Approximate adiabatic gauge potential of non-integrable systems

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1 Introduction

Adiabatic gauge potentials are useful for controlling a quantum system when it's driven externally from one configuration to another. These potentials help us in circumventing standard adiabatic limitations which requires infinitesimally small rates [1, 2, 3]. For example, these potentials can be used for arbitrarily fast annealing protocols and implementing fast dissipationless driving.

The scaling of norm of gauge potential with system's size is quite different for quantum integrable and non-integrable systems. On one hand, for integrable systems , exact gauge potential are supposed to scale like a polynomial in system size. This is due to extensive number of symmetries that exist and as a result, they have a "lot" of degenerate energy levels which comes with their respective "selection rules". This can be easily seen for Transverse Ising model whose analytical expression of gauge potential is known in literature.

On the other hand, for non-integrable systems, using Eigenstate Thermalization Hypothesis (ETH)[4] , we can show that norm of exact gauge potential scale exponentially in system size. This can be verified numerically using exact diagonalization on spin system upto size $L = 15$.

We can exploit this property to distinguish between quantum integrable and non-integrable system. Our method should be better than conventional method (energy level distribution) used in literature for this purpose because unlike the conventional method, we don't have to worry about removing symmetry.

2 Adiabatic gauge potential

2.1 Introduction by example

$$H_0 = \frac{p^2}{2m} + V(x - \lambda(t)) \tag{1}$$

$$H_{CD} = H_0 + \dot{\lambda} A_\lambda$$

where $A_\lambda = p$. Include a picture of glass of water being transported from Dries's PNAS paper. Question: if you have exact gauge potential, does all the excitations during intermediate times is zero.

2.2 Formal introduction

Adiabatic gauge potentials are the generators of a unitary transformation which diagonalize the instantaneous Hamiltonian, attempting to leave its eigenbasis invariant as the parameter is changed. These adiabatic gauge potentials generate *non-adiabatic* corrections to Hamiltonian in the moving basis (λ -dependent basis).

This is something from Anatoli's lecture notes [5]– "an adiabatic basis is a family of adiabatically connected eigenstates, i.e., eigenstates related to a particular initial basis by adiabatic (infinitesimally slow) evolution of the parameter λ. For example, if two levels cross they will exchange order energetically but the adiabatic connection will be non-singular."

$H(\lambda)|n(\lambda)\rangle = E_n(\lambda)|n(\lambda)$. Let's derive diagonal and off-diagonal elements.

- **n-th diagonal element:** $A^n_\lambda = \langle n|A_\lambda|n\rangle = i\hbar\langle n|\partial_\lambda|n\rangle$
- **off- diagonal element:** We use the identity $\langle m|H(\lambda)|n\rangle = 0 \quad , n \neq m$ and then differentiate with respect to λ to obtain:

$$\boxed{\langle m|A_\lambda|n\rangle = -i\hbar\frac{\langle m|\partial_\lambda H|n\rangle}{E_m - E_n}} \tag{2}$$

  where both energies (E_m, E_n) and eigenvectors $(|m\rangle, |n\rangle)$ depend on λ.

2.3 Eigenstate Thermalization Hypothesis

Eigenstate Thermalization Hypothesis(ETH) gives us an ansatz for matrix elements of observables in the basis of energy eigenstates [4]:

$$O_{mn} = O(\bar{E})\delta_{mn} + e^{-S(\bar{E})/2} f_O(\bar{E}, \omega) R_{mn} \tag{3}$$

where $\bar{E} = (E_m + E_n)/2, \omega = E_n - E_m$ and $S(E)$ is the thermodynamic entropy at energy E.

We note that it's applicable only for few-body operators of a non-integrable Hamiltonian. By few-body, we mean n body observables with $n \ll N$, where N is the total number of spins, particles, etc. For example, projection operator to eigenstates of many body Hamiltonian $\hat{P}_\alpha = |\Psi_\alpha\rangle\langle\Psi_\alpha|$ don't satisfy ETH and it also doesn't satisfy predictions of statistical mechanics. Why is that? We expect that microcanonical averaging should be equivalent to canonical averaging:

$$\langle\Psi_\alpha|O|\Psi_\alpha\rangle = \frac{\operatorname{Tr} Oe^{-\beta H}}{\operatorname{Tr} e^{-\beta H}} \tag{4}$$

We can see $O = \hat{P}_\alpha$ doesn't satisfy the above equation (since left hand side is one and the trace of right hand side can be computed in energy basis to find that it's not one). Projection operator is non-local in real space, and we argue that this is the reason it doesn't satisfy ETH and is not experimentally measurable.

2.3.1 Information about $f_O(\bar{E}, \omega)$

$$|f_O(\bar{E}, \omega)| = \begin{cases} e^{-\omega T} & (\omega \gg T), \\ \dfrac{\sqrt{L}}{\omega^2 + \mu_T^2} & (\omega \ll T) \end{cases} \tag{5}$$

where $\mu_T \sim \dfrac{1}{L^2}$ [6, 7, 4]

3 Norm of adiabatic gauge potential

Let's compute the norm by noting that A_λ has only off-diagonal elements in energy basis in our gauge choice:

$$||A_\lambda||^2 = \operatorname{Tr} A_\lambda^2 \tag{6}$$

$$= \sum_n \langle n|A_\lambda^2|n\rangle \tag{7}$$

$$= \sum_n \langle n|A_\lambda|n\rangle^2 + \sum_n \sum_{m\neq n} |\langle m|A_\lambda|n\rangle|^2 \tag{8}$$

$$= \sum_n \sum_{m\neq n} |\langle m|A_\lambda|n\rangle|^2 \tag{9}$$

$$= \hbar^2 \sum_n \sum_{m\neq n} \frac{|\langle m|\partial_\lambda H|n\rangle|^2}{(E_m - E_n)^2} \tag{10}$$

Hence, in general, for both integrable and non-integrable systems we have:

$$\boxed{||A_\lambda||^2 = \hbar^2 \sum_n \sum_{m\neq n} \frac{|\langle m|\partial_\lambda H|n\rangle|^2}{(E_m - E_n)^2}} \tag{11}$$

4 Integrable model

Our goal is to study a integrable model, which is called **Transverse Field Ising model**. It shows quantum phase transition between ferromagnetic and paramagnetic phases. Moreover, it satisfies Ising symmetry $G = \Pi_i \sigma_i^z$ since $[H, G] = 0$, where H is the Hamiltonian. This model can be written in terms of non-interacting spinless fermions $(c_i, c_i^\dagger)$ using Jordan- Wigner transformation.

It's Hamiltonian in spin basis is given by:

$$H = -J \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x - \lambda \sum_j \sigma_j^z \tag{12}$$

where we have chosen periodic boundary conditions and λ is externally-controlled transverse magnetic field.

This model can be written in terms of non-interacting spinless fermions $(c_i, c_i^\dagger)$ using Jordan-Wigner transformation: $\sigma_i^z \sim 1 - 2c_i^\dagger c_i$ and $\sigma_i^+ \sim \prod_{j<i} \sigma_j^z c_j$. Details can be found elsewhere [8] [1] . Here is what we get after this transformation:

$$\mathcal{H} = \sum_k \psi_k^\dagger H_k \psi_k, \quad H_k = -\begin{bmatrix} \lambda - \cos k & \sin k \\ \sin k & -(\lambda - \cos k) \end{bmatrix} \tag{13}$$

where $\psi_k^\dagger = (c_k^\dagger, c_{-k})$ is Nambu spinor basis. We can write H_k in terms of Pauli sigma matrices:

$$H_k = -(\lambda - \cos k)\sigma_k^z - \sin k \sigma_k^x \tag{14}$$

Now using our regulator method (whose details are not given in this report), we can obtain :

$$\boxed{A_\lambda = \sum_{l=1}^{L-1} \alpha_l O_l \quad \text{where} \quad \alpha_l = -\frac{1}{4L} \sum_k \frac{\sin(k)\sin(lk)}{(\cos k - \lambda)^2 + \sin^2 k}} \tag{15}$$

where O_l is given by

$$O_l = 2i \sum_j (c_j^\dagger c_{j+l}^\dagger - \text{h.c}) = \sum_j (\sigma_j^x \sigma_{j+1}^z \dots \sigma_{j+l-1}^z \sigma_{j+l}^y + \sigma_j^y \sigma_{j+1}^z \dots \sigma_{j+l-1}^z \sigma_{j+l}^x) \tag{16}$$

This matches with the result already known in literature [9, 5].

