Counterdiabatic driving

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1 Goal

The goal, as of now, is to distinguish between integrable and non-integrable many-body quantum system by studying their approximate gauge adiabatic potential[1]

Classically, on one hand, integrable systems have a lot of constants of motion, and as a result, they have a few independent degrees of freedom. On the other hand, non-integrable systems contain a large number of independent degrees of freedom. We expect a similar picture for quantum systems.

The central idea is to apply Eigenstate Thermalization Hypothesis (ETH) to operators of exact gauge potential in non-integrable quantum systems, and claim that its' norm scales exponentially in system size. Whereas for integrable systems, exact gauge potential are supposed to scale like a polynomial in system size.

2 Introduction

2.1 Integrable and non-integrable systems

What is an integrable quantum systems? To the best of my knowledge, the general definition of integrability for quantum systems has not been reached conclusively. Despite this, there are some models which are commonly agreed to be integrable and similarly, there are model which are called non-integrable in literature. For our purposes, we would use such models to get some intuition.

Let's list down a few properties of **integrable** quantum systems:

- The many-body density matrix of those systems don't thermalize to Gibbs distribution. In fact, they thermalize to a generalized Gibbs distribution. (see [9, 2] for detail)
- They can be diagonalized using a transformation that is local in space[2]. Examples are non-interacting fermions, 1 D Ising model and 1D transverse field Ising model (TFIM). These can be diagonalized using Bogoliubov, transfer matrix method and Jordan-Wigner transformation, respectively.
- ETH doesn't apply to them
- Distribution of Energy level spacing follows Poisson distribution –energy level attraction.

We note here that many body localized (MBL) system is a new kind of integrable system. To understand its' similarity and difference from integrable system, I am quoting a paragraph from [7] :

"In order to explain the basic phenomenology of MBL systems, including their failure to thermalise, a picture of Local Integrals of Motion (LIOMs) has been put forward. According to this

[1] We expect results to be valid for classical system too. But for now, we would focus on quantum systems.

[2] According to Dries, for 2D transverse quantum Ising model, Jordan Wigner transformation exists to diagonalize the Hamiltonian. However, it's still called a non-integrable model since then the transformation becomes non-local. I need to dig relevant paper for details

picture, the basic mechanism of MBL is similar to integrable models: there emerges an extensive number of operators ("conserved charges") τ_i , which commute amongst themselves $[\tau_i, \tau_j] = 0$ as well as with the Hamiltonian $[H, \tau_i] = 0$.

A special property of MBL systems is that τ_i have eigenvalues ± 1, thus they resemble the bare spin-1/2 operators, and generically there are L such operators in a lattice system of size L. This means that any Hamiltonian eigenvector can be specified by the conserved quantum numbers corresponding to operators τ_i. Because of this extensive number of emergent quantum numbers (that by definition do not change during unitary evolution), the thermalisation of the system is prevented as the MBL state retains the memory of its initial condition. **The difference between integrable models and MBL systems is in the form of individual τ_i : in the integrable case, each τ_i. is an extended sum of local operators, while in the MBL case each τ_i. is a single local operator, up to corrections that vanish exponentially with distance to the core.** The subleading (exponentially suppressed) corrections are important, as they cause the distinction between Anderson and MBL insulators. For example, the presence of tails in LIOMs is responsible for the dephasing dynamics and the spreading of entanglement in MBL systems, which does not occur in Anderson insulators"

In [1], following Hamiltonian is considered for studying MBL:

$$\hat{H} = -\frac{J}{2}\sum_{j=1}^{N-1}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) + V\sum_{j=1}^{N-1}\sigma_j^z\sigma_{j+1}^z + \sum_{j=1}^{N} h_j\sigma_j^z) \tag{1}$$

where h_j is random magnetic field chosen from uniform distribution, i.e. $h_j \in [-W, W]$. In this model, form of τ_i is given as

$$\tau_i^z = \sigma_i^z + \sum_{j,k}\sum_{a,b=x,y,z} f_{i,j,k}^{a,b}\sigma_j^a\sigma_k^b \tag{2}$$

where weights decay exponentially with distance:

$$f_{i,j,k}^{a,b} \propto \exp(-\max\{|i-j|, |i-k|\}/\xi) \tag{3}$$

Let's list down a few properties of **non-integrable** quantum systems:

- They cannot be diagonalized using a transformation that is local in space. This is not a strong argument because it just means that such a transformation has not been found yet.
- ETH does apply to them ([3], [8])
- Distribution of Energy level spacing are correlated and therefore, they show level repulsion. They follow Wigner-Dyson or similar distributions, depending upon the details of Hamiltonian. These properties can be derived using Random Matrix Theory.

We do note that both integrable and non-integrable show quantum phase transition [3]. An example of quantum phase transition in integrable model: TFIM show paramagnetic-ferromagnetic quantum phase transition.

