Finite temperature geometry induced by thermal pure quantum states

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In this note, I derive some prelimenary results on fidelity of TPQ states.

SPARSENESS OF LOCAL GAUGE POTENTIAL

We study the variational gauge potential of the transverse Ising model:

$$\hat{H} = -\sum_{i=1}^{\infty} J\sigma_i^z \sigma_{i+1}^z + h(\cos\theta\sigma_i^x + \sin\theta\sigma_i^z). \qquad (1)$$

The size of the chain is infinite. We set ℓ as the size of a subsystem we look at. The variational gauge potential is expanded using operator basis:

$$\hat{A} = \sum_{n=1}^{4^\ell} a_i \hat{S}_n \qquad (2)$$

where

$$\hat{S}_n \equiv \sigma_1^{\alpha_1} \otimes \sigma_2^{\alpha_2} \otimes \cdots \otimes \sigma_2^{\alpha_2}, \qquad (3)$$

$\alpha_i = 0, x, y, z$ and $\sigma^0 \equiv \hat{1}$. L_2 norm of $\hat{A}$ is

$$\|\hat{A}\|_2 = \sum_{n=1}^{4^\ell} |a_n|^2. \qquad (4)$$

We introduce a p-local operator as

$$\hat{S}_n \equiv \hat{1}_1 \otimes \cdots \otimes \hat{1}_{j-1} \otimes \sigma_j^{\alpha_j} \otimes \sigma_{j+1}^{\alpha_{j+1}} \otimes \cdots \otimes \sigma_{j+p}^{\alpha_{j+p}} \otimes \hat{1}_{j+p+1} \otimes \cdots \otimes \hat{1}_\ell, \qquad (5)$$

where j is any integer s.t. $1 \geq j \geq \ell$, and $\sigma_j^{\alpha_j}$ and $\sigma_{j+p}^{\alpha_{j+p}}$ are not the identical operator $\hat{1}$. That is, the p-local operator is the operator which is defined in the area whose length is p. We observe the ℓ dependence of the L_2 norm and p-local operators. We find that the L_2 norm of the gauge potential increases quadratically in $J > h$ and linearly in $h > J$, respectively. Regarding the p-local operators, we find that the ratio of higher p-local operators decays exponentially. In addition, the absolute value of the 1-local operators increases in $J > h$ and stays constant in in $h > J$. This is consisitent with the results of the L_2 norm.

Weak Ising interaction, Nonintegrable

We set $\theta = 0.4\pi$, and $J = 0.3h$. In Fig.1, L_2 norm increases almost linearly as ℓ increases. The coefficient of the quadratic term is, if it exists, very small. In Fig.4,

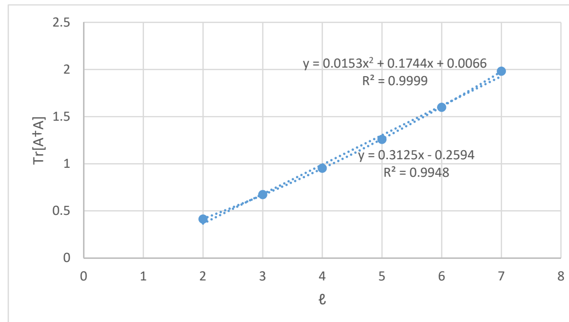
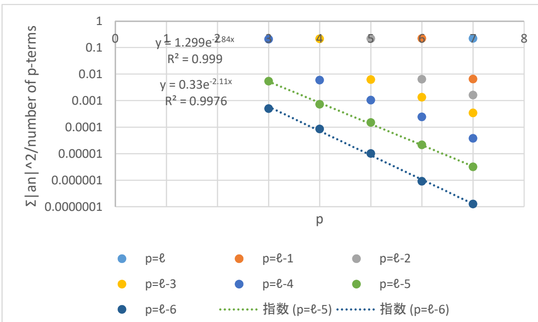


FIG. 1. L_2 norm v.s. ℓ. $\theta = 0.4\pi, J = 0.3h$. It scales almost linearly.

$$\frac{\sum_{n\in p-\text{operator}} |a_n|^2}{(\#\text{ of nonzero } p-\text{operators})} \qquad (6)$$

is plotted against p. In order to see the scaling clearly, we fix $\ell - p$ in each colors. (Usual scaling s.t. fixing ℓ and varying p is shown in Fig.4. This result also shows the exponential decay.) Here, we observe the perfect scaling in which the values decays exponentially with the exponent $\sim -2.1p$ (Blue dots).