[1] Momentum operator chosen to get real valued Hamiltonian is $c_k = \frac{e^{i\pi/4}}{\sqrt{L}} \sum_j c_j e^{-ikj}$, where k is $n\pi/L$ with $n = 0, 1, 2, \dots L-1$

Let's write a first few terms of O_l here:

$$O_{l=1} = \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^y + \sigma_j^y \sigma_{j+1}^x)$$

$$O_{l=2} = \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^y + \sigma_j^y \sigma_{j+1}^z \sigma_{j+2}^x)$$

On computation, we find that with periodic boundary conditions, we get $\mathrm{Tr}\, O_l O_p = \delta_{l,p} 2^{L+1} L$

For large enough system size L, we can compute α_l [5] by computing the sum into an integral and obtain the value of α_l as:

$$\alpha_l = -\frac{1}{8} \begin{cases} \lambda^{l-1} & (\lambda^2 < 1), \\ \lambda^{-l-1} & (\lambda^2 > 1) \end{cases} \tag{17}$$

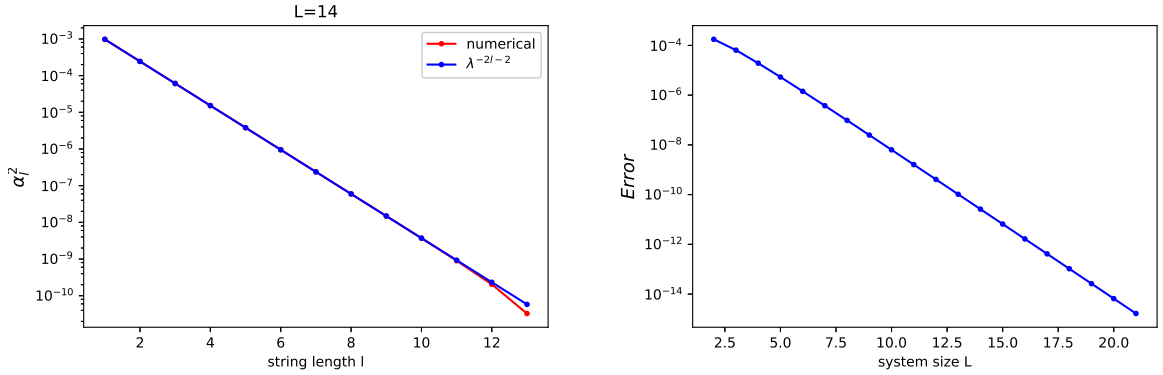

Figure 1: Integrable systems: string length of exact gauge potential as a function of system size

Let's compute norm of gauge potential:

$$||A_\lambda||^2 = \mathrm{Tr}\, A_\lambda^2 \tag{18}$$

$$= \mathrm{Tr} \sum_{l,p} \alpha_p \alpha_l O_l O_p \tag{19}$$

$$= \sum_{l,p} \alpha_p \alpha_l \,\mathrm{Tr}\, O_l O_p \tag{20}$$

$$= 2^{L+1} L \sum_{l=1}^{L-1} \alpha_l^2 \tag{21}$$

Now since α_l for large enough L is exponentially suppressed in l, we can argue that

$$\boxed{||A_\lambda||^2 / 2^L \sim 2L} \tag{22}$$

5 Non-integrable model

If we introduce longitudinal magnetic field in Transverse Ising model, then integrability is broken and we get a non-integrable model. We plan to study both local and global integrability-breaking term.

$$H = -J \sum_{j=1}^{L-1} \sigma_j^x \sigma_{j+1}^x - h \sum_{j=1}^{L} \sigma_j^z - \lambda \sum_{j=1}^{L} \sigma_j^x \tag{23}$$

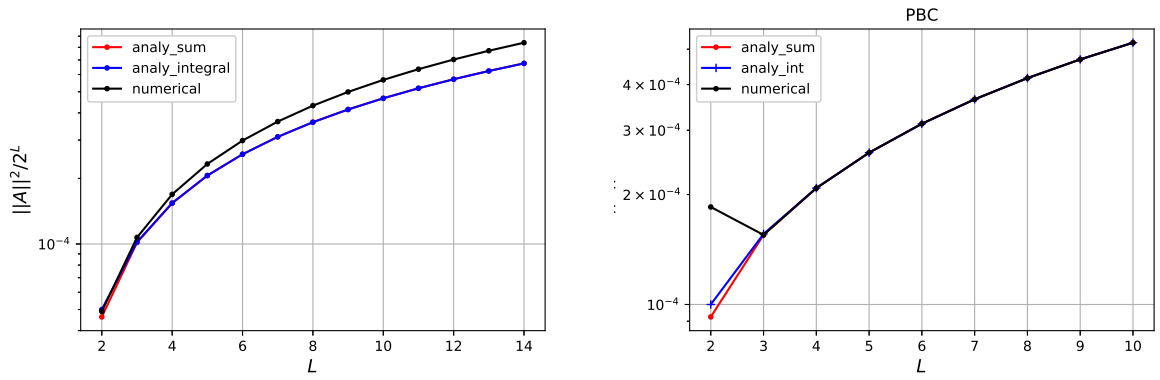

Figure 2: Integrable systems: Norm of exact gauge potential as a function of system size

In this model, $\partial_\lambda H = -\sum_j \sigma_j^x$ is a global operator.

$$H = -J\sum_{j=1}^{L-1}\sigma_j^x - h\sum_{j=1}^{L}\sigma_j^z - \lambda\sigma_0^x \tag{24}$$

In this model, $\partial_\lambda H = -\sigma_0^x$ is a local operator.

5.1 ETH applied to norm

$\partial_\lambda H$ may or may not be a local operator. We are studying such non-integrable models in which it is a local operator or it's composed by adding local operators. Hence, we can apply ETH on the operator $\partial_\lambda H$.

5.1.1 Heuristic argument

$$||A_\lambda||^2 = \hbar^2 \sum_n \sum_{m\neq n} \frac{|\langle m|\partial_\lambda H|n\rangle|^2}{\omega_{mn}^2}$$

where $\omega_{mn} = E_m - E_n$. We would argue that the biggest contribution to norm would come from the smallest ω_{mn} because it's exponentially small in system size. Hence, we find that using ETH for $\partial_\lambda H$:

$$\begin{aligned}
||A_\lambda||^2 &= \hbar^2 \sum_n \sum_{m\neq n} \frac{|\langle m|\partial_\lambda H|n\rangle|^2}{\omega_{mn}^2} \\
&= \hbar^2 \sum_n \sum_{m\neq n} \frac{e^{-S}}{e^{-2S}} \\
&= \hbar^2 \sum_n \sum_{m\neq n} e^{S} \\
&\simeq \hbar^2 2^L e^L
\end{aligned}$$

where we have used the fact that entropy is extensive, i.e. $S \sim L$. Hence, norm averaged over system size is exponential in system size with $\hbar = 1$

$$\boxed{||A_\lambda||^2/2^L \sim e^L} \tag{25}$$

Exponential scaling with system size of gauge potential is due to exponential small eigenvalues. Since these eigenvalues appear in the denominator of gauge potential expression, it's called **small denominator problem** in literature [5].

In Dries's notes, you would find how we are attempting to solve this problem.

5.1.2 Formal calculation

For formal calculation, I would need to introduce a cutoff μ. Otherwise, norm diverges in thermodynamic limit $L \to \infty$, which is clear from above heuristic arguments.

$$\langle n|A_\lambda|m\rangle = \lim_{\mu\to 0}\lim_{L\to\infty} -i\hbar\frac{\langle n|\partial_\lambda H|m\rangle}{(E_n-E_m)^2+\mu^2}(E_n-E_m) \tag{26}$$

where we have chosen a gauge choice in which diagonal elements are zero in energy basis, i.e. $A_\lambda^{nn}=0$.

$$||A_\lambda||^2 = \hbar^2\sum_n ||A_\lambda||_n^2 \tag{27}$$

where $||A_\lambda||_n^2 = \sum_{m\neq n}\frac{(E_m-E_n)^2}{((E_m-E_n)^2+\mu^2)^2}|\langle m|\partial_\lambda H|n\rangle|^2$.

Let's simplify this using ETH:

$$\begin{aligned}
||A_\lambda||_n^2 &= \sum_{m\neq n}\frac{(E_m-E_n)^2}{((E_m-E_n)^2+\mu^2)^2}|\langle m|\partial_\lambda H|n\rangle|^2 \\
&= \sum_{m\neq n}\frac{\omega_{nm}^2}{(\omega_{nm}^2+\mu^2)^2}e^{-S(\bar{E})}|f_O(\bar{E},\omega_{nm})R_{mn}|^2 \\
&= \sum_{m\neq n}\frac{\omega_{nm}^2}{(\omega_{nm}^2+\mu^2)^2}e^{-S(E_n-\omega_{nm}/2)}|f_O(E_n-\omega_{nm}/2,\omega_{nm})|^2|R_{mn}|^2
\end{aligned}$$

where $\bar{E}=(E_m+E_n)/2 = E_n-\omega/2$, $\omega_{nm}=E_n-E_m$ and $S(E)$ is the thermodynamic entropy at energy E. We would need to convert the sum into integral where we use the fact that function f_O is smooth and fluctuations of $|R_{mn}|^2$ average out in the sum.

$$\sum_{m\neq n}\to\int d\omega\Omega(E_n-\omega)=\int d\omega e^{S(E_n-\omega)} \tag{28}$$

where $\Omega(E_n+\omega)$ is density of states.

$$||A_\lambda||_n^2 = \int d\omega e^{S(E_n-\omega)-S(E_n-\omega/2)}\frac{\omega^2}{(\omega^2+\mu^2)^2}|f_O(E_n-\omega/2,\omega)|^2$$

$S(E_n-\omega)-S(E_n-\omega/2) = -\beta\omega/2+\ldots$ and $f_O(E_n-\omega/2,\omega)=f_O(E_n,\omega)+\ldots$ we have

$$||A_\lambda||_n^2 = \int_a^b d\omega e^{-\beta\omega/2}\frac{\omega^2}{(\omega^2+\mu^2)^2}|f_O(E_n,\omega)|^2$$

where a represents the minimum energy difference E_m-E_n in thermodynamic limit (which is $\min\{w_{nm}\}$) and b is the maximum energy difference (for which we have to find m-th state such that we get $\max\{w_{nm}\}$)). $a=e^{-S}\sim e^{-\delta L}$ and $b=\gamma L$, where γ and δ are constants that depend on the details of Hamiltonian.