2.2 What are adiabatic gauge potentials?

Gauge potential

Let's represent a wavefunction in some basis:

$$|\psi\rangle = \sum_n \psi_n |n\rangle_0 \tag{4}$$

[3] Is there any difference between phase transitions shown between integrable and non-integrable models? Apparently no.

where $|n\rangle_0$ is some fixed, parameter independent basis. Now let's do a unitary basis transformation to $|m(\lambda)\rangle$ in the parameter λ dependent space using $U(\lambda)$:

$$|m(\lambda)\rangle = \sum_n U_{mn}|n\rangle \tag{5}$$

Hence, now we can express $|\psi\rangle = \sum_m \tilde{\psi_n}|m(\lambda)\rangle$, where $\tilde{\psi_n} = \langle m(\lambda)|\psi\rangle$

Quantum gauge potentials are defined to be generators of continuous unitary transformation. $\tilde{A_\lambda} = i\hbar U^\dagger \partial_\lambda U$, where $\tilde{A_\lambda}$ is in rotated (λ -dependent basis). In the lab frame, $A_\lambda = U\tilde{A_\lambda}U^\dagger = i\hbar\partial_\lambda$.

Let's take an example of a shifting transformation U to understand gauge potentials:

$$U|x'(\lambda)\rangle = |x+\lambda\rangle \tag{6}$$

We know that unitary transformation $U = \exp(-i\hat{p}\lambda/\hbar)$. Now, $\tilde{A_\lambda} = \hat{p}$ and $A_\lambda = i\hbar\partial_\lambda$.

Now why do we call it a gauge potential? In [6], they call it gauge potential because there is freedom to choose A_λ like how in EM, we have gauge choice. In [6], they say that "one can show that the gauge potentials for canonical shifts of the momentum appear exactly as the electromagnetic vector potential [see Exercise (III.1)]. Gauge potentials generalize these ideas from electromagnetism to arbitrary parameters "

Here I am listing some properties:

- They are Hermitian operator.
- $\langle n(\lambda)|A_\lambda|m(\lambda)\rangle = {}_0\langle n|\tilde{A_\lambda}|m\rangle_0$

Adiabatic gauge potential

The gauge potentials become adiabatic gauge potential when unitary transformation generated by A_λ are used to diagonalize Hamiltonian.

Adiabatic gauge potentials are a special subset of these which diagonalize the instantaneous Hamiltonian, attempting to leave its eigenbasis invariant as the parameter is changed. These adiabatic gauge potentials generate non-adiabatic corrections to Hamiltonian in the moving basis (λ -dependent basis).

This is something from Anatoli's lecture notes [6]– "an adiabatic basis is a family of adiabatically connected eigenstates, i.e., eigenstates related to a particular initial basis by adiabatic (infinitesimally slow) evolution of the parameter λ. For example, if two levels cross they will exchange order energetically but the adiabatic connection will be non-singular."

$H(\lambda)|n(\lambda)\rangle = E_n(\lambda)|n(\lambda)$. Let's derive diagonal and off-diagonal elements.

- **n-th diagonal element:** $A^n_\lambda = \langle n|A_\lambda|n\rangle = \langle n|\partial_\lambda|n\rangle$
- **off- diagonal element:** We use the identity $\langle m|H(\lambda)|n\rangle = 0 \quad , n \neq m$ and then differentiate with respect to λ to obtain:

$$\langle m|A_\lambda|n\rangle = i\hbar\frac{\langle m|\partial_\lambda H|n\rangle}{E_m - E_n} \tag{7}$$

3 Adiabatic gauge potential

Our Hamiltonian would be controlled using a control parameter called λ. Our aim would be drive the system without any transition.

Let Hamiltonian $H_0(\lambda(t))$ satisfy the following equation

$$H_0(\lambda(t))|\psi\rangle = i\partial_t|\psi\rangle \tag{8}$$

Let us go to rotating frame so as to diagonalize our Hamiltonian. Required unitary transformation $U(\lambda)$ would depend on parameter λ. Wave function in moving frame is $|\tilde{\psi}\rangle = U^\dagger|\tilde{\psi}\rangle$. In this basis, Hamiltonian is diagonal: $\tilde{H}_0 = U^\dagger H_0 U = \sum_n \epsilon(\lambda)|n(\lambda)\rangle\langle n(\lambda)|$. [4]

How does the wave function evolve in new basis?

$$i\partial_t|\tilde{\psi}\rangle = (\tilde{H}_0(\lambda(t)) - \dot{\lambda}\tilde{\mathcal{A}}_\lambda)|\psi\rangle \tag{9}$$

Note that gauge potential should be purely imaginary. But this doesn't mean that it has to be necessarily anti-Hermitian for a real Hamiltonian.

♣♣Things to include here
Derive the commutator relation, write the variational approach.

