Adiabatic gauge potential of quantum integrable and non-integrable systems

Mohit

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1 Introduction

Adiabatic gauge potentials are useful for controlling a quantum system when it's driven externally from one configuration to another. These potentials help us in circumventing standard adiabatic limitations which requires infinitesimally small rates [1, 2, 3]. For example, these potentials can be used for arbitrarily fast annealing protocols and implementing fast dissipationless driving.

The scaling of norm of gauge potential with system's size is quite different for quantum integrable and non-integrable systems. On one hand, for integrable systems , exact gauge potential are supposed to scale like a polynomial in system size. This is due to extensive number of symmetries that exist and as a result, they have a "lot" of degenerate energy levels which comes with their respective "selection rules". This can be easily seen for Transverse Ising model whose analytical expression of gauge potential is known in literature.

On the other hand, for non-integrable systems, using Eigenstate Thermalization Hypothesis (ETH)[4] , we can show that norm of exact gauge potential scale exponentially in system size. This can be verified numerically using exact diagonalization on spin system upto size $L = 15$.

We can exploit this property to distinguish between quantum integrable and non-integrable system. Our method should be better than conventional method (energy level distribution) used in literature for this purpose because unlike the conventional method, we don't have to worry about removing symmetry.

2 Adiabatic gauge potential

2.1 Introduction by example

$$H_0 = \frac{p^2}{2m} + V(x - \lambda(t)) \tag{1}$$

$$H_{CD} = H_0 + \dot{\lambda} A_\lambda$$

where $A_\lambda = p$. Include a picture of glass of water being transported from Dries's PNAS paper. Question: if you have exact gauge potential, does all the excitations during intermediate times is zero.

2.2 Formal introduction

Adiabatic gauge potentials are the generators of a unitary transformation which diagonalize the instantaneous Hamiltonian, attempting to leave its eigenbasis invariant as the parameter is changed. These adiabatic gauge potentials generate *non-adiabatic* corrections to Hamiltonian in the moving basis (λ -dependent basis).

This is something from Anatoli's lecture notes [5]– "an adiabatic basis is a family of adiabatically connected eigenstates, i.e., eigenstates related to a particular initial basis by adiabatic (infinitesimally slow) evolution of the parameter λ. For example, if two levels cross they will exchange order energetically but the adiabatic connection will be non-singular."

$H(\lambda)|n(\lambda)\rangle = E_n(\lambda)|n(\lambda)$. Let's derive diagonal and off-diagonal elements.

- **n-th diagonal element:** $A^n_\lambda = \langle n|A_\lambda|n\rangle = i\hbar\langle n|\partial_\lambda|n\rangle$
- **off- diagonal element:** We use the identity $\langle m|H(\lambda)|n\rangle = 0 \quad, n \neq m$ and then differentiate with respect to λ to obtain:

$$\boxed{\langle m|A_\lambda|n\rangle = -i\hbar\frac{\langle m|\partial_\lambda H|n\rangle}{E_m - E_n}} \tag{2}$$

where both energies (E_m, E_n) and eigenvectors $(|m\rangle, |n\rangle)$ depend on λ.

2.3 Eigenstate Thermalization Hypothesis

Eigenstate Thermalization Hypothesis(ETH) gives us an ansatz for matrix elements of observables in the basis of energy eigenstates [4]:

$$O_{mn} = O(\bar{E})\delta_{mn} + e^{-S(\bar{E})/2} f_O(\bar{E}, \omega) R_{mn} \tag{3}$$

where $\bar{E} = (E_m + E_n)/2, \omega = E_n - E_m$ and $S(E)$ is the thermodynamic entropy at energy E.

We note that it's applicable only for few-body operators of a non-integrable Hamiltonian. By few-body, we mean n body observables with $n \ll N$, where N is the total number of spins, particles, etc. For example, projection operator to eigenstates of many body Hamiltonian $\hat{P}_\alpha = |\Psi_\alpha\rangle\langle\Psi_\alpha|$ don't satisfy ETH and it also doesn't satisfy predictions of statistical mechanics. Why is that? We expect that microcanonical averaging should be equivalent to canonical averaging:

$$\langle\Psi_\alpha|O|\Psi_\alpha\rangle = \frac{\mathrm{Tr}\, O e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}} \tag{4}$$

We can see $O = \hat{P}_\alpha$ doesn't satisfy the above equation (since left hand side is one and the trace of right hand side can be computed in energy basis to find that it's not one). Projection operator is non-local in real space, and we argue that this is the reason it doesn't satisfy ETH and is not experimentally measurable.

2.3.1 Information about $f_O(\bar{E}, \omega)$

$$|f_O(\bar{E}, \omega)| = \begin{cases} e^{-\omega T} & (\omega \gg T), \\ \dfrac{\sqrt{L}}{\omega^2 + \mu_T^2} & (\omega \ll T) \end{cases} \tag{5}$$

where $\mu_T^2 \sim \dfrac{1}{L^2}$ [6, 7, 4]

3 Norm of adiabatic gauge potential

Let's compute the norm by noting that A_λ has only off-diagonal elements in energy basis in our gauge choice:

$$||A_\lambda||^2 = \mathrm{Tr}\, A_\lambda^2 \tag{6}$$

$$= \sum_n \langle n|A_\lambda^2|n\rangle \tag{7}$$

$$= \sum_n \langle n|A_\lambda|n\rangle^2 + \sum_n \sum_{m\neq n} |\langle m|A_\lambda|n\rangle|^2 \tag{8}$$

$$= \sum_n \sum_{m\neq n} |\langle m|A_\lambda|n\rangle|^2 \tag{9}$$

$$= \hbar^2 \sum_n \sum_{m\neq n} \frac{|\langle m|\partial_\lambda H|n\rangle|^2}{(E_m - E_n)^2} \tag{10}$$

Hence, in general, for both integrable and non-integrable systems we have:

$$\boxed{||A_\lambda||^2 = \hbar^2 \sum_n \sum_{m\neq n} \frac{|\langle m|\partial_\lambda H|n\rangle|^2}{(E_m - E_n)^2}} \tag{11}$$

4 Integrable model

Our goal is to study a integrable model, which is called **Transverse Field Ising model**. It shows quantum phase transition between ferromagnetic and paramagnetic phases. Moreover, it satisfies Ising symmetry $G = \Pi_i \sigma_i^z$ since $[H, G] = 0$, where H is the Hamiltonian. This model can be written in terms of non-interacting spinless fermions $(c_i, c_i^\dagger)$ using Jordan- Wigner transformation.