The number of p-terms is plotted against p in Fig.**??**. Here, we fix $\ell - p$ in each colors again. (Usual scaling s.t. fixing ℓ and varying p is shown in Fig.5. The result is qualitatively the same.) It increases exponentially as p is increased, and the exponent is $\sim 1.3p$. This exponent 1.3 is smaller than 2.1 in Fig.4; therefore, Tr[A†A] is convergent. This result in Fig. 5 indicates that nonzero $|a_n|^2$ spreads to all the basis operators, because $e^{1.38\ell} \simeq 4^\ell$. Analytically, the number of p-local operators are approximately the combination of $\{hat1, \sigma^x, \sigma^y, \sigma^z\}$, and thus, it scales like 4^ℓ.



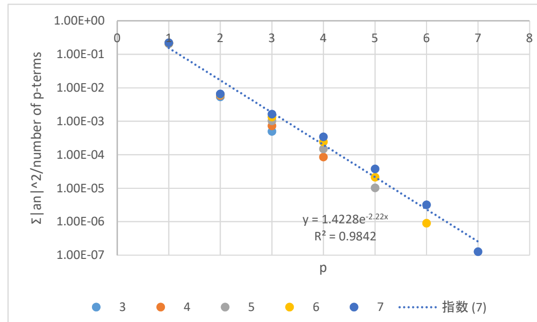
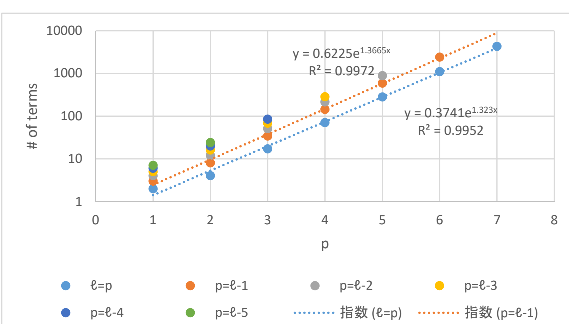


FIG. 2. Average value of the squared coefficients of p-operators v.s. p. $\theta = 0.4\pi, J = 0.3h$. It decreases exponentially as p is increased, and the exopnent is -2.1.

FIG. 4. Average value of the squared coefficients of p-operators v.s. p. $\theta = 0.4\pi, J = 0.3h$

h



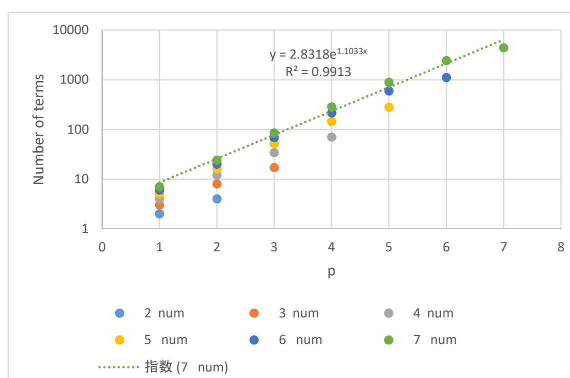


FIG. 3. The number of the p-operators v.s. p. $\theta = 0.4\pi, J = 0.3h$ The values agree well with an exponential fitting. It almost scales like 4^{ℓ}.

FIG. 5. The number of the p-operators v.s. p. $\theta = 0.4\pi, J = 0.3h$

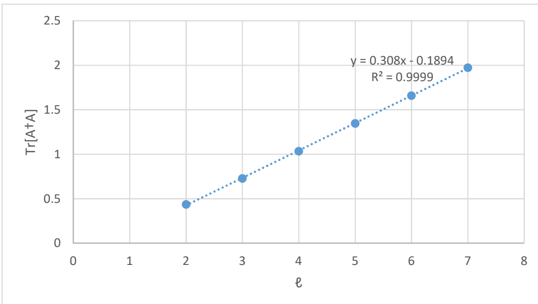


FIG. 6. L_2 norm v.s. ℓ. $\theta = 0.5\pi, J = 0.3h$. It perfectly agrees with the linear fitting.