4 Our model: spin chain with transverse and longitudinal field

Let's consider Ising quantum spin chain with transverse and longitudinal field whose Hamiltonian is given by:

$$H_0 = \sum_{j=1}^{L-1} J(\lambda)\sigma_j^z\sigma_{j+1}^z + \sum_{j=1}^{L}(Z_j(\lambda)\sigma_j^z + X_j(\lambda)\sigma_j^x) \tag{10}$$

We note that for either $Z_j = 0$ or $X_j = 0$, this model is integrable. Apart from these cases, this model is non-integrable. [5]

Let us consider a Counter-diabatic (CD) protocol for turning on an additional x magnetic field from $\lambda_i = 0$ to $\lambda_f = -10J$ in a periodic chain described by $H_0 + \lambda\sigma_0^x$, where H_0 is given by equation 10 with $J = 1$, $Z_j = 2$ and $X_j = 0.8$. Hence, our bare Hamiltonian H_b (which is a special case of H_0) is given by:

$$H_b = \sum_{j=1}^{L-1} \sigma_j^z\sigma_{j+1}^z + \sum_{j=1}^{L}(2\sigma_j^z + 0.8\sigma_j^x) + \lambda\sigma_0^x \tag{11}$$

where λ is a protocol.

Initial Hamiltonian is defined by $\lambda = \lambda_i = 0$ and final Hamiltonian is specified by $\lambda = \lambda_f = -10J$. Our problem is to find an approximate gauge potential such that as we tune our λ from 0 to $-10J$, we should reach the ground state of our final Hamiltonian with minimal "loss" possible after starting from the ground state of our initial Hamiltonian. If our loss is minimal, then fidelity F^2 of our final state will be *high* and energy of state above ground state $E - E_0$ would be *small*, where $F^2 = |\langle\psi(t)|\psi(t)_{GS}\rangle|^2$ and $E - E_0 = \langle\psi(t)|H|\psi(t)\rangle - \langle\psi_{GS}(t)|H|\psi(t)_{GS}\rangle$

We choose λ protocol (figure 1) that goes from $\lambda_i = 0$ to $\lambda_f = -10J$ in time τ as:

$$\lambda(t) = \lambda_0 + (\lambda_f - \lambda_0)\sin^2\left(\frac{\pi}{2}\sin^2\left(\frac{t\pi}{2\tau}\right)\right) \quad , t \in [0, \tau] \tag{12}$$

The *naive* way to drive our system will be take just our bare Hamiltonian H_b and see the performance by computing F^2 and $E - E_0$ as we change duration of protocol τ. This is shown in blue line of figure 2. We note that increasing τ improves our performance no matter how we drive our system because we are going towards adiabatic limit.

For our λ - dependent Hamiltonian H_0, approximate gauge potential is chosen to be

$$A_\lambda^* = \sum_j \alpha_j\sigma_j^y \tag{13}$$

[4]Note that expectation value should remain same in both basis, i.e.$\langle\tilde{\psi}|\tilde{H}_0|\tilde{\psi}\rangle = \langle\psi|H_0|\psi\rangle$

[5]In [5], they have mentioned in their paper which parameter are best for this model to be robustly non-integrable. Since our method also depends on exact diagonalization, we should use their results.

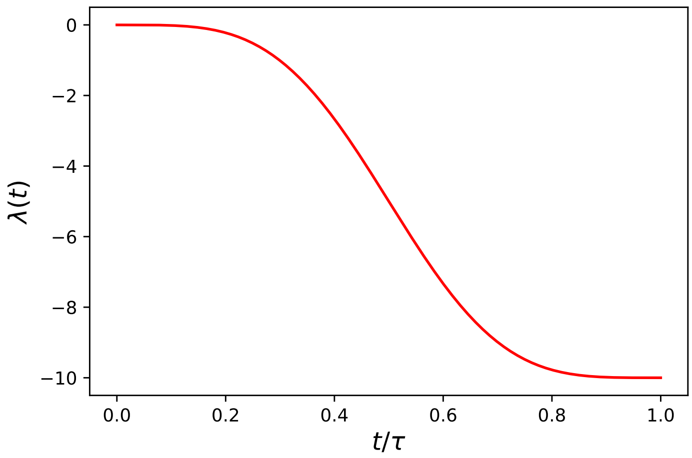

Figure 1: Protocol chosen for going from $\lambda_i = 0$ to $\lambda_f = -10J$ in time τ

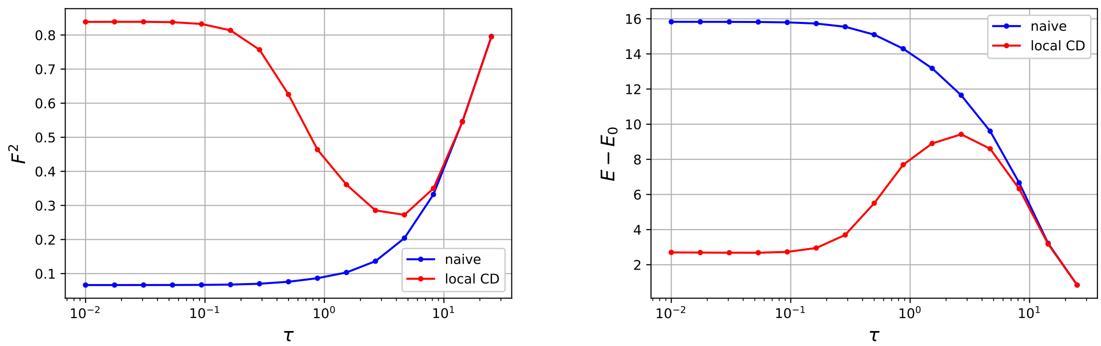

Figure 2: Fidelity F^2 and final energy above ground state $E - E_0$ for L=12 spin chains

where α_j are found using variational approach given in [10]. They find that α_j for H_0 is given by

$$\alpha_j = \frac{1}{2} \frac{Z_j X_j' - X_j Z_j'}{Z_j^2 + X_j^2 + 2J^2} \tag{14}$$

