Regulator based method to find adiabatic gauge potential for quantum many body systems

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1 Goals: what we hope to achieve

Adiabatic gauge potentials are useful for controlling a quantum system when it's driven externally from one configuration to another. These potentials help us in circumventing standard adiabatic limitations which requires infinitesimally small rates [1, 2, 3]. For example, these potentials can be used for arbitrarily fast annealing protocols and implementing fast dissipationless driving.

The goal is to develop a regulator based method to find adiabatic gauge potential for quantum many body systems. If we are successful, then this will be a new method to find these potentials and it will give new insights in quantum control of many body systems.

We will use our method on both quantum integrable and non-integrable systems. For quantum integrable many body systems, exact gauge potential is already known in literature [4, 5]. We hope to derive these results using our new method. For non-integrable systems, exact gauge potentials are very difficult to find. We hope that our method will find an approximate gauge potentials for such systems providing an alternative method to variational approximation scheme recently introduced in [4].

We also hope to use this method to distinguish between quantum integrable and non-integrable systems. Our idea is to use Eigenstate Thermalization Hypothesis (ETH)[6] for this. Since ETH is valid for local operators in non-integrable systems, we expect local approximate gauge potentials to satisfy ETH, and exact gauge potentials (which are non-local) should not satisfy ETH. We can show that using ETH, norm of approximate gauge potential should scale exponentially in system size for non-integrable systems. Whereas for integrable systems (where ETH is not valid), exact gauge potential are supposed to scale like a polynomial in system size. We want to understand this issue into more details using our new method.

2 Introduction

2.1 Gauge potential

Let's represent a wavefunction in some basis as $|\psi\rangle = \sum_n \psi_n |n\rangle_0$ where $|n\rangle_0$ is some fixed, parameter independent basis. Now let's do a unitary basis transformation to $|m(\lambda)\rangle$ in the parameter λ dependent space using $U(\lambda)$ by defining $|m(\lambda)\rangle = \sum_n U_{mn}|n\rangle$. Hence, now we can express $|\psi\rangle = \sum_m \tilde{\psi_n}|m(\lambda)\rangle$, where $\tilde{\psi_n} = \langle m(\lambda)|\psi\rangle$.

Quantum gauge potentials A_λ are defined to be generators of continuous unitary transformation. In the lab frame, A_λ is defined as:

$$\boxed{A_\lambda = i\hbar\partial_\lambda} \tag{1}$$

In rotated frame (λ -dependent basis) , $\tilde{A_\lambda}$ is defined as follows:

$$\boxed{\tilde{A_\lambda} = i\hbar U^\dagger \partial_\lambda U} \tag{2}$$

We can show that gauge potentials in these two frames are related by $A_\lambda = U\tilde{A_\lambda}U^\dagger$

Let's take an example of a shifting transformation U to understand gauge potentials:

$$U|x'(\lambda)\rangle = |x+\lambda\rangle \tag{3}$$

We know that unitary transformation $U = \exp(-i\hat{p}\lambda/\hbar)$. Now, $\tilde{A_\lambda} = \hat{p}$ and $A_\lambda = i\hbar\partial_\lambda$ [1].

Now why do we call it a gauge potential? In [5], they call it gauge potential because there is freedom to choose A_λ like how in EM, we have gauge choice. In [5], they say that "one can

[1] I don't know why there is a minus sign missing as momentum operator is defined as $\hat{p} = -i\hbar\partial_x$

show that the gauge potentials for canonical shifts of the momentum appear exactly as the electromagnetic vector potential [see Exercise (III.1)]. Gauge potentials generalize these ideas from electromagnetism to arbitrary parameters "

Here I am listing down some properties:

- They are Hermitian operator.
- $\langle n(\lambda)|A_\lambda|m(\lambda)\rangle = {}_0\langle n|\tilde{A}_\lambda|m\rangle_0$

2.2 Adiabatic gauge potential

The gauge potentials become adiabatic gauge potential when unitary transformation generated by A_λ are used to diagonalize Hamiltonian.

Adiabatic gauge potentials are a special subset of these which diagonalize the instantaneous Hamiltonian, attempting to leave its eigenbasis invariant as the parameter is changed. These adiabatic gauge potentials generate non-adiabatic corrections to Hamiltonian in the moving basis (λ -dependent basis).

This is something from Anatoli's lecture notes [5]– "an adiabatic basis is a family of adiabatically connected eigenstates, i.e., eigenstates related to a particular initial basis by adiabatic (infinitesimally slow) evolution of the parameter λ. For example, if two levels cross they will exchange order energetically but the adiabatic connection will be non-singular."

$H(\lambda)|n(\lambda)\rangle = E_n(\lambda)|n(\lambda)$. Let's derive diagonal and off-diagonal elements.

- **n-th diagonal element:** $A_\lambda^n = \langle n|A_\lambda|n\rangle = i\hbar\langle n|\partial_\lambda|n\rangle$
- **off- diagonal element:** We use the identity $\langle m|H(\lambda)|n\rangle = 0 \quad, n \neq m$ and then differentiate with respect to λ to obtain:

$$\boxed{\langle m|A_\lambda|n\rangle = -i\hbar\frac{\langle m|\partial_\lambda H|n\rangle}{E_m - E_n}} \tag{4}$$

  where both energies (E_m, E_n) and eigenvectors $(|m\rangle, |n\rangle)$ depend on λ.

This information can be represented in matrix form [7] as follows:

$$i\hbar\partial_\lambda H = [A_\lambda, H] - i\hbar M_\lambda \tag{5}$$

where

$$M_\lambda = -\sum_n \frac{\partial E_n(\lambda)}{\partial\lambda}|n(\lambda)\rangle\langle n(\lambda)| \tag{6}$$

It's to be noted that for finding M_λ, we need to diagonalize Hamiltonian. We can eliminate M_λ by taking commutator on both sides of equation 5 and obtain:

$$[H, i\hbar\partial_\lambda H - [A_\lambda, H]] = 0 \tag{7}$$

Any A_λ satisfying equation 7 is an exact gauge potential. We note that if A_λ satisfies equation 7 , then $A_\lambda + f(H)$, where $f(H)$ is any function that only contains terms involving Hamiltonian H and other operators that commutes with H. We note that $f(H)$ is diagonal in energy basis, i.e. $f(H)^{nm} = \delta_{n,m}f(H)^{nn}$.

2.2.1 Minimum norm gauge choice

For our future purposes, let's study a gauge choice where we assume diagonal elements of A_λ in energy basis is zero, i. e. $\langle n|A_\lambda|n\rangle = A_\lambda^{n,n} = 0$, for $n = 1, 2 \ldots D$, where D is the dimension of Hilbert space. Can we always make such a gauge choice?

As we noted above, a family of A_λ satisfies equation 7 – both A_λ and $A_\lambda + f(H)$ satisfy the equation 7. Using this knowledge, let's suppose we define $A'_\lambda = A_\lambda + f(H)$, where in energy basis, $f(H)$ is diagonal (as we already know), A'_λ is an exact gauge potential with all it's diagonal elements chosen to be zero and A_λ is an exact gauge potential with non-zero diagonal elements. What's the condition on $f(H)$ so that diagonal elements of A'_λ are zero? The required condition is:

$$f(H)^{nn} = -A_\lambda^{nn}, \quad n = 1, 2, \ldots D \tag{8}$$

where D is the dimension of Hilbert space. Hence, if somebody hands me A_λ, here is the method to obtain A'_λ: I can always cook up a function $f(H) = \sum_{n=1}^{D} a_n H^n$ by solving D number of equations which satisfy equation 8 to find out a_n. Once, I know $f(H)$, I can always subtract it from A_λ to obtain A'_λ. Hereby, I show that this gauge choice can always be made without any loss of generality.

Let's try to understand this gauge choice further by computing the Frobenius norm of A_λ.

$$\|A_\lambda\|^2 = \mathrm{Tr}(A_\lambda^2) = \sum_{n,m} |A_\lambda^{n,m}|^2 = \sum_n |A_\lambda^{n,n}|^2 + \sum_{n\neq m} |A_\lambda^{n,m}|^2 = \sum_n |A_\lambda^{n,n}|^2 + \|A'_\lambda\|^2 \tag{9}$$

where $A_\lambda^{n,m} = \langle n|A_\lambda|m\rangle$ and $|m\rangle$ is the energy eigenstate with energy E_m. Thus, we see that in our gauge choice all the diagonal elements ($A_\lambda^{n,n}$) are zero, and therefore, this choice reduces the norm. Is this the minimum norm of A_λ which satisfies equation 7? The answer is yes as explained below.

Let's compute the norm of A_λ another way by exploiting its' equality to $A'_\lambda - f(H)$:

$$||A_\lambda||^2 = \mathrm{Tr}((A'_\lambda - f(H))^2) \tag{10}$$
$$= \mathrm{Tr}(A'^{\,2}_\lambda) + \mathrm{Tr}(f(H)^2) - 2\,\mathrm{Tr}((A'_\lambda f(H)) \tag{11}$$
$$= \mathrm{Tr}(A'^{\,2}_\lambda) + \mathrm{Tr}(f(H)^2) \tag{12}$$
$$= \|A'_\lambda\|^2 + \|f(H)\|^2 \tag{13}$$

where we have used the fact that $f(H)$ is diagonal and A'_λ has no non-zero diagonal elements in energy basis in claiming $\mathrm{Tr}((A'_\lambda f(H)) = 0$. We note that since $f(H)$ is diagonal in energy basis, the only way A_λ will acquire diagonal elements is through $f(H)$. Hence, we see that by choosing diagonal elements of an exact adiabatic gauge potential in energy basis to be zero, we are effectively choosing a gauge potential which has no $f(H)$ term. Hence, this gauge choice is the minimum norm possible.

2.2.2 Time evolution in moving frame

Our Hamiltonian would be controlled using a control parameter called λ and our aim is to find time evolution of wave-function is λ -dependent basis called moving frame.

Our Hamiltonian $H_0(\lambda(t))$ would satisfy the following equation:

$$H_0(\lambda(t))|\psi\rangle = i\partial_t|\psi\rangle \tag{14}$$

Let us go to rotating frame so as to diagonalize our Hamiltonian. Required unitary transformation $U(\lambda)$ would depend on parameter λ. Wave function in moving frame is $|\tilde{\psi}\rangle = U^\dagger|\tilde{\psi}\rangle$. In this basis, Hamiltonian is diagonal: $\tilde{H}_0 = U^\dagger H_0 U = \sum_n \epsilon(\lambda)|n(\lambda)\rangle\langle n(\lambda)|$. [2]

How does the wave function evolve in new basis?

$$i\partial_t|\tilde{\psi}\rangle = (\tilde{H}_0(\lambda(t)) - \dot{\lambda}\tilde{\mathcal{A}}_\lambda)|\psi\rangle \tag{15}$$

Note that gauge potential should be purely imaginary in a basis in which Hamiltonian is real.

[2] Note that expectation value should remain same in both basis, i.e.$\langle\tilde{\psi}|\tilde{H}_0|\tilde{\psi}\rangle = \langle\psi|H_0|\psi\rangle$

2.2.3 Variational principle of adiabatic gauge potential

In [8], variational principle has been discovered to find out an approximate adiabatic gauge potential $\mathcal{X}$.

$$G_\lambda(\mathcal{X}) = \partial_\lambda H + \frac{i}{\hbar}[\mathcal{X}, H] \tag{16}$$

If $\mathcal{X} = A_\lambda$, then $[H, G_\lambda] = 0$. They found that finding the minimum of norm of $G_\lambda(A_\lambda)$ is equivalent to the Euler-Lagrange equation:

$$\frac{\delta S(\mathcal{X})}{\delta \mathcal{X}} = 0 \tag{17}$$

where action $S(\mathcal{X})$ is given by:

$$S(\mathcal{X}) = \mathrm{Tr}[G_\lambda^2(\mathcal{X})] \tag{18}$$

In [8], authors write "Quite often, one is interested in suppressing transitions from a low-temperature manifold of states, in particular, the ground state. Then, targeting the gauge potential, which suppresses transitions everywhere in the spectrum, is overdemanding".

