Mathematics for Machine Learning

TECHNISCHE UNIVERSITÄT DARMSTADT

Solution of the Exercises
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Solution 2 Linear Algebra

Solution 2.1

Solution 2.1.1

1. We have to show, that $\mathbb{R} \setminus \{-1\}$ is closed under $*$, the associativity, the existence of a neutral and inverse elements and the commutativity.
   For the closure of $\mathbb{R} \setminus \{-1\}$ we can use the closure of the addition and multiplication in $\mathbb{R}$. Then we have to show that there are no a and b in $\mathbb{R} \setminus \{-1\}$, so that $a * b = -1$.

   Assuming that $\exists a, b \in \mathbb{R} \setminus \{-1\}$ with $a * b = -1$. Then it is

$$
\begin{aligned}
& a * b = ab + a + b = -1 \\
\iff \quad & ab + a = -1 - b \\
\iff \quad & a(b+1) = -(b+1) \\
\iff \quad & a = -\frac{b+1}{b+1} = -1
\end{aligned}
$$

   So we got a contradiction and that shows that there are no $a, b \in \mathbb{R} \setminus \{-1\}$ so that $a * b = -1$.
   Consider $a, b, c \in \mathbb{R} \setminus \{-1\}$. Then it is

$$
\begin{aligned}
(a * b) * c &= (ab + a + b) * c \\
&= (ab + a + b)c + (ab + a + b) + c \\
&= abc + ac + bc + ab + a + b + c \\
&= abc + ab + ac + a + bc + b + c \\
&= a(bc + b + c) + a + (bc + b + c) \\
&= a * (bc + b + c) = a * (b * c)
\end{aligned}
$$

   That shows the associativity of $*$.

The neutral element is 0, because:

$$a * 0 = a \cdot 0 + a + 0 = 0 + a + 0 = a \text{ and}$$
$$0 * a = 0 \cdot a + 0 + a = 0 + 0 + a = a$$

for any $a \in \mathbb{R} \setminus \{-1\}$.
Consider $a^{-1} = -a/(a+1)$. Then it is

$$\begin{aligned}
a * a^{-1} &= a * -\frac{a}{a+1} \\
&= a(-\frac{a}{a+1}) + a + (-\frac{a}{a+1}) \\
&= \frac{-a^2}{a+1} + a - \frac{a}{a+1} \\
&= \frac{-a^2 - a}{a+1} + \frac{a(a+1)}{a+1} \\
&= \frac{-a^2 - a}{a+1} + \frac{a^2 + a}{a+1} = 0
\end{aligned}$$

The proof of $a^{-1} * a = 0$ works analogously.
The proof of the commutativity is straight forward and based on the commutativity of the addition and multiplication in $\mathbb{R}$. Consider $a, b \in \mathbb{R} \setminus \{-1\}$. Then it is

$$a * b = ab + a + b = ba + b + a = b * a$$

So we have shown all axioms of an Abelian group. □

2. It is

$$\begin{aligned}
3 * x * x &= (3x + 3 + x) * x \\
&= (4x + 3) * x \\
&= (4x + 3)x + (4x + 3) + x \\
&= 4x^2 + 3x + 4x + 3 + x = 4x^2 + 8x + 3
\end{aligned}$$

We can now solve the quadtratic formula $4x^2 + 8x + 3 = 15 \iff 4x^2 + 8x - 12 = 0$ using the completing the square method proposed by HOEHN in [1]:

$$\begin{aligned}
x &= \frac{-8 \pm \sqrt{8^2 - 4 \cdot 4 \cdot (-12)}}{2 \cdot 4} \\
&= \frac{-8 \pm \sqrt{64 + 192}}{8} \\
&= \frac{-8 \pm \sqrt{256}}{8} \\
&= \frac{-8 \pm 16}{8} = -1 \pm 2
\end{aligned}$$

Thus the solution of the equation is $x_1 = -3$ and $x_2 = 1$.

Solution 2.1.2

1. At first we have to be careful, because $(\mathbb{Z}_n, \oplus)$ would not be a group with the unmodified given mapping, because $\mathbb{Z}_n$ would not be closed under $\oplus$: Let $n = 3$, then $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$. So consider $\overline{1}, \overline{2} \in \mathbb{Z}_3$, then:

$$\overline{1} \oplus \overline{2} = \overline{1 + 2} = \overline{3} \notin \mathbb{Z}_3$$

Thus we have to modify the mapping by adding a modulo to the addition:

$$\overline{a} \oplus \overline{b} = \overline{a + b \mod n} \tag{1}$$

Now $\mathbb{Z}_n$ is closed under $\oplus$.