Now for our H_b, α_j is given by

$$\alpha_j = \delta_{j,0} \frac{1}{6 + (\lambda + 0.8)^2} \tag{15}$$

Hence, our Hamiltonian with gauge potential term (CD term) will be :

$$\begin{aligned} H_{CD} &= H_b + \dot{\lambda} A_\lambda^* && (16) \\ &= H_b + \dot{\lambda} \alpha_0 \sigma_0^y && (17) \end{aligned}$$

In red line of figure 2, we do find that Hamiltonian with local CD term H_{CD} does indeed give a better performance by increasing fidelity F^2 and decreasing energy above ground state $E - E_0$ for short protocol duration τ. In Dries's paper [10], they show similar results in their figure 4, where they have used spin chain of $L = 15$.

5 Regulator based method to find Gauge Potential

Here we would introduce a new method to find Gauge Potential A_λ which includes a regulator μ.

Let's start off by writing the off-diagonal elements of exact gauge potential:

$$\langle m|A_\lambda|n\rangle = i\hbar\frac{\langle m|\partial_\lambda H|n\rangle}{E_m - E_n} \tag{18}$$

For a many-body Hamiltonian, number of states in Hilbert space grows exponentially in system size while energy bandwidth grows linearly with system size (since energy is an extensive quantity). Thus, distance between any two *nearby* eigenvalues is exponentially small in system size. In other words, $E_m - E_n \sim e^{-S}$. If there are non-zero off-diagonal elements of $\partial_\lambda H$, then $\langle m|A_\lambda|n\rangle$ is ill-defined. It's called small denominator problem [6].

To resolve this problem, we introduce a regulator/ cutoff μ that regularizes our exact gauge potential in large system size L limit. Once we have taken large L limit, then we take small μ limit. Hence, if this method works, the right way to take limits will be:

$$\langle n|A_\lambda|m\rangle = \lim_{\mu\to 0}\lim_{L\to\infty} i\hbar\frac{\langle n|\partial_\lambda H|m\rangle}{(E_n - E_m)^2+\mu^2}(E_n - E_m) \tag{19}$$

where we have chosen diagonal elements of A_λ to be zero [6]. Now we will use Laplace transform with $s = \mu$:

$$\begin{aligned}
\langle n|A_\lambda|m\rangle &= i\hbar\frac{\langle n|\partial_\lambda H|m\rangle}{(E_n - E_m)^2+\mu^2}(E_n - E_m) && (20)\\
&= i\hbar\int_0^\infty dt\ e^{-\mu t}\langle n|\partial_\lambda H|m\rangle\sin((E_n - E_m)t) && (21)\\
&= \frac{i\hbar}{2i}\int_0^\infty dt\ e^{-\mu t}\langle n|\partial_\lambda H|m\rangle\left(e^{i(E_n-E_m)t} - e^{-i(E_n-E_m)t}\right) && (22)\\
&= \frac{\hbar}{2}\int_0^\infty dt\ e^{-\mu t}\left(\langle n|e^{iE_nt}\partial_\lambda H e^{-iE_mt}|m\rangle - \langle n|e^{-iE_nt}\partial_\lambda H e^{iE_mt}|m\rangle\right) && (23)
\end{aligned}$$

Hence, we can simplify our expression by defining propagator $U = \exp(-iHt/\hbar)$. We note that parameter λ is fixed while we evolve it in the *artificial time* t.

$$\begin{aligned}
A_\lambda &= \frac{\hbar}{2}\int_0^\infty dt\ e^{-\mu t}[U^\dagger(t\hbar)\partial_\lambda H U(t\hbar) - U^\dagger(-t\hbar)\partial_\lambda H U(-t\hbar)] && (24)\\
&= \frac{\hbar}{2}\int_0^\infty dt\ e^{-\mu t}[\partial_\lambda H(t\hbar) - \partial_\lambda H(-t\hbar)] && (25)
\end{aligned}$$

where $\partial_\lambda H(t)$ is time-evolved operator $\partial_\lambda H$ in Heisenberg picture.

