

are called caplet/floorlet or shifted-lognormal implied forward volatilities.

The total cap/floor price at time $t<T_0$ is given by

SLN forward variances: one different forward volatility figure for each caplet/floorlet

$$v_{x,i}(t) = \sigma_x(t;T_{i-1})^2\tau_x(t;T_{i-1})$$

$$\begin{aligned}\mathbf{CF}(t;\mathbf{T},K,\omega) &= N\sum_{i=1}^{n}\mathbf{cf}(t;T_{i-1},T_i,K,\omega)\\ &= N\sum_{i=1}^{n}P_d(t,T_i)\tau_x(T_{i-1},T_i)\mathrm{Black}\left[F_{x,i}(t),K,\lambda_x,v_x(t;T_{i-1}),\omega\right]\\ &:= N\sum_{i=1}^{n}P_d(t,T_i)\tau_x(T_{i-1},T_i)\mathrm{Black}\left[F_{x,i}(t),K,\lambda_x,v_{x,i,n}(t;T_{i-1},T_n),\omega\right].\end{aligned}$$

SLN term variances: one single term volatility figure for the entire cap/floor,

$$v_{x,i,n}(t) = \sigma_x(t;T_n)^2\tau_x(t;T_{i-1})$$

9: Interest rate volatility products

Cap/Floor: market quotes [5]

13:01 30DEC11 ICAP UK69580 VCAP5

EUR Floors - Premium Mids

Please call +44 (0)20 7532 3080 for further details

STK		ATM	1.00	1.50	2.00	2.25	2.50	3.00	3.50	4.00	4.50	5.00	6.00	7.00	10.0
1Y	0.97	10													
18M	0.95	22													
2Y	0.98	39													
3Y	1.30	82	41												
4Y	1.50	157	63	156											
5Y	1.71	249	82	189											
6Y	1.91	350	100	219											
7Y	2.07	452	118	248	424										
8Y	2.20	552	136	277	466										
9Y	2.30	650	154	307	509	625									
10Y	2.39	746	171	333	548	671									
12Y	2.54	930	203	384	621	757	905								
15Y	2.66	1187	260	470	741	897	1067								
20Y	2.67	1573	383	656	1000	1198	1411								
25Y	2.61	1945	537	889	1319	1563	1825								
30Y	2.54	2294	699	1130	1647	1935	2245								

Please note 1 and 2 yrs are vs 3 mth

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OPTIONS Index <VCAP> IR Option RIC Index <IRORICSMENU>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

13:01 30DEC11 ICAP UK69580 VCAP4

EUR Caps - Premium Mids

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STK		ATM	1.00	1.50	2.00	2.25	2.50	3.00	3.50	4.00	4.50	5.00	6.00	7.00	10.0
1Y	0.97	10	10	2											
18M	0.95	22	19	6	2	1	1								
2Y	0.98	39	37	16	9	7	5	3	2	2	1	1	1		
3Y	1.30	82		66	40	32	26	17	12	9	7	5	3	2	1
4Y	1.50	157			108	91	77	56	42	32	25	20	13	9	4
5Y	1.71	249			208	178	153	115	87	67	53	43	29	20	9
6Y	1.91	350			333	290	252	192	147	115	91	74	50	36	17
7Y	2.07	452				412	361	278	216	169	135	110	76	55	27
8Y	2.20	552				538	474	368	287	226	181	147	102	74	37
9Y	2.30	650					590	461	361	286	229	186	130	95	47
10Y	2.39	746					707	556	437	346	278	226	157	116	57
12Y	2.54	930						749	592	471	378	308	215	158	80
15Y	2.66	1187						1020	813	651	528	434	309	232	125
20Y	2.67	1573						1369	1107	902	744	622	456	353	201
25Y	2.61	1945						1661	1363	1127	944	802	604	477	283
30Y	2.54	2294						1924	1597	1336	1132	972	746	598	366

Please note 1 and 2 yrs are vs 3 mth

screen closed

OPTIONS Index <VCAP> IR Option RIC Index <IRORICSMENU>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

13:01 30DEC11 ICAP UK69580 VCAP3A

EUR Caps/Floors - Implied Volatilities

Please call +44 (0)20 7532 3080 for further details

STK		ATM	1.00	1.50	2.00	2.25	2.50	3.00	3.50	4.00	4.50	5.00	6.00	7.00	10.0
1Y	0.97	50.84	50.8	50.1	50.1	50.4	50.7	51.7	52.8	54.0	55.2	56.4	58.8	61.0	66.9
18M	0.95	53.98	53.9	53.5	54.0	54.4	54.7	55.5	56.4	57.3	58.2	59.2	61.0	62.8	67.5
2Y	0.98	58.40	58.4	58.1	58.8	58.9	59.1	59.6	60.1	60.7	61.3	61.9	63.3	64.6	68.1
3Y	1.31	52.04	53.5	51.5	50.7	50.5	50.2	49.9	49.7	49.7	49.7	49.8	50.0	50.4	51.5
4Y	1.51	51.97	55.9	52.0	49.8	48.9	48.1	47.0	46.3	45.8	45.5	45.4	45.3	45.4	45.9
5Y	1.72	49.20	55.9	50.8	47.4	46.0	44.8	43.0	41.8	40.9	40.4	40.1	39.8	39.7	40.2
6Y	1.91	45.87	55.0	49.2	45.2	43.6	42.2	40.0	38.4	37.4	36.7	36.3	35.9	35.9	36.4
7Y	2.07	42.88	53.8	47.7	43.4	41.6	40.1	37.7	36.0	34.9	34.2	33.7	33.4	33.4	34.1
8Y	2.20	40.40	52.6	46.3	41.8	40.0	38.4	36.0	34.2	33.0	32.3	31.8	31.4	31.4	32.2
9Y	2.30	38.41	51.5	45.1	40.6	38.7	37.1	34.6	32.8	31.6	30.8	30.4	30.0	29.9	30.6
10Y	2.39	36.58	50.2	43.9	39.3	37.5	35.9	33.4	31.6	30.3	29.5	29.0	28.6	28.5	29.2
12Y	2.54	33.53	48.0	41.7	37.2	35.3	33.7	31.2	29.4	28.1	27.3	26.8	26.3	26.2	26.9
15Y	2.67	30.96	45.8	39.6	35.2	33.4	31.8	29.4	27.6	26.4	25.6	25.1	24.7	24.8	25.5
20Y	2.68	29.93	43.8	38.0	33.8	32.2	30.8	28.6	27.0	25.9	25.2	24.8	24.5	24.5	25.2
25Y	2.61	30.38	43.3	37.7	33.7	32.2	30.9	28.8	27.3	26.3	25.7	25.3	24.9	24.9	25.4
30Y	2.54	30.83	43.0	37.5	33.7	32.2	31.0	29.0	27.6	26.7	26.0	25.6	25.2	25.2	25.5

Please note 1 and 2 yrs are vs 3 mth

screen closed

OPTIONS Index <VCAP> IR Option RIC Index <IRORICSMENU>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

Market quotes for EUR Caps/Floors as of 30 Dec. 2011 (source: Reuters).

9: Interest rate volatility products

Cap/Floor: market quotes [6]

16:12 30JAN15 ICAP UK69580 VCAP5

EUR Floors - Premium Mids (Eonia disc)

Please call +44 (0)20 7532 3080 for further details

	STK	ATM	0.00	0.13	0.25	0.38	0.50	1.00	1.50	2.00	2.50	3.00	4.00	5.00	10.0
1Y	0.05	5	3												
18M	0.05	9	6												
2Y	0.06	16	10												
3Y	0.20	30	7	20											
4Y	0.26	61	16	34	59										
5Y	0.33	105	27	50	81										
6Y	0.40	163	42	70	106	153									
7Y	0.48	232	58	91	132	185									
8Y	0.56	311	76	114	159	219	279								
9Y	0.64	399	95	137	187	252	318								
10Y	0.71	492	114	160	215	285	357								
12Y	0.83	687													
15Y	0.97	987													
20Y	1.11	1481						1344							
25Y	1.18	1956						1686							
30Y	1.21	2411						2033							

16:12 30JAN15 ICAP UK69580 VCAP4

EUR Caps - Premium Mids (Eonia disc)

Please call +44 (0)20 7532 3080 for further details

	STK	ATM	0.00	0.13	0.25	0.38	0.50	1.00	1.50	2.00	2.50	3.00	4.00	5.00	10.0
1Y	0.05	5		3	1	1									
18M	0.05	9		6	4	2	2								
2Y	0.06	16		12	8	5	4	2	1	1					
3Y	0.20	30			26	19	15	7	5	3	2	2	1	1	
4Y	0.26	61				50	42	23	14	10	7	5	3	2	1
5Y	0.33	105				99	86	53	35	25	19	15	9	6	2
6Y	0.40	163					148	97	67	49	37	29	19	13	4
7Y	0.48	232					228	156	113	84	65	52	34	24	7
8Y	0.56	311						232	172	131	103	83	57	41	12
9Y	0.64	399						320	243	189	151	122	85	62	19
10Y	0.71	492						418	322	254	204	166	116	84	26
12Y	0.83	687						633	498	400	326	269	190	139	40
15Y	0.97	987						974	784	641	530	443	318	235	69
20Y	1.11	1481							1273	1053	885	749	549	412	125
25Y	1.18	1956							1742	1461	1242	1062	792	603	191
30Y	1.21	2411							2187	1852	1586	1367	1032	794	259

16:12 30JAN15 ICAP UK69580 VCAP3

EUR Caps/Floors - Black Volatilities (Eonia disc)

Please call +44 (0)20 7532 3080 for further details

	STK	ATM	0.00	0.13	0.25	0.38	0.50	1.00	1.50	2.00	2.50	3.00	4.00	5.00	10.0
1Y	0.05			374	257	220	202	171	160	155	152	150	148	148	152
18M	0.05			559	283	231	209	172	159	153	149	147	145	144	145
2Y	0.06				288	224	199	160	148	142	138	136	134	133	133
3Y	0.20	134.1		162	122	110	104	96.1	93.7	92.7	92.3	92.2	92.2	92.4	93.7
4Y	0.26	131.3		188	133	115	107	90.7	84.1	80.6	78.4	76.9	75.1	74.1	72.9
5Y	0.33	125.0		223	140	118	107	87.8	79.8	75.4	72.7	70.8	68.4	67.0	64.2
6Y	0.40	118.2		335	150	121	108	85.2	75.9	70.7	67.6	65.4	62.6	60.8	57.3
7Y	0.48	109.9			158	123	108	82.8	72.8	67.3	63.8	61.5	58.5	56.6	52.9
8Y	0.56	102.2			167	124	107	80.7	70.3	64.6	61.0	58.5	55.4	53.5	49.7
9Y	0.64	95.24			176	125	107	78.8	68.1	62.3	58.6	56.1	52.9	50.9	46.9
10Y	0.71	89.16			186	125	106	77.0	66.0	60.2	56.4	53.8	50.4	48.3	43.9
12Y	0.83	80.32						74.1	63.0	57.1	53.2	50.6	47.0	44.7	39.7
15Y	0.97	71.96						71.0	59.9	53.9	50.1	47.4	43.8	41.5	36.2
20Y	1.11	64.74						68.0	56.8	50.6	46.9	44.2	40.7	38.4	32.9
25Y	1.18	61.96						67.2	55.6	49.3	45.6	42.9	39.3	37.0	31.4
30Y	1.21	60.78						67.3	55.0	48.6	44.7	42.0	38.4	36.0	30.2

16:12 30JAN15 ICAP UK69580 VCAP3A

EUR Caps/Floors - Shifted Black Volatilities

Please call +44 (0)20 7532 3080 for further details

	STK	ATM	0.00	0.13	0.25	0.38	0.50	1.00	1.50	2.00	2.50	3.00	4.00	5.00	10.0
1Y	0.05	20.42	19.4	21.9	24.0	26.2	28.1	33.8	37.7	40.9	43.5	45.8	49.6	52.8	63.6
18M	0.05	21.03	20.1	22.4	24.4	26.6	28.4	34.1	38.1	41.2	43.8	46.0	49.6	52.5	62.4
2Y	0.06	21.85	20.8	22.9	24.8	26.9	28.6	34.2	38.2	41.2	43.7	45.8	49.2	52.0	60.9
3Y	0.20	20.20	18.6	19.5	20.8	22.9	24.5	30.2	34.1	37.0	39.3	41.3	44.3	46.6	53.6
4Y	0.26	23.85	22.1	22.6	23.7	25.5	26.9	30.8	33.2	35.0	36.3	37.4	39.2	40.6	45.1
5Y	0.33	26.99	24.9	25.2	26.1	27.5	28.6	31.8	33.6	34.8	35.8	36.6	37.9	38.9	41.8
6Y	0.40	29.40	27.4	27.5	28.1	29.2	30.1	32.3	33.5	34.3	34.9	35.4	36.2	36.8	38.8
7Y	0.48	30.98	29.3	29.2	29.6	30.4	31.1	32.6	33.4	33.8	34.2	34.5	35.1	35.5	36.9
8Y	0.56	32.03	30.8	30.5	30.8	31.4	31.8	32.9	33.3	33.5	33.7	33.8	34.2	34.4	35.4
9Y	0.64	32.58	31.8	31.4	31.6	32.0	32.3	32.9	33.0	33.1	33.1	33.1	33.3	33.4	34.1
10Y	0.71	32.76	32.4	32.0	32.0	32.3	32.5	32.8	32.7	32.6	32.5	32.4	32.3	32.3	32.3
12Y	0.83	32.61						32.5	32.0	31.7	31.4	31.1	30.7	30.5	29.8
15Y	0.97	31.79						31.8	31.1	30.5	30.1	29.7	29.1	28.7	27.5
20Y	1.11	30.34						30.5	29.7	28.9	28.4	27.9	27.2	26.7	25.1
25Y	1.18	29.32						29.6	28.8	28.0	27.4	27.0	26.2	25.6	23.8
30Y	1.21	28.52						28.9	28.0	27.3	26.7	26.2	25.4	24.8	22.8

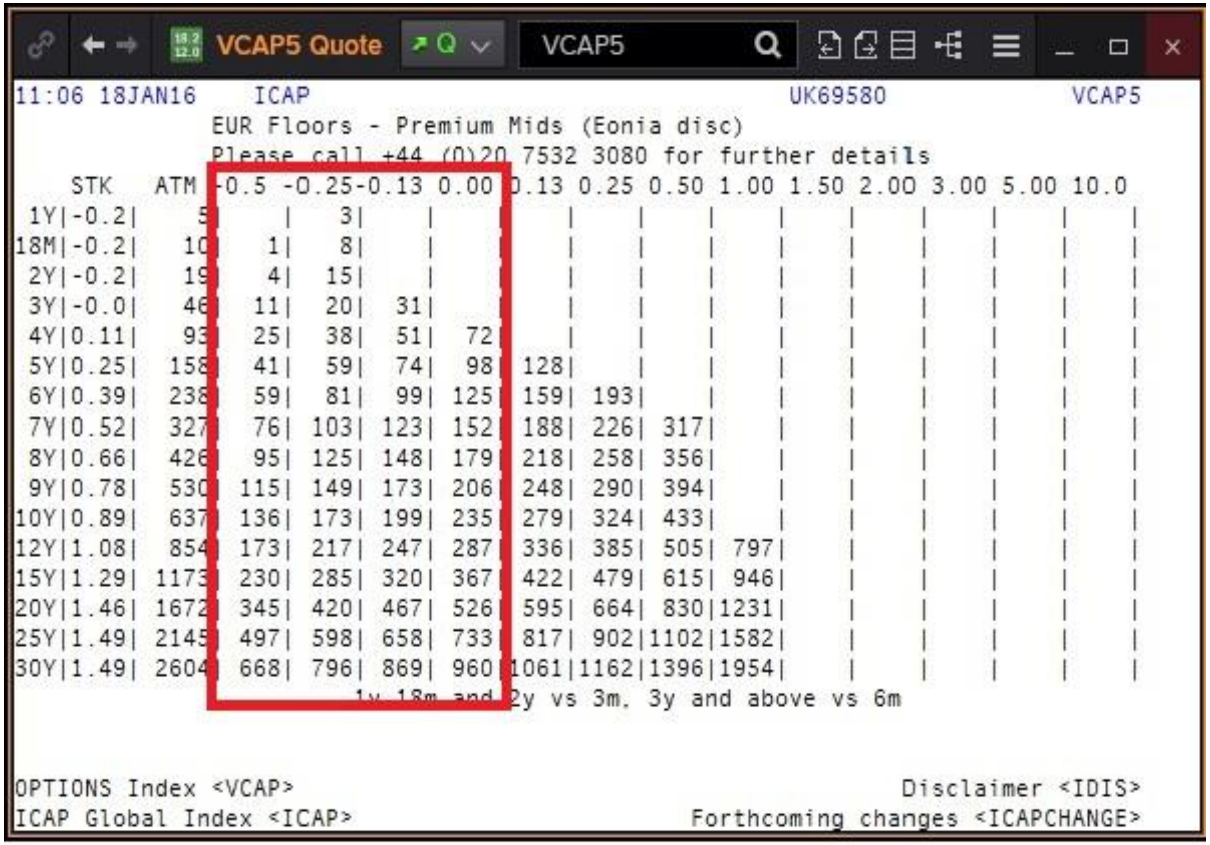
Shift = 1.00%

Market quotes for EUR Caps/Floors as of 30 Jan. 2015 (source: Reuters). Premia are Eonia discounted. Both lognormal and 1% shifted lognormal implied volatilities are shown on different pages.

9: Interest rate volatility products

Cap/Floor: market quotes [7]

- Extension of the smile quotation to negative region appeared in late 2015
- In this new approach, Black model has to be abandoned and replaced by shifted Black Model or Bachelier (normal) with different calibration procedure and hedge ratio computation



11:06 18JAN16 ICAP UK69580 VCAP5

EUR Floors - Premium Mids (Eonia disc)

Please call +44 (0)20 7532 3080 for further details

	STK	ATM	-0.5	-0.25	-0.13	0.00	0.13	0.25	0.50	1.00	1.50	2.00	3.00	5.00	10.0
1Y	-0.2	5		3											
18M	-0.2	10	1	8											
2Y	-0.2	19	4	15											
3Y	-0.0	46	11	20	31										
4Y	0.11	93	25	38	51	72									
5Y	0.25	158	41	59	74	98	128								
6Y	0.39	238	59	81	99	125	159	193							
7Y	0.52	327	76	103	123	152	188	226	317						
8Y	0.66	426	95	125	148	179	218	258	356						
9Y	0.78	530	115	149	173	206	248	290	394						
10Y	0.89	637	136	173	199	235	279	324	433						
12Y	1.08	854	173	217	247	287	336	385	505	797					
15Y	1.29	1173	230	285	320	367	422	479	615	946					
20Y	1.46	1672	345	420	467	526	595	664	830	1231					
25Y	1.49	2145	497	598	658	733	817	902	1102	1582					
30Y	1.49	2604	668	796	869	960	1061	1162	1396	1954					

1y, 18m and 2y vs 3m, 3y and above vs 6m

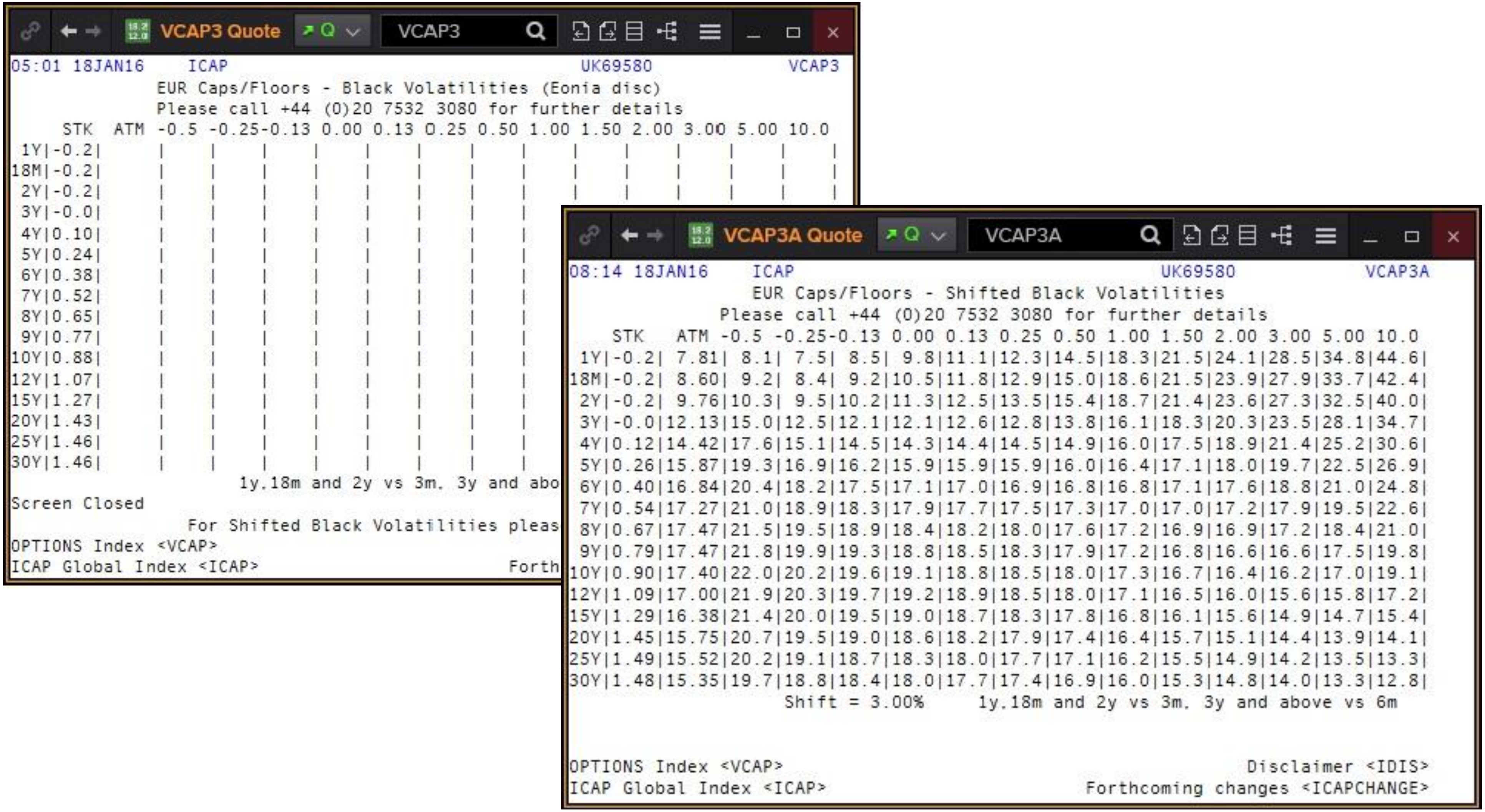
OPTIONS Index <VCAP> Disclaimer <IDIS>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

Market quotes for EUR Caps/Floors as of 18 Jan. 2016 (source: Reuters).
Negative strike Floors has been added.

Cap/Floor: market quotes [8]

Black vs Shifted Black in Caps and Floors Market in 2016



VCAP3 Quote

05:01 18JAN16 ICAP UK69580 VCAP3

EUR Caps/Floors - Black Volatilities (Eonia disc)

Please call +44 (0)20 7532 3080 for further details

	STK	ATM	-0.5	-0.25	-0.13	0.00	0.13	0.25	0.50	1.00	1.50	2.00	3.00	5.00	10.0
1Y	-0.2														
18M	-0.2														
2Y	-0.2														
3Y	-0.0														
4Y	0.10														
5Y	0.24														
6Y	0.38														
7Y	0.52														
8Y	0.65														
9Y	0.77														
10Y	0.88														
12Y	1.07														
15Y	1.27														
20Y	1.43														
25Y	1.46														
30Y	1.46														

1y,18m and 2y vs 3m, 3y and abo

Screen Closed

For Shifted Black Volatilities pleas

OPTIONS Index <VCAP>

ICAP Global Index <ICAP> Forth

VCAP3A Quote

08:14 18JAN16 ICAP UK69580 VCAP3A

EUR Caps/Floors - Shifted Black Volatilities

Please call +44 (0)20 7532 3080 for further details

	STK	ATM	-0.5	-0.25	-0.13	0.00	0.13	0.25	0.50	1.00	1.50	2.00	3.00	5.00	10.0
1Y	-0.2	7.81	8.1	7.5	8.5	9.8	11.1	12.3	14.5	18.3	21.5	24.1	28.5	34.8	44.6
18M	-0.2	8.60	9.2	8.4	9.2	10.5	11.8	12.9	15.0	18.6	21.5	23.9	27.9	33.7	42.4
2Y	-0.2	9.76	10.3	9.5	10.2	11.3	12.5	13.5	15.4	18.7	21.4	23.6	27.3	32.5	40.0
3Y	-0.0	12.13	15.0	12.5	12.1	12.1	12.6	12.8	13.8	16.1	18.3	20.3	23.5	28.1	34.7
4Y	0.12	14.42	17.6	15.1	14.5	14.3	14.4	14.5	14.9	16.0	17.5	18.9	21.4	25.2	30.6
5Y	0.26	15.87	19.3	16.9	16.2	15.9	15.9	15.9	16.0	16.4	17.1	18.0	19.7	22.5	26.9
6Y	0.40	16.84	20.4	18.2	17.5	17.1	17.0	16.9	16.8	16.8	17.1	17.6	18.8	21.0	24.8
7Y	0.54	17.27	21.0	18.9	18.3	17.9	17.7	17.5	17.3	17.0	17.0	17.2	17.9	19.5	22.6
8Y	0.67	17.47	21.5	19.5	18.9	18.4	18.2	18.0	17.6	17.2	16.9	16.9	17.2	18.4	21.0
9Y	0.79	17.47	21.8	19.9	19.3	18.8	18.5	18.3	17.9	17.2	16.8	16.6	16.6	17.5	19.8
10Y	0.90	17.40	22.0	20.2	19.6	19.1	18.8	18.5	18.0	17.3	16.7	16.4	16.2	17.0	19.1
12Y	1.09	17.00	21.9	20.3	19.7	19.2	18.9	18.5	18.0	17.1	16.5	16.0	15.6	15.8	17.2
15Y	1.29	16.38	21.4	20.0	19.5	19.0	18.7	18.3	17.8	16.8	16.1	15.6	14.9	14.7	15.4
20Y	1.45	15.75	20.7	19.5	19.0	18.6	18.2	17.9	17.4	16.4	15.7	15.1	14.4	13.9	14.1
25Y	1.49	15.52	20.2	19.1	18.7	18.3	18.0	17.7	17.1	16.2	15.5	14.9	14.2	13.5	13.3
30Y	1.48	15.35	19.7	18.8	18.4	18.0	17.7	17.4	16.9	16.0	15.3	14.8	14.0	13.3	12.8

Shift = 3.00% 1y,18m and 2y vs 3m, 3y and above vs 6m

OPTIONS Index <VCAP> Disclaimer <IDIS>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

9: Interest rate volatility products
Cap/Floor: market quotes [9]

Also in case of positive strikes, Black model cannot fit market data, because of negative forwards

Black Model vs Market Prices (bps)

	0.000%	0.125%	0.250%	0.375%	0.500%	1.000%
1.0						
1.5						
2.5	-17					
3.5	-16	-10				
4.5	-26	-17				
5.5	-40	-27	-2			
6.5	-58	-41	-10			
7.5	-75	-54	-16	-1		
8.5	-93	-68	-23	-1	1	
9.5	-112	-84	-33	-5	0	
10.5	-134	-103	-45	-11	1	
11.5						
14.5						
19.5						
24.5						
29.5						-88

Shifted Black Model vs Market Prices (bps)

	0.000%	0.125%	0.250%	0.375%	0.500%	1.000%
1.0						
1.5						
2.5	-9					
3.5	0	-1				
4.5	0	-1				
5.5	1	-1	1			
6.5	1	-1	0			
7.5	0	-1	1	-1		
8.5	0	-1	1	-2	1	
9.5	1	-1	0	-2	1	
10.5	1	-1	1	-1	1	
11.5						
14.5						
19.5						
24.5						
29.5						8

9: Interest rate volatility products
Problems

- Cap/Floor parity 1: proof analytically the parity relationships Caplets/Floorlets and Caps/Floors. Deliverable: analytical proof. Is it possible to check the parity relations above using market quotes ?

- Cap/Floor pricing: starting from market quotations for the relevant interest rate yield curves and Cap/Floor volatilities, price three Caps and three Floors with 5Y maturity and 1%, ATM, and 2% strikes (total 6 prices). Deliverable: spreadsheet/VBA with comments.

- Cap/Floor sensitivity: derive analytically the formulas for shifted-lognormal Black delta and vega sensitivities for Caplets/Floorlets and Caps/Floors. Using the market quotations provided in the market data sheets, compute and plot the sensitivities for Caps/Floors quotations. Compare such sensitivities with those calculated using the finite difference method. Deliverable: Analytical proof + spreadsheet with charts and comments.

9: Interest rate volatility products
Swaption [1]

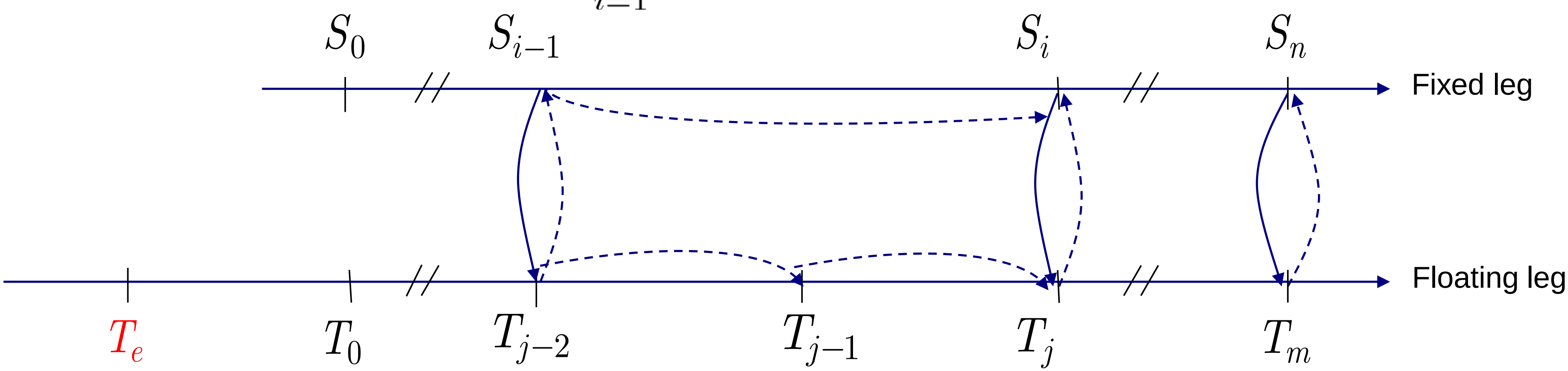
Interest rate swap options or Swaptions are cap/floor options on a swap. There are two standard market payoffs, that differ in the settlement convention: physical or cash.

