Invariant-based pulse design for three-level systems without the rotating-wave approximation

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In this paper, a scheme is put forward to design pulses which drive a three-level system based on the reverse engineering with Lewis-Riesenfeld invariant theory. The scheme can be applied to a three-level system even when the rotating-wave approximation (RWA) can not be used. The amplitudes of pulses and the maximal values of detunings in the system could be easily controlled by adjusting control parameters. We analyze the dynamics of the system by an invariant operator, so additional couplings are unnecessary. Moreover, the approaches to avoid singularity of pulses are studied and several useful results are obtained. We hope the scheme could contribute to fast quantum information processing without RWA.

I. INTRODUCTION

Manipulating physical systems with time-dependent electromagnetic fields, which is important for high-precision quantum information processing, has attracted growing interests in recent years. The adiabatic passage [1–4] is one of typical methods to design and control time-dependent pulses, which has been widely used in numerous previous schemes [5–8]. The adiabatic passage is approved for its robustness against the fluctuations of control parameters, while it is also criticized for the low speed caused by the limit of adiabatic condition. To accelerate evolutions of physical systems, many methods [9–23] have been proposed. Since they are related to the adiabatic passage, but provide alternative paths without the adiabatic condition for evolutions of physical systems, these methods [9–23] are arranged as a new kind of technique named by shortcuts to adiabaticity (STA). In the past several years, STA has drawn much attention of researchers, and has subsequently been used in many physical systems, such as superconducting systems [24, 25], atom-cavity systems [26, 27], and spin-NV center systems [28, 29]. Besides, many schemes [30–37] have been put forward to improve or extend STA. Until now, STA could be used to design pulses perfectly in many different cases.

The previous schemes [9–37] with STA focused on the physical systems under the rotating-wave approximation (RWA). However, many recent schemes about superconducting systems [38–40], optomechanical systems [41], semiconducting systems [42], Bose-Einstein condensates [43], and NV centers [44] have shown that, RWA may be invalid in the cases of ultra-fast operations and ultra-strong couplings. For example, Liu *et al.* [39] have

shown that RWA is broken down in the ultra-strong coupling, where the frequencies of pulses take the value of $10 \times 2\pi$GHz and the coupling strengthes take the value of $1.021 \times 2\pi$GHz. Moreover, Scheuer *et al.* [44] have demonstrated in a NV center that, when using a magnetic field with a frequency of 30MHz, RWA can not be used for a qubit control if the Rabi frequency larger than 15MHz. From these examples [38–44], RWA may be invalid in fast quantum information processing, thus the applications of previous schemes [9–37] with STA would be limited. Therefore, it is worthwhile to study STA without RWA so that pulse design for fast quantum information processing could be more effective.

Last year, two schemes [45, 46] have been proposed, which are about STA without RWA. One scheme [45] is proposed by Chen *et al.*, in which transitionless quantum driving (the counterdiabatic driving) is exploited to investigate the dynamics of both two- and three-level systems. It has shown that population transfers in both two- and three-level systems could be achieved in theory. The scheme [45] is interesting, but it has a few disadvantages. First, using the transitionless quantum driving requires an extra coupling between the initial state and the final state, which may be hard to be realized in several cases. Besides, the control parameters are not flexible enough to control the amplitudes of pulses and maximal values of detunings. Moreover, how to reduce oscillations and avoid the singularity of pulses, which are two questions required to be considered when RWA is broken down, have not been discussed. The other interesting scheme [46] is proposed by Ibáñez *et al.*, which is about pulse design for a two-level system with both transitionless quantum driving and invariant-based method with Lewis-Riesenfeld theory [47]. Their scheme [46] has shown many interesting results. For example, using invariant-based method does not require any extra couplings, which makes the pulse design more feasible in experiments. Moreover, the singularity of pulses can be

avoided by choosing control parameters suitably. Furthermore, they have shown that an invariant-based pulse design can help to achieve a population transfer in a two-level system with a perfect fidelity. These interesting results have demonstrated that the invariant-based method is very promising. However, different systems possess different dynamic features. Invariants for a two-level system without RWA can not properly describe the dynamics of a three-level system without RWA. Moreover, with the dimensions increase, the complexity of invariants would greatly increase. Therefore, the scheme [46] can not be directly applied to a three-level system without RWA. But three-level systems are very important in quantum information processing, as many quantum information tasks can be implemented in physical systems which are equivalent or approximately equivalent to three-level systems [48–53]. So it is necessary to research dynamics of three-level systems without RWA. Considering the advantages of the invariant-based method, if it can be applied to pulse design for three-level systems without RWA, we can realize many interesting quantum information tasks with ultra-fast operations and ultra-strong couplings. That requires us firstly to find out an invariant for three-level systems without RWA.

In this paper, inspired by the schemes [45, 46], we propose a scheme to design pulses for a three-level system without RWA. The scheme is based on a new-found invariant operator, which can help to study the dynamics of a three-level system without RWA. The scheme has some advantages, such as high speed, robustness against fluctuations of parameters, no requirements on extra couplings, etc.. These advantages would be clearly shown in the following sections.

The article is organized as follows. In Sec. II, we briefly review the Lewis-Riesenfeld invariant theory. In Sec. III, we give an invariant for a three-level system without RWA. Based on this invariant, the mathematical expressions of pulses and detunings are determined. In Sec. IV, we complete population transfers for a three-level system without RWA as examples to show the validity of the scheme. Finally, the conclusions are given in Sec. V.