Let's denote $I = e^{-\beta\omega/2}\frac{\omega^2}{(\omega^2+\mu^2)^2}$ and find out how it depends on L. First, we check on upper limit.

$$\lim_{L\to\infty} I(\omega = L) = \lim_{L\to\infty} e^{-\beta L/2}\frac{L^2}{(L^2+\mu^2)^2} \to 0$$

Now on lower limit.

$$\lim_{L\to\infty} I(\omega = e^{-L}) = \lim_{L\to\infty} e^{-\beta e^{-L}/2}\frac{e^{-2L}}{(e^{-2L}+\mu^2)^2} = \lim_{L\to\infty}\frac{e^{-2L}}{(e^{-2L}+\mu^2)^2}$$

$$\lim_{L\to\infty} I(\omega = e^{-L}) = \begin{cases} e^{2L} & (\mu^2 \ll e^{-2L}), \\ \dfrac{e^{-2L}}{\mu^4} & (\mu^2 \gg e^{-2L}) \end{cases} \tag{29}$$

Now, let's compute the norm while assuming $|f_O(E_n,\omega)|^2$ is a constant in ω. Hence, we get:

$$||A_\lambda||_n^2 = |f_O(E_n)|^2 \int_0^\infty d\omega e^{-\beta\omega/2}\frac{\omega^2}{(\omega^2+\mu^2)^2}$$

Let's assume $\beta \ll 1$ (high temperature limit):

$$\begin{aligned} ||A_\lambda||_n^2 &= |f_O(E_n)|^2 \int_0^\infty d\omega\,(1-\beta\omega/2+\ldots)\,\frac{\omega^2}{(\omega^2+\mu^2)^2} \\ &= |f_O(E_n)|^2\left(\frac{\pi}{4\mu} - \frac{\beta}{4} - \frac{\beta}{4}\log(\mu^2+\omega^2)|_0^\infty + \ldots\right) \end{aligned}$$

We see that there is a logarithmic divergence for high temperature limit. We also note that there are two limits, in which we find that there is no ultraviolet divergence: $\beta = 0$ limit gives $\pi/4\mu$ and $\beta \to \infty$ limit gives us zero norm. I don't understand why zero temperature limit gives zero norm.

Hence, ETH claims that norm of gauge potential in infinite temperature will be ($\hbar = 1$):

$$\begin{aligned} ||A_\lambda||^2 &= \sum_n ||A_\lambda||_n^2 \\ &= \frac{\pi}{4\mu}\sum_n |f_O(E_n)|^2 \\ &= \frac{\pi 2^L}{4\mu}\langle |f_O(E_n)|^2\rangle \end{aligned}$$

Hence, we get:

$$\boxed{\frac{||A_\lambda||^2}{2^L} = \frac{\pi}{4\mu}\langle |f_O(E_n)|^2\rangle} \tag{30}$$

6 System-size scaling of minimum and maximum of ω_{ij}

For integrable model, we would study the Hamiltonian of Transverse Field Ising model:

$$H = J\sum_{j=1}^{L-1}\sigma_j^z\sigma_{j+1}^z + \lambda\sum_j \sigma_j^x \tag{31}$$

where we have chosen $J = 1$ and $\lambda = 5$ with open boundary conditions.

For non-integrable model, we would study the Hamiltonian of Ising model with both transverse and longitudinal fields:

$$H = J \sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + h \sum_j \sigma_j^z + \lambda \sum_j \sigma_j^x \tag{32}$$

where we have chosen $J = 1$, $h = (\sqrt{5}+1)/4$ and $\lambda = (\sqrt{5}+5)/8$ with open boundary conditions. These are values of parameters for which this model has been shown to be robustly non-integrable for small systems [10]. Since anti-ferromagnetic phase has more local order compared to ferromagnetic phase, we expect the former to be less affected by finite size effects.

We see that $\partial_\lambda H = \sum_j \sigma_j^x$.

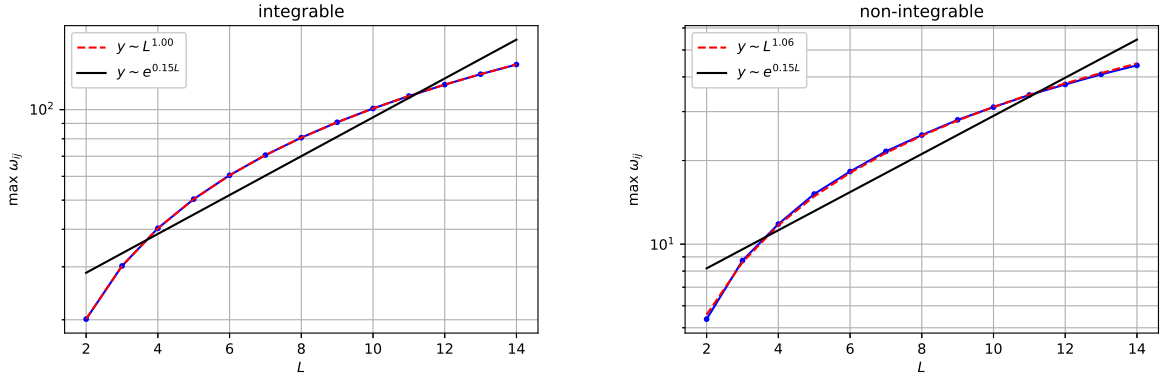

Figure 3: Using ED method, $\max \omega_{ij}(L)$ shows linear scaling with system size

Below we use exact diagonalization (ED) to compute $\max \omega_{ij}$ and $\min \omega_{ij}$ as a function of system size. As expected for both non-integrable and integrable systems, we find that $\max \omega_{ij} \sim L$ (figure 3). For non-integrable system, we find $\min \omega_{ij} \sim e^{-L}$ (right side of figure 4). For integrable systems, since we are using open boundary conditions (which means there is no symmetry of translation in the system), we find that there is no degeneracies for system size checked upto $L = 10$. This can explain why we get the same exponential behavior for this system (left side of figure 4).

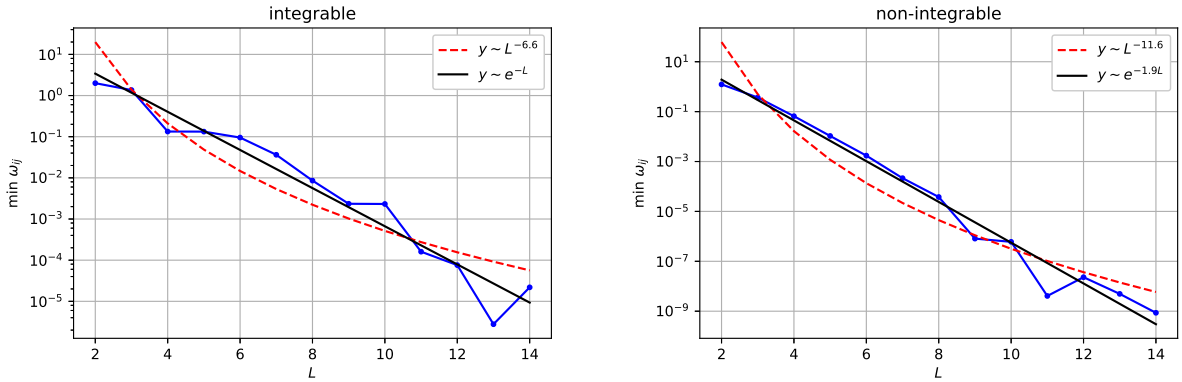

Figure 4: Using ED method, $\min \omega_{ij}(L)$ shows exponential scaling with system size

We should note that in an integrable system with open boundary condition, exact gauge potential shows an almost linear scaling with system size (figure 8). This shows that there are a lot of energy levels in which $\langle n|\partial_\lambda H|m\rangle$ should be zero compared to non-integrable where there are very few degeneracies and very few levels in which $\langle n|\partial_\lambda H|m\rangle$ is zero.

7 Norm computed using ED

Let's look at the expression of off-diagonal elements of gauge potential:

$$\langle m|A_\lambda|n\rangle = -i\hbar\frac{\langle m|\partial_\lambda H|n\rangle}{E_m - E_n}, \quad n \neq m \tag{33}$$

We see that while using ED, we need to be wary of degenerate eigenvalues. Do these degenerate eigenvalues contribute to norm of gauge potential? Answer is no because $\langle m|\partial_\lambda H|n\rangle = 0$ for degenerate pair of eigenvalues as shown in appendix A.

7.1 μ scaling of gauge potential

Our μ-dependent gauge potential A_λ is given by:

$$\langle m|A_\lambda|n\rangle = -i\hbar\frac{\langle m|\partial_\lambda H|n\rangle}{\omega_{mn}^2 + \mu^2}\omega_{mn} \tag{34}$$

where $\omega_{nm} = E_n - E_m$ and eigenstates depend on λ, i.e.$|n\rangle = |n(\lambda)\rangle$. Hence, norm should be (in units of $\hbar = 1$):

$$||A_\lambda||^2 = \sum_n \sum_{m\neq n} \frac{\omega_{nm}^2}{(\omega_{nm}^2 + \mu^2)^2}|\langle m|\partial_\lambda H|n\rangle|^2 \tag{35}$$

Numerically, we find the dependence of gauge potential on μ using Exact Diagonalization method (ED) in figure 5.

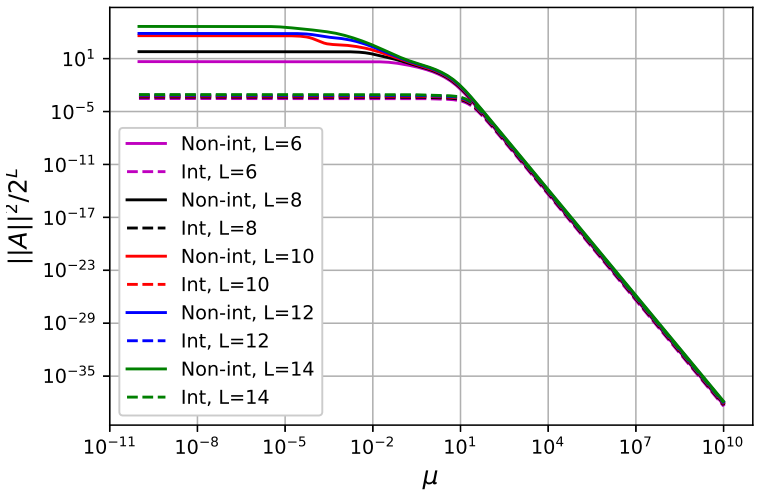

Figure 5: Using ED method, we obtain μ dependence of norm of gauge potential in integrable and non-integrable systems

Let's claim that $||A||^2/2^L = \alpha\mu^\beta$. Then if we take log both sides, we get

$$\log||A||^2/2^L = \log\alpha + \beta\log\mu \tag{36}$$

where β is the slope on a log-log scale. Numerically, we can find β_i for each pair of points using the following relationship (figure 6):

$$\beta_i = \frac{\log y(\mu_{i+1}) - \log y(\mu_i)}{\log\mu_{i+1} - \log\mu_i} \tag{37}$$

where $y = ||A||^2/2^L$.

