On Feynman's proof of the Maxwell equations

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(Received 11 February 1991; accepted 3 August 1991)

Feynman's proof, as recounted by F. J. Dyson [Am. J. Phys. **58**, 209–211 (1990)], that the Lorentz force law and two of Maxwell's equations can apparently be deduced from minimal assumptions about the equation of motion and quantization condition for a nonrelativistic particle, is analyzed. A version of this result is found to hold in classical mechanics, and it is shown that instead of limiting interactions to electromagnetic ones, Feynman's result is a rederivation of the constraints on the velocity-dependent generalized forces that can be accommodated in a canonical Lagrangian formalism. The generality of the result is illustrated by showing that particle motion in a noninertial frame or in a weak gravitational field also satisfies the constraints.

I. INTRODUCTION

Dyson's account of Feynman's argument[1] starts with the quantum mechanics of a nonrelativistic point-particle of mass m with (Cartesian) position operator x_i, $i = 1,2,3$, that is assumed to satisfy the commutation relations

$$m[x_i,\dot{x}_j] = i\hbar\delta_{ij}, \tag{1}$$

$$[x_i,x_j] = 0, \tag{2}$$

and the Heisenberg equation of motion

$$m\ddot{x}_i = F_i(\mathbf{x},\dot{\mathbf{x}},t). \tag{3}$$

From these assumptions Dyson shows[1] that $\mathbf{F}$ has the form of the Lorentz force

$$F_i = E_i + \epsilon_{ijk}\dot{x}_j H_k, \tag{4}$$

where $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{H}(\mathbf{x},t)$ are vector functions of the coordinates $\mathbf{x}$ and time t but not the velocities, that satisfy two of Maxwell's equations:

$$\nabla\cdot\mathbf{H} = 0, \tag{5}$$

$$\frac{\partial \mathbf{H}}{\partial t} + \nabla\times\mathbf{E} = 0. \tag{6}$$

Taken at face value this is a remarkable result because the relatively innocuous assumptions (1)–(3) appear to restrict the interactions of a nonrelativistic particle to electromagnetic ones.[1] However, why should scalar, or gravitational interactions, or motion in a noninertial reference frame not also satisfy these assumptions?

We will focus on these issues in this paper by developing a version of Feynman's argument entirely within nonrelativistic classical mechanics. This approach is a reasonable one because if the equation of motion (3) can be derived from a nonsingular Lagrangian, there will be the usual relation between commutation relations in the quantum theory and Poisson brackets in the classical theory, so that quantum mechanics should not be a necessary ingredient of the argument. We turn to this issue in the next section by studying classical Hamiltonian mechanics. Section III of the paper deals with the origins of Feynman's argument in classical Lagrangian mechanics, by studying the constraints that generalized forces must obey in order for them to be derivable from a Lagrangian. We show that any acceleration-independent generalized force must have the electromagnetic form found in Ref. 1. In Sec. IV we illustrate this result by showing that particle motion in a noninertial frame or in a weak gravitational field can be put into this form too. Finally, in Sec. V, we summarize and discuss these results.

II. FEYNMAN'S ARGUMENT IN CLASSICAL HAMILTONIAN DYNAMICS

We will consider the nonrelativistic dynamics of a classical particle of mass m in three space dimensions, and we will assume that this system is canonical, so that a Hamiltonian exists and Poisson brackets are defined. We will also assume the usual commutator-Poisson bracket correspondence

$$(1/i\hbar)[A,B] \leftrightarrow \{A,B\}, \tag{7}$$

where A,B are operators in the quantum theory and functions of the canonical coordinates $\mathbf{x}$ and momenta $\mathbf{p}$ and possibly the time t in the classical theory. The Poisson bracket is defined to be

$$\{A,B\} \equiv \sum_k \left(\frac{\partial A}{\partial x_k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial x_k}\right). \tag{8}$$

In order to follow the steps in Feynman's argument we chose as our starting point the Poisson bracket

$$m\{x_i,\dot{x}_j\} = \delta_{ij}. \tag{9}$$

From this bracket we deduce that

$$m\dot{x}_i = p_i - A_i(\mathbf{x},t), \tag{10}$$

where $\mathbf{A}$ is some vector function of the coordinates and the time.