We would be using Hadamard formula to simplify $\partial_\lambda H(t)$.

$$\begin{aligned}
\partial_\lambda H(t) &= U^\dagger(t)\partial_\lambda H U(t) && (26)\\
&= \exp(iHt/\hbar)\partial_\lambda H\exp(-iHt/\hbar) && (27)\\
&= \partial_\lambda H + \frac{it}{\hbar}[H,\partial_\lambda H] + \left(\frac{it}{\hbar}\right)^2[H,[H,\partial_\lambda H]] + \left(\frac{it}{3!\hbar}\right)^3[H,[H,[H,\partial_\lambda H]]] + \ldots && (28)
\end{aligned}$$

Similarly, for $\partial_\lambda H(-t)$, we have:

$$\partial_\lambda H(-t) = \partial_\lambda H - \frac{it}{\hbar}[H,\partial_\lambda H] + \left(\frac{it}{\hbar}\right)^2[H,[H,\partial_\lambda H]] - \left(\frac{it}{3!\hbar}\right)^3[H,[H,[H,\partial_\lambda H]]] + \ldots \tag{29}$$

[6] Dries says we can do this without any loss of generality. I am not sure if we can always do this.

Now we see that $\partial_\lambda H(t\hbar) - \partial_\lambda H(-t\hbar)$ contains only odd power of time t:

$$\begin{aligned}
\partial_\lambda H(t\hbar) - \partial_\lambda H(-t\hbar) &= 2\left[it[H,\partial_\lambda H] + \left(\frac{it}{3!}\right)^3 [H,[H,\partial_\lambda H]] + \left(\frac{it}{5!}\right)^5 [H,[H,[H,[H,[H,\partial_\lambda H]]]]] + \dots\right] \\
&= 2\sum_{n=0}^{\infty} \frac{(it)^{2n+1}}{(2n+1)!} C^{2n+1} \qquad (30) \\
&= 2i\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} C^{2n+1} \qquad (31)
\end{aligned}$$

where C^n is n- commutator of H and $\partial_\lambda H$, i.e. $C_n = [H,[H, \text{ n times}\dots,[H,\partial_\lambda H]]]]$

We can simplify our expression if we call $\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} C^{2n+1}$ as $\sin(C^{(1)}t)$, where $C^{(1)} = [H,\partial_\lambda H]$. Thus, we can write:

$$\boxed{A_\lambda = i\hbar \int_0^\infty dt\ e^{-\mu t} \sin([H,\partial_\lambda H]t) = i\hbar \int_0^\infty dt\ e^{-\mu t} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} C^{2n+1}} \qquad (32)$$

Can we further simplify the expression? If we are allowed to change the order of summation and integration [7], then we can do first Laplace transform of t^{2n+1} terms and then later the sum. Hence, we get:

$$\begin{aligned}
A_\lambda &= i\hbar \int_0^\infty dt\ e^{-\mu t} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} C^{2n+1} \qquad (33) \\
&= i\hbar \sum_{n=0}^{\infty} (-1)^n C^{2n+1} \int_0^\infty dt\ e^{-\mu t} \frac{t^{2n+1}}{(2n+1)!} \qquad (34) \\
&= i\hbar \sum_{n=0}^{\infty} (-1)^n \frac{C^{2n+1}}{\mu^{2n+2}} \qquad (35)
\end{aligned}$$

Hence, we get another expression where we have integrated before taking the summation:

$$\boxed{A_\lambda = \frac{i\hbar}{\mu} \sum_{n=0}^{\infty} (-1)^n \frac{C^{2n+1}}{\mu^{2n+1}}} \qquad (36)$$

We note that iC^{2n+1} is Hermitian, which is consistent with the fact that A_λ is Hermitian. Also, we see that while doing the infinite summation we have assumed infinite system size. Why? In general, C^n grows with n in the sense that it would have operators with larger support over lattice sites as n increases. At a certain n_L that is proportional to system length L, we would find that C^{n_L} has operators with support on boundary lattice sites. This is where our summation would be truncated for a finite system. Hence, the correct order of limits should be:

$$\boxed{A_\lambda = \lim_{\mu\to 0} \lim_{L\to\infty} \frac{i\hbar}{\mu} \sum_{n=0}^{n_L} (-1)^n \frac{C^{2n+1}}{\mu^{2n+1}}} \qquad (37)$$

Now one thing which is good is that if we take the wrong order of limit: take $\lim_{\mu\to 0}$ before $\lim_{L\to\infty}$, then A_λ diverges. Thus, now divergence is more explicit than the original expression 19.

How should μ scale as L because it seems that C^n does definitely depend on L? Let's suppose $C^n \propto L^\gamma$, where γ is some constant which we don't know. If we assume that A_λ is well-defined

[7] If there is some singularity, then the order of summation and integration might be important and we might get two different results.

in large system size limit for many-body Hamiltonian (both integrable and non-integrable), then $\mu \propto L^{\gamma}$.

Now our task will be use it to find exact/approximate gauge potential for integrable and non-integrable models.

5.1 Integrable model

Ising model with local transverse magnetic field

We would take the simplest integrable Hamiltonian with Ising interaction and a local x magnetic field:

$$H = J\sum_{j} \sigma_j^z \sigma_{j+1}^z + \lambda \sigma_0^x \tag{38}$$

where boundary conditions are not important. Commutation relation followed by spin operators are:

$$[\sigma_i^a, \sigma_j^b] = 2i\delta_{i,j}\sum_{c} \epsilon_{abc}\sigma_i^c \tag{39}$$

where ϵ_{abc} is the Levi-Civita symbol, δ_{ij} is the Kronecker delta.

