Mathematics for Machine Learning

TECHNISCHE UNIVERSITÄT DARMSTADT

Solution of the Exercises

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Solution 2 Linear Algebra

Solution 2.1

Solution 2.1.1

a) We have to show, that $\mathbb{R} \setminus \{-1\}$ is closed under $*$, the associativity, the existence of a neutral and inverse elements and the commutativity.
For the closure of $\mathbb{R} \setminus \{-1\}$ we can use the closure of the addition and multiplication in $\mathbb{R}$. Then we have to show that there are no a and b in $\mathbb{R} \setminus \{-1\}$, so that $a * b = -1$.

Assuming that $\exists a, b \in \mathbb{R} \setminus \{-1\}$ with $a * b = -1$. Then it is

$$\begin{aligned} & a * b = ab + a + b = -1 \\ \iff & \quad ab + a = -1 - b \\ \iff & \quad a(b+1) = -(b+1) \\ \iff & \quad a = -\frac{b+1}{b+1} = -1 \end{aligned}$$

So we got a contradiction and that shows that there are no $a, b \in \mathbb{R} \setminus \{-1\}$ so that $a * b = -1$.
Consider $a, b, c \in \mathbb{R} \setminus \{-1\}$. Then it is

$$\begin{aligned} (a * b) * c &= (ab + a + b) * c \\ &= (ab + a + b)c + (ab + a + b) + c \\ &= abc + ac + bc + ab + a + b + c \\ &= abc + ab + ac + a + bc + b + c \\ &= a(bc + b + c) + a + (bc + b + c) \\ &= a * (bc + b + c) = a * (b * c) \end{aligned}$$

That shows the associativity of $*$.

The neutral element is 0, because:

$$
\begin{aligned}
&a * 0 = a \cdot 0 + a + 0 = 0 + a + 0 = a \text{ and} \\
&0 * a = 0 \cdot a + 0 + a = 0 + 0 + a = a
\end{aligned}
$$

for any $a \in \mathbb{R} \setminus \{-1\}$.
Consider $a^{-1} = -a/(a+1)$. Then it is

$$
\begin{aligned}
a * a^{-1} &= a * -\frac{a}{a+1} \\
&= a(-\frac{a}{a+1}) + a + (-\frac{a}{a+1}) \\
&= \frac{-a^2}{a+1} + a - \frac{a}{a+1} \\
&= \frac{-a^2 - a}{a+1} + \frac{a(a+1)}{a+1} \\
&= \frac{-a^2 - a}{a+1} + \frac{a^2 + a}{a+1} = 0
\end{aligned}
$$

The proof of $a^{-1} * a = 0$ works analogously.
The proof of the commutativity is straight forward and based on the commutativity of the addition and multiplication in $\mathbb{R}$. Consider $a, b \in \mathbb{R} \setminus \{-1\}$. Then it is

$$a * b = ab + a + b = ba + b + a = b * a$$

So we have shown all axioms of an Abelian group. $\square$

b) It is

$$
\begin{aligned}
3 * x * x &= (3x + 3 + x) * x \\
&= (4x + 3) * x \\
&= (4x + 3)x + (4x + 3) + x \\
&= 4x^2 + 3x + 4x + 3 + x = 4x^2 + 8x + 3
\end{aligned}
$$

We can now solve the quadtratic formula $4x^2 + 8x + 3 = 15 \iff 4x^2 + 8x - 12 = 0$ using the completing the square method proposed by Hoehn in [1]:

$$
\begin{aligned}
x &= \frac{-8 \pm \sqrt{8^2 - 4 \cdot 4 \cdot (-12)}}{2 \cdot 4} \\
&= \frac{-8 \pm \sqrt{64 + 192}}{8} \\
&= \frac{-8 \pm \sqrt{256}}{8} \\
&= \frac{-8 \pm 16}{8} = -1 \pm 2
\end{aligned}
$$

Thus the solution of the equation is $x_1 = -3$ and $x_2 = 1$.

References

[1] Larry Hoehn. A more elegant method of deriving the quadratic formula. *Mathematics Teacher*, 68(5):442–443, 1975.