Next, we take the total time derivative of the bracket (9), and, invoking the product rule for Poisson brackets we obtain

$$\{\dot{x}_i,\dot{x}_j\} + \{x_i,\ddot{x}_j\} = 0, \tag{11}$$

so that, by use of the Jacobi identity

$$\{x_i,\{\dot{x}_j,\dot{x}_k\}\} + \{\dot{x}_j,\{\dot{x}_k,x_i\}\} + \{\dot{x}_k,\{x_i,\dot{x}_j\}\} = 0, \tag{12}$$

and Eq. (9) we find

$$\{x_i,\{x_j,\ddot{x}_k\}\} = \frac{\partial}{\partial p_i}\{x_j,\ddot{x}_k\} = 0. \tag{13}$$

Now, from Eq. (11) we have

$$\{x_j,\ddot{x}_k\} = -\{x_k,\ddot{x}_j\}, \tag{14}$$

so that Eq. (13) may be integrated to give

$$\{x_i,\ddot{x}_j\} = \frac{\partial \ddot{x}_j}{\partial p_i} = -m^{-2}\epsilon_{ijk}H_k(\mathbf{x},t), \tag{15}$$

where $\mathbf{H}$ is another vector function of the coordinates and time. This equation may itself be integrated to give

$$m\ddot{x}_i = E_i(\mathbf{x},t) + \epsilon_{ijk}\dot{x}_j H_k(\mathbf{x},t), \tag{16}$$

where we have used Eq. (10), and $\mathbf{E}$ is a third vector function of the coordinates and time. This establishes the Lorentz form of the force law. However, the fact that the force is independent of the acceleration has been derived here, rather than being assumed as it was in Ref. 1 [cf. Eq. (3)]. This indicates that for this problem the commutation relations are not independent of the equations of motion.

From Eqs. (11) and (15) we find

$$H_i = (m^2/2)\epsilon_{ijk}\{\dot{x}_j,\dot{x}_k\}, \tag{17}$$

and since by the Jacobi identify for Poisson brackets

$$\epsilon_{ijk}\{\dot{x}_i,\{\dot{x}_j,\dot{x}_k\}\} = 0, \tag{18}$$

we deduce

$$\{\dot{x}_i,H_i\} = 0 \quad \text{or} \quad \nabla\cdot\mathbf{H} = 0, \tag{19}$$

where we have used Eq. (10). This establishes one of the equations satisfied by the vector fields. The other is obtained by taking the total time derivative of Eq. (17). Using the product rule for the derivative of a Poisson bracket and substituting from Eq. (16) we find

$$\frac{\partial H_i}{\partial t} + \dot{x}_l\frac{\partial H_i}{\partial x_l} = m\epsilon_{ijk}\{E_j,\dot{x}_k\} - m\{\dot{x}_i H_k,\dot{x}_k\} + m\{\dot{x}_k H_i,\dot{x}_k\}. \tag{20}$$

If we now expand the Poisson brackets that contain products and substitute from Eqs. (10) and (17) we find

$$\frac{\partial \mathbf{H}}{\partial t} + \nabla\times\mathbf{E} = 0, \tag{21}$$

which establishes the second constraint on the vector fields.

This establishes that quantum mechanics is not a necessary ingredient of Feynman's argument. Our argument has followed the steps of the original by replacing commutators with Poisson brackets. However, the results (16), (19), and (21) could have been derived much more quickly by direct integration of the Hamilton's equation

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = m^{-1}[p_i - A_i(\mathbf{x},t)], \tag{22}$$

where we have used Eq. (10), to give the Hamiltonian H. The equation of motion that results from the other Hamilton's equation ($\dot{p}_i = -\partial H/\partial x_i$) would have the properties that have just been derived.

In order to understand the origins of these results we need to go to the Lagrangian formulation of the problem. This is possible because we are dealing with a canonical system ($\det[\partial^2 H/\partial p_i\,\partial p_j] = \det[m^{-1}\delta_{ij}] \neq 0$) so that the momenta may be solved to give the velocities, and a Lagrangian formulation exists.

