Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/
    gadepall/
EE1310/master/filter/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

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2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#read.wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )

#sampling frequency of input signal
sampl_freq = fs

#order of the frequency
order = 4

#cutoff frequency 4khz
cutoff_freq = 4000.0

#digital frequency
Wn = 2*cutoff_freq/sampl_freq

# b and a are numerators and denominator
    polynomials respectively
b,a = signal.butter(order,Wn,'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b,a,input_signal
    )
#output_signal = signal.lfilter(b,a,
    input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal,fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?

Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.

Solution: The following code yields Fig. 3.2.

```
wget https://github.com/NareshBandaru13/
    EE3900-A1/blob/main/codes/3-2.py
```

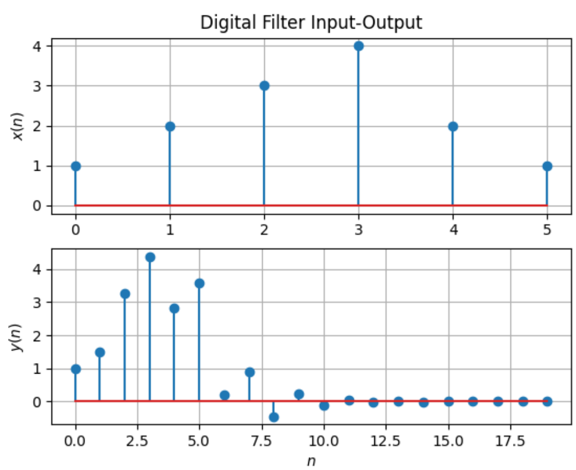

Fig. 3.2

3.3 Repeat the above exercise using a c code. **Solution:** The following code yields x data and y data

```
wget https://github.com/NareshBandaru13/
    EE3900-A1/blob/main/codes/3-3.c
```

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain X(z) for x(n) defined in problem 3.1
Solution:

$$X(z) = \ddagger\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$X(z) = ...0 + x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-}$$
$$= 1 + 2z^{-}1 + 3z^{-}2 + 4z^{-}3 + 2z^{-}4 + 1z^{-}5$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.7}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.8}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.9}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.12}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.13}$$

and from (4.11),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.14}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.16}$$

Solution: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$X(z) = \sum_{n=-\infty}^{-1} 0 + \sum_{n=0}^{\infty} a^n (1) z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az{-1}}$$

Also the series is convergent for $|z| > |a|$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.17}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.

Solution: The following code plots Fig. 4.6.

```
wget https://github.com/NareshBandaru13/
    EE3900−A1/blob/main/codes/4−6.py
```

run the command

```
python3 4−6.py
```

$$H\left(e^{\jmath\omega}\right) = \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2} e^{-\jmath\omega}} \tag{4.18}$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| = \frac{\left|1 + \cos 2\omega - \jmath \sin 2\omega\right|}{\left|1 + \frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \tag{4.19}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{(1 + \frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} \tag{4.20}$$

$$= \sqrt{\frac{2 + 2\cos 2\omega}{\frac{5}{4} + \cos\omega}} \tag{4.21}$$

$$= \sqrt{\frac{2(2\cos^2\omega)4}{5 + 4\cos\omega}} \tag{4.22}$$

$$= \frac{4\left|\cos\omega\right|}{\sqrt{5 + 4\cos\omega}} \tag{4.23}$$

The period of the numerator is π and the denominator is 2π taking the L.C.M of numerator and the denominator we get period as 2π

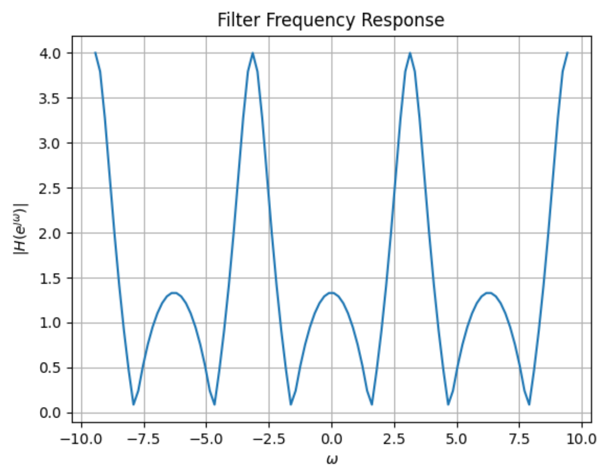

Fig. 4.6: $\left|H\left(e^{\jmath\omega}\right)\right|$

4.7 Express h(n) in terms of $H(e^{2w})$.

Solution:

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega}) e^{\jmath\omega n} d\omega \tag{4.24}$$

$$= \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} h(k) e^{-\jmath\omega k} e^{\jmath\omega n} d\omega \tag{4.25}$$

$$= \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{\jmath\omega(n-k)} d\omega \tag{4.26}$$

Now,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)} d\omega = \begin{cases} \int_{-\pi}^{\pi} d\omega & n-k=0 \\ \left.\frac{\exp\left(\jmath\omega(n-k)\right)}{\jmath(n-k)}\right|_{-\pi}^{\pi} & n-k \neq 0 \end{cases} \tag{4.27}$$

$$= \begin{cases} 2\pi & n-k=0 \\ 0 & n-k \neq 0 \end{cases} \tag{4.28}$$

$$= 2\pi\delta(n-k) \tag{4.29}$$

Thus,

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}\omega = 2\pi \sum_{k=-\infty}^{\infty} h(k)\delta(n-k) \tag{4.30}$$

$$= 2\pi h(n) * \delta(n) \tag{4.31}$$

$$= 2\pi h(n) \tag{4.32}$$

Therefore, $h(n)$ is given by the inverse DTFT (IDTFT) of $H\left(e^{\jmath\omega}\right)$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}\omega \tag{4.33}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.9).
Solution:

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$

$$\begin{array}{r|l} & 2x-4 \\ \hline \frac{1}{2}x+1) & x^2 \qquad +1 \\ & -x^2-2x \\ \hline & -2x+1 \\ & 2x+4 \\ \hline & 5 \end{array}$$

$$\implies 1+z^{-2} = \left(1+\frac{1}{2}z^{-1}\right)\left(-4+2z^{-1}\right)+5 \tag{5.3}$$

$$\implies H(z) = -4+2z^{-1}+\frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.4}$$

On applying the inverse Z-transform on both sides of the equation

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{5.5}$$

$$-4 \overset{\mathcal{Z}}{\rightleftharpoons} -4\delta(n) \tag{5.6}$$

$$2z^{-1} \overset{\mathcal{Z}}{\rightleftharpoons} 2\delta(n-1) \tag{5.7}$$

$$\frac{5}{1+\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.8}$$

$$\tag{5.9}$$

Therefore,

$$h(n) = -4\delta(n)+2\delta(n-1)+5\left(-\frac{1}{2}\right)^n u(n) \tag{5.10}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.11}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: The $H(z)$ can be written as,

$$H(z) = \frac{1}{1+\frac{z^{-1}}{2}} + \frac{z^{-2}}{1+\frac{z^{-1}}{2}} \tag{5.12}$$

From (4.16) we can write it as,

$$h(n) = \left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2) \tag{5.13}$$

5.3 Sketch $h(n)$. Is it bounded? Justify Theoritically.
Solution: Download the code for the plot 5.3 from the below link,

```
wget https://github.com/NareshBandaru13/
    EE3900−A1/blob/main/codes/5−3.py
```

From the plot it seems like $h(n)$ is bounded and becomes smaller in magnitude as n increases. Using (5.13), we can get theoritical expression as,

$$h(n) = \begin{cases} 0 & , n < 0 \\ \left(\frac{-1}{2}\right)^n & , 0 \leq n < 2 \\ 5\left(\frac{-1}{2}\right)^n & , n \geq 2 \end{cases} \tag{5.14}$$

A sequence $\{x_n\}$ is said to be bounded if and only if there exist a positive real number M such that,

$$|x_n| \leq M, \forall n \in \mathcal{N} \tag{5.15}$$

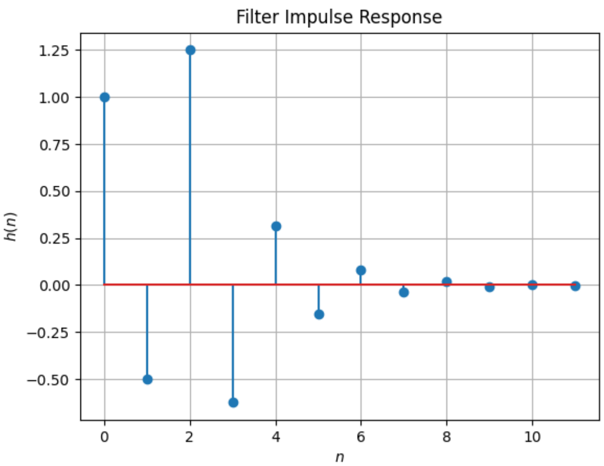

Fig. 5.3: $h(n)$ as inverse of $H(n)$

So to say $h(n)$ is bounded we should able to find the M which satisfies (5.15).
For $n < 0$,

