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Resonance Curve in Rectangular Closed Channel

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Abstract—The dynamics of viscous polytropic gas in closed rectangular resonator is studied numerically and analytically on the base of linearized and reduced 2D Navier—Stokes equations. It is shown that for 2D model the maximum pressure amplitude depends on gas viscosity. The resonance curves for 2D model and for both 1D linear and nonlinear models are computed and compared. It is shown that 2D model takes into account the finiteness of the pressure amplitude at resonance. For 1D nonlinear model the frequency range in which periodic shock wave appears is detected. The range of frequencies, at which pressure beats occur, is found. It is shown that pressure beats disappear after about 50—100 cycles.

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1. INTRODUCTION

In [1—3] a gas oscillations in resonators generated by piston vibrating on the left boundary is investigated numerically and experimentally. The numerical solution obtained in [1] is compared with analytical solution for the case of vibrating the whole resonator, obtained in [4]. To make this comparison more accurate an analytical solution for a resonator with a vibrating piston is needed. Thus, the main purposes of this work are to derive the analytical solution for 2D resonator with vibrating piston; to compute and to compare the resonance curves, obtained for 2D model, as well as for 1D linear and nonlinear models [5, 6].

2. 2D-LINEAR MODEL

Rectangular resonator $-x_0 \le x \le x_0$, $-y_0 \le y \le y_0$ with rigid walls is considered (Fig. 1). On the left boundary the longitudinal velocity is excited:

$$\frac{1}{2y_0}\int\limits_{-y_0}^{y_0} u(-x_0, y, t)dy = U_0 \sin(\omega t).$$

Its exact distribution will be determined later during the solution. The linearized system describing gas oscillations has the form

$$\frac{1}{\rho_0 c_0^2}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1a}$$

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}, \tag{1b}$$

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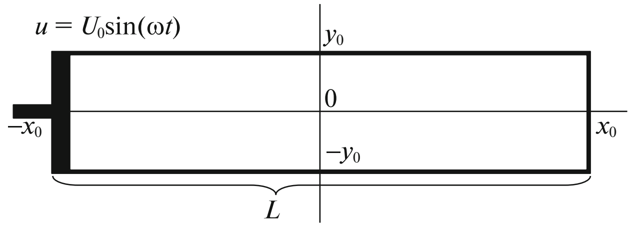

Fig. 1. Rectangular resonator.

$$\frac{\partial p}{\partial y} = 0. \tag{1c}$$

We seek for the periodic solution of system (1) with boundary conditions

$$\text{left: } \frac{1}{2y_0}\int\limits_{-y_0}^{y_0} u(-x_0, y, t)dy = U_0 e^{i\omega t}, \quad v(-x_0, y, t) = 0; \tag{2a}$$

$$\text{right: } u(x_0, y, t) = 0, \quad v(x_0, y, t) = 0; \tag{2b}$$

$$\text{bottom: } u(x, -y_0, t) = 0, \quad v(x, -y_0, t) = 0; \tag{2c}$$

$$\text{upper: } u(x, y_0, t) = 0, \quad v(x, y_0, t) = 0. \tag{2d}$$

Note, that the more is the ratio x_0/y_0 the more accurately linearized system (1) describes the real oscillations of a viscous gas.

The periodic solution of the problem (1)–(2) in the form of a standing wave is

$$p - p_0 = \tilde{p}(x, y)e^{i\omega t}, \quad u = \tilde{u}(x, y)e^{i\omega t}, \quad v = \tilde{v}(x, y)e^{i\omega t}. \tag{3}$$

Substituting (3) into (1)–(2), yields

$$\frac{i\omega}{\rho_0 c_0^2}\tilde{p} + \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0, \tag{4a}$$

$$\tilde{u} + \frac{1}{i\omega\rho_0}\frac{\partial \tilde{p}}{\partial x} = \frac{\mu}{i\omega\rho_0}\frac{\partial^2 \tilde{u}}{\partial y^2}. \tag{4b}$$

