PHYC90045 Introduction to Quantum Computing

Lab 2

Welcome to Lab 2 of PHYC90045 Introduction to Quantum Computing, covering exercises relevant to the material presented in lectures in Week 2.

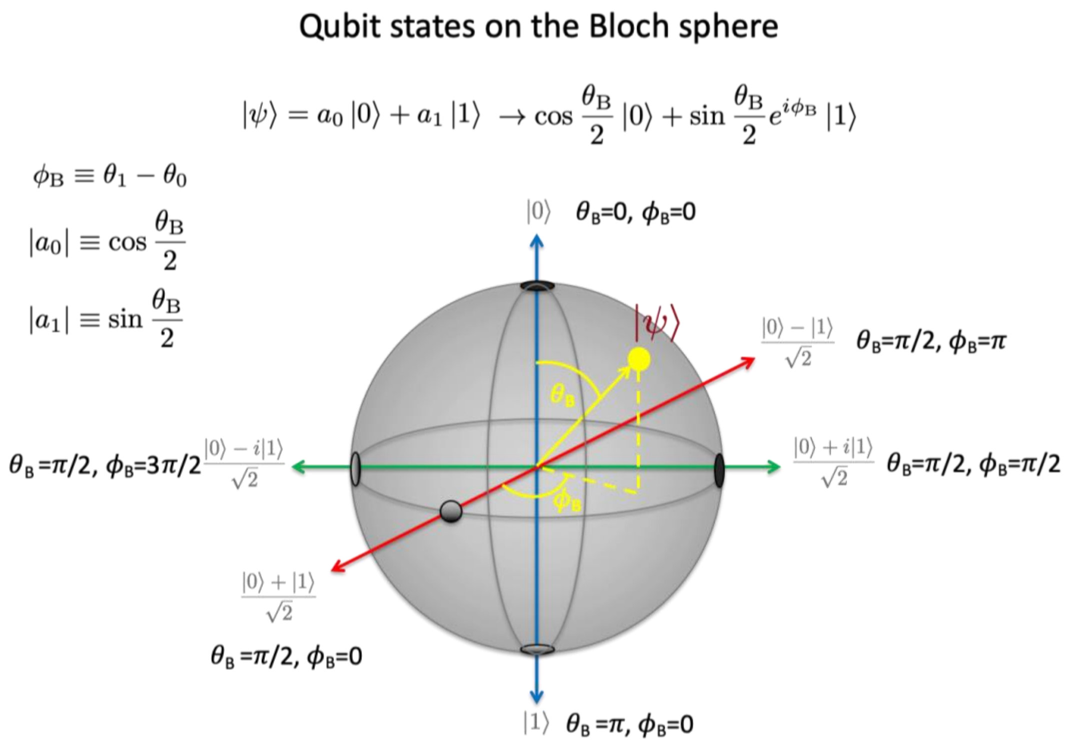
The purpose of this week's exercises is to:

- understand the representation of qubit states on the Bloch Sphere
- understand sequences of logic gates on single qubits
- program sequences of qubit logic gates in QUI
- understand the evolution of qubit states on the Bloch Sphere

The exercises in these notes are to assist your understanding of the subject and may require time outside of the lab to complete.

Qubit states on the Bloch sphere

In lectures we showed how a general qubit state $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$ can be represented as a point on the Bloch sphere. The conversion (ignoring the global phase) is given below:



Exercise 1.1 Consider the state $|\psi\rangle = \frac{1-i}{2}|0\rangle + \frac{1+i}{2}|1\rangle$. We'll go through the whole conversion from this form to the position on the Bloch sphere.

a) Show that the state is appropriately normalised, i.e. that the sum of the probabilities, $|a_0|^2 + |a_1|^2$ is unity.

b) Convert the state to polar form, i.e. $|\psi\rangle = |a_0|e^{i\theta_0}|0\rangle + |a_1|e^{i\theta_1}|1\rangle$.

$|a_0| =$ ________ $\theta_0 =$ ________ $|a_1| =$ ________ $\theta_1 =$ ________

c) Convert the state into "pre-Bloch sphere" form, by pulling out the phase $e^{i\theta_0}$, i.e.

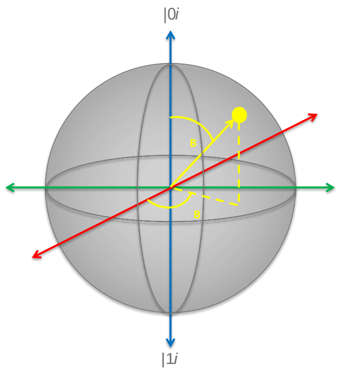
$|\psi\rangle = e^{i\theta_0}\left(|a_0||0\rangle + |a_1|e^{i(\theta_1-\theta_0)}|1\rangle\right)$.