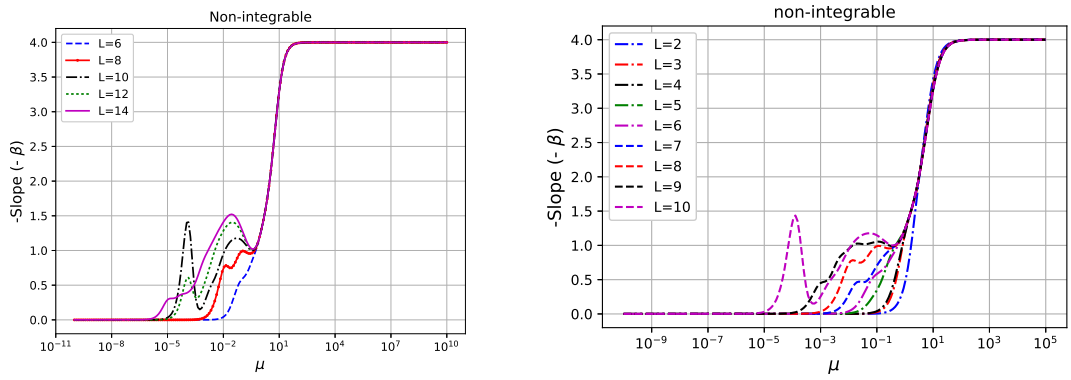

Figure 6: Using ED method, we obtain μ dependence of negative of slope ($-\beta(\mu)$) for non-integrable systems: a) bigger systems b) smaller systems

Trivial/non-physical regimes

Let's study two regimes we see in the figures:

- **small μ regime** when $\mu \ll \min\{w_{nm}\}$: Since density of states is highest in the middle of spectrum, $\min\{w_{nm}\}$ is smallest for two states lying there. In this regime, μ is so small that it doesn't really affect the norm of gauge potential. So, we get exact gauge potential in this regime.

$$||A_\lambda||^2 = \sum_n \sum_{m \neq n} \frac{|\langle m|\partial_\lambda H|n\rangle|^2}{\omega_{nm}^2} \sim 2^L e^L \tag{38}$$

  Exact gauge potential of non-integrable systems consists of all n- body ($1 \leq n \leq L$) operators. It looks something like:

$$A_\lambda = \sum_j [\alpha_j \sigma_j^y + \beta_j(\sigma_j^y \sigma_{j+1}^z + \sigma_j^z \sigma_{j+1}^y) + \gamma_j(\sigma_j^y \sigma_{j+1}^x + \sigma_j^z \sigma_{j+1}^x) + \ldots] \tag{39}$$

  Looking at the above expression, why norm of non-integrable systems is exponentially large in system size? My answer, which needs to be discussed with Dries, is that there are only extensive number of operators ($2L$ in number) as compared to L number of operators in integrable systems. So, the number of operators doesn't make it exponentially large. What makes the big difference is that weight of these operators $\bar{\alpha}_l$ ($\bar{\alpha}_{l=1} = \{\alpha_j\}, \bar{\alpha}_{l=2} = \{\beta_j\}, \bar{\alpha}_{l=3} = \{\gamma_j\}$ etc) is exponentially large in string size l which is in contrast with integrable systems, where these weights are exponential suppressed in string size. I should check this for numerically found exact gauge potential.

  What's the biggest μ of this regime? We know that $\min\{w_{nm}\} \simeq e^{-L}$ due to the exponential number of states in a many-body system whose bandwidth generally increases extensively in system size. Hence, we expect $\mu_c^{(1)} \simeq e^{-L}$.

- **big scaling regime** When $\mu \gg \max\{w_{nm}\}$, approximate gauge potential A_λ^* would be given by:

$$\begin{aligned} A_\lambda^* &= -i\hbar[H, \partial_\lambda H]\frac{1}{\mu^2} \\ &= -i\hbar\frac{1}{\mu^2}C^{(1)} \end{aligned}$$

  where $C^{(1)} = 2i\left(\sum_{j=1}^{L-1} \sigma_j^y \sigma_{j+1}^z + \sum_{j=2}^{L} \sigma_j^y \sigma_{j-1}^z + h\sum_{j=1}^{L} \sigma_j^y\right)$.

$$||A_\lambda||^2 = \frac{\alpha}{\mu^4} \sim \frac{L}{\mu^4} 2^L$$

where $\alpha = \text{Tr}[H, \partial_\lambda H]^2$. For $L = 12$, we obtain $\alpha_2^{Th} = 119.41$ whose details are given in appendix B.

In this regime, approximate gauge potential is local because it has only a single body term and a two body term. What's the smallest μ of this regime? We know that $\max\{w_{nm}\} \simeq L$ because bandwidth generally increases extensively in system size. Hence, we expect $\mu_c^{(2)} \simeq L$.

These regimes are non-physical because it's not useful to choose μ either exponentially small (because we will have exact gauge potential which has small denominator problem) or extensively big (because in that case, μ is the biggest energy scale and then nothing interesting really happens).

7.2 Phase diagram of gauge potential

What we find here is a "phase diagram" where we get different 'types' of gauge potential depending upon the value of μ used to construct it. Trivial/non-physical regimes are: exact gauge potential in small μ regime and trivial local approximate gauge potential in big μ regime.

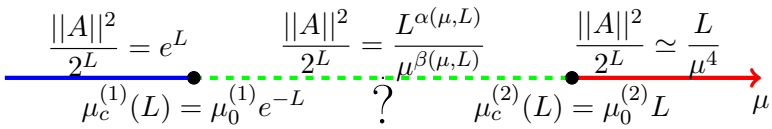

Figure 7: Phase diagram of gauge potential of non-integrable systems as a function of μ

In the above picture, the middle part ($\mu_c^{(1)}(L) < \mu < \mu_c^{(2)}(L)$) still needs to be found out. How does α and β depend on μ and L? In order to answer these questions, we need to study more to characterize the different "phases" for intermediate value of μ based upon following parameters. For a given phase, we would have a certain range of allowed μ. Within this range of μ,

- how does the norm of gauge potential varies as a function of system size (L)? Polynomial in L or exponential in L? We know that in exact regime (small μ), it grows as an exponential and in big μ regime, it grows as linear in system size. What happens in regimes outside these extreme regimes?
- what kind of operators makes up our gauge potential? We know that in exact regime (small μ), gauge potential consists all all n- body ($1 \leq n \leq L$) operators are there and in big μ regime, it contains only one and two body operators. What happens in regimes outside these extreme regimes?
- How does the weight of these operators increase as string length of these operators increase? We know that in exact regime (small μ) for non-integrable model, these weights increase exponentially as the string length increases and in big μ regime, the weight is uniform for the one and two body operators. For integrable systems, these weights are exponentially suppressed as the string length increases. What happens in regimes outside these extreme regimes?

We need to figure out a systematic way to find out the boundary of our phase diagram.

7.3 Finding boundary of phase diagram: system size scaling

Numerically, we find that norm of exact gauge potential of non-integrable system scale exponentially in system size while it scales as polynomial in system size for integrable system (figure 8).

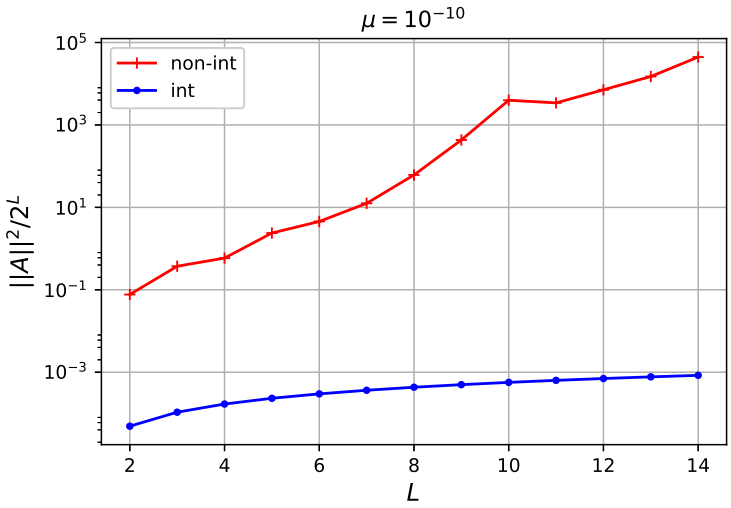

Figure 8: Exact gauge potential as a function of system size: non-integrable systems (exponential scaling) and integrable (polynomial scaling)

Scaling of slope of gauge potential

Here we take different slices in y-axis of slope of gauge potential (figure 5) at certain β_c. In small μ regime, since we expect $\beta \simeq 0$, we start off with a value of $\beta_c = 10^{-10}$ and then we keep on increasing our value of β_c in subsequent iterations.

In big μ regime, since we expect $\beta \simeq 4$, we start off with a value of $\beta_c = 3$ and then we keep on decreasing our value of β_c in subsequent iterations.

We find that to reach the same value of β_c, the system needs cutoff μ_c whose value is exponentially small in system size in small μ regime while in big μ regime, μ_c whose value is extensively big in system size (figure 9). This provides evidence that $\mu_c^{(1)} \propto e^{-L}$ and $\mu_c^{(2)} \propto L$.