The boundary conditions are

$$\text{left: } \frac{1}{2y_0}\int\limits_{-y_0}^{y_0} \tilde{u}(-x_0, y)dy = U_0, \quad \tilde{v}(-x_0, y) = 0;$$

$$\text{right: } \tilde{u}(x_0, y) = 0, \quad \tilde{v}(x_0, y) = 0;$$

$$\text{bottom: } \tilde{u}(x, -y_0) = 0, \quad \tilde{v}(x, -y_0) = 0;$$

$$\text{upper: } \tilde{u}(x, y_0) = 0, \quad \tilde{v}(x, y_0) = 0.$$

Longitudinal velocity in arbitrary cross section. Entering the average value $\tilde{u}_{x0}(x) = -\frac{1}{i\omega\rho_0}\frac{\partial \tilde{p}}{\partial x}$, equation for velocity will take the form $\tilde{u} - \frac{\mu}{i\omega\rho_0}\frac{\partial^2 \tilde{u}}{\partial y^2} = \tilde{u}_{x0}(x)$. Its solution has the form $\tilde{u} = \tilde{u}_{x0}(x) + A(x)\cosh(i\lambda y) + B(x)\sinh(i\lambda y)$, where

$$\lambda = \sqrt{-\frac{i\omega\rho_0}{\mu}} = i\sqrt{\frac{i\omega\rho_0}{\mu}} = i\sqrt{\frac{\omega\rho_0}{\mu}}\frac{i+1}{\sqrt{2}} = \sqrt{\frac{\omega\rho_0}{2\mu}}(i-1) = \frac{i-1}{\delta}.$$

So, we have $\tilde{u} = \tilde{u}_{x0}(x) + A(x)\cosh(\beta y) - B(x)\sinh(\beta y)$, where $\beta = \frac{i+1}{\delta}$, $\delta = \sqrt{\frac{2\mu}{\omega\rho_0}}$.

The condition of the symmetry

$$\left.\frac{\partial \tilde{u}}{\partial y}\right|_{y=0} = \beta \left\{A(x)\sinh(\beta y) - B(x)\cosh(\beta y)\right\}|_{y=0} = 0,$$

gives $B(x) = 0$ and $\tilde{u} = \tilde{u}_{x0}(x) + A(x)\cosh(\beta y)$. Due to slip condition at $y = \pm y_0$

$$\tilde{u}_{x0} + A(x)\cosh(\pm\beta y_0) = 0,$$

one get $A(x) = -\frac{\tilde{u}_{x0}(x)}{\cosh(\beta y_0)}$ and $\tilde{u} = \tilde{u}_{x0}(x)\left(1 - \frac{\cosh(\beta y)}{\cosh(\beta y_0)}\right)$.

Introducing the cross-sectional averaged velocity $U(x) = \frac{1}{2y_0}\int\limits_{-y_0}^{y_0} \tilde{u}(x, y)dy$, and taking into account

$$\int\limits_{-y_0}^{y_0} \frac{\cosh(\beta y)}{\cosh(\beta y_0)}dy = \frac{\sinh(\beta y)|_{-y_0}^{y_0}}{\beta\cosh(\beta y_0)} = \frac{2}{\beta}\tanh(\beta y_0),$$

one gets $U(x) = (1 - f)\tilde{u}_{x_0}(x)$, where $f = \frac{\tanh(\beta y_0)}{\beta y_0}$. Thus the amplitude of the longitudinal velocity is

$$\tilde{u} = \frac{U(x)}{1-f}\left(1 - \frac{\cosh(\beta y)}{\cosh(\beta y_0)}\right).$$

Shear stress on the wall. Amplitude of the shear stress is

$$\tilde{\sigma} = \mu \left.\frac{\partial \tilde{u}}{\partial y}\right|_{y_0} = -\mu\frac{U(x)}{1-f}\beta^2 y_0\frac{\sinh(\beta y_0)}{\beta y_0\cosh(\beta y_0)} = -2i\mu\frac{U(x)}{1-f}\frac{y_0}{\delta^2}f.$$