II. LEWIS-RIESENFELD INVARIANT THEORY

In this section, let us briefly introduce Lewis-Riesenfeld theory [47]. We consider a quantum system which has a time-dependent Hamiltonian $H(t)$. Now, we introduce an invariant Hermitian operator $I(t)$, which satisfies ($\hbar = 1$)

$$i\frac{\partial}{\partial t}I(t) - [H(t), I(t)] = 0. \qquad (1)$$

If $|\psi(t)\rangle$ is a solution of the time-dependent Schrödinger equation $i\partial_t|\psi(t)\rangle = H(t)|\psi(t)\rangle$, $I(t)|\psi(t)\rangle$ is a solution as well. Moreover, $|\psi(t)\rangle$ can be expanded by eigenvec-
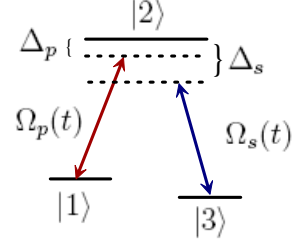

FIG. 1: The energy levels of the three-level system.

tors of $I(t)$ as

$$|\psi(t)\rangle = \sum_k C_k e^{i\theta_k}|\phi_k(t)\rangle, \qquad (2)$$

where, $|\phi_k(t)\rangle$ is the kth eigenvector of $I(t)$, and $C_k = \langle\phi_k(0)|\psi(0)\rangle$ is the corresponding coefficient. θ_k is the Lewis-Riesenfeld phase for $|\phi_k(t)\rangle$, which satisfies

$$\dot{\theta}_k = \langle\phi_k(t)|i\partial_t - H(t)|\phi_k(t)\rangle, \qquad (3)$$

with $\theta_k(t_i) = 0$ (t_i is the initial time).

III. INVARIANT-BASED PULSE DESIGN FOR THREE-LEVEL SYSTEMS WITHOUT THE ROTATING-WAVE APPROXIMATION

Let us start with a three-level system with two ground states $|1\rangle$, $|3\rangle$, and an excited state $|2\rangle$ shown in Fig. 1. Without RWA, the Hamiltonian of this system can be written by

$$H(t) = \begin{bmatrix} -\omega_p - \Delta_p(t) & \Omega_p(t)\cos(\omega_p t) & 0 \\ \Omega_p^*(t)\cos(\omega_p t) & 0 & \Omega_s^*(t)\cos(\omega_s t) \\ 0 & \Omega_s(t)\cos(\omega_s t) & -\omega_s - \Delta_s(t) \end{bmatrix}, (4)$$

in basis $\{|1\rangle, |2\rangle, |3\rangle\}$, where, $\Omega_p(t)$, $\Omega_s(t)$ are the pump and Stokes pulses driving the transitions $|1\rangle \leftrightarrow |2\rangle$ and $|3\rangle \leftrightarrow |2\rangle$, respectively. $\Omega_p^*(t)$, $\Omega_s^*(t)$ are the complex conjugates of $\Omega_p(t)$, $\Omega_s(t)$, respectively. ω_p and ω_s are the frequencies of pump and Stokes pulses, respectively. $\Delta_p(t)$ and $\Delta_s(t)$ denote the detunings of the pump and Stokes pulses from their relevant transitions, respectively. By analyzing Eqs. (1) and (4) with some undetermined coefficients, we find out a Lewis-Riesenfeld invariant as follows:

$$I(t) = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12}^* & I_{22} & I_{23} \\ I_{13}^* & I_{23}^* & I_{33} \end{bmatrix}, \qquad (5)$$

and the matrix elements of $I(t)$ are given as

$$I_{11} = \cos 2\lambda(\cos^2\alpha\cos^2\beta - \sin^2\alpha)$$
$$+ \cos\epsilon\cos\beta\sin 2\alpha\sin 2\lambda,$$

$$I_{12} = (\cos\alpha\cos 2\lambda\cos\beta + e^{-i\epsilon}\sin\alpha\sin 2\lambda)\sin\beta,$$

$$I_{13} = \frac{1}{4}\cos 2\lambda(3 + \cos 2\beta)\sin 2\alpha$$
$$- \cos\beta(\cos\epsilon\cos 2\alpha + i\sin\epsilon)\sin 2\lambda,$$

$$I_{22} = \cos 2\lambda\sin^2\beta,$$

$$I_{23} = (\sin\alpha\cos 2\lambda\cos\beta - e^{i\epsilon}\cos\alpha\sin 2\lambda)\sin\beta,$$

$$I_{33} = \cos 2\lambda(\sin^2\alpha\cos^2\beta - \cos^2\alpha)$$
$$- \cos\epsilon\cos\beta\sin 2\alpha\sin 2\lambda. \qquad (6)$$

In Eq. (6), α, β, ϵ, λ are four auxiliary time-dependent parameters, and they are required to satisfy

$$\dot{\alpha} = \dot{\lambda}\cos\beta\cos\epsilon. \qquad (7)$$

The invariant $I(t)$ has three eigenvectors as follows:

$$|\phi_+(t)\rangle = \begin{bmatrix} \cos\alpha\cos\beta\cos\lambda + e^{i\epsilon}\sin\alpha\sin\lambda \\ \sin\beta\cos\lambda \\ \sin\alpha\cos\beta\cos\lambda - e^{i\epsilon}\cos\alpha\sin\lambda \end{bmatrix},$$

$$|\phi_-(t)\rangle = \begin{bmatrix} \cos\alpha\cos\beta\sin\lambda - e^{i\epsilon}\sin\alpha\cos\lambda \\ \sin\beta\sin\lambda \\ \sin\alpha\cos\beta\sin\lambda + e^{i\epsilon}\cos\alpha\cos\lambda \end{bmatrix},$$

$$|\phi_0(t)\rangle = \begin{bmatrix} \cos\alpha\sin\beta \\ -\cos\beta \\ \sin\alpha\sin\beta \end{bmatrix}, \qquad (8)$$

which corresponds to eigenvalues 1, -1 and 0 of $I(t)$.