This model satisfies Ising symmetry $G = \Pi_i \sigma_i^x$ since $[H, G] = 0$.

Let's find out A_λ for this Hamiltonian for which we need to compute different odd-powered commutator $[H, \partial_\lambda H]$, where $\partial_\lambda H = \sigma_0^x$. Here we begin:

$$\begin{aligned}
C^{(1)} &= 2iJ\sigma_0^y(\sigma_{-1}^z + \sigma_1^z)\\
C^{(2)} &= 8J^2(\sigma_1^z\sigma_0^x\sigma_{-1}^z + \sigma_0^x) - 4J\lambda\sigma_0^z(\sigma_{-1}^z + \sigma_1^z)\\
C^{(3)} &= (16J^2 + 4\lambda^2)[H, \partial_\lambda] = \alpha^2[H, \partial_\lambda H]\\
[H,[H,[H,[H,[H,\partial_\lambda H]]]]] &= \alpha^2[H,[H,[H,\partial_\lambda H]]] = \alpha^4[H,\partial_\lambda H]
\end{aligned}$$

Hence, $C^{2n+1} = \alpha^{2n}C^{(1)}$, where $\alpha^2 = 4(4J^2 + \lambda^2)$. Now, we would compute A_λ using two methods and compare our results. Using 32, we get :

$$\begin{aligned}
A_\lambda &= i\hbar C^{(1)}\int_0^\infty dt\ e^{-\mu t}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\alpha^{2n}\\
&= i\hbar C^{(1)}\int_0^\infty dt\ e^{-\mu t}\sum_{n=0}^{\infty}\frac{(-1)^n \alpha^{2n+1} t^{2n+1}}{\alpha(2n+1)!}\\
&= \frac{i\hbar C^{(1)}}{\alpha}\int_0^\infty dt\ e^{-\mu t}\sum_{n=0}^{\infty}\frac{(-1)^n (\alpha t)^{2n+1}}{(2n+1)!}\\
&= \frac{i\hbar C^{(1)}}{\alpha}\int_0^\infty dt\ e^{-\mu t}\sin(\alpha t)\\
&= \frac{i\hbar C^{(1)}}{\alpha}\frac{\alpha}{\alpha^2+\mu^2} = \frac{i\hbar C^{(1)}}{\alpha^2+\mu^2} = \frac{-2\hbar J}{\alpha^2+\mu^2}\sigma_0^y(\sigma_{-1}^z + \sigma_1^z)
\end{aligned}$$

Using 36, we get:

$$\begin{aligned}
A_\lambda &= \frac{i\hbar C^{(1)}}{\mu}\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^{2n}}{\mu^{2n+1}} \\
&= \frac{i\hbar}{\mu\alpha}C^{(1)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\alpha}{\mu}\right)^{2n+1} \\
&= \frac{i\hbar}{\mu\alpha}C^{(1)}\frac{\mu\alpha}{\mu^2+\alpha^2} \quad ,\text{if } \ \alpha^2/\mu^2<1 \\
&= C^{(1)}\frac{i\hbar}{\mu^2+\alpha^2} = \frac{-2\hbar J}{\alpha^2+\mu^2}\sigma_0^y(\sigma_{-1}^z+\sigma_1^z) \quad ,\text{if } \ \alpha^2/\mu^2<1
\end{aligned}$$

Hence, now we can use analytical continuation to claim that our result is also true when $\alpha^2/\mu^2>1$ since there is no divergence when $\alpha^2/\mu^2=1$. Hence, both methods give the same answer as it should.

After taking $\mu\to 0$ limit, we get an expression for exact gauge potential:

$$\boxed{A_\lambda = \frac{-\hbar J}{8J^2+2\lambda^2}\sigma_0^y(\sigma_{-1}^z+\sigma_1^z)} \tag{40}$$

Now one thing is good that $A_\lambda \propto \sigma_0^y$ since Hamiltonian is real in $\{\sigma_z^i\}$ basis and σ_0^y is imaginary, which is consistent with calculation shown in [10].

One thing which I don't understand is why A_λ is non-zero in $\lambda\to 0$ limit[8]. Algebraically, it's because $C^{(n)}\propto\partial_\lambda H$, which is constant in λ for our model. What are the diagonal elements of A_λ? Is it consistent with our assumption that diagonal elements are zero? I think it's probably because we set diagonal element to be zero as diagonal element depends on derivatives of eigenvalues $|n(\lambda)\rangle$. I think I should be able to come up with a perturbation argument here.

I can similarly write an exact expression for additional $\sum_{j=1}^{L}h_j\sigma_j^z$ term in the Hamiltonian, although this term breaks Ising symmetry G.

Transverse Field Ising model

We would study another integrable model, which is called Transverse Field Ising model. This model shows quantum phase transition between ferromagnetic and paramagnetic phases. This model satisfies Ising symmetry $G=\Pi_i\sigma_i^z$ since $[H,G]=0$, where H is the Hamiltonian. It's Hamiltonian is given by:

$$H = -J\sum_j\sigma_j^x\sigma_{j+1}^x - \lambda\sum_j\sigma_j^z \tag{41}$$

where we have not specified boundary conditions and λ is externally-controlled transverse magnetic field.

