From Classical Nonlinear Integrable Systems to Quantum Shortcuts to Adiabaticity

Manaka Okuyama and Kazutaka Takahashi

Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan

Using shortcuts to adiabaticity, we solve the time-dependent Schrödinger equation that is reduced to a classical nonlinear integrable equation. For a given time-dependent Hamiltonian, the counterdiabatic term is introduced to prevent nonadiabatic transitions. Using the fact that the equation for the dynamical invariant is equivalent to the Lax equation in nonlinear integrable systems, we obtain the counterdiabatic term exactly. The counterdiabatic term is available when the corresponding Lax pair exists and the solvable systems are classified in a unified and systematic way. Multisoliton potentials obtained from the Korteweg–de Vries equation and isotropic XY spin chains from the Toda equations are studied in detail.

Introduction.—Ideal control of quantum systems has attracted interest recently from both theoretical and practical viewpoints. Rapid technological advances make it possible to manipulate quantum systems precisely, and designing the optimal Hamiltonian is a realistic important problem. The meaning of optimality is not obvious, and various methods have been proposed theoretically in different contexts. In the methods using shortcuts to adiabaticity, the Hamiltonian is designed so that the state follows an adiabatic passage of a reference Hamiltonian [1–5]. This technique was realized in several experiments [6–9] and is expected to be applied to the adiabatic quantum computation called quantum annealing [10].

Although the formulation of the method is general, explicit constructions of the Hamiltonian are restricted to simple systems such as two- and three-level systems [3,11], harmonic oscillators [12], and scale-invariant systems [13]. Most of systems fall into these categories, and various applications have been studied in many works [5,14–19]. The question addressed in this Letter is whether there are other classes of Hamiltonians in which the exact solutions are available. We propose a new method using the results from nonlinear integrable systems such as Korteweg–de Vries (KdV) solitons [20] and Toda lattices [21]. In nonlinear systems, a quantum mechanical viewpoint is used to solve the equations of motion. We find that the method developed there can be directly applied to the quantum adiabatic dynamics. Several previous works used the result from the integrable systems to construct a driver Hamiltonian [13,22,23]. Here we utilize the integrability of classical nonlinear systems to develop the fundamental principle to design the quantum Hamiltonian in a systematic way. It is applied not only to a specific system but also to a broad class of integrable systems including many-body systems.

Counterdiabatic driving and dynamical invariant.—In counterdiabatic driving, known as one of the strategies in shortcuts to adiabaticity, we control the adiabatic states of the time-dependent Hamiltonian $\hat{H}_{\rm ad}(t)$. The adiabatic state is defined by a linear combination of the instantaneous eigenstates $|n(t)\rangle$ of $\hat{H}_{\rm ad}(t)$ as $|\psi(t)\rangle = \sum_n c_n e^{-i\theta_n(t)}|n(t)\rangle$, where c_n is a time-independent constant and $\theta_n(t) = \int dt(\langle n(t)|\hat{H}_{\rm ad}(t)|n(t)\rangle - i\langle n(t)|(\partial/\partial t)|n(t)\rangle)$ represents a time-dependent phase [24]. This state does not satisfy the Schrödinger equation owing to nonadiabatic transitions. Those unwanted transitions are suppressed by introducing an additional term called the counterdiabatic term, $\hat{H}_{\rm cd}(t)$. The adiabatic state of $\hat{H}_{\rm ad}(t)$ satisfies the Schrödinger equation:

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = [\hat{H}_{\rm ad}(t) + \hat{H}_{\rm cd}(t)]|\psi(t)\rangle. \qquad (1)$$

The spectral representation of $\hat{H}_{\rm cd}(t)$ is given by [1–3]

$$\hat{H}_{\rm cd}(t) = i\sum_n (1 - |n(t)\rangle\langle n(t)|)\frac{\partial|n(t)\rangle}{\partial t}\langle n(t)|. \qquad (2)$$