Section-averaged continuity and momentum equations. Section-averaged equations (4) are

$$\frac{i\omega}{\rho_0 c_0^2}\tilde{p} + \frac{\partial U}{\partial x} = 0, \quad U(x) + \frac{1}{i\omega\rho_0}\frac{\partial\tilde{p}}{\partial x} = \frac{1}{i\omega\rho_0}\frac{\tilde{\sigma}}{y_0} \text{ or } U(x) + \frac{1}{i\omega\rho_0}\frac{\partial\tilde{p}}{\partial x} = -\frac{U(x)}{1-f}f,$$

finally

$$\frac{i\omega}{\rho_0 c_0^2}\tilde{p} + \frac{\partial U}{\partial x} = 0, \tag{5a}$$

$$\frac{U(x)}{1-f} + \frac{1}{i\omega\rho_0}\frac{\partial\tilde{p}}{\partial x} = 0. \tag{5b}$$

Longitudinal velocity. Differentiating the first equation in respect to x and eliminating the pressure gradient, the equation of averaged velocity takes the form

$$\frac{\partial^2 U}{\partial x^2} - \alpha^2 U = 0, \quad \text{where} \quad \alpha = \frac{i\omega/c_0}{\sqrt{1-f}}. \tag{6}$$

The general solution of the equation (6) is $U = A\cosh(\alpha x) + B\sinh(\alpha x)$. The boundary conditions are $U_{-x_0} = A\cosh(\alpha x_0) - B\sinh(\alpha x_0) = U_0$, $U_{x_0} = A\cosh(\alpha x_0) + B\sinh(\alpha x_0) = 0$, what yields $A = \frac{U_0}{2\cosh(\alpha x_0)}$, $B = -\frac{U_0}{2\sinh(\alpha x_0)}$. The amplitudes of section-averaged and longitudinal velocities are

$$U(x) = \frac{U_0}{2}\left(\frac{\cosh(\alpha x)}{\cosh(\alpha x_0)} - \frac{\sinh(\alpha x)}{\sinh(\alpha x_0)}\right), \quad \frac{\tilde{u}}{U_0} = \frac{0.5}{1-f}\left(\frac{\cosh(\alpha x)}{\cosh(\alpha x_0)} - \frac{\sinh(\alpha x)}{\sinh(\alpha x_0)}\right)\left(1 - \frac{\cosh(\beta y)}{\cosh(\beta y_0)}\right). \tag{7}$$

Pressure amplitude. From (5a) one can obtain

$$\tilde{p} = -\frac{\rho_0 c_0^2}{i\omega}\frac{\partial U}{\partial x} = -\frac{\rho_0 c_0 U_0}{2}\frac{1}{\sqrt{1-f}}\left(\frac{\sinh(\alpha x)}{\cosh(\alpha x_0)} - \frac{\cosh(\alpha x)}{\sinh(\alpha x_0)}\right),$$

finally

$$\frac{\tilde{p}}{\rho_0 c_0 U_0} = -\frac{0.5}{\sqrt{1-f}}\left(\frac{\sinh(\alpha x)}{\cosh(\alpha x_0)} - \frac{\cosh(\alpha x)}{\sinh(\alpha x_0)}\right). \tag{8}$$

Transverse velocity. The transverse velocity distribution is found by integrating the continuity equation (4a)

$$\frac{i\omega}{\rho_0 c_0^2} y\tilde{p} + \frac{\partial}{\partial x}\int_0^y \tilde{u}dy + \int_0^y \frac{\partial \tilde{v}}{\partial y}dy = 0$$

i.e.