In [5], authors write "let us note that the trace norm in the action is similar to the infinite temperature norm as we are summing over all the eigenstates of H with the equal weight. Very often we are interested only in the low energy manifold, as for example in trying to find the approximate counter-diabatic driving required to keep the system close to the ground state. If we are dealing with quantum or classical systems with unbounded spectra, the Frobenius norm of the operators may also be ill-defined, requiring some cutoff regularization. In such situations we may instead define the finite temperature action:

$$S(\mathcal{X}) = \langle G_\lambda^2(\mathcal{X})\rangle - \langle G_\lambda(\mathcal{X})\rangle^2 \tag{19}$$

where $\langle \ldots \rangle$ stands for the averaging with respect to the thermal density matrix $\rho = \frac{1}{Z}\exp[-\beta H]$".

I think a properly defined regulator would help us in defining a cutoff for averaging of $G_\lambda(\mathcal{X})$ when we have finite/zero temperature.

2.3 Eigenstate Thermalization Hypothesis

Eigenstate Thermalization Hypothesis(ETH) gives us an ansatz for matrix elements of observables in the basis of eigenstates of a non-integrable Hamiltonian [6]:

$$O_{mn} = O(\bar{E})\delta_{mn} + e^{-S(\bar{E})/2} f_O(\bar{E}, \omega) R_{mn} \tag{20}$$

where $\bar{E} = (E_m + E_n)/2, \omega = E_n - E_m$ and $S(E)$ is the thermodynamic entropy at energy E.

3 Regulator based method to find Gauge Potential

Here we would introduce a new method to find Gauge Potential A_λ which includes a regulator μ.

Let's start off by writing the off-diagonal elements of exact gauge potential:

$$\langle m|A_\lambda|n\rangle = -i\hbar \frac{\langle m|\partial_\lambda H|n\rangle}{E_m - E_n} \tag{21}$$

For a many-body Hamiltonian, number of states in Hilbert space grows exponentially in system size while energy bandwidth grows linearly with system size (since energy is an extensive quantity). Thus, distance between any two *nearby* eigenvalues is exponentially small in system size. In other words, $E_m - E_n \sim e^{-S}$. If there are non-zero off-diagonal elements of $\partial_\lambda H$, then $\langle m|A_\lambda|n\rangle$ is ill-defined. It's called small denominator problem [5].

To resolve this problem, we introduce a regulator/ cutoff μ that regularizes our exact gauge potential in large system size L limit. Once we have taken large L limit, then we take small μ limit. Hence, if this method works, the right way to take limits will be:

$$\langle n|A_\lambda|m\rangle = \lim_{\mu\to 0}\lim_{L\to\infty} -i\hbar\frac{\langle n|\partial_\lambda H|m\rangle}{(E_n-E_m)^2+\mu^2}(E_n-E_m) \tag{22}$$

where we have chosen a gauge choice in which diagonal elements are zero in energy basis, i.e. $A_\lambda^{nn}=0$.

Now let's first recall here the Laplace transform of $\sin(\omega t)$:

$$\int_0^\infty dt\ e^{-st}\sin(\omega t) = \frac{\omega}{\omega^2+s^2} \tag{23}$$

Thus, we see that with $s=\mu$, we can convert our expression of A_λ into an integral. From here on to simplify our expression of integration, we will choose our units such that $\hbar = 1$. [3]

$$\begin{aligned}
\langle n|A_\lambda|m\rangle &= -i\frac{\langle n|\partial_\lambda H|m\rangle}{(E_n-E_m)^2+\mu^2}(E_n-E_m) && (24)\\
&= -i\int_0^\infty dt\ e^{-\mu t}\langle n|\partial_\lambda H|m\rangle\sin((E_n-E_m)t) && (25)\\
&= -\frac{1}{2}\int_0^\infty dt\ e^{-\mu t}\left(\langle n|e^{iE_nt}\partial_\lambda He^{-iE_mt}|m\rangle - \langle n|e^{-iE_nt}\partial_\lambda He^{iE_mt}|m\rangle\right) && (26)
\end{aligned}$$

Hence, we can simplify our expression by defining propagator $U = \exp(-iHt)$. We note that parameter λ is fixed while we evolve it in the *artificial time* t.

$$\begin{aligned}
A_\lambda &= -\frac{1}{2}\int_0^\infty dt\ e^{-\mu t}[U^\dagger(t)\partial_\lambda HU(t) - U^\dagger(-t)\partial_\lambda HU(-t)] && (27)\\
&= -\frac{1}{2}\int_0^\infty dt\ e^{-\mu t}[\partial_\lambda H(t) - \partial_\lambda H(-t)] && (28)
\end{aligned}$$

where $\partial_\lambda H(t)$ is time-evolved operator $\partial_\lambda H$ in Heisenberg picture. We note that characteristic time scale in the integration is $1/\mu$.

Now using Hadamard (or sometimes called Baker-Campbell-Hausdorff) formula, we find a simplified expression. Detailed calculations are given in appendix C .

$$A_\lambda = -i\int_0^\infty dt\ e^{-\mu t}\sin([H,\partial_\lambda H]t) = -i\hbar\int_0^\infty dt\ e^{-\mu t}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}C^{(2n+1)} \tag{29}$$

where $C^{(n)}$ is n- commutator of H and $\partial_\lambda H$, i.e. $C^{(n)} = [H,[H,\ \text{n times}\ldots,[H,\partial_\lambda H]]]]$. We define the first term as $C^{(1)} = [H,\partial_\lambda H]$, second term as $C^{(2)} = [H,[H,\partial_\lambda H]] = [H,C^{(1)}]$ and so on and forth. Properties of $C^{(n)}$ are noted in appendix B.

Can we further simplify the expression? If we are allowed to change the order of summation and integration [4], then we can do first Laplace transform of t^{2n+1} terms and then later the sum. Hence, we get:

$$A_\lambda = -i\int_0^\infty dt\ e^{-\mu t}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}C^{(2n+1)} \tag{30}$$

[3] If we don't choose $\hbar=1$, then for Laplace integral variable, we should use not use t. Instead we should use $t/\hbar$ so that we have $\sin((E_n-E_m)t/\hbar)$ and $e^{-\mu t/\hbar}$. Otherwise, we will end up with terms $\sin((E_n-E_m)t)$ and $e^{-\mu t}$, which are dimensionally incorrect.

[4] If there is some singularity, then the order of summation and integration might be important and we might get two different results.

$$= -i\sum_{n=0}^{\infty}(-1)^n C^{(2n+1)}\int_0^{\infty} dt\; e^{-\mu t}\frac{t^{2n+1}}{(2n+1)!} \tag{31}$$

$$= -i\sum_{n=0}^{\infty}(-1)^n \frac{C^{(2n+1)}}{\mu^{2n+2}} \tag{32}$$

Hence, we get another expression where we have integrated before taking the summation:

$$A_\lambda = \frac{-i\hbar}{\mu}\sum_{n=0}^{\infty}(-1)^n \frac{C^{(2n+1)}}{\mu^{2n+1}} \tag{33}$$

Once we are done with the integration, we can bring back $\hbar$ factor. We note that $iC^{(2n+1)}$ is Hermitian, which is consistent with the fact that A_λ is Hermitian.

Now let's think about $\lim_{L\to\infty}$ limit which we need to take. I would claim that while doing the infinite summation we have already taken that limit as we have assumed infinite system size. Why is that? In general, $C^{(n)}$ grows with n in the sense that it would have operators with larger support over lattice sites as n increases. At a certain n_L that is proportional to system length L, we would find that C^{n_L} has operators with support on boundary lattice sites. This is where our summation would be truncated for a finite system. Hence, the correct order of limits should be:

$$\boxed{A_\lambda = \lim_{\mu\to 0}\lim_{L\to\infty}\frac{-i\hbar}{\mu}\sum_{n=0}^{n_L}(-1)^n \frac{C^{(2n+1)}}{\mu^{2n+1}}} \tag{34}$$

Now one thing which is good is that if we take the wrong order of limit: take $\lim_{\mu\to 0}$ before $\lim_{L\to\infty}$, then A_λ diverges. Thus, now divergence is more explicit than the original expression 22.

How does μ scale as L? It's an important question whose answer we don't know. Allow me to make a heuristic argument: In general, it seems that operators involved in the expression of $C^{(n)}$ would have support which depend on L. Let's suppose the support of these operators grow as L^γ, i.e., $C^{(n)} \propto L^\gamma$, where γ is some constant which we don't know. If we assume that A_λ is well-defined in large system size limit for many-body Hamiltonian (both integrable and non-integrable), then $\mu \propto L^\gamma$.

4 Physical meaning of regulator

4.1 Norm of adiabatic gauge potential

Norm of A_λ is obviously dictated by μ. Let's compute it by noting that A_λ has only off-diagonal elements in energy basis in our gauge choice:

$$||A_\lambda||^2 = \mathrm{Tr}\, A_\lambda^2 \tag{35}$$

$$= \sum_n \langle n|A_\lambda^2|n\rangle \tag{36}$$

$$= \sum_{m,n}\langle n|A_\lambda|m\rangle\langle m|A_\lambda|n\rangle \tag{37}$$

$$= \sum_n \langle n|A_\lambda|n\rangle^2 + \sum_n\sum_{m\neq n}|\langle m|A_\lambda|n\rangle|^2 \tag{38}$$

$$= \sum_n\sum_{m\neq n}|\langle m|A_\lambda|n\rangle|^2 \tag{39}$$

$$= \hbar^2\sum_n\sum_{m\neq n}\frac{(E_m-E_n)^2}{((E_m-E_n)^2+\mu^2)^2}|\langle m|\partial_\lambda H|n\rangle|^2 \tag{40}$$

Hence, in general, for both integrable and non-integrable systems we have:

$$\boxed{||A_\lambda||^2 = \hbar^2 \sum_n \sum_{m\neq n} \frac{(E_m - E_n)^2}{((E_m - E_n)^2 + \mu^2)^2} |\langle m|\partial_\lambda H|n\rangle|^2} \tag{41}$$

After thermodynamic limit ($L \to \infty$), if we take $\mu = 0$ limit, then we have exact gauge potential A_λ, which has infinite norm. However, after thermodynamic limit, if we take $\mu \neq 0$, then we have approximate gauge potential. If we take $\mu \to \infty$ limit, then norm of approximate gauge potential is zero.