It's Hamiltonian in spin basis is given by:

$$H = -J \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x - \lambda \sum_j \sigma_j^z \tag{12}$$

where we have chosen periodic boundary conditions and λ is externally-controlled transverse magnetic field.

This model can be written in terms of non-interacting spinless fermions $(c_i, c_i^\dagger)$ using Jordan-Wigner transformation: $\sigma_i^z \sim 1 - 2c_i^\dagger c_i$ and $\sigma_i^+ \sim \prod_{j<i} \sigma_j^z c_j$. Details can be found elsewhere [8] [1] . Here is what we get after this transformation:

$$\mathcal{H} = \sum_k \psi_k^\dagger H_k \psi_k, \quad H_k = -\begin{bmatrix} \lambda - \cos k & \sin k \\ \sin k & -(\lambda - \cos k) \end{bmatrix} \tag{13}$$

where $\psi_k^\dagger = (c_k^\dagger, c_{-k})$ is Nambu spinor basis. We can write H_k in terms of Pauli sigma matrices:

$$H_k = -(\lambda - \cos k)\sigma_k^z - \sin k \sigma_k^x \tag{14}$$

Now using our regulator method (whose details are not given in this report), we can obtain :

$$\boxed{A_\lambda = \sum_{l=1}^{L-1} \alpha_l O_l \quad \text{where} \quad \alpha_l = -\frac{1}{4L} \sum_k \frac{\sin(k)\sin(lk)}{(\cos k - \lambda)^2 + \sin^2 k}} \tag{15}$$

where O_l is given by

$$O_l = 2i \sum_j (c_j^\dagger c_{j+l}^\dagger - \text{h.c}) = \sum_j (\sigma_j^x \sigma_{j+1}^z \ldots \sigma_{j+l-1}^z \sigma_{j+l}^y + \sigma_j^y \sigma_{j+1}^z \ldots \sigma_{j+l-1}^z \sigma_{j+l}^x) \tag{16}$$

This matches with the result already known in literature [9, 5].

[1] Momentum operator chosen to get real valued Hamiltonian is $c_k = \frac{e^{i\pi/4}}{\sqrt{L}} \sum_j c_j e^{-ikj}$, where k is $n\pi/L$ with $n = 0, 1, 2, \ldots L - 1$

Let's write a first few terms of O_l here:

$$O_{l=1} = \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^y + \sigma_j^y \sigma_{j+1}^x)$$

$$O_{l=2} = \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^y + \sigma_j^y \sigma_{j+1}^z \sigma_{j+2}^x)$$

On computation, we find that with periodic boundary conditions, we get $\mathrm{Tr}\, O_l O_p = \delta_{l,p} 2^{L+1} L$

For large enough system size L, we can compute α_l [5] by computing the sum into an integral and obtain the value of α_l as:

$$\alpha_l = -\frac{1}{8} \begin{cases} \lambda^{l-1} & (\lambda^2 < 1), \\ \lambda^{-l-1} & (\lambda^2 > 1) \end{cases} \tag{17}$$

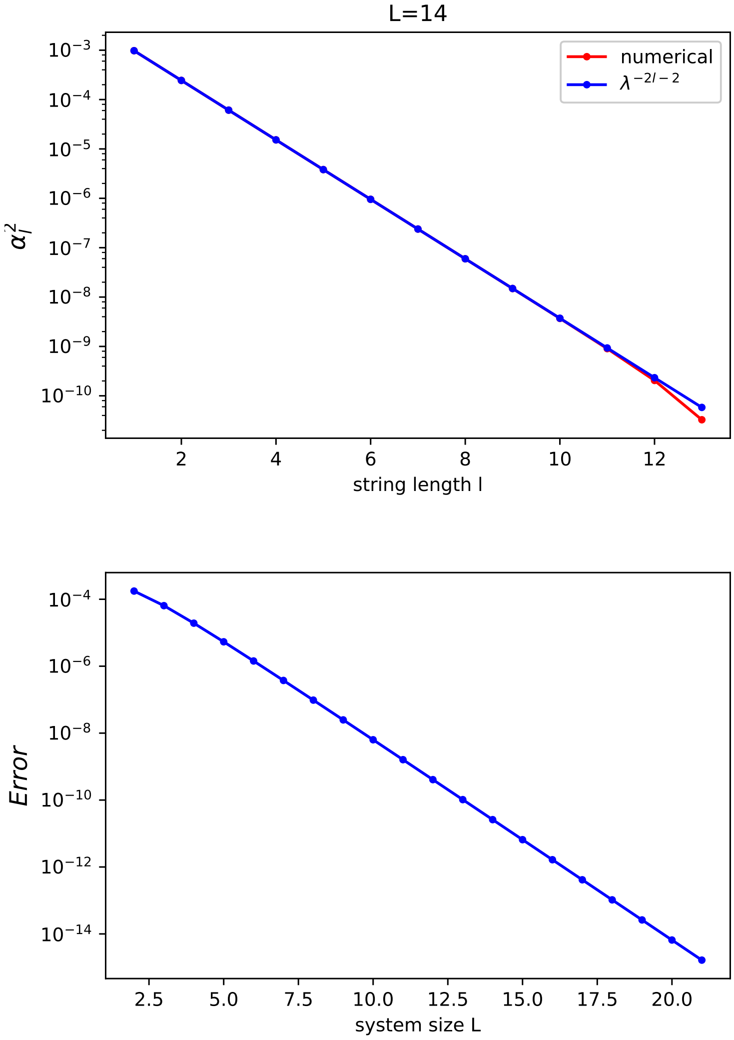

Figure 1: Integrable systems: string length of exact gauge potential as a function of system size