$$\tilde{v} = -U_0\frac{0.5}{1-f}\alpha\left(\frac{\sinh(\alpha x)}{\cosh(\alpha x_0)} - \frac{\cosh(\alpha x)}{\sinh(\alpha x_0)}\right)\int_0^y\left(1 - \frac{\cosh(\beta y)}{\cosh(\beta y_0)}\right)dy - \frac{i\omega y}{\rho_0 c_0^2}\tilde{p}.$$

Eliminating pressure term one gets

$$\tilde{v} = -U_0\frac{0.5}{1-f}\alpha\left(\frac{\sinh(\alpha x)}{\cosh(\alpha x_0)} - \frac{\cosh(\alpha x)}{\sinh(\alpha x_0)}\right)\left(y - y_0\frac{\sinh(\beta y)}{\beta y_0\cosh(\beta y_0)}\right) + y\frac{U_0}{2}\alpha\left(\frac{\sinh(\alpha x)}{\cosh(\alpha x_0)} - \frac{\cosh(\alpha x)}{\sinh(\alpha x_0)}\right)$$

finally

$$\tilde{v} = \frac{U_0}{2}\alpha y_0\frac{f}{1-f}\left(\frac{\sinh(\alpha x)}{\cosh(\alpha x_0)} - \frac{\cosh(\alpha x)}{\sinh(\alpha x_0)}\right)\left(\frac{\sinh(\beta y)}{\sinh(\beta y_0)} - \frac{y}{y_0}\right). \tag{9}$$

Real quantities. For arbitrary phase ωt the velocity and pressure are determined from (7)–(9):

$$\frac{u}{U_0} = \Im m\left[\frac{0.5}{1-f}\left(\frac{\cosh(\alpha x)}{\cosh(\alpha x_0)} - \frac{\sinh(\alpha x)}{\sinh(\alpha x_0)}\right)\left(1 - \frac{\cosh(\beta y)}{\cosh(\beta y_0)}\right)e^{i\omega t}\right],$$

$$\frac{v}{U_0} = \Im m\left[\alpha y_0\frac{0.5f}{1-f}\left(\frac{\sinh(\alpha x)}{\cosh(\alpha x_0)} - \frac{\cosh(\alpha x)}{\sinh(\alpha x_0)}\right)\left(\frac{\sinh(\beta y)}{\sinh(\beta y_0)} - \frac{y}{y_0}\right)e^{i\omega t}\right],$$

$$\frac{p - p_0}{\rho_0 c_0 U_0} = -\Im m\left[\frac{0.5}{\sqrt{1-f}}\left(\frac{\sinh(\alpha x)}{\cosh(\alpha x_0)} - \frac{\cosh(\alpha x)}{\sinh(\alpha x_0)}\right)e^{i\omega t}\right],$$

where

$$\alpha = \frac{i\omega/c_0}{\sqrt{1-f}}, \quad f = \frac{\tanh(\beta y_0)}{\beta y_0}, \quad \beta = \frac{i+1}{\delta}, \quad \delta = \sqrt{\frac{2\mu}{\omega\rho_0}}.$$

Although the drive frequency ω in this solution is arbitrary, it is common to consider excitation at the lowest resonance frequency ω_1. For determining ω_1 as a function of y_0/δ we have to find the value of ω at which the longitudinal component of the acoustic particle velocity is maximized in the center of the resonator. To evaluate $|\tilde{u}(0,0)|$ for this purpose, which corresponds to maximizing the quantity $\tilde{u} = \frac{0.5U_0}{1-f}\left(\frac{\cosh(\alpha x)}{\cosh(\alpha x_0)} - \frac{\sinh(\alpha x)}{\sinh(\alpha x_0)}\right)\left(1 - \frac{\cosh(\beta y)}{\cosh(\beta y_0)}\right)$ as a function of ω/Ω. Values of ω_1/Ω obtained in this way are shown in Fig. 2, and they are seen to differ from unity only slightly over the domain of interest. With ω_1/Ω, the solutions can be plotted as functions of x/x_0 and y/y_0 depending on the sole parameter y_0/δ.

