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Editors' Choice

Quantum-Classical Correspondence of Shortcuts to Adiabaticity

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We formulate the theory of shortcuts to adiabaticity in classical mechanics. For a reference Hamiltonian, the counterdiabatic term is constructed from the dispersionless Korteweg–de Vries (KdV) hierarchy. Then the adiabatic theorem holds exactly for an arbitrary choice of time-dependent parameters. We use the Hamilton–Jacobi theory to define the generalized action. The action is independent of the history of the parameters and is directly related to the adiabatic invariant. The dispersionless KdV hierarchy is obtained from the classical limit of the KdV hierarchy for the quantum shortcuts to adiabaticity. This correspondence suggests some relation between the quantum and classical adiabatic theorems.

Shortcuts to adiabaticity (STA) is a method controlling dynamical systems. The implementation of the method results in dynamics that are free from nonadiabatic transitions for an arbitrary choice of time-dependent parameters in a reference Hamiltonian. It was developed in quantum systems[1–4] and its applications have been studied in various fields of physics and engineering.[5] It is important to notice that this method, decomposing the Hamiltonian into the reference term and the counterdiabatic term, is applied to any dynamical systems and offers a novel insight into the systems.

It is an interesting problem to find the corresponding method in classical mechanics from several points of view as we discuss in the following. Jarzynski studied STA for the classical system by using the adiabatic invariant.[6] He found the form of the counterdiabatic term in a certain system based on a generator of the adiabatic transport. Although several applications have been discussed,[7–9] the complete formulation of the classical STA is still under investigation.

In the quantum system, the adiabatic theorem is described by the adiabatic state constructed from the instantaneous eigenstate of a reference Hamiltonian H_0. When the time-dependence of the parameters in the Hamiltonian is weak, the solution of the Schrödinger equation can be approximated by the adiabatic state. On the other hand, the adiabatic theorem in the classical system is described by the phase volume defined in periodic systems. The closed trajectory in phase space for a fixed parameter gives the adiabatic invariant

$$J = \int dx\, dp\, \theta(E_0 - H_0), \tag{1}$$

where E_0 denotes the instantaneous energy. J is defined instantaneously and the adiabatic theorem states that J is approximately conserved when the parameter change is slow.

Thus the quantum and classical adiabatic theorems look very different and the relation between them is not obvious. The quantum STA is introduced so that the quantum adiabatic theorem holds exactly and we expect that the same holds for the classical case. These two formulations will allow us to make a link between two theorems. In this letter, we develop the theory of the classical STA. First, we formulate the classical STA in a general way so that the counterdiabatic term can be calculated, in principle, from the derived formula. Second, we show that the adiabatic theorem holds exactly in the classical STA and the adiabatic invariant is obtained directly from the nonperiodic trajectory. Third, we show that the quantum STA reduces to the classical STA by taking the limit $\hbar \to 0$, which suggests some relation between quantum and classical adiabatic dynamics.

We consider classical systems with one degree of freedom for simplicity. The system is characterized by the Hamiltonian $H = H(x, p; \alpha(t)) = H_0 + H_{\mathrm{CD}}$. The dynamical variables in phase space are denoted by x and p. To consider the counterdiabatic driving, we use a time-dependent parameter $\alpha(t)$. It is a straightforward task to generalize the present formulation to systems including several parameters. In the adiabatic time evolution, $\alpha(t)$ represents a slowly-varying function. Here, we do not impose any conditions on $\alpha(t)$. In the quantum counterdiabatic driving, the state is determined instantaneously. The same must be implemented for classical dynamics and we impose the condition that, when the solution of the equation of motion $(x, p) = (x(t;\alpha(t)), p(t;\alpha(t)))$ is substituted, the reference Hamiltonian $H_0(x, p, \alpha(t))$ is equal to the instantaneous energy as

$$H_0(x(t;\alpha(t)), p(t;\alpha(t)), \alpha(t)) = E_0(\alpha(t)). \tag{2}$$

Then, by considering the time derivative, we find that the counterdiabatic term $H_{\mathrm{CD}} = H - H_0 = \dot{\alpha}(t)\xi(x, p, \alpha)$, added to the Hamiltonian, satisfies