Let's compute norm of gauge potential:

$$||A_\lambda||^2 = \mathrm{Tr}\, A_\lambda^2 \tag{18}$$

$$= \mathrm{Tr} \sum_{l,p} \alpha_p \alpha_l O_l O_p \tag{19}$$

$$= \sum_{l,p} \alpha_p \alpha_l \operatorname{Tr} O_l O_p \tag{20}$$

$$= 2^{L+1} L \sum_{l=1}^{L-1} \alpha_l^2 \tag{21}$$

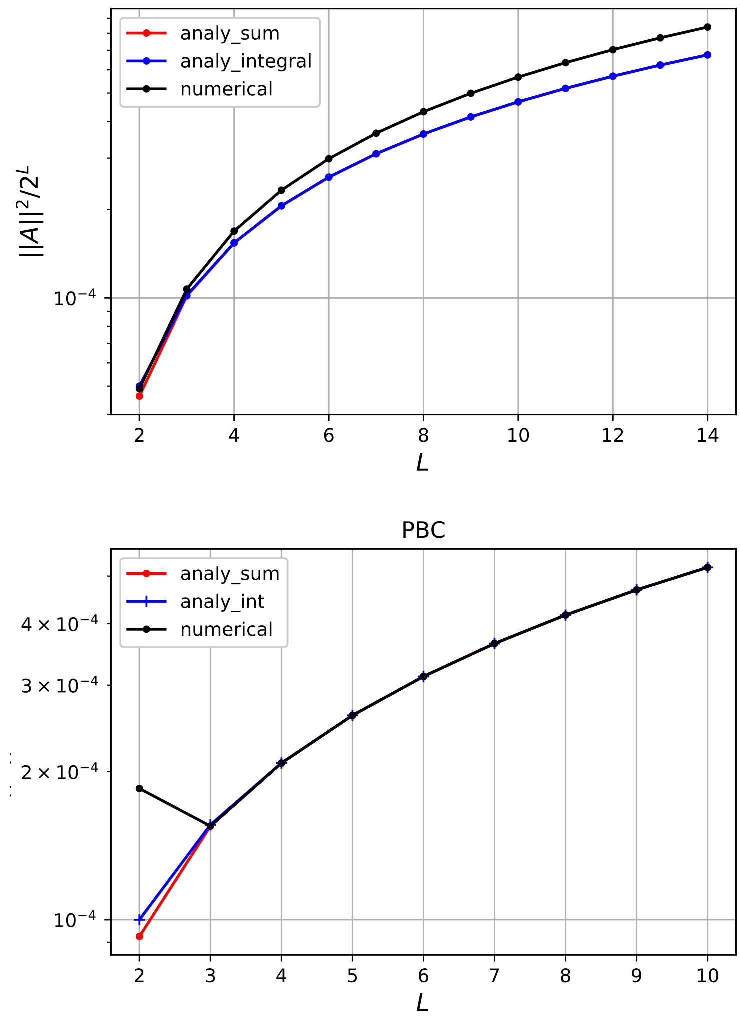

Figure 2: Integrable systems: Norm of exact gauge potential as a function of system size

Now since α_l for large enough L is exponentially suppressed in l, we can argue that

$$\boxed{||A_\lambda||^2/2^L \sim 2L} \tag{22}$$

5 Non-integrable model

If we introduce longitudinal magnetic field in Transverse Ising model, then integrability is broken and we get a non-integrable model. We plan to study both local and global integrability-breaking term.

$$H = -J \sum_j \sigma_j^x (\sigma_{j+1}^x + \sigma_{j-1}^x) - h \sum_j \sigma_j^z - \lambda \sum_j \sigma_j^x \tag{23}$$

In this model, $\partial_\lambda H = -\sum_j \sigma_j^x$ is a global operator.

$$H = -J\sum_j \sigma_j^x(\sigma_{j+1}^x + \sigma_{j-1}^x) - h\sum_j \sigma_j^z - \lambda\sigma_0^x \tag{24}$$

In this model, $\partial_\lambda H = -\sigma_0^x$ is a local operator.

5.1 ETH applied to norm

$\partial_\lambda H$ may or may not be a local operator. We would be studying such non-integrable models in which it is a local operator. Hence, we can apply ETH on the operator $\partial_\lambda H$.

5.1.1 Heuristic argument

$$||A_\lambda||^2 = \hbar^2 \sum_n \sum_{m\neq n} \frac{|\langle m|\partial_\lambda H|n\rangle|^2}{\omega_{mn}^2}$$

where $\omega_{mn} = E_m - E_n$. We would argue that the biggest contribution to norm would come from the smallest ω_{mn} because it's exponentially small in system size. Hence, we find that using ETH for $\partial_\lambda H$:

$$\begin{aligned}
||A_\lambda||^2 &= \hbar^2 \sum_n \sum_{m\neq n} \frac{|\langle m|\partial_\lambda H|n\rangle|^2}{\omega_{mn}^2} \\
&= \hbar^2 \sum_n \sum_{m\neq n} \frac{e^{-S}}{e^{-2S}} \\
&= \hbar^2 \sum_n \sum_{m\neq n} e^{S} \\
&\simeq \hbar^2 2^L e^L
\end{aligned}$$

where we have used the fact that entropy is extensive, i.e. $S \sim L$. Hence, norm averaged over system size is exponential in system size with $\hbar = 1$

$$\boxed{||A_\lambda||^2/2^L \sim e^L} \tag{25}$$

Exponential scaling with system size of gauge potential is due to exponential small eigenvalues. Since these eigenvalues appear in the denominator of gauge potential expression, it's called **small denominator problem** in literature [5].

In Dries's notes, you would find how we are attempting to solve this problem.

5.1.2 Formal calculation

For formal calculation, I would need to introduce a cutoff μ. Otherwise, norm diverges in thermodynamic limit $L \to \infty$, which is clear from above heuristic arguments.

$$\langle n|A_\lambda|m\rangle \quad = \quad \lim_{\mu\to 0}\lim_{L\to\infty} -i\hbar\frac{\langle n|\partial_\lambda H|m\rangle}{(E_n - E_m)^2 + \mu^2}(E_n - E_m) \tag{26}$$

where we have chosen a gauge choice in which diagonal elements are zero in energy basis, i.e. $A_\lambda^{nn} = 0$.

$$||A_\lambda||^2 = \hbar^2 \sum_n ||A_\lambda||_n^2 \tag{27}$$

where $||A_\lambda||_n^2 = \sum_{m\neq n} \dfrac{(E_m - E_n)^2}{((E_m-E_n)^2+\mu^2)^2} |\langle m|\partial_\lambda H|n\rangle|^2$.