$|\psi\rangle =$

d) From the definition of the Bloch sphere form, $|\psi\rangle = \cos\frac{\theta_B}{2}|0\rangle + \sin\frac{\theta_B}{2}e^{i\phi_B}|1\rangle$, determine the Bloch sphere angles $\phi_B = \theta_1 - \theta_0$ and θ_B, and write out the state in this form:

$\phi_B =$ ________ $\theta_B =$ ________ $|\psi\rangle =$ ________ $|0\rangle +$ ________ $|1\rangle$

e) Given the definitions of the Bloch angles in the schematic (right), plot a point corresponding where this state resides on the Bloch sphere. Compare with the figure on page 1.



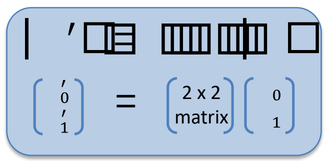
Exercise 1.2 Make up your own (normalised) state, and repeat all the steps a)-e) above.

Logical operations on qubits

In lectures, we introduced the notion of an operator U acting on a quantum state to transform it into a new state, i.e.

$$|\psi'\rangle = U\,|\psi\rangle$$

In quantum computing these operations correspond to logic operations, or gates, on qubits. The matrix representation gives us a useful representation of these logic operations:



Here is some practice with the maths involving the entire QUI logic gate library for single qubits, in both matrix and ket form.

Exercise 2.1 Compute by hand the single gate operations H, X, Y, Z, S, and T on the state $|0\rangle$, and complete the table below. Compare with the QUI in each case.

Gate	Operator (matrix rep)	Operation (matrix rep)	Operation (ket rep)	Final state $\|0\rangle$ amplitude	Final state $\|1\rangle$ amplitude	Final state probabilities p_0 and p_1
H	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$H\|0\rangle$ $= \frac{1}{\sqrt{2}}\|0\rangle + \frac{1}{\sqrt{2}}\|1\rangle$ $= \|a_0\|e^{i\theta_0}\|0\rangle + \|a_1\|e^{i\theta_1}\|1\rangle$	$\|a_0\| = \frac{1}{\sqrt{2}}$ $\theta_0 = 0$	$\|a_1\| = \frac{1}{\sqrt{2}}$ $\theta_1 = 0$	$p_0 = 0.5$ $p_1 = 0.5$
X	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$X\|0\rangle = \|1\rangle$	$\|a_0\| = 0$ $\theta_0 = 0$	$\|a_1\| = 1$ $\theta_1 = 0$	$p_0 = 0$ $p_1 = 1$
Y	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$					
Z	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$					
S	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$					
T	$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$					

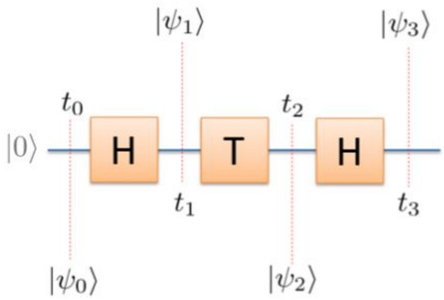
Exercise 2.2 Compute by hand the single gate operations H, X, Y, Z, S, and T on the state $|1\rangle$, and complete the table below. Compare with the QUI in each case.

Gate	Operator (matrix rep)	Operation (matrix rep)	Operation (ket rep)	Final state $\|0\rangle$ amplitude	Final state $\|1\rangle$ amplitude	Final state probabilities p_0 and p_1
H	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$H\|1\rangle$ $= \frac{1}{\sqrt{2}}\|0\rangle - \frac{1}{\sqrt{2}}\|1\rangle$ $= \|a_0\|e^{i\theta_0}\|0\rangle + \|a_1\|e^{i\theta_1}\|1\rangle$	$\|a_0\| = \frac{1}{\sqrt{2}}$ $\theta_0 = 0$	$\|a_1\| = \frac{1}{\sqrt{2}}$ $\theta_1 = \pi$	$p_0 = \frac{1}{2}$ $p_1 = \frac{1}{2}$
X	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$X\|1\rangle = \|0\rangle$	$\|a_0\| = 1$ $\theta_0 = 0$	$\|a_1\| = 0$ $\theta_1 = 0$	$p_0 = 1$ $p_1 = 0$
Y	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$					
Z	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$					
S	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$					
T	$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$					