III. FEYNMAN'S ARGUMENT IN LAGRANGIAN MECHANICS

An equation of motion of the form (3) can only be derived from a Lagrangian that is a function of the coordinates, the velocities and possibly the time, $L \equiv L(\mathbf{x},\dot{\mathbf{x}},t)$, using the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0, \tag{23}$$

if the generalized force F_i can be written as[2-4]

$$F_i = -\frac{\partial V}{\partial x_i} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}_i}\right), \tag{24}$$

where $V \equiv V(\mathbf{x},\dot{\mathbf{x}},t)$ is a generalized potential, so that the Lagrangian has the form

$$L = \tfrac{1}{2}m\dot{x}_i\dot{x}_i - V(\mathbf{x},\dot{\mathbf{x}},t). \tag{25}$$

Such a generalized force can be at most a linear function of the acceleration,[2] and it must satisfy the identities ("Helmholtz conditions"):[2]

$$\frac{\partial F_i}{\partial \ddot{x}_j} = \frac{\partial F_j}{\partial \ddot{x}_i}; \tag{26}$$

$$\frac{\partial F_i}{\partial \dot{x}_j} + \frac{\partial F_j}{\partial \dot{x}_i} = \frac{d}{dt}\left(\frac{\partial F_i}{\partial \ddot{x}_j} + \frac{\partial F_j}{\partial \ddot{x}_i}\right); \tag{27}$$

$$\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial F_i}{\partial \dot{x}_j} - \frac{\partial F_j}{\partial \dot{x}_i}\right). \tag{28}$$

Also, it was stated by Helmholtz[5] that it is only if a generalized force satisfies the identities (26)–(28) that a Lagrangian formulation is possible.[4] A proof was later provided by Mayer,[6] and extended by Hirsch[7] and Boehm.[8] Further discussion of the historical development of this inverse problem of the calculus of variations is given by Santilli[9] and Hojman and Shepley.[10]

One of the assumptions in Feynman's argument is that the generalized force is acceleration independent [cf. Eq. (3)]. In this case we find from Eq. (27) that

$$\frac{\partial^2 F_i}{\partial \dot{x}_j\,\partial \dot{x}_k} = 0, \tag{29}$$

and hence

$$\frac{\partial F_i}{\partial \dot{x}_j} = \epsilon_{ijk}H_k(\mathbf{x},t), \tag{30}$$

where $\mathbf{H}(\mathbf{x},t)$ is a vector function of the coordinates and time. Therefore, from Eqs. (29) and (30) the generalized force must have the velocity-dependent Lorentz form (4)

$$F_i(\mathbf{x},\dot{\mathbf{x}},t) = E_i(\mathbf{x},t) + \epsilon_{ijk}\dot{x}_j H_k(\mathbf{x},t), \tag{31}$$

where $\mathbf{E}(\mathbf{x},t)$ is another vector function of the coordinates and time.

From the identity (28), we deduce that

$$\left(\nabla\times\mathbf{E} + \frac{\partial \mathbf{H}}{\partial t}\right)_k - \dot{x}_k(\nabla\cdot\mathbf{H}) = 0, \tag{32}$$

and so, because of the independence of the coordinates and velocities, the functions $\mathbf{E}$ and $\mathbf{H}$ satisfy the two "Maxwell" equations

$$\nabla\cdot\mathbf{H} = 0 \tag{33}$$

and

$$\nabla\times\mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = \mathbf{0}. \tag{34}$$

Note that these last two equations are just the requirements for the existence of vector and scalar potentials.

From this analysis it is clear that any equation of motion that involves acceleration-independent generalized forces, that can be derived from a Lagrangian, can be written in the Lorentz form (4) with "electric" and "magnetic" fields that satisfy the "Maxwell" equations (5) and (6). So, Feynman's argument does not uniquely pick-out the electromagnetic interaction. However, although there are various well-known velocity-dependent generalized forces in nonrelativistic mechanics that can be derived from Lagrangians, the only familiar example of the form (31) with the constraints (5) and (6) on the force is the electromagnetic one. In the following sections we will study some of the other examples, but first we will complete the analysis by returning to the Hamiltonian formulation of the problem.