(Associativity) Let $\overline{a}, \overline{b}, \overline{c} \in \mathbb{Z}_n$. Then it is

$$\begin{aligned}(\overline{a} \oplus \overline{b}) \oplus \overline{c} &= \overline{a+b \mod n} \oplus \overline{c} \\ &= \overline{(a+b \mod n) + c \mod n} \\ &= \overline{a+b+c \mod n} \\ &= \overline{a \mod n + (b+c \mod n)} \\ &= \overline{a} \oplus \overline{b+c \mod n} = \overline{a} \oplus (\overline{b} \oplus \overline{c})\end{aligned}$$

(Neutral Element) The neutral element is $\overline{0} \in \mathbb{Z}_n$. Let $\overline{a} \in \mathbb{Z}_n$, then it is:

$$\overline{a} \oplus \overline{0} = \overline{a+0 \mod n} = \overline{a \mod n} = \overline{a}$$

and

$$\overline{0} \oplus \overline{a} = \overline{0+a \mod n} = \overline{a \mod n} = \overline{a}$$

(Inverse Element) Let $\overline{a} \in \mathbb{Z}_n$. Then the inverse element of $\overline{a}$ is $\overline{a}^{-1} = \overline{n-a}$. It is

$$\begin{aligned}\overline{a} \oplus \overline{a}^{-1} &= \overline{a} \oplus \overline{n-a} \\ &= \overline{a + (n-a) \mod n} \\ &= \overline{n \mod n} = \overline{0}\end{aligned}$$

The proof of $\overline{a}^{-1} \oplus \overline{a}$ works analogously.

(Commutativity) For the proof of the commutativity in $\mathbb{Z}_n$ we use the commutativity of the addition in $\mathbb{Z}$. Let $\overline{a}, \overline{b} \in \mathbb{Z}_n$. Then it is

$$\overline{a} \oplus \overline{b} = \overline{a+b \mod n} = \overline{b+a \mod n} = \overline{b} \oplus \overline{a}$$

We have shown, that $(\mathbb{Z}_n, \oplus)$ is an Abelian Group. □

2. Firstly we also have to modify the mapping for mathematical correctness:

$$\overline{a} \otimes \overline{b} = \overline{a \cdot b \mod n} \tag{2}$$

In table 1 we can see that for each elements $\overline{a}, \overline{b} \in \mathbb{Z}_5 \setminus \{\overline{0}\}$ the result of $\overline{a} \otimes \overline{b} \in \mathbb{Z}_5 \setminus \{\overline{0}\}$. The neutral element

Table 1: The times table of $\mathbb{Z}_5 \setminus \{\overline{0}\}$ under $\otimes$

$\otimes$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{2}$	$\overline{2}$	$\overline{4}$	$\overline{1}$	$\overline{3}$
$\overline{3}$	$\overline{3}$	$\overline{1}$	$\overline{4}$	$\overline{2}$
$\overline{4}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

of this mapping is $\overline{1}$, because $\overline{a} \otimes \overline{1} = \overline{a \cdot 1 \mod 5} = \overline{a}$. We can read the inverse elements of each element in $\mathbb{Z}_5 \setminus \{\overline{0}\}$ out of table 1, where the result is $\overline{1}$. And the commutativity follows directly from the commutativity of the multiplication in $\mathbb{Z}$. □

3. Consider $\overline{4} \in \mathbb{Z}_8 \setminus \{\overline{0}\}$. If we multiply $\overline{4}$ with all elements of $\mathbb{Z}_8 \setminus \{\overline{0}\}$, we get the following results:

$$\overline{4} \otimes \overline{1} = \overline{4} \qquad \overline{4} \otimes \overline{2} = \overline{0} \qquad \overline{4} \otimes \overline{3} = \overline{4} \qquad \overline{4} \otimes \overline{4} = \overline{0}$$
$$\overline{4} \otimes \overline{5} = \overline{4} \qquad \overline{4} \otimes \overline{6} = \overline{0} \qquad \overline{4} \otimes \overline{7} = \overline{4}$$

Hence we see, that there is no inverse element of $\overline{4}$ in $\mathbb{Z}_8 \setminus \{\overline{0}\}$. Thus $(\mathbb{Z}_8 \setminus \{\overline{0}\})$ is not a group. □

4. The key point for showing that $(\mathbb{Z}_n \setminus \{\overline{0}\}, \otimes)$ is a group, is to show the existence of an inverse element for each element $\overline{a} \in \mathbb{Z}_n \setminus \{\overline{0}\}$.
Assuming $n \in \mathbb{N} \setminus \{0\}$ is prime. Hence a and n are relatively prime and with that we know from Bézout theorem that there are two integers u and v such that $au + nv = 1$. That implies $\overline{au} \otimes \overline{nv} = \overline{1}$ and also $\overline{a} \otimes \overline{u} = \overline{1}$. Thus the inverse element of $\overline{a}$ is $\overline{u}$.
Assuming $(\mathbb{Z}_n \setminus \{\overline{0}\}, \otimes)$ is a group. Also assuming n is not prime, i.e. $\exists k, l \in \mathbb{N}$ such that $1 < k, l < n$ and $n = kl$. We see that $\overline{k}, \overline{l} \in \mathbb{Z}_n \setminus \{\overline{0}\}$. But it is $\overline{k} \otimes \overline{l} = \overline{k \cdot l \mod n} = \overline{n \mod n} = \overline{0} \notin \mathbb{Z}_n \setminus \{\overline{0}\}$. Hence $\mathbb{Z}_n \setminus \{\overline{0}\}$ would not be closed under $\otimes$ and would therefore be no group. Thus n has to be prime. □