Let's simplify this using ETH:

$$\begin{aligned}
||A_\lambda||_n^2 &= \sum_{m\neq n} \frac{(E_m - E_n)^2}{((E_m-E_n)^2+\mu^2)^2} |\langle m|\partial_\lambda H|n\rangle|^2 \\
&= \sum_{m\neq n} \frac{\omega_{nm}^2}{(\omega_{nm}^2+\mu^2)^2} e^{-S(\bar{E})} |f_O(\bar{E},\omega_{nm}) R_{mn}|^2 \\
&= \sum_{m\neq n} \frac{\omega_{nm}^2}{(\omega_{nm}^2+\mu^2)^2} e^{-S(E_n-\omega_{nm}/2)} |f_O(E_n-\omega_{nm}/2,\omega_{nm})|^2 |R_{mn}|^2
\end{aligned}$$

where $\bar{E} = (E_m + E_n)/2 = E_n - \omega/2$, $\omega_{nm} = E_n - E_m$ and $S(E)$ is the thermodynamic entropy at energy E. We would need to convert the sum into integral where we use the fact that function f_O is smooth and fluctuations of $|R_{mn}|^2$ average out in the sum.

$$\sum_{m\neq n} \to \int d\omega \Omega(E_n - \omega) = \int d\omega e^{S(E_n-\omega)} \tag{28}$$

where $\Omega(E_n + \omega)$ is density of states.

$$||A_\lambda||_n^2 = \int d\omega e^{S(E_n-\omega)-S(E_n-\omega/2)} \frac{\omega^2}{(\omega^2+\mu^2)^2} |f_O(E_n-\omega/2,\omega)|^2$$

$S(E_n - \omega) - S(E_n - \omega/2) = -\beta\omega/2 + \dots$ and $f_O(E_n - \omega/2, \omega) = f_O(E_n, \omega) + \dots$ we have

$$||A_\lambda||_n^2 = \int_a^b d\omega e^{-\beta\omega/2} \frac{\omega^2}{(\omega^2+\mu^2)^2} |f_O(E_n,\omega)|^2$$

where a represents the minimum energy difference $E_m - E_n$ in thermodynamic limit (which is $\min\{w_{nm}\}$) and b is the maximum energy difference (for which we have to find m-th state such that we get $\max\{w_{nm}\}$)). $a = e^{-S} \sim e^{-\delta L}$ and $b = \gamma L$, where γ and δ are constants that depend on the details of Hamiltonian.

Let's denote $I = e^{-\beta\omega/2} \dfrac{\omega^2}{(\omega^2+\mu^2)^2}$ and find out how it depends on L. First, we check on upper limit.

$$\lim_{L\to\infty} I(\omega = L) = \lim_{L\to\infty} e^{-\beta L/2} \frac{L^2}{(L^2+\mu^2)^2} \to 0$$

Now on lower limit.

$$\lim_{L\to\infty} I(\omega = e^{-L}) = \lim_{L\to\infty} e^{-\beta e^{-L}/2} \frac{e^{-2L}}{(e^{-2L}+\mu^2)^2} = \lim_{L\to\infty} \frac{e^{-2L}}{(e^{-2L}+\mu^2)^2}$$

$$\lim_{L\to\infty} I(\omega = e^{-L}) = \begin{cases} e^{2L} & (\mu^2 \ll e^{-2L}), \\ \dfrac{e^{-2L}}{\mu^4} & (\mu^2 \gg e^{-2L}) \end{cases} \tag{29}$$

Now, let's compute the norm while assuming $|f_O(E_n,\omega)|^2$ is a constant in ω. Hence, we get:

$$||A_\lambda||_n^2 = |f_O(E_n)|^2 \int_0^\infty d\omega e^{-\beta\omega/2} \frac{\omega^2}{(\omega^2+\mu^2)^2}$$

Let's assume $\beta \ll 1$ (high temperature limit):

$$\begin{aligned} ||A_\lambda||_n^2 &= |f_O(E_n)|^2 \int_0^\infty d\omega\,(1-\beta\omega/2+\dots)\frac{\omega^2}{(\omega^2+\mu^2)^2} \\ &= |f_O(E_n)|^2\left(\frac{\pi}{4\mu}-\frac{\beta}{4}-\frac{\beta}{4}\log(\mu^2+\omega^2)|_0^\infty+\dots\right) \end{aligned}$$

We see that there is a logarithmic divergence for high temperature limit. We also note that there are two limits, in which we find that there is no ultraviolet divergence: $\beta = 0$ limit gives $\pi/4\mu$ and $\beta \to \infty$ limit gives us zero norm. I don't understand why zero temperature limit gives zero norm.

Hence, ETH claims that norm of gauge potential in infinite temperature will be ($\hbar = 1$):

$$\begin{aligned} ||A_\lambda||^2 &= \sum_n ||A_\lambda||_n^2 \\ &= \frac{\pi}{4\mu}\sum_n |f_O(E_n)|^2 \\ &= \frac{\pi 2^L}{4\mu}\langle|f_O(E_n)|^2\rangle \end{aligned}$$

Hence, we get:

$$\boxed{\frac{||A_\lambda||^2}{2^L} = \frac{\pi}{4\mu}\langle|f_O(E_n)|^2\rangle} \tag{30}$$

6 System-size scaling of minimum and maximum of ω_{ij}

https://stackoverflow.com/questions/14854339/in-scipy-how-and-why-does-curve-fit-calculate-the-covariance-of-the-parameter-es

https://stackoverflow.com/questions/14581358/getting-standard-errors-on-fitted-parameters-using-the-optimize-leastsq-method-i