The generalized force can only be acceleration independent if

$$\frac{\partial^2 V}{\partial \dot{x}_i\, \partial \dot{x}_j} = 0, \tag{35}$$

so that the potential is also a linear function of the velocities. From Eq. (35) we may introduce a scalar potential $\phi(\mathbf{x},t)$ and a vector potential $\mathbf{A}(\mathbf{x},t)$ through

$$V(\mathbf{x},\dot{\mathbf{x}},t) \equiv \phi - \dot{x}_i A_i, \tag{36}$$

so that

$$E_i \equiv -\frac{\partial A_i}{\partial t} - \frac{\partial \phi}{\partial x_i} \tag{37}$$

and

$$H_i \equiv \epsilon_{ijk} \frac{\partial A_k}{\partial x_j}. \tag{38}$$

The constraints (5) and (6) are therefore automatically satisfied if the vector and scalar potentials are suitably differentiable. Moreover, the nonuniqueness of the Lagrangian under[3,11]

$$L \to L - \frac{d\psi(\mathbf{x},t)}{dt}, \tag{39}$$

where ψ is an arbitrary function of $\mathbf{x}$ and t, induces a "gauge" transformation

$$\phi \to \phi + \frac{\partial \psi}{\partial t}, \tag{40}$$

$$\mathbf{A} \to \mathbf{A} - \nabla \psi, \tag{41}$$

on the potentials. It is clear that the fields (37) and (38) are invariant under this gauge transformation.

Since the Lagrangian

$$L = \tfrac{1}{2} m \dot{x}_i \dot{x}_i - \phi(\mathbf{x},t) + \dot{x}_i A_i(\mathbf{x},t), \tag{42}$$

is nonsingular $[\det(\partial^2 L/\partial \dot{x}_i\, \partial \dot{x}_j) = \det(m\delta_{ij}) \neq 0]$ we have a canonical system whose Hamiltonian

$$H(\mathbf{x},\mathbf{p},t) = p_i \dot{x}_i - L, \tag{43}$$

has the "electromagnetic" (minimally coupled) form

$$H = (1/2m)(p_i - A_i)^2 + \phi, \tag{44}$$

where the canonical momenta are

$$p_i \equiv \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + A_i. \tag{45}$$

This shows that minimal coupling is not an arbitrary condition that one is free to introduce into the dynamics, nor is it unique to electromagnetic interactions.

By use of Hamilton's equation

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = m^{-1}(p_i - A_i), \tag{46}$$

we may derive the Poisson bracket

$$\{x_i, \dot{x}_j\} = m^{-1}\delta_{ij}, \tag{47}$$

which is the classical analog of the commutator (1), and so we again find that the assumptions underlying Feynman's argument are not independent. Therefore, in the quantum theory, the commutation relations are determined by the equations of motion.[12-14]

IV. EXAMPLES OF SYSTEMS THAT SATISFY THE "ELECTROMAGNETIC" CONSTRAINTS

A. Motion in a noninertial frame

The equation of motion of a particle of mass m in a frame of reference whose origin has a linear acceleration $\mathbf{a}$ and an angular velocity $\boldsymbol{\Omega}$ with respect to an inertial reference frame, is[15,16]

$$m\ddot{\mathbf{x}} = \mathbf{F} - m\mathbf{a} + m\mathbf{x}\times\dot{\boldsymbol{\Omega}} + 2m\dot{\mathbf{x}}\times\boldsymbol{\Omega} + m\boldsymbol{\Omega}\times(\mathbf{x}\times\boldsymbol{\Omega}), \tag{48}$$

where $\mathbf{x}$ is the position vector of the particle with respect to the origin of the noninertial frame. The first term on the right-hand side of Eq. (48) is the force that would be acting on the particle in an inertial reference frame, while the next four terms are the "inertial" forces that must be introduced in the noninertial frame. Of these, the fourth and fifth are the Coriolis and centrifugal forces, respectively.

Since we are pursuing the "electromagnetic" analogy for the inertial forces we will henceforth set the physical force to zero ($\mathbf{F} = 0$). Note that the inertial forces are independent of the particle's acceleration, and linear in its velocity. The equation of motion (48) can be derived from a Lagrangian $L = T - V$, where T is the kinetic energy and V is a velocity-dependent generalized potential[15]

$$V(\mathbf{x},\dot{\mathbf{x}},t) = m\mathbf{a}\cdot\mathbf{x} - m\dot{\mathbf{x}}\cdot\boldsymbol{\Omega}\times\mathbf{x} - \tfrac{1}{2}m(\boldsymbol{\Omega}\times\mathbf{x})^2, \tag{49}$$

which is a linear function of the velocities. We may now introduce a scalar potential

$$\phi(\mathbf{x},t) = m\mathbf{a}\cdot\mathbf{x} - \tfrac{1}{2}m(\boldsymbol{\Omega}\times\mathbf{x})^2, \tag{50}$$

and a vector potential

$$\mathbf{A}(\mathbf{x},t) \equiv m\boldsymbol{\Omega}\times\mathbf{x}, \tag{51}$$

so that $V = \phi - \dot{\mathbf{x}}\cdot\mathbf{A}$.