$$\frac{\partial H_0(x, p, \alpha)}{\partial \alpha} = \{\xi(x, p, \alpha), H_0(x, p, \alpha)\} + \frac{dE_0(\alpha)}{d\alpha}, \tag{3}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. This is obtained by using the equation of motion (Section A of Ref. 10). If we find ξ that satisfies this equation, we can realize an ideal time evolution that is characterized by $E_0(\alpha(t))$ at each time t. This equation corresponds to the equation derived in Ref. 6 and may be related to the equation for the dynamical invariant in the quantum STA.[4,11] The commutator in the quantum system is replaced by the Poisson bracket in the classical limit. We note that ξ is not uniquely determined from Eq. (3).[6] This arbitrariness is discussed in the formulation discussed below.

It is a simple task to show that the adiabatic theorem holds in this counterdiabatic driving. Taking the derivative of J in Eq. (1) with respect to α and using Eq. (3), we obtain

$$\frac{dJ}{d\alpha} = -\int dx\, dp \{\xi, H_0\}\delta(E_0 - H_0). \tag{4}$$

This integral is evaluated by the surface contributions and goes to zero in systems with a smooth trajectory as we see in the following examples (Section A of Ref. 10). This result means that J is determined by the initial condition and is independent of t. The proof clearly indicates that the counterdiabatic term is introduced so that the adiabatic theorem holds exactly. We note that the time variable t does not appear in Eq. (3) explicitly, which allows us to handle the adiabatic invariant defined geometrically in phase space. This result is the same as that in Ref. 6.

The solution of Eq. (3) can be studied systematically as was done in Ref. 12 for the quantum system. As an example, we set the reference Hamiltonian in a standard form

$$H_0(x, p, \alpha(t)) = p^2 + U(x, \alpha(t)). \tag{5}$$

Then we show that the solution of Eq. (3) is given by the dispersionless Korteweg–de Vries (KdV) hierarchy[13,14] (Section B of Ref. 10). The corresponding method in the quantum STA was developed in Ref. 12 and the KdV hierarchy was found. The reference Hamiltonian and the counterdiabatic term represent the Lax pair in the corresponding nonlinear integrable system.[15] The dispersionless KdV hierarchy is known as the “classical” limit of the KdV hierarchy.[16]

When $\xi(x, p, \alpha)$ is linear in p, the potential is of the form

$$U(x, \alpha(t)) = \frac{1}{\gamma^2(t)} u\left(\frac{x - x_0(t)}{\gamma(t)}\right), \tag{6}$$

where u is an arbitrary function, and α represents both x_0 and γ, the former represents a translation and the latter a dilation. The counterdiabatic term is given by

$$H_{\mathrm{CD}} = \dot{x}_0 p + \frac{\dot{\gamma}}{\gamma}(x - x_0)p. \tag{7}$$

This is known as the scale-invariant driving and was found in previous works.[6,7,17] A new result is obtained when we set that ξ is third order in p. We find that the counterdiabatic term is given by

$$H_{\mathrm{CD}} = \dot{\alpha}\xi = \dot{\alpha}\left(pU(x, \alpha) + \frac{2}{3}p^3\right), \tag{8}$$

and the potential satisfies the dispersionless KdV equation

$$\frac{\partial U(x, \alpha)}{\partial \alpha} + U(x, \alpha)\frac{\partial U(x, \alpha)}{\partial x} = 0. \tag{9}$$

This equation can be obtained by removing the third derivative term, the dispersion term, in the KdV equation.[18] The form of the potential is different between the quantum and classical STA for the same form of the counterdiabatic term in Eq. (8). This property is contrasted with that in the scale-invariant system where the quantum and classical STA give the same result. In the same way, we can find the correspondence between the KdV and dispersionless KdV hierarchies at each odd order in p. Although it is a difficult problem to implement the higher order terms in an actual experiment, some deformation of the counterdiabatic term is possible to represent the term by a potential function.[12]