1. Physical Delivery Swaptions

Options expiring at time T_e to enter into a swap starting at time $T_0 \geq T_e$ and maturing at T_m , with payoff

$$\begin{aligned}\mathbf{Swaption}(T_e;\mathbf{T},\mathbf{S},K,\omega) &= \mathrm{Max}\left[\mathbf{Swap}(T_e;\mathbf{T},\mathbf{S},K,\omega);0\right]\\ &= N A_d(T_e,\mathbf{S})\mathrm{Max}\left\{\omega\left[R_x^{\mathrm{Swap}}(T_e,\mathbf{T},\mathbf{S})-K\right];0\right\},\end{aligned}$$

$$A_d(T_e,\mathbf{S}) = \sum_{i=1}^{n} P_d(T_e;S_i)\tau_K(S_{i-1},S_i).$$



2. Cash Delivery Swaptions

Options expiring at time T_e to receive the T_e – value, but with a cash-delivery annuity, of a swap starting at time $T_0 \geq T_e$ and maturing at T_n , with payoff

$$\mathbf{Swaption}(T_e; \mathbf{T}, \mathbf{S}, K, \omega) = N P_d(T_e; T_0) \tilde{A}\left[T_e, R_x^{\text{Swap}}(T_e, \mathbf{T}, \mathbf{S})\right] \times \text{Max}\left\{\omega\left[R_x^{\text{Swap}}(T_e, \mathbf{T}, \mathbf{S}) - K\right]; 0\right\},$$

$$\tilde{A}\left[T_e, R_x^{\text{Swap}}(T_e, \mathbf{T}, \mathbf{S})\right] = \sum_{i=1}^{n} \tilde{P}_x(S_0, S_i)\tau_K(S_{i-1}, S_i),$$

Cash delivery annuity

$$\tilde{P}_x(S_0, S_i) = \prod_{j=1}^{i} \tilde{P}_x(S_{j-1}, S_j),$$

$$\tilde{P}_x(S_{j-1}, S_j) = \frac{1}{1 + R_x^{\text{Swap}}(T_e, \mathbf{T}, \mathbf{S})\tau_K(S_{j-1}, S_j)}.$$

Swaption [2]

The cash delivery annuity is often simplified, for $\tau_K(S_{j-1}, S_j) \simeq \tau_K \ \forall\, j$, as follows

$$\tilde{A}\left[T_e, R_x^{\text{Swap}}(T_e, \mathbf{T}, \mathbf{S})\right] \simeq \sum_{i=1}^{n} \frac{\tau_K}{\left[1 + R_x^{\text{Swap}}(T_e, \mathbf{T}, \mathbf{S})\tau_K\right]^i}$$

$$= \begin{cases} \frac{1}{R_x^{\text{Swap}}}\left[1 - \frac{1}{(1+\tau_K R_x^{\text{Swap}})^n}\right], & \text{if } R_x^{\text{Swap}} > 0, \\ n\tau_K, & \text{if } R_x^{\text{Swap}} = 0. \end{cases}$$

9: Interest rate volatility products
Swaption [3]

Remarks

- The physical delivery annuity
  - is a good numeraire (it is a portfolio of tradable assets),
  - depends on the discount factors at fixed cash flow dates, and thus on the shape of the yield curve.
  - Thus different counterparties may encounter differences in reconciliation.

- The cash delivery annuity:
  - is not a good numeraire
  - is a simplification of the physical delivery annuity, where the swap cash flows are discounted using a flat yield curve at the swap rate level.
  - Since the swap rate is quoted, the annuity is the same for all the market participants, and price uncertainty is reduced.

- Physical delivery swaptions, once exercised, are transformed into the underlying swap. Such swap will generate a credit exposure. In case of financial counterparties, the swap is typically cleared (migrated to a CCP), thus reducing the credit exposure but introducing large funding cost for initial margin. Cash delivery swaptions avoids such complications.

Usages of Swaptions

Swaptions are typically traded by large corporations, banks, financial institutions and hedge funds.

- Corporations exposed to rising interest rates in future periods may hedge this risk buying a payer swaption.

- Banks holding a mortgage portfolio may buy receiver swaptions with different expiries to protect against lowering interest rates in future periods, that might lead to early prepayment of the mortgages.

- Hedge funds believing that interest rates will not rise above a certain level may sell a payer swaption, betting to make money by collecting the premium.

- Large financial institutions will sell the swaptions to the entities above.

9: Interest rate volatility products
Swaption [5]

The price of physical delivery swaptions at time $t<T_0$ is given by the Black formula

$$\begin{aligned}\mathbf{Swaption}(t;\mathbf{T},\mathbf{S},K,\omega) &= N\mathbb{E}_t^{\boxed{Q}}\left\{D_d(t,T_e)A_d(T_e,\mathbf{S})\mathrm{Max}\left[\omega[R_x^{\mathrm{Swap}}(T_e,\mathbf{T},\mathbf{S})-K];0\right]\right\}\\ &= N\mathbb{E}_t^{\boxed{Q^{\mathbf{S}}}}\left\{\frac{A_d(t,\mathbf{S})}{\cancel{A_d(T_e,\mathbf{S})}}\cancel{A_d(T_e,\mathbf{S})}\mathrm{Max}\left[\omega[R_x^{\mathrm{Swap}}(T_e,\mathbf{T},\mathbf{S})-K];0\right]\right\}\\ &= NA_d(t,\mathbf{S})\mathbb{E}_t^{Q^{\mathbf{S}}}\left\{\mathrm{Max}\left[\omega[\boxed{R_x^{\mathrm{Swap}}(T_e,\mathbf{T},\mathbf{S})}-K];0\right]\right\}\\ &= NA_d(t,\mathbf{S})\mathrm{Black}\left[R_x^{\mathrm{Swap}}(t,\mathbf{T},\mathbf{S}),K,\lambda_x,v_x(t,\mathbf{T},\mathbf{S}),\omega\right],\end{aligned}$$

Change of pricing measure, from risk neutral (numeraire = bank account) to swap measure (numeraire = annuity)

where we have assumed a shifted lognormal martingale (driftless) dynamics for the evolution of the swap rate under its corresponding discounting swap measure Q^{S} associated to the numeraire $A_d(t,\boldsymbol{S})$,

$$\boxed{dR_x^{\mathrm{Swap}}(t;\mathbf{T},\mathbf{S}) = \left[R_x^{\mathrm{Swap}}(t;\mathbf{T},\mathbf{S})+\lambda_x\right]\sigma_x(t,\mathbf{T},\mathbf{S})dW^{Q^{\mathbf{S}}}(t),}$$

and $v_x^{\lambda}(t,\boldsymbol{T},\boldsymbol{S})$ is the (forward) swap rate lognormal variance such that

$$v_x(t,\mathbf{T},\mathbf{S}) = \int_t^{T_e}\sigma_x(u,\mathbf{T},\mathbf{S})^2\,du,\qquad \sigma_x(t,T_e,\mathbf{T},\mathbf{S}) = \sqrt{\frac{v_x(t,\mathbf{T},\mathbf{S})}{\tau_x(t,T_e)}}$$

9: Interest rate volatility products
Swaption [6]

The price of cash delivery swaptions at time $t<T_0$ is given by a similar Black formula

$$
\begin{aligned}
&\mathbf{Swaption}(t;\mathbf{T},\mathbf{S},K,\omega) \\
&= N\mathbb{E}_t^{\boxed{Q}}\left\{D_d(t,T_e)P_d(T_e;T_0)\tilde{A}(T_e,R_x^{\text{Swap}}(T_e))\text{Max}\left[\omega[R^{\text{Swap}}(T_e,\mathbf{T},\mathbf{S})-K];0\right]\right\} \\
&= N\mathbb{E}_t^{\boxed{Q^{\mathbf{S}}}}\left\{A_d(t,\mathbf{S})\boxed{\frac{P_d(T_e;T_0)\tilde{A}(T_e;R_x^{\text{Swap}}(T_e))}{A_d(T_e,\mathbf{S})}}\text{Max}\left[\omega[R_x^{\text{Swap}}(T_e,\mathbf{T},\mathbf{S})-K];0\right]\right\} \\
&\simeq N\cancel{A_d(t,\mathbf{S})}\frac{P_d(t;T_0)\tilde{A}(t;R_x^{\text{Swap}}(t))}{\cancel{A_d(t,\mathbf{S})}}\mathbb{E}_t^{Q^{\mathbf{S}}}\left\{\text{Max}\left[\omega[R_x^{\text{Swap}}(T_e,\mathbf{T},\mathbf{S})-K];0\right]\right\} \\
&= NP_d(t;T_0)\tilde{A}(t;R_x^{\text{Swap}}(t))\mathbb{E}_t^{Q^{\mathbf{S}}}\left\{\text{Max}\left[\omega\boxed{[R_x^{\text{Swap}}(T_e,\mathbf{T},\mathbf{S})}-K];0\right]\right\} \\
&= NP_d(t;T_0)\tilde{A}(t;R_x^{\text{Swap}}(t))\text{Black}\left[R_x^{\text{Swap}}(t,\mathbf{T},\mathbf{S}),K,\lambda_x,v_x(t,\mathbf{T},\mathbf{S}),\omega\right],
\end{aligned}
$$

where we have assumed that the (discounted) annuity ratio in the box has lower variance and may be freezed at its initial value at time *t*. The final results follows from the standard assumption of a shifted lognormal swap rate under the forward swap measure Q^S.

9: Interest rate volatility products
Swaption [7]

Remarks about physical vs cash delivery swaptions

- Black implied volatilities are equal, because of the Black approximation for cash delivery Swaptions, based on the same Black dynamics of the underlying forward swap rate. Hence, prices are different only because of the different annuity.

- Arbitrage may be possible (see F. Mercurio, "*No-arbitrage conditions for cash-settled swaptions*", Risk, Feb. 2008).

- Normally just one type of swaption is quoted on the market (e.g. cash delivery swaptions in EUR, physical delivery swaptions in USD), and is typically used as hedge for exotic portfolios exposed to swap rate vega sensitivity (more about this later).

- In principle, we may consider physical delivery swaptions as plain vanilla and cash delivery swaptions as exotics. Cash delivery swaption prices may be mapped onto physical delivery swaptions prices (see Andersen and Piterbarg, 2010) (more about this later).

Collar and zero-wide collar

- A **Collar** is a strategy based on a long payer and a short receiver Swaption, both out of the money of a fixed value. The payoff can be written as

$$\begin{aligned}\mathbf{Collar}(t;\mathbf{T},\mathbf{S},K_{ATM},\Delta) = {} & \mathbf{Swaption}(t;\mathbf{T},\mathbf{S},K_{ATM}+\Delta,\omega=1) \\ & - \mathbf{Swaption}(t;\mathbf{T},\mathbf{S},K_{ATM}-\Delta,\omega=-1)\end{aligned}$$

  where K_{ATM} is the ATM strike, equal to the forward swap rate.

- Setting $\Delta = 0$ leads to a **Zero-Wide Collar**, corresponding to a call-put parity strategy. Hence we have two distinct cases
  - Physical delivery Zero-Wide Collar: the price is strictly equal to zero because of the call-put parity relationship.
  - Cash delivery Zero-Wide Collar: the price is different from zero in a strict mathematical sense (because of the different annuity), leading to a call-put parity violation. The classic Black Formula approximation restore call-put parity, leading to a null price.

- Thus, cash delivery Zero Wide Collars prices may work as indicators of call-put parity violation, or, in other words, of a possible breach of the classical Black formula approximation for cash delivery swaptions.

Swaption: no arbitrage conditions [1]

Swaption prices satisfy a number of no-arbitrage conditions.

1. **Monotonicity (vertical spread)**: for any strike K_1, K_2

$$\frac{\mathbf{Swaption}(t; K_1, 1) - \mathbf{Swaption}(t; K_2, 1)}{K_2 - K_1} > 0,$$

$$\frac{\mathbf{Swaption}(t; K_1, -1) - \mathbf{Swaption}(t; K_2, -1)}{K_2 - K_1} < 0$$

2. **Convexity (butterfly spread) 1**: for any strike $K_0 < K_1 < K_2$

$$\mathbf{Swaption}(t; K_0, -1) - \frac{K_2 - K_0}{K_2 - K_1}\mathbf{Swaption}(t; K_1, -1) + \frac{K_1 - K_0}{K_2 - K_1}\mathbf{Swaption}(t; K_2, -1) > 0,$$

Swaption: no arbitrage conditions [2]

3. **Convexity 2 (butterfly spread) 2**: for any strike K_1, K_2

$$\frac{\mathbf{Swaption}(t; K_2, 1)}{K_2} - \frac{\mathbf{Swaption}(t; K_1, 1)}{K_1} > 0,$$

4. **Triangular arbitrage**: for any expiry $T_1 < T_2 < T_3$ and strike K

$$\mathbf{Swaption}(t; T_1, T_2, K, -1) + \mathbf{Swaption}(t; T_2, T_3, K, -1) \\ - \mathbf{Swaption}(t; T_1, T_3, K, -1) > 0.$$

Swaption: no arbitrage conditions [3]

Remarks

- Conditions 1-3 hold to each call/put payoff, regardless the asset class and the remapping model used for implied volatilities. Condition 4 is specific for swaptions, see Johnson & Nonas 2009.

- No-arbitrage conditions are typically respected by market quotations since brokers and market makers check them to avoid arbitrages.

- No-arbitrage conditions should be checked whenever interpolatng quotes to get **volatility cubes**. In particular, the use of **SABR implied volatility** functional form for low strikes may produce local no-arbitrage violations.

9: Interest rate volatility products

Swaption: pricing table

	Swaption pricing formulas: physical delivery
Classical (single-curve, positive rates)	$NA(t,\mathbf{S})\text{Black}\left[R^{\text{Swap}}(t,\mathbf{T},\mathbf{S}),K,v(t,\mathbf{T},\mathbf{S}),\omega\right]$
Modern (multi-curve, positive/negative rates)	$NA_d(t,\mathbf{S})\text{Black}\left[R_x^{\text{Swap}}(t,\mathbf{T},\mathbf{S}),K,\lambda_x,v_x(t,\mathbf{T},\mathbf{S}),\omega\right]$

	Swaption pricing formulas: cash delivery
Classical (single-curve, positive rates)	$NP(t;T_0)\tilde{A}(t;R^{\text{Swap}}(t))\text{Black}\left[R^{\text{Swap}}(t,\mathbf{T},\mathbf{S}),K,v_x(t,\mathbf{T},\mathbf{S}),\omega\right]$
Modern (multi-curve, positive/negative rates)	$NP_d(t;T_0)\tilde{A}(t;R_x^{\text{Swap}}(t))\text{Black}\left[R_x^{\text{Swap}}(t,\mathbf{T},\mathbf{S}),K,\lambda_x,v_x(t,\mathbf{T},\mathbf{S}),\omega\right]$

9: Interest rate volatility products

Swaption: market quotes [1]

13:00 30DEC11 ICAP UK69580 VCAP2A
EUR ATM Swaption Straddles - Fwd Premium Mids
Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	11.0	25.5	46.5	70.5	95.0	118	140	163	186	209	303	397	491	585
2M Opt	16.0	36.5	69.5	103	135	167	198	230	264	295	429	561	692	823
3M Opt	19.5	46.0	85.5	125	163	202	241	280	322	361	523	680	835	989
6M Opt	30.0	68.5	119	172	230	284	336	390	446	502	717	926	1129	1327
9M Opt	39.0	86.0	144	212	284	347	410	473	539	605	863	1110	1350	1582
1Y Opt	46.5	102	170	248	329	401	473	546	621	697	989	1268	1534	1785
18M Opt	65.5	137	220	309	403	489	574	659	744	830	1171	1488	1781	2070
2Y Opt	87.5	177	275	375	476	569	663	757	850	943	1320	1664	1990	2301
3Y Opt	123.5	239	356	471	585	696	804	909	1014	1104	1509	1887	2245	2596
4Y Opt	148.5	278	407	536	665	784	901	1013	1122	1222	1643	2041	2428	2796
5Y Opt	162.0	306	445	579	715	843	967	1086	1202	1314	1761	2162	2566	2933
7Y Opt	178.0	340	494	642	788	929	1064	1198	1328	1461	1943	2356	2774	3139
10Y Opt	193.5	377	551	719	880	1038	1193	1346	1494	1640	2158	2601	3009	3351
15Y Opt	217.5	426	627	819	1002	1184	1361	1534	1703	1876	2433	2908	3298	3664
20Y Opt	232.0	458	674	884	1085	1284	1476	1665	1848	2020	2633	3117	3563	3950
25Y Opt	239.0	470	697	914	1127	1330	1530	1728	1917	2107	2692	3170	3634	4004
30Y Opt	242.5	472	698	915	1131	1335	1540	1741	1944	2154	2741	3240	3717	4166

screen closed 1pm.Screen will open 3rd jan 2012. Happy New Year
<VCAP1> <VCAP3>
Options Index <VCAP> IR Option RIC Index <IRORICSMENU>

13:00 30DEC11 ICAP UK69580 VCAP1A
EUR ATM Swaption Straddles - Implied Volatilities
Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	46.1	43.9	49.9	50.7	49.2	46.8	44.8	43.7	42.9	42.4	39.8	41.3	44.2	47.2
2M Opt	49.7	45.7	53.4	52.9	49.7	47.0	45.0	43.8	43.4	42.5	40.1	41.6	44.4	47.4
3M Opt	51.0	47.6	53.5	51.9	48.5	46.1	44.4	43.3	43.0	42.4	39.8	41.3	43.8	46.6
6M Opt	57.5	49.9	51.0	48.8	46.8	44.4	42.8	41.8	41.4	41.1	38.4	39.6	41.9	44.2
9M Opt	59.9	49.6	48.3	46.8	45.3	42.9	41.5	40.5	40.0	39.7	37.4	38.6	40.7	42.9
1Y Opt	59.2	48.5	46.4	45.0	43.5	41.4	40.2	39.5	39.1	38.9	36.7	37.9	39.9	41.7
18M Opt	60.1	47.1	43.8	41.6	40.3	38.8	37.9	37.3	36.9	36.6	35.0	36.1	37.7	39.5
2Y Opt	58.8	46.0	42.3	39.8	38.2	36.8	36.0	35.4	35.0	34.7	33.6	34.6	36.2	37.7
3Y Opt	48.4	39.9	36.9	35.1	34.1	33.4	32.8	32.3	31.9	31.4	30.5	31.6	33.1	34.5
4Y Opt	40.2	34.1	32.3	31.5	31.1	30.5	30.0	29.6	29.3	29.0	28.4	29.5	30.9	32.0
5Y Opt	33.8	30.6	29.5	28.9	28.5	28.1	27.7	27.5	27.4	27.4	27.2	28.0	29.3	30.0
7Y Opt	28.7	27.0	26.2	25.6	25.3	25.1	25.0	25.1	25.3	25.7	25.9	26.4	27.2	27.5
10Y Opt	24.0	23.3	23.1	23.2	23.3	23.6	24.0	24.4	24.9	25.4	25.5	25.7	25.5	25.1
15Y Opt	24.8	25.1	25.6	26.1	26.5	27.0	27.4	27.8	28.3	28.8	27.5	26.0	24.5	23.7
20Y Opt	29.9	30.0	30.2	30.6	30.8	31.1	31.4	31.6	31.7	31.6	27.7	24.9	23.5	22.6
25Y Opt	32.9	32.5	32.6	32.4	32.1	31.5	30.9	30.4	29.8	29.0	24.3	22.0	21.0	20.1
30Y Opt	29.5	27.3	26.7	26.0	24.9	24.9	24.5	24.2	24.0	24.0	21.0	19.5	18.8	18.5

screen closed 1pm.Screen will open 3rd jan 2012. Happy New Year
<VCAP> <VCAP2>
Options Index <VCAP> IR Option RIC Index <IRORICSMENU>
ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

13:05 30DEC11 BGC Partners UK38000 BGCOPTIONS07
EUR Smile Summary Page 1 - lognormal volatility % difference v ATM

.	-200bp	-100bp	-50bp	-25bp	ATM	25bp	50bp	100bp	200bp
3M2Y	-	+39.24	+7.85	+2.29	0	-0.34	+0.26	+2.30	+6.17
3M5Y	-	+19.08	+6.52	+2.74	0	-1.95	-3.32	-4.86	-5.62
3M10Y	+62.92	+18.39	+7.64	+3.49	0	-2.88	-5.18	-8.23	-10.04
3M20Y	+50.44	+16.91	+7.29	+3.39	0	-2.92	-5.40	-9.01	-11.89
3M30Y	+66.15	+21.22	+9.13	+4.25	0	-3.69	-6.84	-11.56	-15.52
1Y2Y	-	+24.34	+6.19	+2.31	0	-1.33	-2.07	-2.56	-2.16
1Y5Y		+14.42	+5.37	+2.35	0	-1.82	-3.22	-5.10	-6.73
1Y10Y	+39.18	+11.74	+4.85	+2.21	0	-1.84	-3.35	-5.54	-7.64
1Y20Y	+35.50	+11.39	+4.75	+2.17	0	-1.81	-3.28	-5.38	-7.17
1Y30Y	+44.98	+13.78	+5.73	+2.62	0	-2.19	-3.99	-6.58	-8.93
5Y2Y	+18.01	+5.47	+2.19	+0.98	0	-0.79	-1.41	-2.29	-3.06
5Y5Y	+17.42	+5.71	+2.36	+1.07	0	-0.88	-1.60	-2.64	-3.61
5Y10Y	+17.19	+5.77	+2.39	+1.09	0	-0.89	-1.60	-2.60	-3.40
5Y20Y	+21.51	+6.83	+2.79	+1.26	0	-1.02	-1.84	-2.95	-3.83
5Y30Y	+26.97	+8.08	+3.24	+1.45	0	-1.15	-2.05	-3.23	-4.04
10Y2Y	+11.42	+3.75	+1.52	+0.69	0	-0.55	-0.99	-1.58	-2.03
10Y5Y	+13.63	+4.54	+1.85	+0.83	0	-0.67	-1.19	-1.86	-2.27
10Y10Y	+17.61	+5.68	+2.31	+1.04	0	-0.83	-1.48	-2.32	-2.88
10Y20Y	+23.27	+6.74	+2.67	+1.19	0	-0.94	-1.65	-2.58	-3.20
10Y30Y	+22.08	+6.28	+2.52	+1.13	0	-0.92	-1.65	-2.69	-3.67

MASTER INDEX PAGE.....<BGCPINDEX> BGCP CONTACT...................<BGCTEL>

13:05 30DEC11 BGC Partners UK38000 BGCOPTIONS08
EUR Smile Summary Page 2 - lognormal volatility % difference v ATM

.	-200bp	-100bp	-50bp	-25bp	ATM	25bp	50bp	100bp	200bp
20Y2Y	+34.20	+8.17	+3.24	+1.46	0	-1.20	-2.20	-3.70	-5.42
20Y5Y	+44.64	+9.40	+3.65	+1.63	0	-1.31	-2.36	-3.89	-5.45
20Y10Y	+56.14	+9.80	+3.79	+1.69	0	-1.38	-2.51	-4.22	-6.21
20Y20Y	+32.81	+7.37	+2.88	+1.29	0	-1.05	-1.89	-3.14	-4.49
20Y30Y	+26.78	+6.57	+2.58	+1.16	0	-0.94	-1.69	-2.79	-3.93
30Y2Y	+40.38	+8.42	+3.28	+1.46	0	-1.18	-2.14	-3.54	-5.04
30Y5Y	+32.09	+7.70	+3.00	+1.34	0	-1.06	-1.90	-3.06	-4.10
30Y10Y	+26.02	+6.67	+2.63	+1.18	0	-0.96	-1.73	-2.84	-4.00
30Y20Y	+19.54	+5.27	+2.06	+0.92	0	-0.72	-1.29	-2.04	-2.66
30Y30Y	+17.72	+4.82	+1.90	+0.85	0	-0.67	-1.21	-1.94	-2.59

MASTER INDEX PAGE.....<BGCPINDEX> BGCP CONTACT...................<BGCTEL>

ATM premia and volatilities from ICAP (top), smile volatility spreads on ATM from BGC (bottom), as of 30 Dec. 2011.

9: Interest rate volatility products

Swaption: market quotes [2]

16:16 31OCT16 ICAP UK69580 VCAP2A

EUR ATM Swaption Straddles - Fwd Premium Mids (Eonia disc)

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	3.5	8.0	16.5	28.0	42.5	56.5	72.0	88.0	106	123	201	281	363	441
2M Opt	5.0	11.5	23.5	40.5	60.0	80.5	104	125	149	173	285	398	506	611
3M Opt	7.0	15.5	29.5	49.5	74.5	99.0	126	153	184	211	347	481	605	729
6M Opt	11.5	26.0	46.5	75.0	111	146	182	218	256	296	484	662	827	988
9M Opt	16.0	35.5	61.5	99.0	140	184	230	276	323	371	598	812	1009	1196
1Y Opt	20.5	45.5	77.5	120	169	217	271	327	380	435	690	932	1153	1363
18M Opt	29.5	64.0	107	157	217	277	344	413	478	547	849	1128	1395	1647
2Y Opt	38.5	84.0	139	197	267	340	418	496	573	650	991	1305	1609	1890
3Y Opt	61.0	126	200	278	364	454	550	645	739	828	1215	1578	1923	2250
4Y Opt	83.0	169	260	358	455	557	661	769	872	979	1404	1806	2186	2553
5Y Opt	102.5	209	317	429	540	654	769	884	997	1114	1568	2000	2404	2801
7Y Opt	136.0	274	409	542	674	806	940	1073	1204	1326	1816	2296	2729	3167
10Y Opt	169.0	339	505	666	823	978	1132	1280	1425	1565	2119	2641	3130	3605
15Y Opt	202.0	404	600	793	986	1170	1345	1516	1684	1844	2487	3030	3562	4060
20Y Opt	223.0	446	663	871	1078	1277	1467	1647	1825	1997	2703	3260	3832	4351
25Y Opt	234.5	470	696	915	1123	1326	1521	1703	1891	2061	2801	3398	4032	4598
30Y Opt	241.0	484	710	929	1134	1338	1532	1717	1906	2086	2846	3472	4133	4766

screen closed

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Options Index <VCAP> Disclaimer <IDIS>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

16:16 31OCT16 ICAP UK69580 VCAP2

EUR ATM Swaption Straddles - Implied Spot Premium Mids (Eonia disc)

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	3.5	8.0	16.5	28.0	42.5	56.5	72.0	88.0	106	123	201	281	363	441
2M Opt	5.0	11.5	23.5	40.5	60.5	80.5	104	126	149	174	285	398	506	612
3M Opt	7.0	15.5	29.5	49.5	75.0	99.0	126	154	184	211	347	481	606	729
6M Opt	11.5	26.0	46.5	75.0	112	146	182	218	257	297	485	663	829	990
9M Opt	16.0	36.0	61.5	99.0	141	184	230	277	324	372	600	814	1012	1199
1Y Opt	21.0	45.5	77.5	120	170	218	272	328	382	436	692	936	1157	1368
18M Opt	29.5	64.5	107	158	218	278	346	416	481	550	853	1134	1403	1656
2Y Opt	39.0	84.5	140	199	269	342	421	500	577	655	998	1315	1620	1904
3Y Opt	61.5	128	202	281	367	459	555	651	746	836	1227	1594	1942	2273
4Y Opt	84.0	171	263	362	460	564	669	778	883	991	1420	1827	2212	2584
5Y Opt	104.0	211	321	434	546	662	778	895	1009	1127	1586	2024	2432	2833
7Y Opt	136.5	275	410	543	676	808	943	1076	1207	1330	1821	2303	2737	3176
10Y Opt	164.5	330	491	648	799	951	1100	1244	1384	1521	2059	2567	3042	3504
15Y Opt	183.0	365	543	718	892	1058	1217	1372	1524	1669	2250	2742	3223	3673
20Y Opt	188.5	378	561	737	912	1080	1241	1393	1544	1689	2286	2758	3241	3681
25Y Opt	187.5	376	557	731	898	1060	1216	1361	1511	1647	2239	2716	3223	3676
30Y Opt	183.0	367	539	706	862	1016	1164	1305	1448	1585	2162	2638	3140	3621

screen closed

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Options Index <VCAP> Disclaimer <IDIS>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

16:16 31OCT16 ICAP UK69580 VCAP1A

EUR ATM Swaption Straddles - Shifted Black Volatilities

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	5.7	7.0	11.0	16.1	22.9	25.1	27.0	28.5	30.1	31.3	31.1	32.8	35.1	37.3
2M Opt	5.9	7.1	11.1	16.4	22.8	25.1	27.4	28.5	29.8	31.0	31.0	32.8	34.6	36.5
3M Opt	6.4	7.7	11.2	16.1	22.7	24.7	26.6	28.0	29.4	30.3	30.4	31.9	33.4	35.1
6M Opt	7.6	9.2	12.4	17.2	23.8	25.6	27.0	27.8	28.8	29.9	30.1	31.4	32.6	34.1
9M Opt	8.4	10.2	13.2	18.2	23.8	25.6	27.0	27.9	28.8	29.8	29.8	31.0	32.1	33.3
1Y Opt	9.5	11.1	14.2	18.7	24.2	25.5	26.9	28.0	28.8	29.6	29.4	30.6	31.6	32.7
18M Opt	10.8	12.6	15.5	19.3	24.3	25.4	26.7	27.8	28.4	29.3	29.0	29.9	31.0	32.2
2Y Opt	12.1	13.9	17.0	20.2	24.7	25.7	26.8	27.6	28.4	29.2	28.8	29.6	30.7	31.8
3Y Opt	15.0	16.1	18.6	21.4	25.0	25.7	26.5	27.2	27.9	28.5	27.9	28.7	29.6	30.6
4Y Opt	16.6	17.4	19.4	21.9	24.7	25.1	25.6	26.3	26.9	27.7	27.1	27.9	28.8	29.8
5Y Opt	17.1	17.9	19.6	21.7	24.2	24.6	25.1	25.6	26.2	27.0	26.6	27.4	28.1	29.1
7Y Opt	16.7	17.5	19.0	20.7	23.0	23.4	23.9	24.5	25.2	25.8	25.4	26.2	26.8	27.7
10Y Opt	15.5	16.8	18.2	20.0	22.2	22.7	23.4	24.0	24.6	25.4	25.1	25.7	26.1	26.9
15Y Opt	15.0	16.4	18.1	20.2	22.9	23.6	24.2	24.9	25.7	26.4	25.9	25.6	25.8	26.2
20Y Opt	15.3	16.9	18.7	20.9	23.8	24.4	25.0	25.5	26.2	26.9	26.1	25.4	25.6	26.0
25Y Opt	15.0	16.6	18.4	20.6	23.3	23.9	24.4	24.8	25.5	26.1	25.4	25.0	25.5	25.7
30Y Opt	14.4	16.0	17.6	19.7	22.3	22.8	23.3	23.7	24.5	25.2	24.8	24.7	24.9	25.2

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Options Index <VCAP> Disclaimer <IDIS>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

16:16 31OCT16 ICAP UK69580 VCAP1

EUR ATM Swaption Straddles - Black Volatilities (Eonia disc)

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt						375	211	153	124	107	74.6	70.1	72.1	74.6
2M Opt						351	206	148	120	104	74.1	70.0	71.0	73.0
3M Opt						323	192	141	116	100	72.2	68.0	68.4	70.1
6M Opt						279	173	130	107	95.1	70.4	66.3	66.5	67.8
9M Opt						233	156	121	102	91.1	68.7	65.1	65.2	66.2
1Y Opt						198	141	113	97.3	87.2	66.9	63.9	64.0	64.9
18M Opt					250	151	117	99.9	88.1	80.9	64.4	61.7	62.4	63.5
2Y Opt					171	124	102	89.9	81.6	75.9	62.5	60.4	61.3	62.4
3Y Opt				145	109	91.9	82.0	75.5	71.0	67.4	58.2	57.3	58.3	59.6
4Y Opt		259	124	97.7	82.2	73.8	68.5	65.2	62.5	60.9	55.0	55.0	56.2	57.8
5Y Opt		108	87.2	75.5	68.0	63.6	60.7	58.7	57.2	56.5	52.6	53.3	54.4	56.2
7Y Opt	85.3	64.6	59.6	56.2	54.0	52.6	51.9	51.5	51.3	51.1	49.4	50.8	51.9	53.7
10Y Opt	53.5	50.6	49.9	49.3	48.9	48.9	49.3	49.6	50.1	50.6	50.3	51.3	52.6	54.3
15Y Opt	52.3	53.0	54.4	55.9	57.6	59.0	59.8	60.8	61.6	62.3	60.5	58.9	59.4	61.2
20Y Opt	82.1	83.7	88.8	90.4	95.4	98.9	98.0	96.7	97.1	96.2	82.6	73.9	75.9	80.9
25Y Opt														
30Y Opt														

screen closed

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Options Index <VCAP> Disclaimer <IDIS>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

ATM forward/spot premia (top) and Black/shifted Black implied volatilities from ICAP, as of 31 Oct. 2016.