Solving Eq. (1) with $H(t)$ in Eq. (4) and $I(t)$ in Eq. (5), we obtain the following results

$$\Omega_p(t)\cos(\omega_p t) = i\dot{\lambda}e^{-i\epsilon}\sin\alpha\sin\beta$$
$$+ \frac{1}{2}\cos\alpha(-2i\dot{\beta} + \dot{\theta}\sin 2\beta),$$

$$\Omega_s(t)\cos(\omega_s t) = -i\dot{\lambda}e^{-i\epsilon}\cos\alpha\sin\beta$$
$$+ \frac{1}{2}\sin\alpha(-2i\dot{\beta} + \dot{\theta}\sin 2\beta),$$

$$\omega_p + \Delta_p(t) = -\dot{\epsilon}\sin^2\alpha + \dot{\theta}(\cos^2\alpha\sin^2\beta - \cos^2\beta)$$
$$- \dot{\lambda}\sin\epsilon\sin 2\alpha\cos\beta,$$

$$\omega_s + \Delta_s(t) = -\dot{\epsilon}\cos^2\alpha + \dot{\theta}(\sin^2\alpha\sin^2\beta - \cos^2\beta)$$
$$+ \dot{\lambda}\sin\epsilon\sin 2\alpha\cos\beta. \qquad (9)$$

In Eq (9), θ is the Lewis-Riesenfeld phase of $|\phi_0(t)\rangle$, which could be solved by

$$\dot{\theta} = \langle\phi_0(t)|i\partial_t - H(t)|\phi_0(t)\rangle = -\frac{\dot{\epsilon} + 2\dot{\lambda}\sin\epsilon\cos\beta\cot 2\alpha}{\sin^2\beta}. \qquad (10)$$

Besides, the Lewis-Riesenfeld phases of $|\phi_+(t)\rangle$ and $|\phi_-(t)\rangle$ are both zero.

With the results above, we can use the following formula to calculate the evolution of the system

$$|\psi(t)\rangle = (\langle\phi_+(0)|\psi(0)\rangle)|\phi_+(t)\rangle + (\langle\phi_-(0)|\psi(0)\rangle)|\phi_-(t)\rangle$$
$$+ e^{i\theta}(\langle\phi_0(0)|\psi(0)\rangle)|\phi_0(t)\rangle. \qquad (11)$$

IV. POPULATION TRANSFERS FOR A THREE-LEVEL SYSTEM

Using the results shown in Sec. III, we would like to perform population transfers for a three-level system to check the validity of the scheme. For simplicity, the condition $\dot{\lambda} = \dot{\alpha} = 0$ is set, so $\dot{\epsilon} = -\dot{\theta}\sin^2\beta$ can be obtained from Eq. (10), and Eq. (9) reduces to

$$\Omega_p(t)\cos(\omega_p t) = \frac{1}{2}\cos\alpha(-2i\dot{\beta} + \dot{\theta}\sin 2\beta),$$

$$\Omega_s(t)\cos(\omega_s t) = \frac{1}{2}\sin\alpha(-2i\dot{\beta} + \dot{\theta}\sin 2\beta),$$

$$\omega_p + \Delta_p(t) = -\dot{\theta}\cos 2\beta,$$

$$\omega_s + \Delta_s(t) = -\dot{\theta}\cos 2\beta. \qquad (12)$$

Then, two time-independent coefficients κ_p and κ_s are introduced, such that

$$\omega_p + \Delta_p(t) = -\kappa_p\dot{\theta} + (\kappa_p - \cos 2\beta)\dot{\theta},$$

$$\omega_s + \Delta_s(t) = -\kappa_s\dot{\theta} + (\kappa_s - \cos 2\beta)\dot{\theta}. \qquad (13)$$

In addition, we introduce a positive time-independent parameter ω, which has the scale of frequency. Assuming $\dot{\theta} = -\omega$, Eq. (13) can be replaced by

$$\omega_p + \Delta_p(t) = \kappa_p\omega - (\kappa_p - \cos 2\beta)\omega,$$

$$\omega_s + \Delta_s(t) = \kappa_s\omega - (\kappa_s - \cos 2\beta)\omega. \qquad (14)$$

Furthermore, Eq. (14) can be rewritten by

$$\omega_p = \kappa_p\omega,$$

$$\omega_s = \kappa_s\omega,$$

$$\Delta_p(t) = -(\kappa_p - \cos 2\beta)\omega,$$

$$\Delta_s(t) = -(\kappa_s - \cos 2\beta)\omega. \qquad (15)$$

For a brief discussion, we consider that the pump and Stokes pulses have the same frequency $\omega_p = \omega_s$, but different polarization directions. Besides, the two-photon resonance condition, where $\Delta_p(t) = \Delta_s(t) = \Delta(t)$, is considered. With assumptions shown above, a simple choice is to set $\kappa_p = \kappa_s = 1$, such that

$$\omega_p = \omega_s = \omega,$$

$$\Delta(t) = -2\omega \sin^2 \beta,$$

$$\Omega_p(t)\cos(\omega t) = -\frac{\cos\alpha}{2}(2i\dot{\beta} + \omega\sin 2\beta),$$

$$\Omega_s(t)\cos(\omega t) = -\frac{\sin\alpha}{2}(2i\dot{\beta} + \omega\sin 2\beta). \quad (16)$$

In the following, we design the parameters from different viewpoints and analyze the physical feasibility of the population transfers.

A. Pulse design with smooth functions

We suppose that a population transfer starts at $t = 0$ and ends at $t = T$. And the initial state of the system is $|\psi(0)\rangle = |1\rangle$. Considering the following requirements:

(i) The pump and Stokes pulses could be smoothly turned on and turned off.

(ii) To avoid the singularity of the pump and Rabi frequencies of Stokes pulses.

(iii) To avoid overlarge detunings or pulses.