Before studying the solutions of the dispersionless KdV equation, we reformulate the classical STA by using the Hamilton–Jacobi theory. The standard classical adiabatic theorem is described in periodic systems since the validity of the approximation is written in terms of the period T as $T|\dot{\alpha}/\alpha| \ll 1$. Although the adiabatic invariant is treated in ergodic systems,[19–21] its generalization is a delicate and difficult problem. To find the quantum-classical correspondence of the adiabatic systems, we need to extend the formulation to general systems. This can be done by the Hamilton–Jacobi theory. The adiabatic invariant is related to the action $S = \int_0^t dt'\, L = \int_0^t dt'(\dot{x}p - H)$. This is a function of $x(t)$, t, and the whole history of $\alpha(t)$: $S = S(x(t), t, \{\alpha(t)\})$. The property that S is independent of the history of $x(t)$ is shown by using the equation of motion. In the counterdiabatic driving, trajectories in phase space are determined from Eq. (2) and the Hamiltonian satisfies Eq. (3) that has no explicit time dependence. These properties imply that the dynamics is characterized at each t, irrespective of past history. By considering the variation $\alpha(t') \to \alpha(t') + \delta\alpha(t')$ of S at an arbitrary t' between 0 and t, and using the equation of motion, we obtain the deviation of the action as (Section C of Ref. 10)

$$\delta S = \int_0^t dt'\, \delta\alpha(t')\left(-\frac{\partial H_0}{\partial \alpha} + \{\xi, H_0\}\right) - [\delta\alpha(t')\xi]_0^t. \tag{10}$$

We use Eq. (3) to find that the function defined as

$$\Omega = S(x(t), t, \{\alpha(t)\}) + \int_0^t dt'\, E_0(\alpha(t')) \tag{11}$$

is independent of the history of $\alpha(t)$. This function is a simple generalization of the Hamilton’s characteristic function, or the abbreviated action, which is usually defined for constant E_0 by the Legendre transformation. It satisfies

$$\frac{\partial \Omega}{\partial x} = p(x, \alpha), \quad \frac{\partial \Omega}{\partial \alpha} = -\xi(x, p(x, \alpha), \alpha). \tag{12}$$

The momentum p is represented as a function of x and α as we see from Eq. (2). These derivatives have no explicit t dependence. This implies that Ω is a function of x and α, and not of t, just like the property of the Legendre transformation. The explicit t dependence of Ω can be removed by adding a time-dependent term, which does not change the trajectory, to the Hamiltonian. As a result we can set $\Omega = \Omega(x(t), \alpha(t))$.

The Hamilton–Jacobi equation is given by $\partial S/\partial t + H = 0$ with $p = \partial S/\partial x$. We substitute Ω to this equation. Noting that the time derivative is replaced in the present system with $\partial_t \to \partial_t + \dot{\alpha}\partial_\alpha$, we find that the Hamilton–Jacobi equation for the counterdiabatic driving is decomposed as

$$H_0\left(x, p = \frac{\partial \Omega(x, \alpha)}{\partial x}, \alpha\right) = E_0(\alpha), \tag{13}$$

$$\xi\left(x, p = \frac{\partial \Omega(x, \alpha)}{\partial x}, \alpha\right) = -\frac{\partial \Omega(x, \alpha)}{\partial \alpha}. \tag{14}$$

These equations are solved as a function of x and α, and have no explicit time dependence. We note that the counterdiabatic term is given by $H_{\mathrm{CD}} = \dot{\alpha}\xi(x, p, \alpha)$. Thus the counterdiabatic driving is characterized by Ω.

The definition of the action shows that Ω is written as

$$\begin{aligned}\Omega(x(t), \alpha(t)) &= \int_0^t dt'(\dot{x}p(x, \alpha) - \dot{\alpha}\xi(x, p(x, \alpha), \alpha)) \\ &= \int_0^t dt'\left[\frac{\partial H_0}{\partial p}p + \dot{\alpha}\left(\frac{\partial \xi}{\partial p}p - \xi\right)\right],\end{aligned} \tag{15}$$

where we use the equation of motion in the second line. This expression shows that Ω, as a function of t and $\alpha(t)$, satisfies the following equations:

$$\left(\frac{\partial\Omega}{\partial t}\right)_\alpha = \frac{\partial H_0}{\partial p}p, \quad \left(\frac{\partial\Omega}{\partial\alpha}\right)_t = \frac{\partial\xi}{\partial p}p - \xi. \tag{16}$$