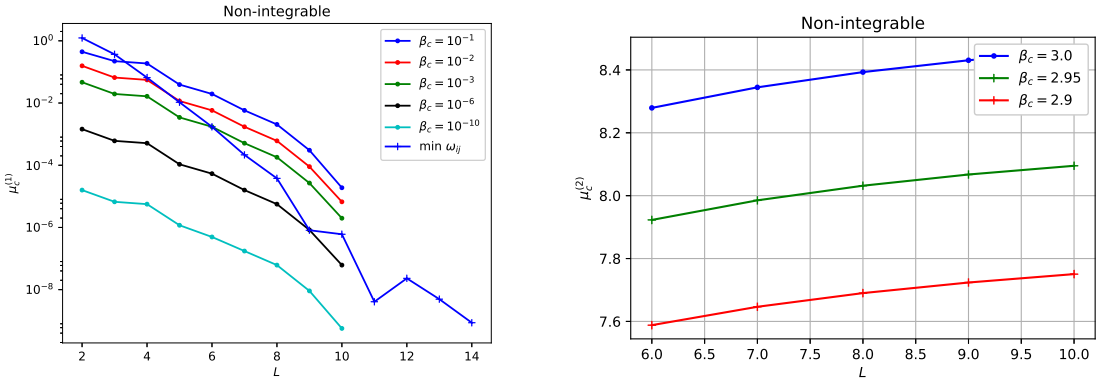

Figure 9: $\mu_c^{(1)}$ and $\mu_c^{(2)}$ dependence of system size L: former seems exponentially small and the latter a linear one

Scaling of gauge potential

In small μ regime, since we expect $\mu_c^{(1)} \propto e^{-L}$, we parametrize our $\mu_c^{(1)}(L) = \mu_0 e^{-L}$, where μ_0 is our parameter (left side of figure 10). Here, we are coming from far left side of phase diagram

which means our first guess of μ_0 (10^{-30}) is really smaller than $\min \omega_{ij}$ and then we will crank up our value of μ_0 in subsequent guesses. In this regime of exact gauge potential, the value of μ^0 and scaling of μ with system size should not matter. As we increase the value of μ_0, we would find that the value of μ is big enough that the value of norm of gauge potential is "appreciably" different than exact gauge potential (when $\mu_0 \sim 10^{-30}$). Since, this change of behavior is continuous [2], it's difficult to exactly determine the boundary of this phase diagram from this approach (maybe I can look into derivative of some order parameter (norm?) and then find a sharper transition?). Hence, from left side of figure 10, it's reasonable to claim that $\mu_c^{(1)} = \mu_0^* e^{-L}$, where μ_0^* lies between 1 and 10.

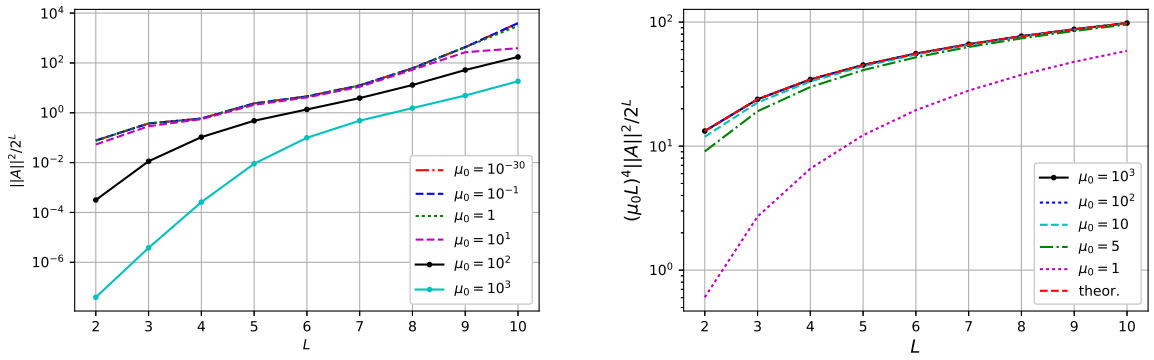

Figure 10: a) **Scaling of $\mu_c^{(1)}$:** we parameterize μ as $\mu_c^{(1)}(L) = \mu_0 e^{-L}$. We find μ_0^* to be between 1 and 10 b)**Scaling of $\mu_c^{(2)}$:** we parameterize μ as $\mu_c^{(2)}(L) = \mu_0 L$. We find μ_0^* to be between 10 and 100.

Similarly, in big μ regime, since we expect $\mu_c^{(2)} \propto L$, we parametrize our $\mu_c^{(2)}(L) = \mu_0 L$, where μ_0 is our parameter. In this regime, we have:

$$||A_\lambda||^2 = \frac{1}{\mu(L)^4} \operatorname{Tr}[H, \partial_\lambda H]^2 \tag{40}$$

where $\mu(L) = \mu_0 L$.

Here, we are coming from far right side of phase diagram which means our first guess of μ_0 (10^3) is really bigger than $\max \omega_{ij}$ and then we will reduce our value of μ_0 in subsequent guesses. Unlike the regime of exact gauge potential, in this regime of this approximate local gauge potential, the value of μ^0 and scaling of μ with system size does determine the value of norm of approximate gauge potential. That's why y-axis of right side of figure 10 has been scaled properly by multiplying norm with $\mu(L)^4$. As we decrease the value of μ_0, we would find that the value of μ is small enough that the value of norm of gauge potential is "appreciably" different than local approximate gauge potential (when $\mu_0 \sim 10^3$). Since, this change of behavior is continuous, it's difficult to exactly determine the boundary of this phase diagram from this approach. Hence, from right side of figure 10), it's reasonable to claim that $\mu_c^{(2)} = \mu_0^* L$, where μ_0^*, where μ_0^* lies between 10 and 100.

Before we wind off this section, let's try $\mu_c^{(2)}(L) = \mu_0 L^2$ so as to see what happens if we parametrize with a wrong scaling (figure 11).

Can this second analysis on its' own tell us if our system size dependence of critical points ($\mu_c^{(1)}$ and $\mu_c^{(2)}$) is correct? No, we also need to look into the first analysis of slope of gauge potential with respect to μ, which is presented in figure 9.

[2] Is it a crossover phenomenon or second order phase transition (continuous in order parameter and first derivative of order parameter but discontinuous in second derivative)?

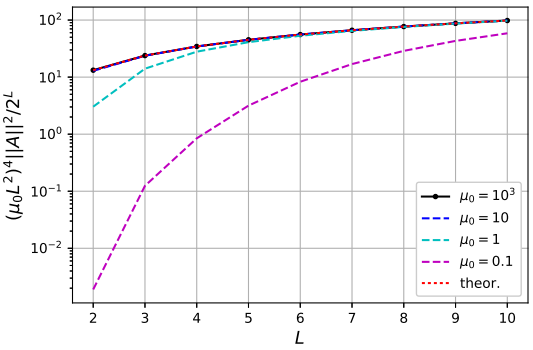

Figure 11: Scaling of $\mu_c^{(2)}$: we parameterize μ as $\mu_c^{(2)}(L) = \mu_0 L^2$. We find μ_0^* to be between 1 and 10.

8 Reducing transition and dissipation rate with approximate gauge potential

Let's try to understand more about μ by studying transition rate and dissipation rate for a quantum many body systems using Fermi's Golden rule. Let's consider a Hamiltonian:

$$\mathcal{H}_{\mathcal{X}} = \mathcal{H}(\lambda) + \dot{\lambda}\mathcal{X} \tag{41}$$

where $\lambda = \lambda_0 + \epsilon(t)$ and $\epsilon(t)$ is an infinitesimal white noise which satisfies $\overline{\epsilon(t)\epsilon(t')} = \kappa\delta(t-t')$ [3]. A system under white noise would lead to transitions between all eigenstates. We will see how much there is reduction in transition and dissipation rate with our approximate gauge potential.

We can simplify the above expression:

$$\mathcal{H}_{\mathcal{X}} \approx \mathcal{H}(\lambda_0) + \epsilon\partial_\lambda\mathcal{H} + \dot{\epsilon}\mathcal{X} \tag{42}$$

Our expressions will involve G_λ which is given as $G_\lambda(\mathcal{X}) = \partial_\lambda H + \frac{i}{\hbar}[\mathcal{X}, H]$ where $\mathcal{X} = A_\lambda(\mu)$.

8.1 Transition rate

Now we would drive the system periodically in addition to the white noise we have in the system. So, in this protocol, time dependence of λ is given as $\lambda(t) = \lambda_0 \sin(\omega t) + \epsilon(t)$. We can use Fermi's golden rule (using results from [11]) to derive transition rate $\langle\Gamma_n\rangle$ [5] as follows:

$$\Gamma_n(\omega) = \kappa \sum_{m\neq n} |\langle n|G_\lambda|m\rangle|^2 \delta(E_n - E_m - \hbar\omega) \tag{43}$$

where ω is frequency of the periodic external drive. More details are given in appendix C.

Using the fact that $\langle m|A_\lambda|n\rangle = -i\hbar\frac{\langle m|\partial_\lambda H|n\rangle}{\omega_{mn}^2 + \mu^2}\omega_{mn}$, we find

$$\begin{aligned}
\langle n|G_\lambda|m\rangle &= \langle n|\partial_\lambda H + \frac{i}{\hbar}[A_\lambda, H]|m\rangle \\
&= \langle n|\partial_\lambda H|m\rangle + \frac{i}{\hbar}\langle n|[A_\lambda, H]|m\rangle \\
&= \partial_\lambda H^{nm} + \frac{i}{\hbar}(E_m - E_n)\langle n|A_\lambda|m\rangle
\end{aligned}$$

[3] If the system is in equilibrium, then fluctuation -dissipation theorem dictates $\kappa = T$

$$= \partial_\lambda H^{nm} \left(1 - \frac{(E_n - E_m)^2}{(E_n - E_m)^2 + \mu^2}\right)$$

Hence, in units of $\hbar = 1$, we have:

$$\boxed{\Gamma_n(\omega) = \kappa \sum_{m \neq n} |\partial_\lambda H^{nm}|^2 \left(1 - \frac{\omega_{nm}^2}{\omega_{nm}^2 + \mu^2}\right)^2 \delta(\omega_{nm} - \omega)} \tag{44}$$

where δ function is broadened for finite-sized system to include several eigenstates so as to have a smooth dependence of $\Gamma_n(\omega)$. This means for each ω, we only include the eigenstates in the sum which falls within some tolerance value. We define a tolerance ϵ as $|\omega - \omega_{nm}| < \epsilon$ to characterize this smoothing procedure.