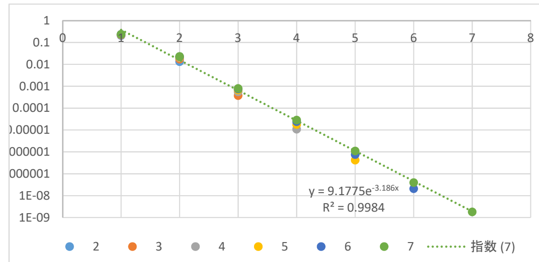


FIG. 7. Average value of the squared coefficients of p-operators v.s. p. $\theta = 0.5\pi, J = 0.3h$. It deacreases exponentially.

Weak Ising interaction, Integrable

We set $\theta = 0.5\pi$, and $J = 0.3h$. In Fig.6, the norm increases linearly. In Fig.7, the average coefficient decreases exponentially. In Fig.7, the average coefficient increases exponentially, but the exponent is slightly smaller than the nonintegrable case. These results are qualitatively the same as the nonintegrable case, $\theta = 0.4\pi$, and $J = 0.3h$.

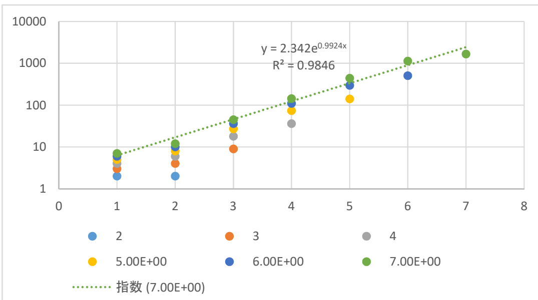


FIG. 8. The number of the p-operators v.s. p. $\theta = 0.5\pi, J = 0.3h$. It increases exponentially. The exponent is smaller than that in Fig.5.

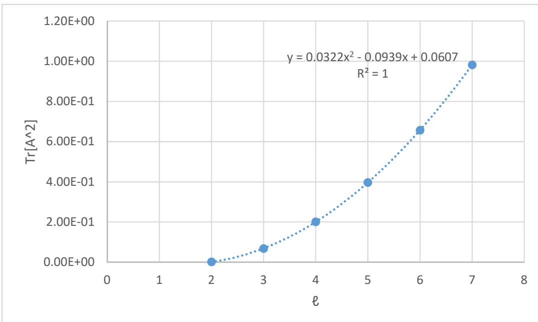


FIG. 9. L_2 norm v.s. ℓ. $\theta = 0.4\pi, J = 3h$. It increases quadratically.

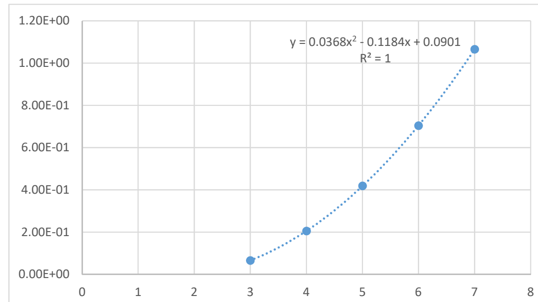


FIG. 10. L_2 norm v.s. ℓ. $\theta = 0.5\pi, J = 3h$. It increases quadratically.

Strong Ising interaction, Nonintegrable and Integrable

We set $J = 3h$. In this strong Ising regime, the norm shows quadratic growth, irrespective of integrable/nonintegrable. We show the result of the nonintegrable model in Fig.9, and that of the integrable one in Fig.10. Regarding to the average values of coefficients of the p-body operator, they decreases exponentially in both the integrable model and the nonintegrable one. We show these results in Fig.11 and Fig.12.

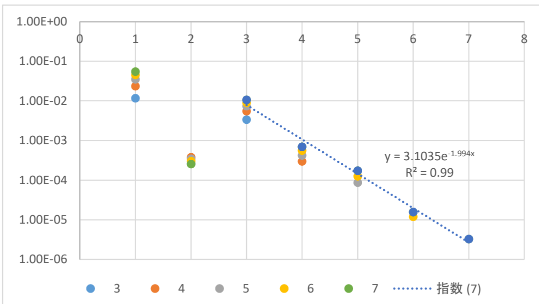


FIG. 11. Average value of the squared coefficients of p-operators v.s. p. $\theta = 0.5\pi, J = 3h$. It decreases exponentially.