Second equation states that Ω is independent of $\alpha(t)$ when the counterdiabatic term is linear in p. This is the case of the scale-invariant driving where H_0 is of the form (5) with (6). The Hamilton–Jacobi equation (13) reads

$$\left(\frac{\partial\Omega}{\partial x}\right)^2 + \frac{1}{\gamma^2}u\left(\frac{x-x_0}{\gamma}\right) = E_0(\alpha), \tag{17}$$

and we find that E_0 and Ω take the form $E_0(\alpha(t)) = \epsilon_0/\gamma^2(t)$ and $\Omega = \Omega((x-x_0)/\gamma)$, respectively. By setting $\gamma(0) = 1$, we can regard ϵ_0 as the initial energy at $t = 0$. The counterdiabatic term is calculated as

$$H_{\rm CD} = -\dot{x}_0\frac{\partial\Omega}{\partial x_0} - \dot{\gamma}\frac{\partial\Omega}{\partial\gamma} = \dot{x}_0 p + \frac{\dot{\gamma}}{\gamma}(x-x_0)p, \tag{18}$$

where we use the property that the derivatives of Ω with respect to x_0 and γ are translated to that with x in the present system. Ω is also a function of $E_0(\alpha(0)) = \epsilon_0$ and its definition shows that the derivative of Ω with respect to ϵ_0 gives the relation

$$\frac{\partial\Omega}{\partial\epsilon_0} = \int_0^t \frac{dt'}{\gamma^2(t')} = \tau(t), \tag{19}$$

where the last equality is the definition of the rescaled time $\tau(t)$. We conclude that Ω as a function of t and α in the scale-invariant system satisfies the relation

$$\Omega(x(t;\alpha(t)),\alpha(t)) = \Omega(x(\tau(t);\alpha(0)),\alpha(0)). \tag{20}$$

The left-hand side represents Ω at t obtained in the protocol $\alpha(t)$ and the right-hand side represents Ω at $\tau(t)$ in the fixed protocol $\alpha(0)$. When the latter system gives a closed trajectory, Ω at the period is equal to the adiabatic invariant in Eq. (1) and is written as $\Omega = \oint p\,dx$ (Section D of Ref. 10 for an example of the harmonic oscillator). This relation shows that the adiabatic invariant is directly obtained from the corresponding nonperiodic trajectory.

For nonscale-invariant systems, Eq. (20) is not satisfied. We treat the dispersionless KdV system as an example. H_0 is given by Eq. (5) and the potential U satisfies the dispersionless KdV equation (9). We can rederive Eq. (8) in the present formalism by assuming that E_0 is constant. Substituting $U = E_0 - (\partial_x\Omega)^2$ to Eq. (9), we find (Section C of Ref. 10)

$$\frac{\partial\Omega}{\partial\alpha} + U\frac{\partial\Omega}{\partial x} + \frac{2}{3}\left(\frac{\partial\Omega}{\partial x}\right)^3 = 0. \tag{21}$$

This equation shows that the counterdiabatic term obtained from Eq. (14) is given by Eq. (8).

Equation (9) is solved by the hodograph method as

$$U(x,\alpha) = f(x - \alpha U(x,\alpha)), \tag{22}$$

where f is an arbitrary function.[22,23] As a simple example we consider the case $f(x) = x^2$. Then, by solving the quadratic equation, we obtain

$$U(x,\alpha) = \frac{2\alpha x + 1 - \sqrt{4\alpha x + 1}}{2\alpha^2}. \tag{23}$$

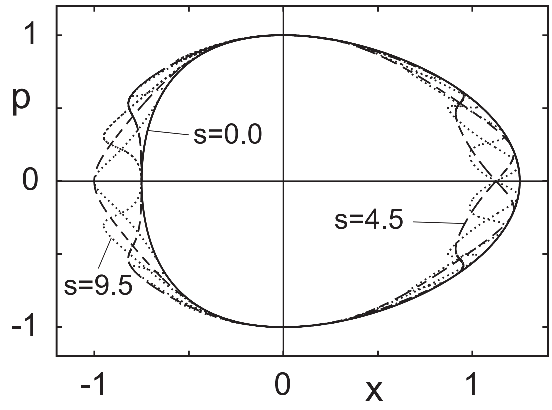

Fig. 1. Trajectories in phase space for the dispersionless KdV system in Eq. (5) with (23). We take the initial condition as $\alpha(0) = 1/4$, $E_0(\alpha(0)) = 1$, and $(x(0), p(0)) = (-3/4, 0)$. The solid line represents $s = 0.0$, the dashed line $s = 4.5$, and the dotted line $s = 9.5$.