In small μ limit, $\Gamma_n(\omega)$ is zero for all energy ω. In μ is really big, when A_λ is almost zero, $G_\lambda \approx \partial_\lambda H = \sum_i \sigma_i^x$. This would mean driving the system without any counter-dibatic term.

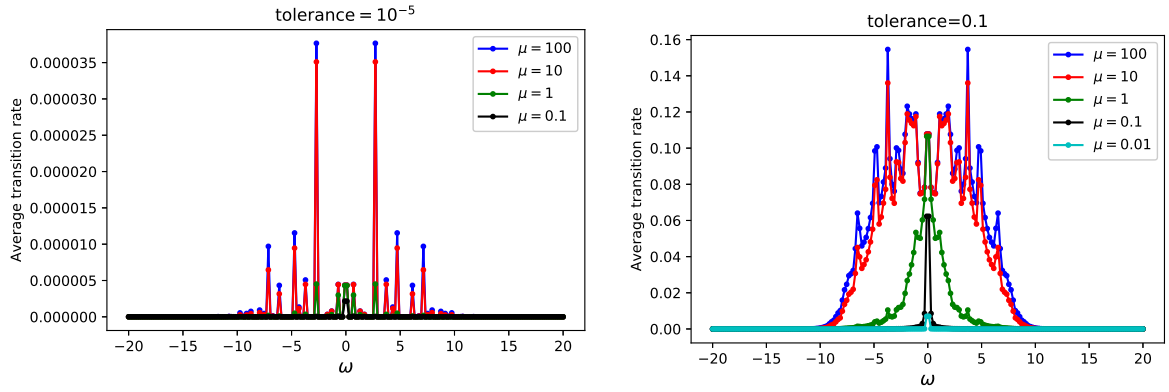

Figure 12: Transition rate with $\kappa = 1$: for L=10, we find that increasing μ reduces the transition rate across the whole range of energy ω

In small μ limit, $\Gamma_n(\omega)$ is zero for all external energy ω through noise. In μ is really big, when A_λ is almost zero, $G_\lambda \approx \partial_\lambda H = \sum_i \sigma_i^x$. This would mean driving the system using white noise without any counter-dibatic term.

From figure 12, we see that when $\mu > 1$, transition rate is not reducing much for $\omega = 0$. Hence, we find that central peak is almost the same while other peaks are reducing a lot. Only when value of μ is reduced to less than 1, we find that central peak started reducing. Also, it seems that central peak only starts reducing when other peaks have almost reduced to zero leaving central peak as the only peak in this regime. This should be investigated further. Can we make a claim that it is in the big μ regime when central peak doesn't get reduced whiel other peaks gets reduced substantially? Is there any information in seeing how width scales with μ?

8.2 Dissipation rate

Dissipation rate $\frac{d\sigma_E^2}{dt}$ for a particular eigenstate $|n\rangle$ is defined as:

$$\frac{d\sigma_E^2}{dt} = \int d\omega \omega^2 \Gamma_n(\omega) \tag{45}$$

$$= -\kappa \langle n|[G_\lambda, H]^2|n\rangle \tag{46}$$

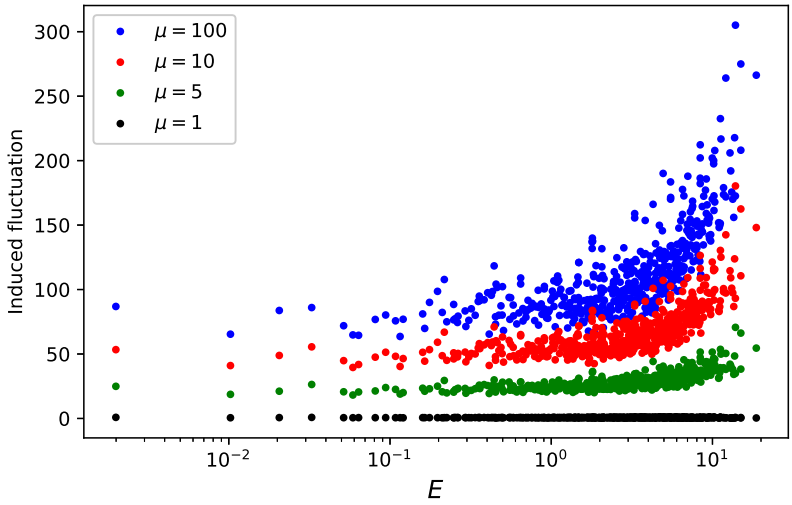

Figure 13: a) Dissipation rate with $-\kappa = 1$: for L=10, we find that increasing μ reduces the dissipation rate across the whole range of energy ω b) Average dissipation rate

After doing some calculations (whose details are given in appendix D), we find that:

$$\boxed{\frac{d\sigma_E^2(n)}{dt} = -\kappa \sum_{m \neq n} \frac{\mu^4 (E_n - E_m)^2}{((E_n - E_m)^2 + \mu^2)^2} |\langle n|\partial_\lambda H|m\rangle|^2} \tag{47}$$

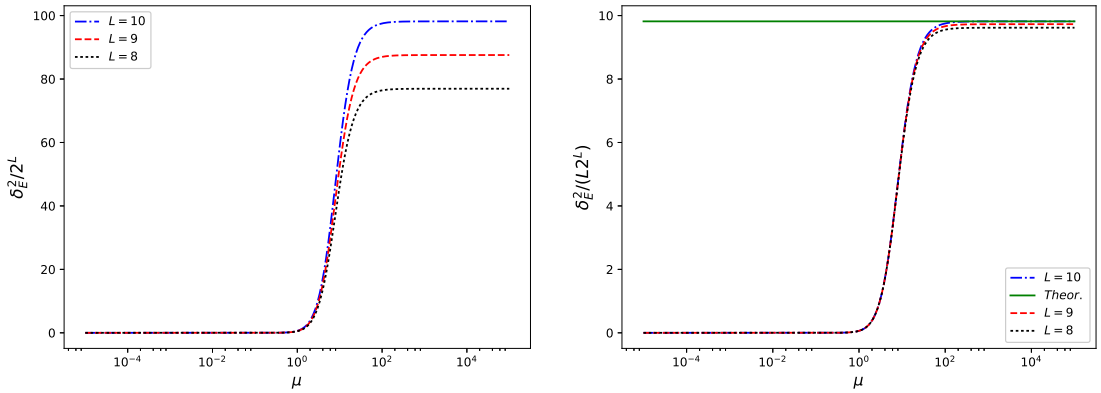

Figure 14: Average dissipation rate a) size-dependence b) system-size scaled

In small μ limit, $\frac{d\sigma_E^2}{dt}$ is zero. In the regime when μ is really big, A_λ is almost zero and $G_\lambda \approx \partial_\lambda H = \sum_i \sigma_i^x$. This would mean driving the system without any counter-dibatic term. Hence $[G_\lambda, H]^2 = (C^{(1)})^2$ and $\frac{d\sigma_E^2}{dt} = -\kappa \langle n|(C^{(1)})^2|n\rangle$.

Average transition rate is $\delta_E^2/2^L$ where $\delta_E^2 = \sum_{n=1}^{2^L} \frac{d\sigma_E^2(n)}{dt}$.

9 Physical meaning of cutoff regulator

9.1 Equation of motion for approximate gauge potential

Equation of motion for exact gauge potential A_λ is:

$$[H, G_\lambda(\mu = 0)] = 0 \tag{48}$$

where $G_\lambda(\mu = 0) = \partial_\lambda H + \frac{i}{\hbar}[A_\lambda, H]$

Approximate gauge potential is

$$A_\lambda(\mu) = -i\hbar \sum_{n=0}^{\infty} (-1)^n \frac{C^{(2n+1)}}{\mu^{2n+2}} \tag{49}$$

where $C^{(n)}$ is n- commutator of H and $\partial_\lambda H$, i.e. $C^{(n)} = [H, [H,\ \text{n times} \ldots, [H, \partial_\lambda H]]]]$. We define the first term as $C^{(1)} = [H, \partial_\lambda H]$, second term as $C^{(2)} = [H, [H, \partial_\lambda H]] = [H, C^{(1)}]$ and so on and forth. Properties of $C^{(n)}$ are noted in appendix E.