Solution 2.1.3

(Closure) Consider two matrices of $\mathcal{G}$. Then it is

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+x & l+xm+y \\ 0 & 1 & m+z \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{G}$$

(Associativity) Consider three matrices of $\mathcal{G}$. Then it is

$$\begin{aligned}
\left(\begin{pmatrix} 1 & x_1 & y_1 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_2 & y_2 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & x_3 & y_3 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & x_2+x_1 & y_2+x_1z_2+y_1 \\ 0 & 1 & z_2+z_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_3 & y_3 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & x_3+x_2+x_1 & y_3+z_3x_2+z_3x_1+y_2+x_1z_2+y_1 \\ 0 & 1 & z_3+z_2+z_1 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & (x_3+x_2)+x_1 & (y_3+z_3x_2+y_2)+(z_3+z_2)x_1+y_1 \\ 0 & 1 & (z_3+z_2)+z_1 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & x_1 & y_1 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_3+x_2 & y_3+z_3x_2+y_2 \\ 0 & 1 & z_3+z_2 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & x_1 & y_1 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & x_2 & y_2 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_3 & y_3 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \right)
\end{aligned}$$

(Neutral Element) The neutral element is the identity matrix, because it is

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

(Inverse Element) For the existence of the inverse element, we can use theorem 4.1. It is

$$\det(X) = 1 \cdot 1 \cdot 1 + x \cdot z \cdot 0 + y \cdot 0 \cdot 0 - y \cdot 1 \cdot 0 - x \cdot 0 \cdot 1 - 1 \cdot z \cdot 0 = 1 + 0 + 0 - 0 - 0 = 1$$

As we can see it is $\det(X) = 1 \neq 0$ and with the help of theorem 4.1 we can imply that every matrix in $\mathcal{G}$ is invertible.

Hence all axioms for a group is fullfilled.
The group is not abelian, because it is

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 7 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution 2.1.4

At first we recall the definition of the matrix product in section 2.2.1. The matrix multiplication is defined for two matrices A and B as $C = AB \in \mathbb{R}^{k\times n}$ with $A \in \mathbb{R}^{k\times m}$ and $B \in \mathbb{R}^{m\times n}$. For the multiplication itself we can use the memory hook "row times column". For the first defined multiplication we write down the entire calculation, but for the further exercises we just provide the solutions.

1. For this both matrices we note, that the first matrix has the dimensions 3×2 and the second matrix has the dimensions 3×3. So as mentioned above, the product of this two matrices is not defined in this order.

2. The product of this two matrices is again a 3×3-matrix. Let

$$C = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

   Then we have to calculate the following terms:

$$\begin{aligned} c_{1,1} &= 1\cdot 1 + 2\cdot 0 + 3\cdot 1 = 1 + 0 + 3 = 4 \\ c_{1,2} &= 1\cdot 1 + 2\cdot 1 + 3\cdot 0 = 1 + 2 + 0 = 3 \\ c_{1,3} &= 1\cdot 0 + 2\cdot 1 + 3\cdot 1 = 0 + 2 + 3 = 5 \\ c_{2,1} &= 4\cdot 1 + 5\cdot 0 + 6\cdot 1 = 4 + 0 + 6 = 10 \\ c_{2,2} &= 4\cdot 1 + 5\cdot 1 + 6\cdot 0 = 4 + 5 + 0 = 9 \\ c_{2,3} &= 4\cdot 0 + 5\cdot 1 + 6\cdot 1 = 0 + 5 + 6 = 11 \\ c_{3,1} &= 7\cdot 1 + 8\cdot 0 + 9\cdot 1 = 7 + 0 + 9 = 16 \\ c_{3,2} &= 7\cdot 1 + 8\cdot 1 + 9\cdot 0 = 7 + 8 + 0 = 15 \\ c_{3,3} &= 7\cdot 0 + 8\cdot 1 + 9\cdot 1 = 0 + 8 + 9 = 17 \end{aligned}$$

   And we get

$$C = \begin{pmatrix} 4 & 3 & 5 \\ 10 & 9 & 11 \\ 16 & 15 & 17 \end{pmatrix}$$

3.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 5 & 7 & 9 \\ 11 & 13 & 15 \\ 8 & 10 & 12 \end{pmatrix}$$

4.

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 13 & 5 \\ -21 & 2 \end{pmatrix}$$

5.

$$\begin{pmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{pmatrix} = \begin{pmatrix} 12 & 3 & -3 & -12 \\ -3 & 1 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 13 & 12 & 3 & 2 \end{pmatrix}$$

References

[1] Larry Hoehn. A more elegant method of deriving the quadratic formula. *Mathematics Teacher*, 68(5):442–443, 1975.