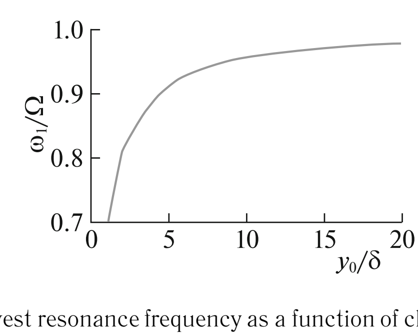

Fig. 2. Lowest resonance frequency as a function of channel width.

3. 1D-LINEAR MODEL

For the case of 1D-resonator $0 \le x \le L = 2x_0$ from (4) follows:

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + k^2 \tilde{u} = 0, \quad \frac{\partial^2 \tilde{p}}{\partial x^2} + k^2 \tilde{p} = 0, \quad \text{where} \quad k = \frac{\omega}{c_0} \ \text{is wave number.}$$

Boundary conditions: $\tilde{u}(0) = U_0$, $\tilde{u}(L) = 0$. The solution of this problem is

$$\frac{u}{U_0} = \sin(\omega t - kx) - \sin(\omega t - kL)\frac{\sin(kx)}{\sin(kL)},$$

$$\frac{p - p_0}{\rho_0 c_0 U_0} = \sin(\omega t - kx) - \cos(\omega t - kL)\frac{\cos(kx)}{\sin(kL)},$$

where $L = 2x_0$ is length of closed resonator.

4. 1D-NONLINEAR MODEL

If piston amplitudes are not low, the Lagrange's approach is used. Further an 1D closed resonator $0 \le x \le L + l\cos(\omega t)$ is considered with piston oscillating on the right, $L = 2x_0$ is cycle-averaged length of resonator, $l = U_0/\omega$ is piston displacement amplitude. Instead of system (1) the nonlinear equations in Lagrangian coordinate are used [5]:

$$\rho x_\xi = \rho_0, \quad \rho_0 u_t = -p_\xi + \sigma_\xi, \quad dp = c^2 d\rho,$$

where $x = x(\xi, t)$ is the Eulerian coordinate of gas particle at time t, ξ is the Lagrangian coordinate, $u = x_t$ is the velocity, ρ, ρ_0 are gas density and its initial value respectively, p is the pressure, c is the local sound speed,

$$\sigma = (2\mu + \lambda) \left.\frac{\partial u}{\partial x}\right|_{t=const} = \left(\frac{4}{3}\mu + \zeta\right) \left.\frac{\partial u}{\partial x}\right|_{t=const} = \left(\frac{4}{3}\mu + \zeta\right) \left.\frac{\partial u}{\partial \xi}\right|_{t=const} \Bigg/ \left.\frac{\partial x}{\partial \xi}\right|_{t=const}$$

is viscous stress, λ is Lame parameter, $\zeta = \lambda + 2/3\mu$ is second viscosity (according to the Stokes hypothesis $\zeta = 0$). Using the sound speed $c^2 = \gamma p/\rho = c_0^2/x_\xi^{\gamma-1}$, eliminating density and pressure $dp = c^2 d\rho$, one obtains the nonlinear wave equation for viscous polytropic gas [6]:

$$x_{tt} = \frac{c_0^2}{x_\xi^{\gamma+1}} x_{\xi\xi} + \frac{\mu}{\rho_0}\left(\frac{4}{3} + \frac{\zeta}{\mu}\right)\left(\frac{x_{t\xi}}{x_\xi}\right)_\xi. \tag{10}$$

One have to find the solution of (10) in the domain $\xi \in [0, L + l]$. The boundary conditions are $x(0, t) = 0$, $x(L, t) = L + l\cos(\omega t)$, initial conditions: $x(\xi, 0) = \xi$, $x_t(\xi, 0) = 0$.