We take the negative branch of the equation so that the trajectories are bound. This potential is well-defined for $x > -1/4\alpha$ and we set the parameters so that this relation is satisfied throughout the time evolution. By taking the limit $\alpha \to 0$, we have the harmonic oscillator $U(x, 0) = x^2$.

The equation of motion is solved numerically and we show the trajectories in phase space and $\Omega(x(t;\alpha(t)),\alpha(t))$ in Figs. 1 and 2, respectively. We use the protocol $\alpha(t) = \alpha(0)[1 - \sin^2(st)]$. We see from Fig. 2 that Eq. (20) is not satisfied since Ω is not necessarily a monotone increasing function and the time rescaling cannot give the result at $s = 0$. The counterdiabatic driving determines Ω instantaneously as we showed in the above analysis. This can be confirmed numerically in the present system. In Figs. 1 and 2, we consider an oscillating $\alpha(t)$ with the parameter s and we see that Ω with s is equal to Ω with $s = 0$ when $t = 2\pi n/s$ with integer n. We also find that Ω is equal to the adiabatic invariant J when t is equal to the period of the closed trajectory which is defined for a fixed α:

$$\Omega(x(T;\alpha(T)),\alpha(T)) = J. \tag{24}$$

This relation holds for an arbitrary choice of $\alpha(t)$ and can be proved by assuming that T is independent of the initial energy. We use the theory of action-angle variables for the proof. See Sect. E of Ref. 10. The period T can also be calculated there and we find $T = \pi$ in the present case. Thus the adiabatic invariant can be calculated directly from the real nonperiodic trajectory.

The Hamilton–Jacobi theory of the classical STA makes a link between the classical and quantum systems. In scale-invariant Hamiltonian in Eq. (5) with (6), the counterdiabatic term in classical system becomes the same as that in quantum system if we use the symmetrization as $px \to (\hat{p}\hat{x} + \hat{x}\hat{p})/2$. In the KdV systems, the form of the Hamiltonian is unchanged but the potential function U satisfies an equation that is different from Eq. (9).

In the quantum system the state is described by the wavefunction which satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = (\hat{H}_0 + \hat{H}_{\rm CD})\psi(x,t). \tag{25}$$

For example, we consider the Hamiltonian (5) and (8) with the replacement $pU \to (\hat{p}\hat{U} + \hat{U}\hat{p})/2$. This setting gives a

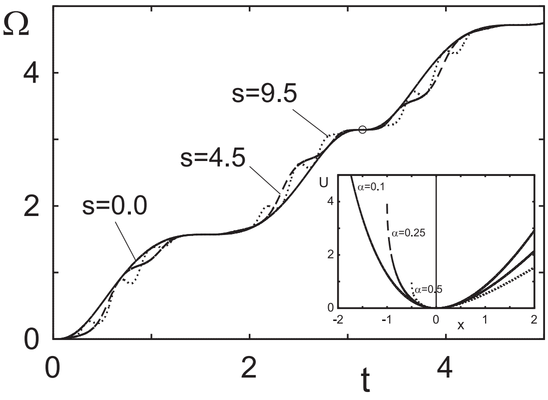

Fig. 2. The characteristic function $\Omega(x(t;\alpha(t)),\alpha(t))$ in the case of Fig. 1. The circle denotes the point $(T,\Omega(T))=(\pi,\pi)$ where T denotes the period of the closed trajectory for a constant α. Inset: The potential function $U(x,\alpha)$ in Eq. (23).