Various strategies exist for designing the Hamiltonian in shortcuts to adiabaticity. The common property is described by the existence of the Lewis-Riesenfeld dynamical invariant $\hat{F}(t)$ satisfying

$$i\frac{\partial \hat{F}(t)}{\partial t} = [\hat{H}(t), \hat{F}(t)], \qquad (3)$$

where $\hat{H}(t)$ is the total Hamiltonian. It was shown that the eigenvalues of $\hat{F}(t)$ are independent of t, and the solutions of the Schrödinger equation are given by adiabatic states of $\hat{F}(t)$ [25]. This means that the Hamiltonian is divided into adiabatic [$\hat{H}_{\rm ad}(t)$] and counterdiabatic [$\hat{H}_{\rm cd}(t)$] components, and $\hat{F}(t)$ commutes with $\hat{H}_{\rm ad}(t)$. $\hat{H}_{\rm cd}(t)$ is written using the eigenstates of $\hat{F}(t)$ as Eq. (2). In inverse engineering [4], the problem is formulated using Eq. (3) instead of Eq. (2), as we show below. Furthermore, the optimality of the counterdiabatic driving is shown using a method called the quantum brachistochrone [26,27]. The optimal solution is represented by the invariant, which is

constructed for given constraints of the Hamiltonian. Thus, finding the invariant is the key to designing the optimal Hamiltonian.

We mainly treat a class of systems such that $\hat{F}(t) = \hat{H}_{ad}(t)$. The counterdiabatic term $\hat{H}_{cd}(t) = \hat{H}(t) - \hat{H}_{ad}(t)$ is obtained by solving Eq. (3). In this case, the eigenvalues of $\hat{H}_{ad}(t)$ are time independent. $\hat{H}_{cd}(t)$ is expressed by a linear combination of possible operators, and the time dependence of the coefficients is obtained by solving Eq. (3). The series is generally infinite, and solving the equation is a formidable task. However, in the integrable systems that we discuss in this Letter, the counterdiabatic term is expressed in a compact form.

Lax form for KdV equation.—First, we study one-dimensional systems in a time-dependent potential $u(x, t)$. The adiabatic Hamiltonian is given by

$$\hat{H}_{ad}(t) = \hat{p}^2 + u(\hat{x}, t). \tag{4}$$

The potential is left undetermined, and we study conditions that a compact form of the counterdiabatic term is obtained. As the simplest case, if we assume that $\hat{H}_{cd}(t)$ includes terms up to first order in $\hat{p}$, the general form is given by $\hat{H}_{cd}(t) = v(t)\hat{p} + \epsilon(t)$, where $v(t)$ and $\epsilon(t)$ are arbitrary functions. The potential must have the form $u(x, t) = u_0[x - x_0(t)]$, where u_0 is an arbitrary function with a single variable, and $x_0(t) = \int v(t)dt$. A similar result is obtained for $\hat{H}_{cd}(t)$ including second-order terms in $\hat{p}$ [28]. These results are well known and are described by the method of scale-invariant driving [13].

A novel solution is obtained when we consider the third-order term. The counterdiabatic term takes the form

$$\hat{H}_{cd}(t) = a(t)\left[\hat{p}^3 + \frac{3}{4}(\hat{p}u(\hat{x}, t) + u(\hat{x}, t)\hat{p})\right] + c_1(t)\hat{p}, \tag{5}$$

where $a(t)$ and $c_1(t)$ are arbitrary time-dependent functions. We neglected trivial contributions that are proportional to $\hat{H}_{ad}(t)$ and the identity operator. The condition for the potential $u(x, t)$ reads

$$\frac{\partial u}{\partial t} = -\frac{a(t)}{4}\left(6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}\right) - 2c_1(t)\frac{\partial u}{\partial x}. \tag{6}$$

Using the change of variables, we set $a(t) = -4$ and $c_1(t) = 0$ in the following analysis [28]. Then, this equation is reduced to the standard form of the KdV equation [20]. The KdV equation is known as a typical nonlinear integrable system and has multisoliton solutions [29].

It is no accident that the KdV equation is obtained in the present formulation. The integrability of nonlinear systems is generally represented by the Lax equation,

$$\frac{\partial L(t)}{\partial t} = [M(t), L(t)], \tag{7}$$

where the set of operators $(L(t), M(t))$ is called the Lax pair [30]. The Lax equation is equivalent to the equation for

the dynamical invariant Eq. (3). The eigenvalues of $L(t)$ are independent of t. Then, $M(t)$ is defined as $\partial_t \psi(t) = M(t)\psi(t)$, where $\psi(t)$ is an eigenfunction of $L(t)$. The Lax pair for the KdV equation is known to have the form

$$L(t) = -\frac{\partial^2}{\partial x^2} + u(x, t),$$

$$M(t) = -4\frac{\partial^3}{\partial x^3} + 3\frac{\partial}{\partial x}u(x, t) + 3u(x, t)\frac{\partial}{\partial x}. \tag{8}$$