5. RESULTS

Although the drive frequency ω in 2D model is arbitrary, it is common to consider excitation at the lowest resonance frequency ω_1. For determining ω_1 as a function of y_0/δ we find the value of ω at which the longitudinal component of the velocity in the center of resonator $|\tilde{u}(0, 0)|$ has maximum. In other words, we maximize the quantity $\tilde{u} = \frac{0.5U_0}{1-f}\frac{1}{\cosh(\alpha x_0)}\left(1 - \frac{1}{\cosh(\beta y_0)}\right)$ as a function of ω. In Fig. 2 the dependence of lowest resonance frequency ω_1/Ω on the resonator width y_0/δ is shown. Here $\Omega = \pi c_0/(2x_0)$ is the fundamental resonant frequency, corresponding to the resonator filled with inviscid fluid.

For all three models computations of pressure amplitude for frequencies in the range $\omega/\Omega \in [0.96, 1.06]$ are carried out at parameters: $L = 2x_0 = 106$ cm, $2y_0 = 2.4$ cm , $\nu = 0.15$ cm^2/s, $\rho_0 = 0.00129$ g/cm^3, $\mu = \rho_0\nu$, $l = 0.03$ cm, $U_0 = l\omega$.

For three models the dependencies of the dimensionless pressure amplitude $\frac{1}{2}\frac{p_{max} - p_{min}}{\rho_0 c_0 U_0}$ on the frequency ratio ω/Ω are depicted in Fig. 3. The values $p_{max} = \max\limits_t p(0, t)$ and $p_{min} = \min\limits_t p(0, t)$ are

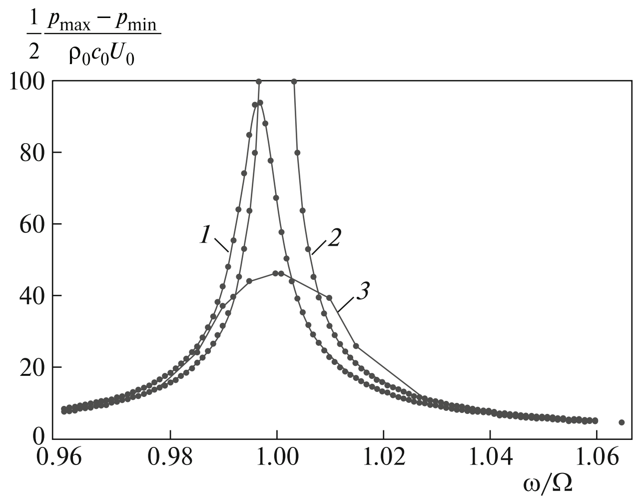

Fig. 3. Resonance curves: (*1*) 2D analytical solution, (*2*) 1D linear inviscid model, (*3*) 1D nonlinear model.

pressure maximum and minimum at $x = 0$ respectively. The 2D analytical solution (*1*) yields finite resonant value of pressure amplitude equal to 95 at $\omega/\Omega = 0.996$. For the 1D linear model (*2*) pressure amplitude grows infinitely at fundamental resonant frequency $\omega/\Omega = 1$. For the 1D nonlinear model (*3*) pressure amplitude reaches maximum equal to the value 45 at $\omega/\Omega = 1$. Its maximum is about two times lower than maximum for 2D model at the air viscosity $\nu = 0.15$ cm^2/s. For the 2D model the maximum pressure amplitude depends on gas viscosity. At hypothetical viscosity $\nu = 0.015$ cm^2/s the curve 1 becomes close to the curve *2*.

For the 1D nonlinear model shock waves appear at frequencies in the range of $\omega/\Omega \in [0.98, 1.02]$. Moreover, in the range of $\omega/\Omega \in [0.99, 1.01]$ no pressure beats on piston are observed. Out of this range, the beats of pressure occur in the transition period during 50–100 cycles. It is remarkable that for 1D nonlinear model the pressure oscillations are not symmetric about p_0.

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