β and its time derivative $\dot{\beta}$ can be designed as follows:

$$\beta = \frac{A}{2}[1 - \cos(\frac{2\pi t}{T})]\cos^2(\omega t),$$

$$\dot{\beta} = \frac{\pi A}{T}\sin(\frac{2\pi t}{T})\cos^2(\omega t)$$
$$-\frac{A\omega}{2}[1 - \cos(\frac{2\pi t}{T})]\sin(2\omega t), \quad (17)$$

where A is a time-independent coefficient controlling the maximal value of β. From Eq. (17), we have

$$\beta(0) = \beta(T) = \dot{\beta}(0) = \dot{\beta}(T) = 0, \quad (18)$$

so the pump and Stokes pulses could be smoothly turned on and turned off. Moreover, when A is not too large, the maximal values of detunings and amplitudes of pulses could be controlled in desired ranges. Besides, substituting Eq. (17) into Eq. (16), one can find that when $\cos(\omega t) \to 0$, we have $\sin 2\beta/\cos(\omega t) \to 0$ and $\dot{\beta}/\cos(\omega t) \to$ Const. Therefore, the singularity of Rabi frequencies of the pump and Stokes pulses can be eliminated.

On the other hand, using Eqs. (11) and (18), the final state of the system can be obtained

$$|\psi(T)\rangle = \begin{bmatrix} \cos^2\alpha + e^{i\epsilon}\sin^2\alpha \\ 0 \\ (1 - e^{i\epsilon})\sin\alpha\cos\alpha \end{bmatrix}. \quad (19)$$

By choosing $\alpha = \pi/4$, we have $|\psi(T)\rangle = |3\rangle$ when

$$\epsilon(T) = \omega\int_0^T \sin^2\beta\, dt = \pi. \quad (20)$$

Solving Eq. (20) with numerical methods, some samples of the relations between A and ωT are given in Table I.
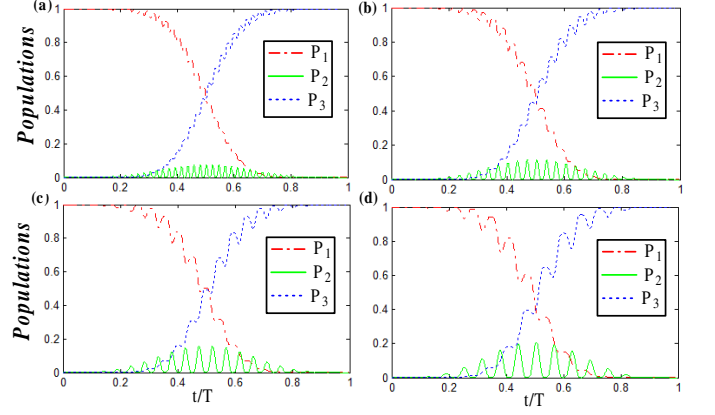

FIG. 2: Populations P_1 (the dashed-dotted red line), P_2 (the solid green line), and P_3 (the dotted blue) versus t/T with different parameters: (a) $A = 0.4$, $\omega = 45.7220\pi/T$; (b) $A = 0.5$, $\omega = 29.7323\pi/T$; (c) $A = 0.6$, $\omega = 21.0533\pi/T$; (d) $A = 0.7$, $\omega = 15.8274\pi/T$.

Table I. A with corresponding ωT.

A	ωT
0.2	179.04π
0.3	80.28π
0.4	45.72π
0.5	29.73π
0.6	21.05π
0.7	15.83π

When $A = 0.2$ ($A = 0.3$), a population transfer would go through about 90 (40) pulse periods, which makes the Rabi frequencies of pump and Stokes pulses oscillate very quickly. Therefore, we focus on the cases when $A = 0.4, 0.5, 0.6, 0.7$ in the following. We define the population of state $|j\rangle$ as $P_j(t) = |\langle j|\psi(t)\rangle|^2$ ($j = 1, 2, 3$). In addition, since $\alpha = \pi/4$ is chosen, we have $\Omega_s(t) = \Omega_p(t) = \Omega(t)$. The populations P_1, P_2, and P_3 versus t/T with different parameters are shown in Fig. 2. Besides, the real (imaginary) part $\text{Re}[\Omega(t)]$ ($\text{Im}[\Omega(t)]$) of $\Omega(t)$ and the detuning $\Delta(t)$ versus t/T with different parameters are shown in Fig. 3.

According to Fig. 2, population transfers could be achieved with $A = 0.4, 0.5, 0.6, 0.7$. This proves the invariant given in Eq. (5) is correct, and the parameters designed in this section are valid. Moreover, it is easy to find out that the maximal population of the intermediate state $|2\rangle$ increases slightly when A increases.

According to Fig. 3, the oscillations of pulses reduce when A increases, since A with larger value make the evolution of the system go through fewer pulse periods (as shown in Table I). When detunings and Rabi frequencies have too many oscillations, they may be difficult to be realized in experiments. To reduce the oscillations, one may increase A. However, from Fig. 3, the maximal values of ratios $\Omega(t)/\omega$ and $\Delta(t)/\omega$ increase when A increases. When A is too large, the pulses and detunings
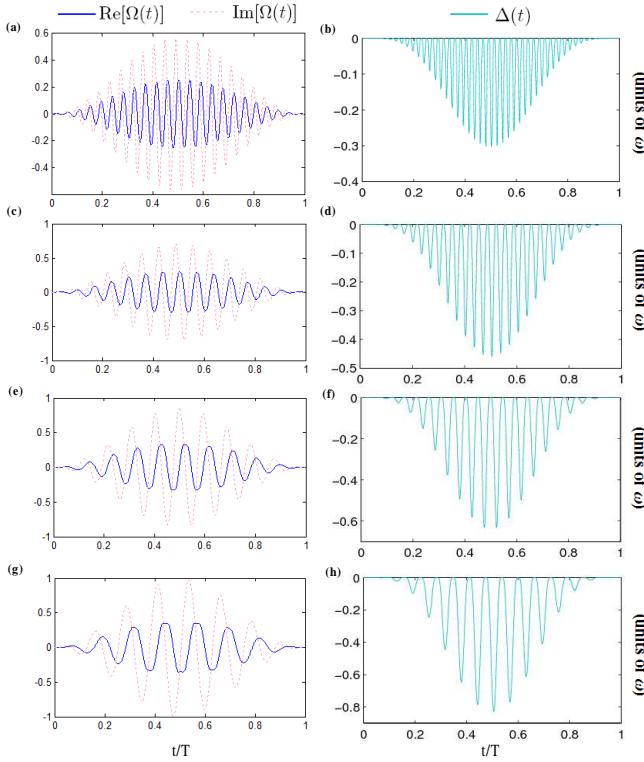
FIG. 3: The real part $\text{Re}[\Omega(t)]$ (the solid blue line) and the imaginary part $\text{Im}[\Omega(t)]$ (the dotted pink line) of $\Omega(t)$, and the detuning $\Delta(t)$ (the light blue line) versus t/T with different parameters: (a)-(b) $A = 0.4$, $\omega = 45.7220\pi/T$; (c)-(d) $A = 0.5$, $\omega = 29.7323\pi/T$; (e)-(f) $A = 0.6$, $\omega = 21.0533\pi/T$; (g)-(h) $A = 0.7$, $\omega = 15.8274\pi/T$.