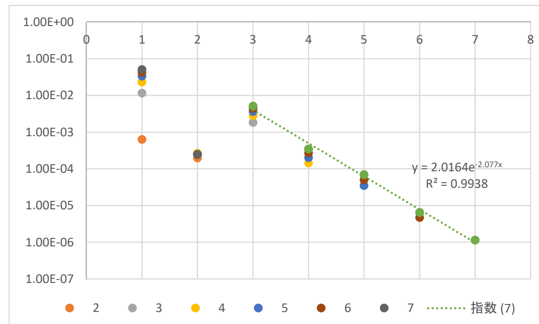


FIG. 12. Average value of the squared coefficients of p-operators v.s. p. $\theta = 0.4\pi, J = 3h$. It decreases exponentially.

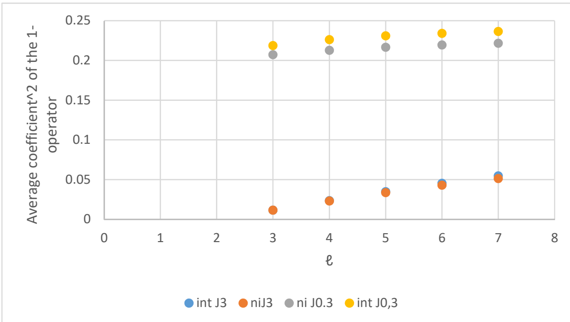


FIG. 13. Average value of the squared coefficients of 1-operators v.s. ℓ. int J3:$\theta = 0.5\pi, J = 3h$. ni J3:$\theta = 0.4\pi, J = 3h$. int J0.3:$\theta = 0.5\pi, J = 0.3h$. int J0.3:$\theta = 0.4\pi, J = 0.3h$. The coefficient of the 1-operators for $J = 0.3h$ is much larger than that for $J = 3h$

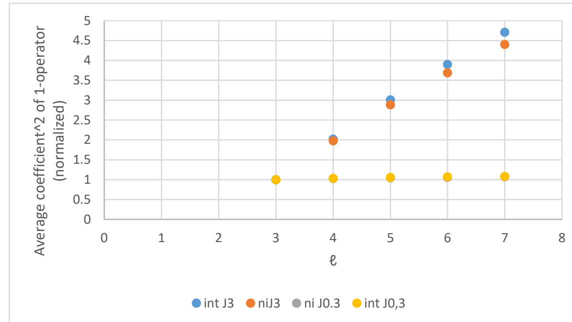


FIG. 14. Average value of the squared coefficients of 1-operators v.s. ℓ. Each value is normalized by the value of $\ell = 3$. int J3:$\theta = 0.5\pi, J = 3h$. ni J3:$\theta = 0.4\pi, J = 3h$. int J0.3:$\theta = 0.5\pi, J = 0.3h$. int J0.3:$\theta = 0.4\pi, J = 0.3h$. The value increases linearly for $J = 3h$, while it stays constant for $J = 0.3h$.

RENORMALIZATION EFFECT

We observe that the p-operators of higher p have a vanishingly small contributions in the last section. Hence, the L_2 norm is mostly determined by the 1-operators. Here we see how they behave. We find the qualitative difference between $J = 0.3h$ and $J = 3h$, irrespective of the integrability.

In Fig.13, we show the ℓ-scaling of the average value of the squared coefficients of the 1-operators. The value for $J = 0.3h$ is large and it does not change as we change ℓ. On the other hand, the value for $J = 3h$ is small and seems to increase linearly as we increase ℓ. To see whether it increases or not, we show the result of the result which is normalized by the value for $\ell = 3$. Since the contributions of the higher p-operators vanish, what determine the scaling of the L_2 norm is the scaling of the squared coefficients of the 1-operators. When the latter stays constant, the former scales $O(\ell)$, because the number of the terms of the 1-operator is ℓ. When the latter scales $O(\ell)$, the former scales $O(\ell^2)$. As this qualitative difference is not observed in the exact gauge potential, we expect that this is the renormalized effect in the local gauge potential.