Sequences of logic gates

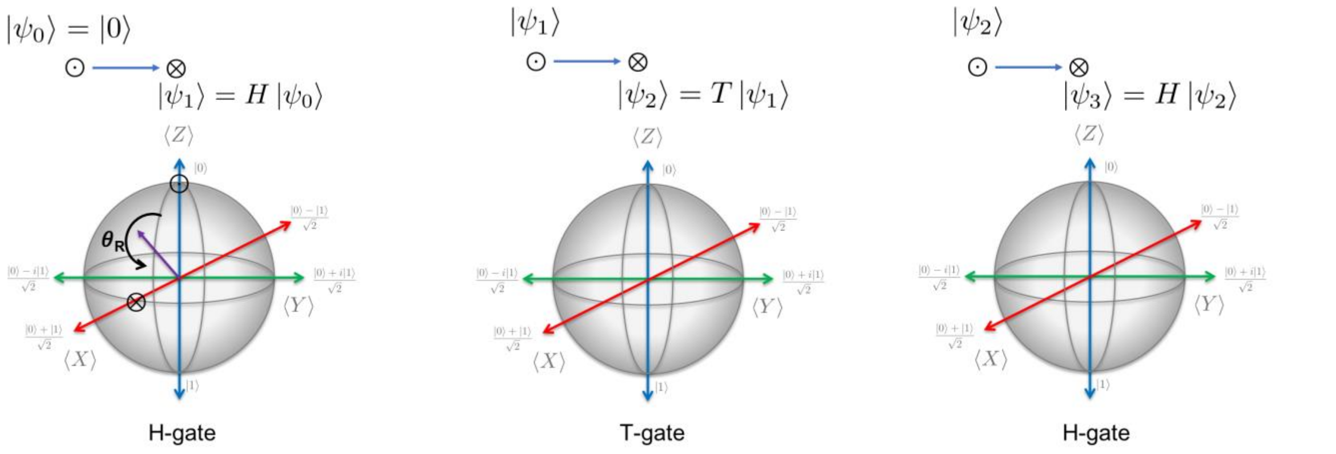
In the following exercises we will look at the mathematics of the quantum state evolution in more detail. In order to understand what is happening we will compute some examples by hand and compare with the QUI output.

Exercise 3.1 Program the following sequence of single qubit gates H-T-H (shown below). Compute by hand the states at each time step in the matrix representation, covert to ket representation and fill out the table below. Now compare the amplitudes you obtained with the QUI output at each time step and check they agree.



Matrix representation	Ket representation	Amplitudes
$\|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\|\psi_0\rangle = \|0\rangle$	$\|a_0\| = 1$ $\|a_1\| = 0$ $\theta_0 = 0$ $\theta_1 = 0$
$\|\psi_1\rangle = H\|\psi_0\rangle$		
$\|\psi_2\rangle = T\|\psi_1\rangle$		
$\|\psi_3\rangle = H\|\psi_2\rangle$		

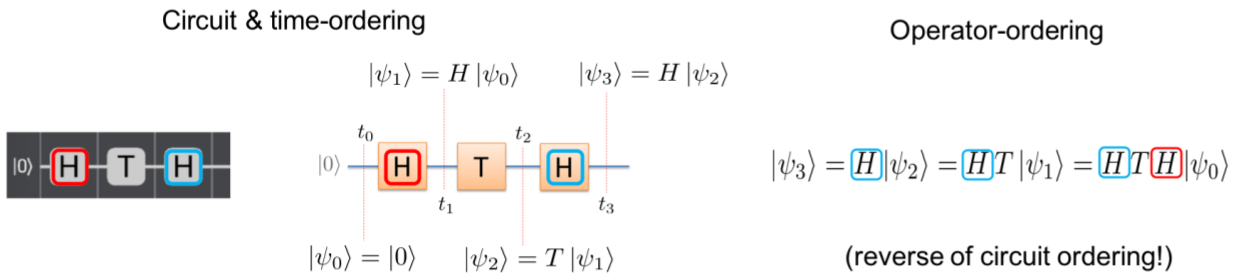
Exercise 3.2 Examine the QUI Bloch sphere animations by hovering the mouse over each gate in the program circuit and complete the following representations of these gates in the sequence H-T-H as per below (i.e. plot ⊙ and ⊗):