may go beyond the acceptable ranges. Therefore, when designing pulses with smooth functions for a real experiment, one should choose a suitable A to make the pulses and detunings in acceptable ranges.

B. Pulse design with modifications around singular points

In this part, we try to reduce the oscillations by choosing a flat varying β. And we try to avoid the singularity of pulses by modifying the pulses around their singular points. The modifications are based on the fact that the Cauchy principal value of

$$\int_{x_0-\varsigma}^{x_0+\varsigma} \frac{1}{\cos x} dx \tag{21}$$

is zero, where x_0 denotes a singular point of function $1/\cos x$, and ς is an arbitrary small value. Therefore, populations vary little in the time intervals around the singular points of $1/\cos(\omega t)$. We can make some modifications of pulses around the singular points of $1/\cos(\omega t)$.

Suppose that a population transfer starts at $t = 0$ and ends at $t = T$, and the initial state of the system is $|\psi(0)\rangle = |1\rangle$. We maintain the condition $\alpha = \pi/4$ in this section. Instead of β and $\dot{\beta}$ shown in part A, we choose $\bar{\beta}$ and $\dot{\bar{\beta}}$, respectively, as follows:

$$\bar{\beta} = \frac{B}{2}[1 - \cos(\frac{2\pi t}{T})],$$
$$\dot{\bar{\beta}} = \frac{\pi B}{T} \sin(\frac{2\pi t}{T}), \tag{22}$$

where, B is a time-independent coefficient controlling the maximal value of $\bar{\beta}$. In this case, pulses could still be smoothly turned on and turned off. But different from the β and $\dot{\beta}$ designed in part A, the parameters that we chose here could not eliminate the singularity of pulses. However, when $B = A$, we have

$$\bar{\epsilon}(T) = \omega \int_0^T \sin^2 \bar{\beta} dt \geq \omega \int_0^T \sin^2 \beta dt = \epsilon(T). \tag{23}$$

So in the case of $B = A$, the maximal value of β is approximately equal to that of $\bar{\beta}$, but the population transfer could be completed faster as it goes through fewer pulse periods. That means the oscillations of pulses decrease a lot compared with the results of part A. Moreover, the singular points of pulses that we need to deal with are not too many. These results could also be got by comparing Table II with Table I.

Table II. B with corresponding ωT.

B	ωT
0.4	17.33π
0.5	11.34π
0.6	8.09π
0.7	6.13π

Now, let us show how to deal with the singular points of pulses by modifying the pulses around them. We take $B = 0.5$ as an example. In this case, a population transfer goes through more than 5 pulse periods but fewer than 6 pulse periods. Since $\alpha = \pi/4$ is set, we have

$$\bar{\Omega}_p(t) = \bar{\Omega}_s(t) = \bar{\Omega}(t) = -\frac{1}{2\sqrt{2}\cos(\omega t)}(2i\dot{\bar{\beta}} + \omega \sin 2\bar{\beta}), \tag{24}$$

where, $\bar{\Omega}_p(t)$ and $\bar{\Omega}_s(t)$ are respectively the pump and Stokes pulses decided by $\bar{\beta}$. There are eleven singular points of $\bar{\Omega}(t)$ in this case. They are $t_n = \frac{(2n-1)\pi}{2\omega}$ ($n = 1, 2, 3..., 11$). We modify $\bar{\Omega}(t)$ around these eleven singular points by $\tilde{\Omega}(t)$ as follows:

$$\tilde{\Omega}(t) = \begin{cases} \bar{\Omega}(t_n - \delta t) \\ +\frac{\bar{\Omega}(t_n+\delta t)-\bar{\Omega}(t_n-\delta t)}{2\delta t}(t - t_n + \delta t), & t \in \Xi_n, \\ \bar{\Omega}(t), & others, \end{cases} \tag{25}$$

where $\Xi_n = (t_n - \delta t, t_n + \delta t)$ is the modifying interval around the singular point t_n, and δt is a parameter which
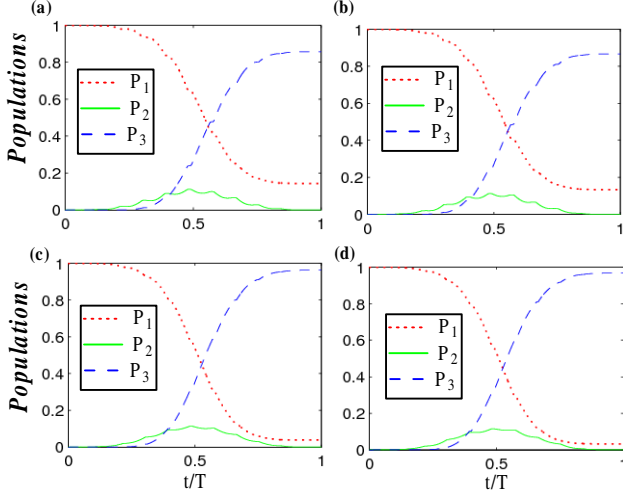
FIG. 4: Populations P_1 (the dotted red line), P_2 (the solid green line), and P_3 (the dashed blue) versus t/T with $B = 0.5$ in different cases: (a) $\delta t = 0.01\ T$; (b) $\delta t = 0.01\ T$ with Im[$\tilde{\Omega}(t)$] been neglected; (c) $\delta t = 0.005\ T$; (d) $\delta t = 0.005\ T$ with Im[$\tilde{\Omega}(t)$] been neglected.