SMALLEST EIGENVALUE OF LOCAL GAUGE POTENTIAL

We consider how the smallest eigenvalue scales in the local gauge potential. We use the same setup as the note by Dries. We consider the system whose Hamiltonian is

$$\hat{H}_S + \hat{H}_E + \hat{H}_{SE} \tag{7}$$

and

$$\hat{H}_{SE} = \sum_i \gamma_i S_i \otimes E_i. \tag{8}$$

We optimize the gauge potential of the form

$$\hat{A} = \hat{A}_S \otimes \hat{1}_E \tag{9}$$

Hereafter, we use quantum informational notation.

$$\hat{A} = \sum_{i,j} a_{ij} |i\rangle\langle j| \tag{10}$$

$$= \sum_{i,j} a_{ij} |i\rangle_1 \otimes |j\rangle_2 \tag{11}$$

$$\equiv |A\rangle. \tag{12}$$

The nice property of the latter notation is that we can simply write the commutator as follows.

$$[\hat{B}, \hat{A}] = \hat{B}\hat{A} - \hat{B}\hat{A} \tag{13}$$

$$= (\hat{B}_1 \otimes \hat{1}_2 - \hat{1}_1 \otimes \hat{B}_2)|A\rangle. \tag{14}$$

Namely, the commutator becomes usual operator acting on a vector. The optimization equation is

$$i(\hat{H}_1 - \hat{H}_2)|V\rangle = \left[(\hat{H}_1 - \hat{H}_2)^2 + \sum_j \gamma_j^2 (\hat{S}_j \otimes \hat{1} - \hat{1} \otimes \hat{S}_j)^2 \right] |A\rangle \tag{15}$$

where $\hat{H}_1 \equiv \hat{H} \otimes \hat{1}, \hat{H}_2 \equiv \hat{1} \otimes \hat{H}$, and $|V\rangle \equiv \partial_\lambda \hat{H}_S$. Using the solution of the eigenvalue problem,

$$\left[(\hat{H}_1 - \hat{H}_2)^2 + \sum_j \gamma_j^2 (\hat{S}_j \otimes \hat{1} - \hat{1} \otimes \hat{S}_j)^2 \right] |O_n\rangle = \omega_n^2 |O_n\rangle, \tag{16}$$

the solution of (15) is

$$|A\rangle = i \sum_n \frac{\langle O_n|(\hat{H}_1 - \hat{H}_2)|V\rangle}{\omega_n^2} |O_n\rangle. \tag{17}$$

Since exponentially small eigenvalue determine the exact gauge potentila, we are interested in the smallest eigenvalue. Multiplying $\langle O_n|$ to (16), we get

$$\omega_n^2 = \langle O_n|(\hat{H}_1 - \hat{H}_2)^2|O_n\rangle + \sum_j \gamma_j^2 \langle O_n|(\hat{S}_j \otimes \hat{1} - \hat{1} \otimes \hat{S}_j)^2|O_n\rangle. \tag{18}$$

Here, the first term and the second terms of rhs are nonnegative, respectively; therefore we examine each terms separately. I feel that the second term cannot be very small.

By the way, the eigenvector for the system without boundary is

$$(\hat{H}_1 - \hat{H}_2)^2 |E_k\rangle \otimes |E_l\rangle = (E_k - E_l)^2 |E_k\rangle \otimes |E_l\rangle, \tag{19}$$

The first term of rhs in (18) is

$$\langle O_n|(\hat{H}_1 - \hat{H}_2)^2|O_n\rangle = \sum_{k,l} b_{kl}^2 (E_k - E_l)^2. \tag{20}$$

Since $(E_k - E_l)^2$ can be exponentially small, the first term can be exponentially small.