$$|h(n)| \leq 0 \tag{5.16}$$

For $0 \leq n < 2$,

$$|h(n)| = \left|\frac{-1}{2}\right|^n \tag{5.17}$$

$$= \left(\frac{1}{2}\right)^n \leq 1 \tag{5.18}$$

And for $n \geq 2$,

$$|h(n)| = \left|5\left(\frac{-1}{2}\right)\right|^n \tag{5.19}$$

$$= \left(\frac{5}{2}\right)^n \leq \frac{5}{4} \tag{5.20}$$

From above three cases, we can get M as,

$$M = max\left\{0, 1, \frac{5}{4}\right\} \tag{5.21}$$

$$= \frac{5}{4} \tag{5.22}$$

Therefore, $h(n)$ is bounded using (5.15) with $M = \frac{5}{4}$ i.e.,

$$|h(n)| \leq \frac{5}{4} \forall n \in \mathcal{N} \tag{5.23}$$

5.4 Convergent? Justify using the ratio test.
Solution: We can say a given real sequence $\{x_n\}$ is convergent if

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| < 1 \tag{5.24}$$

This is known as Ratio test.
In this case the limit will become,

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{5\left(\frac{-1}{2}\right)^{n+1}}{5\left(\frac{-1}{2}\right)^n}\right| \tag{5.25}$$

$$= \lim_{n\to\infty}\left|\frac{-1}{2}\right| \tag{5.26}$$

$$= \frac{1}{2} \tag{5.27}$$

As $\frac{1}{2} < 1$, from root test we can say that $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.28}$$

Is the system defined by (3.2) stable for the impulse response in (5.11)?

Solution: From (5.13),

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty}\left(\left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2)\right) \tag{5.29}$$

$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) \tag{5.30}$$

$$= \frac{4}{3} \tag{5.31}$$

$\therefore$ the system is stable.

5.6 Verify the above result using a python code.
Solution: Download the python code from the below link

```
wget https://github.com/NareshBandaru13/
    EE3900-A1/blob/main/codes/5-6.py
```

Then run the following command,

```
python3 5-6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.32}$$

This is the definition of $h(n)$.
Solution: Download the code for the plot 5.7 from the below link,

```
wget https://github.com/NareshBandaru13/
    EE3900−A1/blob/main/codes/5−7.py
```

Note that this is same as 5.3.
For $n < 0$, $h(n) = 0$ and,

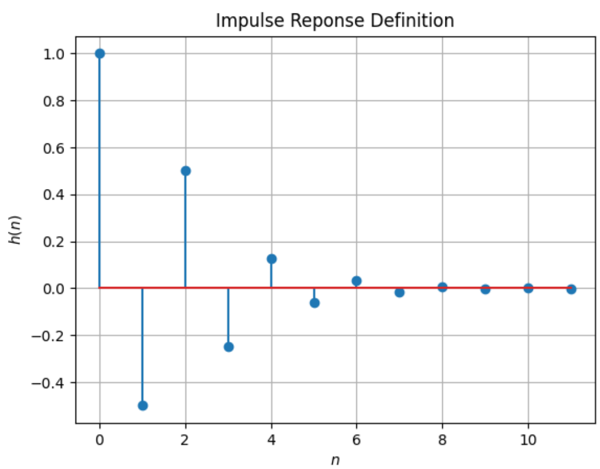

Fig. 5.7: From the definition of $h(n)$

$$h(0) = \delta(0) \tag{5.33}$$
$$= 1 \tag{5.34}$$

For $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) \tag{5.35}$$
$$\implies h(1) = -\frac{1}{2}h(0) \tag{5.36}$$
$$= -\frac{1}{2} \tag{5.37}$$

$n = 2$,

$$h(2) + \frac{1}{2}h(1) = 0 + \delta(0) \tag{5.38}$$
$$h(2) = 1 + \frac{1}{4} \tag{5.39}$$
$$= \frac{5}{4} \tag{5.40}$$

And for $n > 2$ RHS will be 0 so,

$$h(n) = -\frac{1}{2}h(n-1) \tag{5.41}$$

Overall

$$h(n) = \begin{cases} 0 & , n < 0 \\ 1 & , n = 0 \\ -\frac{1}{2} & , n = 1 \\ \frac{5}{4} & , n = 2 \\ -\frac{1}{2}h(n-1) & , n > 2 \end{cases} \tag{5.42}$$

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.43}$$

Comment. The operation in (5.43) is known as *convolution*.
Solution: Download the code for plot 5.8 from the below link