4.1.1 ETH applied to norm

$\partial_\lambda H$ may or may not be a local operator. We would be studying such non-integrable models in which it is a local operator. Hence, we can apply ETH on the operator $\partial_\lambda H$.

$$\begin{aligned}
||A_\lambda||^2_{nn} &= \sum_{m\neq n} \frac{(E_m - E_n)^2}{((E_m - E_n)^2 + \mu^2)^2} |\langle m|\partial_\lambda H|n\rangle|^2 \\
&= \sum_{m\neq n} \frac{\omega^2}{(\omega^2 + \mu^2)^2} e^{-S(\bar{E})} |f_O(\bar{E}, \omega) R_{mn}|^2 \\
&= \sum_{m\neq n} \frac{\omega^2}{(\omega^2 + \mu^2)^2} e^{-S(E_n - \omega/2)} |f_O(E_n - \omega/2, \omega)|^2 |R_{mn}|^2
\end{aligned}$$

where $\bar{E} = (E_m + E_n)/2 = E_n - \omega/2$,$\omega = E_n - E_m$ and $S(E)$ is the thermodynamic entropy at energy E. We would need to convert the sum into integral where we use the fact that function f_O is smooth and fluctuations of $|R_{mn}|^2$ average out in the sum.

$$\sum_{m\neq n} \to \int d\omega \Omega(E_n - \omega) = \int d\omega e^{S(E_n - \omega)} \tag{42}$$

where $\Omega(E_n + \omega)$ is density of states.

$$||A_\lambda||^2_{nn} = \int d\omega e^{S(E_n - \omega) - S(E_n - \omega/2)} \frac{\omega^2}{(\omega^2 + \mu^2)^2} |f_O(E_n - \omega/2, \omega)|^2$$

$S(E_n - \omega) - S(E_n - \omega/2) = -\beta\omega/2 + \ldots$ and $f_O(E_n - \omega/2, \omega) = f_O(E_n, \omega) + \ldots$ we have

$$||A_\lambda||^2_{nn} = \int_0^\infty d\omega e^{-\beta\omega/2} \frac{\omega^2}{(\omega^2 + \mu^2)^2} |f_O(E_n, \omega)|^2$$

4.2 Fermi golden rule: transition rate

Before we talk about Fermi golden rule, let's study $||G_\lambda||^2$, where $G_\lambda(\mathcal{X}) = \partial_\lambda H + \frac{i}{\hbar}[\mathcal{X}, H]$.

$$\begin{aligned}
||G_\lambda||^2 &= \mathrm{Tr}(\partial_\lambda H + \frac{i}{\hbar}[\mathcal{X}, H])^2 && (43) \\
&= \mathrm{Tr}(\partial_\lambda H)^2 - \frac{1}{\hbar^2}\mathrm{Tr}([\mathcal{X}, H]^2) + \frac{2i}{\hbar}\mathrm{Tr}(\partial_\lambda H[\mathcal{X}, H]) && (44) \\
&= ||\partial_\lambda H||^2 - \frac{1}{\hbar^2}||[H, \mathcal{X}]||^2 + \frac{2i}{\hbar}\mathrm{Tr}(\partial_\lambda H[\mathcal{X}, H]) && (45)
\end{aligned}$$

$$= ||\partial_\lambda H||^2 - \frac{1}{\hbar^2}||[H, \mathcal{X}]||^2 \tag{46}$$

where we have used cyclic property of trace operation to show that last term $\mathrm{Tr}(\partial_\lambda H[\mathcal{X}, H]) = 0$.

Let's try to understand more about μ by studying transition rate and dissipation rate for a quantum many body systems using Fermi's Golden rule. Let's consider a Hamiltonian:

$$\mathcal{H}_{\mathcal{X}} = \mathcal{H}(\lambda) + \dot{\lambda}\mathcal{X} \tag{47}$$

where $\lambda = \lambda_0 + \epsilon(t)$ and $\epsilon(t)$ is an infinitesimal white noise $\overline{\epsilon(t)\epsilon(t')} = \kappa\delta(t - t')$. If the system is in equilibrium, then fluctuation -dissipation theorem dictates $\kappa = T$. Hence, we can write

$$\mathcal{H}_{\mathcal{X}} \approx \mathcal{H}(\lambda_0) + \epsilon\partial_\lambda\mathcal{H} + \dot{\epsilon}\mathcal{X} \tag{48}$$

Now using Fermi's golden rule in [5], transition rate $\langle\Gamma_n\rangle$ (averaged over all eigenstates) is derived using results from [9]. $\langle\Gamma_n\rangle$ is given by:

$$\langle\Gamma_n\rangle = \kappa\left(\sum_n \langle n|G_\lambda^2|n\rangle - \sum_n \langle n|\partial_\lambda H|n\rangle^2\right) \tag{49}$$

$$= \kappa\left(||G_\lambda||^2 - \sum_n \langle n|\partial_\lambda H|n\rangle^2\right) \tag{50}$$

$$= \kappa\sum_n\sum_{m\neq n}\left(|\langle m|\partial_\lambda H|n\rangle|^2 - \frac{1}{\hbar^2}(E_n - E_m)^2|\langle m|\mathcal{X}|n\rangle|^2\right) \tag{51}$$

where $G_\lambda(\mathcal{X}) = \partial_\lambda H + \frac{i}{\hbar}[\mathcal{X}, H]$. Now, we will choose $\mathcal{X} = A_\lambda(\mu)$ to find our final expression:

$$\boxed{\langle\Gamma_n\rangle = \kappa\sum_n\sum_{m\neq n}|\langle m|\partial_\lambda H|n\rangle|^2\left(1 - \frac{(E_n - E_m)^4}{((E_n - E_m)^2 + \mu^2)^2}\right)} \tag{52}$$

We note that if take $\mu \to 0$ limit, then transition rate goes to zero.

4.3 Fermi golden rule: dissipation rate

They define dissipation rate and find that it is equal to:

$$\frac{d\sigma_E^2}{dt} = -\kappa\,\mathrm{Tr}([G_\lambda, H]^2) \tag{53}$$

We note that in energy basis, $[G_\lambda, H]$ has only off-diagonal elements because $\langle n|[G_\lambda, H]|n\rangle = 0$

Ignoring κ and negative sign, we get:

$$\frac{d\sigma_E^2}{dt} = \mathrm{Tr}([G_\lambda, H]^2) \tag{54}$$

$$= \sum_n \langle n|[G_\lambda, H]^2|n\rangle \tag{55}$$

$$= \sum_{m,n} \langle n|[G_\lambda, H]|m\rangle\langle m|[G_\lambda, H]|n\rangle \tag{56}$$

$$= \sum_n \langle n|[G_\lambda, H]|n\rangle^2 + \sum_n\sum_{m\neq n}\langle n|[G_\lambda, H]|m\rangle\langle m|[G_\lambda, H]|n\rangle \tag{57}$$

$$= \sum_n \langle n|[G_\lambda, H]|n\rangle^2 + \sum_n\sum_{m\neq n}|\langle n|[G_\lambda, H]|m\rangle|^2 \tag{58}$$

$$= \sum_n \sum_{m\neq n} |\langle n|[G_\lambda, H]|m\rangle|^2 \tag{59}$$

Now

$$[G_\lambda, H] = [\partial_\lambda H, H] + \frac{i}{\hbar}[[A_\lambda, H], H]$$

Now let's recall:

$$\begin{aligned} \langle n|A_\lambda|m\rangle &= -i\hbar\frac{\langle n|\partial_\lambda H|m\rangle}{(E_n - E_m)^2 + \mu^2}(E_n - E_m) \quad (60)\\ &= -i\hbar\frac{\langle n|[H, \partial_\lambda H]|m\rangle}{(E_n - E_m)^2 + \mu^2} \quad (61) \end{aligned}$$

Off-diagonal elements of $[G_\lambda, H]$ is given by:

$$\begin{aligned} [G_\lambda, H] &= [\partial_\lambda H, H] + \frac{i}{\hbar}[[A_\lambda, H], H] \quad (62)\\ &= -[H, \partial_\lambda H] - \frac{i}{\hbar}[[H, A_\lambda], H] \quad (63)\\ &= -[H, \partial_\lambda H] + \frac{i}{\hbar}[H, [H, A_\lambda]] \quad (64)\\ &= -[H, \partial_\lambda H] + \frac{1}{(E_n - E_m)^2 + \mu^2}[H, [H, [H, \partial_\lambda H]]] \quad (65)\\ & \quad (66) \end{aligned}$$

Hence, we find that :

$$\begin{aligned} \langle n|[G_\lambda, H]|m\rangle &= -\langle n|[H, \partial_\lambda H]|m\rangle + \langle n|\frac{1}{(E_n - E_m)^2 + \mu^2}[H, [H, [H, \partial_\lambda H]]]|m\rangle \\ &= \left(-(E_n - E_m) + \frac{(E_n - E_m)^3}{(E_n - E_m)^2 + \mu^2}\right)\langle n|\partial_\lambda H|m\rangle \\ &= -\frac{\mu^2}{(E_n - E_m)^2 + \mu^2}\langle n|\partial_\lambda H|m\rangle \end{aligned}$$

Hence, we get :

$$\boxed{\frac{d\sigma_E^2}{dt} = -\sum_n \sum_{m\neq n} \frac{\mu^4}{((E_n - E_m)^2 + \mu^2)^2}|\langle n|\partial_\lambda H|m\rangle|^2} \tag{67}$$

We note that if take $\mu \to 0$ limit, then dissipation rate goes to zero.

4.3.1 ETH for complex systems

Now, let's apply ETH to this expression

$$\begin{aligned} \frac{d\sigma_E^2}{dt} &= -\sum_n \sum_{m\neq n} \frac{\mu^4}{((E_n - E_m)^2 + \mu^2)^2}|\langle n|\partial_\lambda H|m\rangle|^2 \quad (68)\\ &= -\sum_n \sum_{m\neq n} \frac{\mu^4}{((E_n - E_m)^2 + \mu^2)^2}e^{-S}|f_O R_{mn}|^2 \quad (69) \end{aligned}$$

4.3.2 Formula involving commutators

If G_λ is exact with $\mu = 0$, then $[G_\lambda, H] = 0$. Let's find out μ dependent dissipation rate. We recall that $G_\lambda(A_\lambda(\mu)) = \partial_\lambda H + \frac{i}{\hbar}[A_\lambda(\mu), H]$

$$[G_\lambda(A_\lambda(\mu)), H] = [\partial_\lambda H, H] + \frac{i}{\hbar}[[A_\lambda(\mu), H], H] \tag{70}$$
$$= -C^{(1)} - \frac{i}{\hbar}[H, [A_\lambda(\mu), H]] \tag{71}$$
$$= -C^{(1)} + \frac{i}{\hbar}[H, [H, A_\lambda(\mu)]] \tag{72}$$

Now, let's simplify $[H, [H, A_\lambda(\mu)]]$:

$$[H, [H, A_\lambda(\mu)]] = \frac{-i\hbar}{\mu}\sum_{n=0}^{\infty}(-1)^n [H, [H, \frac{C^{(2n+1)}}{\mu^{2n+1}}]] \tag{73}$$
$$= -i\hbar \sum_{n=0}^{\infty}(-1)^n \frac{C^{(2n+3)}}{\mu^{2n+2}} \tag{74}$$

Hence, we find that:

$$[G_\lambda(A_\lambda(\mu)), H] = -C^{(1)} + \sum_{n=0}^{\infty}(-1)^n \frac{C^{(2n+3)}}{\mu^{2n+2}} \tag{75}$$

Let's suppose $C^{(2n+1)} = \alpha^{2n} C^{(1)}$. Then we find that $C^{(2n+3)} = \alpha^{2n+2} C^{(1)}$. Using this, we can simplify our expression for dissipation rate as follows:

$$[G_\lambda(A_\lambda(\mu)), H] = -C^{(1)} + \sum_{n=0}^{\infty}(-1)^n \frac{C^{(2n+3)}}{\mu^{2n+2}} \tag{76}$$
$$= -C^{(1)} + \frac{C^{(1)}\alpha^2}{\mu^2}\sum_{n=0}^{\infty}\left(-\frac{\alpha^2}{\mu^2}\right)^n \tag{77}$$
$$= -C^{(1)} + \frac{C^{(1)}}{\mu^2}\frac{\alpha^2}{1 + \alpha^2/\mu^2} \tag{78}$$
$$= \left(-1 + \frac{\alpha^2}{\mu^2 + \alpha^2}\right) C^{(1)} \tag{79}$$
$$= -\frac{\mu^2}{\mu^2 + \alpha^2} C^{(1)} \tag{80}$$

Hence, we get

$$\boxed{\frac{1}{\kappa}\frac{d\sigma_E^2}{dt} = -\frac{\mu^4}{(\mu^2 + \alpha^2)^2}\,\mathrm{Tr}(C^{(1)})^2} \tag{81}$$

We note that if take $\mu \to 0$ limit, then dissipation rate goes to zero.