By contrast, the second term is

$$\sum_j \gamma_j^2 \langle O_n|(\hat{S}_j \otimes \hat{1} - \hat{1} \otimes \hat{S}_j)^2|O_n\rangle. \tag{21}$$

We can understand that the eigenvectors of (16) are the energy eigenvectors after local quench at site L of two chains. The operator $(\hat{S}_j \otimes \hat{1} - \hat{1} \otimes \hat{S}_j)^2$ acts only on boundaries. To make it concrete, suppose that the Hamiltonian is

$$\hat{H} = -\sum_i J\sigma_i^z \sigma_{i+1}^z + h(\cos\theta \sigma_i^x + \sin\theta \sigma_i^z). \tag{22}$$

Then, (21) becomes

$$J^2 \langle O_n|(\hat{\sigma}_L^z \otimes \hat{1} - \hat{1} \otimes \hat{\sigma}_L^z)^2|O_n\rangle. \tag{23}$$

We rewrite $|O_n\rangle$ as

$$\begin{aligned} &|O_n\rangle \\ &= a_1|\psi_1\rangle_{1/L}|\uparrow\rangle_L|\phi_1\rangle_{2/L}|\uparrow\rangle_L + a_2|\psi_2\rangle_{1/L}|\uparrow\rangle_L|\phi_2\rangle_{2/L}|\downarrow\rangle_L \\ &+ a_3|\psi_3\rangle_{1/L}|\downarrow\rangle_L|\phi_3\rangle_{2/L}|\uparrow\rangle_L + a_4|\psi_4\rangle_{1/L}|\downarrow\rangle_L|\phi_4\rangle_{2/L}|\downarrow\rangle_L \end{aligned} \tag{24}$$

where we decompose the wave function into the product of that on site L and that on the rest. Then (23) is simply expressed as

$$J^2 \langle O_n|(\hat{\sigma}_L^z \otimes \hat{1} - \hat{1} \otimes \hat{\sigma}_L^z)^2|O_n\rangle = 4J^2(|a_2|^2 + |a_3|^2). \tag{25}$$

Therefore, both $|a_2|^2$ and $|a_3|^2$ of some eigenstates $|O_n\rangle$ have to be exponentially scmall in order to make ω_n^2 be exponentially small. When both $|a_2|^2$ and $|a_3|^2$ are exponentially small, the state is polarized state:

$$\begin{aligned} &|O_n\rangle \\ &\simeq a_1|\psi_1\rangle_{1/L}|\uparrow\rangle_L|\phi_1\rangle_{2/L}|\uparrow\rangle_L + a_4|\psi_4\rangle_{1/L}|\downarrow\rangle_L|\phi_4\rangle_{2/L}|\downarrow\rangle_L. \end{aligned} \tag{26}$$

However, (25) is likely to converge to some constant; therefore the smallest ω_n^2 is expected to have a constant lower bound.

We numerically check 18 of (22). In order to see how 21 behaves, I employ the Hamiltonian of the form

$$\begin{aligned}\hat{H} = &- h\sum_{i=1}^{L}(\cos\theta\sigma_i^z + \sin\theta\sigma_i^x) \\ &- J\sum_{i=2}^{L-1}\sigma_i^z\sigma_{i+1}^z - J_1\sigma_1^z\sigma_2^z - J_L\sigma_i^z\sigma_L^z. \end{aligned} \tag{27}$$

Here, the system is from the site 2 to the site L-1, and the site 1 and L are the environment. J_1 and J_L are the system-environment couplings γ. In Fig. 15, I plot J_1 $(= J_L)$ v.s. the smallest eigenvalue. I take $L = 8, J = 3, h = 1, \theta = 0.7\pi$. When $J_1 = J_L = 0$, the system and the environment are decoupled. Therefore, the smallest eigenvalue in this case is the smallest eigenvalue of exact gauge potential for the system size $L-2$, which is exponentially small. After we turn on J_1, the value of the smallest eigenvalue increases in propotional to J^2, which is in consistent with (18).

However, when J_1 becomes large, the value sarturates. In Fig. 16, I plot J_1 $(= J_L)$ v.s. the smallest eigenvalue for wider range of J_1 and different L's First, look at the result Then, for L=8 and 9, the values increase in propotional to J^2, but it saturates at $J_1 \simeq 1$. The saturation value decreases as we increase L. As it is hard to see whether it decreases exponentially or polynomially, we cannot conclude how (21) behaves. We need further theory to reveal the physics behind. By the way, for L=6 and 7. the values of the smallest eigenvalues have lower bound at small $J_1 < 0.1$. This lower bound is the smallest eigenvalue of exact gauge potential, which corresponds to the first term in (18). Since the first term becomes exponentially small as we increase L, this lower bound is not important here.