Now, let's compute:

$$\begin{aligned}
[H, G_\lambda(\mu)] &= [H, \partial_\lambda H + \frac{i}{\hbar}[A_\lambda, H]] \\
&= [H, \partial_\lambda H] - \frac{i}{\hbar}[H, [H, A_\lambda]] \\
&= C^{(1)} + i^2 \sum_{n=0}^{\infty} (-1)^n \frac{C^{(2n+3)}}{\mu^{2n+2}} \\
&= C^{(1)} - (\frac{C^{(3)}}{\mu^2} - \frac{C^{(5)}}{\mu^4} + \ldots) \\
&= C^{(1)} - \frac{C^{(3)}}{\mu^2} + \frac{C^{(5)}}{\mu^4} - \ldots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{C^{(2n+1)}}{\mu^{2n}} \\
&= \frac{i}{\hbar}\mu^2 A_\lambda(\mu)
\end{aligned}$$

Hence, we get:

$$\boxed{[H, \partial_\lambda H] + \frac{i}{\hbar}[H, [A_\lambda(\mu), H]] = \frac{i}{\hbar}\mu^2 A_\lambda(\mu)} \tag{50}$$

This equation reminds us of the equation wrote by Dries using local gauge potential. He argued that Hamiltonian is $H_S + H_B + H_{\text{int}}$, where H_S is Hamiltonian of system, H_B is Hamiltonian of boundary and $H_{\text{int}} = \sum \gamma_j S_j \otimes B_j$. Local gauge potential is given by this ansatz $A = A_S \otimes 1_B$. Then equation of motion is given by:

$$[\partial_\lambda H_S, H_S] + \frac{i}{\hbar}[[A_S, H_S], H_S] + \frac{i}{\hbar}\sum_j \gamma_j^2 [[A_S, S_j], S_j] = 0 \tag{51}$$

Let's rewrite the way we like it:

$$[H_S, \partial_\lambda H_S] + \frac{i}{\hbar}[H_S, [A_S, H_S]] = \frac{i}{\hbar}\sum_j \gamma_j^2 [[A_S, S_j], S_j]$$

If γ^2 goes to zero, then we have exact gauge potential for the remaining system excluding boundaries. Dries found that for $\gamma = 1$, norm of gauge potential is L^2.

Let's work on this $\mu^2 A_\lambda(\mu) = \gamma^2 \sum_j [[A_S, S_j], S_j]$ by framing it as an eigenvalue equation in the form of $\hat{O} A_\lambda(\mu) = \mu^2 A_\lambda(\mu)$.

Big picture stuff: It's possible that by working on this, we can show that μ in intermediate regime leads to an approximate gauge potential which is local in the way Dries's ansatz is local.

9.2 Computing gauge potential by truncating the summation

Approximate gauge potential is

$$A_\lambda(\mu) = -i\hbar \sum_{n=0}^{\infty} (-1)^n \frac{C^{(2n+1)}}{\mu^{2n+2}} \tag{52}$$

Let's truncate the series by modulating cutoff μ_n as a function of summation index n. This summation index is proportional to lattice index, which means that each higher order of commutator has bigger support on the lattice, i.e. $C^{(n+1)}$ contains operators which are more spread over lattice than operators in $C^{(n-1)}$. We choose the simplest way to truncate the series by arguing that we have no control over boundary and therefore, cutoff μ_n should be as follows:

$$\mu_n = \begin{cases} \mu & (\text{if } n \leq L), \\ \infty & (\text{if } n > L) \end{cases} \tag{53}$$

This approximation means that $C^{(2n+1)}/\mu_n^{2n+2} \ll 1$ for $n > L$. As a result, approximate gauge potential after this truncation is:

$$A_\lambda^L(\mu) = -i\hbar \sum_{n=0}^{L} (-1)^n \frac{C^{(2n+1)}}{\mu^{2n+2}} \tag{54}$$

The advantage with this series truncation is that this approximate gauge potential can be obtained without using exact diagonalization; it requires only extensive number of steps while computing the sum. However, we don't know if this approximation is useful at all. Does the norm of this approximate gauge potential is polynomial for all value of μ? Does it reduce dissipation and transition rate for all value of μ? These questions should be investigated numerically.

It's equation of motion is given by (where L is even):

$$\begin{aligned}
[H, G_\lambda(\mu)^L] &= [H, \partial_\lambda H + \frac{i}{\hbar}[A_\lambda^L, H]] \\
&= [H, \partial_\lambda H] - \frac{i}{\hbar}[H, [H, A_\lambda^L]] \\
&= C^{(1)} + i^2 \sum_{n=0}^{L} (-1)^n \frac{C^{(2n+3)}}{\mu^{2n+2}} \\
&= C^{(1)} - (\frac{C^{(3)}}{\mu^2} - \frac{C^{(5)}}{\mu^4} + \ldots - \frac{C^{(2L+1)}}{\mu^{2L}} + \frac{C^{(2L+3)}}{\mu^{2L+2}}) \\
&= C^{(1)} - \frac{C^{(3)}}{\mu^2} + \frac{C^{(5)}}{\mu^4} - \ldots + \frac{C^{(2L+1)}}{\mu^{2L}} - \frac{C^{(2L+3)}}{\mu^{2L+2}} \\
&= \sum_{n=0}^{L+1} (-1)^n \frac{C^{(2n+1)}}{\mu^{2n}} \\
&= \frac{i}{\hbar} \mu^2 A_\lambda^{L+1}(\mu)
\end{aligned}$$

This expression is true for both even and odd values of L.

Hence, we get:

$$\boxed{[H, \partial_\lambda H] + \frac{i}{\hbar}[H, [A_\lambda^L(\mu), H]] = \frac{i}{\hbar} \mu^2 A_\lambda^{L+1}(\mu)} \tag{55}$$

Question that is staring right in our face is whether $\mu^2 A_\lambda^{L+1}(\mu) \stackrel{?}{=} \gamma^2 \sum_j [[A_S, S_j], S_j]$? In other words, whether $\mu^2 \stackrel{?}{=} \gamma^2$ and whether $A_\lambda^{L+1}(\mu) \stackrel{?}{=} \sum_j [[A_S, S_j], S_j]$?

Both operators represent system-boundary interaction terms. If γ^2 goes to zero, then we have exact gauge potential for the remaining system excluding boundaries. However, if μ^2 goes to zero, then we will have exact gauge potential for the remaining system provided system size L is really big [4]. Is it possible that we could work with bigger systems numerically since we don't have to use exact diagonlization here?

If μ^2 becomes a really big number, then we know norm of gauge potential is zero. I don't know for sure if norm of gauge potential goes to zero, if γ^2 becomes really big.

9.2.1 Single body problem

We would start off with a simple problem and find its' adiabatic gauge potential using our method.

$$H = \Delta\sigma^z + \lambda\sigma^x \tag{56}$$

$$C^{(1)} = 2i\Delta\sigma^y \tag{57}$$
$$C^{(2)} = -4\Delta(-\Delta\sigma^x + \lambda\sigma^z) \tag{58}$$
$$C^{(3)} = \alpha^2 C^{(1)} \tag{59}$$

where $\alpha^2 = 4(\Delta^2 + \lambda^2)$

Now, we would compute A_λ :

$$\begin{aligned}
A_\lambda &= \frac{-i\hbar C^{(1)}}{\mu} \sum_{n=0}^{n_L} (-1)^n \frac{\alpha^{2n}}{\mu^{2n+1}} \\
&= \frac{-i\hbar}{\mu\alpha} C^{(1)} \sum_{n=0}^{n_L} (-1)^n \left(\frac{\alpha}{\mu}\right)^{2n+1} \\
&= \frac{-i\hbar}{\mu\alpha} \left(\frac{\alpha}{\mu}\right) C^{(1)} \sum_{n=0}^{n_L} (-1)^n \left(\frac{\alpha^2}{\mu^2}\right)^n \\
&= \frac{-i\hbar}{\mu^2} C^{(1)} \sum_{n=0}^{n_L} (-1)^n r^n \\
&= \frac{-i\hbar}{\mu^2} C^{(1)} \frac{1 + r^{n_L+1}}{1 + r}
\end{aligned}$$

where $r = \dfrac{\alpha^2}{\mu^2}$.

Now in approximate regime when μ is really big and r is smaller than 1, then we have in the limit $n_L \to \infty$:

$$A_\lambda = -i\hbar C^{(1)} \left(\frac{1 + e^{(n_L+1)\log r}}{\mu^2 + \alpha^2}\right) \approx -i\hbar C^{(1)} \frac{1}{\mu^2 + \alpha^2} \approx -i\hbar \frac{C^{(1)}}{\mu^2}$$

Essentially, only the first term contributes in the sum when we are in this regime. Now let's see what happens in exact regime when μ is small :

[4] A better approximation (which would give exact gauge potential in small μ limit for even small systems) is if we tune our μ such that operators of length bigger than system size L don't appear in the summation.

$$
\begin{aligned}
A_\lambda &= \frac{-i\hbar}{\mu^2} C^{(1)} \sum_{n=0}^{n_L} (-1)^n r^{-n} \\
&= \frac{-i\hbar}{\mu^2} C^{(1)} \frac{1 + r^{-n_L - 1}}{1 + r^{-1}} \\
&= -i\hbar C^{(1)} \left(\frac{1 + e^{-(n_L+1)\log r}}{\mu^2 + \alpha^2} \right) \approx -i\hbar \frac{C^{(1)}}{\mu^2 + \alpha^2} \approx -i\hbar \frac{C^{(1)}}{\alpha^2}
\end{aligned}
$$

where $r = \dfrac{\mu^2}{\alpha^2}$.

Big picture stuff: For non-integrable systems, it's difficult to compute the infinite summation. Truncating that summation might help in giving an approximate gauge potential which is tractable. For each system size, we can sum this series a big n_L number of times and see if we get a gauge potential whose averaged norm is L^2.

10 Bethe integrable models

One of the goals of this project is to use adiabatic gauge potential as a theoretical tool to distinguish between quantum integrable and non-integrable system. Our method, if it works, should be better than conventional method (energy level distribution) used in literature for this purpose because unlike the conventional method, we don't have to worry about removing symmetry.

Let's study XXZ model which is given below:

$$
H = J \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \lambda \sum_{j=1}^{L} \sigma_j^z \sigma_{j+1}^z \tag{60}
$$

where periodic conditions are assumed and λ is the anisotropic parameter (and also our control parameter). Hence, $\partial_\lambda H = \sum_j \sigma_j^z \sigma_{j+1}^z$

We are going to find scaling of norm of exact adiabatic gauge potential with system size.

A Do degenerate eigenvalues contribute to norm of gauge potential?