9: Interest rate volatility products

Swaption: market quotes [3]

11:07 31OCT16 ICAP UK69580 ICAPSKEW1A

EUR Gamma - Shifted Black Vol Skews - for shifts see VCAP1B

	Receivers						Payers				
	-200	-150	-100	-50	-25	ATM	+25	+50	+100	+150	+200
1m2y		34.39	19.89	8.65	3.63	6.95	2.70	6.08	11.55	15.85	
1m5y		97.22	34.17	8.29	0.08	22.53	6.84	13.16	22.93	30.17	
1m10y		105.0	44.01	12.48	2.07	30.82	6.60	13.79	25.48	34.36	
1m20y		56.71	26.05	7.48	1.83	32.27	1.78	4.93	11.35	16.77	
1m30y		63.06	28.46	8.45	2.41	36.76	1.03	3.66	9.55	14.73	
3m2y		27.11	15.14	5.93	2.01	7.70	2.45	5.33	10.09	13.84	
3m5y		66.35	22.07	3.90	-0.75	22.68	3.99	8.00	14.52	19.48	
3m10y		48.02	16.66	2.34	-0.45	30.28	2.25	4.90	9.75	13.69	
3m20y		43.27	19.30	5.45	1.52	31.86	0.54	2.14	6.04	9.64	
3m30y		40.89	17.55	5.27	1.81	35.08	-0.44	0.01	1.99	4.24	
6m2y		23.57	12.56	4.22	0.93	9.25	2.36	5.04	9.55	13.13	
6m5y		56.84	19.11	3.56	-0.11	23.91	2.48	5.36	10.40	14.39	
6m10y		35.78	11.64	1.84	0.09	30.07	0.88	2.16	4.84	7.24	
6m20y		35.91	15.79	4.63	1.49	31.50	-0.11	0.61	2.97	5.44	
6m30y		31.08	12.92	4.19	1.65	34.25	-0.94	-1.36	-1.28	-0.62	
9m2y		21.47	11.11	3.36	0.51	10.18	2.14	4.60	8.82	12.20	
9m5y		47.57	15.93	3.11	0.24	23.94	1.46	3.42	7.18	10.31	
9m10y		28.28	8.97	1.78	0.40	30.02	0.19	0.71	2.07	3.48	
9m20y		26.34	11.17	3.47	1.29	31.17	-0.60	-0.71	-0.13	0.88	
9m30y		29.85	12.55	4.20	1.69	33.56	-1.05	-1.62	-1.83	-1.42	

<ICAPSKEW> <ICAPSKEW2>

11:07 31OCT16 ICAP UK69580 ICAPSKEW2A

EUR Vega - Shifted Black Vol Skews - for shifts see VCAP1B

	Receivers						Payers				
	-200	-150	-100	-50	-25	ATM	+25	+50	+100	+150	+200
1y2y		15.58	7.40	1.65	-0.04	11.11	1.42	3.13	6.25	8.83	
1y5y		37.91	12.19	2.27	0.26	24.41	0.82	2.06	4.66	6.98	
1y10y		24.07	7.81	1.86	0.58	29.83	-0.13	0.01	0.68	1.54	
1y20y		22.80	9.72	3.22	1.29	30.76	-0.79	-1.22	-1.35	-1.00	
1y30y		28.25	11.97	4.09	1.69	32.99	-1.11	-1.78	-2.26	-2.08	
2y2y		9.25	3.39	-0.02	-0.46	13.92	0.97	2.10	4.27	6.16	
2y5y		20.14	6.84	1.54	0.43	24.91	0.02	0.29	1.18	2.17	
2y10y		16.90	6.69	2.28	0.96	29.38	-0.70	-1.20	-1.81	-2.06	
2y20y		17.38	7.82	2.95	1.31	29.81	-1.04	-1.87	-3.05	-3.77	
2y30y		21.68	9.34	3.52	1.56	31.99	-1.26	-2.29	-3.79	-4.76	
5y2y		4.07	1.86	0.58	0.22	17.93	-0.10	-0.12	0.02	0.29	
5y5y		8.07	4.02	1.55	0.68	24.36	-0.54	-0.97	-1.57	-1.93	
5y10y		9.98	5.10	2.06	0.93	27.23	-0.79	-1.45	-2.50	-3.27	
5y20y		12.49	6.31	2.54	1.15	27.59	-0.97	-1.79	-3.07	-4.02	
5y30y		16.06	7.81	3.08	1.39	29.36	-1.16	-2.13	-3.63	-4.73	
10y2y		2.44	1.36	0.57	0.26	16.74	-0.22	-0.41	-0.69	-0.89	
10y5y		5.78	3.15	1.31	0.60	22.30	-0.50	-0.92	-1.57	-2.01	
10y10y		7.71	4.28	1.84	0.86	25.50	-0.76	-1.44	-2.58	-3.50	
10y20y		9.87	5.24	2.20	1.02	25.84	-0.88	-1.66	-2.93	-3.93	
10y30y		12.18	6.10	2.46	1.12	27.10	-0.94	-1.74	-3.00	-3.91	

<ICAPSKEW1> <ICAPSKEW3>

Smile volatility spreads on ATM from ICAP, as of 31 Oct. 2015.
(ATM volatilities show minor differences w.r.t. page VCAP1A because of the different snapshot time)

- ATM premia are quoted in terms of straddles (payer + receiver ATM cash-delivery swaptions with same expiry/tenor).

- Since the straddle has zero delta sensitivity (proof this), the straddle allows a pure volatility trading, both for investing purposes (bet on volatility) or hedging purposes (hedge the vega without affecting the delta).

9: Interest rate volatility products
Swaption: market quotes [4]

Swaption market in low rates regime.

12:28 01MAR13 ICAP VCAP8

EUR ATM Swaption Forwards (Eonia disc)

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
1M Opt	0.2020	0.4220	0.5530	0.7200	0.9110	1.1040	1.2860	1.4520	1.5990	1.7350
2M Opt	0.2070	0.4350	0.5730	0.7460	0.9390	1.1310	1.3120	1.4770	1.6240	1.7590
3M Opt	0.2150	0.4510	0.5940	0.7700	0.9660	1.1590	1.3390	1.5030	1.6490	1.7810
6M Opt	0.2500	0.5070	0.6650	0.8520	1.0520	1.2440	1.4220	1.5820	1.7240	1.8500
9M Opt	0.2980	0.5700	0.7420	0.9370	1.1410	1.3310	1.5050	1.6610	1.7990	1.9210
1Y Opt	0.3530	0.6430	0.8240	1.0280	1.2310	1.4190	1.5880	1.7390	1.8730	1.9890
18M Opt	0.4820	0.8030	1.0070	1.2190	1.4170	1.5970	1.7570	1.8980	2.0210	2.1270
2Y Opt	0.6410	0.9880	1.2060	1.4180	1.6080	1.7760	1.9240	2.0550	2.1660	
3Y Opt	1.0480	1.4180	1.6320	1.8190	1.9820	2.1230	2.2460	2.3520	2.4410	
4Y Opt	1.5120	1.8590	2.0350	2.1890	2.3200	2.4350	2.5320	2.6130	2.6820	
5Y Opt	1.9430	2.2360	2.3780	2.4990	2.6040	2.6920	2.7650	2.8270	2.8710	
7Y Opt	2.5550	2.7730	2.8640	2.9360	2.9960	3.0450	3.0750	3.0870	3.0900	
10Y Opt	3.0750	3.2080	3.2420	3.2500	3.2380	3.2190	3.1910	3.1570	3.1210	
15Y Opt	3.0700	3.0600	3.0050	2.9550	2.9110	2.8750	2.8450	2.8180	2.7930	
20Y Opt	2.6470	2.6640	2.6460	2.6280	2.6100	2.5930	2.5780	2.5650	2.5560	
25Y Opt	2.4760	2.4900	2.4830	2.4810	2.4840	2.4930	2.5050	2.5190	2.5340	
30Y Opt	2.4630	2.5620	2.5840	2.6050	2.6250	2.6450	2.6650	2.6870	2.7110	

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ICAP Global Index <ICAP> Forthcoming changes <ICA

← Forward swap rates, 2013

12:33 01MAR13 ICAP UK69580 VCAP1

EUR ATM Swaption Straddles - Black Volatilities (Eonia disc)

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	157	100	93.9	79.7	67.3	56.6	49.2	44.1	40.1	36.6	29.9	28.4	28.0	27.8
2M Opt	157	105	93.0	80.0	66.6	56.1	49.1	44.1	40.5	37.6	31.2	29.8	29.4	29.1
3M Opt	150	101	92.3	76.2	63.5	53.9	47.5	43.1	40.1	37.7	31.0	29.6	29.2	28.6
6M Opt	146	93.8	82.6	69.9	59.7	52.3	46.6	42.7	39.7	37.5	31.1	29.7	29.4	29.2
9M Opt	139	87.2	75.6	65.9	56.2	49.8	45.1	41.4	38.7	36.9	30.9	29.6	29.3	29.2
1Y Opt	129	80.2	70.8	61.6	53.3	47.4	43.3	40.3	38.0	36.2	30.8	29.6	29.3	29.2
18M Opt	111	71.0	60.6	53.7	48.0	43.5	40.1	37.6	35.7	34.3	29.8	28.9	28.8	28.8
2Y Opt	94.0	64.0	54.8	48.3	44.1	40.4	37.6	35.6	34.1	32.9	29.0	28.4	28.5	28.4
3Y Opt	68.1	50.8	44.4	40.4	37.9	35.4	33.5	32.1	31.0	30.1	27.5	27.1	27.4	27.4
4Y Opt	50.3	40.1	36.8	34.4	32.7	31.1	30.0	29.2	28.5	28.0	26.3	26.0	26.3	26.4
5Y Opt	40.0	34.0	31.9	30.4	29.2	28.3	27.6	27.0	26.6	26.4	25.3	25.2	25.6	25.6
7Y Opt	30.0	27.1	26.1	25.4	24.9	24.5	24.3	24.3	24.3	24.3	23.7	23.5	23.7	23.7
10Y Opt	23.5	22.3	22.1	21.9	22.0	22.1	22.3	22.6	22.9	23.0	22.6	22.2	22.1	21.8
15Y Opt	21.8	21.7	22.1	22.5	22.8	23.0	23.2	23.5	23.6	23.6	22.5	21.5	20.6	19.9
20Y Opt	23.4	23.3	23.5	23.6	23.8	23.9	24.0	24.0	24.0	23.7	21.7	20.0	18.9	18.4
25Y Opt	23.6	23.5	23.7	23.7	23.7	23.4	23.2	22.9	22.7	22.4	19.7	17.8	17.4	16.8
30Y Opt	22.3	21.3	21.2	21.1	20.9	20.6	20.4	20.1	19.9	19.7	17.1	16.2	15.8	15.3

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Black volatilities, 2013 →

01/03/2013 13:33:35 (GMT+1)

9: Interest rate volatility products
Swaption: market quotes [5]

Swaption market in negative rates regime

16:12 30DEC15 ICAP VCAP8

EUR ATM Swaption Forwards (Eonia disc)

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
1M Opt	-0.177	-0.030	0.0720	0.2040	0.3460	0.4910	0.6350	0.7710	0.8980	1.0110
2M Opt	-0.181	-0.024	0.0840	0.2190	0.3630	0.5090	0.6530	0.7910	0.9160	1.0280
3M Opt	-0.183	-0.016	0.0980	0.2370	0.3830	0.5290	0.6740	0.8100	0.9340	1.0460
6M Opt	-0.178	0.0170	0.1480	0.2950	0.4450	0.5930	0.7370	0.8720	0.9940	1.1030
9M Opt	-0.159	0.0630	0.2070	0.3590	0.5110	0.6610	0.8040	0.9360	1.0550	1.1610
1Y Opt	-0.126	0.1190	0.2700	0.4260	0.5790	0.7290	0.8700	0.9990	1.1150	1.2170
18M Opt	-0.022	0.2540	0.4150	0.5720	0.7250	0.8720	1.0080	1.1290	1.2370	1.3310
2Y Opt	0.1220	0.4100	0.5710	0.7270	0.8780	1.0190	1.1470	1.2600	1.3590	1.4430
3Y Opt	0.4430	0.7340	0.8890	1.0380	1.1770	1.3010	1.4100	1.5		
4Y Opt	0.7670	1.0480	1.1950	1.3300	1.4510	1.5530	1.6430	1.7		
5Y Opt	1.0850	1.3500	1.4810	1.5960	1.6920	1.7750	1.8420	1.8		
7Y Opt	1.6530	1.8490	1.9290	1.9980	2.0510	2.0910	2.1260	2.1		
10Y Opt	2.1660	2.2440	2.2640	2.2840	2.2970	2.2980	2.2920	2.2		
15Y Opt	2.2680	2.2790	2.2490	2.2150	2.1790	2.1380	2.0990	2.0		
20Y Opt	1.8970	1.8850	1.8530	1.8200	1.7890	1.7700	1.7520	1.7		
25Y Opt	1.6450	1.6520	1.6480	1.6470	1.6390	1.6340	1.6300	1.6		
30Y Opt	1.5900	1.6060	1.6060	1.6060	1.6060	1.6140	1.6130	1.6		

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← Forward swap rates, 2016

Black volatilities, 2016 →

16:11 30DEC15 ICAP UK69580 VCAP1

EUR ATM Swaption Straddles - Black Volatilities (Eonia disc)

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt			459	191	120	90.9	74.6	64.8	58.6	54.1	43.8	42.1	43.0	43.3
2M Opt			383	165	113	86.1	72.9	63.3	57.2	53.4	43.7	41.9	41.7	41.9
3M Opt			341	154	108	83.3	71.8	63.7	58.3	53.9	43.9	41.8	41.5	41.5
6M Opt			218	125	95.5	78.1	69.5	62.3	57.9	53.7	44.7	42.7	42.5	42.7
9M Opt			177	110	86.0	71.9	65.0	59.1	56.1	52.9	44.5	42.7	42.5	42.8
1Y Opt			139	96.4	79.5	68.5	62.7	58.2	54.9	52.2	44.3	42.4	42.6	42.8
18M Opt		171	105	79.7	69.3	62.2	57.5	54.1	51.6	50.0	43.1	41.7	42.0	42.4
2Y Opt		114	86.9	71.3	63.2	57.2	53.8	50.9	48.8	47.6	41.5	40.3	40.9	41.4
3Y Opt	146	77.5	66.0	58.7	54.1	50.1	47.7	45.8	44.6	44.0	39.2	38.3	39.1	39.6
4Y Opt	86.5	60.9	54.1	50.1	47.4	45.1	43.5	42.4	41.4	40.9	37.3	37.0	37.5	38.0
5Y Opt	65.0	50.9	47.0	44.3	42.6	41.1	40.1	39.3	38.6	38.4	35.5	35.6	36.2	36.6
7Y Opt	45.0	39.9	38.4	37.2	36.4	35.8	35.4	35.1	35.0	35.0	33.5	33.9	34.4	34.9
10Y Opt	34.4	33.4	33.4	33.1	32.9	33.0	33.2	33.3	33.6	33.9	33.4	33.8	34.5	34.8
15Y Opt	32.3	32.4	32.9	33.6	34.2	34.8	35.4	35.9	36.4	36.8	35.7	34.5	34.4	35.3
20Y Opt	39.1	40.0	40.7	41.3	41.8	42.2	42.3	42.2	42.0	42.1	38.6	35.9	36.5	36.6
25Y Opt	47.0	46.9	46.5	45.8	45.1	44.6	44.0	43.2	42.5	42.1	38.0	36.9	37.1	36.3
30Y Opt	47.4	47.1	45.7	44.5	43.0	41.9	41.1	40.1	39.5	39.9	39.2	37.0	36.0	36.0

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Options Index <VCAP> Disclaimer <IDIS>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

9: Interest rate volatility products
Swaption: market quotes [6]

Swaption market Black shift evolution

← Black shifts, 2013

08:07 01MAR13 ICAP UK69580 VCAP1B

EUR - Shifts for Shifted Black ATM Swaptions Vols & Swaptions Skews as %

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
2M Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
3M Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
6M Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
9M Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
1Y Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
18M Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
2Y Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
3Y Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
4Y Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.1		
5Y Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.1		
7Y Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.1		
10Y Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.1		
15Y Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.1		
20Y Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.1		
25Y Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.1		
30Y Opt	0.20	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.1		

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ICAP Global Index <ICAP> Forthcoming changes

16:12 23FEB16 ICAP UK69580 VCAP1B

EUR - Shifts for Shifted Black ATM Swaptions Vols & Swaptions Skews as %

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
2M Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
3M Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
6M Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
9M Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
1Y Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
18M Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
2Y Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
3Y Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
4Y Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
5Y Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
7Y Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
10Y Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
15Y Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
20Y Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
25Y Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50
30Y Opt	3.00	2.65	2.30	1.95	1.60	1.58	1.56	1.54	1.52	1.50	1.50	1.50	1.50	1.50

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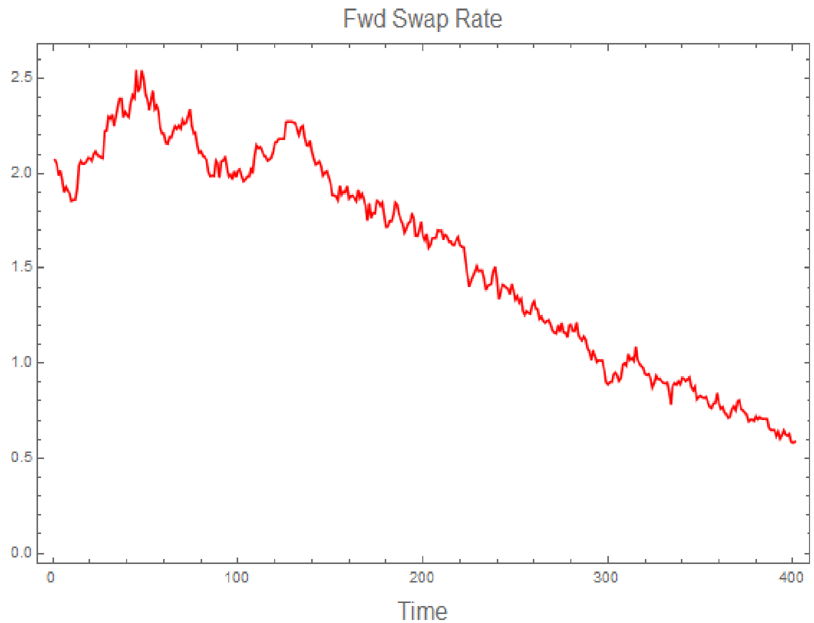
ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

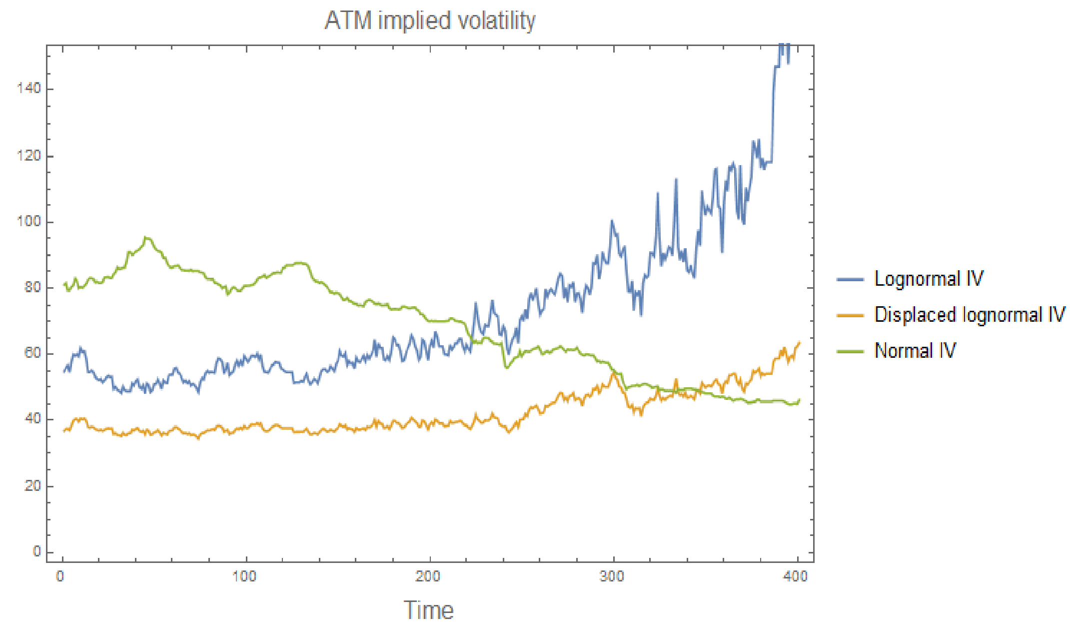
Black shifts, 2016 →

9: Interest rate volatility products

Swaption: market quotes [7]



Forward swap rate (3Y X 3Y).



ATM volatility for Black, shifted Black and normal formulas.

9: Interest rate volatility products
Swaption: cash delivery [1]

- Since summer 2016, new market data are available for EUR cash delivery Swaptions:
  - cash delivery zero-wide Collars trade at non-zero premium, thus showing a non-negligible call-put parity violation;
  - different prices are available for physical and cash delivery swaptions, not compatible with a unique Black implied volatility (taking into account the different annuity);
  - two distinct (Black implied) volatility cubes are available (different ATM, same smile).

- Hence the market abandoned the classic (multi curve) shifted Black pricing formula for cash delivery Swaptions, acknowledging that the cash delivery annuity is not a valid numeraire,

- More refined approaches, that better represent the payoff features, are based on the replication approach (see e.g. Lutz 2015). They include new parameters to be calibrated to the new market data available to calibrate. Hence cash delivery swaptions are actually treated as exotics.

9: Interest rate volatility products
Swaption: cash delivery [2]

Cash Settled zero wide collars

EUR Zero Wide Collars - All vs 0wc Phys LCH @ 0

PH SH F H W

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1Y	0	0	0	0	0.1	0.2	0.3	0.4	0.6	0.7	2.2	2.5	3.1	3.2
18M	0	0	-0.1	0	0.1	0.4	0.6	0.8	1.2	1.6	3.1	4.6	5.3	5
2Y	0	0	0	0.1	0.3	0.6	1.1	1.6	2.2	3.2	3.4	6.8	8	7.9
3Y	0	0.1	0.4	0.8	1.4	2	2.6	3.1	3.8	4.6	7.1	12.5	15.5	15.7
4Y	0	0.2	0.5	1	1.6	2.2	2.9	3.6	4.5	5.6	11.4	16.5	21.7	23.5
5Y	0	0.2	0.6	1.1	1.6	2.2	3	4	5.1	6.3	15.8	20.9	28.1	32.4
6Y	0	0.4	0.9	1.5	2.3	3.1	4	5.3	6.7	8.6	19.8	22.9	33.2	39.6
7Y	0	0.5	1.2	1.9	2.5	3.7	5.1	6.7	8.3	10.4	23	25.5	39.2	49.1
8Y	0	0.5	1.3	2	2.8	4	5.5	7.3	9.6	12	26.4	31.7	46.2	59.4
9Y	0	0.6	1.3	2.2	2.9	4.2	5.9	8	10.4	12.9	29.2	36.2	51.7	68.8
10Y	0	0.6	1.4	2.3	3	4.6	6.3	8.6	11.3	14.1	31.3	40.8	61.4	80.8
12Y	0	0.7	1.7	2.7	3.4	5	7	9.6	12.3	15.1	34.4	46.9	66.6	90.5
15Y	0	1	2.3	3.3	3.7	5.6	8.4	11	14	16.7	42	57.5	83.6	112.8
20Y	0	1.3	2.7	3.9	4.1	6.1	9.1	12.1	15.2	19.2	46.1	71.5	105.5	142.7
25Y	0	1.5	3.1	4.3	4.6	7	10	13.6	16.6	20.5	51.3	88.4	124.6	165.7
30Y	0	1.6	3.4	4.6	4.9	7.8	11.4	16.5	20.7	24.2	55.8	97.9	135.7	176.2

9: Interest rate volatility products

Swaption: cash delivery [3]

Separate pages for cash and physical delivery swaptions since Dec. 2017

VCAP2A Quote | VCAP2A

08:03 16JAN18 ICAP UK69580 VCAP2A

EUR ATM Swaption Straddles - Fwd Premium Mids (Eonia disc)

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	2.5	7.5	15.0	24.5	35.0	44.0	52.5	61.0	69.5	76.5	118	158	198	236
2M Opt	3.5	10.5	21.0	35.5	52.0	65.5	78.5	91.0	105	116	179	238	297	349
3M Opt	5.0	13.5	26.5	45.5	62.5	79.0	94.5	110	124	139	212	280	346	405
6M Opt	9.0	24.0	45.0	72.0	99.5	124	149	173	197	218	328	427	519	604
9M Opt	13.5	35.5	64.0	98.5	134	167	198	232	261	289	430	554	671	780
1Y Opt	18.5	48.5	83.5	123	166	203	242	280	316	351	516	664	804	933
18M Opt	30.5	72.5	120	172	221	267	316	365	412	458	668	851	1025	1186
2Y Opt	44.5	99.0	158	218	276	334	390	449	505	558	800	1024	1226	1408
3Y Opt	70.5	146	223	299	374	450	522	595	664	729	1019	1296	1540	1769
4Y Opt	91.5	185	277	367	455	544	631	716	797	874	1200	1515	1788	2050
5Y Opt	109.0	217	323	426	526	628	725	819	910	999	1356	1699	1998	2281
7Y Opt	134.5	267	395	519	638	756	870	983	1093	1198	1604	1993	2326	2652
10Y Opt	161.5	320	472	618	763	903	1040	1170	1298	1425	1904	2347	2730	3092
15Y Opt	186.0	369	544	716	886	1048	1201	1353	1502	1651	2201	2682	3114	3504
20Y Opt	201.0	398	590	777	961	1138	1304	1468	1627	1780	2382	2860	3301	3696
25Y Opt	212.0	421	622	817	1006	1185	1356	1521	1686	1842	2467	2937	3388	3778
30Y Opt	220.0	437	643	845	1040	1221	1392	1548	1704	1863	2503	2982	3438	3842

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Options Index <VCAP> Disclaimer <IDIS>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

← Cash delivery Swaptions

VCAP2P Quote | VCAP2P

ICAP VCAP2P

ption Straddles (PHYSICAL LCH) - Fwd Premium Mids (Eonia disc)

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
		7.5	15.0	24.5	35.5	44.5	53.5	62.5	71.0	78.5	123	166	208	247
		10.5	21.0	36.0	52.5	66.5	80.0	93.0	108	119	187	250	312	365
		13.5	26.5	46.0	63.0	80.0	96.5	112	127	143	221	294	363	424
		24.0	45.5	72.5	101	126	152	177	202	224	342	447	544	631
		35.5	64.5	99.5	135	169	202	237	267	297	448	579	701	814
		48.5	84.0	124	168	206	246	286	323	360	536	693	839	972
		73.0	121	173	223	271	321	372	421	470	692	885	1066	1231
		99.5	159	220	279	338	396	457	516	572	828	1061	1270	1457
		147	225	301	378	455	530	606	677	745	1050	1336	1588	1822
		186	279	370	459	550	640	728	812	892	1231	1555	1836	2104
5Y Opt	109.0	218	325	429	531	635	734	832	925	1017	1386	1736	2044	2335
7Y Opt	134.5	268	397	523	643	763	879	994	1106	1215	1629	2024	2364	2701
10Y Opt	161.5	322	474	620	767	908	1047	1179	1308	1436	1921	2373	2766	3143
15Y Opt	186.0	369	546	717	888	1051	1204	1358	1508	1658	2216	2712	3161	3571
20Y Opt	201.0	398	591	778	963	1141	1307	1474	1634	1789	2405	2902	3363	3777
25Y Opt	212.0	421	623	819	1008	1188	1360	1527	1695	1853	2493	2980	3448	3873
30Y Opt	220.0	437	643	847	1042	1224	1396	1554	1713	1875	2529	3022	3506	3953

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Options Index <VCAP> Disclaimer <IDIS>

Physical delivery Swaptions →

9: Interest rate volatility products

Problems [1]

- Swaption parity: proof analytically the parity relationships between Swaptions and Swaps prices, both for cash and physical settled swaptions. Check numerically the parity relationships above using market quotes, where available. Use spreadsheet with static data or possibly Reuters/Bloomberg real time. Deliverable: analytical proof + spreadsheet with charts and comments.

- Swaption pricing: starting from market quotations for the relevant interest rate yield curves and Swaption volatilities, price two ATM swaptions 5Yx10Y and 10Yx5Y, both cash and physical settled. Deliverable: spreadsheet/VBA with comments.

- Swaption volatility/price cubes: starting from market quotations for the relevant interest rate yield curves and Swaption volatilities, build the cube of prices and the corresponding implicit sifthed-lognormal Black volatility cubes for different shift values. Use spreadsheet/VBA with static data or possibly Reuters/Bloomberg real time. Deliverable: spreadsheet/VBA with charts and comments.

9: Interest rate volatility products
Problems [2]

- Swaption no-arbitrage conditions: using the price cube above, check the no-arbitrage conditions for Swaptions (vertical spread, butterfly, triangular arbitrage). Use spreadsheet with static data or possibly Reuters/Bloomberg real time and possibly VBA. Deliverable: spreadsheet/VBA with charts and comments.

- Swaption sensitivities: derive analytical formulas for shifted-lognormal Black delta and vega sensitivities for cash and physical settled swaptions. Compute and plot delta and vega for the market price cube. Compare the analytical sensitivities with those calculated using the finite difference method. Use spreadsheet with static data or possibly Reuters/Bloomberg real time and possibly VBA. Deliverable: analytical proof + spreadsheet/VBA with charts and comments.

- Swaption straddles: derive analytical formulas for shifted-lognormal price, delta and vega sensitivities for cash and physical settled swaption straddles. Check when the straddle has zero delta. Compute and plot prices, delta and vega for the market cube. Compare the analytical sensitivities with those calculated using the finite difference method. Use spreadsheet with static data or possibly Reuters/Bloomberg real time and possibly VBA. Deliverable: analytical proof + spreadsheet with charts and comments.

9: Interest rate volatility products

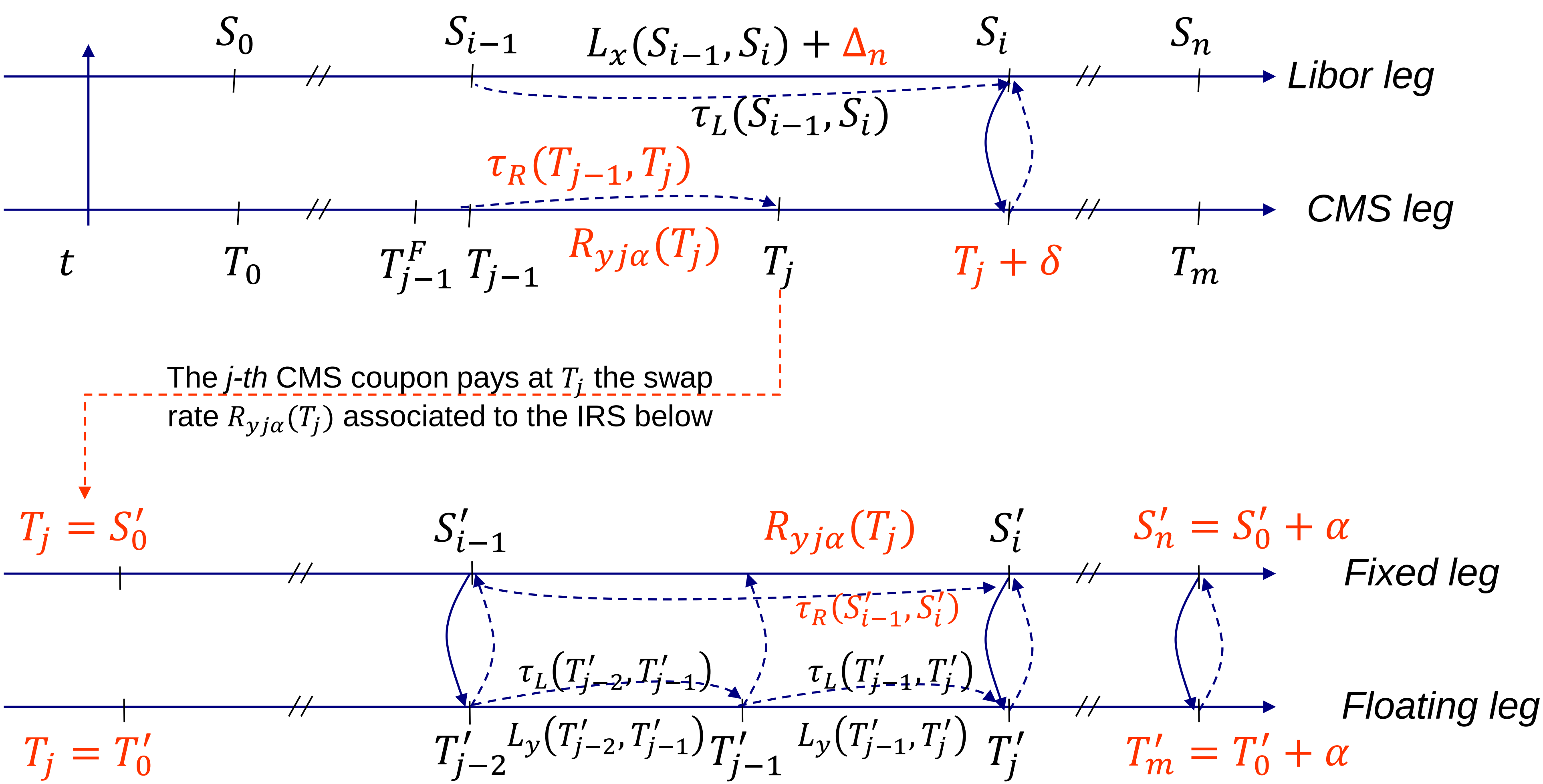
Constant Maturity Swap [1]

Constant Maturity Swaps (CMS) are, typically, floating vs floating swaps with one leg indexed to a swap rate of a given tenor, and the other leg indexed to a Libor rate plus spread. We thus have the following schedules and payoffs

$$\mathbf{S} = \{S_0, ..., S_n\}, \text{ plain vanilla floating leg schedule,}$$
$$\mathbf{T} = \{T_0, ..., T_m\}, \text{ CMS floating leg schedule,}$$
$$S_0 = T_0, S_n = T_m,$$
$$\mathbf{Swaplet}_{\text{float}}(S_i; S_{i-1}, S_i, K) = N\left[L_x(S_{i-1}, S_i) + \Delta_n\right]\tau_L(S_{i-1}, S_i),$$
$$\mathbf{CMSlet}(T_j + \delta; T_{j-1}, T_j, \alpha, \delta) = N R_{yj\alpha}(T_j)\tau_R(T_{j-1}, T_j),$$
$$R_{yj\alpha}(t) = \text{swap rate with Libor tenor } y, \text{ start in } T_j, \text{ end in } T_j + \alpha.$$

Equilibrium CMS spread

Constant Maturity Swaps (CMS) structure



Constant Maturity Swap [3]

Usage of CMS

CMS may be used to hedge against steepening/flatting movements of the yield curve.