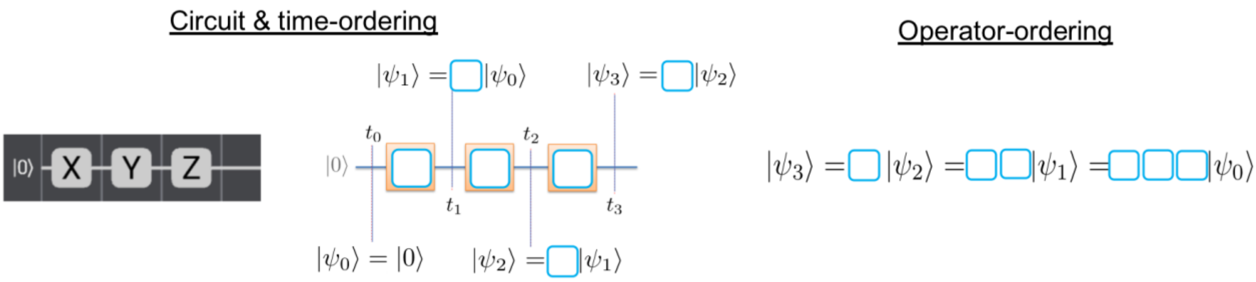
Exercise 3.3 Writing the above example as a string of operations on the initial state would look like the following:

$$|\psi_3\rangle = H\,|\psi_2\rangle = H\,T\,|\psi_1\rangle = H\,T\,H\,|\psi_0\rangle$$

This looks exactly like the circuit ordering, but that's because this example is palindromic (looks the same from either direction) – see below.

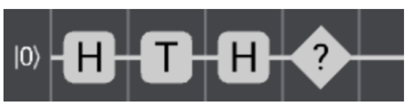


Complete the same analysis for the circuit comprising the combination X-Y-Z:



The reversal of operator and time orderings is something to keep in mind.

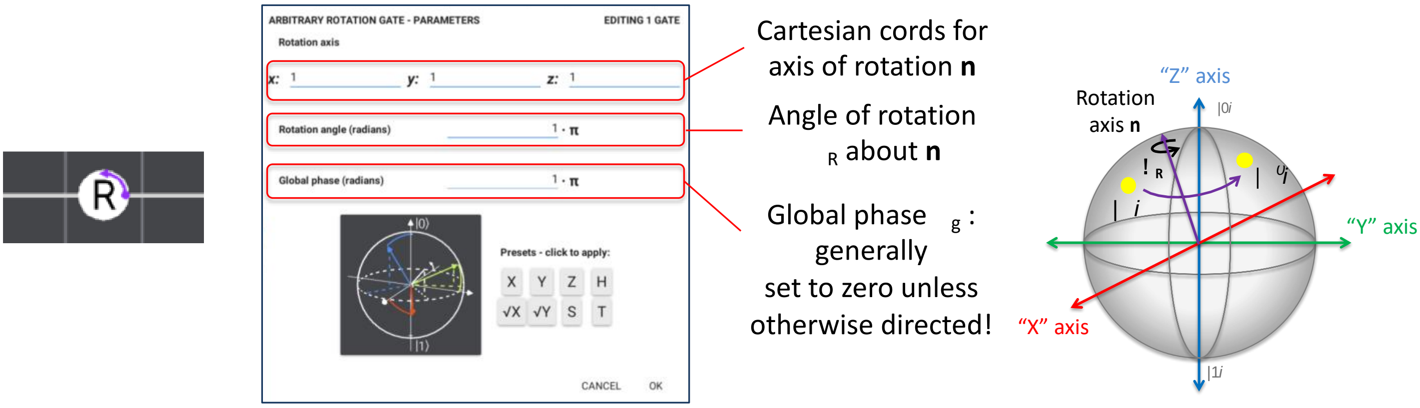
Exercise 3.4 Add a measurement gate at the end of the HTH sequence. Hit the compute button many times (say N = 100) and record the number of 0 and 1 outcomes and fill in the table below. Compare the estimated probabilities with those expected.



$\vert\psi\rangle = HTH\vert 0\rangle$ components	Exact probability	Measurement record	# outcomes, n	Estimated Prob = n/N
$\vert 0\rangle$	0.854	$\cancel{\vert\vert\vert\vert}\ \cancel{\vert\vert\vert\vert}\ \cdots$		
$\vert 1\rangle$	0.146	$\cancel{\vert\vert\vert\vert}\ \cdots$		

Arbitrary rotation gate, R

Consider the R-gate in the QUI, with edit menu given below (right clock on the circuit symbol to bring up this menu):



Exercise 4.1 Set QUI to 1-qubit, initialised in the default zero state. Add an R-gate in the first time block. Setting the global phase to zero, explore the action of the R-gate for a range of rotation axes and rotation angles. Hovering the mouse over the R-gate make sure you understand how the qubit state evolves on the Bloch sphere. You can start the system from states different from the default by adding gates prior to the R-gate. If you would like to see the Bloch animations go crazy, try some very high rotation angles.

Exercise 4.2 Going back to the HTH example, can you determine R-gate parameters to place the system, initially in the default zero state, into the same final state?