As we see from the comparison between the Lax form and the dynamical invariant, $L(t)$ and $M(t)$ correspond to $\hat{H}_{ad}(t)$ in Eq. (4) and $-i\hat{H}_{cd}(t)$ in Eq. (5), respectively. In integrable systems, there exist infinite conserved quantities, which reflect the property that the eigenvalues of the dynamical invariant are independent of t.

In the same way, considering higher-order terms, we can find a hierarchical structure in the KdV equations. It is generally known that a different KdV equation is obtained at each odd order [30]. For example, at fifth order, the counterdiabatic term is given by $H_{cd}(t) = -16p^5 + 20(p^3u + up^3) - 30upu - 5(pu_{xx} + u_{xx}p)$, where the subscript denotes differentiation in terms of the variable, and the potential satisfies $u_t - 10(u_{xxx}u + 2u_{xx}u_x) + 30u^2u_x + u_{xxxxx} = 0$.

Deformation of the counterdiabatic term.—Although we have obtained the exact form of the counterdiabatic term Eq. (5), it contains a cubic term in $\hat{p}$ and is not convenient for our purpose of quantum control. This drawback is circumvented by using gauge transformation and state-dependent deformation [13,31–33]. We demonstrate this by treating the single- and double-soliton potentials.

The single-soliton solution of the KdV equation is given by $u(x, t) = -2\kappa^2/\cosh(\kappa x - 4\kappa^3 t)$, where κ is a real positive parameter. The adiabatic Hamiltonian [Eq. (4)] with this hyperbolic Scarf potential has a single bound state whose eigenenergy is given by $E_0 = -\kappa^2$. To deform the counterdiabatic term, we rewrite Eq. (5) with $a(t) = -4$ and $c_1(t) = 0$ as

$$\hat{H}_{cd}(t) = -4[\hat{p} + iW(\hat{x}, t)]\hat{p}[\hat{p} - iW(\hat{x}, t)]$$
$$- [\hat{p}\,\tilde{u}(\hat{x}, t) + \tilde{u}(\hat{x}, t)\hat{p}] - 4E_0\hat{p}, \tag{9}$$

where $W(x, t) = \kappa\tanh(\kappa x - 4\kappa^3 t)$ and $\tilde{u}(x, t) = W^2(x, t) + \partial_x W(x, t) + E_0$. The adiabatic Hamiltonian in this case has supersymmetry [34–38], and the potential is written using the superpotential $W(x, t)$ as $u(x, t) = W^2(x, t) - \partial_x W(x, t) + E_0$. Supersymmetry also tells us that the ground-state eigenfunction $\psi(x, t)$ satisfies the differential equation $[-i\partial_x - iW(x, t)]\psi(x, t) = 0$. This means that the first term in Eq. (9) is neglected if we treat only the ground state. Furthermore, the superpartner potential $\tilde{u}(x, t)$ goes to zero in this case. Thus, we arrive at a simple form of the counterdiabatic term: $\hat{H}_{cd}(t) = -4E_0\hat{p} = 4\kappa^2\hat{p}$, which only affects the phase of the wave function and does not change the probability density. This form can be derived

from the formula for scale-invariant systems [13]. In the single-soliton potential, the system has translational symmetry, and we can directly obtain this simple form.

For multisoliton potentials, the system is not scale invariant, and our method becomes useful. The double-soliton potential is represented as

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\ln\left[1 + A_1 e^{2\eta_1} + A_2 e^{2\eta_2}\right.$$
$$\left. + \left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}\right)^2 A_1 A_2 e^{2(\eta_1+\eta_2)}\right], \quad (10)$$

where κ_1, κ_2, A_1, and A_2 are real positive parameters, and $\eta_i = \kappa_i x - 4\kappa_i^3 t$ with $i = 1, 2$. This system has two bound states. When $\kappa_1 > \kappa_2$, the ground-state energy is $E_0 = -\kappa_1^2$ and the first excited-state energy is $E_1 = -\kappa_2^2$. The ground state is localized in the deeper well, and the first excited state is in the other well.