```
wget https://github.com/NareshBandaru13/
    EE3900−A1/blob/main/codes/5−8.py
```

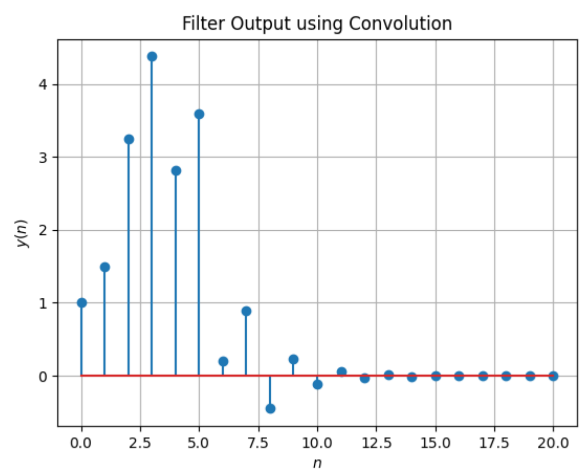

Fig. 5.8: $y(n)$ using the convolution definition

5.9 Express the above convolution using a Toeplitz matrix.
Solution: Download the python code from the below link for the plot 5.9,

```
wget https://github.com/NareshBandaru13/
    EE3900−A1/blob/main/codes/5−9.py
```

Then run the following command,

```
python3 5−9.py
```

From (5.43),we express $y(n)$ as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.44}$$

To understand how we can use a Toeplitz matrix, we will see what we are doing in (5.43)

$$y(0) = x(0)h(0) \tag{5.45}$$
$$y(1) = x(0)h(1) + x(1)h(0) \tag{5.46}$$
$$y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) \tag{5.47}$$

.

.

The same thing can be written as,

$$y(0) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.48}$$

$$y(1) = \begin{pmatrix} h(1) & h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.49}$$

$$y(2) = \begin{pmatrix} h(2) & h(1) & h(0) & 0 & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.50}$$

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Using Toeplitz matrix of $h(n)$ we can simplify it as,

$$y(n) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \\ h(1) & h(0) & 0 & . & . & .0 \\ h(2) & h(1) & h(0) & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & .h(m-1) \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.51}$$

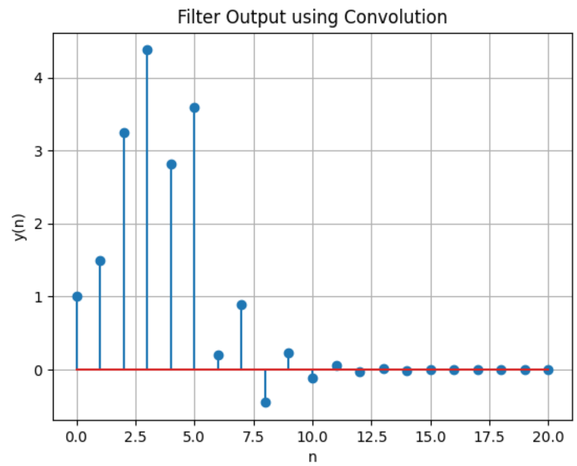

Fig. 5.9: Convolution of $x(n)$ and $h(n)$ using toeplitz matrix

Now from (**??**) we will take n

$$x(n) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{5.52}$$

And from (5.14)

$$h(n) = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ . \\ . \end{pmatrix} \tag{5.53}$$

Now using (5.51),

$$y(n) = x(n) * h(n) \tag{5.54}$$

$$= \begin{pmatrix} 1 & 0 & 0 & . & . & .0 \\ -0.5 & 1 & 0 & . & . & .0 \\ 1.25 & -0.5 & 1 & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & . \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.55}$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ . \\ . \\ . \end{pmatrix} \tag{5.56}$$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.57}$$

Solution: Substitute $k := n - k$ in (5.43), we will get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.58}$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \tag{5.59}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.60}$$

$$\tag{5.61}$$