5 Single body problem

We would start off with a simple problem and find its' adiabatic gauge potential using our method.

$$H = \Delta\sigma^z + \lambda\sigma^x \tag{82}$$

$$C^{(1)} = 2i\Delta\sigma^y \tag{83}$$

$$C^{(2)} = -4\Delta(-\Delta\sigma^x + \lambda\sigma^z) \tag{84}$$
$$C^{(3)} = \alpha^2 C^{(1)} \tag{85}$$

where $\alpha^2 = 4(\Delta^2 + \lambda^2)$

Hence, we find that

$$A_\lambda = \frac{\hbar}{2}\frac{\Delta}{\Delta^2 + \lambda^2}\sigma^y \tag{86}$$

Similarly, we find that

$$A_\Delta = -\frac{\hbar}{2}\frac{\lambda}{\Delta^2 + \lambda^2}\sigma^y \tag{87}$$

6 Many body problem: integrable model

Our goal is to study a integrable model, which is called **Transverse Field Ising model**. It shows quantum phase transition between ferromagnetic and paramagnetic phases. Moreover, it satisfies Ising symmetry $G = \Pi_i \sigma_i^z$ since $[H, G] = 0$, where H is the Hamiltonian. This model can be written in terms of non-interacting spinless fermions $(c_i, c_i^\dagger)$ using Jordan- Wigner transformation. It's Hamiltonian in spin basis is given by:

$$H = -J\sum_j \sigma_j^x \sigma_{j+1}^x - \lambda \sum_j \sigma_j^z \tag{88}$$

where we have not specified boundary conditions and λ is externally-controlled transverse magnetic field.

This model's exact gauge potential is already known in literature [4, 5] and it's given by:

$$A_\lambda = \sum_l \alpha_l O_l \quad \text{where} \quad \alpha_l = -\frac{1}{4L}\sum_k \frac{\sin(k)\cos(lk)}{(\cos k - h)^2 + \sin^2 k} \tag{89}$$

and where O_l is given by

$$O_l = 2i\sum_j (c_j^\dagger c_{j+l}^\dagger - \text{h.c}) = \sum_j (\sigma_j^x \sigma_{j+1}^z \dots \sigma_{j+l-1}^z \sigma_{j+l}^y + \sigma_j^y \sigma_{j+1}^z \dots \sigma_{j+l-1}^z \sigma_{j+l}^x) \tag{90}$$

Let's write a first few terms of O_l here:

$$O_{l=1} = \sum_j (\sigma_j^x \sigma_{j+1}^y + \sigma_j^y \sigma_{j+1}^x)$$
$$O_{l=2} = \sum_j (\sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^y + \sigma_j^y \sigma_{j+1}^z \sigma_{j+2}^x)$$

It will be good to find either exact or approximate gauge potential using our regulator method. However, before we study this model, we will study much simpler models to learn about this new method.

6.1 Ising model with local transverse magnetic field

We would take the simplest integrable Hamiltonian with Ising interaction and a local x magnetic field:

$$H = J\sum_j \sigma_j^z \sigma_{j+1}^z + \lambda\sigma_0^x \tag{91}$$

where boundary conditions are not important. Commutation relation followed by spin operators are:

$$[\sigma_i^a, \sigma_j^b] = 2i\delta_{i,j} \sum_c \epsilon_{abc}\sigma_i^c \tag{92}$$

where ϵ_{abc} is the Levi-Civita symbol, δ_{ij} is the Kronecker delta.

This model satisfies Ising symmetry $G = \Pi_i \sigma_i^x$ since $[H, G] = 0$.

Let's find out A_λ for this Hamiltonian for which we need to compute different odd-powered commutator $[H, \partial_\lambda H]$, where $\partial_\lambda H = \sigma_0^x$. Here we begin:

$$C^{(1)} = 2iJ\sigma_0^y(\sigma_{-1}^z + \sigma_1^z) \tag{93}$$

$$C^{(2)} = 8J^2(\sigma_1^z\sigma_0^x\sigma_{-1}^z + \sigma_0^x) - 4J\lambda\sigma_0^z(\sigma_{-1}^z + \sigma_1^z) \tag{94}$$

$$C^{(3)} = (16J^2 + 4\lambda^2)[H, \partial_\lambda H] = \alpha^2 C^{(1)} \tag{95}$$

$$C^{(5)} = [H, [H, C^{(3)}]] = \alpha^2[H, [H, C^{(1)}]]] = \alpha^2 C^{(3)} = \alpha^4 C^{(1)} \tag{96}$$

Hence, $C^{(2n+1)} = \alpha^{2n}C^{(1)}$, where $\alpha^2 = 4(4J^2 + \lambda^2)$. Now, we would compute A_λ using two methods and compare our results. Using equation (29), we get :

$$\begin{aligned}
A_\lambda &= -i\hbar C^{(1)} \int_0^\infty dt\ e^{-\mu t} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}\alpha^{2n} \\
&= -i\hbar C^{(1)} \int_0^\infty dt\ e^{-\mu t} \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1} t^{2n+1}}{\alpha(2n+1)!} \\
&= \frac{-i\hbar C^{(1)}}{\alpha} \int_0^\infty dt\ e^{-\mu t} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha t)^{2n+1}}{(2n+1)!} \\
&= \frac{-i\hbar C^{(1)}}{\alpha} \int_0^\infty dt\ e^{-\mu t} \sin(\alpha t) \\
&= \frac{-i\hbar C^{(1)}}{\alpha} \frac{\alpha}{\alpha^2 + \mu^2} = \frac{-i\hbar C^{(1)}}{\alpha^2 + \mu^2} = \frac{2\hbar J}{\alpha^2 + \mu^2}\sigma_0^y(\sigma_{-1}^z + \sigma_1^z)
\end{aligned}$$

Using 33, we get:

$$\begin{aligned}
A_\lambda &= \frac{-i\hbar C^{(1)}}{\mu} \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n}}{\mu^{2n+1}} \\
&= \frac{-i\hbar}{\mu\alpha} C^{(1)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{\alpha}{\mu}\right)^{2n+1} \\
&= \frac{-i\hbar}{\mu\alpha} C^{(1)} \frac{\mu\alpha}{\mu^2 + \alpha^2} \quad \text{, if } \ \alpha^2/\mu^2 < 1 \\
&= C^{(1)} \frac{-i\hbar}{\mu^2 + \alpha^2} = \frac{2\hbar J}{\alpha^2 + \mu^2}\sigma_0^y(\sigma_{-1}^z + \sigma_1^z) \quad \text{, if } \ \alpha^2/\mu^2 < 1
\end{aligned}$$

Hence, now we can use analytical continuation to claim that our result is also true when $\alpha^2/\mu^2 > 1$ since there is no divergence when $\alpha^2/\mu^2 = 1$. Hence, both methods give the same answer as they should.

After taking $\mu \to 0$ limit, we get an expression for exact gauge potential:

$$\boxed{A_\lambda = \frac{\hbar J}{8J^2 + 2\lambda^2}\sigma_0^y(\sigma_{-1}^z + \sigma_1^z)} \tag{97}$$

This expression is correct because it satisfies equation 7. And it's unique upto any term that commutes with Hamiltonian.

Why A_λ is non-zero in $\lambda \to 0$ limit? It need not be zero because additional term in Hamiltonian is $\dot{\lambda} A_\lambda$, which goes to zero in $\lambda \to 0$ limit.

I can similarly write an exact expression for additional $\sum_{j=1}^{L} h_j \sigma_j^z$ term in the Hamiltonian, although this term breaks Ising symmetry G.

Now for future purposes, let's study the following Hamiltonian:

$$H = -J \sum_j \sigma_j^x \sigma_{j+1}^x - \lambda \sigma_0^z \tag{98}$$

Apart from change in sign of J and λ, we have rotated our axes through y axis such that it interchanges z and x axes. In other words, we have done following transformation on spins ignoring the sign changes of couplings J and λ:

$$\sigma_j^x \to \sigma_j^z, \sigma_j^z \to \sigma_j^x, \sigma_j^y \to \sigma_j^y$$

Hence, we get the following expression for commutator after the required sign changes:

$$\begin{aligned}
C^{(1)} &= -2iJ\sigma_0^y(\sigma_{-1}^x + \sigma_1^x) \\
C^{(2)} &= -8J^2(\sigma_1^x\sigma_0^z\sigma_{-1}^x + \sigma_0^z) + 4J\lambda\sigma_0^x(\sigma_{-1}^x + \sigma_1^x) \\
C^{(3)} &= (16J^2 + 4\lambda^2)[H, \partial_\lambda H] = \alpha^2 C^{(1)} \\
C^{(5)} &= [H, [H, C^{(3)}]] = \alpha^2[H, [H, C^{(1)}]]] = \alpha^2 C^{(3)} = \alpha^4 C^{(1)}
\end{aligned}$$

Thus, we conclude that we get an expression for exact gauge potential:

$$\boxed{A_\lambda = \frac{\hbar J}{8J^2 + 2\lambda^2}\sigma_0^y(\sigma_{-1}^x + \sigma_1^x)} \tag{99}$$

6.2 Ising model with local transverse magnetic fields at two sites

$$H = -J \sum_j \sigma_j^x \sigma_{j+1}^x - \lambda(\sigma_0^z + \sigma_1^z) \tag{100}$$

$$\begin{aligned}
C^{(1)} &= -2iJ\sigma_0^y(\sigma_{-1}^x + \sigma_1^x) - 2iJ\sigma_1^y(\sigma_0^x + \sigma_2^x) \\
C^{(2)} &= -8J^2(\sigma_1^x\sigma_{-1}^x + 1)\sigma_0^z + 4J\lambda\sigma_0^x(\sigma_{-1}^x + \sigma_1^x) \\
&\quad - 8J^2(\sigma_0^x\ \sigma_2^x + 1)\sigma_1^z + 4J\lambda\sigma_1^x(\sigma_0^x + \sigma_2^x) - 8J\lambda\sigma_1^y\sigma_0^y \\
C^{(3)} &= \alpha^2 C^{(1)} + T_2 + T_3
\end{aligned}$$

where two body operator $T_2 = -24iJ\lambda^2(\sigma_0^y\sigma_1^x + \sigma_1^y\sigma_0^x)$, three body operator $T_3 = 32iJ^2\lambda(\sigma_{-1}^x\sigma_0^z\sigma_1^y + \sigma_0^y\sigma_1^z\sigma_2^x)$, $C_{T_2}^{(2)} = [H, [H, T_2]]$ and $C_{T_3}^{(2)} = [H, [H, T_3]]$.