For example a customer is exposed to rising Euribor3M rates below 2 years (flattening). He may hedge this risk buying a CMS receiving Euribor3M and paying the 5Y swap rate.

Hedge funds will use the same CMS to bet on the falling Euribor3M relative the the 5Y swap rate.

Asset and Liability Management in commercial banks have huge portfolios of loans. They are exposed to the prepayment risk, which is also related to Swap Curve increase or decrease. CMS can provide a natural hedge of the positions.

9: Interest rate volatility products
Constant Maturity Swap [3]

The price of a CMS coupon (CMSlet) is given by

Martingale under the swap measure $Q_{dj\alpha}$ associated to the annuity $A_{dj\alpha}$

$$\begin{aligned}
\mathbf{CMSlet}(t; T_{j-1}, T_j, \alpha, \delta) \\
&= N P_d(t, T_j + \delta)\tau_R(T_{j-1}, T_j)\mathbb{E}^{Q_{T_j+\delta}}\left[R_{yj\alpha}(T_j)\right] \\
&= N A_{dj\alpha}(t, \mathbf{S}')\tau_R(T_{j-1}, T_j)\mathbb{E}^{Q_{dj\alpha}}\left[\frac{P_d(T_j, T_j+\delta)}{A_{dj\alpha}(T_j, \mathbf{S}')} R_{yj\alpha}(T_j)\right] \\
&\simeq N A_{dj\alpha}(t, \mathbf{S}')\tau_R(T_{j-1}, T_j)\mathbb{E}^{Q_{dj\alpha}}\left\{g\left[R_{yj\alpha}(T_j)\right] R_{yj\alpha}(T_j)\right\},
\end{aligned}$$

$$A_{dj\alpha}(t, \mathbf{S}') == \sum_{i=1}^{n} P_d(t, S_i')\tau_R(S_{i-1}', S_i').$$

Function of the swap rate only

Remarks:

- The discount/annuity ratio is a (complex) function of the whole yield term structure.
- We approximate the yield curve dynamics with a single stochastic factor, the swap rate $R_{yj\alpha}(t)$.
- The choice of the function *g* introduces model dependency.
- At the last step, the payoff has become

$$\mathbf{CMSlet}(T_j; T_{j-1}, T_j, \alpha, \delta) = g\left[R_{yj\alpha}(T_j)\right] R_{yj\alpha}(T_j)$$

The price of the CMS is given by

$$
\begin{aligned}
&\mathbf{CMS}(t; \mathbf{S}, \mathbf{T}, \alpha, \delta) \\
&\quad = \omega \left[\sum_{j=1}^{m} \mathbf{CMSlet}(t; T_{j-1}, T_j, \alpha, \delta) - \sum_{i=1}^{n} \mathbf{Swaplet}_{\text{float}}(t; S_{i-1}, S_i) \right] \\
&\quad \simeq \omega N \left[\sum_{j=1}^{m} A_{dj\alpha}(t, \mathbf{S}') \tau_R(T_{j-1}, T_j) \mathbb{E}^{Q_{dj\alpha}} \left\{ g\left[R_{yj\alpha}(T_j) \right] R_{yj\alpha}(T_j) \right\} \right. \\
&\quad \left. - \sum_{j=1}^{m} P_d(t, S_i) \left[F_{x,i}(t) + \Delta_n \right] \tau_L(S_{i-1}, S_i) \right] .
\end{aligned}
$$

9: Interest rate volatility products
Constant Maturity Swap [5]

Each CMSlet is priced by replication using the Breeden-Litzenberger formula.

Given:

- an European payoff $V[R(T)]$
- the probability distribution $p(k)dk = P(k < R(T) < k + dk)$ of $R(T)$, such that

$$\mathbb{E}\left\{V\left[R(T)\right]\right\} = \int_{-\infty}^{+\infty} V(k)p(k)dK,$$

$$p(k)dk = P(k \leq R(T) \leq k + dk) = \frac{1}{P(t,T)} \frac{\partial^2 \mathbf{Call}\left(R(t), T, k\right)}{\partial k^2} dk$$

Then, integrating by parts, we have

$$P(t,T)\mathbb{E}_t\left\{V\left[R(T)\right]\right\} = P(t,T)\left\{V(\tilde{K}) + V'(\tilde{K})\left[R(t) - \tilde{K}\right]\right\}$$
$$+ \int_{-\infty}^{\tilde{K}} \mathbf{Put}\left[R(t), T, k\right] V''(k)dk + \int_{\tilde{K}}^{+\infty} \mathbf{Call}\left[R(t), T, k\right] V''(k)dk.$$

The price can be expressed as an integral of calls and put prices with maturity *T*, weights $V''(k)$ and any possible strike.

We apply the replication formula to our CMSlet by setting

$$V\left[R(T)\right] := g\left[R(T)\right]R(T)$$

We obtain

$$\begin{aligned}
&\mathbf{CMSlet}(t; T_{j-1}, T_j, \alpha, \delta) \\
&\quad \simeq N A_{dj\alpha}(t)\mathbb{E}^{Q_{dj\alpha}}\left\{g\left[R_{yj\alpha}(T_j)\right] R_{yj\alpha}(T_j)\right\} \tau_R(T_{j-1}, T_j) \\
&\quad = N A_{dj\alpha}(t) g\left[R_{yj\alpha}(t) R_{yj\alpha}(t)\right] \\
&\quad + \int_0^{\tilde{K}} \mathbf{Swaption}\left[t; R_{yj\alpha}(t), T_j, k, \omega = -1\right]\left[2g'(k) + k g''(k)\right] dk \\
&\quad + \int_{\tilde{K}}^{+\infty} \mathbf{Swaption}\left[t; R_{yj\alpha}(t), T_j, k, \omega = +1\right]\left[2g'(k) + k g''(k)\right] dk,
\end{aligned}$$

Receiver swaption

Payer swaption

9: Interest rate volatility products
Constant Maturity Swap [7]

Remarks

- Possible choices of $\widetilde{K}$:
  - $\widetilde{K} = 0 \rightarrow$ replication using payer swaptions only
  - $\widetilde{K} = +\infty \rightarrow$ replication using receiver swaptions only
  - $\widetilde{K} = R_{yj\alpha}(t) \rightarrow$ split at ATM level, replication using both receiver and payer swaptions, where they are more liquid on the market.

- Lower bound $\widetilde{K} = 0$ because swap rates are supposed to be strictly positive.

- The integral depends on a continuity of swaption prices, where the market quotes a finite discrete set. Thus we need a no-arbitrage interpolation/extrapolation framework to compute the continuum swaption set (e.g. SABR).

- The integral is unbounded from above and unstable for $\widetilde{K} \rightarrow 0$, thus we need appropriate numerical integration techniques to compute it.

- Possible choices of the function *g*: see e.g. Brigo and Mercurio (2006)

Constant Maturity Swap [8]

Remarks (cont'd)

- Since the CMS price depends on the swaption smile $\rightarrow$ it is sensible to the SABR parameter β, which commands the SABR wings.

- The SABR model can be fully calibrated to:
  - the market quotations for Swaptions for SABR parameters α, ν, ρ,
  - to the market quotations for CMS (see next slide) for the SABR parameter β.

- The 4-dimensional joint calibration can be slaved to 3+1 parameters nested calibrations, see e.g. F. Mercurio, A. Pallavicini, "*Swaption skews and convexity adjustments*", July 21, 2006.

9: Interest rate volatility products

Constant Maturity Swap [9]

13:00 30DEC11 ICAP UK69580 CMS01

Constant Maturity Swaps

	2Y Index	5Y Index	10Y Index	20Y Index	30Y Index
5Y Swaps	47.4/53.4	95.6/104.6	141.5/151.5	144.6/164.6	134.2/159.2
10Y Swaps	39.5/45.5	72.4/81.4	102.4/112.4	92.2/112.2	90.5/115.5
15Y Swaps	33.4/39.4	56.8/65.8	79.0/89.0	61.1/81.1	70.1/95.1
20Y Swaps	29.5/35.5	51.7/60.7	62.7/82.7	57.5/77.5	67.0/97.0

Pr

*

16:09 30JAN15 ICAP UK69580 CMS01

Constant Maturity Swaps (Eonia disc)

	2Y Index	5Y Index	10Y Index	20Y Index	30Y Index
5Y Swaps	18.6/24.6	40.6/49.6	78.7/88.7	105.2/125.2	110.4/135.4
10Y Swaps	21.9/27.9	42.0/51.0	73.0/83.0	89.9/109.9	95.1/120.1
15Y Swaps	20.0/26.0	36.9/45.9	66.8/76.8	78.8/98.8	84.6/109.6
20Y Swaps	18.4/24.4	33.5/42.5	63.6/83.6	74.8/94.8	76.8/106.8

Incorrect EOM levels published on 30/09. Pls contact ICAP for revised close

Prices Quoted Q A/360 vs 3M Euribor

* Please call for swap fixing

Conventions

- Libor leg: Euribor3M, quarterly, act/360.
- CMS leg: quarterly, 30/360, payment lag $\delta = 3M$.
- CMS rate: equilibrium swap rate of a standard IRS, fix annual30/360 vs Euribor6M quarterly act/360.

9: Interest rate volatility products

Constant Maturity Swap [10]

Problems

- Proof the **Breeden-Litzenberger formula**
  See Breden-Litzenberger, "*Prices of State-Contingent Claims Implicit in Option Prices*", D. T. Breeden and R H. Litzenberger, The Journal of Business, Vol. 51, No. 4 (Oct., 1978), pp. 621-651

9: Interest rate volatility products

CMS Cap/Floor [1]

CMS caps/floors are caps/floors on constant maturity swap rates. Each caplet/floorlet in the portfolio is characterized by schedule and payoff

$$
\begin{aligned}
&\mathbf{T} = \{T_0, ..., T_m\}\,, \ \text{cap/floor schedule}, \\
&\mathbf{CMScf}(T_j + \delta; T_{j-1}, T_j, \alpha, \delta, K, \omega) \\
&\qquad\qquad = N\tau_R(T_{j-1}, T_j)\mathrm{Max}\left\{\omega\left[R_{yj\alpha}(T_j) - K\right]; 0\right\},
\end{aligned}
$$

where ω = *+/-1* for caplet/floorlet, respectively.

The price of the CMS cap/floor is given, at time *t*, by

$$
\begin{aligned}
&\mathbf{CMScf}(t; T_{j-1}, T_j, \alpha, \delta, K, \omega) \\
&\qquad = NP_d(t, T_j + \delta)\tau_R(T_{j-1}, T_j)\mathbb{E}^{Q_{T_j+\delta}}\left[\mathrm{Max}\left\{\omega\left[R_{yj\alpha}(T_j) - K\right]; 0\right\}\right]
\end{aligned}
$$

9: Interest rate volatility products

CMS Spread Option [1]

CMS spread options are caps/floors on the difference (spread) between two CMS rates. Each caplet/floorlet in the portfolio is characterized by schedule and payoff

$$
\begin{aligned}
&\mathbf{T} = \{T_0, ..., T_m\}\,, \ \text{cap/floor schedule},\\
&\mathbf{CMSso}(T_j + \delta; T_{j-1}, T_j, \alpha, \beta, \delta, K, \omega)\\
&\qquad\qquad = N\tau_R(T_{j-1}, T_j)\mathrm{Max}\left\{\omega\left[R_{yj\alpha}(T_j) - R_{xj\beta}(T_j) - K\right]; 0\right\},
\end{aligned}
$$

where ω = *+/-1* for caplet/floorlet, respectively.

The price of the CMS spread option is given, at time *t*, by

$$
\begin{aligned}
\mathbf{CMSso}(t; T_{j-1}, T_j, \alpha, \beta, \delta, K, \omega) &= NP_d(t; T_j + \delta)\tau_R(T_{j-1}, T_j)\\
&\times \mathbb{E}_t^{Q^{T_j+\delta}}\left\{\mathrm{Max}\left\{\omega\left[R_{yj\alpha}(T_j) - R_{xj\beta}(T_j) - K\right]; 0\right\}\right\},
\end{aligned}
$$

CMS Spread Option [2]

Usage of CMS Spread Option

CMS Spread Options may be used to hedge against steepening/flatting movements of the yield curve, considering different pillars of the Swap Curve

For example a customer is exposed to rising Swap rates below 2 years (flattening). He may hedge this risk buying a CMS Spread Options 30Y-2Y.

CMS Spread Options may be used to hedge ALM books

CMS Spread Options may also be used in structured product allowing the investor to take a view on the long term curve evolution.

9: Interest rate volatility products
CMS Spread Option [3]

Standard market practice: Bivariate Lognormal Model (BLM):

- each CMS rate follows a distinct driftless lognormal dynamics under its own swap measure
- the two dynamics are correlated

$$
\begin{aligned}
dR_{yj\alpha}(t) &= R_{yj\alpha}(t)\sigma_{yj\alpha}(t)dW^{Q_{dj\alpha}}(t), \\
dR_{xj\beta}(t) &= R_{xj\beta}(t)\sigma_{xj\beta}(t)dW^{Q_{dj\beta}}(t).
\end{aligned}
$$

Moving to the forward measure Q^{T_j} we obtain

$$
\begin{aligned}
dR_{yj\alpha}(t) &= \mu_{yj\alpha}(t)R_{yj\alpha}(t)dt + \sigma_{yj\alpha}(t)R_{yj\alpha}(t)dW_{yj\alpha}^{Q^{T_j}}(t), \\
dR_{xj\beta}(t) &= \mu_{xj\beta}(t)R_{xj\beta}(t)dt + \sigma_{xj\beta}(t)R_{xj\beta}(t)dW_{xj\beta}^{Q^{T_j}}(t), \\
dW_{yj\alpha}^{Q^{T_j}}(t)dW_{xj\beta}^{Q^{T_j}}(t) &= \rho_{yxj\alpha\beta}(t)dt,
\end{aligned}
$$

where the drifts are obtained by the Girsanov theorem (see Brigo&Mercurio, 2006, eq. 6.39).

9: Interest rate volatility products
CMS Spread Option [4]

Recurring to the standard drift freezing approximation

$$\mu_{yj\alpha}(T) \simeq \mu_{yj\alpha}(t), \quad \mathbb{E}_t^{Q^{T_j}}\left[R_{yj\alpha}(T_j)\right] \simeq R_{yj\alpha}(t)e^{\mu_{yj\alpha}(t)(T_j-t)},$$

$$\mu_{xj\beta}(T) \simeq \mu_{xj\beta}(t), \quad \mathbb{E}_t^{Q^{T_j}}\left[R_{xj\beta}(T_j)\right] \simeq R_{xj\beta}(t)e^{\mu_{xj\beta}(t)(T_j-t)},$$

we obtain

$$\mathbf{CMSso}(t; T_{j-1}, T_j, \alpha, \beta, \delta, K, \omega) = NP_d(t; T_j + \delta)\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}} f(t, u)du,$$

$$f(t, u) := \omega\left[h^+_{yxj\alpha\beta}(t, u)\Phi(\omega d^+_{yxj\alpha\beta}(t, u)) - h^-_{yxj\alpha\beta}(t, u)\Phi(\omega d^-_{yxj\alpha\beta}(t, u))\right]$$

$$h^+_{yxj\alpha\beta}(t, u) := R_{yj\alpha}(u)\exp\left[\left(\mu_{yj\alpha}(u) - \frac{1}{2}\rho^2_{yxj\alpha\beta}\sigma^2_{yj\alpha}\right)(T_j - u) + u\rho_{yxj\alpha\beta}\sigma_{xj\beta}\sqrt{T_j - u}\right],$$

$$h^-_{yxj\alpha\beta}(t, u) := K - R_{xj\beta}(u)\exp\left[\left(\mu_{xj\beta}(u) - \frac{1}{2}\sigma^2_{xj\beta}\right)(T_j - u) + u\sigma_{xj\beta}\sqrt{T_j - u}\right],$$

$$d^\pm_{yxj\alpha\beta}(t, u) := \frac{\ln\frac{R_{yj\alpha}(u)e^{\mu_{yj\alpha}(u)(T_j-u)}}{h^-_{yxj\alpha\beta}(t,u)} \pm \frac{1}{2}\rho^2_{yxj\alpha\beta}\sigma^2_{yj\alpha}(T_j - u) + u\rho_{yxj\alpha\beta}\sigma_{yj\alpha}\sqrt{T_j - u}}{\sigma_{yj\alpha}\sqrt{(1 - \rho^2_{yxj\alpha\beta})(T_j - u)}}$$

9: Interest rate volatility products

CMS Spread Option [5]

Remarks:

- Swap rates correlations must be implied from market quotes, where available.
- A single correlation does not account for all the market prices, hence we obtain a correlation surface $\rho_{yxj\alpha\beta}(t, T_j, K)$
- Use forward starting and digital CMS spread options to check the correlation surface.

13:03 30DEC11 ICAP UK69580 ICAPSPREADS1

EUR 10/2 CMS Spread Options

	FWD	ATM	Flr -0.25	Flr -0.10	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
1y	1.25	30.5	0.7	0.8	1.0	78.0	59.9	42.7	27.4	7.8
2y	1.24	91.2	5.2	6.1	6.8	184.9	144.3	106.1	72.1	26.8
3y	1.16	163.9	11.7	13.9	15.6	273.8	213.1	156.9	108.0	43.2
4y	1.05	249.2	23.9	28.2	31.6	343.1	265.4	195.1	135.1	56.5
5y	0.94	342.2	41.4	48.5	54.2	398.7	306.6	224.8	156.4	67.9
7y	0.78	541.1	89.6	103.8	114.9	499.6	382.8	282.0	199.6	94.0
10y	0.62	860.7	191.1	217.0	237.1	637.7	491.3	368.5	270.1	144.3
15y	0.39	1461.5	441.0	494.2	530.7	846.8	666.6	511.1	393.6	241.6
20y	0.33	2046.2	678.7	752.2	806.8	1096.2	879.9	698.9	552.4	353.1
2x4	0.85	150.0	19.1	22.7	25.5	154.5	117.8	86.2	60.9	28.6
5x10	0.27	465.3	150.5	169.4	183.9	237.3	183.5	142.8	113.1	76.1
5X15	0.10	1024.6	400.5	443.3	476.3	448.3	360.2	292.8	242.4	176.2
10x15	-0.10	547.5	249.3	274.4	292.9	210.2	176.1	149.6	129.0	100.0
10X20	-0.02	1118.5	483.7	530.7	564.7	469.2	398.0	338.3	288.6	212.0
Digitals										
5y	0.94				54.1		356.0		253.3	
10y	0.62				191.2		555.6		361.0	
15y	0.39				369.1		682.1		437.4	

13:04 30DEC11 ICAP UK69580 ICAPSPREADS2

EUR 30/2 CMS Spread Options

	FWD	ATM	Flr -0.25	Flr -0.10	Flr 0.00	Flr 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
1y	1.42	37.1	0.5	0.7	0.9	1.4	72.6	55.4	39.5	15.6
2y	1.37	110.1	4.4	5.4	6.3	9.1	167.6	130.1	96.0	44.1
3y	1.24	201.5	15.0	17.9	20.2	27.6	243.0	188.5	140.0	67.3
4y	1.08	308.8	34.1	40.1	44.9	59.5	298.8	231.3	172.2	84.9
5y	0.95	420.7	57.9	68.3	76.2	100.2	343.0	265.0	197.3	98.2
7y	0.77	646.3	116.0	136.1	151.2	195.6	426.5	329.3	246.0	124.6
10y	0.59	1001.5	233.9	270.1	296.8	373.1	545.9	424.2	320.3	168.4
15y	0.42	1600.4	456.1	531.8	561.5	685.7	762.2	587.4	454.3	257.2
20y	0.45	2165.7	638.0	718.0	775.7	936.2	1040.7	843.7	673.7	414.9
2x4	0.79	186.6	31.3	36.7	40.8	53.2	126.2	97.1	72.8	39.0
5x10	0.18	528.6	181.8	208.3	227.4	280.7	197.6	154.8	119.5	68.1
5X15	0.14	1100.9	408.2	459.8	496.8	598.6	411.3	331.8	264.6	163.3
10x15	0.09	571.7	226.3	251.5	269.4	317.9	213.7	176.9	145.1	95.2
10X20	0.63	1083.2	309.5	342.0	365.2	672.8	583.3	502.6	430.6	311.3
Digitals										
5y	0.95				79.0		335.5		254.9	
10y	0.59				264.2		531.1		390.7	
15y	0.42				438.8		695.0		513.9	

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13:02 30DEC11 ICAP UK69580 ICAPSPREADS3

EUR 30/10 CMS Spread Options

	FWD	ATM	Flr -0.25	Flr -0.10	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
1y	0.17	19.7	2.1	3.6	5.3	6.8	1.8	0.5	0.1	0.0
2y	0.13	59.8	12.1	16.9	21.5	19.5	7.7	3.5	1.9	0.8
3y	0.08	109.8	28.5	37.7	46.3	32.6	14.9	7.9	4.8	2.4
4y	0.04	166.3	50.0	64.6	77.7	45.7	22.8	13.0	8.4	4.3
5y	0.01	228.2	74.9	94.9	112.5	61.3	32.6	19.6	13.0	7.0
7y	-0.01	365.4	130.2	160.7	186.5	101.8	59.4	37.7	25.9	14.2
10y	-0.03	603.0	228.2	275.0	313.1	181.1	115.7	77.2	53.3	27.4
15y	0.04	1037.4	370.9	439.1	492.7	392.6	282.6	206.5	151.9	82.2
20y	0.14	1485.6	485.6	569.6	634.8	660.2	505.1	388.3	299.4	176.2
2x4	-0.04	104.2	37.8	47.4	55.8	26.5	15.2	9.6	6.5	3.5
5x10	-0.07	373.0	153.2	180.0	200.5	120.0	83.2	57.7	40.2	20.4
5X15	0.06	811.0	296.7	344.9	381.1	330.9	249.5	186.5	138.4	74.9
10x15	0.21	435.8	143.5	165.0	180.6	210.9	166.3	128.9	98.2	54.6
10X20	0.36	874.8	259.0	296.5	323.8	477.7	388.1	311.0	246.0	148.8
Digitals										
5y	0.01				128.9		97.8		22.9	
10y	-0.03				303.2		234.7		87.2	
15y	0.04				443.9		407.3		204.6	

<ICAPSPREADS2> <ICAPSPREADS>

9: Interest rate volatility products

CMS Spread Option [6]

16:14 30JAN15 ICAP UK69580 ICAPSPREADS1

EUR 10/2 CMS Spread Options (Eonia disc)

	FWD	ATM	Flr -0.25	Flr -0.10	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
1y	0.58	17.1	0.0	0.0	0.0	25.5	11.3	4.6	2.0	0.4
2y	0.61	56.9	0.3	0.8	1.4	68.7	37.5	19.5	10.3	3.1
3y	0.62	104.6	1.8	3.2	4.8	117.7	69.3	39.0	21.8	7.1
4y	0.63	156.7	5.2	8.0	10.9	168.6	103.1	60.1	34.4	11.4
5y	0.62	214.5	10.8	15.6	20.1	218.6	137.1	82.0	48.2	16.7
7y	0.58	348.7	32.8	43.1	52.1	313.2	202.9	126.2	77.5	30.0
10y	0.51	629.1	110.1	131.8	149.8	468.4	320.4	215.1	145.6	72.8
15y	0.44	1285.6	345.4	385.5	418.2	800.2	598.1	454.2	357.6	251.5
20y	0.39	2030.9	621.3	683.7	733.6	1162.6	917.4	741.7	621.0	480.8
2x4	0.65	99.2	4.8	7.2	9.5	99.9	65.7	40.5	24.0	8.2
5x10	0.41	413.1	99.3	116.3	129.7	249.8	183.3	133.1	97.5	56.1
5X15	0.35	1062.4	334.7	370.0	398.1	581.7	461.1	372.2	309.5	234.8
10x15	0.28	646.8	235.4	253.8	268.4	331.9	277.8	239.1	212.0	178.7
10X20	0.26	1378.5	511.3	551.9	583.9	694.3	597.1	526.6	475.4	408.0
Digitals										
5y	0.62				31.3		294.5		115.0	
10y	0.51				141.6		540.9		242.2	
15y	0.44				261.4		737.2		339.1	

<ICAPSPREADS> Screen Closed <ICAPSPREADS2>

16:14 30JAN15 ICAP UK69580 ICAPSPREADS2

EUR 30/2 CMS Spread Options (Eonia disc)

	FWD	ATM	Flr -0.25	Flr -0.10	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
1y	1.08	24.3	0.1	0.2	0.3	63.9	45.7	28.9	15.4	2.9
2y	1.08	75.3	1.8	2.4	2.9	153.2	112.4	75.3	45.4	14.2
3y	1.07	136.9	6.5	7.8	9.0	243.6	180.8	124.2	78.8	30.4
4y	1.05	204.2	13.6	16.0	18.0	330.1	246.4	171.5	111.7	46.7
5y	1.02	276.3	20.7	24.7	28.0	408.7	305.9	214.7	141.9	60.8
7y	0.93	446.0	43.5	52.7	60.0	548.3	412.4	293.5	198.6	88.6
10y	0.81	770.6	114.7	135.1	151.0	751.4	573.2	418.8	295.2	145.5
15y	0.72	1427.5	299.8	340.7	371.7	1141.3	897.6	687.0	516.6	297.5
20y	0.66	2111.4	511.3	573.1	619.4	1530.5	1227.6	965.9	752.1	467.3
2x4	1.02	128.5	11.8	13.6	15.1	176.9	134.0	96.2	66.3	32.5
5x10	0.61	473.8	94.0	110.4	123.0	342.7	267.3	204.1	153.3	84.7
5X15	0.57	1115.7	279.1	316.0	343.8	732.7	591.7	472.3	374.7	236.8
10x15	0.52	641.0	185.1	205.6	220.7	390.0	324.4	268.3	221.4	152.0
10X20	0.63	1083.2	309.5	342.0	365.2	672.8	583.3	502.6	430.6	311.3
Digitals										
5y	1.02				27.9		404.4		273.9	
10y	0.81				141.1		696.7		467.2	
15y	0.72				280.9		951.5		647.6	

<ICAPSPREADS1> Screen Closed <ICAPSPREADS>

16:14 30JAN15 ICAP UK69580 ICAPSPREADS3

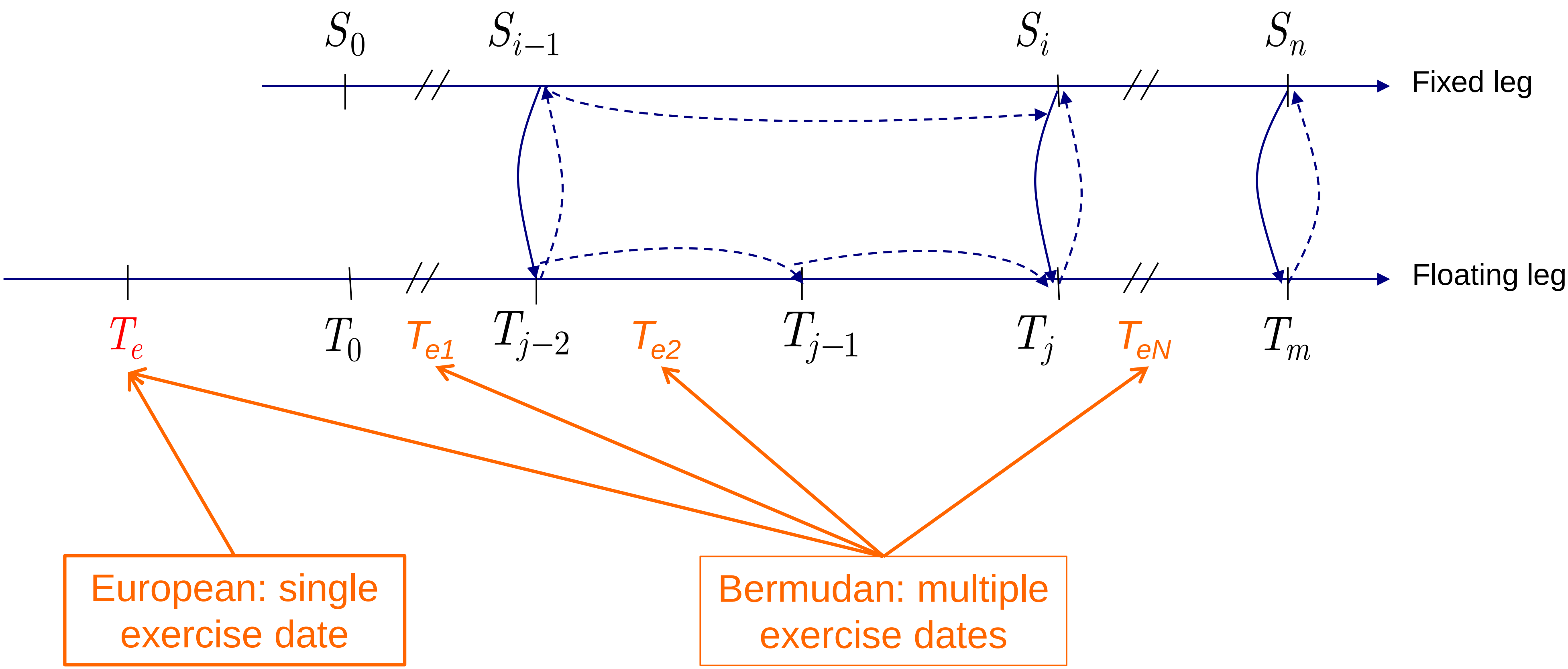
EUR 30/10 CMS Spread Options

	FWD	ATM	Flr -0.25	Flr -0.10	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
1y	0.51	15.6	0.1	0.2	0.4	21.5	8.1	1.9	0.4	0.0
2y	0.48	46.5	0.7	1.6	2.7	50.0	21.5	6.9	1.9	0.1
3y	0.45	83.2	2.3	4.7	7.3	76.8	35.1	12.8	4.1	0.4
4y	0.42	125.7	6.9	11.5	16.2	103.0	48.8	19.2	6.9	0.9
5y	0.40	174.4	16.1	23.2	30.2	129.4	63.1	26.7	10.9	2.3
7y	0.35	286.2	42.9	56.4	68.6	182.7	93.4	43.3	20.4	6.0
10y	0.30	484.4	98.9	123.7	145.0	270.6	148.2	76.4	40.5	14.3
15y	0.28	890.1	219.3	265.4	302.9	467.7	288.4	172.7	104.8	41.6
20y	0.27	1340.0	357.5	426.7	481.3	688.4	457.3	298.9	197.0	87.9
2x4	0.37	78.0	6.2	9.9	13.5	53.0	27.3	12.3	5.0	0.8
5x10	0.21	306.1	82.9	100.5	114.9	141.1	85.1	49.7	29.6	11.9
5X15	0.22	710.0	203.3	242.2	272.7	338.3	225.3	146.1	93.9	39.3
10x15	0.23	404.1	120.4	141.7	157.8	197.1	140.2	96.3	64.3	27.4
10X20	0.23	852.4	258.5	303.0	336.3	417.8	309.1	222.5	156.5	73.7
Digitals										
5y	0.40				41.7		241.3		51.3	
10y	0.30				148.7		448.0		122.4	
15y	0.28				280.5		665.1		241.4	

<ICAPSPREADS2> Screen Closed <ICAPSPREADS>

9: Interest rate volatility products
Bermudan Swaption [1]

Bermudan Swaptions are contracts that allows to enter, at multiple exercise dates $\{T_e$, T_{e1} , T_{e2} …… $T_{eN}\}$, into a swap starting at time $T_0 \geq T_e$ and maturing at T_m (co-terminal swaption.