FIG. 5: (a)-(b) The real part Re[$\tilde{\Omega}(t)$] (the solid red line) and the imaginary part Im[$\tilde{\Omega}(t)$] (the dotted blue line) of $\tilde{\Omega}(t)$ versus t/T with $B = 0.5$ in different cases: (a) $\delta t = 0.01\ T$; (b) $\delta t = 0.005\ T$. (c) The detuning $\Delta(t)$ (the light blue line, independent of δt) versus t/T with $B = 0.5$.

controls the length of modifying intervals around singular points.

By using Eq. (25), we perform numerical simulations with $\delta t = 0.01\ T$ and $\delta t = 0.005\ T$. In Figs. 4 (a) and (c), we plot populations P_1, P_2, and P_3 versus t/T with $B = 0.5$ in the cases of $\delta t = 0.01\ T$ and $\delta t = 0.005\ T$, respectively. In Figs. 5 (a) and (b), we plot the real part Re[$\tilde{\Omega}(t)$] and the imaginary part Im[$\tilde{\Omega}(t)$] of $\tilde{\Omega}(t)$ versus t/T with $B = 0.5$ in the cases of $\delta t = 0.01\ T$ and $\delta t = 0.005\ T$, respectively. The detuning $\Delta(t)$ which is independent of δt, is plotted in Fig. 5 (c).

Seen from Figs. 4 (a) and (c), we find that population transfers are imperfect, while the final population of $|3\rangle$ ($P_3(T)$) increases when δt reduces. For $\delta t = 0.01\ T$, we have $P_3(T) = 0.8516$, while for $\delta t = 0.005\ T$, we have $P_3(T) = 0.9618$. However, according to Figs. 5 (a) and (b), increasing $P_3(T)$ by reducing δt results in the increments of amplitudes of pulses. In addition, we find that Im[$\tilde{\Omega}(t)$] influences the population transfers little for both $\delta t = 0.01\ T$ and $\delta t = 0.005\ T$, as Im[$\tilde{\Omega}(t)$] $\ll$ Re[$\tilde{\Omega}(t)$]. We plot the population of each state versus t/T with $\delta t = 0.01\ T$ and $\delta t = 0.005\ T$ in Figs. 4 (b) and (d), respectively, when Im[$\tilde{\Omega}(t)$] is neglected; the numerical result shows $P_3(T) = 0.8675$ ($P_3(T) = 0.9680$) with $\delta t = 0.01\ T$ ($\delta t = 0.005\ T$).

C. Pulse design with reversely solved parameters

In part B, we investigate pulse design with modifications around singular points. The results show that the oscillations of pulses could be reduced a lot. However, population transfers may be imperfect if modifying in-
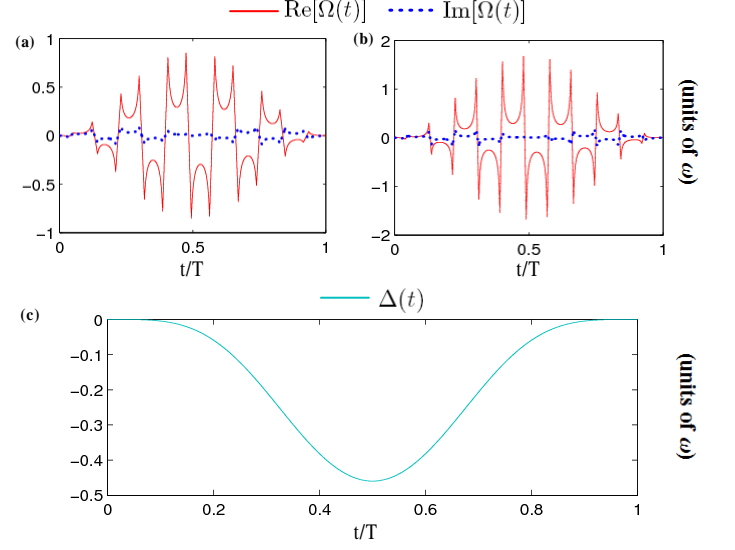
tervals are not narrow enough. To decrease the length of modifying intervals, we need to intensify the amplitudes of pulses. Moreover, the forms of pulses may be complex for the experimental realization. That motivates us to consider how to design pulses with suitable forms, amplitudes and fewer oscillations. In this part, we do not choose parameter β directly, while we consider the Rabi frequencies of pulses first.

Let us start from Eq. (16). Here, the condition $\alpha = \pi/4$ is still adopted, such that

$$\Omega_p(t) = \Omega_s(t) = \Omega(t) = -\frac{1}{2\sqrt{2}\cos(\omega t)}(2i\dot{\beta} + \omega\sin 2\beta).$$
(26)