counterdiabatic driving when the potential satisfies the KdV equation[18)]

$$\frac{\partial U(x,\alpha)}{\partial \alpha}+U(x,\alpha)\frac{\partial U(x,\alpha)}{\partial x}+\frac{\partial^3 U(x,\alpha)}{\partial x^3}=0. \tag{26}$$

By substituting the wavefunction

$$\psi(x,t)=e^{-iE_0t/\hbar}A(x,\alpha(t))e^{i\Omega(x,\alpha(t))/\hbar}, \tag{27}$$

where A and Ω are real functions, to the Schrödinger equation and taking the limit $\hbar\to 0$, we can obtain Eq. (21). Thus the classical STA is obtained from the quantum version by taking the classical limit. To find this relation, it is crucial to develop the classical STA by the Hamilton–Jacobi theory as we discuss in this letter.

In the quantum STA, the wavefunction is given by the adiabatic state of $H_0(\hat{x},\hat{p},\alpha(t))$. By using the instantaneous eigenstate of H_0, $|n(\alpha(t))\rangle$, and the instantaneous energy of H_0, $E_n(\alpha(t))$, we can write the wave function as

$$|\psi_n^{(\mathrm{ad})}(t)\rangle=\exp\left(-\frac{i}{\hbar}\int_0^t dt'\,E_n(\alpha(t'))\right)|\tilde{\psi}_n(\alpha(t))\rangle, \tag{28}$$

where

$$|\tilde{\psi}_n(\alpha(t))\rangle=\exp\left(-\int_{\alpha(0)}^{\alpha(t)}d\alpha'\langle n(\alpha')|\frac{\partial}{\partial\alpha'}|n(\alpha')\rangle\right)\times|n(\alpha(t))\rangle. \tag{29}$$

The point is that $|\tilde{\psi}_n(\alpha(t))\rangle$, the adiabatic state without the dynamical phase, is written in terms of α, not of t. We define the unitary operator $\hat{V}$ as

$$|\tilde{\psi}_n(\alpha(t))\rangle=\hat{V}(\alpha(t))|\tilde{\psi}_n(\alpha(0))\rangle. \tag{30}$$

This operator was introduced to develop the path integral formulation of the adiabatic theorem.[24)] Then we can show that the counterdiabatic term is written as

$$\hat{H}_{\mathrm{CD}}(t)=i\hbar\dot{\alpha}(t)\frac{\partial\hat{V}(\alpha)}{\partial\alpha}\hat{V}^{\dagger}(\alpha). \tag{31}$$

We note that, in this expression, the unitary operator $\hat{V}$ can be replaced by the total time evolution operator to find the formula by Demirplak and Rice.[1)] By using $\hat{V}$, we can write the formula of $\hat{H}_{\mathrm{CD}}$ in a more suggestive form. We write $\hat{V}$ as

$$\hat{V}(\alpha)=\exp\left[\frac{i}{\hbar}\hat{\Omega}(\alpha)\right], \tag{32}$$

and this operator $\hat{\Omega}$ is the quantum analogue of the characteristic function $\Omega(x,\alpha)$ in the classical system. For example, in the scale-invariant Hamiltonian (5) with (6), $\hat{\Omega}$ is given by

$$\hat{\Omega}(\alpha(t))=-\hat{p}(x_0(t)-x_0(0))-\frac{1}{2}[\hat{p}(\hat{x}-x_0(t))+(\hat{x}-x_0(t))\hat{p}]\ln\frac{\gamma(t)}{\gamma(0)}. \tag{33}$$

The counterdiabatic term in this case is written as

$$\hat{H}_{\mathrm{CD}}(t)=-\dot{\alpha}(t)\frac{\partial\hat{\Omega}(\alpha)}{\partial\alpha}. \tag{34}$$

This expression formally coincides with Eq. (14).

To summarize, we have developed the classical STA by using the Hamilton–Jacobi theory. The system is characterized by the generalized characteristic function $\Omega(x,\alpha)$ and the counterdiabatic term is obtained from this function. The equation for the counterdiabatic term can be studied systematically and is solvable when the system falls in the dispersionless KdV hierarchy. Our formulation also gives a relation to the adiabatic theorem. We can also show that the classical STA is reduced from the quantum STA by taking the standard semiclassical approximation.

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