This model can be written in terms of spinless fermions $(c_i, c_i^\dagger)$ using Jordan- Wigner transformation.

This model's exact gauge potential is already known in literature [4, 6] and it's given by:

$$A_\lambda = \sum_l \alpha_l O_l \tag{42}$$

where O_l is given by

$$O_l = 2i\sum_j(c_j^\dagger c_{j+l}^\dagger - \text{h.c}) \tag{43}$$

[8]Alright, it need not be because additional term in Hamiltonian is $\dot\lambda A_\lambda$, which goes to zero in $\lambda\to 0$ limit.

It will be good to find either exact or approximate gauge potential using our regulator method.

Jordan Wigner transformation:

A Spin 1/2 particle in a time-dependent magnetic field

I would include a derivation from lecture notes to gain an intuition here. I also plan to understand Berry's paper and reproduce some of his calculations in this appendix.

B Free interacting fermions in an external potential

$$H_0 = -J\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \sum_{j=1}^{L} V_j(\lambda)c_j^\dagger c_j \tag{44}$$

$$\mathcal{A}_\lambda^* = i\sum_{j=1}^{L-1}\alpha_j(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j) \tag{45}$$

I should include pictures drawn using sympy here.

C Classical adiabatic gauge potential

Let's start by considering classical systems. For such systems, we specify the system by defining Hamiltonian $H(\lambda)$ in terms of canonical variables $q_i(\lambda, t)$ and $p_j(\lambda, t)$. where λ is an externally controlled parameter. These variables satisfy the canonical relations:

$$\{q_i, p_j\} = \delta_{ij} \tag{46}$$

where $\{\ldots\}$ denotes the Poisson bracket.

Canonical transformations are transformations of q_i and p_j to new variables $\bar{q_i}$ and $\bar{p_j}$ such that it preserves Poisson bracket. Hence,

$$\{\bar{q_i}, \bar{p_j}\} = \delta_{ij} \tag{47}$$

What are gauge potentials? Gauge potential A_λ are the generators of continuous canonical transformations in parameter λ space , which can be defined as :

$$q_j(\lambda + \delta\lambda) = q_j - \frac{\partial A_\lambda}{\partial p_j}\delta\lambda \Rightarrow \frac{\partial q_j}{\partial \lambda} = -\frac{\partial A_\lambda}{\partial p_j} = \{A_\lambda, q_j\} \tag{48}$$

$$p_j(\lambda + \delta\lambda) = p_j + \frac{\partial A_\lambda}{\partial q_j}\delta\lambda \Rightarrow \frac{\partial p_j}{\partial \lambda} = \frac{\partial A_\lambda}{\partial q_j} = \{A_\lambda, p_j\} \tag{49}$$

We can verify that these transformations are canonical upto order $\delta\lambda^2$ because we can show that:

$$\{q_j(\lambda + \delta\lambda), p_j(\lambda + \delta\lambda)\} = \delta_{ij} + O(\delta\lambda^2) \tag{50}$$

Let's try to understand by taking an example of continuous canonical transformation. We would shift the position coordinate by X_i. Here our parameter λ is X_i

$$q_i(X_i, t) = q_i(0, t) - X_i \tag{51}$$

$$p_i(X_i, t) = p_i(0, t) \tag{52}$$

Using equation 49, we see that $\frac{\partial A_{X_i}}{\partial q_j} = 0$ and $-\frac{\partial A_{X_i}}{\partial p_j} = -\delta_{ij}$. Hence, $A_{X_i} = p_j + C_j$, where C_j are arbitrary constants of integration. This is the gauge choice we have got in defining these gauge potentials.

D Transverse Field Ising model: calculations in spin basis

We would study another integrable model, which is called Transverse Field Ising model. This model shows quantum phase transition between ferromagnetic and paramagnetic phases. It's Hamiltonian is given by:

$$H = J\sum_j \sigma_j^x\sigma_{j+1}^x + h\sum_j \sigma_j^z + \lambda\sigma_0^z \tag{53}$$

This model satisfies Ising symmetry $G = \Pi_i \sigma_i^z$ since $[H, G] = 0$.

Since this model's exact gauge potential is already known in literature [4, 6], it will be good to find either exact or approximate gauge potential using our regulator method.