Now on further computation, we find that:

$$C_{T_2}^{(2)} = \delta_1 C^{(1)} + \delta_2 T_2 + \delta_3 T_3 \tag{101}$$

$$C_{T_3}^{(2)} = \epsilon_1 C^{(1)} + \epsilon_2 T_2 + \epsilon_3 T_3 \tag{102}$$

where $\delta_1 = 96J^2\lambda^2$, $\delta_2 = 16\lambda^2$, $\delta_3 = 9\lambda^2$, $\epsilon_1 = 64J\lambda^3$, $\epsilon_2 = \frac{32}{3}J\lambda$ and $\epsilon_3 = \alpha^2$.

6.2.1 Infinite summation

Using theorem proved in appendix, we get

$$C^{(2n+1)} = \alpha^{2n} C^{(1)} + \alpha^{2n-2} \sum_{q=0}^{n-1} \alpha^{-2q} (C_{T_2}^{(2q)} + C_{T_3}^{(2q)})$$

This shows that for any subsequent $C^{(2n+1)}$, we won't have any *new* term apart from $C^{(1)}, T_2$ and T_3.

In appendix, we have proved that if $C_T^{(2)} = \delta T + O$, then $C_T^{(2n)} = \delta^n T + \delta^{n-1} \sum_{q=0}^{n-1} \delta^{-q} C_O^{(2q)}$. In our case, $C_{T_2}^{(2)} = \delta_2 T_2 + O$ where $O = \delta_1 C^{(1)} + \delta_3 T_3$. Now, let's compute $C_O^{(2q)}$:

$$\begin{aligned} C_O^{(2)} &= \delta_1 C^{(3)} + \delta_3 C_{T_3}^{(2)} \\ Similarly, C_O^{(2q)} &= \delta_1 C^{(2q+1)} + \delta_3 C_{T_3}^{(2q)} \end{aligned}$$

Hence, we see that $C_{T_2}^{(2n)} = \delta_2^n T_2 + \delta_2^{n-1} \sum_{q=0}^{n-1} \delta^{-q} (\delta_1 C^{(2q+1)} + \delta_3 C_{T_3}^{(2q)})$. Similarly, $C_{T_3}^{(2n)} = \epsilon_3^n T_3 + \epsilon_3^{n-1} \sum_{q=0}^{n-1} \epsilon_3^{-q} (\epsilon_1 C^{(2q+1)} + \epsilon_2 C_{T_2}^{(2q)})$.

We are left with a set of three coupled recurrence equations given by:

$$\boxed{\begin{aligned} C^{(2n+1)} &= \alpha^{2n} C^{(1)} + \alpha^{2n-2} \sum_{q=0}^{n-1} \alpha^{-2q} (C_{T_2}^{(2q)} + C_{T_3}^{(2q)}) \\ C_{T_2}^{(2q)} &= \delta_2^q T_2 + \delta_2^{q-1} \sum_{p=0}^{q-1} \delta_2^{-p} (\delta_1 C^{(2p+1)} + \delta_3 C_{T_3}^{(2p)}) \\ C_{T_3}^{(2q)} &= \epsilon_3^q T_3 + \epsilon_3^{q-1} \sum_{p=0}^{q-1} \epsilon_3^{-p} (\epsilon_1 C^{(2p+1)} + \epsilon_2 C_{T_2}^{(2p)}) \end{aligned}} \tag{103}$$

Now let's concentrate on $C_{T_2}^{(2q)}$ here:

$$\begin{aligned} C_{T_2}^{(2q)} &= \delta_2^q T_2 + \delta_2^{q-1} \sum_{p=0}^{q-1} \delta_2^{-p} (\delta_1 C^{(2p+1)} + \delta_3 C_{T_3}^{(2p)}) && (104) \\ &= \delta_2^q T_2 + \delta_2^{q-1} \sum_{p=0}^{q-1} \delta_2^{-p} \delta_1 C^{(2p+1)} + \delta_2^{q-1} \sum_{p=0}^{q-1} \delta_2^{-p} \delta_3 C_{T_3}^{(2p)} && (105) \\ &= \delta_2^q T_2 + \delta_2^{q-1} \sum_{p=0}^{q-1} \delta_2^{-p} \delta_1 C^{(2p+1)} + O && (106) \\ &= \delta_2^q T_2 + \delta_2^{q-1} \sum_{p=0}^{q-1} \delta_2^{-p} \delta_1 (\alpha^{2p} C^{(1)} + \alpha^{2p-2} \sum_{l=0}^{p-1} \alpha^{-2l} (C_{T_2}^{(2l)} + C_{T_3}^{(2l)})) + O && (107) \end{aligned}$$

Hence, we see that we get infinite summation expression, which seems intractable to me.

6.2.2 Linear recurrence coupled equations

We know that:

$$C^{(3)} = \alpha^2 C^{(1)} + T_2 + T_3 \tag{108}$$

$$C^{(2)}_{T_2} = \delta_1 C^{(1)} + \delta_2 T_2 + \delta_3 T_3 \tag{109}$$
$$C^{(2)}_{T_3} = \epsilon_1 C^{(1)} + \epsilon_2 T_2 + \epsilon_3 T_3 \tag{110}$$

Using the results proved in appendix, we can use the above information to write three coupled homogeneous recurrence equations, which are linear in $C^{(2n+1)}$, $C^{(2n)}_{T_2}$ and $C^{(2n)}_{T_3}$:

$$\boxed{\begin{aligned} C^{(2n+1)} &= \alpha^2 C^{(2n-1)} + C^{(2n-2)}_{T_2} + C^{(2n-2)}_{T_3} \\ C^{(2n)}_{T_2} &= \delta_1 C^{(2n-1)} + \delta_2 C^{(2n-2)}_{T_2} + \delta_3 C^{(2n-2)}_{T_3} \\ C^{(2n)}_{T_3} &= \epsilon_1 C^{(2n-1)} + \epsilon_2 C^{(2n-2)}_{T_2} + \epsilon_3 C^{(2n-2)}_{T_3} \end{aligned}} \tag{111}$$

The advantage is that these equations have constant coefficients.

6.2.3 Exact solution

Let's claim that

$$A^*_\lambda = \alpha^* C^{(1)} + \beta^* T_2 + \delta^* T_3 \tag{112}$$

where $\alpha^*, \beta^*, \delta^*$ are unknowns.

$$G_\lambda = \partial_\lambda H + \frac{i}{\hbar}[\mathcal{X}, H] \tag{113}$$

If $\mathcal{X} = A_\lambda$, then $[H, G_\lambda] = 0$. Let's reformulate it in a way that's easier to compute.

$$[H, \partial_\lambda H] = \frac{i}{\hbar}[H, [H, A^*_\lambda]] \tag{114}$$
$$C^{(1)} = \frac{i}{\hbar}[H, [H, \alpha^* C^{(1)} + \beta^* T_2 + \delta^* T_3]] \tag{115}$$
$$C^{(1)} = \frac{i}{\hbar}\left(\alpha^*[H, [H, C^{(1)}]] + \beta^*[H, [H, T_2]] + \delta^*[H, [H, T_3]]\right) \tag{116}$$
$$C^{(1)} = \frac{i}{\hbar}\left(\alpha^* C^{(3)} + \beta^* C^{(2)}_{T_2} + \delta^* C^{(2)}_{T_3}\right) \tag{117}$$

Now, we can use previous results which we obtained to see if we are on the right track.

$$C^{(3)} = \alpha^2 C^{(1)} + T_2 + T_3$$
$$C^{(2)}_{T_2} = \delta_1 C^{(1)} + \delta_2 T_2 + \delta_3 T_3$$
$$C^{(2)}_{T_3} = \epsilon_1 C^{(1)} + \epsilon_2 T_2 + \epsilon_3 T_3$$

Using the above information to simplify our equations, we get:

$$\begin{bmatrix} \alpha^2 & \delta_1 & \epsilon_1 \\ 1 & \delta_2 & \epsilon_2 \\ 1 & \delta_3 & \epsilon_3 \end{bmatrix} \begin{bmatrix} \alpha^* \\ \beta^* \\ \delta^* \end{bmatrix} = -i\hbar \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{118}$$

Using mathematica, we get

$$\alpha^* = -i\hbar \frac{(\delta_3\epsilon_2 - \delta_2\epsilon_3)}{\alpha^2\delta_3\epsilon_2 - \alpha^2\delta_2\epsilon_3 + \delta_2\epsilon_1 - \delta_3\epsilon_1 - \delta_1\epsilon_2 + \delta_1\epsilon_3} \tag{119}$$
$$\beta^* = i\hbar \frac{(\epsilon_2 - \epsilon_3)}{\alpha^2\delta_3\epsilon_2 - \alpha^2\delta_2\epsilon_3 + \delta_2\epsilon_1 - \delta_3\epsilon_1 - \delta_1\epsilon_2 + \delta_1\epsilon_3} \tag{120}$$
$$\delta^* = -i\hbar \frac{(\delta_2 - \delta_3)}{\alpha^2\delta_3\epsilon_2 - \alpha^2\delta_2\epsilon_3 + \delta_2\epsilon_1 - \delta_3\epsilon_1 - \delta_1\epsilon_2 + \delta_1\epsilon_3} \tag{121}$$

where $\alpha^2 = 4(4J^2 + \lambda^2)$, $\delta_1 = 96J^2\lambda^2$, $\delta_2 = 16\lambda^2$, $\delta_3 = 9\lambda^2$, $\epsilon_1 = 64J\lambda^3$, $\epsilon_2 = \frac{32}{3}J\lambda$ and $\epsilon_3 = \alpha^2$.

A Linear recurrence coupled equations

We have a set of three coupled linear recurrence difference equations:

$$\boxed{\begin{aligned} a_{2n+1} &= \alpha^2 a_{2n-1} + b_{2n-2} + c_{2n-2} \\ b_{2n} &= \delta_1 a_{2n-1} + \delta_2 b_{2n-2} + \delta_3 c_{2n-2} \\ c_{2n} &= \epsilon_1 a_{2n-1} + \epsilon_2 b_{2n-2} + \epsilon_3 c_{2n-2} \end{aligned}} \quad , \forall n > 0 \tag{122}$$

where $a_n = \{a_1, a_3, \ldots\}$, $b_n = \{b_0, b_2, \ldots\}$ and $c_n = \{c_0, c_2, \ldots\}$.

We are looking for a function f such that $a_n = f(a_1, b_0, c_0)$. Similarly, we will get $b_n = g(a_1, b_0, c_0)$ and $c_n = h(a_1, b_0, c_0)$.