The equation of motion (48) may now be written in Lorentz form as

$$m\ddot{\mathbf{x}} = \mathbf{E}(\mathbf{x},t) + \dot{\mathbf{x}}\times\mathbf{H}(\mathbf{x},t), \tag{52}$$

with

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi = -m\dot{\boldsymbol{\Omega}}\times\mathbf{x} - m\mathbf{a} + m\boldsymbol{\Omega}\times(\mathbf{x}\times\boldsymbol{\Omega}) \tag{53}$$

and

$$\mathbf{H} = \nabla\times\mathbf{A} = 2m\boldsymbol{\Omega}. \tag{54}$$

The "Maxwell" equations (5) and (6) are satisfied by virtue of Eqs. (53) and (54). The similarity between the Coriolis and magnetic forces, which underlies Larmor's theorem, has been previously noted,[17,18] and has been used to provide an explanation of the operation of the gyrocompass in analogy with the magnetic compass.[19]

B. Gravitational and gravitomagnetic interactions

A second example of a velocity-dependent interaction that has a Lorentz form is the force acting on a nonrelativistic test particle of mass m in the weak gravitational field of a rotating, massive body, of mass M, and angular momentum $\mathbf{J}$. Using the space-time coordinates

$$x^{\mu} = (ct,\mathbf{x}), \tag{55}$$

the metric outside such a body can be written approximately as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{56}$$

with

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1), \tag{57}$$

$$h_{00} = h_{11} = h_{22} = h_{33} = -2GM/xc^2, \tag{58}$$

$$h_{0i} = -(2G/c^3)\epsilon_{ijk}J_k(\hat{x}_j/x^2), \tag{59}$$

where G is Newton's gravitational constant, c is the speed of light, and $\mathbf{x}$ is the radius vector with origin at the center of mass of the source of the gravitational field. At the surface of the earth $|\mathbf{x}| = R_{\oplus}$, for instance, we have $GM_{\oplus}/R_{\oplus}c^2 \approx 6\times 10^{-10}$ and $GJ_{\oplus}/R^2_{\oplus}c^3 \approx 5\times 10^{-16}$ where $M_{\oplus}$, $R_{\oplus}$, and $J_{\oplus}$ are the earth's mass, radius, and angular momentum, respectively.

The action of a test particle in a gravitational field is

$$S = -mc\int ds, \tag{60}$$

where the interval ds is given by

$$ds^2 = g_{\mu\nu}dx^{\mu}\,dx^{\nu}. \tag{61}$$

We introduce a scalar potential

$$\phi \equiv (mc^2/2)h_{00} = -GmM/x, \tag{62}$$

and a vector potential

$$A_i \equiv -mch_{0i} = (2Gm/c^2)\epsilon_{ijk}J_k(\hat{x}_j/x^2), \tag{63}$$

so that in the nonrelativistic $[O(v^2/c^2)]$, weak-field $[O(GM/xc^2)]$ limit the Lagrangian that describes the test particle motion is

$$L = \tfrac{1}{2}m\dot{x}_i\dot{x}_i - \phi + \dot{x}_iA_i, \tag{64}$$

where we have dropped the (constant) rest mass term. The equation of motion of the test body therefore has the Lorentz form

$$m\ddot{\mathbf{x}} = \mathbf{E} + \dot{\mathbf{x}}\times\mathbf{H}, \tag{65}$$

with

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi = -\frac{GmM}{x^2}\hat{\mathbf{x}}, \tag{66}$$

$$H_i = (\nabla\times\mathbf{A})_i = (2Gm/c^2x^5)(x^2\delta_{ij} - 3x_ix_j)J_j, \tag{67}$$

so that the "Maxwell" equations

$$\nabla\cdot\mathbf{H} = 0, \tag{68}$$

$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{H}}{\partial t} = 0, \tag{69}$$

are satisfied outside the source of the gravitational field. (The second of these is an identity because of the time independence of the gravitational field.)

