Random Numbers

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Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/NareshBandaru13/
    Random−Numbers/blob/main/ex1/1/
    exrand.c
wget https://github.com/NareshBandaru13/
    Random−Numbers/blob/main/ex1/1/coeffs
    .h
```

Use the below command in the terminal to run the code

```
gcc exrand.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The graph 1.2 is obtained by running the below code

```
https://github.com/NareshBandaru13/Random
    −Numbers/blob/main/ex1/2/main.py
```

Run the following command in the terminal to run the code.

```
python3 main.py
```

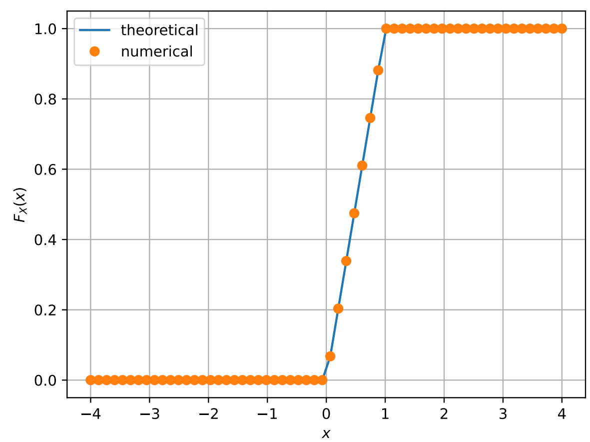

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is uniform random variable so

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(x)dx \\ &= 0 + \int_0^x 1dx \\ &= \begin{cases} 1 & x > 1 \\ x & 0 \leq x \leq 1 \\ 0 & x < 0 \end{cases} \end{aligned}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.2}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.3}$$

Write a C program to find the mean and

variance of U.
Solution:

```
wget https://github.com/NareshBandaru13/
    Random−Numbers/blob/main/ex1/4/
    exrand.c
wget https://github.com/NareshBandaru13/
    Random−Numbers/blob/main/ex1/4/coeffs
    .h
```

Use below command to run file,

```
gcc exrand.c −lm
./a.out
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_X(x)$$

Given

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_X(x)$$

$$\begin{aligned} E[U] &= \int_{-\infty}^{\infty} x^k f_X(x) dx \\ &= \int_0^1 x \times 1 dx \\ &= \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} \end{aligned}$$

if k=2

$$\begin{aligned} E[U^2] &= \int_0^1 x^2 \times 1 dx \\ &= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} variance &= E[u - E[u]]^2 \\ &= E[U^2] - E^2[U] \\ &= \frac{1}{3} - \frac{1}{4} = 0.0833 \end{aligned}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
wget https://github.com/NareshBandaru13/
    Random−Numbers/blob/main/ex2/1/
    exrand.c
wget https://github.com/NareshBandaru13/
    Random−Numbers/blob/main/ex2/1/coeffs
    .h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
gcc exrand.c −lm
.\a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in plot,Properties of the CDF:
- $F_X(x) = P(X \le x)$
- $Q_X(x) = P(X > x)$
- $F_X(x) = 1 - Q_X(x)$ This can be used to calculate F (x).

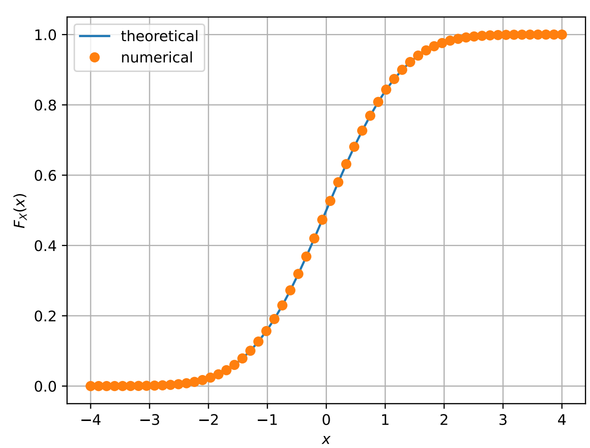

Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?

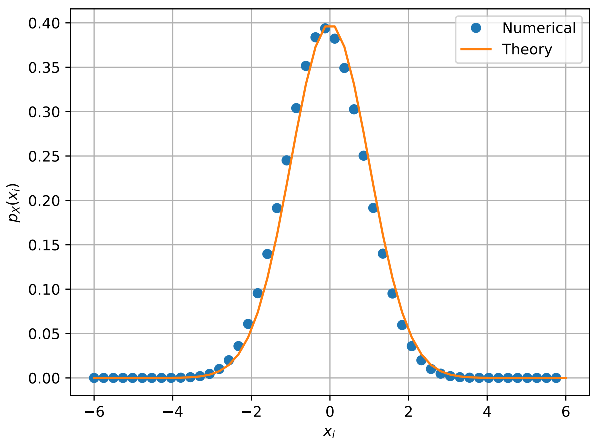

Fig. 2.3: The PDF of X

Solution: The PDF of X is plotted using the code below