We will solve this using generating function approach. We define three functions:

$$A(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1} \tag{123}$$

$$B(z) = \sum_{n=0}^{\infty} b_{2n} z^{2n} \tag{124}$$

$$C(z) = \sum_{n=0}^{\infty} c_{2n} z^{2n} \tag{125}$$

Using this, we get from one equation:

$$a_{2n+1} z^{2n+1} = \alpha^2 a_{2n-1} z^{2n+1} + b_{2n-2} z^{2n+1} + c_{2n-2} z^{2n+1} \tag{126}$$

$$\sum_{n=1}^{\infty} a_{2n+1} z^{2n+1} = \alpha^2 \sum_{n=1}^{\infty} a_{2n-1} z^{2n+1} + \sum_{n=1}^{\infty} b_{2n-2} z^{2n+1} + \sum_{n=1}^{\infty} c_{2n-2} z^{2n+1} \tag{127}$$

$$A(z) - a_1 z = \alpha^2 z^2 A(z) + z^3 B(z) + z^3 C(z) \tag{128}$$

$$A(z)(1 - \alpha^2 z^2) = a_1 z + z^3 B(z) + z^3 C(z) \tag{129}$$

Hence,we get :

$$\boxed{A(z) = \beta(z)(a_1 + z^2 B(z) + z^2 C(z))} \tag{130}$$

where $\beta(z) = z/(1 - \alpha^2 z^2)$

Similarly, from the second equation we get:

$$b_{2n} z^{2n} = \delta_1 a_{2n-1} z^{2n} + \delta_2 b_{2n-2} z^{2n} + \delta_3 c_{2n-2} z^{2n} \tag{131}$$

$$\sum_{n=1}^{\infty} b_{2n} z^{2n} = \delta_1 \sum_{n=1}^{\infty} a_{2n-1} z^{2n} + \delta_2 \sum_{n=1}^{\infty} b_{2n-2} z^{2n} + \delta_3 \sum_{n=1}^{\infty} c_{2n-2} z^{2n} \tag{132}$$

$$B(z) - b_0 = \delta_1 z A(z) + \delta_2 z^2 B(z) + \delta_3 z^2 C(z) \tag{133}$$

$$B(z)(1 - \delta_2 z^2) = b_0 + \delta_1 z A(z) + \delta_3 z^3 C(z) \tag{134}$$

Finally, we find from third equation:

$$C(z)(1 - \epsilon_3 z^2) = c_0 + \epsilon_1 z A(z) + \epsilon_2 z^3 B(z) \tag{135}$$

Using the value of $A(z)$ we get:

$$B(z)(1 - \delta_2 z^2) = b_0 + \delta_3 z^3 C(z) + \delta_1 z \beta(z)(a_1 + z^2 B(z) + z^2 C(z))$$

$$B(z)(1 - \delta_2 z^2 - \delta_1 z^3 \beta(z)) = b_0 + \delta_3 z^3 C(z) + \delta_1 z \beta(z)(a_1 + z^2 C(z))$$

$$
\begin{aligned}
B(z)(1-\delta_2 z^2-\delta_1 z^3\beta(z)) &= b_0+\delta_1 z\beta(z)a_1+\delta_3 z^3 C(z)+\delta_1\beta(z)z^3C(z)) \\
B(z)(1-\delta_2 z^2-\delta_1 z^3\beta(z)) &= b_0+\delta_1 z\beta(z)a_1+(\delta_3+\delta_1\beta(z))z^3C(z) \\
B(z) &= \tau(z)(\omega_0+\omega_1 z^3 C(z))
\end{aligned}
$$

where $\tau(z)=1/(1-\delta_2 z^2-\delta_1 z^3\beta(z))$, $\omega_0=b_0+\delta_1 z\beta(z)a_1$ and $\omega_1=\delta_3+\delta_1\beta(z)$.

Similarly, we get for $C(z)$:

$$
\begin{aligned}
C(z)(1-\epsilon_3 z^2) &= c_0+\epsilon_2 z^3 B(z)+\epsilon_1 z A(z) \\
C(z)(1-\epsilon_3 z^2) &= c_0+\epsilon_2 z^3 B(z)+\epsilon_1 z\beta(z)(a_1+z^2B(z)+z^2C(z)) \\
C(z)(1-\epsilon_3 z^2-\epsilon_1 z^3\beta(z)) &= c_0+\epsilon_1 z\beta(z)a_1+\epsilon_2 z^3B(z)+\epsilon_1\beta(z)z^3B(z) \\
C(z)(1-\epsilon_3 z^2-\epsilon_1 z^3\beta(z)) &= c_0+\epsilon_1 z\beta(z)a_1+(\epsilon_2+\epsilon_1\beta(z))z^3B(z) \\
C(z)(1-\epsilon_3 z^2-\epsilon_1 z^3\beta(z)) &= \theta_0+\theta_1 z^3 B(z)
\end{aligned}
$$

where $\theta_0=c_0+\epsilon_1 z\beta(z)a_1$ and $\theta_1=\epsilon_2+\epsilon_1\beta(z)$

Now, we will use the fact that $B(z)=\tau(z)(\omega_0+\omega_1 z^3C(z))$ to find out value of $C(z)$:

$$
\begin{aligned}
C(z)(1-\epsilon_3 z^2-\epsilon_1 z^3\beta(z)) &= \theta_0+\theta_1 z^3 B(z) \\
C(z)(1-\epsilon_3 z^2-\epsilon_1 z^3\beta(z)) &= \theta_0+\theta_1 z^3\tau(z)(\omega_0+\omega_1 z^3C(z)) \\
C(z)(1-\epsilon_3 z^2-\epsilon_1 z^3\beta(z)) &= \theta_0+\theta_1 z^3\tau(z)\omega_0+\theta_1\omega_1 z^6\tau(z)C(z) \\
C(z)(1-\epsilon_3 z^2-\epsilon_1 z^3\beta(z)-\theta_1\omega_1 z^6\tau(z)) &= \theta_0+\theta_1 z^3\tau(z)\omega_0 \\
C(z) &= \eta(z)(\theta_0+\theta_1 z^3\tau(z)\omega_0)
\end{aligned}
$$

where $\eta(z)=1/(1-\epsilon_3 z^2-\epsilon_1 z^3\beta(z)-\theta_1\omega_1 z^6\tau(z))$.

Hence,

$$
\boxed{C(z)=\eta(z)\theta_0+\omega_0\theta_1 z^3\eta(z)\tau(z)} \tag{136}
$$

Hence, the value of $B(z)$ is :

$$
\begin{aligned}
B(z) &= \omega_0\tau(z)+\omega_1 z^3\tau(z)C(z) \\
&= \omega_0\tau(z)+\omega_1 z^3\tau(z)\eta(z)(\theta_0+\theta_1 z^3\tau(z)\omega_0)
\end{aligned}
$$

Hence,

$$
\boxed{B(z)=\omega_0\tau(z)+\omega_1 z^3\tau(z)\eta(z)\theta_0+\omega_0\omega_1\theta_1 z^2\eta(z)\tau(z)^2} \tag{137}
$$

Now, once we figure out power series expansions of following unknowns, we can plug it back to $B(z)$ and $C(z)$:

$$
\begin{aligned}
\beta(z) &= z/(1-\alpha^2 z^2) \\
\tau(z) &= 1/(1-\delta_2 z^2-\delta_1 z^3\beta(z)) \\
\omega_0 &= b_0+\delta_1 z\beta(z)a_1 \\
\omega_1 &= \delta_3+\delta_1\beta(z) \\
\eta(z) &= 1/(1-\epsilon_3 z^2-\epsilon_1 z^3\beta(z)-\theta_1\omega_1 z^6\tau(z)) \\
\theta_0 &= c_0+\epsilon_1 z\beta(z)a_1 \\
\theta_1 &= \epsilon_2+\epsilon_1\beta(z)
\end{aligned}
$$

Now, I have got $\beta(z)=z/(1-\alpha^2z^2)=\sum_{n=0}^{\infty}\alpha^{2n}z^{2n+1}$. Hence, we get:

$$
\beta(z)=\sum_{n=0}^{\infty}\alpha^{2n}z^{2n+1}
$$

$$\omega_0 = b_0 + \delta_1 a_1 \sum_{n=0}^{\infty} \alpha^{2n} z^{2n+2}$$

$$\omega_1 = \delta_3 + \delta_1 \sum_{n=0}^{\infty} \alpha^{2n} z^{2n+1}$$

$$\theta_0 = c_0 + \epsilon_1 \sum_{n=0}^{\infty} \alpha^{2n} z^{2n+2} a_1$$

$$\theta_1 = \epsilon_2 + \epsilon_1 \sum_{n=0}^{\infty} \alpha^{2n} z^{2n+1}$$

Now we need to figure out $\tau(z)$ and then we will figure out $\eta(z)$.

$$\begin{aligned}
\tau(z)^{-1} &= 1 - \delta_2 z^2 - \delta_1 z^3 \beta(z) \\
&= 1 - \delta_2 z^2 - \delta_1 \frac{z^4}{1-\alpha^2 z^2} \\
&= \frac{1}{1-\alpha^2 z^2}(1 - \alpha^2 z^2 - \delta_2 z^2 + \alpha^2 \delta_2 z^4 - \delta_1 z^4) \\
&= \frac{1}{1-\alpha^2 z^2}(1 - (\alpha^2 + \delta_2) z^2 + (\alpha^2 \delta_2 - \delta_1) z^4) \\
&= \frac{\alpha^2 \delta_2 - \delta_1}{1-\alpha^2 z^2} \zeta(z)
\end{aligned}$$

where $\zeta(z) = (\frac{1}{\alpha^2\delta_2-\delta_1} - \frac{\alpha^2+\delta_2}{\alpha^2\delta_2-\delta_1} z^2 + z^4)$. Let's simplify it further by writing it as $\zeta(z) = (z-z_1)(z-z_2)(z-z_3)(z-z_4)$, where z_1, z_2, z_3, z_4 are roots of $\zeta(z)$. Once we do that, then we can write its' reciprocal as sum of partial fractions.

Now let's attack $\eta(z)$:

$$\begin{aligned}
\eta(z)^{-1} &= 1 - \epsilon_3 z^2 - \epsilon_1 z^3 \beta(z) - \theta_1 \omega_1 z^6 \tau(z) \\
&= 1 - \epsilon_3 z^2 - \epsilon_1 z^3 \beta(z) - \theta_1 \omega_1 z^6 (1 - \delta_2 z^2 - \delta_1 z^3 \beta(z))
\end{aligned}$$

Now, we plug these values to find value of $A(z)$:

$$A(z) = a_1 \beta(z) + z^2 \beta(z) B(z) + z^2 \beta(z) C(z) \tag{138}$$

Note that I need to find reciprocal of a polynomial here. `http://www.sciencedirect.com/science/article/pii/S0377042705006230#bib20`

B Properties of n-commutators

In this section, we would be proving some results of n-commutators $C^{(n)}$.

Theorem 1. *$C^{(n)} = [H, C^{(n-1)}], \forall n > 0$ where $C^{(0)} = \partial_\lambda H$ and $C^{(1)} = [H, \partial_\lambda H]$*

Proof. We define the first two terms as $C^{(0)} = \partial_\lambda H$ and $C^{(1)} = [H, \partial_\lambda H]$.

Now, $C^{(2)} = [H, [H, \partial_\lambda H]] = [H, C^{(1)}]$. Similarly, $C^{(3)} = [H, C^{(2)}]$. Hence, we can claim using induction argument:

$$C^{(n)} = [H, C^{(n-1)}] \tag{139}$$

□

Theorem 2. *If $C^{(3)} = \alpha^2 C^{(1)} + T$, then $C^{(2n+1)} = \alpha^{2n} C^{(1)} + \alpha^{2n-2} \sum_{q=0}^{n-1} \alpha^{-2q} C_T^{(2q)}$, $\forall n > 0$, where T is an operator, $C_T^{(0)} = T$, $C_T^{(2)} = [H, [H, T]]$ and so on.*

Proof.

$$C^{(5)} = [H, [H, C^{(3)}]] \tag{140}$$
$$= [H, [H, \alpha^2 C^{(1)} + T]] \tag{141}$$
$$= \alpha^2 C^{(3)} + [H, [H, T]] \tag{142}$$
$$= \alpha^4 C^{(1)} + \alpha^2 T + C_T^{(2)} \tag{143}$$

$$C^{(7)} = [H, [H, C^{(5)}]] \tag{144}$$
$$= [H, [H, \alpha^4 C^{(1)} + \alpha^2 T + C_T^{(2)}]] \tag{145}$$
$$= \alpha^6 C^{(1)} + \alpha^4 T + \alpha^2 C_T^{(2)} + C_T^{(4)} \tag{146}$$

Hence, in general, we can claim that

$$\begin{aligned} C^{(2n+1)} &= \alpha^{2n} C^{(1)} + \alpha^{2n-2} T + \alpha^{2n-4} C_T^{(2)} + \ldots + C_T^{(2n-2)} \\ &= \alpha^{2n} C^{(1)} + \sum_{q=0}^{n-1} \alpha^{2n-2-2q} C_T^{(2q)} \\ &= \alpha^{2n} C^{(1)} + \alpha^{2n-2} \sum_{q=0}^{n-1} \alpha^{-2q} C_T^{(2q)} \end{aligned}$$

□

If $C_T^{(2)} = \beta^2 T$, then we have a corollary result given below.