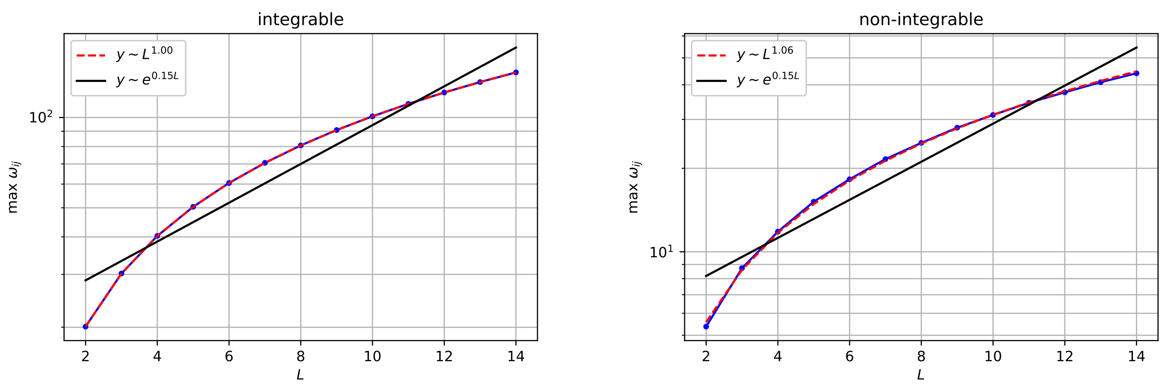

Figure 3: Using ED method, $\min \omega_{ij}(L)$

If there is degeneracy, $\min \omega_{ij}(L)$ should be zero. Why don't I see any degenerate level?

I find that because of open boundary conditions, I don't get any degenerate states for integrable model[2]. The question is then how do I get an almost linear scaling of norm for integrable models?

[2]Can I see this analytically for a simple model with only $J\sum_i \sigma_{i+1}^z\sigma_i^z$ term?

We had reasoned that $\langle n|\partial_\lambda H|m\rangle$ is zero because of extensive number of degenerate levels. It doesn't seem like that here.

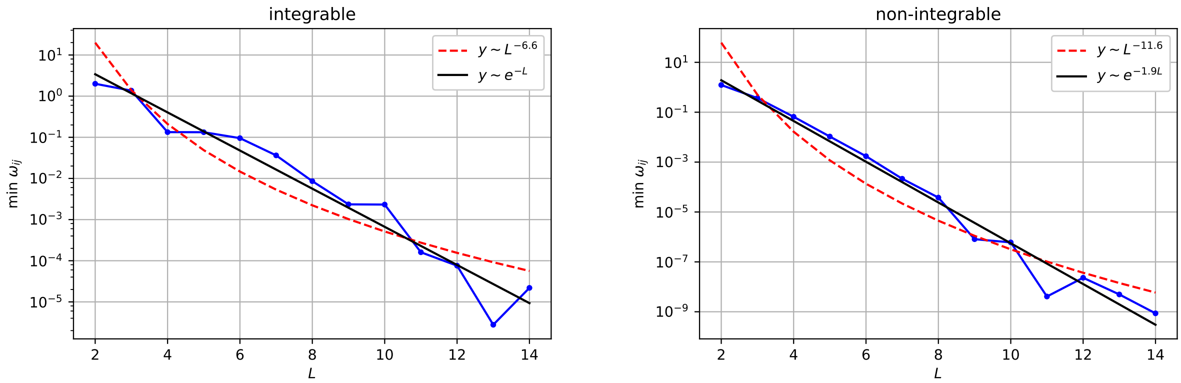

Figure 4: Using ED method, $\max \omega_{ij}(L)$

7 Norm computed using ED

Let's look at the expression of off-diagonal elements of gauge potential:

$$\langle m|A_\lambda|n\rangle = -i\hbar \frac{\langle m|\partial_\lambda H|n\rangle}{E_m - E_n}, \quad n \neq m \tag{31}$$

We see that while using ED, we need to be wary of degenerate eigenvalues. Do these degenerate eigenvalues contribute to norm of gauge potential? Answer is no because $\langle m|\partial_\lambda H|n\rangle = 0$ for degenerate pair of eigenvalues as shown in appendix A.

For integrable model, we would study the Hamiltonian of Transverse Field Ising model:

$$H = J\sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + \lambda \sum_j \sigma_j^x \tag{32}$$

where we have chosen $J = 1$ and $\lambda = 5$ with open boundary conditions.

For non-integrable model, we would study the Hamiltonian of Ising model with both transverse and longitudinal fields:

$$H = J\sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + h\sum_j \sigma_j^z + \lambda \sum_j \sigma_j^x \tag{33}$$

where we have chosen $J = 1$, $h = (\sqrt{5}+1)/4$ and $\lambda = (\sqrt{5}+5)/8$ with open boundary conditions. These are values of parameters for which this model has been shown to be robustly non-integrable for small systems [10].

We see that $\partial_\lambda H = \sum_j \sigma_j^x$.

Since anti-ferromagnetic phase has more local order compared to ferromagnetic phase, we expect the former to be less affected by finite size effects.

7.1 μ scaling of gauge potential

Our μ-dependent gauge potential A_λ is given by:

$$\langle m|A_\lambda|n\rangle = -i\hbar \frac{\langle m|\partial_\lambda H|n\rangle}{\omega_{mn}^2 + \mu^2}\omega_{mn} \tag{34}$$

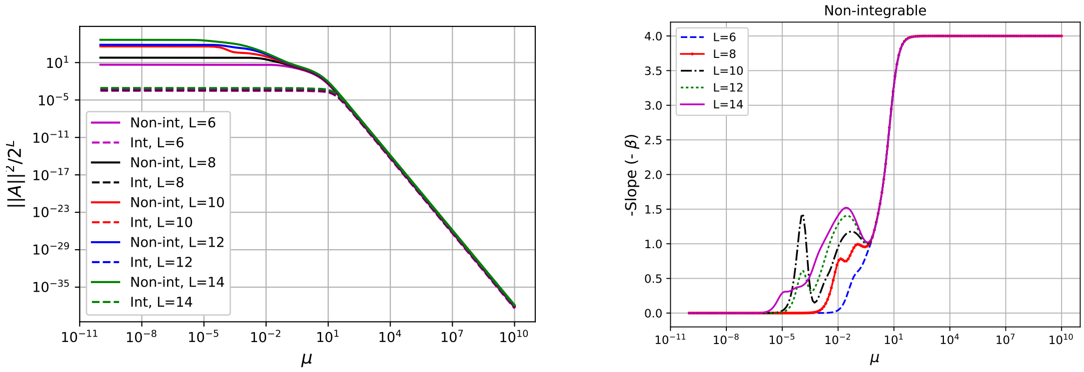

Figure 5: a) Using ED method, we obtain μ dependence of norm of gauge potential in integrable and non-integrable systems b) μ dependence of negative of slope $(-\beta(\mu))$ is shown for non-integrable systems