```
https://github.com/NareshBandaru13/Random
    -Numbers/blob/main/ex2/3/main.py
```

Use the below command to run the code:

```
python3 main.py
```

Properties of PDF:

- PDF is symmetric about $x \approx 0$
- graph is similar to bell shaped
- mean of graph is situated at the symmetrical point

2.4 Find the mean and variance of X by writing a C program.
Solution: Running the below code gives Mean = 0.000326 Variance= 1.000906

```
https://github.com/NareshBandaru13/Random
    -Numbers/blob/main/ex2/4/exrand.c
```

Command used:

```
gcc exrand.c -lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty$$

by property of probability

$$\int_{-\infty}^{\infty} p_X(x)dx = 1$$

$$F_X(x) = \int_{-\infty}^{x} p_X(x)dx$$
$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$E[X] = \int_{-\infty}^{\infty} xp(x)dx$$
$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$

$\frac{x}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ is an odd function so integral is zero i.e $E[X] = 0$.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p(x)dx$$
$$= x\int_{-\infty}^{\infty} xp(x)dx - \int_{-\infty}^{\infty}\left(\int xp(x)dx\right)dx$$
$$= \left[-x\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$
$$= 0 - (-1) \qquad = 1$$

we know that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = \sqrt{2\pi}$$

by series expansion $\frac{x}{e^{\frac{x^2}{2}}} = \frac{x}{1+\frac{x^2}{2}+\frac{x^4}{8}+...}$
putting $x = \infty$,we get $\frac{1}{\infty} = 0$
Similarly when $x = -\infty$ we get 0
$var(x) = E[x^2] - E[x] = 1 - 0 = 1$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution:
Running the below code generates samples of V from file uni.dat(U).

```
https://github.com/NareshBandaru13/Random
    -Numbers/blob/main/ex3/main.py
```

Use the below command in the terminal to run the code:

```
python3 main.py
```

Now these samples are used to plot by running the below code

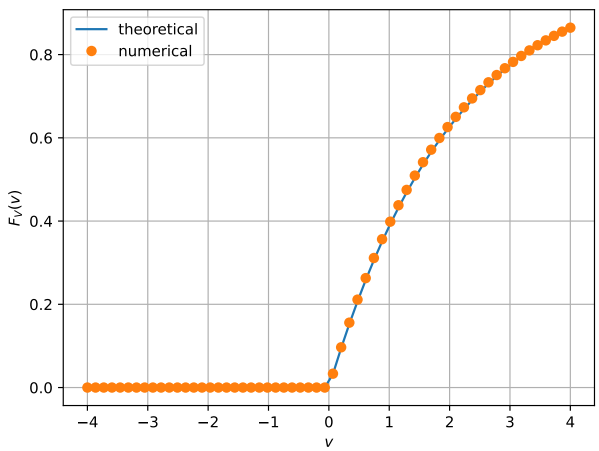

Fig. 3.1: CDF for (3)

```
https://github.com/NareshBandaru13/Random
    -Numbers/blob/main/ex3/cdf.py
```

Use the below command to run the code:

```
python3 cdf.py
```

3.2 Theorotical expression for $F_V(x)$

$$\begin{aligned}
F_V(x) &= P\{V \leq x\} \\
&= P\{-2 \times \ln(1-U) \leq x\} \\
&= P\{U \leq 1 - e^{(-\frac{x}{2})}\} \\
&= F_U\{1 - e^{(-\frac{x}{2})}\} \\
&= \begin{cases} 1 - e^{(-\frac{x}{2})} & 0 \leq x < \infty \\ 0 & x < 0 \end{cases}
\end{aligned}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

we get the code

```
https://github.com/NareshBandaru13/Random
    -Numbers/blob/main/ex4/1/main.py
```

run the command

```
python3 main.py
```

4.2 Find the CDF of T. we have code

```
https://github.com/NareshBandaru13/Random
    -Numbers/blob/main/ex4/2/main1.py
```

run the command

```
python3 main1.py
```

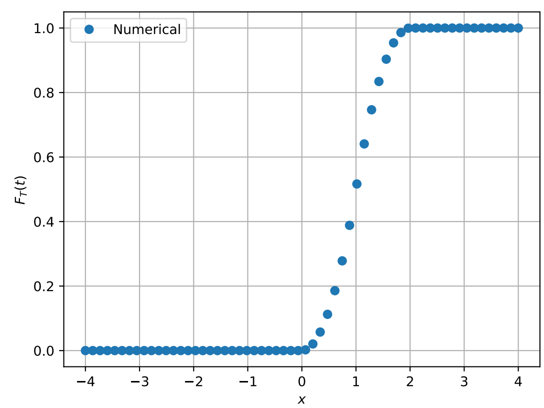

Fig. 4.2: numerical cdf

4.3 Find the PDF of T we have code

```
https://github.com/NareshBandaru13/Random
    -Numbers/blob/main/ex4/3/main2.py
```