9: Interest rate volatility products
Bermudan Swaption [2]

- Bermudan Swaptions are an example of derivative embedding early exercise (as American options).

- European Swaptions can be seen as Bermudan Swaption with single exercise date.

- Bermudan Swaptions are OTC contracts, and no market quotations or broker pages are available. Consensus Platform (e.g. Totem) are one possible source of prices (monthly data) that helps inferring the market practice.

- Usage of Bermudan swaptions:
  - Bermudan Swaption contracts are typically traded OTC for hedging purposes of callable floating rate notes.
  - ALM units may trade Bermudan Swaptions to hedge the prepayment of mortgages portfolios.

9: Interest rate volatility products
Bermudan Swaption [3]

- Bermudan Swaptions cannot be priced analytically, since their value depends, at each exercise date, on the choice of the option holder whether it is more convenient to exercise the option (retrieving the payoff) or to continue with the contract (hold or continuation value).

- Thus, their price depends on the pricing model chosen for their valuation, embedding as a consequence some amount of model risk. Such models have to be calibrated to the corresponding European Swaptions that mature at $\{T_e , T_{e1} , T_{e2} \ldots\ldots T_{eN}\}$ on Swaps with different tenor and the same expiry date T_m

- The Bermudan Swaption price admits the following bounds:
  - Lower bound: the price of the maximum underlying European Swaption
  - Upper bound: the sum of the prices of all the underlying European Swaptions.

9: Interest rate volatility products

Interest rates and derivatives: resumè

Interest rate quantities		
Interest rates	Interest rate derivatives	Probability Measures
Short rate	Bank account	Risk neutral measure
Zero rate	Zero Coupon Bond	Forward measure
Spot rate	Deposit	
Forward rate	FRA	
Futures rate	Futures	
Swap rate	Swap (IRS)	Forward swap measure
	OIS	
	Basis swap (IRBS)	
	Bond	
	Cap/Floor	
	Swaption	
	Constant Maturity Swap (CMS)	
	CMS Cap/Floor	
	CMS Spread Option	

Summary

10. Multiple volatility cubes

- Modern multi-curve, multi-volatility market practice
- Main issues
- Caps/Floors volatility cube
- Swaptions volatility cube

10: Multiple volatility cubes
Introduction [1]

The OTC market quotes interest rate vanilla options, i.e. European Caps/Floors and Swaptions, with specific:

- schedules (e.g. semi-annual floating vs annual fixed),
- underlying rates (e.g. Euribor6M),
- maturities and swap tenors (e.g. 5Y, 10Y),
- Strikes (e.g. 1%).

However the market participants need to manage past or new interest rate trades with customized characteristics, such as:

- different schedules,
- different undelying rates,
- broken periods and non-standard residual maturities/tenors,
- different strikes.

- Example 1: a Swaption 5x10Y ATM payer annual fixed vs semiannual floating traded 2Y5M ago, is now a 2Y7Mx10Y out of the money swaption with an ongoing period (if rates decreased during the swaption's lifetime).

- Example 2: an European Cap/Floor/Swaption on Euribor1M.

10: Multiple volatility cubes
Introduction [2]

In the interest rate options market it is common practice to trade instruments in terms of option's premia. Thus options' premia are the reference market quantities.

On the other side, interest rate volatility cubes implied in market premia (using Black formula) allows us to express the price non-quoted instruments in terms of quoted market instruments, with the aim of:

- getting arbitrage-free relative prices,
- determine how variations in market quotes affect the price of non-quoted instruments (delta and vega sensitivity),
- determine how to neutralize the delta and vega risk a portfolio of non-quoted instruments (hedging).

We usually distinguish between:

- Cap/Floor volatility 2D surfaces (maturity x strike for each underlying rate tenor),
- Swaption volatility 3D cubes (maturity x tenor x strike for each underlying rate tenor).

Since Caps&Floors can be seen as portfolios of swaptions with a short tenor (typically 6M), we talk, in general, of volatility cubes.

10: Multiple volatility cubes
Modern multi-curve, multi-volatility market practice

The market practice for pricing & hedging volatility derivatives works as follows.

1. Multiple interest rate yield curves $C_1, \cdots, C_N$ are given.

2. Build multiple distinct volatility cubes $\Sigma_1, \cdots, \Sigma_N$ using distinct selections of vanilla interest rate options, each homogeneous in the underlying rate tenor $f_1, \cdots, f_N$, typically 1M, 3M, 6M, 12M for Euribor rate and swap rate volatilities;

3. compute_the FRA/Swap rates and volatilities with tenor *f* on the corresponding curves C_f and volatility surfaces Σ_f, and calculate the corresponding cashflows;

4. compute the corresponding discount factors using the discounting curve C_d and work out prices by summing the discounted cashflows;

5. compute the delta and vega sensitivities and hedge the resulting delta and vega risk using the suggested amounts (hedge ratios) of the corresponding set of vanillas.

10: Multiple volatility cubes
Volatility cube construction

A volatility cube is a three-dimensional matrix of triplets {date, tenor, volatility} corresponding to a set of homogeneous interest rate options with different maturities and tenors. Homogeneity relates to the market rate the instruments are based upon.

Volatility cube bootstrapping is an iterative optimization procedure to build a volatiity cube from market quotations of interest rate options.

The bootstrapping procedure produces a volatility cube such that either each input instrument must be repriced within a predetermined precision (best fit approach).

A volatility cube is associated to some interpolation rules to determine volatilities for any maturity/tenor/strike.

The bootstrapping approach differs between Caps/Floors and Swaptions, see the following slides.

10: Multiple volatility cubes
Main issues

- **Main issues:**
  - Interpretation of the market quotes
  - Selection of bootstrapping instruments
  - Bootstrap elementary instruments (caplets/floorlets)
  - SABR calibration
  - Volatility cube construction
  - Pricing (using interpolation)
  - Multiple delta & vega hedging

- **Some references** (see bibliography)**:**
  - Brigo & Mercurio, 2006.
  - Andersen & Piterbarg, 2010.
  - Hagan et al (2002)
  - Bianchetti and Carlicchi (2010/2011)
  - Kienitz (2013)

10: Multiple volatility cubes
Market Quotations [1]

Market quotations		
Underlying	**Caps/Floors**	**Swaptions**
Euribor1M	Absent	Absent
Euribor3M	Quoted* Maturity: [1Y-2Y] Smile: [-1%%-10%] Quarterly	Quoted ATM: maturity [1M-30Y] x Tenor [1Y] Smile: ATM +/- 2% Semi-annual, cash settlement
Euribor6M	Quoted* Maturity: [3Y-30Y] Smile: [1%-10%] Semi-annual	Quoted ATM: maturity [1M-30Y] x Tenor [2Y-30Y] Smile: ATM +/- 2% Semi-annual, cash settlement
Euribor12M	Absent	Absent

**For short maturities (<1Y) we find liquid Futures options. Some brokers also show pages with premia of Euribor3M cap/floors since 2016, and in parallel also Euribor6M below 2Y started being quoted.

10: Multiple volatility cubes
Market Quotations [2]

Since brokers typically quote both market options premia and Black implied volatilities, there are two alternative choices for bootstrapping inputs.

1. Use quoted premia as inputs: $\rightarrow$ we need our yield curves for discounting and forwarding and our numerical algorithms for inverting the Black formula and find the implied volatilities.
2. Use quoted volatilities as inputs: $\rightarrow$ We save Black formula inversion, but such inputs depend on Broker's yield curves, numerical algorithms, and model (i.e. shifted Black or normal), which may differ from our own ones.

We stress that yield curves are critical for volatility bootstrapping: different yield curves lead to different spot prices and implied volatilities. $\rightarrow$ Swaption premia are expressed as forward premia to mitigate the effect of discounting.

We also make the usual assumptions:

- the OTC interbank market is fully collateralized under an ideal CSA with standard CSAs (daily margination, overnight collateral rate, no frictions);
- the market uses coherently an OIS discounting curve;
- market quotes observable on the OTC interbank market, e.g. Broker quotes, reflect such characteristics.

10: Multiple volatility cubes
Swaptions volatility cube [1]

- Brokers on the OTC market typically quote European swaptions on standard swaps as follows:

  - ATM swaptions: spot/forward premia and shifted-Black implied volatilities (two-dimensional matrix expiry x tenor), given some shift,

  - OTM swaptions: volatility spreads on the top of ATM implied volatilities for selected combinations of maturity/tenor and strikes (yet in the form of spread on the ATM swap rates), and premia.

  In case of non ATM swaptions, the OTM option is quoted, being it payer or receiver. The other option may be recovered via the call-put parity:

$$(S-K)^{+}-(K-S)^{+}=S-K$$

  this leads to the relation between payer and receiver swaptions (by multiplying by the annuity)

$$\boldsymbol{Swaption}(t,K,1)-\boldsymbol{Swaption}(t,K,-1)=\boldsymbol{Swap}(t,K)$$

10: Multiple volatility cubes

Swaptions volatility cube [2]

13:00 30DEC11 ICAP UK69580 VCAP2A

EUR ATM Swaption Straddles - Fwd Premium Mids

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	11.0	25.5	46.5	70.5	95.0	118	140	163	186	209	303	397	491	585
2M Opt	16.0	36.5	69.5	103	135	167	198	230	264	295	429	561	692	823
3M Opt	19.5	46.0	85.5	125	163	202	241	280	322	361	523	680	835	989
6M Opt	30.0	68.5	119	172	230	284	336	390	446	502	717	926	1129	1327
9M Opt	39.0	86.0	144	212	284	347	410	473	539	605	863	1110	1350	1582
1Y Opt	46.5	102	170	248	329	401	473	546	621	697	989	1268	1534	1785
18M Opt	65.5	137	220	309	403	489	574	659	744	830	1171	1488	1781	2070
2Y Opt	87.5	177	275	375	476	569	663	757	850	943	1320	1664	1990	2301
3Y Opt	123.5	239	356	471	585	696	804	909	1014	1104	1509	1887	2245	2596
4Y Opt	148.5	278	407	536	665	784	901	1013	1122	1222	1643	2041	2428	2796
5Y Opt	162.0	306	445	579	715	843	967	1086	1202	1314	1761	2162	2566	2933
7Y Opt	178.0	340	494	642	788	929	1064	1198	1328	1461	1943	2356	2774	3139
10Y Opt	193.5	377	551	719	880	1038	1193	1346	1494	1640	2158	2601	3009	3351
15Y Opt	217.5	426	627	819	1002	1184	1361	1534	1703	1876	2433	2908	3298	3664
20Y Opt	232.0	458	674	884	1085	1284	1476	1665	1848	2020	2633	3117	3563	3950
25Y Opt	239.0	470	697	914	1127	1330	1530	1728	1917	2107	2692	3170	3634	4004
30Y Opt	242.5	472	698	915	1131	1335	1540	1741	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

screen closed 1pm.Screen will open 3rd jan 2012. [illegible]

Straddles defined in Chapter 9, pag 460

13:00 30DEC11 ICAP UK69580 VCAP1A

EUR ATM Swaption Straddles - Implied Volatilities

Please call +44 (0)20 7532 3050 for further details

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M Opt	46.1	43.9	49.9	50.7	49.2	46.8	44.8	43.7	42.9	42.4	39.8	41.3	44.2	47.2
2M Opt	49.7	45.7	53.4	52.9	49.7	47.0	45.0	43.8	43.4	42.5	40.1	41.6	44.4	47.4
3M Opt	51.0	47.6	53.5	51.9	48.5	46.1	44.4	43.3	43.0	42.4	39.8	41.3	43.8	46.6
6M Opt	57.5	49.9	51.0	48.8	46.8	44.4	42.8	41.8	41.4	41.1	38.4	39.6	41.9	44.2
9M Opt	59.9	49.6	48.3	46.8	45.3	42.9	41.5	40.5	40.0	39.7	37.4	38.6	40.7	42.9
1Y Opt	59.2	48.5	46.4	45.0	43.5	41.4	40.2	39.5	39.1	38.9	36.7	37.9	39.9	41.7
18M Opt	60.1	47.1	43.8	41.6	40.3	38.8	37.9	37.3	36.9	36.6	35.0	36.1	37.7	39.5
2Y Opt	58.8	46.0	42.3	39.8	38.2	36.8	36.0	35.4	35.0	34.7	33.6	34.6	36.2	37.7
3Y Opt	48.4	39.9	36.9	35.1	34.1	33.4	32.8	32.3	31.9	31.4	30.5	31.6	33.1	34.5
4Y Opt	40.2	34.1	32.3	31.5	31.1	30.5	30.0	29.6	29.3	29.0	28.4	29.5	30.9	32.0
5Y Opt	33.8	30.6	29.5	28.9	28.5	28.1	27.7	27.5	27.4	27.4	27.2	28.0	29.3	30.0
7Y Opt	28.7	27.0	26.2	25.6	25.3	25.1	25.0	25.1	25.3	25.7	25.9	26.4	27.2	27.5
10Y Opt	24.0	23.3	23.1	23.2	23.3	23.6	24.0	24.4	24.9	25.4	25.5	25.7	25.5	25.1
15Y Opt	24.8	25.1	25.6	26.1	26.5	27.0	27.4	27.8	28.3	28.8	27.5	26.0	24.5	23.7
20Y Opt	29.9	30.0	30.2	30.6	30.8	31.1	31.4	31.6	31.7	31.6	27.7	24.9	23.5	22.6
25Y Opt	32.9	32.5	32.6	32.4	32.1	31.5	30.9	30.4	29.8	29.0	24.3	22.0	21.0	20.1
30Y Opt	29.5	27.3	26.7	26.0	24.9	24.9	24.5	24.2	24.0	24.0	21.0	19.5	18.8	18.5

screen closed 1pm.Screen will open 3rd jan 2012. Happy New Year

10: Multiple volatility cubes
Swaptions volatility cube [3]

13:05 30DEC11 BGC Partners UK38000 BGCOPTIONS07
EUR Smile Summary Page 1 - lognormal volatility % difference v ATM

.	-200bp	-100bp	-50bp	-25bp	ATM	25bp	50bp	100bp	200bp
3M2Y	-	+39.24	+7.85	+2.29	0	-0.34	+0.26	+2.30	+6.17
3M5Y	-	+19.08	+6.52	+2.74	0	-1.95	-3.32	-4.86	-5.62
3M10Y	+62.92	+18.39	+7.64	+3.49	0	-2.88	-5.18	-8.23	-10.04
3M20Y	+50.44	+16.91	+7.29	+3.39	0	-2.92	-5.40	-9.01	-11.89
3M30Y	+66.15	+21.22	+9.13	+4.25	0	-3.69	-6.84	-11.56	-15.52
1Y2Y	-	+24.34	+6.19	+2.31	0	-1.33	-2.07	-2.56	-2.16
1Y5Y		+14.42	+5.37	+2.35	0	-1.82	-3.22	-5.10	-6.73
1Y10Y	+39.18	+11.74	+4.85	+2.21	0	-1.84	-3.35	-5.54	-7.64
1Y20Y	+35.50	+11.39	+4.75	+2.17	0	-1.81	-3.28	-5.38	-7.17
1Y30Y	+44.98	+13.78	+5.73	+2.62	0	-2.19	-3.99	-6.58	-8.93
5Y2Y	+18.01	+5.47	+2.19	+0.98	0	-0.79	-1.41	-2.29	-3.06
5Y5Y	+17.42	+5.71	+2.36	+1.07	0	-0.88	-1.60	-2.64	-3.61
5Y10Y	+17.19	+5.77	+2.39	+1.09	0	-0.89	-1.60	-2.60	-3.40
5Y20Y	+21.51	+6.83	+2.79	+1.26	0	-1.02	-1.84	-2.95	-3.83
5Y30Y	+26.97	+8.08	+3.24	+1.45	0	-1.15	-2.05	-3.23	-4.04
10Y2Y	+11.42	+3.75	+1.52	+0.69	0	-0.55	-0.99	-1.58	-2.03
10Y5Y	+13.63	+4.54	+1.85	+0.83	0	-0.67	-1.19	-1.86	-2.27
10Y10Y	+17.61	+5.68	+2.31	+1.04	0	-0.83	-1.48	-2.32	-2.88
10Y20Y	+23.27	+6.74	+2.67	+1.19	0	-0.94	-1.65	-2.58	-3.20
10Y30Y	+22.08	+6.28	+2.52	+1.13	0	-0.92	-1.65	-2.69	-3.67

13:05 30DEC11 BGC Partners UK38000 BGCOPTIONS08
EUR Smile Summary Page 2 - lognormal volatility % difference v ATM

.	-200bp	-100bp	-50bp	-25bp	ATM	25bp	50bp	100bp	200bp
20Y2Y	+34.20	+8.17	+3.24	+1.46	0	-1.20	-2.20	-3.70	-5.42
20Y5Y	+44.64	+9.40	+3.65	+1.63	0	-1.31	-2.36	-3.89	-5.45
20Y10Y	+56.14	+9.80	+3.79	+1.69	0	-1.38	-2.51	-4.22	-6.21
20Y20Y	+32.81	+7.37	+2.88	+1.29	0	-1.05	-1.89	-3.14	-4.49
20Y30Y	+26.78	+6.57	+2.58	+1.16	0	-0.94	-1.69	-2.79	-3.93
30Y2Y	+40.38	+8.42	+3.28	+1.46	0	-1.18	-2.14	-3.54	-5.04
30Y5Y	+32.09	+7.70	+3.00	+1.34	0	-1.06	-1.90	-3.06	-4.10
30Y10Y	+26.02	+6.67	+2.63	+1.18	0	-0.96	-1.73	-2.84	-4.00
30Y20Y	+19.54	+5.27	+2.06	+0.92	0	-0.72	-1.29	-2.04	-2.66
30Y30Y	+17.72	+4.82	+1.90	+0.85	0	-0.67	-1.21	-1.94	-2.59

10: Multiple volatility cubes
Swaptions volatility cube [4]

- The swaption implied volatility cube is obtained from market quotes as follows.

  - Starting from ATM market premia, obtain the ATM implied volatility surface using the shifted-Black formula with the preferred shifts. Such ATM volatility surface could differ from the broker's surface because of possibly different shifts.

  - Obtain the market implied volatility cube adding the market volatility spreads to the corresponding ATM volatilities. Notice that this is valid if shifts are the same.

  - Calibrate a shifted-SABR model for each smile section.

  - Obtain an expanded shifted-SABR volatility cube using:
    - SABR interpolation on each smile section,
    - another interpolation scheme for different maturities/tenors.

10: Multiple volatility cubes
Swaptions volatility cube [5]

Since customised swaptions are traded OTC on non-standard underlying swaps, e.g. on Euribor1M, 3M, we need some recipe to derive the implied volatility cubes for different underling rate tenors.

This task is accomplished as follows.

- Analogously to the caplet/floorlet case (see later), let us assume the relevant Swap rates to have a shifted lognormal dynamics

$$dR_x^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) = \left[R_x^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) + \lambda_x\right]\sigma_x(t,\mathbf{T},\mathbf{S})dW_x^{Q^{\mathbf{S}}}(t),$$

- and that the basis $b_x(t)$ w.r.t the corresponding OIS rate $R_d^{OIS}(t;\boldsymbol{T})$ rates is static,

$$b_x(t) := R_x^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) - R_d^{\text{OIS}}(t;\mathbf{T}).$$

- Then, OIS rates inherit a shifted lognormal dynamics, too

$$dR_d^{\text{OIS}}(t;\mathbf{T}) = \left[R_d^{\text{OIS}}(t;\mathbf{T}) + b_x(t) + \lambda_x\right]\sigma_x(t,\mathbf{T},\mathbf{S})dW_x^{Q^{\mathbf{S}}}(t).$$

10: Multiple volatility cubes
Swaptions volatility cube [6]

- Let us now consider a second swap rate $R_y^{Swap}(t;\boldsymbol{T}_y,\boldsymbol{S})$ with different floating leg structure $\boldsymbol{T}_y \neq \boldsymbol{T}_x$ (e.g. Euribor3M vs Euribor6M) but same fixed leg as the previous $R_x^{Swap}(t;\boldsymbol{T}_x,\boldsymbol{S})$. The corresponding rates built on the overnight curve will coincide and will have the same implied volatility as before.

- Hence, equating the istantaneous variances and freezing forward swap rates, we get a good proxy to convert ATM volatilities,

$$\left[R_x^{\mathrm{Swap}}(0;\mathbf{T}_x,\mathbf{S})+\lambda_x\right]\sigma_x^2(0,\mathbf{T}_x,\mathbf{S}) \simeq \left[R_y^{\mathrm{Swap}}(0;\mathbf{T}_y,\mathbf{S})+\lambda_y\right]\sigma_y^2(0,\mathbf{T}_y,\mathbf{S}).$$

- Notice that if $\lambda_x = \lambda_y = 0$, this is a basis point volatility equivalence.

- Using the relationship above we may obtain the ATM volatility surface for swaptions on non-quoted tenors (e.g. on Euribor3M) from the quoted ATM volatility surface (e.g. on Euribor6M).

$$\sigma_y(0,\mathbf{T}_y,\mathbf{S}) = \sigma_x(0,\mathbf{T}_x,\mathbf{S})\sqrt{\frac{R_x^{\mathrm{Swap}}(0;\mathbf{T}_x,\mathbf{S})+\lambda_x}{R_y^{\mathrm{Swap}}(0;\mathbf{T}_y,\mathbf{S})+\lambda_y}}.$$

10: Multiple volatility cubes
Swaptions volatility cube [7]

- The ITM/OTM volatilities on non-quoted tenors are obtained from the SABR model calibrated for the quoted tenor as follows:
  - keep the same $\{\beta, \rho, \nu\}$ parameters, describing the smile shape,
  - calibrate the α parameter, describing the smile level, to match the non-quoted ATM volatilities computed as before.

- An alternative approach, common among market practitioners, is to use a simple basis point volatility equivalence as follows

$$\sigma_x^2(0, \mathbf{T}_x, \mathbf{S}, \Delta K)\left[R_x^{\text{Swap}}(0; \mathbf{T}_x, \mathbf{S}) + \lambda_x\right]\left[R_x^{\text{Swap}}(0; \mathbf{T}_x, \mathbf{S}) + \lambda_x + \Delta K\right]$$
$$\simeq \sigma_y^2(0, \mathbf{T}_y, \mathbf{S}, \Delta K)\left[R_y^{\text{Swap}}(0; \mathbf{T}_y, \mathbf{S}) + \lambda_x\right]\left[R_y^{\text{Swap}}(0; \mathbf{T}_y, \mathbf{S}) + \lambda_y + \Delta K\right]$$

  where K is the strike such that $K = R^{Swap}(0, \boldsymbol{T}, \boldsymbol{S}) + \Delta K$.

- If the fixed leg is different, the conversion formula is more involved, but it is anyway possible to derive a solution

10: Multiple volatility cubes
Swaptions volatility cube [8]

- We stress that the multi-tenor swaption volatility bootstrapping is a consequence of the existence of the interest rate basis. In fact, in the zero-basis, single-curve limit, floating swap legs with different tenors have the same value, and par swap rates are function of fixed swap legs only, so there is no need to convert swaption volatilities: swap rates vs 1M, 3M, 6M represent the same underlying.

$$R_{1M}^{\text{Swap}}(0;\mathbf{T}_{1M},\mathbf{S}) = R_{3M}^{\text{Swap}}(0;\mathbf{T}_{3M},\mathbf{S}) = R_{6M}^{\text{Swap}}(0;\mathbf{T}_{6M},\mathbf{S}) = R_{12M}^{\text{Swap}}(0;\mathbf{T}_{12M},\mathbf{S}).$$

- The above equality does not hold if the fixed leg schedule is modified

10: Multiple volatility cubes
Caps/Floors volatility cube [1]

- Brokers on the OTC market typically quote European Caps/Floors on standard FRAs as follows (we refer to the EUR case for simplicity):

  - Short terms (1-2 Y): spot premia and shifted-Black implied volatilities (two-dimensional matrix expiry x strike), given some shift, on Euribor3M;

  - Medium/log terms (3-30 Y): as above, on Euribor6M.

- Since the quotations above mix different rate tenors, and also customised Caps/Floors are traded OTC on non-standard FRAs (e.g. on Euribor1M and 12M), we need some recipe to derive the implied volatility surfaces for different underling rate tenors. This task is accomplished as described in the following.

10: Multiple volatility cubes
Caps/Floors volatility cube [2]

13:01 30DEC11 ICAP UK69580 VCAP5

EUR Floors - Premium Mids

Please call +44 (0)20 7532 3080 for further details

	STK	ATM	1.00	1.50	2.00	2.25	2.50	3.00	3.50	4.00	4.50	5.00	6.00	7.00	10.0
1Y	0.97	10													
18M	0.95	22													
2Y	0.98	39													
3Y	1.30	82	41												
4Y	1.50	157	63	156											
5Y	1.71	249	82	189											
6Y	1.91	350	100	219											
7Y	2.07	452	118	248	424										
8Y	2.20	552	136	277	466										
9Y	2.30	650	154	307	509	625									
10Y	2.39	746	171	333	548	671									
12Y	2.54	930	203	384	621	757	905								
15Y	2.66	1187	260	470	741	897	1067								
20Y	2.67	1573	383	656	1000	1198	1411								
25Y	2.61	1945	537	889	1319	1563	1825								
30Y	2.54	2294	699	1130	1647	1935	2245								

13:01 30DEC11 ICAP UK69580 VCAP4

EUR Caps - Premium Mids

Please call +44 (0)20 7532 3080 for further details

	STK	ATM	1.00	1.50	2.00	2.25	2.50	3.00	3.50	4.00	4.50	5.00	6.00	7.00	10.0
1Y	0.97	10	10	2											
18M	0.95	22	19	6	2	1	1								
2Y	0.98	39	37	16	9	7	5	3	2	2	1	1	1		
3Y	1.30	82		66	40	32	26	17	12	9	7	5	3	2	1
4Y	1.50	157			108	91	77	56	42	32	25	20	13	9	4
5Y	1.71	249			208	178	153	115	87	67	53	43	29	20	9
6Y	1.91	350			333	290	252	192	147	115	91	74	50	36	17
7Y	2.07	452				412	361	278	216	169	135	110	76	55	27
8Y	2.20	552				538	474	368	287	226	181	147	102	74	37
9Y	2.30	650					590	461	361	286	229	186	130	95	47
10Y	2.39	746					707	556	437	346	278	226	157	116	57
12Y	2.54	930						749	592	471	378	308	215	158	80
15Y	2.66	1187						1020	813	651	528	434	309	232	125
20Y	2.67	1573						1369	1107	902	744	622	456	353	201
25Y	2.61	1945						1661	1363	1127	944	802	604	477	283
30Y	2.54	2294						1924	1597	1336	1132	972	746	598	366

Please note 1 and 2 yrs are vs 3 mth

10: Multiple volatility cubes

Caps/Floors volatility cube [3]

13:01 30DEC11 ICAP UK69580 VCAP3A

EUR Caps/Floors - Implied Volatilities

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	STK	ATM	1.00	1.50	2.00	2.25	2.50	3.00	3.50	4.00	4.50	5.00	6.00	7.00	10.0
1Y	0.97	50.84	50.8	50.1	50.1	50.4	50.7	51.7	52.8	54.0	55.2	56.4	58.8	61.0	66.9
18M	0.95	53.98	53.9	53.5	54.0	54.4	54.7	55.5	56.4	57.3	58.2	59.2	61.0	62.8	67.5
2Y	0.98	58.40	58.4	58.1	58.8	58.9	59.1	59.6	60.1	60.7	61.3	61.9	63.3	64.6	68.1
3Y	1.31	52.04	53.5	51.5	50.7	50.5	50.2	49.9	49.7	49.7	49.7	49.8	50.0	50.4	51.5
4Y	1.51	51.97	55.9	52.0	49.8	48.9	48.1	47.0	46.3	45.8	45.5	45.4	45.3	45.4	45.9
5Y	1.72	49.20	55.9	50.8	47.4	46.0	44.8	43.0	41.8	40.9	40.4	40.1	39.8	39.7	40.2
6Y	1.91	45.87	55.0	49.2	45.2	43.6	42.2	40.0	38.4	37.4	36.7	36.3	35.9	35.9	36.4
7Y	2.07	42.88	53.8	47.7	43.4	41.6	40.1	37.7	36.0	34.9	34.2	33.7	33.4	33.4	34.1
8Y	2.20	40.40	52.6	46.3	41.8	40.0	38.4	36.0	34.2	33.0	32.3	31.8	31.4	31.4	32.2
9Y	2.30	38.41	51.5	45.1	40.6	38.7	37.1	34.6	32.8	31.6	30.8	30.4	30.0	29.9	30.6
10Y	2.39	36.58	50.2	43.9	39.3	37.5	35.9	33.4	31.6	30.3	29.5	29.0	28.6	28.5	29.2
12Y	2.54	33.53	48.0	41.7	37.2	35.3	33.7	31.2	29.4	28.1	27.3	26.8	26.3	26.2	26.9
15Y	2.67	30.96	45.8	39.6	35.2	33.4	31.8	29.4	27.6	26.4	25.6	25.1	24.7	24.8	25.5
20Y	2.68	29.93	43.8	38.0	33.8	32.2	30.8	28.6	27.0	25.9	25.2	24.8	24.5	24.5	25.2
25Y	2.61	30.38	43.3	37.7	33.7	32.2	30.9	28.8	27.3	26.3	25.7	25.3	24.9	24.9	25.4
30Y	2.54	30.83	43.0	37.5	33.7	32.2	31.0	29.0	27.6	26.7	26.0	25.6	25.2	25.2	25.5

Please note 1 and 2 yrs are vs 3 mth

10: Multiple volatility cubes
Caps/Floors volatility cube [4]

- Market Quotes (see previous slides) refers to Cap Volatilities, i.e. the same volatility has to be used to price all the caplets composing a cap, with constant notional and strike.

- For practical Pricing and Hedging purposes, it is more convenient to report as market data the Caplet Volatilities i.e. a volatility that is associated to each single caplet, since Cap Volatility cannot be used to price typical products as
  - Accreting/Amortizing Caps or Floors
  - Contracts with non constant strike.

- The procedure to recover Caplet Volatility σ_i^{Caplet} from raw market data is called forward volatility bootstrap since it foresees a recursive approach to be followed for all the strikes. The basic equation underlying is

$$P_T^{Cap} = \sum_i Black(F_i, \sigma_T^{Cap}, T_i, K) = \sum_i Black(F_i, \sigma_i^{Caplet}, T_i, K)$$

where σ_T^{Cap} is the quoted market data for a Cap expiring at T.

10: Multiple volatility cubes
Caps/Floors volatility cube [5]

- Theoretically, assuming that all the contracts have the same underlying (e.g. Euribor 6M) Caplet Volatility Boostrap foresees the following steps.
  - Selecting a Strike K from the Market Data Matrix
  - selecting the first expiry for the Cap or Floor
  - set $\sigma_1^{Caplet} = \sigma_1^{Cap}$
  - selecting the second expiry and infer σ_2^{Caplet} from the formula

$$P_T^{Cap} = \sum_i Black(F_i, \sigma_T^{Cap}, T_i, K) = \sum_i Black(F_i, \sigma_i^{Caplet}, T_i, K)$$

  with T=2
  - Proceed with all the remaining expiries and strikes, with suitable interpolation rules where needed.

- As a result, we will have a Caplet Volatility (or Forward Volatility) smile that can be used to price a broader range of product.

- Anyway, there are some subtleties that are needed to implement a working boostrap for practical purposes.

10: Multiple volatility cubes
Caps/Floors volatility cube [6]

Let us go back to the old fashioned ***single-curve*** framework and consider three times $T_1 < T_2 < T_3$.