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[1] M. Kolodrubets, N. Gritsev, and A. Polkovnikov, Phys. Rev. B **88**, 064304 (2013).

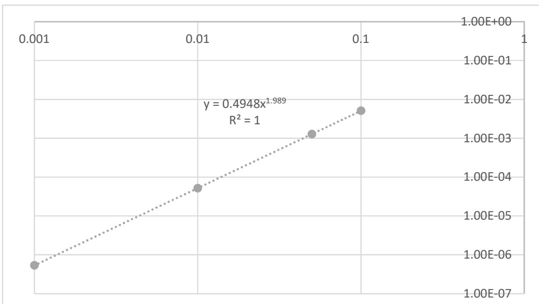
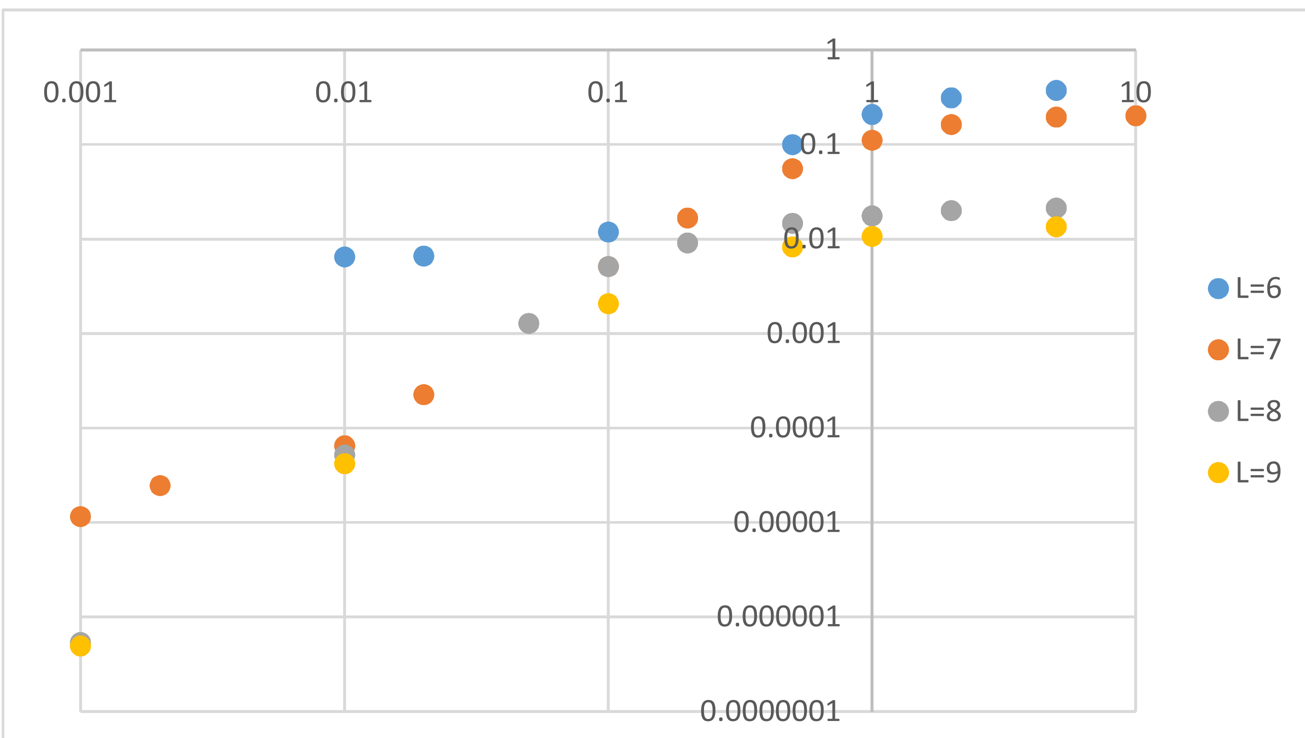


FIG. 15. The smallest eigenvalue (vertical) v.s. J_1 (horizontal). In this small J_1 region, the value of the smallest eigenvalue increase quadratically, which is consistent with Eq.(23).



1
0.001
0.01
0.1
1
10
0.1
0.01
0.001
0.0001
0.00001
0.000001
0.0000001
L=6
L=7
L=8
L=9