run the command

```
python3 main2.py
```

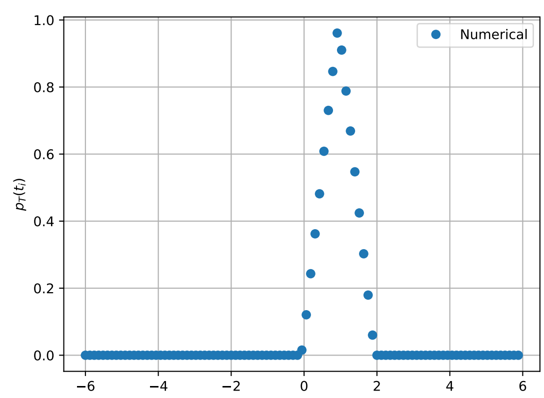

Fig. 4.3: Numerical cdf

4.4 Find the theoretical expressions for the PDF and CDF of T. **Solution:**

$$\begin{aligned} F_T(t) &= P\{T \le t\} \\ &= P\{U1 + U2 \le t\} \end{aligned}$$

let us take two cases if $0 \le t \le 1$ and $1 < t \le 2$

cdf.png

Fig. 4.4: def plot

The above graph is produced by

```
https://github.com/NareshBandaru13/Random
    −Numbers/blob/main/ex4/2/find.py
```

Run the code in terminal

```
python3 find.py
```

from the figures it is evident that $P(U1 + U2 < t, 0 \le t < 1) = \frac{t^2}{2}$ $P(U1 + U2 < t, 1 \le t \le 2) = 1 - \frac{(2-t)^2}{2}$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \le 2 \\ 1 & t > 2 \end{cases}$$

$$P_T(t) = \frac{d(F_T(t))}{dt}$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2 - t & 0 < t \le 2 \\ 0 & t > 2 \end{cases}$$

4.5 Verify your results through a plot Take the code for cdf

```
https://github.com/NareshBandaru13/Random
    −Numbers/blob/main/ex4/5/main1.py
```

Run in terminal

```
python3 main1.py
```

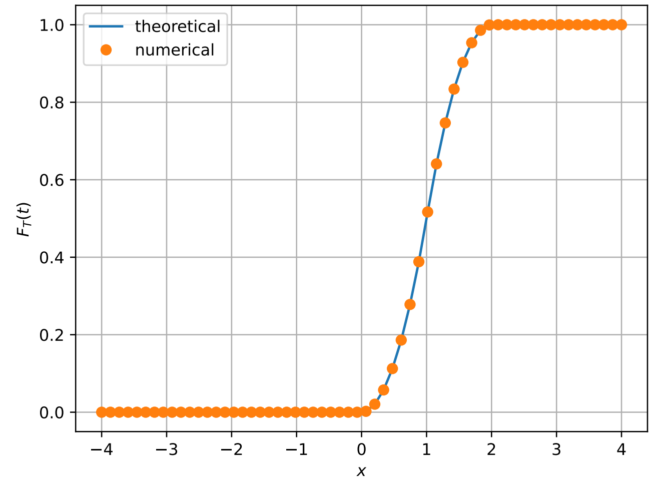

Fig. 4.5: t-cdf

Take the code for pdf

```
https://github.com/NareshBandaru13/Random
    −Numbers/blob/main/ex4/5/main2.py
```

Run in terminal

```
python3 main2.py
```

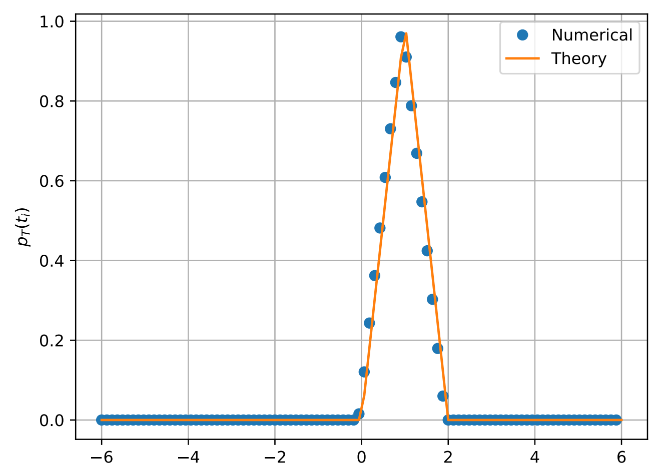

Fig. 4.5: t-pdf