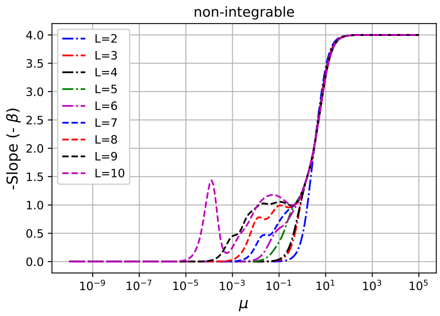

Figure 6: a) Using ED method, we obtain μ dependence of norm of gauge potential in integrable and non-integrable systems b) μ dependence of negative of slope $(-\beta(\mu))$ is shown for non-integrable systems

where $\omega_{nm} = E_n - E_m$ and eigenstates depend on λ, i.e.$|n\rangle = |n(\lambda)\rangle$. Hence, norm should be (in units of $\hbar = 1$):

$$||A_\lambda||^2 = \sum_n \sum_{m \neq n} \frac{\omega_{nm}^2}{(\omega_{nm}^2 + \mu^2)^2} |\langle m|\partial_\lambda H|n\rangle|^2 \tag{35}$$

Numerically, we find the dependence of gauge potential on μ using Exact Diagonalization method (ED) in figure 5. Let's claim that $||A||^2/2^L = \alpha\mu^\beta$. Then if we take log both sides, we get

$$\log ||A||^2/2^L = \log\alpha + \beta \log\mu \tag{36}$$

where β is the slope on a log-log scale. Numerically, we can find β_i for each pair of points using the following relationship (figure 5):

$$\beta_i = \frac{\log y(\mu_{i+1}) - \log y(\mu_i)}{\log\mu_{i+1} - \log\mu_i} \tag{37}$$

where $y = ||A||^2/2^L$.

Trivial/non-physical regimes

Let's study two regimes we see in the figures:

- **small μ regime** when $\mu \ll \min\{w_{nm}\}$: Since density of states is highest in the middle of spectrum, $\min\{w_{nm}\}$ is smallest for two states lying there. In this regime, μ is so small that it doesn't really affect the norm of gauge potential. So, we get exact gauge potential in this regime.

$$||A_\lambda||^2 = \sum_n \sum_{m\neq n} \frac{|\langle m|\partial_\lambda H|n\rangle|^2}{\omega_{nm}^2} \sim 2^L e^L \tag{38}$$

  Exact gauge potential of non-integrable systems consists of all n- body ($1 \leq n \leq L$) operators. It looks something like:

$$A_\lambda = \sum_j [\alpha_j \sigma_j^y + \beta_j(\sigma_j^y\sigma_{j+1}^z + \sigma_j^z\sigma_{j+1}^y) + \gamma_j(\sigma_j^y\sigma_{j+1}^x + \sigma_j^z\sigma_{j+1}^x) + \ldots] \tag{39}$$

  Looking at the above expression, why norm of non-integrable systems is exponentially large in system size? My answer, which needs to be discussed with Dries, is that there are only extensive number of operators ($2L$ in number) as compared to L number of operators in integrable systems. So, the number of operators doesn't make it exponentially large. What makes the big difference is that weight of these operators $\bar{\alpha}_l$ ($\bar{\alpha}_{l=1} = \{\alpha_j\}, \bar{\alpha}_{l=2} = \{\beta_j\}, \bar{\alpha}_{l=3} = \{\gamma_j\}$ etc) is exponentially large in string size l which is in contrast with integrable systems, where these weights are exponential suppressed in string size. I should check this for numerically found exact gauge potential.

  What's the biggest μ of this regime? We know that $\min\{w_{nm}\} \simeq e^{-L}$ due to the exponential number of states in a many-body system whose bandwidth generally increases extensively in system size. Hence, we expect $\mu_c^1 \simeq e^{-L}$. In figure 7, we take different slices in y-axis of slope of gauge potential figure 5.

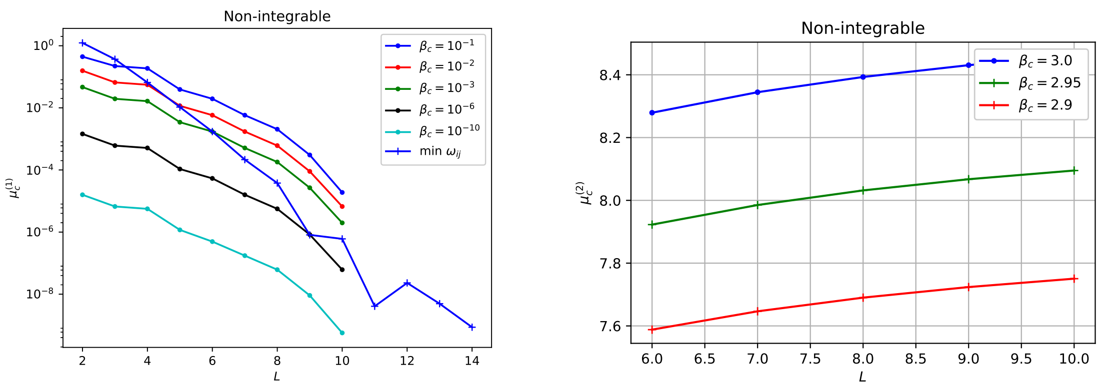

Figure 7: $\mu_c^{(1)}$ and $\mu_c^{(2)}$ dependence of system size L: former seems exponentially small and the latter a linear one

- **big scaling regime** When $\mu \gg \max\{w_{nm}\}$, approximate gauge potential A_λ^* would be given by:

$$\begin{aligned} A_\lambda^* &= -i\hbar[H, \partial_\lambda H]\frac{1}{\mu^2} \\ &= -i\hbar\frac{1}{\mu^2}C^{(1)} \end{aligned}$$

  where $C^{(1)} = 2i\left(\sum_{j=1}^{L-1}\sigma_j^y\sigma_{j+1}^z + \sum_{j=2}^{L}\sigma_j^y\sigma_{j-1}^z + h\sum_{j=1}^{L}\sigma_j^y\right)$.

$$||A_\lambda||^2 = \frac{\alpha}{\mu^4} \sim \frac{L}{\mu^4} 2^L$$

where $\alpha = \mathrm{Tr}[H, \partial_\lambda H]^2$. For $L = 12$, we obtain $\alpha_2^{Th} = 119.41$ whose details are given in appendix B.