The no-arbitrage relationships between consecutive forward rates may be exploited to link short to long tenor volatilities:

$$(1+\tau_{13}F_{13}(t)) = (1+\tau_{12}F_{12}(t)) \cdot (1+\tau_{23}F_{23}(t))$$

Under hypothesis of shifted lognormal model for all «short tenor» rates $F_{i,i+1}$

$$dF_{i,i+1}(t) = \left(F_{i,i+1}(t) + \Delta_{i,i+1}\right)\sigma_{i,i+1} dW_{i,i+1}(t) + (\ldots)dt$$

$$dW_{i,i+1}(t) dW_{i+1,i+2}(t) = \rho^{i,i+1}_{i+1,i+2} dt$$

We apply Itô's calculus

$$\begin{aligned}
dF_{13}(t)\tau_{13} &= (1+\tau_{23}F_{23}(t))\tau_{12} dF_{12}(t) + (1+\tau_{12}F_{12}(t))\tau_{23} dF_{23}(t) + (\ldots)dt \\
&\ldots \\
&= (1+\tau_{23}F_{23}(t))(F_{12}(t)+\Delta_{12})\tau_{12}\sigma_{12} dW_{12}(t) + (1+\tau_{12}F_{12}(t))(F_{23}(t)+\Delta_{23})\tau_{23}\sigma_{23} dW_{23}(t) + (\ldots)dt \\
&\approx (F_{13}(t)+\Delta_{13})\tau_{13}\sigma_{13} dW_{13}(t)
\end{aligned}$$

10: Multiple volatility cubes
Caps/Floors volatility cube [7]

We want to infer **an approximated** shifted lognormal implied volatility for F_{13} by equating instantaneous variances (freezing stochastic quantities and so on...)

$$\mathrm{Var}\left[(F_{13}(t)+\Delta_{13})\,\tau_{13}\sigma_{13}dW_{13}(t)\right]\approx$$

$$\mathrm{Var}\left[(1+\tau_{23}F_{23}(t))(F_{12}(t)+\Delta_{12})\,\tau_{12}\sigma_{12}dW_{12}(t)+(1+\tau_{12}F_{12}(t))(F_{23}(t)+\Delta_{23})\,\tau_{23}\sigma_{23}dW_{23}(t)\right]$$

That is

$$\left[(F_{13}(0)+\Delta_{13})\tau_{13}\sigma_{13}\right]^2\approx$$

$$\left[(1+\tau_{23}F_{23}(0))(F_{12}(0)+\Delta_{12})\tau_{12}\sigma_{12}\right]^2+\left[(1+\tau_{12}F_{12}(0))(F_{23}(0)+\Delta_{23})\tau_{23}\sigma_{23}\right]^2+$$

$$+\;2\rho_{23}^{12}\left[(1+\tau_{23}F_{23}(0))(F_{12}(0)+\Delta_{12})\tau_{12}\sigma_{12}\right]\cdot\left[(1+\tau_{12}F_{12}(0))(F_{23}(0)+\Delta_{23})\tau_{23}\sigma_{23}\right]$$

This expression is to be solved for σ_{13} and can be easily extended to the case of the composition of more than two short tenor rates. For instance a 3M rate can be obtain as composition of three consecutive 1M rates.

10: Multiple volatility cubes
Caps/Floors volatility cube [8]

Correlations among consecutive rates may be modelled by simple forms, as Rebonato four-parameter one, which should in the end be calibrated to market prices.

$$\rho_{i+1,i+2}^{i,i+1} = \rho_\infty + (\rho_0 - \rho_\infty + bT_i)e^{-c \cdot T_i}$$

- ρ_0 represents the correlation between consecutive rates on the short part of the curve
- ρ_∞ represents the correlation between consecutive rates on the long part of the curve and is going to be very close to 1
- c rules the speed of convergence to asymptotical value
- b controls the presence of local maximum/minimum point

10: Multiple volatility cubes
Caps/Floors volatility cube [9]

- The **Euribor6M** surface is generated from market quotes as follows:

  1. For each strike Forward Cap/Floor premia are implied from quoted spot cap/floor premia. E.g. *Cap(9Y,10Y) = Cap(0,10Y) - Cap(0,9Y)*. This Cap contains two caplets.

  2. For each strike Forward (Caplet/Floorlet) volatilities are implied from forward Cap/Floor premia, using volatility interpolation (e.g. piecewise constant, or a smooth functional form like the famous Rebonato 4-parameter $f(t)=(a+bt)e^{-ct}+d$).

  3. Short term Euribor6M volatility (0Y-2Y) is obtained from Euribor3M surface.

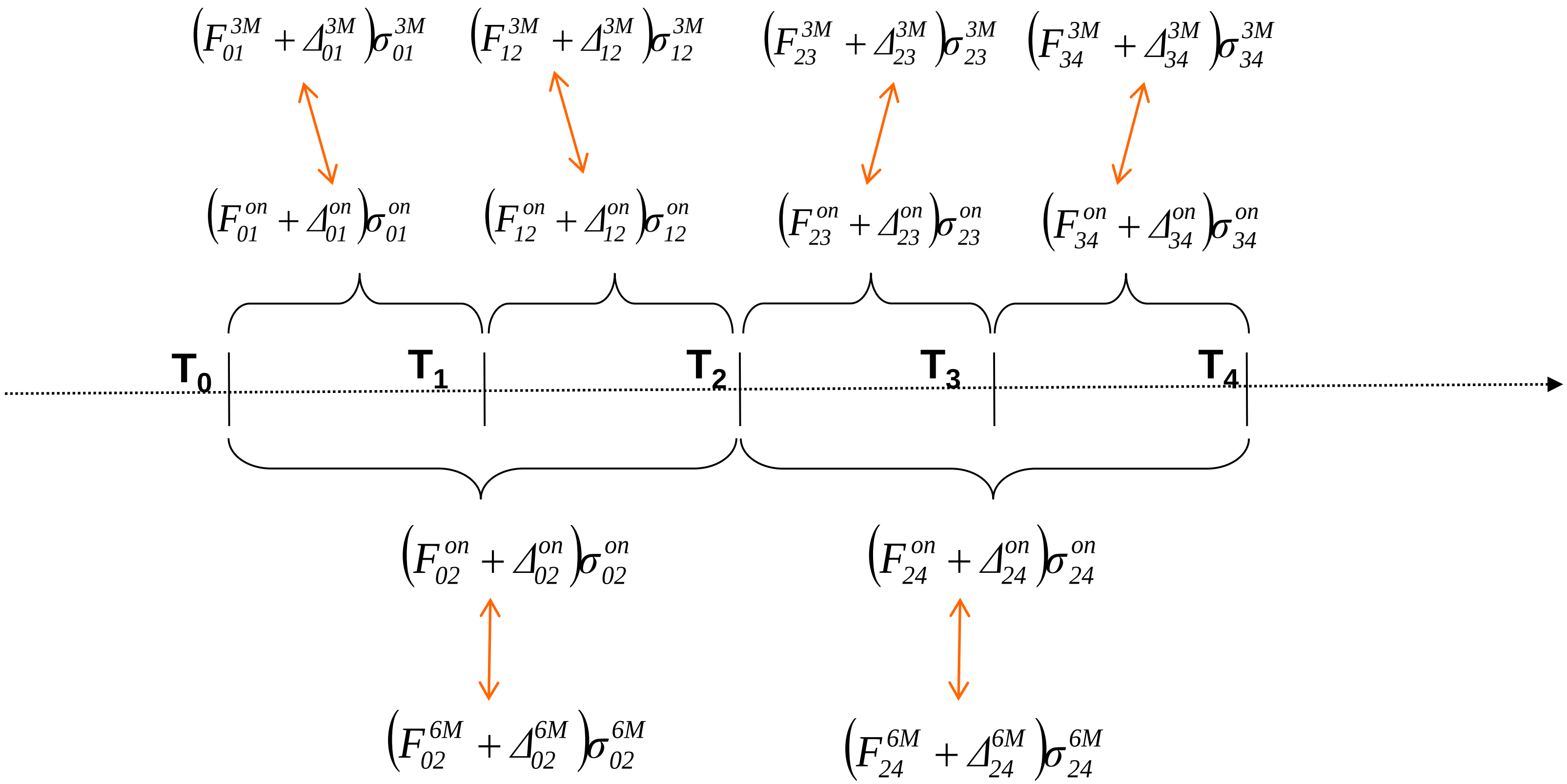
  4. Finally, for each maturity, SABR (or another interpolation/smoothing model) is used for Caplet/Floorlet volatility smile interpolation on non-quoted strike points.

- The **Euribor3M** surface is generated from market quotes similarly to Euribor6M.

- The **Euribor1M** and **Euribor12M** volatility surfaces is generated from the Euribor3M and Euribor6M surfaces, respectively, as described below.

10: Multiple volatility cubes
Caps/Floors volatility cube [10]

The conversion between shorter and longer tenor can be done, following Kienitz (2013), by giving the overnight curve the role of pivot curve on which single-curve relationships can be naturally applied.



The volatility involved are the ATM ones.

10: Multiple volatility cubes
Caps/Floors volatility cube [11]

We make the following assumptions :

- all the relevant Xibor rates to have a shifted lognormal dynamics

$$dF^x{}_{i,j}(t) = (F^x{}_{i,j}(t) + \Delta^x{}_{i,j})\sigma^x{}_{i,j} dW^x{}_{i,j}$$

- the spreads between Xibor rates and the corresponding O/N based rates are static

$$F^x{}_{i,j}(t) = F^{on}_{i,j}(t) + b^x{}_{i,j}$$

$$b^x{}_{i,j} \approx F^x{}_{i,j}(0) - F^{on}_{i,j}(0)$$

- As a consequence, O/N based rates have a shifted lognormal dynamics, too

$$dF^{on}_{i,i+1}(t) \approx (F^{on}_{i,i+1}(t) + \Delta^{on}{}_{i,i+1})\sigma^x{}_{i,i+1} dW^x{}_{i,i+1}(t)$$

$$\Delta^{on}{}_{i,i+1} := b^x{}_{i,i+1} + \Delta^x{}_{i,i+1}$$

and the machinery illustrated for the single-curve case could be applied.

10: Multiple volatility cubes
Caps/Floors volatility cube [12]

- Following the approach above, we are able to link 3M-ibor (shifted lognormal) ATM volatilities σ^{3M}_{01} , σ^{3M}_{12} to the 6M-ibor (shifted lognormal) ATM volatility σ^{6M}_{02} , and so on.....

- The procedure can be applied either from shorter-to-longer (e.g. from 6M to 1Y tenor) or from longer-to-shorter (e.g. from 6M to 3M). In the latter case we can set for simplicity $\sigma^{3M}_{01} = \sigma^{3M}_{12}$ in order to keep a one-to.one relationship between unknowns and constraints.

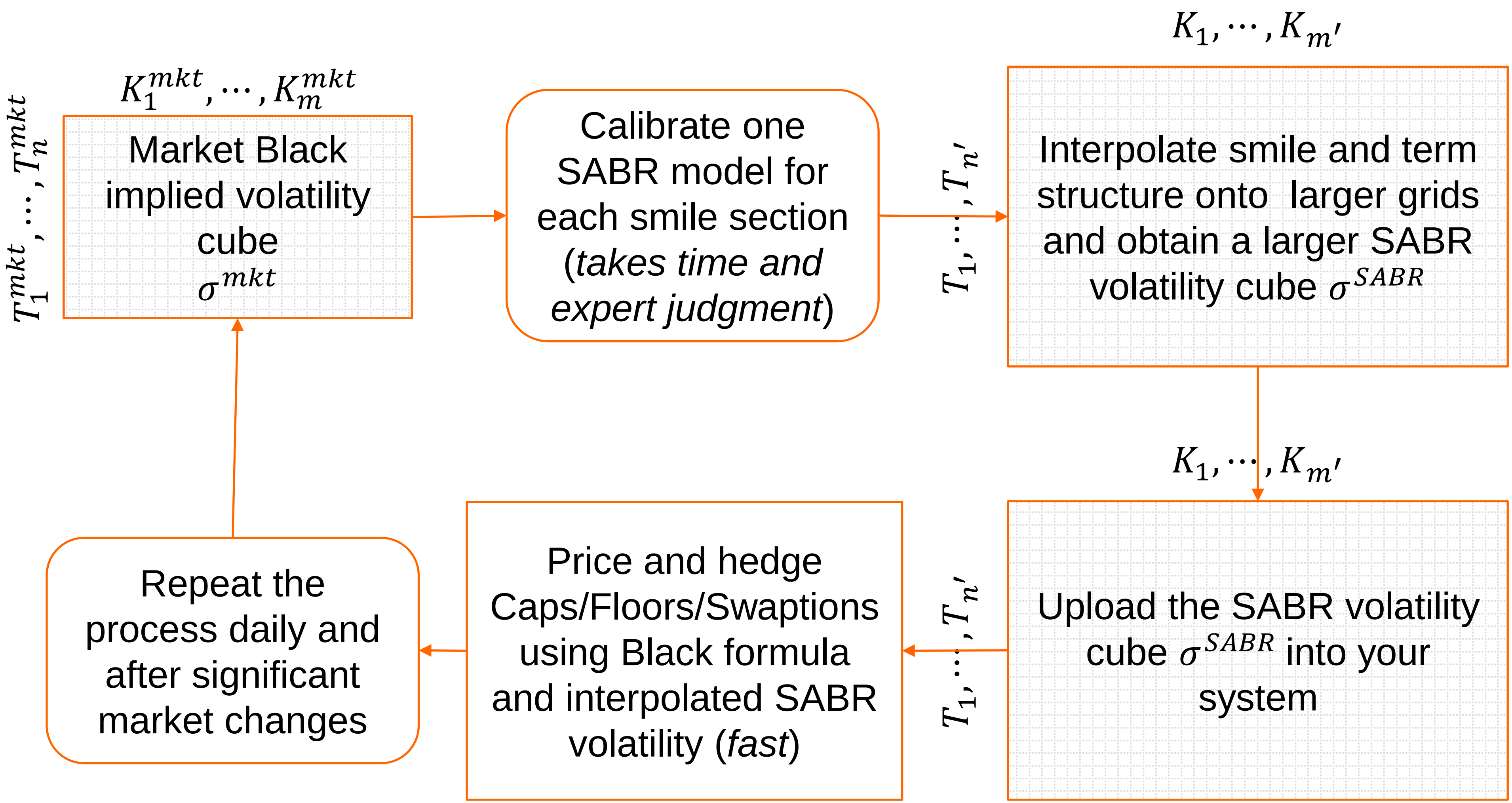
- Kienitz's suggestion for smile:
  - start from the most liquid tenor (6M for Euribor)
  - compute implied volatility per-strike
  - calibrate a SABR for each expiry
  - «deduce» SABR parameters for unquoted tenors. For instance one could use the same parameters for F^{6M}_{02} and for F^{3M}_{01}, F^{3M}_{12} , or linking parameters for longer tenor rate to the average of the ones of the underlying short tenor rates.
  - SABR parameter «alpha» should be adjusted such as to match ATM implied vol computed as before (composition with corellation via O/N curve)

Remark: the assumption of static basis between forward rates belonging to different tenor curves is quite strong and all covariance effects are somehow incorporated into the correlation function used for volatility compositions and which should be calibrated to (more or less) sparse market data.

10: Multiple volatility cubes

Practical SABR usage for Caps/Floors/Swaptions vols

Typically the SABR model is used as an no-arbitrage interpolator to enlarge the strike dimension of market volatilities.



10: Multiple volatility cubes
Problems

- SABR calibration: calibrate the SABR parameters to the swaption volatility cube. Check different objective functions and calibration algorithms in terms of calibration convergence and performance. Use spreadsheet for input/output and Matlab for calibration. Deliverable: written relation with charts + Matlab code with comments.

- SABR volatility cube: using the SABR calibration above, build a SABR extended volatility cube with expiry/tenor grid as of market quotes, and extended strike grid from -300 bps to +1000 bps w.r.t. the ATM with step 25 bps. Using this cube, chech the no-arbitrage conditions for Swaption prices (vertical spread, butterfly, triangular arbitrage). Use spreadsheet. Deliverable: spreadsheet with charts and comments.

Summary

11. Term structure modelling

- Introduction
  - Exotic derivatives
  - Term structure modelling
- Short rate models
  - Overview
  - Vasicek
  - Shifted Vasicek
  - Multi-curve extension
  - Monte Carlo simulation
- Libor Market Models

11: Term Structure Modelling
Introduction: exotic derivatives, modern market practice [1]

Broadly speaking, an exotic interest rate derivative is a financial product whose payoff depends on multiple points of the term structure (yield curve) in a non-separable way.

Examples of IR derivatives:

- Vanilla Trades (quoted values and closed formulas available).
  - Caps/floors are portfolios of caplets/floorlets each depending on a single forward rate in the term structure.
  - Swaptions depends on a single swap rate.

- Derivatives requiring more complex pricing approaches, although market quotes may be available.
  - CMS and CMS options depends on a single swap rate.
  - Spread options depends on two rates.

- Exotic Derivatives
  - A Bermudan Swaption depends on multiple swap rates, one for each exercise date.

Introduction: exotic derivatives, modern market practice [2]

Exotic Derivatives

- Typically exhibit custom / non standardized payoffs, possibly including early exercise (callable) or path dependent features.
- Usually traded OTC, there are no market quotes available.
- OTC prices can be sometimes showed by brokers typically between collateralized counterparties.

To price such products, we need a model for the whole term structure of each yield curve required for pricing. Hence the name «term structure modelling».

IR Models used to price Exotic Derivatives typically also include a set of model parameters, calibrated on available market quotes for vanilla derivatives.

Concept of "Model Price" as opposite to "Market Price"

Introduction: exotic derivatives, modern market practice [3]

General pricing procedure for exotic interest rate derivatives

#	Step	Examples
1	Assume multiple yield curves and volatility cubes as input.	
2	Choose the fundamental dynamical variables	discrete/instantaneous spot/forward rates
3	Assume a dynamics for the time evolution of the fundamental dynamical variables (dynamical model).	HJM, short rate, LMM
4	Derive (arbitrage free) pricing formulas for plain vanilla instruments.	Caps/Floors/Swaptions
5	Calibrate the model parameters to the market quotes of a chosen set of plain vanilla instruments (calibration instruments) using the appropriate numerical approach.	single-/multimensional minimization of RMSE
6	Price other derivatives using the calibrated model and appropriate numerical approach	Monte Carlo simulation, PDE
7	Derive sensitivities and hedge ratios with respect to a choosen subset of calibration instruments (hedging instruments) using the appropriate numerical approach.	Finite differences, AAD

Introduction: term structure modelling [1]

Summary of interest rate products and of their corresponding pricing models, by increasing product/model complexity.

Product	Pricing Model	Risk factors
Linear interest derivatives	Single forward rate model	Yield curves
Plain vanilla interest rate options	Single forward rate market model	Yield curves, volatilities
CMS, CMS Caps/Floors	Replication model	Yield curves, volatilities
Rate Spread Options	Bivariate models	Yield curves, volatilities, correlations
Exotic derivatives	Term Structure Modelling	Yield curves, volatilities, correlations, Model parameters

Introduction: term structure modelling [2]

Summary of multiple yield curves, volatility and correlation cubes quoted on the market

Object	Types and examples
Yield curves	○ Discount curves (e.g. OIS, Euribor6M) ○ Forward curves (e.g. Euribor1M, 3M, 6M, 12M) ○ Repo curves ○ Bond curves (Sovereign, Corporate, etc.)
Volatility cubes	○ Cap/Floor term/forward volatility cube (e.g. Euribor1M, 3M, 6M, 12M) ○ Swaption volatility cube (e.g. Euribor6M, Euribor3M)
Correlation cube	○ CMS spreads correlations (e.g. 10Y-2Y, 30Y-2Y, 30Y-10Y)

Introduction: term structure modelling [3]

Fundamental interest rate variables and their associated Term Structure Modelling.

Fundamental stochastic variables	Term structure model
Short rates	Short Rate Models
Instantaneous forward rates	HJM Models
Discrete FRA rates	Libor Market Models
Forward Swap rates	Swap Market Models

A Model il fully specified once

- We define the Financial Variable that is being modeled
- a Stochastic Differential Equation or PDE for the Financial Variable is specified
- a parameterization is defined.

Introduction: term structure modelling [4]

Remind of the fundamental stochastic variables (see previous slides)

Fundamental stochastic variables	Notation	Properties
Short rates	$r_x(t)$	$r_x(t) = f_x(t,t)$
Bank accounts	$B_x(t) = e^{\int_0^t r_x(u)du}$	$D_x(t,T) = \dfrac{B_x(t)}{B_x(T)} = e^{-\int_t^T r_x(u)du}$
Zero coupon bond	$P_x(t,T) = \mathbb{E}_t^Q[D_x(t,T)]$	$P_x(t,t) = P_x(T,T) = 1$
Spot rates	$L_x(t,T)$	$L_x(t,T) = \dfrac{1}{\tau(t,T)}\left[\dfrac{1}{P_x(t,T)} - 1\right]$
FRA rates	$F_x(t;T,S)$	$F_x(t;T,S) = \dfrac{1}{\tau(T,S)}\left[\dfrac{P_x(t,T)}{P_x(t,S)} - 1\right]$ $F_x(T,T,S) = L_x(T,S)$
Forward Swap rates	$R_x(t,\boldsymbol{T},\boldsymbol{S})$	
Instantaneous forward rates	$f_x(t,T)$	$f_x(t,T) = -\dfrac{\partial Log P_x(t,T)}{\partial T}$

Introduction: term structure modelling [5]

Main issues in multi-curve term structure modeling

- The term structures of interest rates are yield curves $\mathcal{C}_x(t,T) = \{t \longrightarrow P_x(t,T)\}$, a complex object. The index *x* denotes the yield curve type (e.g. OIS, Euribor6M, etc.)
  - In principle, infinite-dimensional (it's a continuum of interest rates with different start/end dates).
  - In practice, finite but very high dimensional (30-100 discrete pillars, for yield curves rich of market quotations up to 60Y-70Y (e.g. Euribor6M).
  - There are consistency and no-arbitrage relations among different pillars: not all yield curve shapes are admitted or financially sound.
  - Requires consistency among different yield curves: not all basis curve shapes are admitted or financially sound

- Term Structure Modelling calibration
  - Current market term structures (yield curves) $\mathcal{C}_x^{Mkt}(t,T) = \{t \longrightarrow P_x^{Mkt}(t,T)\}$.
  - Current market volatilities (Caps/Floors and/or Swaptions, with smiles).
  - Current CMS, CMS Cap/Floor, and CMS Spread Options (correlations), if used for hedging purposes.

11: Term Structure Modelling
Introduction: term structure modelling [6]

- Multi-curve extension
  - Deterministic basis
  - Stochastic basis

- Examples of research articles on multi-curve extension of Term Structure Modelling:
  - HJM models:
    see e.g N. Moreni, A. Pallavicini, "*Parsimonious HJM Modelling for Multiple Yield-Curve Dynamics*", Oct. 2010, SSRN working paper, *http://ssrn.com/abstract=1699300*
  - HJM Model with credit risk:
    see e.g. S. Crepey, Z. Grbac, H. Nguyen, "*A defaultable HJM multiple curve term structure model*", 9 Oct. 2011.
  - Short rate models:
    see e.g. C. Kenyon, "*Post Shock Short-Rate Pricing*", Risk Magazine, Oct. 2010.
  - Libor Market Models:
    see e.g. F. Mercurio, "*A Libor Market Model with Stochastic Basis*", Risk Magazine, Dec. 2010 and refs. therein.

Short rate models: overview [1]

- Main features of the most common models
  - Fundamental stochastic quantity: short rate $r_x(t)$ of type (i.e. tenor) *x*.
  - Simple stochastic dynamics allowed
  - Small number of parameters, with financial interpretation
  - Possible calibration of current IR term structure (yield curve)
  - Possible tractability with (semi)analytical formulas for plain vanilla options,
  - Possible (semi)analytical calibration to ATM market options.
  - Straightforward Monte Carlo simulation
  - Straightforward implementation on trees or finite difference methods.

- Drawbacks
  - In most cases, the model are too simple to describe properly the term structure behavior
  - Tractability is lost if the model is enhanced (N factor models)
  - In most cases, not realistic forward smile implied by the model
  - No smile, need introduction of additional features (for instance stochastic volatility) for smile calibration

Short rate models: overview [2]

- Steps in short rate modelling
  - Choose a rate $r_x(t)$ and its dynamics (SDE) of the general form

$$\begin{cases} dr(t) = \mu\left(t, r(t)\right) dt + \sigma\left(t, r(t)\right) \cdot dW^Q(t), \\ r(0) = r_0, \end{cases}$$

    where we omit the index *x* for brevity, and
    - μ is the drift
    - σ is the volatility (D-dimensional row vector)
    - W is the Brownian motion (D-dimensional column vector)

  - Compute ZCBs

$$P(t,T) = \mathbb{E}_t^Q\left[D(t,T)\right] = \mathbb{E}_t^Q\left[\exp\left(-\int_t^T r(u)du\right)\right]$$

    either analytically (if allowed by the model), or numerically, e.g. by Monte Carlo.
  - Calibrate the model to the current term structure of interest rates $C_x^{Mkt}(t,T)$.
  - Compute Caps/Floors/Swaptions either analytically or numerically
  - Calibrate the model to the current market implied volatility structure.

Short rate models: overview [3]

- SDE solution
  In case the SDE is linear with deterministic volatility

$$\begin{cases} dr(t) = [\alpha(t)r(t) + \beta(t)]\, dt + \sigma(t) \cdot dW^Q(t), \\ r(0) = r_0, \end{cases}$$

where α, β, ν are deterministic functions of time regular enough to guarantee Cauchy theorem. We have the general solution

$$r(T) = r(t)D_\beta^{-1}(t,T) + \int_t^T D_\beta^{-1}(u,T)\alpha(u)du + \int_t^T D_\beta^{-1}(u,T)\sigma(u) \cdot dW^Q(u),$$

$$D_\beta(t,T) = \exp\left(-\int_t^T \beta(u)du\right),$$

- Mean and variance at time *t* of the future distribution $r(T), T > t$ are given by

$$\mathbb{E}_t^Q[r(T)] = r(t)D_\beta^{-1}(t,T) + \int_t^T D_\beta^{-1}(u,T)\alpha(u)du,$$

$$\mathrm{Var}_t^Q[r(T)] = \int_t^T D_\beta^{-2}(u,T)\sigma^2(u).$$

11: Term Structure Modelling

Short rate models: overview [4]

Model	Dynamics	Properties
Vasicek	Ornstein-Uhlembeck (Gaussian mean reverting)	○ Partial yield curve calibration ○ Analytical ZCB, Cap/Floor/Swaption ○ Negative rates ○ Large steps Monte Carlo simulation
Hull-White (HW)	As above	○ Exact yield curve calibration ○ As above
Cox-Ingersoll-Ross (CIR)	Non Central Chi-Squared mean reverting	○ Exact yield curve calibration ○ Analytical ZCB, Cap/FloorSwaption ○ Non negative rates
Dothan	Lognormal	○ Analytical ZCB ○ Positive rates ○ Large steps Monte Carlo simulation
Black-Karasinski (BK)	Exponential Ornstein-Uhlembeck	○ No analytical formulas ○ Positive rates ○ Large steps Monte Carlo simulation

See Brigo D. and Mercurio M, “Interest rate models”, 2nd edition 2006), table 3.1

Short rate models: Vasicek [1]

Oldrich Vasicek

- Dynamics

The short rate dynamics in the Vasicek model [*] is given by

$$\begin{cases} dr(t) = k\left[\theta - r(t)\right]dt + \sigma dW^Q(t), \\ r(0) = r_0, \\ \sigma \in \mathbb{R}^+;\ k, \theta, r_0 \in \mathbb{R}, \end{cases}$$

where

  - $\theta =$ mean reversion (asymptotic) level
  - $k =$ mean reversion speed
  - $\sigma =$ instantaneous volatility (1D)
  - $W =$ Brownian motion (1D)

- Remarks
  - The number of stochastic factors is 1.
  - The number of parameters is 3.
  - $k > 0 \rightarrow$ mean reverting model: the mean reversion drives the dynamics towards the asymptotic level and the variance is asymptotically finite. $k < 0 \rightarrow$ repulsive model, asymptotical divergent mean and variance.

[*] O. Vasicek (1977), "*An equilibrium characterization of the term structure*", Journal of Financial Economics 5: 177–188.

11: Term Structure Modelling
Short rate models: Vasicek [2]

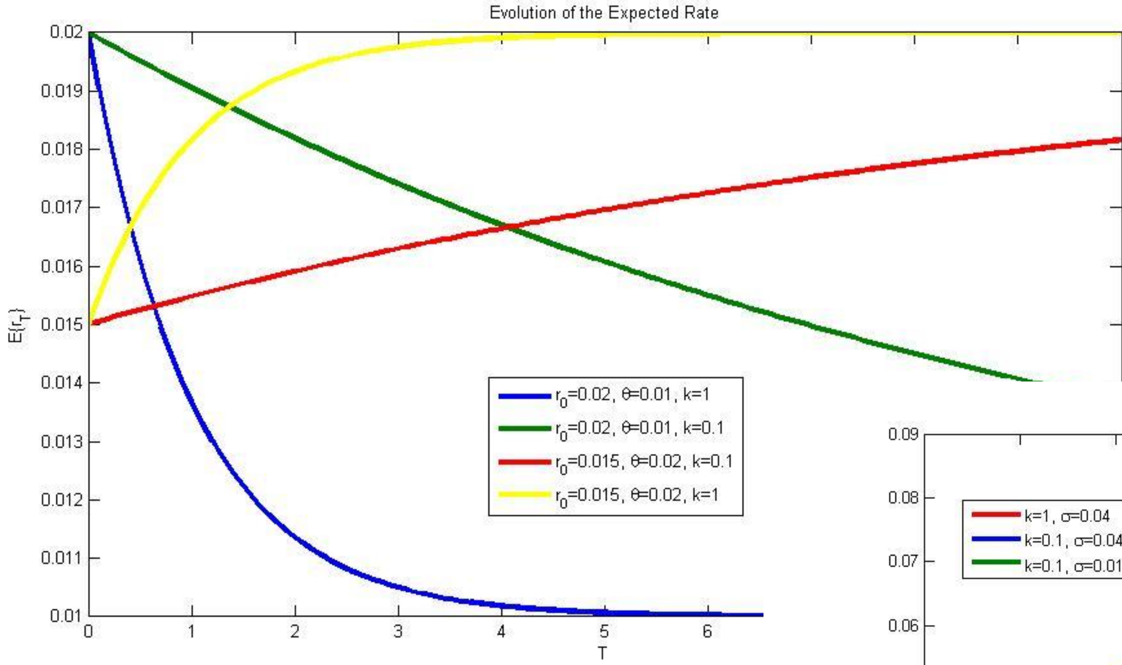
- Define the useful quantity
$$B(t,T,k) := \frac{1}{k}\left[1 - e^{-k(T-t)}\right],$$

- SDE integration: the short rate is a Gaussian Variable
$$\begin{aligned} r(T) &= r(t)e^{-k(T-t)} + \theta\left(1 - e^{-k(T-t)}\right) + \sigma \int_t^T e^{-k(T-u)} dW^Q(u) \\ &= r(t) + k\left[\theta - r(t)\right] B(t,T,k) + \sigma \int_t^T \left[1 - kB(t,T,k)\right] dW^Q(u) \end{aligned}$$

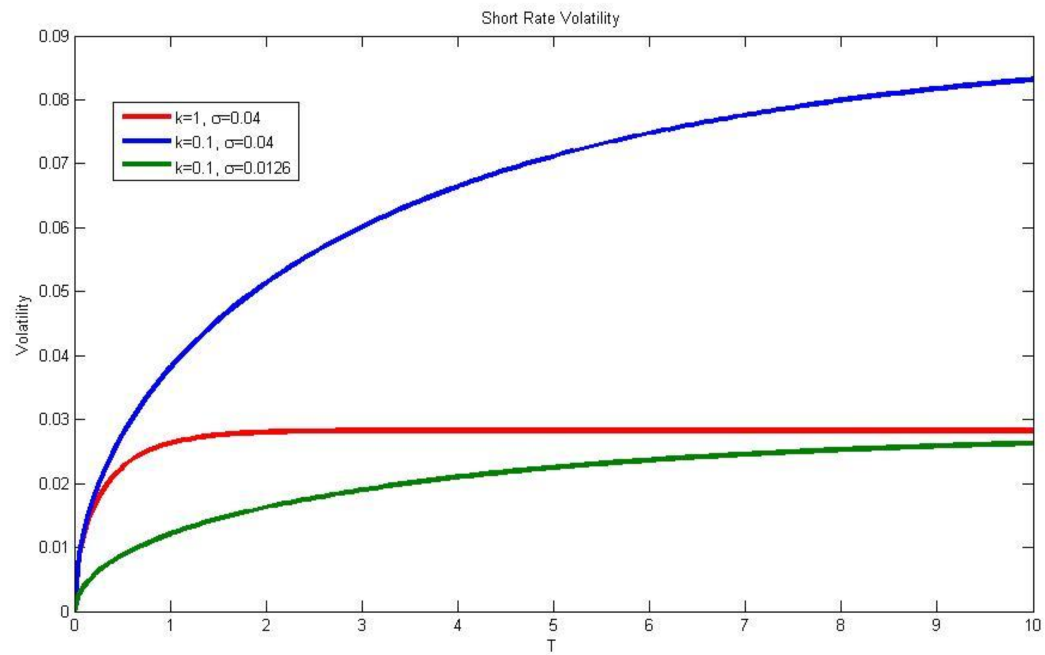
- Mean and variance at time *t* of the future distribution $r(T)$ at time $T > t$ are given by for $k > 0$)
$$\begin{aligned} \mathbb{E}_t^Q\left[r(T)\right] &= r(t)e^{-k(T-t)} + \theta\left(1 - e^{-k(T-t)}\right) \\ &= r(t) + k\left[\theta - r(t)\right] B(t,T,k) \xrightarrow[T\to\infty]{} \theta, \end{aligned}$$
$$\mathrm{Var}_t^Q\left[r(T)\right] = \frac{\sigma^2}{2k}\left(1 - e^{-2k(T-t)}\right) = \sigma^2 B(t,T,2k) \xrightarrow[T\to\infty]{} \frac{\sigma^2}{2k}.$$

For $k < 0$ both mean and variance diverge.