Theorem 3. *If $C^{(3)} = \alpha^2 C^{(1)} + T$ and $C_T^{(2)} = \beta^2 T$, then $C^{(2n+1)} = \alpha^{2n} C^{(1)} + \left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha^2 - \beta^2} \right) T$, $\forall n > 0$, where T is a term involving some operators and $C_T^{(2)} = [H, [H, T]]$*

Proof. Since $C_T^{(2)} = \beta^2 T$, we have $C_T^{(2n)} = \beta^{2n} T$, $n > 0$. Using this, we get:

$$\begin{aligned} C^{(2n+1)} &= \alpha^{2n} C^{(1)} + \alpha^{2n-2} \sum_{q=0}^{n-1} \alpha^{-2q} C_T^{(2q)} \\ &= \alpha^{2n} C^{(1)} + \alpha^{2n-2} T \sum_{q=0}^{n-1} \left(\frac{\beta^2}{\alpha^2} \right)^q \\ &= \alpha^{2n} C^{(1)} + \left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha^2 - \beta^2} \right) T \end{aligned}$$

□

Theorem 4. *If $C_T^{(2)} = \delta T + O$, then $C_T^{(2n)} = \delta^n T + \delta^{n-1} \sum_{q=0}^{n-1} \delta^{-q} C_O^{(2q)}$, $\forall n > 0$, where O is an operator, $C_T^{(0)} = T$, $C_T^{(2)} = [H, [H, T]]$ and so on. Similarly, $C_O^{(0)} = O$, $C_O^{(2)} = [H, [H, O]]$ and so on*

Proof.

$$C_T^{(4)} = [H, [H, C_T^{(2)}]] \tag{147}$$
$$= [H, [H, \delta T + O]] \tag{148}$$

$$= \delta C_T^{(2)} + C_O^{(2)} \tag{149}$$

$$= \delta^2 T + \delta O + C_O^{(2)} \tag{150}$$

$$C_T^{(6)} = [H, [H, C_T^{(4)}]] \tag{151}$$

$$= [H, [H, \delta^2 T + \delta O + C_O^{(2)}]] \tag{152}$$

$$= \delta^2 C_T^{(2)} + \delta C_O^{(2)} + C_O^{(4)} \tag{153}$$

$$= \delta^3 T + \delta^2 O + \delta C_O^{(2)} + C_O^{(4)} \tag{154}$$

Hence, in general, we can claim that

$$\begin{aligned} C_T^{(2n)} &= \delta^n T + \delta^{n-1} O + \delta^{n-2} C_O^{(2)} + \ldots + C_O^{(2n-2)} \\ &= \delta^n T + \sum_{q=0}^{n-1} \delta^{n-1-q} C_O^{(2q)} \\ &= \delta^n T + \delta^{n-1} \sum_{q=0}^{n-1} \delta^{-q} C_O^{(2q)} \end{aligned}$$

□

Theorem 5. *If $C^{(3)} = \alpha^2 C^{(1)} + T$, then $C^{(2n+1)} = \alpha^2 C^{(2n-1)} + C_T^{(2n-2)}$, $\forall n > 0$, where T is an operator, $C_T^{(0)} = T$, $C_T^{(2)} = [H, [H, T]]$ and so on.*

Proof. For $n = 2$, we have

$$C^{(5)} = [H, [H, C^{(3)}]] \tag{155}$$

$$= [H, [H, \alpha^2 C^{(1)} + T]] \tag{156}$$

$$= \alpha^2 C^{(3)} + C_T^{(2)} \tag{157}$$

For $n = 3$, we have

$$C^{(7)} = [H, [H, C^{(5)}]] \tag{158}$$

$$= [H, [H, \alpha^2 C^{(3)} + C_T^{(2)}]] \tag{159}$$

$$= \alpha^2 C^{(5)} + C_T^{(4)} \tag{160}$$

Hence, in general, we can claim that

$$C^{(2n+1)} = \alpha^2 C^{(2n-1)} + C_T^{(2n-2)}$$

□

Theorem 6. *If $C_T^{(2)} = \delta T + O$, then $C_T^{(2n)} = \delta C_T^{(2n-2)} + C_O^{(2n-2)}$, $\forall n > 0$, where O is an operator, $C_T^{(0)} = T$, $C_T^{(2)} = [H, [H, T]]$ and so on. Similarly, $C_O^{(0)} = O$, $C_O^{(2)} = [H, [H, O]]$ and so on*

Proof.

$$C_T^{(4)} = [H, [H, C_T^{(2)}]] \tag{161}$$

$$= [H, [H, \delta T + O]] \tag{162}$$

$$= \delta C_T^{(2)} + C_O^{(2)} \tag{163}$$

$$
\begin{aligned}
C_T^{(6)} &= [H, [H, C_T^{(4)}]] && (164)\\
&= [H, [H, \delta C_T^{(2)} + C_O^{(2)}]] && (165)\\
&= \delta C_T^{(4)} + C_O^{(4)} && (166)
\end{aligned}
$$

Hence, in general, we can claim that

$$
C_T^{(2n)} = \delta C_T^{(2n-2)} + C_O^{(2n-2)}
$$

□

C Adiabatic gauge potential formula general derivation

$$
\begin{aligned}
\langle n|A_\lambda|m\rangle &= -i\frac{\langle n|\partial_\lambda H|m\rangle}{(E_n - E_m)^2 + \mu^2}(E_n - E_m) && (167)\\
&= -i\int_0^\infty dt\; e^{-\mu t}\langle n|\partial_\lambda H|m\rangle \sin((E_n - E_m)t) && (168)\\
&= \frac{-i}{2i}\int_0^\infty dt\; e^{-\mu t}\langle n|\partial_\lambda H|m\rangle \left(e^{i(E_n-E_m)t} - e^{-i(E_n-E_m)t}\right) && (169)\\
&= -\frac{1}{2}\int_0^\infty dt\; e^{-\mu t}\left(\langle n|e^{iE_nt}\partial_\lambda H e^{-iE_mt}|m\rangle - \langle n|e^{-iE_nt}\partial_\lambda H e^{iE_mt}|m\rangle\right) && (170)
\end{aligned}
$$

Hence, we can simplify our expression by defining propagator $U = \exp(-iHt)$. We note that parameter λ is fixed while we evolve it in the *artificial time* t.

$$
\begin{aligned}
A_\lambda &= -\frac{1}{2}\int_0^\infty dt\; e^{-\mu t}[U^\dagger(t)\partial_\lambda H U(t) - U^\dagger(-t)\partial_\lambda H U(-t)] && (171)\\
&= -\frac{1}{2}\int_0^\infty dt\; e^{-\mu t}[\partial_\lambda H(t) - \partial_\lambda H(-t)] && (172)
\end{aligned}
$$

where $\partial_\lambda H(t)$ is time-evolved operator $\partial_\lambda H$ in Heisenberg picture.

We would be using Hadamard (or sometimes called Baker-Campbell-Hausdorff) formula to simplify $\partial_\lambda H(t)$.

$$
\begin{aligned}
\partial_\lambda H(t) &= U^\dagger(t)\partial_\lambda H U(t) && (173)\\
&= \exp(iHt)\partial_\lambda H \exp(-iHt) && (174)\\
&= \partial_\lambda H + it[H, \partial_\lambda H] + \left(\frac{it}{2!}\right)^2[H, [H, \partial_\lambda H]] + \left(\frac{it}{3!}\right)^3[H, [H, [H, \partial_\lambda H]]] + \ldots && (175)
\end{aligned}
$$

Similarly, for $\partial_\lambda H(-t)$, we have:

$$
\partial_\lambda H(-t) = \partial_\lambda H - it[H, \partial_\lambda H] + \left(\frac{it}{2!}\right)^2[H, [H, \partial_\lambda H]] - \left(\frac{it}{3!}\right)^3[H, [H, [H, \partial_\lambda H]]] + \ldots \quad (176)
$$

Now we see that $\partial_\lambda H(t\hbar) - \partial_\lambda H(-t\hbar)$ contains only odd power of time t:

$$
\partial_\lambda H(t\hbar) - \partial_\lambda H(-t\hbar) = 2\left[it[H, \partial_\lambda H] + \left(\frac{it}{3!}\right)^3[H, [H, \partial_\lambda H]] + \left(\frac{it}{5!}\right)^5[H, [H, [H, [H, [H, \partial_\lambda H]]]]] + \ldots\right]
$$

$$= \quad 2\sum_{n=0}^{\infty}\frac{(it)^{2n+1}}{(2n+1)!}C^{(2n+1)} \tag{177}$$
$$= \quad 2i\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}C^{(2n+1)} \tag{178}$$

where $C^{(n)}$ is n- commutator of H and $\partial_\lambda H$, i.e. $C^{(n)} = [H,[H, \text{ n times}\ldots,[H,\partial_\lambda H]]]]$. We define the first term as $C^{(1)} = [H,\partial_\lambda H]$, second term as $C^{(2)} = [H,[H,\partial_\lambda H]] = [H,C^{(1)}]$ and so on and forth. Properties of $C^{(n)}$ are noted in appendix B.

We can simplify our expression if we call $\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}C^{(2n+1)}$ as $\sin(C^{(1)}t)$. Thus, we can write:

$$A_\lambda = -i\int_0^\infty dt\ e^{-\mu t}\sin([H,\partial_\lambda H]t) = -i\hbar\int_0^\infty dt\ e^{-\mu t}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}C^{(2n+1)} \tag{179}$$

D Classical adiabatic gauge potential

Let's start by considering classical systems. For such systems, we specify the system by defining Hamiltonian $H(\lambda)$ in terms of canonical variables $q_i(\lambda,t)$ and $p_j(\lambda,t)$. where λ is an externally controlled parameter. These variables satisfy the canonical relations:

$$\{q_i,p_j\} = \delta_{ij} \tag{180}$$

where $\{\ldots\}$ denotes the Poisson bracket.

Canonical transformations are transformations of q_i and p_j to new variables $\bar{q_i}$ and $\bar{p_j}$ such that it preserves Poisson bracket. Hence,

$$\{\bar{q_i},\bar{p_j}\} = \delta_{ij} \tag{181}$$

What are gauge potentials? Gauge potential A_λ are the generators of continuous canonical transformations in parameter λ space , which can be defined as :

$$q_j(\lambda+\delta\lambda) = q_j - \frac{\partial A_\lambda}{\partial p_j}\delta\lambda \Rightarrow \frac{\partial q_j}{\partial\lambda} = -\frac{\partial A_\lambda}{\partial p_j} = \{A_\lambda,q_j\} \tag{182}$$
$$p_j(\lambda+\delta\lambda) = p_j + \frac{\partial A_\lambda}{\partial q_j}\delta\lambda \Rightarrow \frac{\partial p_j}{\partial\lambda} = \frac{\partial A_\lambda}{\partial q_j} = \{A_\lambda,p_j\} \tag{183}$$

We can verify that these transformations are canonical upto order $\delta\lambda^2$ because we can show that:

$$\{q_j(\lambda+\delta\lambda),p_j(\lambda+\delta\lambda)\} = \delta_{ij} + O(\delta\lambda^2) \tag{184}$$

Let's try to understand by taking an example of continuous canonical transformation. We would shift the position coordinate by X_i. Here our parameter λ is X_i

$$q_i(X_i,t) \quad = \quad q_i(0,t) - X_i \tag{185}$$
$$p_i(X_i,t) \quad = \quad p_i(0,t) \tag{186}$$

Using equation 183, we see that $\frac{\partial A_{X_i}}{\partial q_j} = 0$ and $-\frac{\partial A_{X_i}}{\partial p_j} = -\delta_{ij}$. Hence, $A_{X_i} = p_j + C_j$, where C_j are arbitrary constants of integration. This is the gauge choice we have got in defining these gauge potentials.