Suppose that $\Omega(t) = \Omega_r(t) + i\Omega_i(t)$, Eq. (16) can be replaced by

$$\Omega_r(t)\cos(\omega t) = -\frac{\omega}{2\sqrt{2}}\sin 2\beta,$$
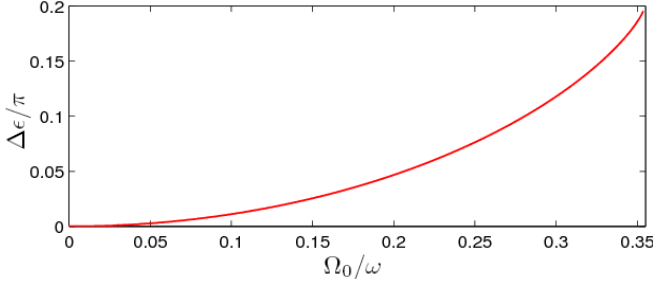$$\Omega_i(t)\cos(\omega t) = -\frac{1}{\sqrt{2}}\dot{\beta},$$
(27)

where $\Omega_r(t)$ and $\Omega_i(t)$ are two real functions, representing the real part and the imaginary part of $\Omega(t)$. Parameters

$$\beta = -\frac{1}{2}\arcsin[\frac{2\sqrt{2}}{\omega}\Omega_r(t)\cos(\omega t)],$$
$$\Omega_i(t) = \frac{\dot{\Omega}_r(t) - \Omega_r(t)\omega\tan(\omega t)}{\sqrt{\omega^2 - 8\Omega_r^2(t)\cos^2(\omega t)}},$$
(28)

can be solved from Eq. (27). To make $\Omega_i(t)$ a bounded function, it requires

$$\lim_{t \to t_m}[\dot{\Omega}_r(t) - \Omega_r(t)\omega\tan(\omega t)]$$

FIG. 6: $\Delta\epsilon/\pi$ versus Ω_0/ω.

$$= \lim_{t \to t_m} \frac{\dot{\Omega}_r(t)\cos(\omega t) - \Omega_r(t)\omega\sin(\omega t)}{\cos(\omega t)} = \text{Const}, \quad (29)$$

where, $t_m = (m+1/2)\pi$ $(m = 0, 1, 2, ...)$. Furthermore, Eq. (29) can be replaced by

$$\lim_{t \to t_m} [\dot{\Omega}_r(t)\cos(\omega t) - \Omega_r(t)\omega\sin(\omega t)]$$
$$= \lim_{t \to t_m} -\Omega_r(t) = 0, \quad (30)$$

where, $\dot{\Omega}_r(t)$ is supposed to be a bounded function. It means that, to avoid the singularity of $\Omega_i(t)$, we require $\Omega_r(t_m) = 0$.

Now, let us start from investigating pulses in a whole pulse period. For example, the time interval $\pi/2\omega \leq t \leq 5\pi/2\omega$ is considered. To fulfill the condition $\Omega_r(t_m) = 0$, we simply choose

$$\Omega_r(t) = \Omega_0 \cos^3(\omega t), \quad (31)$$

which is not difficult to be realized in experiments. It is easy to obtain

$$\Omega_i(t) = \frac{-4\Omega_0 \cos^2(\omega t)\sin(\omega t)}{\sqrt{1 - \frac{8\Omega_r^2(t)}{\omega^2}\cos^2(\omega t)}}. \quad (32)$$

To avoid β and $\Omega_i(t)$ taking complex values, it is better to set $0 \leq \Omega_0 < \omega/2\sqrt{2}$. The increment $\Delta\epsilon$ of ϵ in this pulse period can be calculated by

$$\Delta\epsilon = \omega \int_{\pi/2\omega}^{5\pi/2\omega} \sin^2\beta dt, \quad (33)$$

via a numerical integration. We plot $\Delta\epsilon/\pi$ versus Ω_0/ω in Fig. 6. Moreover, $\Omega_i(t)/\omega$ versus $t/(\pi/2\omega)$ and Ω_0/ω are plotted in Fig. 7.

From Fig. 6, to make the pulses have fewer oscillations, one can increase the ratio Ω_0/ω to make a population transfer go through fewer pulse periods. On the other hand, according to Fig. 7, the amplitude of $\Omega_i(t)$ increases when Ω_0/ω increases. For Ω_0 that satisfies $0 \leq \Omega_0 < \omega/2\sqrt{2}$, we have $|\Omega_i(t)| < 0.64\omega$ $(\forall t \in [\pi/2\omega, 5\pi/2\omega])$. To make the operations simple, $\Omega_0/\omega = 0.3396$ is chosen, such that $\Delta\epsilon = \pi/6$. Suppose that the parameters in every pulse period repeat
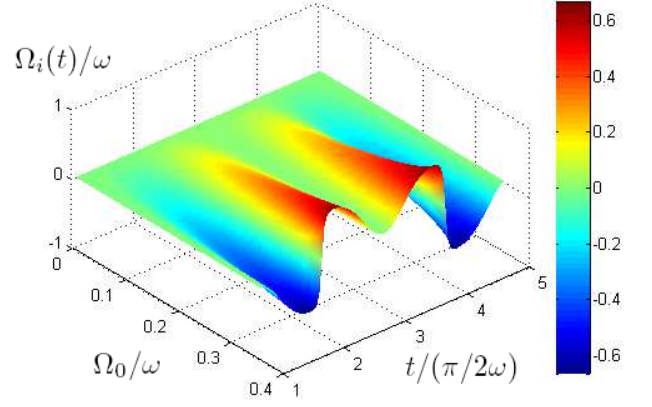
FIG. 7: $\Omega_i(t)/\omega$ versus $t/(\pi/2\omega)$ and Ω_0/ω.
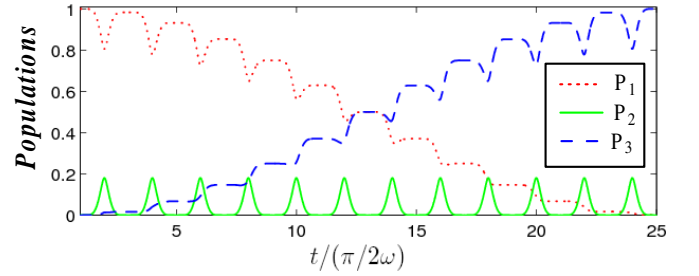
FIG. 8: Populations P_1 (the dotted red line), P_2 (the solid green line), and P_3 (the dashed blueline) versus $t/(\pi/2\omega)$ with $\Omega_0/\omega = 0.3396$.

the results in $[\pi/2\omega, 5\pi/2\omega]$. In this case, if a population transfer starts at $t = \pi/2\omega$, it could be finished at $t = 25\pi/2\omega$, i.e., the population transfer goes through 6 pulse periods.