The similarity between the Lagrangian (64) and that of a particle in a rotating frame has been previously pointed out.[20] In particular, there is a velocity-dependent "Coriolis" force acting on the particle that is the same as if it was in a frame rotating with an angular velocity

$$\boldsymbol{\Omega} = (1/2m)\nabla\times\mathbf{A}. \tag{70}$$

This analogy suggests the possibility of a Foucault pendulum experiment to measure the $\mathbf{A}$ field of the earth.[21] At the earth's poles we have $|\Omega| = 2GJ_{\oplus}/R^3_{\oplus}c^2 \approx 5\times 10^{-14}\,\mathrm{s}^{-1} \approx 0.3''\,\mathrm{yr}^{-1}$.

The analogy between the equation of motion (65) and the Lorentz force on a charged particle is also well known.[22-24] The fields (66) and (67) are often referred to as gravitoelectric and gravitomagnetic fields, respectively.[25] This analogy with electromagnetism has been used in discussions[26] of the precession of an orbiting gyroscope,[27] in the Lense–Thirring field[28,29] of the earth, although the Lagrangian (64) does not include the special relativistic terms that produce a Thomas precession, and which must also be included for a correct treatment.[26] This is because we must include terms $O(v^4/c^4)$ in the Lagrangian once we consider speeds $> 1\ \mathrm{m\,s^{-1}}$.

C. Quantum mechanics

So far, our investigation has been entirely within classical mechanics. However, quantization of nonrelativistic particle motion in a noninertial frame or in a weak gravitational field can proceed by the usual rules.[30] The Hamiltonian, (44), that is to be used in the Schrödinger equation has the minimally coupled form familiar from electromagnetism. This is the only form that is compatible with an acceleration-independent generalized force, rather than an additional prescription to be imposed on the dynamics. Therefore, we should expect the quantum mechanics of a particle in a noninertial frame or a weak gravitational field to exhibit phenomena that have analogies with the quantum mechanics of a charged particle in an electromagnetic field. This analogy with electromagnetism has been exploited in the analysis of the Sagnac effect in a neutron interferometer that is caused by the rotation of the earth,[31,32] and in studies of the effect of gravity on Cooper pairs in a superconductor.[33,34] There has also been some discussion of a noninertial version[35,36] of the Aharonov–Bohm effect.[37]

V. SUMMARY DISCUSSION AND CONCLUSIONS

In this paper we have shown that Feynman's argument is actually a rederivation of the properties of any acceleration-independent force that allow it to be derived from a Lagrangian. We have illustrated this point by showing that the inertial forces in a rotating reference frame, and the gravitational forces on a nonrelativistic test-particle moving in the weak gravitational field of a rotating mass both have the minimally coupled electromagnetic form. Although the analogies with electromagnetism that underly Larmor's theorem in the first case, and gravitoelectricity and gravitomagnetism in the second are well known, the generality of the constraints satisfied by acceleration-independent generalized forces that permit a Lagrangian formulation is not.

Our analysis in Sec. III has proceeded by considering the Helmholtz conditions for an acceleration-independent force that can be derived from a nonsingular Lagrangian. We then found that the commutation relations (1) and (2) are determined. In a recent paper, Hojman and Shepley[38] have analyzed Feynman's argument from essentially the opposite viewpoint and obtained correspondingly stronger results. They have shown that the existence of a Lagrangian follows from Eq. (2), and that its explicit form comes from Eq. (1), which determines the Helmholtz conditions, and hence the form of the equation of motion. Their analysis, which is therefore complementary to ours, includes several examples, but not the physically motivated ones that we have discussed.

The authors of Refs. 39–42 have shown that Feynman's argument[1] is not a derivation of Maxwell's equations because of a conflict with Lorentz invariance. Specifically, although the two Maxwell equations (5) and (6) that allow the existence of vector and scalar potentials, are comparable with both Galilean and Lorentz invariance, the other two Maxwell equations are not.[43] However, a consistent version of electrodynamics that is based on a generalization of Galilean invariance has been developed.[44] In this approach, the nonrelativistic dynamics of a point particle of mass m is required to be invariant under the generalized Galilean transformation of the canonical variables

$$\begin{aligned} \bar{q}_i &= q_i, \\ \bar{p}_i &= p_i + m\mathbf{v}_i. \end{aligned} \tag{71}$$

It has been shown that the most general form of Hamiltonian that is consistent with invariance under this transformation has the minimally coupled form (44).[45] This result, which amounts to an alternative derivation of Feynman's argument, has been extended to a suitably generalized Lorentz invariance.[46] However, the authors of Refs. 45 and 46 regarded their results as a restriction of a particle's interactions to electromagnetic ones.