Let's consider $H|n(\lambda)\rangle = E_n|n(\lambda)\rangle$. Hence, we have $\langle m(\lambda)|H|n(\lambda)\rangle = 0$ for $n \neq m$. We can exploit this property to get some insight:

$$
\begin{aligned}
\partial_\lambda \langle m|H|n\rangle &= 0 \\
\langle \partial_\lambda m|H|n\rangle + \langle m|H|\partial_\lambda n\rangle + \langle m|\partial_\lambda H|n\rangle &= 0 \\
\langle \partial_\lambda m|n\rangle E_n + E_m \langle m|\partial_\lambda n\rangle + \langle m|\partial_\lambda H|n\rangle &= 0 \\
(E_n - E_m)\langle \partial_\lambda m|n\rangle + \langle m|\partial_\lambda H|n\rangle &= 0
\end{aligned}
$$

Hence, we find that if there are two degenerate energy levels n and m such that $E_n = E_m$, then $\langle m|\partial_\lambda H|n\rangle = 0$. Hence, the contribution to norm of gauge potential from this pair of energy levels will be zero. I should check this numerically if results of my code respect this property.

B Computing $\mathrm{Tr}[H, \partial_\lambda H]$

B.1 Integrable model

$$
H = \sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + \lambda \sum_j \sigma_j^x \tag{61}
$$

We know that that $\partial_\lambda H = \sum_j \sigma_j^x$. Let's denote $C^{(1)} = [H, \partial_\lambda H]$.

$$\begin{aligned}[H, \partial_\lambda H] &= \sum_{j=1}^{L-1} [\sigma_j^z \sigma_{j+1}^z, \sum_i \sigma_i^x] \\ &= 2i \left(\sum_{j=1}^{L-1} \sigma_j^y \sigma_{j+1}^z + \sum_{j=2}^{L} \sigma_j^y \sigma_{j-1}^z \right)\end{aligned}$$

We find that $\mathrm{Tr}\,|C^{(1)}|^2 = 8(L-1)2^L$

B.2 Non-integrable model

Non-integrable model's Hamiltonian is given by:

$$H = J \sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + h \sum_j \sigma_j^z + \lambda \sum_j \sigma_j^x \tag{62}$$

$$\begin{aligned}[H, \partial_\lambda H] &= [\sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + h \sum_{j=1}^{L} \sigma_j^z, \sum_i \sigma_i^x] \\ &= 2i \left(\sum_{j=1}^{L-1} \sigma_j^y \sigma_{j+1}^z + \sum_{j=2}^{L} \sigma_j^y \sigma_{j-1}^z + h \sum_{j=1}^{L} \sigma_j^y \right)\end{aligned}$$

We find that $\mathrm{Tr}\,|C^{(1)}|^2 = 2^L 4(h^2 L + 2J(L-1))$.

C details on transition rate

$$\Gamma_n(\omega) = \kappa \sum_{m \neq n} |\langle n|\partial_\lambda H - i\omega \mathcal{X}|m\rangle|^2 \delta(E_n - E_m - \omega) \tag{63}$$

$$= \kappa \sum_{m \neq n} |\langle n|\partial_\lambda H - \frac{i}{\hbar}(E_n - E_m)\mathcal{X}|m\rangle|^2 \delta(E_n - E_m - \omega) \tag{64}$$

$$= \kappa \sum_{m \neq n} |\langle n|\partial_\lambda H + \frac{i}{\hbar}(E_m - E_n)\mathcal{X}|m\rangle|^2 \delta(E_n - E_m - \omega) \tag{65}$$

$$= \kappa \sum_{m \neq n} |\langle n|G_\lambda|m\rangle|^2 \delta(E_n - E_m - \omega) \tag{66}$$

$$\tag{67}$$

D details on dissipation rate

$$\frac{d\sigma_E^2}{dt} = \int d\omega \omega^2 \Gamma_n(\omega) \tag{68}$$

$$= \int d\omega \kappa \sum_{m \neq n} \omega^2 |\langle n|G_\lambda|m\rangle|^2 \delta(E_m - E_n - \omega) \tag{69}$$

$$= -\kappa\, \mathrm{Tr}([G_\lambda, H]^2) \tag{70}$$

Ignoring κ and negative sign, we get:

$$\frac{d\sigma_E^2}{dt} = \langle n|[G_\lambda, H]^2|n\rangle \tag{71}$$
$$= \sum_m \langle n|[G_\lambda, H]|m\rangle\langle m|[G_\lambda, H]|n\rangle \tag{72}$$
$$= \langle n|[G_\lambda, H]|n\rangle^2 + \sum_{m\neq n} \langle n|[G_\lambda, H]|m\rangle\langle m|[G_\lambda, H]|n\rangle \tag{73}$$
$$= \sum_{m\neq n} |\langle n|[G_\lambda, H]|m\rangle|^2 \tag{74}$$

Now

$$[G_\lambda, H] = [\partial_\lambda H, H] + \frac{i}{\hbar}[[A_\lambda, H], H]$$

Now let's recall:

$$\langle n|A_\lambda|m\rangle = -i\hbar\frac{\langle n|\partial_\lambda H|m\rangle}{(E_n - E_m)^2 + \mu^2}(E_n - E_m) \tag{75}$$
$$= -i\hbar\frac{\langle n|[H, \partial_\lambda H]|m\rangle}{(E_n - E_m)^2 + \mu^2} \tag{76}$$

Off-diagonal elements of $[G_\lambda, H]$ is given by:

$$[G_\lambda, H] = [\partial_\lambda H, H] + \frac{i}{\hbar}[[A_\lambda, H], H] \tag{77}$$
$$= -[H, \partial_\lambda H] - \frac{i}{\hbar}[[H, A_\lambda], H] \tag{78}$$
$$= -[H, \partial_\lambda H] + \frac{i}{\hbar}[H, [H, A_\lambda]] \tag{79}$$
$$= -[H, \partial_\lambda H] + \frac{1}{(E_n - E_m)^2 + \mu^2}[H, [H, [H, \partial_\lambda H]]] \tag{80}$$

Hence, we find that :

$$\langle n|[G_\lambda, H]|m\rangle = -\langle n|[H, \partial_\lambda H]|m\rangle + \langle n|\frac{1}{(E_n - E_m)^2 + \mu^2}[H, [H, [H, \partial_\lambda H]]]|m\rangle$$
$$= \left(-(E_n - E_m) + \frac{(E_n - E_m)^3}{(E_n - E_m)^2 + \mu^2}\right)\langle n|\partial_\lambda H|m\rangle$$
$$= -\frac{\mu^2(E_n - E_m)}{(E_n - E_m)^2 + \mu^2}\langle n|\partial_\lambda H|m\rangle$$

Hence, we get :

$$\boxed{\frac{d\sigma_E^2}{dt} = -\kappa\sum_{m\neq n}\frac{\mu^4(E_n - E_m)^2}{((E_n - E_m)^2 + \mu^2)^2}|\langle n|\partial_\lambda H|m\rangle|^2} \tag{81}$$

E Properties of n-commutators

In this section, we would be proving some results of n-commutators $C^{(n)}$.

Theorem 1. $C^{(n)} = [H, C^{(n-1)}], \forall n > 0$ *where* $C^{(0)} = \partial_\lambda H$ *and* $C^{(1)} = [H, \partial_\lambda H]$

Proof. We define the first two terms as $C^{(0)} = \partial_\lambda H$ and $C^{(1)} = [H, \partial_\lambda H]$.

Now, $C^{(2)} = [H, [H, \partial_\lambda H]] = [H, C^{(1)}]$. Similarly, $C^{(3)} = [H, C^{(2)}]$. Hence, we can claim using induction argument:

$$C^{(n)} = [H, C^{(n-1)}] \tag{82}$$

□

F Integrable systems with open boundary condition

Here I show that for integrable systems with open boundary conditions, we find some kind of structure, which is absent when computed using periodic boundary conditions.

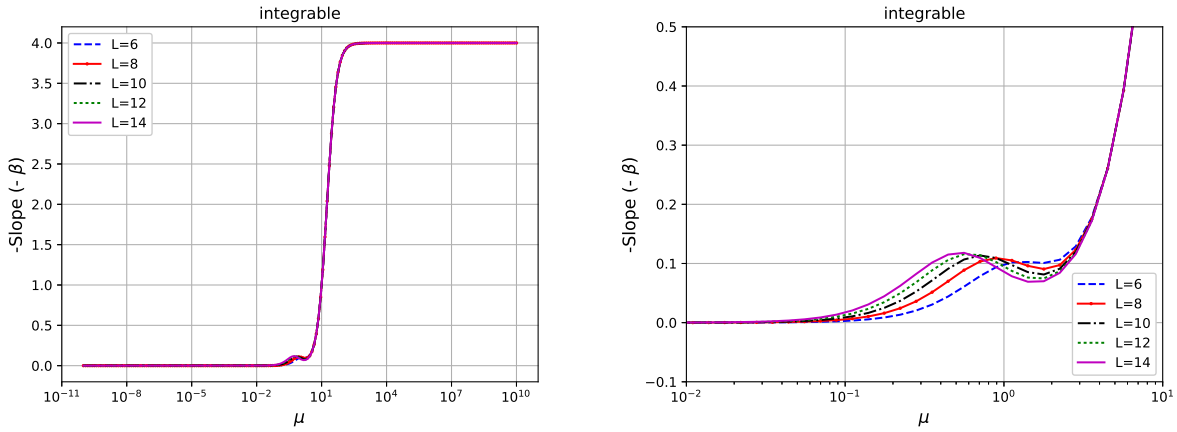

Figure 15: μ dependence of negative of slope $(-\beta(\mu))$ is shown for integrable systems

If there is degeneracy, $\min \omega_{ij}(L)$ should be zero. Why don't I see any degenerate level?

I find that because of open boundary conditions, I don't get any degenerate states for integrable model[5]. The question is then how do I get an almost linear scaling of norm for integrable models? We had reasoned that $\langle n|\partial_\lambda H|m\rangle$ is zero because of extensive number of degenerate levels. It doesn't seem like that here. Notes on how to find error in fitting using least-square method in python [6]

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