In this regime, approximate gauge potential is local because it has only a single body term and a two body term.

These regimes are non-physical because it's not useful to choose μ either exponentially small (because we will have exact gauge potential which small denominator problem) or extensively big (because in that case, μ is the biggest energy scale and then nothing interesting really happens).

Phase diagram of gauge potential

What we find here is a "phase diagram" where we get different 'types' of gauge potential depending upon the value of μ used to construct it. Trivial/non-physical regimes are: exact gauge potential in small μ regime and trivial local approximate gauge potential in big μ regime. We need to study more to characterize the different "phases" for intermediate value pf μ based upon following parameters. For a given phase, we would have a certain range of allowed μ. Within this range of μ,

– how does the norm of gauge potential varies as a function of system size (L)? Polynomial in L or exponential in L? We know that in exact regime (small μ), it grows as an exponential and in big μ regime, it grows as linear in system size. What happens in regimes outside these extreme regimes?

– what kind of operators makes up our gauge potential? We know that in exact regime (small μ), gauge potential consists all all n- body ($1 \leq n \leq L$) operators are there and in big μ regime, it contains only one and two body operators. What happens in regimes outside these extreme regimes?

– How does the weight of these operators increase as string length of these operators increase? We know that in exact regime (small μ) for non-integrable model, these weights increase exponentially as the string length increases and in big μ regime, the weight is uniform for the one and two body operators. For integrable systems, these weights are exponentially suppressed as the string length increases. What happens in regimes outside these extreme regimes?

We need to figure out a systematic way to find out the boundary of our phase diagram.

7.2 L scaling of gauge potential

Numerically, we find that norm of exact gauge potential of non-integrable system scale exponentially in system size while it scales as polynomial in system size for integrable system.

In small μ regime, since we expect $\mu_c^{(1)} = e^{-L}$, we parametrize our $\mu = \mu_0 e^{-L}$, where μ_0 is our parameter. (figure 9). Here, we are coming from far left side of phase diagram. Numerically, we find that μ_0 lies between 1 and 10. We don't know how to make the window tighter.

Similarly, in big μ regime, since we expect $\mu_c^{(2)} = L$, we parametrize our $\mu = \mu_0 L$, where μ_0 is our parameter. Here, we are coming from the far right side of phase diagram. We don't know for sure when we cross the critical point, if we find a different scaling of system size apart from linear scaling.

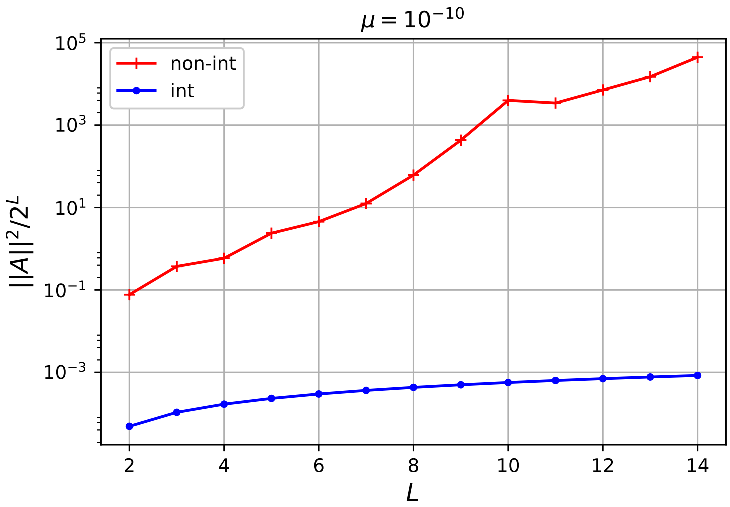

Figure 8: Exact gauge potential as a function of system size: non-integrable systems (exponential scaling) and integrable (polynomial scaling)

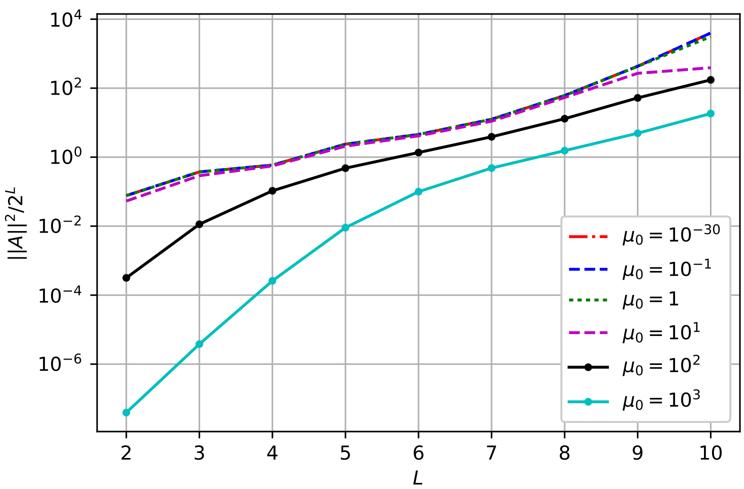

Figure 9: something

A Do degenerate eigenvalues contribute to norm of gauge potential?