It is also possible to consider the deformation of the counterdiabatic term [Eq. (5)] using the representation Eq. (9) [39]. When we consider the ground state, we can again neglect the first term of Eq. (9). Using a gauge transformation, we obtain the counterdiabatic potential

$$V_{\rm cd}(x,t) = -\tilde{u}^2(x,t) + 4(\kappa_1^2 - \kappa_2^2)\tilde{u}(x,t), \quad (11)$$

where $\tilde{u}(x,t) = -2\kappa_2^2/\cosh^2(\eta_2+\delta_2)$, with $\delta_2 = \ln[A_2(\kappa_1 - \kappa_2)/(\kappa_1+\kappa_2)]/2$. Deformation of the excited states can be obtained in a similar way.

We plot the potential and ground-state wave function in Fig. 1. The counterdiabatic potential is represented by the single soliton $\tilde{u}(x,t)$ with the argument $\eta_2 + \delta_2$. It makes a deep well around the shallower well in the original potential and moves so that the moving state, trapped in the deep well of the original potential, is not disturbed by the shallow well.

The above demonstrations show that we can consider particle transport by using the soliton potentials. A particle is conveyed by a soliton potential, and we can find conditions such that the state is not disturbed by noises represented by other solitons. The use of the soliton potential shows that the control of systems is possible by changing the potential locally, which can be a useful manipulating method. The exact solutions from the KdV equation also have a great advantage that the small deviations can be treated perturbatively to study stability [40]. This method can be a principle to design the particle transport in general complicated systems.

Free fermion, spin chain, and Toda lattice.—Next, we demonstrate the design of the Hamiltonian by using a result from the Lax form in nonlinear integrable systems. A famous example is the Toda lattice system. This one-dimensional classical lattice system has nearest-neighbor hopping in an exponential form, and the Lax form is written in a matrix form [21,41,42]. The matrix has a tridiagonalized form and can be interpreted as the quantum
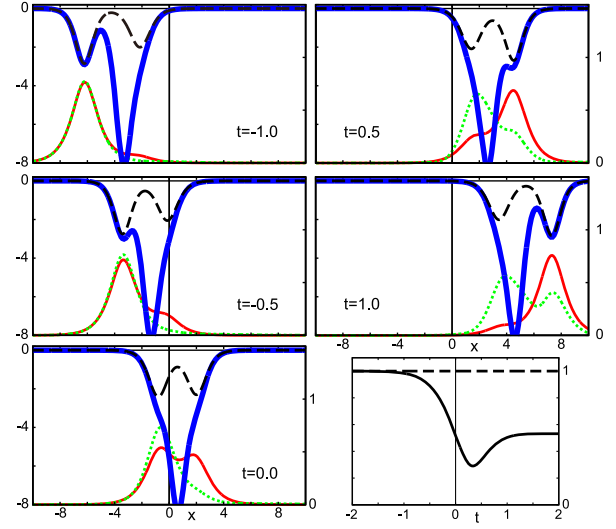

FIG. 1.   Time evolution of double-soliton potential. Plotted are the soliton potential $u(x,t)$ (black dashed line, left scale), total potential $u(x,t) + V_{\rm cd}(x,t)$ (blue thick line, left), and ground-state wave function $|\psi(x,t)|$ (red solid line, right). Green dotted lines represent the wave function $|\psi_0(x,t)|$ without the counterdiabatic potential $V_{\rm cd}(x,t)$ calculated numerically (right-hand scale). Solid line in the lower right-hand panel represents the overlap between states with and without the counterdiabatic term $|\langle\psi_0(t)|\psi(t)\rangle|$. We take $A_1 = A_2 = 3.0$, $\kappa_1 = 1.2$, and $\kappa_2 = 1.0$.