The population of each state is plotted in Fig. 8. Furthermore, in Figs. 9 (a), (b), and (c), $\Omega_r(t)$, $\Omega_i(t)$ and the detuning $\Delta(t)$ versus $t/(\pi/2\omega)$ during the first pulse period are plotted.

As shown in Fig. 8, the population transfer can be achieved with the designed pulses in this part. However, P_3 increases up to near unity with greater and greater oscillations. The maximal hump of the oscillations appears at $t = 24\pi/2\omega$ with $P_3 = 0.806$. Therefore, the real interaction time T' should not approach $24\pi/2\omega$ for a real experiment. To obtained $P_3 \geq 0.9999$, we require $T' \geq 24.71\pi/2\omega$ $(\delta T = |T' - T| \leq 1.29\%)$. On the other hand, the pulses become weaker and weaker, and they could be turned off smoothly at $t = 25\pi/2\omega$. Besides, the curve of P_3 has a platform at $t = 25\pi/2\omega$. Therefore, P_3 still keeps near unity when $T' \geq 25\pi/2\omega$. To summarize, using the approach proposed in this part may have less robustness against the operation errors of the interaction time compared with the approaches of parts A and B. On the other hand, seen from Fig. 9, the real parts and the imaginary parts of pulses are much more smooth compared with that of pulses which were designed in part
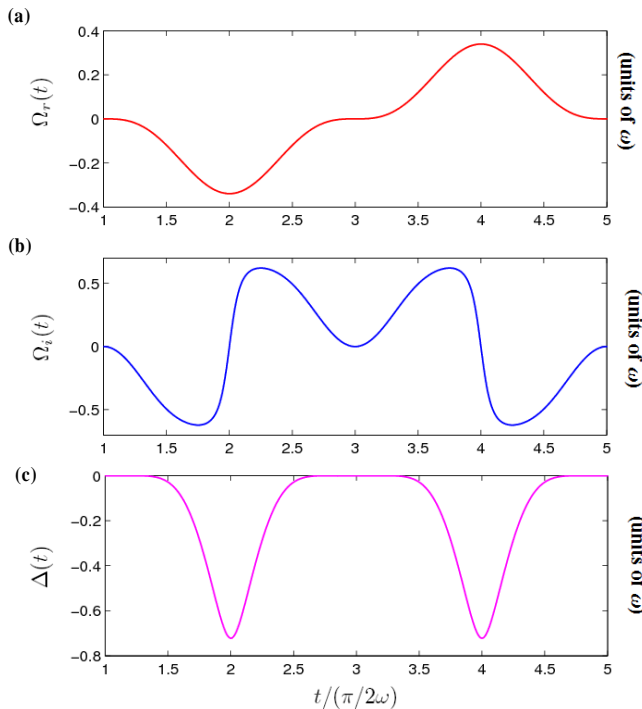
FIG. 9: (a) $\Omega_r(t)$ versus $t/(\pi/2\omega)$; (b) $\Omega_i(t)$ versus $t/(\pi/2\omega)$; (c) $\Delta(t)$ versus $t/(\pi/2\omega)$. ($\Omega_0/\omega = 0.3396$)

B. Moreover, since the population transfer goes through only 6 pulse periods, the oscillations of pulses and detunings are much fewer than that of pulses and detunings which were designed in part A. Therefore, the approach of pulse design shown in this section may be more attractive.

For the situation where the total interaction time T is not the integral multiple of a pulse period, e.g., $T = 2p\pi/\omega + \tau$, $(p = 0, 1, 2, ...)$, we can deal with the evolution of the system in the pth pulse period by similar way for 1st pulse period. And then, we only need to add pulse design for interval $[T - \tau, T]$ to make population transfers successful at $t = T$.

V. CONCLUSION

In conclusion, we have proposed an invariant-based scheme for pulse design without RWA. First, we found out an invariant for a three-level system without RWA. Then, we exploited the invariant to investigate pulse design for the population transfers in a three-level sys-

tem. From three different viewpoints, we gave three approaches to design pulses in parts A, B and C of Sec. IV. In part A of Sec. IV, we tried to design pulses with smooth functions. The population transfers could be realized without singularity of pulses. But the pulses would involve many oscillations. In part B of Sec. IV, we tried to reduce the oscillations of pulses by modifying the pulses around their singular points. The oscillations could be reduced a lot, while the population transfers became imperfect and the pulse forms might be complex for the experimental realization. In part C of Sec. IV, instead of choosing control parameters directly, we first chose pulses with feasible forms accompanied with some undetermined coefficients. Then we reversely solved the control parameters. With the help of numerical calculations, we determined all the coefficients of pulses. With the approach shown in part C of Sec. IV, feasible pulses could be designed for every pulse period, and the oscillations of pulses could be well restricted.

Overall, the scheme has shown several novel results and advantages:

(i) To our knowledge, invariants for a three-level system without RWA have not been investigated in the previous schemes. Therefore, the invariant shown in Eq. (5) may be a new one.

(ii) Based on pulse design with the invariant shown in Eq. (5), we do not need any extra couplings.

(iii) The amplitudes of pulses and the maximal values of detunings could be well controlled in the present scheme. But it is difficult for schemes with transitionless quantum driving to do so.

(iv) The pulses designed by the scheme can be smoothly turned on and turned off. Therefore, the scheme should be robust against the fluctuations of parameters.

(v) Compared with adiabatic processes, the system is not required to satisfy the adiabatic condition, thus possessing higher evolution speed.

With these advantages, the scheme may be useful for fast quantum information processing without RWA.

Acknowledgement

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