Short rate models: Vasicek [4]

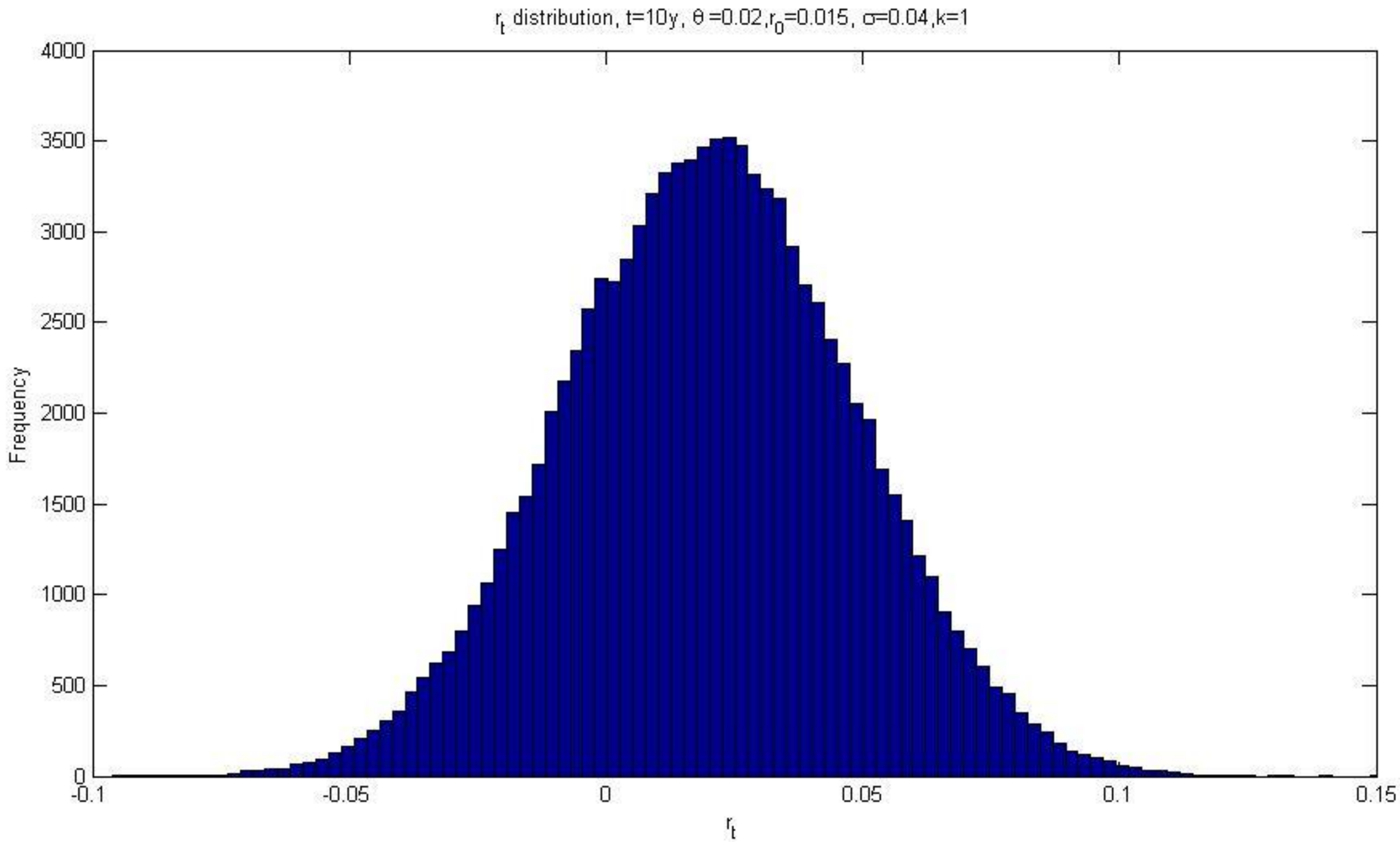


Expected Rate and Volatility as a function of time



Short rate models: Vasicek [5]

Distribution of short rates in Vasicek Model



Short rate models: Vasicek [6]

- ZCB price

$$P(t,T) = \mathbb{E}_t^Q\left[D(t,T)\right] = A(t,T)e^{-B(t,T,k)r(t)},$$

$$A(t,T) := \exp\left\{\left(\theta - \frac{\sigma^2}{2k^2}\right)\left[B(t,T,k) - (T-t)\right] - \frac{\sigma^2}{4k}B(t,T,k)^2\right\}.$$

Notice that, since the short rate has a normal distribution, the ZCB price has a lognormal distribution.

- Short rate vs spot rate
  Given the continuous compounded spot rate $R(t,T)$ such that

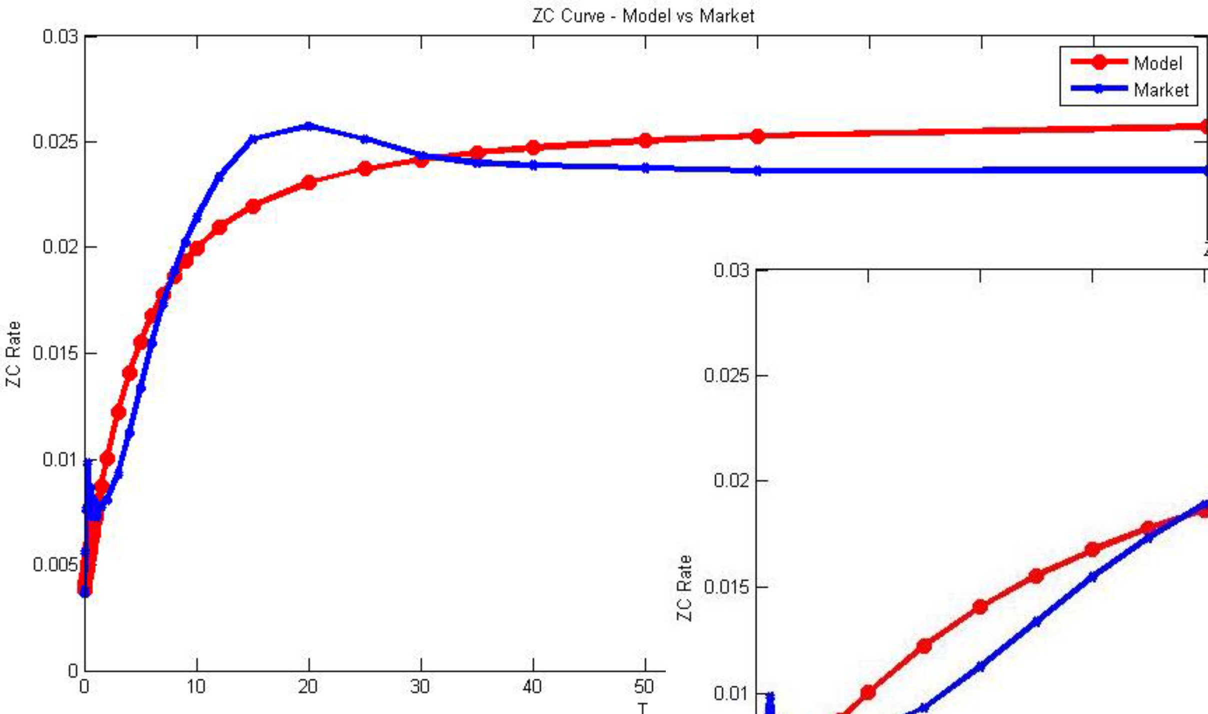
$$P(t,T) = e^{-R(t,T)(T-t)},$$

we obtain

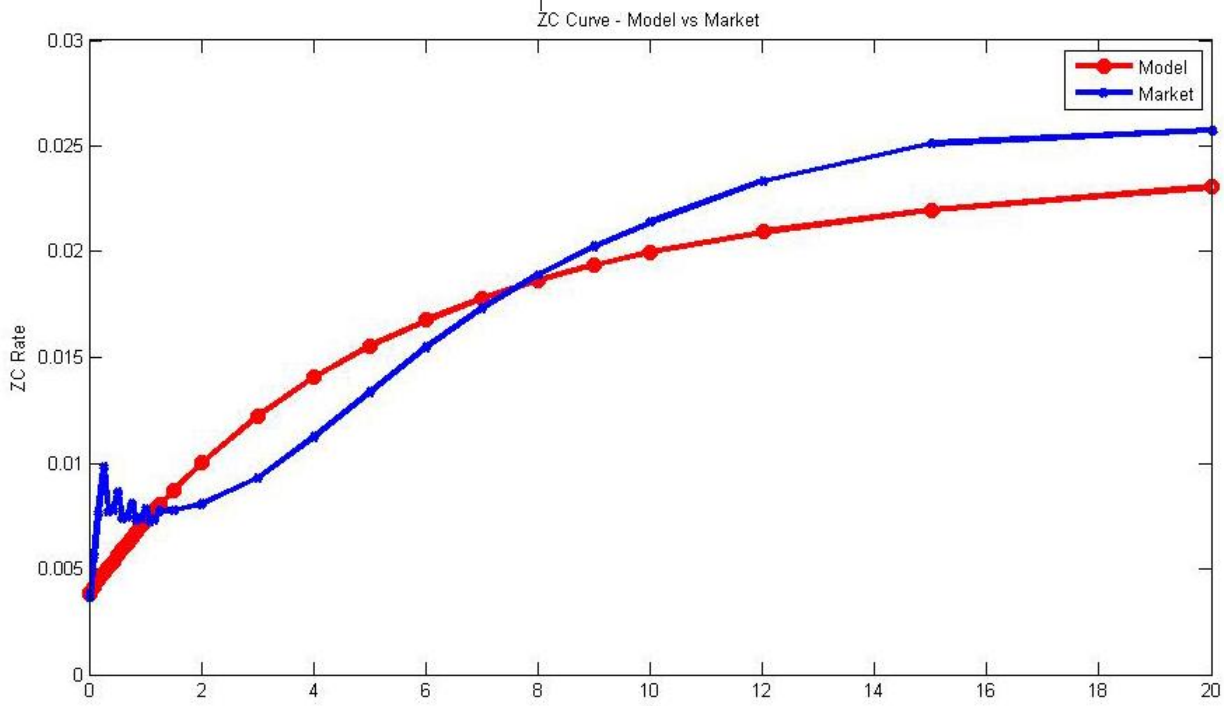
$$R(t,T) = \frac{B(t,T,k)}{T-t}r(t) - \frac{\ln A(t,T)}{T-t},$$

which is an affine transformation of the short rate $r(t)$.

- Calibration: yield curve
  The current term structure of interest rates (market yield curves $C(0,T) = \{T \longrightarrow P^M(0,T)\}$) cannot be exactly recovered.



Calibration of parameters
on Market Term Structure



Short rate models: Vasicek [8]

- Cap/Floor pricing

  The Vasicek model admits analytical formulas to price Caplets/Floorlets, and thus Caps/Floors. Under the risk neutral measure *Q* we have

$$
\begin{aligned}
\mathbf{cf}(t;T_{i-1},T_i,K,\omega) &= N\tau_i\mathbb{E}_t^Q\left[D_d(t,T_i)\mathrm{Max}\left\{\omega\left[L_x(T_{i-1},T_i)-K\right];0\right\}\right] \\
&= N\tau_i\mathbb{E}_t^Q\left[D_d(t,T_{i-1})\mathbb{E}_{T_{i-1}}^Q\left[D_d(T_{i-1},T_i)\right]\mathrm{Max}\left\{\omega\left[L_x(T_{i-1},T_i)-K\right];0\right\}\right] \\
&= N\tau_i\mathbb{E}_t^Q\left[D_d(t,T_{i-1})P(T_{i-1},T_i)\mathrm{Max}\left\{\omega\left[L_x(T_{i-1},T_i)-K\right];0\right\}\right] \\
&= N\mathbb{E}_t^Q\left[D_d(t,T_{i-1})P(T_{i-1},T_i)\mathrm{Max}\left\{\omega\left[\frac{1}{P(T_{i-1},T_i)}-1-K\tau_i\right];0\right\}\right] \\
&= N\mathbb{E}_t^Q\left[D_d(t,T_{i-1})\mathrm{Max}\left\{\omega\left[1-(1+K\tau_i)P(T_{i-1},T_i)\right];0\right\}\right] \\
&= N(1+K\tau_i)\mathbb{E}_t^Q\left[D_d(t,T_{i-1})\mathrm{Max}\left\{-\omega\left[P(T_{i-1},T_i)-\frac{1}{1+K\tau_i}\right];0\right\}\right].
\end{aligned}
$$

The last expectation is proportional to the price of a Zero Coupon Bond Option

$$
\mathbf{cf}(t;T_{i-1},T_i,K,\omega) = (1+K\tau_i)\,\mathbf{ZCBO}(t;T_{i-1},T_i,K',\omega'),
$$

$$
K' = \frac{1}{1+K\tau_i}, \quad \omega' = -\omega.
$$

(see next page)

- Zero Coupon Bond Option (ZCBO) pricing

  An european Zero Coupon Bond Option with maturity T_{i-1} is an option written on a Zero Coupon Bond $P(T_{i-1}, T_i)$. The payoff is given by

$$\mathbf{ZCBO}(T_{i-1}; T_{i-1}, T_i, K, \omega) = N\mathrm{Max}\left\{\omega\left[P(T_{i-1}, T_i) - K\right]; 0\right\}.$$

  Since the ZCB price is lognormal, the ZCBO price is given by a Black-like formula

$$\begin{aligned}
&\mathbf{ZCBO}(t; T_{i-1}, T_i, K, \omega) \\
&\qquad = NP(t, T_{i-1})\mathbb{E}_t^{Q_{i-1}}\left\{\mathrm{Max}\left\{\omega\left[P(T_{i-1}, T_i) - K\right]; 0\right\}\right\}, \\
&\qquad = NP(t, T_{i-1})\omega\left\{P(T_{i-1}, T_i)N(\omega d_B) - KN\left[\omega(d_B - \sigma_B)\right]\right\}, \\
&d_B = \frac{\ln\frac{P(T_{i-1}, T_i)}{K} + \frac{1}{2}\sigma_B^2}{\sigma_B}, \\
&\sigma_B = \sigma B(T_{i-1}, T_i, k)\sqrt{B(t, T_{i-1}, 2k)}.
\end{aligned}$$

  Notice that Caplet/Floorlet prices corresponds to Put/Call ZCBO prices, respectively.

Short rate models: Vasicek [10]

- Swaption pricing
  Swaptions can be priced as portfolio of Coupon Bond Options (CBO) using the Jamshidian decomposition, see Brigo & Mercurio (2006).

- Calibration: volatilities
  The mean reversion speed k and the instantaneous volatility σ can be calibrated using market prices of ATM Caps/Floors or Swaptions.

Short rate models: Hull-White [1]

- **Dynamics**

  The short rate dynamics in the Hull-White model [*] is given, under the risk neutral measure Q, by

$$
\begin{cases}
dr(t) = k\left[\vartheta(t) - r(t)\right]dt + \sigma dW^{Q}(t), \\
r(0) = r_0, \\
\vartheta : \mathbb{R}^{+} \rightarrow \mathbb{R}, k, \sigma \in \mathbb{R}^{+};\ r_0 \in \mathbb{R},
\end{cases}
$$

John Hull

  where $\vartheta(t)$ = mean reversion level, deterministic function of time.

- **Remarks**
  - The number of stochastic factors is 1.
  - The number of parameters is infinite.
  - If $\vartheta(t)$ is, e.g. piecewise constant with M pillars, the number of parameters is M+2.
  - For $\vartheta(t) = \theta\ \forall\ t < T$ we recover the Vasicek model.

Alan White

[*] J. Hull, A. White, "*Branching Out*", *Risk,* 7, 34-37 (1994).

Short rate models: Hull-White [2]

- Calibration: yield curve

  The current term structure of interest rates (market yield curves $C_x(0,T) = \{T \longrightarrow P_x^M(0,T)\}$) can be be exactly recovered by choosing

$$\vartheta(t) = k\left[\frac{\partial f^{Mkt}(0,t)}{\partial t} + k f^{Mkt}(0,t) + \sigma^2 B(0,t,2k)\right],$$

$$f^{Mkt}(0,t) := -\frac{\partial \ln P^{Mkt}(0,t)}{\partial t}.$$

  Actually this formula is not necessary to compute prices, see below.

- SDE integration

$$r(T) = r(t)e^{-k(T-t)} + \frac{1}{k}\int_t^T \vartheta(u)e^{-k(T-u)}du + \sigma\int_t^T e^{-k(T-u)}dW^Q(u)$$

$$= \varphi(T) + [r(t) - \varphi(t)]\,[1 - kB(t,T,k)] + \sigma\int_t^T [1 - kB(t,T,k)]\,dW^Q(u),$$

$$\varphi(t) := k\left[f^{Mkt}(0,t) + \frac{1}{2}\sigma^2 B^2(0,t,k)\right].$$

Short rate models: Hull-White [3]

- Mean and variance at time t of the future distribution $r(T)$ at time $T > t$ are given by

$$\mathbb{E}_t^Q\left[r(T)\right] = \varphi(T) + \left[r(t) - \varphi(t)\right]\left[1 - kB(t,T,k)\right],$$
$$\mathrm{Var}_t^Q\left[r(T)\right] = \sigma^2 B(t,T,2k).$$

- ZCB price

  We obtain a modified Vasicek formula as follows

$$P(t,T) = \mathbb{E}_t^Q\left[D(t,T)\right] = A(t,T)e^{-B(t,T,k)r(t)},$$
$$A(t,T) := \frac{P^{Mkt}(0,T)}{P^{Mkt}(0,t)}\exp\left\{B(t,T,k)f^{Mkt}(0,t) + \frac{\sigma^2}{2}B(0,t,2k)B(t,T,k)^2\right\},$$
$$B(t,T,k) := \frac{1}{k}\left[1 - e^{-k(T-t)}\right].$$

- Cap/Floor/Swaption pricing

  There are analytical formulas for cap/floor/swaption prices, see e.g. Brigo & Mercurio (2006).

Short rate models: shifted Vasicek [1]

- Dynamics
  The short rate dynamics in the shifted-Vasicek model [*] is given by

$$\begin{cases} r(t) = x(t) + \varphi(t), \\ dx(t) = -kx(t)dt + \sigma dW^Q(t), \\ x(0) = 0,\ \varphi(0) = \varphi_0, \\ k, \sigma \in \mathbb{R}^+, \varphi_0 \in \mathbb{R}, \end{cases}$$

  where $\varphi(t)$ = shift, deterministic function of time.
  Differentiating both sides we obtain

$$\begin{cases} dr(t) = k\left[\vartheta(t) - r(t)\right] dt + \sigma dW^Q(t), \\ \vartheta(t) := \frac{1}{k}\frac{\partial\varphi(t)}{\partial t} + \varphi(t), \\ r(0) = \varphi_0. \end{cases}$$

  Hence, the shifted-Vasicek model includes the Hull-White model.

- Remarks: the same remarks for Hull-White apply to shifted Vasicek.

[*] see Brigo & Mercurio, 2006, eq. 3.38.

Short rate models: shifted Vasicek [2]

- SDE integration

  Just substitute $r(t) = x(t) + \varphi(t)$ in the Hull-White SDE solution and obtain

$$x(T) = x(t)\left[1 - kB(t,T,k)\right] + \sigma \int_t^T \left[1 - kB(t,T,k)\right] dW^Q(u),$$

- Mean and variance at time *t* of the future distribution $x(T)$ at time $T > t$ are given by

$$\mathbb{E}_t^Q\left[x(T)\right] = x(t)\left[1 - kB(t,T,k)\right],$$

$$\mathrm{Var}_t^Q\left[x(T)\right] = \sigma^2 B(t,T,2k).$$

- ZCB price

  We obtain a modified Vasicek formula as follows

$$P(t,T) = \mathbb{E}_t^Q\left[D(t,T)\right] = A(t,T)e^{-B(t,T,k)x(t)},$$

$$A(t,T) := \frac{P^{Mkt}(0,T)}{P^{Mkt}(0,t)} \exp\left\{\frac{1}{2}\left[V(t,T) + V(0,t) - V(0,T)\right]\right\},$$

$$V(t,T) := \frac{\sigma^2}{k^2}\left[(T-t) - B(t,T,k) - \frac{1}{2}kB^2(t,T,2k)\right].$$

Short rate models: shifted Vasicek [3]

- Forward measure

  Moving to the T-forward measure Q_T associated to the numeraire $P(t,T)$, we obtain

$$dx(t) = -\left[kx(t) + \sigma^2 B(t,T,k)\right] dt + \sigma dW^T(t),$$

$$x(t) = x(s)\left[1 - kB(s,t,k)\right] + M^T(s,t) + \sigma \int_s^t \left[1 - kB(u,t,k)\right] dW^{Q^T}(u),$$

$$M^T(s,t) = \frac{\sigma^2}{2k}\left[2B(s,t,k) + B(t,T,k) - B(t-s,T-s,k)\right].$$

  See Brigo & Mercurio (2006).

- Calibration: volatilities

  The mean reversion speed k and the instantaneous volatility σ can be calibrated using market prices of ATM Caps/Floors or Swaptions.

Short rate models: limits of single factor models [1]

- **Correlation**

  1-factor models discussed so far allows the ZCB pricing formula and the affine transformation,

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

$$R(t,T) = -\frac{\ln A(t,T)}{T-t} + \frac{B(t,T,k)}{T-t} r(t) := a(t,T) + b(t,T) r(t),$$

  showing that the entire term structure of interest rates depends on one single (stochastic) short rate.

  The correlation between two zero coupon rates $R(t,T_1), R(t,T_2)$ is given by

$$corr\left\{R(t,T_1); R(t,T_2)\right\} = corr\left\{a(t,T_1) + b(t,T_1)r(t); a(t,T_2) + b(t,T_2)r(t)\right\} = 1.$$

  Thus, when the initial point of the yield curve (the short rate $r(t)$) changes, the shock propagates rigidly to all maturities *T*.

  Such feature makes 1-factor models suited to price interest rate products depending on one single pillar ot the term structure.

Short rate models: limits of single factor models [2]

- **Principal component analysis**
  Principal component analysis shows that two components already explain more than 80% of the historical variance of interest rates. Thus, just a few factors are sufficient to guarantee a good description of the dynamics of the yield curve.

- **Smile**
  Multi-factor models allow for a humped volatility shape of forward rates. With time dependent parameters, a term structure of volatility can be calibrated. Anyway, Smile cannot be calibrated.

- **Multi Curve**
  Although the original Models were developed in a single curve framework, multi curve extension is possible (next slides)

11: Term Structure Modelling
Short rate models: multi-factor Gaussian GN++

- Dynamics

  The short rate dynamics in the GN++ model [*] is given by

$$\begin{cases} r(t) = \sum_{i=1}^{N} x_i(t) + \varphi(t), \\ dx_i(t) = -k_i x_i(t)dt + \sigma_i(t)dW_i^Q(t), \\ dW_i^Q(t)dW_j^Q(t) = \rho_{ij}dt, \\ x_i(0) = 0 \; \forall\, i = 1, \cdots, N, \;\; \varphi(0) = \varphi_0, \\ k_i \in \mathbb{R}^+, \;\; \rho_{ij} \in [-1; 1]\,, \forall\, i, j = 1, \cdots, N, \varphi_0 \in \mathbb{R}, \end{cases}$$

  where $\phi(t), \sigma(t) =$ deterministic functions of time.

- Remarks
  - The number of stochastic factors is N.
  - For *N=1* we recover the shifted Vasicek model.
  - The number of parameters is infinite.
  - If $\varphi(t)$ is, e.g. piecewise constant with M pillars, the number of parameters is $M + 2N + \frac{1}{2}N(N-1)$.

[*] see Brigo & Mercurio, 2006, sec. 4.2 for the G2++ version.

Short rate models: multi-curve extension [1]

In the multi-curve world, we must typically evaluate at time t floating coupons occurring at future time T indexed to a generic Libor rate of type x, $L_x(T_1, T_2)$, fixed at time T_1^{fix}, referred to the time interval $[T_1, T_2]$, with $t < T_1^{fix} < T_1 < T_2$, and $T \geq T_1^{fix}$, such that

$$V(t;\boldsymbol{T}) = \mathbb{E}_t^Q\left[D_d(t,T)L_x(T_1,T_2)\right](T_2 - T_1),$$

$$D_d(t,T) = e^{-\int_t^T r_d(u)du},$$

$$L_x(T_1,T_2) = \frac{1}{(T_2 - T_1)}\left[\frac{1}{P_x(T_1,T_2)} - 1\right],$$

$$P_x(T_1,T_2) = \mathbb{E}_{T_1}^Q\left[D_x(T_1,T_2)\right] = \mathbb{E}_{T_1}^Q\left[e^{-\int_{T_1}^{T_2} r_x(u)du}\right].$$

What is the connection between the discount and underlying short rates, r_d and r_x, respectively ?

11: Term Structure Modelling
Short rate models: multi-curve extension [2]

We remind that $P_x(t,T)$ is a risky bond, issued by a risky (defaultable) average Libor issuer *x*, such that

$$P_x(t,T) = \mathbb{E}_t^Q\left[D_x(t,T)\right] = \mathbb{E}_t^Q\left[D_d(t,T)\mathbf{1}_{[\tau_x(t)>T]}\right],$$

where $\tau_x(t) > t$ is the (stochastic) default time of the issuer *x* as seen from time *t* (we set zero recovery for simplicity).

Assuming that the default time $\tau_x(t)$ is modelled as a Poisson jump process independent of the rate process, we obtain

$$\begin{aligned} P_x(t,T) &= \mathbb{E}_t^Q\left[D_d(t,T)\right]\mathbb{E}_t^Q\left[\mathbf{1}_{[\tau_x(t)>T]}\right] \\ &= \mathbb{E}_t^Q\left[e^{-\int_t^T [r_d(u)+\lambda_x(u)]du}\right], \end{aligned}$$

$$\mathbb{E}_t^Q\left[\mathbf{1}_{[\tau_x(t)>T]}\right] = \mathbb{Q}\left[\tau_x(t) > T\right] = e^{-\int_t^T \lambda_x(u)du},$$

where λ is the stochastic default intensity and $\mathbb{Q}[\tau_x(t) > T]$ is the risk neutral survival probability of the issuer *x* in the time interval $[t,T]$.

Short rate models: multi-curve extension [3]

Thus we have the following financial interpretation of the short rates, r_d and r_x, as

$$e^{-\int_t^T r_x(u)du} = e^{-\int_t^T [r_d(u)+\lambda_x(u)]du},$$

$$r_x(t) = r_d(t) + \lambda_x(t).$$

Actually the default event of the average Libor issuer is never observed, because the Libor panel is made of single defaultable issuers but it is dynamically renewed to alive issuers.

Short rate models: multi-curve extension [4]

- Deterministic basis

  The simplest modeling approach is to assume a deterministic basis, such that the two short rates r_d and r_x are given by

$$\begin{aligned} r_d(t) &= x_d(t) + \varphi_d(t), \\ r_x(t) &= x_d(t) + \varphi_x(t), \\ r_x(t) - r_d(t) &= \varphi_x(t) - \varphi_d(t). \end{aligned}$$

Short rate basis, deterministic

  Notice that we do not specify the dynamics of $x_d(t)$.

- The generic floating coupon is given, choosing $T = T_2$ for simplicity, by

$$\begin{aligned} V(t;\boldsymbol{T}) &= \mathbb{E}_t^Q\left[D_d(t,T_2)L_x(T_1,T_2)\right](T_2 - T_1), \\ &= \mathbb{E}_t^Q\left[D_d(t,T_2)\left(\frac{1}{\mathbb{E}_{T_1}^Q\left[D_x(T_1,T_2)\right]} - 1\right)\right] = \cdots \end{aligned}$$

Short rate models: multi-curve extension [5]

$$\cdots = \mathbb{E}_t^Q\left[\frac{D_d(t,T_2)}{\mathbb{E}_{T_1}^Q\left[D_x(T_1,T_2)\right]}\right] - \mathbb{E}_t^Q\left[D_d(t,T_2)\right]$$

$$= \mathbb{E}_t^Q\left[\frac{e^{-\int_t^{T_2}[x_d(u)+\varphi_d(u)]du}}{\mathbb{E}_{T_1}^Q\left[e^{-\int_{T_1}^{T_2}[x_d(u)+\varphi_x(u)]du}\right]}\right] - P_d(t,T_2)$$

$$= \frac{\phi_d(t,T_2)}{\phi_x(T_1,T_2)}\mathbb{E}_t^Q\left[\frac{e^{-\int_t^{T_2}x_d(u)du}}{\mathbb{E}_{T_1}^Q\left[e^{-\int_{T_1}^{T_2}x_d(u)du}\right]}\right] - P_d(t,T_2),$$

where

$$\phi(t,T) := e^{-\int_t^T \varphi(u)du}.$$

11: Term Structure Modelling

Short rate models: multi-curve extension [6]

Using the tower rule we obtain

$$
\begin{aligned}
V(t;\boldsymbol{T}) &= \frac{\phi_d(t,T_2)}{\phi_x(T_1,T_2)}\mathbb{E}_t^Q\left[\frac{e^{-\int_t^{T_2} x_d(u)du}}{\mathbb{E}_{T_1}^Q\left[e^{-\int_{T_1}^{T_2} x_d(u)du}\right]}\right] - P_d(t,T_2)\\
&= \frac{\phi_d(t,T_2)}{\phi_x(T_1,T_2)}\mathbb{E}_t^Q\left[\frac{e^{-\int_t^{T_1} x_d(u)du}\cancel{\mathbb{E}_{T_1}^Q\left[e^{-\int_{T_1}^{T_2} x_d(u)du}\right]}}{\cancel{\mathbb{E}_{T_1}^Q\left[e^{-\int_{T_1}^{T_2} x_d(u)du}\right]}}\right] - P_d(t,T_2)\\
&= \frac{\phi_d(t,T_2)}{\phi_x(T_1,T_2)}\mathbb{E}_t^Q\left[e^{-\int_t^{T_1} x_d(u)du}\right] - P_d(t,T_2)\\
&= \frac{\phi_d(T_1,T_2)}{\phi_x(T_1,T_2)}P_d(t,T_1) - P_d(t,T_2).
\end{aligned}
$$

For $\varphi_x = \varphi_d$ we recover the single-curve result

$$V(t;\boldsymbol{T}) = P(t,T_1) - P(t,T_2).$$

- Stochastic basis

  Another possible modeling approach is to assume a stochastic additive basis, such that the two short rates r_d and r_x are given by

$$
\begin{aligned}
r_d(t) &= x_d(t) + \varphi_d(t), \\
r_x(t) &= x_d(t) + s(t) + \varphi_x(t), \\
r_x(t) - r_d(t) &= s(t) + \varphi_x(t) - \varphi_d(t), \\
dW_d^Q(t) dW_s^Q(t) &= 0.
\end{aligned}
$$

  Short rate basis, stochastic

  Notice that we do not specify the dynamics of $x_d(t)$ and $s(t)$, but only that they are uncorrelated.