Hamiltonian for a one-dimensional lattice system with nearest-neighbor hopping:

$$\hat{H}_{\rm ad}(t) = \sum_{n=1}^{N} J_n(t)(\hat{f}_n^\dagger \hat{f}_{n+1} + \hat{f}_{n+1}^\dagger \hat{f}_n)$$
$$+ \sum_{n=1}^{N} h_n(t)\left(\hat{f}_n^\dagger \hat{f}_n - \frac{1}{2}\right),$$
$$\hat{H}_{\rm cd}(t) = i\sum_{n=1}^{N} J_n(t)(\hat{f}_n^\dagger \hat{f}_{n+1} - \hat{f}_{n+1}^\dagger \hat{f}_n). \quad (12)$$

The fermion operators $\hat{f}_n$ and $\hat{f}_n^\dagger$ satisfy anticommutation relations [43]. Substituting these expressions into Eq. (3) with $\hat{F}(t) = \hat{H}_{\rm ad}(t)$, we find the Toda equations:

$$\frac{dJ_n(t)}{dt} = J_n(t)[h_{n+1}(t) - h_n(t)],$$
$$\frac{dh_n(t)}{dt} = 2[J_n^2(t) - J_{n-1}^2(t)]. \quad (13)$$

The solutions of the Toda equations are discussed below.

Although this transformation is a straightforward task, the resulting Hamiltonian has the time-dependent hopping amplitude and is generally difficult to realize. The matrix representation of the Lax form reminds us that the Hamiltonian is interpreted as a quantum spin system. By using the Jordan-Wigner transformation [44,45], we can map the free fermion onto the *XY* spin model [43]:

$$\hat{H}_{\text{ad}}(t) = \sum_{n=1}^{N} \frac{J_n(t)}{2}(\hat{\sigma}_n^x \hat{\sigma}_{n+1}^x + \hat{\sigma}_n^y \hat{\sigma}_{n+1}^y) + \sum_{n=1}^{N} \frac{h_n(t)}{2}\hat{\sigma}_n^z,$$

$$\hat{H}_{\text{cd}}(t) = \sum_{n=1}^{N} \frac{J_n(t)}{2}(\hat{\sigma}_n^x \hat{\sigma}_{n+1}^y - \hat{\sigma}_n^y \hat{\sigma}_{n+1}^x), \qquad (14)$$

where $\{\hat{\boldsymbol{\sigma}}_n = (\hat{\sigma}_n^x, \hat{\sigma}_n^y, \hat{\sigma}_n^z)\}$ represent the Pauli matrices at each site n.

It is instructive to see that the isotropic, nonuniform interaction $J_n(t)$ is necessary to obtain the adiabatic control. For a uniform system with site-independent couplings, the energy eigenvalues and eigenstates of the XY spin Hamiltonian $\hat{H}_{\text{ad}}(t)$ are obtained exactly by using the Jordan-Wigner transformation [46–49]. In that case, we can find $\hat{H}_{\text{cd}}(t)$ by using Eq. (2). The problem is that $\hat{H}_{\text{cd}}(t)$ has a noncompact form including many-body interaction terms [46,47]. We circumvent this difficulty by using the integrability of the Toda lattice.

It is interesting to compare the Lax pair for the spin model with that for the Toda lattice. Each term in Eq. (14) commutes with the total magnetization in the z direction: $\hat{M} = \sum_n \hat{\sigma}_n^z$. In the single spin-flip sectors with $M = \pm(N-2)$, the matrix representation of the Hamiltonian is the same as that of the Lax pair for the Toda lattice. There are no counterparts for other sectors, and it is possible to find new Lax pairs for classical nonlinear systems by analyzing those matrix representations.

We study several typical solutions of the Toda equations (13) to see the behavior of the states in the corresponding spin system. For $N = 3$ with the open boundary condition, one of the solutions is given as follows. $J_1(t)$ and $h_1(t)$ are given by

$$J_1(t) = v v_1 \frac{\sqrt{v_1^2 + v_2^2 \cosh(2vt)}}{v_1^2 \cosh(2vt) + v_2^2},$$

$$h_1(t) = v \frac{v_1^2 \sinh(2vt)}{v_1^2 \cosh(2vt) + v_2^2}, \qquad (15)$$

where v_1 and v_2 are constants and $v = \sqrt{v_1^2 + v_2^2}$. $J_2(t)$ is obtained by interchanging v_1 and v_2 in $J_1(t)$, $h_3(t)$ is obtained by interchanging v_1 and v_2 in $-h_1(t)$, and $h_2(t) = -h_1(t) - h_3(t)$. We note that $(J_1(0), J_2(0)) = (v_1, v_2)$, $(h_1(0), h_2(0), h_3(0)) = (0,0,0)$, $(J_1(\infty), J_2(\infty)) = (0,0)$, and $(h_1(\infty), h_2(\infty), h_3(\infty)) = (v, 0, -v)$. These limiting values show that the Hamiltonian of the XY model can be continuously deformed to the noninteracting Hamiltonian with the same eigenvalues by using the time evolution of the Toda equations. This result is generalized to an arbitrary number of spins N [50]. We note that continuous change is possible for the isotropic XY model and not for anisotropic models, including the Ising model, which is considered to be related to the universal adiabatic computing property of the XY Hamiltonian [51].