Let's consider $H|n(\lambda)\rangle = E_n|n(\lambda)\rangle$. Hence, we have $\langle m(\lambda)|H|n(\lambda)\rangle = 0$ for $n \neq m$. We can exploit this property to get some insight:

$$\begin{aligned}
\partial_\lambda \langle m|H|n\rangle &= 0 \\
\langle \partial_\lambda m|H|n\rangle + \langle m|H|\partial_\lambda n\rangle + \langle m|\partial_\lambda H|n\rangle &= 0 \\
\langle \partial_\lambda m|n\rangle E_n + E_m \langle m|\partial_\lambda n\rangle + \langle m|\partial_\lambda H|n\rangle &= 0 \\
(E_n - E_m)\langle \partial_\lambda m|n\rangle + \langle m|\partial_\lambda H|n\rangle &= 0
\end{aligned}$$

Hence, we find that if there are two degenerate energy levels n and m such that $E_n = E_m$, then $\langle m|\partial_\lambda H|n\rangle = 0$. Hence, the contribution to norm of gauge potential from this pair of energy levels will be zero. I should check this numerically if results of my code respect this property.

B Computing $\mathrm{Tr}[H, \partial_\lambda H]$

B.1 Integrable model

$$H = \sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + \lambda \sum_j \sigma_j^x \tag{40}$$

We know that that $\partial_\lambda H = \sum_j \sigma_j^x$. Let's denote $C^{(1)} = [H, \partial_\lambda H]$.

$$\begin{aligned}[H, \partial_\lambda H] &= \sum_{j=1}^{L-1} [\sigma_j^z \sigma_{j+1}^z, \sum_i \sigma_i^x] \\ &= 2i \left(\sum_{j=1}^{L-1} \sigma_j^y \sigma_{j+1}^z + \sum_{j=2}^{L} \sigma_j^y \sigma_{j-1}^z \right)\end{aligned}$$

We find that $\mathrm{Tr}\, |C^{(1)}|^2 = 8(L-1)2^L$

B.2 Non-integrable model

Non-integrable model's Hamiltonian is given by:

$$H = J \sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + h \sum_j \sigma_j^z + \lambda \sum_j \sigma_j^x \tag{41}$$

$$\begin{aligned}[H, \partial_\lambda H] &= [\sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + h \sum_{j=1}^{L} \sigma_j^z, \sum_i \sigma_i^x] \\ &= 2i \left(\sum_{j=1}^{L-1} \sigma_j^y \sigma_{j+1}^z + \sum_{j=2}^{L} \sigma_j^y \sigma_{j-1}^z + h \sum_{j=1}^{L} \sigma_j^y \right)\end{aligned}$$

We find that $\mathrm{Tr}\, |C^{(1)}|^2 = 2^L 4(h^2 L + 2J(L-1))$.

B.3 Integrable systems with open boundary condition

Here I show that for integrable systems with open boundary conditions, we find some kind of structure, which is absent when computed using periodic boundary conditions.

References

[1] Mustafa Demirplak and Stuart A Rice. Adiabatic population transfer with control fields. *The Journal of Physical Chemistry A*, 107(46):9937–9945, 2003.

[2] Mustafa Demirplak and Stuart A Rice. Assisted adiabatic passage revisited. *The Journal of Physical Chemistry B*, 109(14):6838–6844, 2005.

[3] MV Berry. Transitionless quantum driving. *Journal of Physics A: Mathematical and Theoretical*, 42(36):365303, 2009.

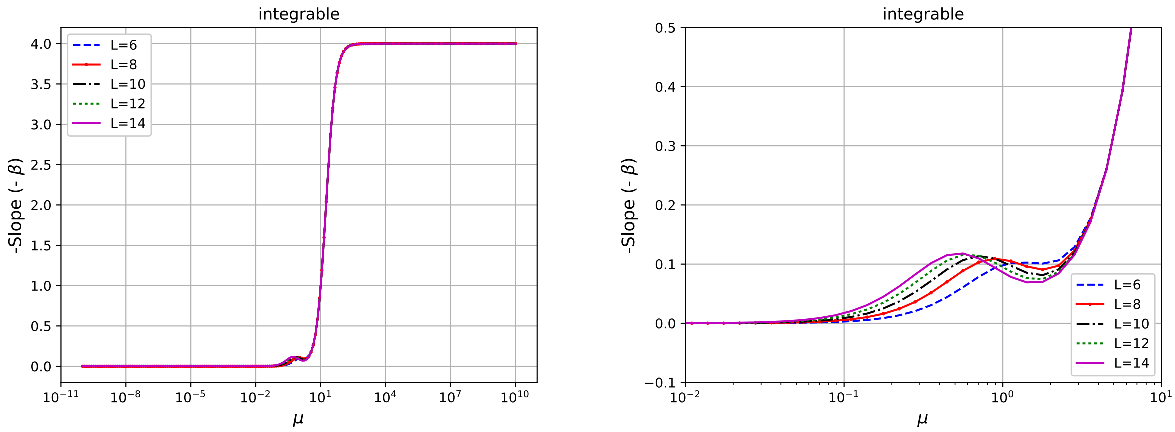

Figure 10: μ dependence of negative of slope $(-\beta(\mu))$ is shown for integrable systems

[4] Luca D'Alessio, Yariv Kafri, Anatoli Polkovnikov, and Marcos Rigol. From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics. *Advances in Physics*, 65(3):239–362, 2016.

[5] Michael Kolodrubetz, Pankaj Mehta, and Anatoli Polkovnikov. Geometry and non-adiabatic response in quantum and classical systems. *arXiv preprint arXiv:1602.01062*, 2016.

[6] Ehsan Khatami, Guido Pupillo, Mark Srednicki, and Marcos Rigol. Fluctuation-dissipation theorem in an isolated system of quantum dipolar bosons after a quench. *Physical review letters*, 111(5):050403, 2013.

[7] Mark Srednicki. The approach to thermal equilibrium in quantized chaotic systems. *Journal of Physics A: Mathematical and General*, 32(7):1163, 1999.

[8] Subir Sachdev. *Quantum phase transitions*. Wiley Online Library, 2007.

[9] Adolfo del Campo, Marek M Rams, and Wojciech H Zurek. Assisted finite-rate adiabatic passage across a quantum critical point: exact solution for the quantum ising model. *Physical review letters*, 109(11):115703, 2012.

[10] Hyungwon Kim and David A Huse. Ballistic spreading of entanglement in a diffusive nonintegrable system. *Physical review letters*, 111(12):127205, 2013.