The Helmholtz conditions (26)–(28) must also be satisfied by any particle equation of motion that has the form

$$F_i(\mathbf{x},\dot{\mathbf{x}},\ddot{\mathbf{x}},t) = 0, \tag{72}$$

in order for it to be derivable from a Lagrangian that is a function of the coordinates, the velocities, and the time.[4] In particular this is true for relativistic particles. For instance, for a free particle we can verify that the equation of motion

$$F_i = \frac{d}{dt}\left(\frac{m\dot{x}_i}{\sqrt{1-\dot{x}^2/c^2}}\right) = 0, \tag{73}$$

satisfies the constraints. It would be interesting to see, in analogy with Feynman's argument in the nonrelativistic case, what limitations the constraints (26), (27), and (28) impose on the types of interactions that a relativistic particle can experience. (It is easy to show that an interaction that depends only on the coordinates and linearly on the velocities must have the electromagnetic form.) Likewise, another interesting direction would be to study the Helmholtz conditions and gauge couplings in field theory.

We have seen how the Helmholtz conditions require the Hamiltonian of a point-particle to have the minimally coupled or "Abelian" gauge form when the force acting on the particle is assumed to be acceleration independent. Perhaps a "non-Abelian" gauge structure would emerge if the Helmholtz conditions were applied to a spinning particle? Some preliminary work in this direction has been carried out, although from the erroneous point-of-view that it was a derivation of the Yang–Mills equations.[47] (See Ref. 48 for criticism of this interpretation.)

In conclusion, we find that although Feynman's argument is not a derivation of the Maxwell equations, on closer study it does illuminate the connection between the existence of a Lagrangian, and the allowable forms of the equations of motion and the commutation relations. Perhaps the most remarkable aspect of this connection is that minimal coupling is not only a requirement of the Lagrangian formulation of acceleration-independent forces, but also is not unique to the electromagnetic interaction. This observation provides insight into the analogies between charged particle dynamics and motion in noninertial frames and weak gravitational fields.

ACKNOWLEDGMENT

This paper is dedicated to the memory of John S. Bell. The results presented here developed out of discussions that I had with John on this subject during our last meeting in July of 1990.

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A modern analysis of the Stern–Gerlach experiment

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(Received 21 September 1990; accepted 3 December 1990)

While the Stern–Gerlach experiment is an old and familiar problem, no analysis of this experiment is presented in the pedagogical literature using modern quantum mechanical techniques. Expositions of the Stern–Gerlach experiment are usually based on "old quantum theory," i.e., semiclassical Bohr–Sommerfeld quantum mechanics. The experiment is also a popular one in discussions of the postulate of quantum measurement, which asserts that the process of measuring an observable forces the state vector of the system into an eigenvector of that observable, and the value measured will be the eigenvalue of that eigenvector. The most enduring of the philosophical arguments associated with quantum mechanics ultimately revolves around the mechanism of quantum measurement. Since the postulate of quantum measurement is expressed in terms well outside the realm of the old quantum mechanics, the traditional analysis of the Stern–Gerlach experiment cannot reveal any information about the relationship between dynamics and measurement. Therefore, a modern analysis of the Stern–Gerlach experiment must be made if this experiment is to reveal any information about the relationship between dynamics and the mechanism of quantum measurement. This paper presents such an analysis, and develops some of the implications for the theory of quantum measurement.

The Stern–Gerlach experiment[1] has long been a staple of elementary modern physics courses. Performed just before the advent of modern quantum mechanics,[2] the Stern–Gerlach experiment firmly established the quantization of intrinsic spin. Expressed in the semiclassical treatment of the old quantum mechanics, no detailed modern analysis of the Stern–Gerlach experiment has ever found its way into the standard pedagogical texts.

As students move into intermediate quantum mechanics, they encounter the Stern–Gerlach experimental configuration again, only now as a spin polarizer being employed to help communicate the relationship between the eigen-