In infinite systems, the boundary effect is neglected and the soliton solutions are obtained by using the inverse scattering method [41]. The solutions allow us to control the spin state. As we find in the KdV solitons, the flipped spin is transported by controlling the spin interaction and the magnetic field as in Fig. 2 [52]. The quantum state transfer has been discussed in many contexts and the use of spin chain is one of the typical applications [53]. Our result based on the soliton dynamics can be useful as a robust control method.

For practical applications, one of the disadvantages of counterdiabatic driving is that we need to introduce an additional term $\hat{H}_{\text{cd}}(t)$. It is possible to perform the gauge transformation to keep the total Hamiltonian in a simple form. This change of representation is interpreted as inverse engineering. The concept of shortcuts to adiabaticity extends beyond counterdiabatic driving, and we can consider other formulations. In inverse engineering, the Hamiltonian is kept in the original form and the invariant is constructed from Eq. (3) [54].

Nonisospectral Hamiltonian.—Although our discussion is restricted to systems with isospectral Hamiltonian, it can be generalized to nonisospectral systems. The isospectral Hamiltonian is obtained by assuming $\hat{F}(t) = \hat{H}_{\text{ad}}(t)$. In classical nonlinear integrable systems, the Lax pair is generalized to nonisospectral systems [55]. The integrability is maintained when we consider the case $\hat{F}(t) = \gamma^2(t)\hat{H}_{\text{ad}}(t)$, where $\gamma(t)$ is a time-dependent function. In systems with the Hamiltonian Eq. (4), if we impose this condition, we can find a generalized KdV hierarchy. In the simplest ansatz where the counterdiabatic term is linear in $\hat{p}$, we can find the potential form $u(x,t) = \gamma^{-2}(t)u_0\{[x - x_0(t)]/\gamma(t)\}$ with the counterdiabatic term $\hat{H}_{\text{cd}}(t) \propto \hat{x}\hat{p} + \hat{p}\hat{x}$. This is known as the scale-invariant driving [13]. The analysis can be applied to higher-order hierarchies to give results that cannot be described by the scale-invariant driving [56]. Thus, the previous known results can be described in the present formulation in a systematic way and further generalizations are possible.

Conclusions.—We have proposed counterdiabatic driving for nonlinear integrable systems. The Lax form is known for various systems other than the KdV and Toda systems. Possible applications are the modified KdV equation, sine-Gordon equation, nonlinear Schrödinger equation, Ablowitz-Kaup-Newell-Segur formalism [57],
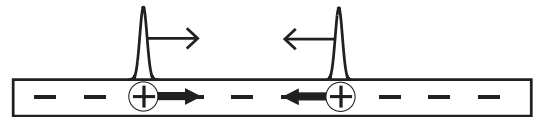

FIG. 2. Transport of flipped spin by soliton interactions. $+$ or $-$ represents spin state at each site and the soliton wave represents the coupling functions. The figure represents the transport of two flipped spins by double solitons, and other cases can be described in a similar way [52].

and so on. We stress that the infinite set of Lax pairs is known in nonlinear integrable systems, and one can find a corresponding counterdiabatic driving in each system.

Our approach establishes a fruitful connection between the classical nonlinear systems and the dynamical quantum systems. In the quantum systems, the counterdiabatic term was available only for limited simple systems. In the present work, the solvable systems can be obtained in a unified and systematic way by using the knowledge from the classical nonlinear integrable systems. For example, in the potential systems where $\hat{H}_{ad}(t)$ is written as Eq. (4), the counterdiabatic term is obtained from the KdV hierarchy. In the example of the Toda systems, we can find the corresponding quantum spin system and fermion system by knowing the matrix representation of the Lax form. The generalization is possible, and we can use results from systems such as the Calogero-Sutherland and Calogero-Moser models and Volterra lattice [58–61]. Several applications of such systems will be reported elsewhere.

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