Let's find out A_λ for this Hamiltonian for which we need to compute different odd-powered commutator $[H, \partial_\lambda H]$, where $\partial_\lambda H = \sigma_0^z$. Here we begin:

$$C^{(1)} = -2iJ\sigma_0^y(\sigma_{-1}^x + \sigma_1^x) \tag{54}$$

$$C^{(2)} = 8J^2(\sigma_0^z + \sigma_{-1}^x\sigma_0^z\sigma_1^x) - 4J\lambda(\sigma_{-1}^x + \sigma_1^x)\sigma_0^x - 4hJ((\sigma_{-1}^x + \sigma_1^x)\sigma_0^x - (\sigma_{-1}^y + \sigma_1^y)\sigma_0^y) \tag{55}$$

$$\begin{aligned}C^{(3)} = &-8i\big(2h^2J\sigma_{-1}^x\sigma_0^y + 2h^2J\sigma_{-1}^y\sigma_0^x + 2h^2J\sigma_0^x\sigma_1^y + 2h^2J\sigma_0^y\sigma_1^x - hJ^2\sigma_{-2}^z\sigma_{-1}^z\sigma_0^y\\ &-3hJ^2\sigma_{-1}^x\sigma_0^z\sigma_1^y - 3hJ^2\sigma_{-1}^y\sigma_0^z\sigma_1^x - hJ^2\sigma_0^y\sigma_1^z\sigma_2^z\\ &+2hJ\lambda\sigma_{-1}^x\sigma_0^y + 2hJ\lambda\sigma_{-1}^y\sigma_0^x + 2hJ\lambda\sigma_0^x\sigma_1^y + 2hJ\lambda\sigma_0^y\sigma_1^x + 4J^3\sigma_{-1}^x\sigma_0^y\\ &+4J^3\sigma_0^y\sigma_1^x + J\lambda^2\sigma_{-1}^x\sigma_0^y + J\lambda^2\sigma_0^y\sigma_1^x\big)\end{aligned}$$

$$\begin{aligned}C^{(3)} = &-8i\big(2h^2J(\sigma_{-1}^x + \sigma_1^x)\sigma_0^y + 2h^2J(\sigma_{-1}^y + \sigma_1^y)\sigma_0^x - hJ^2(\sigma_{-2}^z\sigma_{-1}^z + \sigma_1^z\sigma_2^z)\sigma_0^y\\ &-3hJ^2(\sigma_{-1}^x\sigma_1^y + \sigma_{-1}^y\sigma_1^x)\sigma_0^z + 2hJ\lambda(\sigma_{-1}^x + \sigma_1^x)\sigma_0^y + 2hJ\lambda(\sigma_{-1}^y + \sigma_1^y)\sigma_0^x\\ &+4J^3(\sigma_{-1}^x + \sigma_1^x)\sigma_0^y + J\lambda^2(\sigma_{-1}^x + \sigma_1^x)\sigma_0^y\big)\end{aligned}$$

After rearranging terms of $(\sigma_{-1}^x + \sigma_1^x)\sigma_0^y$, we get :

$$\begin{aligned}C^{(3)} = &\alpha^2C^{(1)} - 16ihJ(h+\lambda)(\sigma_{-1}^y + \sigma_1^y)\sigma_0^x + 8ihJ^2(\sigma_{-2}^z\sigma_{-1}^z + \sigma_1^z\sigma_2^z)\sigma_0^y\\ &24ihJ^2(\sigma_{-1}^x\sigma_1^y + \sigma_{-1}^y\sigma_1^x)\sigma_0^z\end{aligned}$$

where $\alpha^2 = 4(4J^2 + 2h^2 + \lambda^2 + 2h\lambda) = (4J^2 + h^2 + (h+\lambda)^2)$

References

[1] Dmitry A Abanin and Zlatko Papić. Recent progress in many-body localization. *arXiv preprint arXiv:1705.09103*, 2017.

[2] Amy C Cassidy, Charles W Clark, and Marcos Rigol. Generalized thermalization in an integrable lattice system. *Physical review letters*, 106(14):140405, 2011.

[3] Luca D'Alessio, Yariv Kafri, Anatoli Polkovnikov, and Marcos Rigol. From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics. *Advances in Physics*, 65(3):239–362, 2016.

[4] Adolfo del Campo, Marek M Rams, and Wojciech H Zurek. Assisted finite-rate adiabatic passage across a quantum critical point: exact solution for the quantum ising model. *Physical review letters*, 109(11):115703, 2012.

[5] Hyungwon Kim and David A Huse. Ballistic spreading of entanglement in a diffusive nonintegrable system. *Physical review letters*, 111(12):127205, 2013.

[6] Michael Kolodrubetz, Pankaj Mehta, and Anatoli Polkovnikov. Geometry and non-adiabatic response in quantum and classical systems. *arXiv preprint arXiv:1602.01062*, 2016.

[7] TE O'Brien, Dmitry A Abanin, Guifre Vidal, and Z Papić. Explicit construction of local conserved operators in disordered many-body systems. *Physical Review B*, 94(14):144208, 2016.

[8] Marcos Rigol, Vanja Dunjko, and Maxim Olshanii. Thermalization and its mechanism for generic isolated quantum systems. *Nature*, 452(7189):854–858, 2008.

[9] Marcos Rigol, Vanja Dunjko, Vladimir Yurovsky, and Maxim Olshanii. Relaxation in a completely integrable many-body quantum system: an ab initio study of the dynamics of the highly excited states of 1d lattice hard-core bosons. *Physical review letters*, 98(5):050405, 2007.

[10] Dries Sels and Anatoli Polkovnikov. Minimizing irreversible losses in quantum systems by local counterdiabatic driving. *Proceedings of the National Academy of Sciences*, page 201619826, 2017.