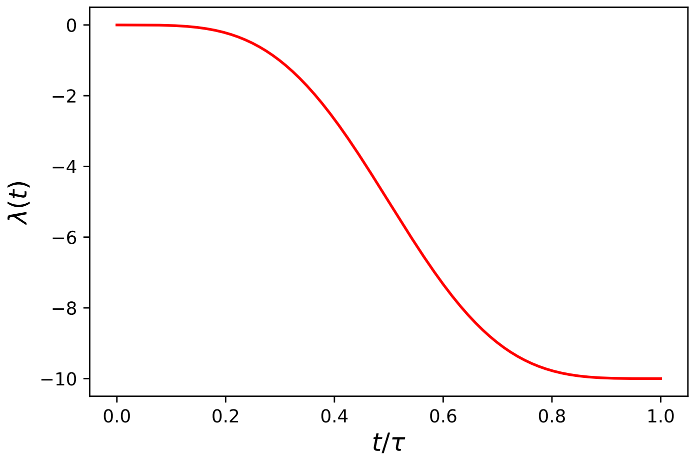

Figure 1: Protocol chosen for going from $\lambda_i = 0$ to $\lambda_f = -10J$ in time τ

E An example of variational approximation scheme: non-integrable Ising spin chain

Let's consider Ising quantum spin chain with transverse and longitudinal field whose Hamiltonian is given by:

$$H_0 = \sum_{j=1}^{L-1} J(\lambda)\sigma_j^z \sigma_{j+1}^z + \sum_{j=1}^{L}(Z_j(\lambda)\sigma_j^z + X_j(\lambda)\sigma_j^x) \tag{187}$$

We note that for either $Z_j = 0$ or $X_j = 0$, this model is integrable. Apart from these cases, this model is non-integrable. [5]

Let us consider a Counter-diabatic (CD) protocol for turning on an additional x magnetic field from $\lambda_i = 0$ to $\lambda_f = -10J$ in a periodic chain described by $H_0 + \lambda\sigma_0^x$, where H_0 is given by equation 187 with $J = 1$, $Z_j = 2$ and $X_j = 0.8$. Hence, our bare Hamiltonian H_b (which is a special case of H_0) is given by:

$$H_b = \sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + \sum_{j=1}^{L}(2\sigma_j^z + 0.8\sigma_j^x) + \lambda\sigma_0^x \tag{188}$$

where λ is a protocol.

Initial Hamiltonian is defined by $\lambda = \lambda_i = 0$ and final Hamiltonian is specified by $\lambda = \lambda_f = -10J$. Our problem is to find an approximate gauge potential such that as we tune our λ from 0 to $-10J$, we should reach the ground state of our final Hamiltonian with minimal "loss" possible after starting from the ground state of our initial Hamiltonian. If our loss is minimal, then fidelity F^2 of our final state will be *high* and energy of state above ground state $E - E_0$ would be *small*, where $F^2 = |\langle\psi(t)|\psi(t)_{GS}\rangle|^2$ and $E - E_0 = \langle\psi(t)|H|\psi(t)\rangle - \langle\psi_{GS}(t)|H|\psi(t)_{GS}\rangle$

We choose λ protocol (figure 1) that goes from $\lambda_i = 0$ to $\lambda_f = -10J$ in time τ as:

$$\lambda(t) = \lambda_0 + (\lambda_f - \lambda_0)\sin^2\left(\frac{\pi}{2}\sin^2\left(\frac{t\pi}{2\tau}\right)\right) \quad , t \in [0, \tau] \tag{189}$$

The *naive* way to drive our system will be take just our bare Hamiltonian H_b and see the performance by computing F^2 and $E - E_0$ as we change duration of protocol τ. This is shown in

[5] In [10], they have mentioned in their paper which parameter are best for this model to be robustly non-integrable. Since our method also depends on exact diagonalization, we should use their results.

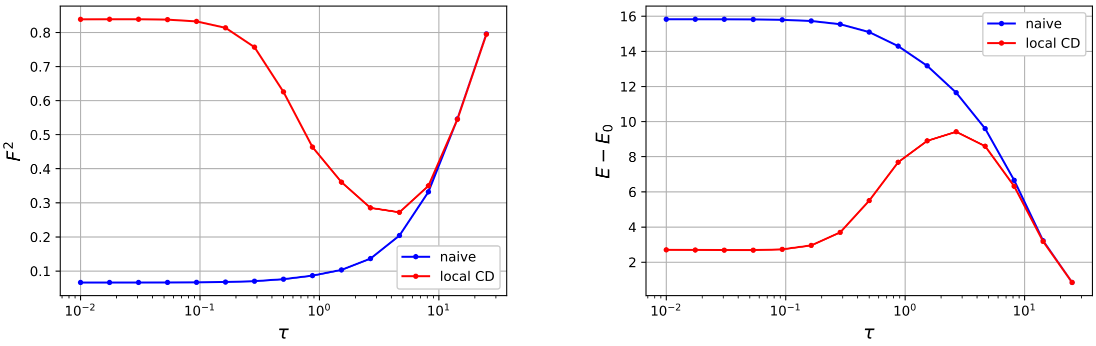

Figure 2: Fidelity F^2 and final energy above ground state $E - E_0$ for L=12 spin chains

blue line of figure 2. We note that increasing τ improves our performance no matter how we drive our system because we are going towards adiabatic limit.

For our λ - dependent Hamiltonian H_0, approximate gauge potential is chosen to be

$$A^*_\lambda = \sum_j \alpha_j \sigma^y_j \tag{190}$$

where α_j are found using variational approach given in [8]. They find that α_j for H_0 is given by

$$\alpha_j = \frac{1}{2} \frac{Z_j X'_j - X_j Z'_j}{Z_j^2 + X_j^2 + 2J^2} \tag{191}$$

Now for our H_b, α_j is given by

$$\alpha_j = \delta_{j,0} \frac{1}{6 + (\lambda + 0.8)^2} \tag{192}$$

Hence, our Hamiltonian with gauge potential term (CD term) will be :

$$\begin{aligned} H_{CD} &= H_b + \dot{\lambda} A^*_\lambda && (193) \\ &= H_b + \dot{\lambda} \alpha_0 \sigma^y_0 && (194) \end{aligned}$$

In red line of figure 2, we do find that Hamiltonian with local CD term H_{CD} does indeed give a better performance by increasing fidelity F^2 and decreasing energy above ground state $E - E_0$ for short protocol duration τ. In Dries's paper [8], they show similar results in their figure 4, where they have used spin chain of $L = 15$.

F Transverse Field Ising model: calculations in spin basis

We would study another integrable model, which is called Transverse Field Ising model. This model shows quantum phase transition between ferromagnetic and paramagnetic phases. It's Hamiltonian is given by:

$$H = J \sum_j \sigma^x_j \sigma^x_{j+1} + h \sum_j \sigma^z_j + \lambda \sigma^z_0 \tag{195}$$

This model satisfies Ising symmetry $G = \Pi_i \sigma^z_i$ since $[H, G] = 0$.

Since this model's exact gauge potential is already known in literature [4, 5], it will be good to find either exact or approximate gauge potential using our regulator method.

Let's find out A_λ for this Hamiltonian for which we need to compute different odd-powered commutator $[H, \partial_\lambda H]$, where $\partial_\lambda H = \sigma^z_0$. Here we begin:

$$C^{(1)} = -2iJ\sigma_0^y(\sigma_{-1}^x + \sigma_1^x) \tag{196}$$
$$C^{(2)} = 8J^2(\sigma_0^z + \sigma_{-1}^x\sigma_0^z\sigma_1^x) - 4J\lambda(\sigma_{-1}^x + \sigma_1^x)\sigma_0^x - 4hJ((\sigma_{-1}^x + \sigma_1^x)\sigma_0^x - (\sigma_{-1}^y + \sigma_1^y)\sigma_0^y) \tag{197}$$

$$\begin{aligned}C^{(3)} =& -8i\big(2h^2J\sigma_{-1}^x\sigma_0^y + 2h^2J\sigma_{-1}^y\sigma_0^x + 2h^2J\sigma_0^x\sigma_1^y + 2h^2J\sigma_0^y\sigma_1^x - hJ^2\sigma_{-2}^z\sigma_{-1}^z\sigma_0^y\\ &-3hJ^2\sigma_{-1}^x\sigma_0^z\sigma_1^y - 3hJ^2\sigma_{-1}^y\sigma_0^z\sigma_1^x - hJ^2\sigma_0^y\sigma_1^z\sigma_2^z\\ &+2hJ\lambda\sigma_{-1}^x\sigma_0^y + 2hJ\lambda\sigma_{-1}^y\sigma_0^x + 2hJ\lambda\sigma_0^x\sigma_1^y + 2hJ\lambda\sigma_0^y\sigma_1^x + 4J^3\sigma_{-1}^x\sigma_0^y\\ &+4J^3\sigma_0^y\sigma_1^x + J\lambda^2\sigma_{-1}^x\sigma_0^y + J\lambda^2\sigma_0^y\sigma_1^x\big)\end{aligned}$$

$$\begin{aligned}C^{(3)} =& -8i\big(2h^2J(\sigma_{-1}^x + \sigma_1^x)\sigma_0^y + 2h^2J(\sigma_{-1}^y + \sigma_1^y)\sigma_0^x - hJ^2(\sigma_{-2}^z\sigma_{-1}^z + \sigma_1^z\sigma_2^z)\sigma_0^y\\ &-3hJ^2(\sigma_{-1}^x\sigma_1^y + \sigma_{-1}^y\sigma_1^x)\sigma_0^z + 2hJ\lambda(\sigma_{-1}^x + \sigma_1^x)\sigma_0^y + 2hJ\lambda(\sigma_{-1}^y + \sigma_1^y)\sigma_0^x\\ &+4J^3(\sigma_{-1}^x + \sigma_1^x)\sigma_0^y + J\lambda^2(\sigma_{-1}^x + \sigma_1^x)\sigma_0^y\big)\end{aligned}$$

After rearranging terms of $(\sigma_{-1}^x + \sigma_1^x)\sigma_0^y$, we get :

$$\begin{aligned}C^{(3)} =& \alpha^2C^{(1)} - 16ihJ(h+\lambda)(\sigma_{-1}^y + \sigma_1^y)\sigma_0^x + 8ihJ^2(\sigma_{-2}^z\sigma_{-1}^z + \sigma_1^z\sigma_2^z)\sigma_0^y\\ &24ihJ^2(\sigma_{-1}^x\sigma_1^y + \sigma_{-1}^y\sigma_1^x)\sigma_0^z\end{aligned}$$

where $\alpha^2 = 4(4J^2 + 2h^2 + \lambda^2 + 2h\lambda) = (4J^2 + h^2 + (h+\lambda)^2)$

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