- The generic floating coupon is given by

$$
\begin{aligned}
V(t; \boldsymbol{T}) &= \mathbb{E}_t^Q \left[D_d(t, T_2) L_x(T_1, T_2) \right] (T_2 - T_1), \\
&= \mathbb{E}_t^Q \left[\frac{D_d(t, T_2)}{\mathbb{E}_{T_1}^Q \left[D_x(T_1, T_2) \right]} \right] - P_d(t, T_2) \cdots
\end{aligned}
$$

Short rate models: multi-curve extension [8]

$$\cdots = \mathbb{E}_t^Q \left[\frac{e^{-\int_t^{T_2}[x_d(u)+\varphi_d(u)]du}}{\mathbb{E}_{T_1}^Q\left[e^{-\int_{T_1}^{T_2}[x_d(u)+s(u)+\varphi_x(u)]du}\right]} \right] - P_d(t,T_2)$$

$$= \frac{\phi_d(t,T_2)}{\phi_x(T_1,T_2)} \mathbb{E}_t^Q \left[\frac{e^{-\int_t^{T_2} x_d(u)du}}{\mathbb{E}_{T_1}^Q\left[e^{-\int_{T_1}^{T_2}[x_d(u)+s(u)]du}\right]} \right] - P_d(t,T_2)$$

$$= \frac{\phi_d(t,T_2)}{\phi_x(T_1,T_2)} \mathbb{E}_t^Q \left[\frac{e^{-\int_t^{T_1} x_d(u)du}\, \cancel{\mathbb{E}_{T_1}^Q\left[e^{-\int_{T_1}^{T_2} x_d(u)du}\right]}}{\cancel{\mathbb{E}_{T_1}^Q\left[e^{-\int_{T_1}^{T_2} x_d(u)du}\right]}\, \mathbb{E}_{T_1}^Q\left[e^{-\int_{T_1}^{T_2} s(u)du}\right]} \right] - P_d(t,T_2)$$

$$= \frac{\phi_d(t,T_2)}{\phi_x(T_1,T_2)} \frac{\mathbb{E}_t^Q\left[e^{-\int_t^{T_1} x_d(u)du}\right]}{\mathbb{E}_t^Q\left[P_s(T_1,T_2)\right]} - P_d(t,T_2) = \cdots$$

Processes $x_d(t)$ and $s(t)$ are uncorrelated

$$\cdots = \frac{\phi_d(T_1, T_2)}{\phi_x(T_1, T_2)\mathbb{E}_t^Q\left[P_s(T_1, T_2)\right]} P_d(t, T_1) - P_d(t, T_2).$$

If $s(t)$ is an affine process we have

$$V(t; \boldsymbol{T}) = \frac{P_d(t, T_1)}{\phi_x(T_1, T_2) A_s(T_1, T_2)\mathbb{E}_t^Q\left[e^{-B_s(T_1, T_2, k_s)s(T_1)}\right]} - P_d(t, T_2).$$

For $s(t) = 0$ we recover the previous result for deterministic basis.

Short rate models: Monte Carlo simulation [1]

Given a generic derivative with payoff at maturity $\Pi(T, r(T))$, we may compute the price using a Monte Carlo simulation.

- Under the risk neutral measure *Q*,

$$\Pi(t) = \mathbb{E}_t^Q \left[D(t,T)\Pi(r(T)) \right],$$

$$D(t,T) = \exp\left[-\int_t^T r(u)du \right],$$

  the calculation of the integral in the stochastic discount factor requires the simulation of the short rate $r(u)$ with a short time step in the interval $[t, T]$.

- Under the T-forward measure Q_T,

$$\Pi(t) = P(t,T)\mathbb{E}_t^{Q_T} \left[\Pi(r(T)) \right],$$

  instead, we may exploit the analytical ZCB pricing formulas, and simulate the short rate $r(T)$ directly at time *T*. As a consequence, typically it is preferred to use T-forward Measure, especially whether large-steps are possible

Short rate models: Monte Carlo simulation [2]

- Monte Carlo simulation under the T-forward measure Q_T
- Choose a T larger than Payoff schedule (usually the last payment or fixing date)
- identify the dates $\{T_i\}$ necessary to compute the payoff (fixing dates and payment dates)
- sample the short rate value at each T_i
  - if the model is tractable, and large steps simulation are possible, only $\{T_i\}$ can be simulated
  - if the model needs short step simulation (e.g. Cox-Ingersoll-Ross) a step by step simulation is in order (more time consuming)
- At each fixing date compute the discount factors from the short rates
- compute the Payoff $Payoff(T_p)$ (path-wise or early exercise feature can be taken into account) for each Payment date T_p
- compute the Numeraire $P\left(T_p, T\right)$
- compute the Numeraire rebased payoff at each payment date and for each MC scenario
- Take the average over N Scenarios

$$PV(t) = P(t,T)\frac{1}{N}\sum_{j}^{N}\frac{Payoff(T_p)}{P(T_p,T)}$$

Example: path dependent payoffs

- $Payoff\left(T_p\right) = max(Libor6M(T_2) - Libor6M(T_1), 0)$
- three dates, 2 Fixing Dates T_2 and T_1 and one payment date T_p
- choose T as T_p (which is larger than the Fixing Dates)
- Loop on MC Scenarios
  - need to sample the short rate at those three dates
  - compute the relevant Discount Factors
  - $P\left(T_p, T\right) = P\left(T_p, T_p\right) = 1$
  - Discount factors needed to compute Libor6M at the fixing dates, i.e. $P(T_1, T_1 + 6M)$ and $P(T_2, T_2 + 6M)$
  - Compute the Libors
  - Evaluate the Payoff under each MC Scenario
- Take the average over MC scenarios
- Discount to valuation time t to obtain $PV(t)$

Example: early Exercise Payoff - Bermudan Swaption

Early Exercise Payoffs are evaluated via recursive methods

- Last Exercise Date: T_N^{Ex}
- Compute the Payoff at the last exercise date (i.e. Swaption Payoff on Swap with tenor τ)

$$Payoff\,(T_N^{Ex}) = \max(R_{6M}(T_N^{Ex},\tau),0)\,Annuity$$

- Compute the Numeraire $P(T_N^{Ex},T)$
- Evaluate the Bermudan on the previous Exercise Date T_{N-1}^{Ex} by comparing
  - Exercise Value $Payoff\,(T_{N-1}^{Ex}) = \max(R_{6M}(T_N^{Ex},\tau+1),0)\,Annuity$
  - Continuation Value, estimated as the Payoff at T_N^{Ex} rescaled by the Numeraire

$$V_{cont}(T_{N-1}^{Ex}) = \frac{Payoff\,(T_N^{Ex})\,P(T_{N-1}^{Ex},T)}{P(T_N^{Ex},T)}$$

- Bermudan Value at T_{N-1}^{Ex} is $V(T_{N-1}^{Ex}) = \max(V_{cont}(T_{N-1}^{Ex}), Payoff\,(T_{N-1}^{Ex}))$
- Loop Recursively until the first Exercise Date and discount back to zero.

Example: early Exercise Payoff - Bermudan Swaption (cont'd)

Within Monte Carlo Simulation, recursive approach is slightly more cumbersome

- Simulate short rates for all the relevant dates (Exercise Dates)
- Compute the Payoff at the last exercise date
- Compute the Numeraire
- Evaluate the Bermudan on the previous Exercise Date by applying the algorithm described in the previous slide taking into account the maximum between holding and exercise value
  - Determine the exercise condition (or exercise boundary) by approximating the continuation value $V_{Cont}(T_{N-1}^{Ex})$ by a polynomial of state variables (in our case the short rate is the natural candidate). A calibration of coefficient is necessary.
  - Once calibrated the Polynomial coefficients, one can efficiently evaluate if, for a simulated value of r_t is more efficient to hold or to exercise the contract (Longstaff-Schwarz or Least Squares Method)
  - Typically, a first simulation is run to calibrate the coefficient with a small number of scenarios, then a larger N simulation is run to price the contract.
- Perform the Polynomial calibration and PV estimation recursively, to price the Bermudan

Short rate models: problems [1]

- **SDE 1**: proof the general SDE solution for linear SDEs

$$\begin{cases} dr(t) = \left[\alpha(t)r(t) + \beta(t)\right] dt + \sigma(t) dW^Q(t), \\ r(0) = r_0, \end{cases}$$

$$r(T) = r(t)D_\beta^{-1}(t,T) + \int_t^T D_\beta^{-1}(u,T)\alpha(u)du + \int_t^T D_\beta^{-1}(u,T)\sigma(u)dW^Q(u),$$

$$\mathbb{E}_t^Q\left[r(T)\right] = r(t)D_\beta^{-1}(t,T) + \int_t^T D_\beta^{-1}(u,T)\alpha(u)du,$$

$$\mathrm{Var}_t^Q\left[r(T)\right] = \int_t^T D_\beta^{-2}(u,T)\sigma^2(u)dW^Q(u).$$

Short rate models: problems [2]

- **SDE 2**: proof the SDE solution for Vasiceck Model

$$\begin{cases} dr(t) = k\left[\theta - r(t)\right]dt + \sigma dW^Q(t), \\ r(0) = r_0, \\ k, \sigma \in \mathbb{R}^+;\ \theta, r_0 \in \mathbb{R}, \end{cases}$$

$$r(T) = r(t) + k\left[\theta - r(t)\right]B(t,T,k) + \sigma \int_t^T \left[1 - kB(t,T,k)\right]dW^Q(u),$$

$$\mathbb{E}_t^Q\left[r(T)\right] = r(t) + k\left[\theta - r(t)\right]B(t,T,k),$$

$$\mathrm{Var}_t^Q\left[r(T)\right] = \sigma^2 B(t,T,2k).$$

Short rate models: problems [3]

- **SDE 3**: proof the SDE solution for Hull-White Model

$$\begin{cases} dr(t) = k\left[\vartheta(t) - r(t)\right]dt + \sigma dW^Q(t), \\ r(0) = r_0, \\ k, \sigma \in \mathbb{R}^+;\ r_0 \in \mathbb{R}, \end{cases},$$

$$\vartheta(t) = k\left[\frac{\partial f^M(t,T)}{\partial T} + kf^M(t,T) + \sigma^2 B(t,T,2k)\right],$$

$$r(T) = \varphi(T) + \left[r(t) - \varphi(t)\right]\left[1 - kB(t,T,k)\right] + \sigma\int_t^T \left[1 - kB(t,T,k)\right]dW^Q(u),$$

$$\mathbb{E}_t^Q\left[r(T)\right] = \varphi(T) + \left[r(t) - \varphi(t)\right]\left[1 - kB(t,T,k)\right],$$

$$\mathrm{Var}_t^Q\left[r(T)\right] = \sigma^2 B(t,T,2k).$$

- **PCA**: show that just a few principal components already explain much of the historical variance of interest rates. Use historical series of swap market data. Deliverable: spreadsheet with charts and comments. Hint: look at John Hull's website `http://www-2.rotman.utoronto.ca/~hull`

11: Term Structure Modelling
Libor Market Models [1]

The Libor Market Model, also known as BGM (Brace, Gatarek, Musiela) model, assumes discrete FRA rates as the fundamental variables. The Swap Market Model (SMM) assumes discrete Swap rates as the fundamental variables.

There are many flavours of LMM/SMM:
- Classic LMM/SMM
- Shifted LMM/SMM
- SABR LMM/SMM
- Stochastic volatility LMM/SMM
- Multi-curve LMM/SMM

Alan Brace

Dariusz Gatarek

Marek Musiela

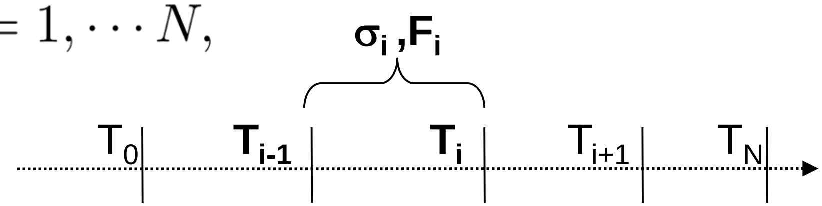
See
- D. Brigo, F. Mercurio, "*Interest Rate Models - Theory and Practice*", 2nd ed., Springer, 2006.
- L. B. G. Andersen, V. V. Piterbarg, "*Interest Rate Modeling*", Atlantic Financial Press, 2010.

11: Term Structure Modelling
Libor Market Models [2]

The classic Libor Market model assumes the discrete FRA rates as the fundamental variables.

- **Dynamics**
  The LMM dynamics in the classic LMM is given by

$$\begin{cases} dF_i(t) = F_i(t)\sigma_i(t)dW_i^{Q_i}(t), \ i = 1, \cdots N, \\ dW_i^{Q_i}(t)dW_j^{Q_i}(t) = \rho_{ij}(t)dt, \\ F_i(0) \in \mathbb{R}, \sigma(t) \in \mathbb{R}^+ \ \forall t. \end{cases}$$



  where
  - $\{T_1, \dots, T_N\}$ = vector of N regular FRA rate dates,
  - $\{\sigma_1(t), \dots, \sigma_N(t)\}$ = vector of time-dependent instantaneous volatilities,
  - $\{W_1^{Q_1}(t), \dots, W_N^{Q_N}(t)\}$ = vector of standard Brownian motions under forward measures $\{Q_1, \dots, Q_N\}$,
  - $\{F_1(0), \dots, F_N(0)\}$ = vector of initial conditions,
  - N = number of FRA rates = number of risk factors.

  Total number of parameters: N volatilities + N(N-1)/2 correlations = N(N+1)/2.

- **Dynamics: change of measure**

  Since we are dealing with the general case of products depending on multiple forward rates, we must model all forward rates under the same T_k forward measure, Q_k. We have

$$dF_i(t) = \mu_i^k(t)dt + \sigma_i(t)F_i(t)dW^{Q_k}(t),$$

$$\mu_i^k(t) = \begin{cases} \sigma_i(t)F_i(t)\sum_{j=k+1}^{i}\frac{\rho_{ij}\sigma_j(t)F_j(t)\tau_j}{1+F_j(t)\tau_j}dt, \textit{if } k < i, t < T_i, \\ 0, \textit{if } k = i, t < T_{i-1}, \\ -\sigma_i(t)F_i(t)\sum_{j=i+1}^{k}\frac{\rho_{ij}\sigma_j(t)F_j(t)\tau_j}{1+F_j(t)\tau_i}dt, \textit{if } k > i, t < T_{i-1}. \end{cases}$$

  Q_N is also called terminal measure.

  Thus when multiple forward rates are modelled jointly, all rates but one have a dynamics that depends also on the level of other forward rates.

  These dynamics are able to capture the no-arbitrage relationships linking together the different parts of the term structure.

Libor Market Models [4]

Regions of the correlation matrix involved in the LMM drift term $\mu_i^k(t)$. Green: $k < i$, red: $k > i$

Step	0.5	Numeraire	6.5

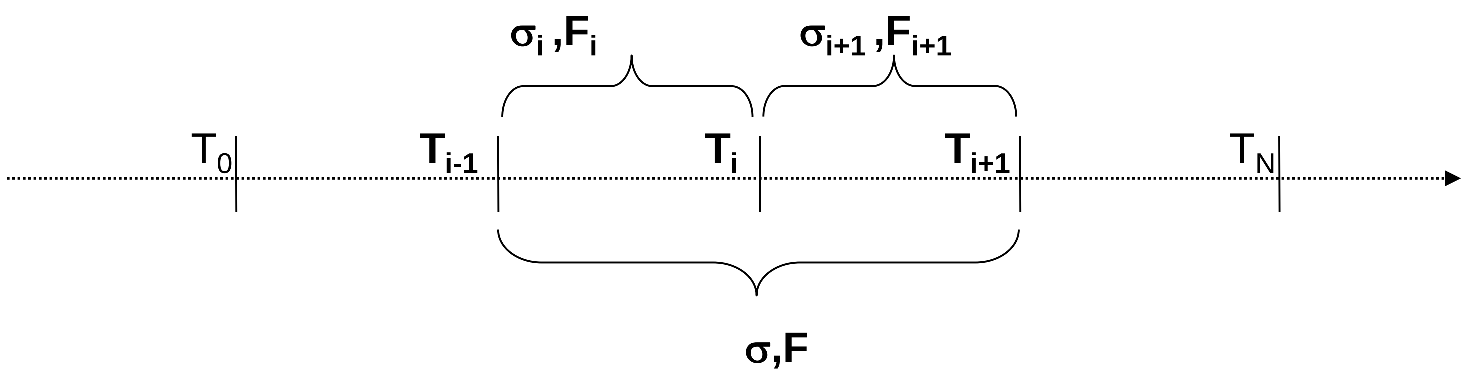
	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
0.5	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0
1.0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0
1.5	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0
2.0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0
2.5	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0
3.0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0
3.5	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0
4.0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	0	0	0	0	0	0	0
4.5	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0
5.0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	0	0	0	0	0	0	0
5.5	0	0	0	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0
6.0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0
6.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7.0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
7.5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
8.0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
8.5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0
9.0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0
9.5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0
10.0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1

Step	0.5	Numeraire	3

	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
0.5	0	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1.0	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1.5	0	0	0	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2.0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2.5	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3.5	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
4.0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
4.5	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0
5.0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
5.5	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0
6.0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0
6.5	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0
7.0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0
7.5	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0
8.0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0
8.5	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0
9.0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0
9.5	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0
10.0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1

11: Term Structure Modelling
Libor Market Models [5]

- **Remark on FRA rate tenor**
  The LMM describes the joint dynamics of FRA rates with the same tenor. Nested FRA rates inherits other dynamics than LMM dynamics.



$$F_i(t) = F(t; T_{i-1}, T_i) = \frac{1}{\tau(T_{i-1}, T_i)} \left[\frac{P(t; T_{i-1})}{P(t; T_i)} - 1 \right],$$

$$F_{i+1}(t) = F(t; T_i, T_{i+1}) = \frac{1}{\tau(T_i, T_{i+1})} \left[\frac{P(t; T_i)}{P(t; T_{i+1})} - 1 \right],$$

$$F(t) := F(t; T_{i-1}, T_{i+1}) = \frac{1}{\tau(T_{i-1}, T_{i+1})} \left[\frac{P(t; T_{i-1})}{P(t; T_{i+1})} - 1 \right].$$

But the long FRA rate $F(t)$, in a single curve setting may be written as follows

$$\begin{aligned} F(t) := F(t; T_{i-1}, T_{i+1}) &= \frac{1}{\tau(T_{i-1}, T_{i+1})}\left[\frac{P(t;T_{i-1})}{P(t;T_{i+1})} - 1\right] \\ &= \frac{1}{\tau(T_{i-1}, T_{i+1})}\left[\frac{P(t;T_{i-1})}{P(t;T_{i})}\frac{P(t;T_{i})}{P(t;T_{i+1})} - 1\right] \\ &= \frac{[1+\tau(T_{i-1},T_i)F_i(t)]\,[1+\tau(T_i,T_{i+1})F_{i+1}(t)] - 1}{\tau(T_{i-1}, T_{i+1})} \\ &\simeq \frac{F_i(t)+F_{i+1}(t)}{2} + \frac{\tau(T_{i-1},T_i)}{2}F_i(t)F_{i+1}(t). \end{aligned}$$

Thus, if $F_i(t), F_{i+1}(t)$ are lognormal w.r.t. their respective Q_i, Q_{i+1} forward measures, associated to $P(t;T_i), P(t;T_{i+1})$ numeraires, $F(t)$ cannot be lognormal w.r.t. the forward measure Q_{i+1}.

Proof: apply Ito’s lemma to $F(t)$.

- **Remark on swap rates**

  Following the previous remark, also swap rates, even in the single curve limit

$$\mathbf{Swap}(t;\mathbf{T},\mathbf{S},K,\omega)=N\omega\left[R_{\alpha\beta}(t)-K\right]A(t),$$

$$R_{\alpha\beta}(t)=\frac{\sum_{i=\alpha+1}^{\beta}P(t,T_i)F_i(t)\tau_i}{\sum_{j=\alpha+1}^{\beta}P(t,S_j)\tau_j}.$$

  (where we have shortened the notation) clearly do not inherit a lognormal dynamics from the underlying FRA rates.

  Notice that the swap rate $R_{\alpha\beta}(t)$ may be written in terms of FRA rates $\{F_1(t),\dots,F_N(t)\}$ as follows

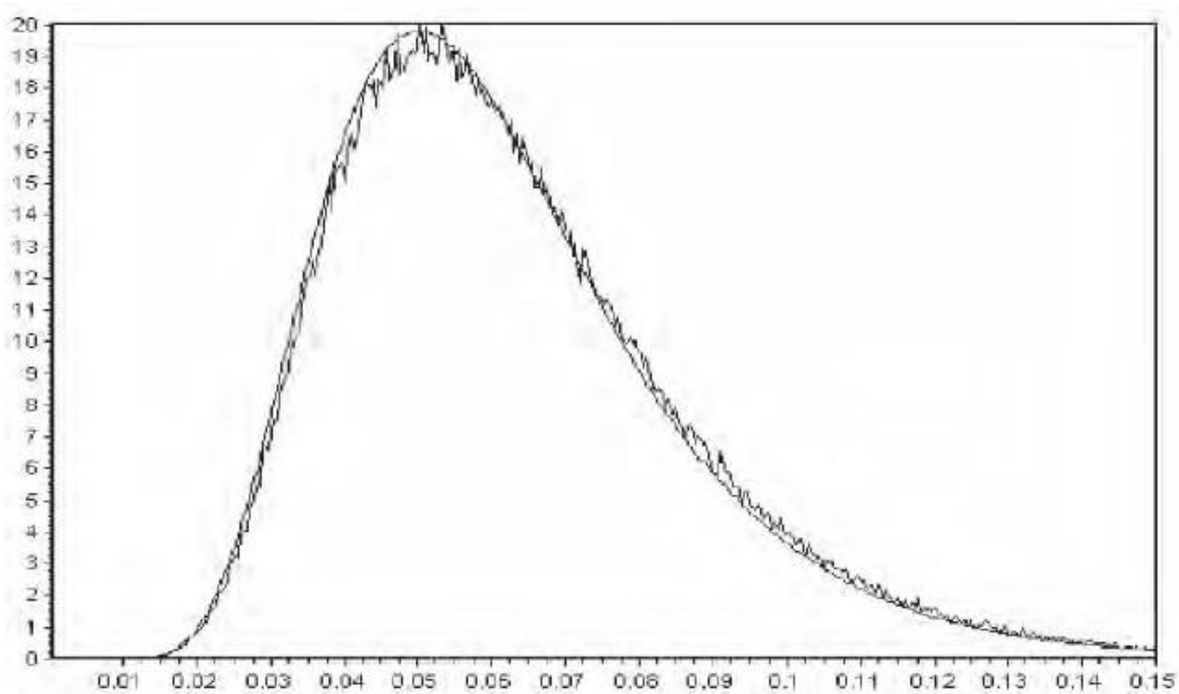
$$R_{\alpha\beta}(t)=\frac{\sum_{i=\alpha+1}^{\beta}P(t,T_i)F_i(t)\tau_i}{\sum_{j=\alpha+1}^{\beta}P(t,S_j)\tau_j}=\sum_{i=\alpha+1}^{\beta}w_i(t)F_i(t),$$

$$w_i(t):=\frac{\tau_iP(t,T_i)}{\sum_{k=\alpha+1}^{\beta}\tau_kP(t,T_k)}=\frac{\tau_i\prod_{j=\alpha+1}^{i}\frac{1}{1+\tau_jF_j(t)}}{\sum_{k=\alpha+1}^{\beta}\tau_k\prod_{j=\alpha+1}^{k}\frac{1}{1+\tau_jF_j(t)}}$$

How to compare the distribution of the swap rate implied by the LMM ?

- Assume the LMM for FRA rates
- Change from forward measure to a given swap measure $Q^{\alpha,\beta}$ (use Girsanov)
- Find the implied dynamics of the swap rate $R_{\alpha\beta}(t)$ (use Ito)
- Generate the distribution of $R_{\alpha\beta}(t)$ numerically (use Monte Carlo)
- Compare with the corresponding lognormal distribution of $R_{\alpha\beta}(t)$ under $Q^{\alpha,\beta}$.

The two distributions are NOT exactly the same, but the difference is very small, at least in standard market conditions (what about negative rates ?)



Source: Brigo and Mercurio (2006), fig. 8.7

Pricing of Caps/Floors

The distributional assumptions for FRA rates in the LMM are consistent with those underlying the market Caplet/Floorlet Black Formula. Hence, under the LMM we price Caps/Floors using the standard Black formula.

Pricing of Swaptions

Under the LMM, swap rates are not strictly lognormal, thus we cannot use, in principle, the standard Black formula for Swaptions. In practice swap rates are quasi-lognormal under the LMM. Thus the Black formula should be a good approximation. In fact, Rebonato and Jackel (1998) found

$$\mathbf{Swaption}(t; T_\alpha, K, \omega) = N \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i) \text{Black}\left[S_{\alpha\beta}(t), K, V_{\alpha\beta}^{LMM}(t; T_\alpha), \omega\right],$$

$$V_{\alpha\beta}^{LMM}(t; T_\alpha) \simeq \sum_{i=\alpha+1}^{\beta} \sum_{j=\alpha+1}^{\beta} \frac{w_i(t) w_j(t) F_i(t) F_j(t)}{S_{\alpha\beta}(t)^2} \rho_{ij} \int_0^{T_\alpha} \sigma_i(t) \sigma_j(t) dt.$$

Freezing

Covariance

- **Calibration**
  - FRA rates

    The initial FRA rates $\{F_1(0), \dots, F_N(0)\}$ parameters in the LMM are exactly calibrated to the term structure of interest rates by choosing the initial condition $F_i(0) = F_i^{Mkt}(0)$, the corresponding FRA rates quoted on the market.
  - Volatilities

    The LMM instantaneous volatilities $\{\sigma_1(t), \dots, \sigma_N(t)\}$ can be easily calibrated to market Caps/Floors volatilities thanks to the Black formula. There are many possible choices for the volatility time function, e.g. piecewise constant.
  - Correlations

    The LMM instantaneous correlation matrix ρ_{ij} can be calibrated to market Swaptions volatilities using the Rebonato-Jackel approximation. Since there are $\frac{1}{2}N(N-1)$ correlation matrix elements, there are many possible choices to reduce the number of parameters, e.g. the Rebonato 3-parameters functional form

$$\rho_{ij} = \rho_\infty + (1-\rho_\infty)e^{-\left[\beta-\alpha\left(\mathrm{Max}(i,j)-1\right)\right]|i-j|}.$$

Libor Market Models: problems

- **Correlation**: plot a 3D chart of the Rebonato 3-parameters functional form for the correlation matrix and discuss it’s financial characteristcs. Deliverable: spreadsheet with charts and comments.

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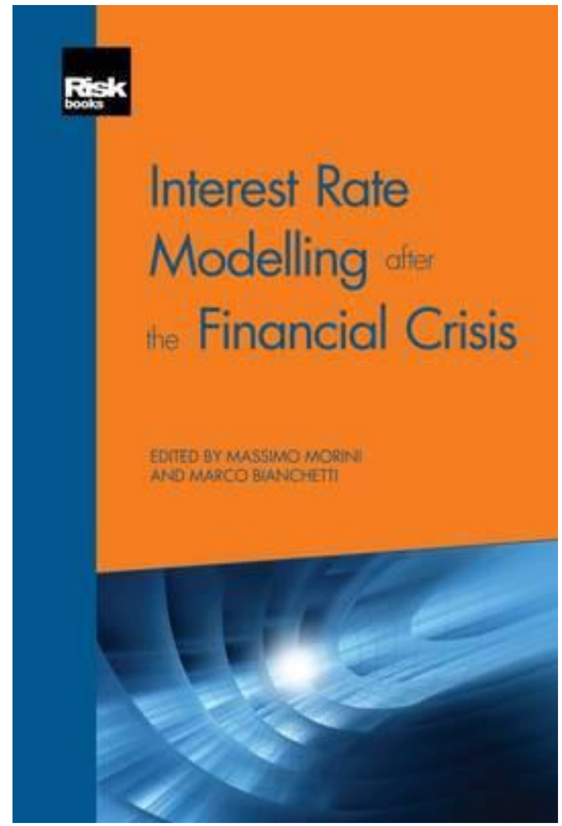

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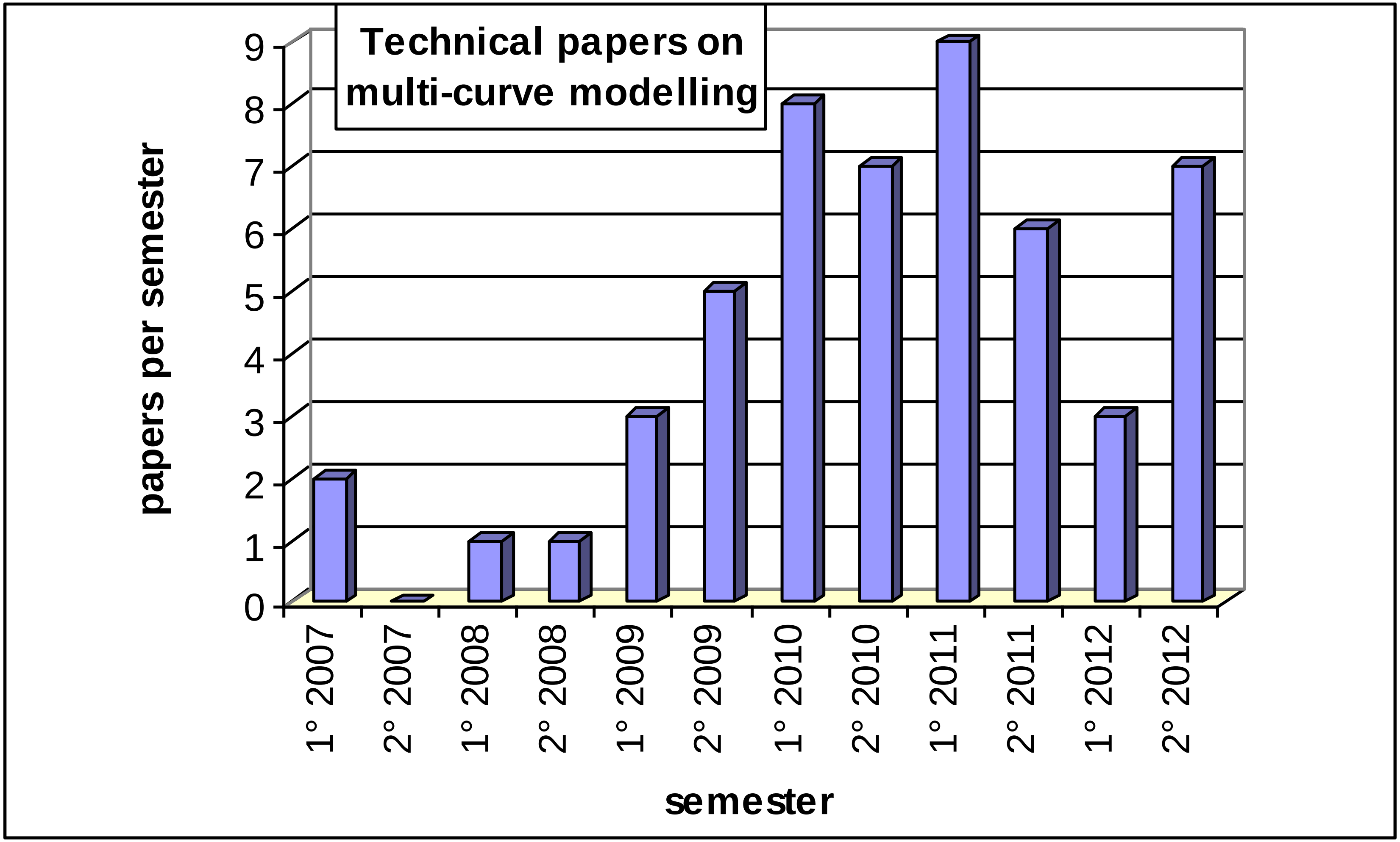

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