Homework 5 in 18.06 Due on Gradescope Sunday April 9, at 11:59 p.m.

1. If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are orthonormal vectors in $\mathbf{R}^5$, what combination $\_\_\_\_\ \boldsymbol{q}_1 + \_\_\_\_\ \boldsymbol{q}_2$ is closest to a given vector $\boldsymbol{b}$?

This is Problem 4.4.8 on page 186 of ILA6.

2. What multiple of $\boldsymbol{a} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ should be subtracted from $\boldsymbol{b} = \left[\begin{smallmatrix}4\\0\end{smallmatrix}\right]$ to make the result $\boldsymbol{B}$ orthogonal to $\boldsymbol{a}$? Sketch a figure to show $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{B}$.

Complete the Gram-Schmidt process in Problem 13 by computing $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\|$ and $\boldsymbol{q}_2 = \boldsymbol{B}/\|\boldsymbol{B}\|$ and factoring into QR:

$$\begin{bmatrix}1 & 4\\ 1 & 0\end{bmatrix} = \begin{bmatrix}\boldsymbol{q}_1 & \boldsymbol{q}_2\end{bmatrix}\begin{bmatrix}\|\boldsymbol{a}\| & ?\\ 0 & \|\boldsymbol{B}\|\end{bmatrix}.$$

This is Problem 4.4.13 ***AND*** Problem 4.4.14 on page 187 of ILA6.

3. Find $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ (orthonormal) as combinations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ (independent columns). Then write A as QR:

$$A = \begin{bmatrix}1 & 2 & 4\\ 0 & 0 & 5\\ 0 & 3 & 6\end{bmatrix}.$$

This is Problem 4.4.23 on page 188 of ILA6.

4. Choose c so that Q is an orthogonal matrix:

$$Q = c\begin{bmatrix}1 & -1 & -1 & -1\\ -1 & 1 & -1 & -1\\ -1 & -1 & 1 & -1\\ -1 & -1 & -1 & 1\end{bmatrix}.$$

Project $\boldsymbol{b} = (1,1,1,1)$ onto the first column. Then project $\boldsymbol{b}$ onto the plane of the first two columns.

This is Problem 4.4.31 on page 189 of ILA6.

5. If you add row 1 $= \begin{bmatrix}a & b & c\end{bmatrix}$ to row 2 $= \begin{bmatrix}p & q & r\end{bmatrix}$ to get $\begin{bmatrix}p+a & q+b & r+c\end{bmatrix}$ in row 2, show from formula (1) for $\det A$ that the 3 by 3 determinant *does not change*. Here is another approach to the rule for adding two rows:

$$\det\begin{bmatrix}\text{row 1}\\ \textbf{row 1+row 2}\\ \text{row 3}\end{bmatrix} = \det\begin{bmatrix}\text{row 1}\\ \textbf{row 1}\\ \text{row 3}\end{bmatrix} + \det\begin{bmatrix}\text{row 1}\\ \textbf{row 2}\\ \text{row 3}\end{bmatrix} = \mathbf{0} + \det\begin{bmatrix}\text{row 1}\\ \text{row 2}\\ \text{row 3}\end{bmatrix}$$

This is Problem 5.1.7 on page 204 of ILA6.

6. Do these matrices have determinant $0, 1, 2$, or 3?

$$A = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \quad B = \begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix} \quad C = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \quad D = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}.$$

This is Problem 5.1.9 on page 204 of ILA6.

7. Show that $\det A = 0$, regardless of the five numbers marked by x's :

$$A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}.$$

What are the cofactors of row 1?
What is the rank of A?
What are the 6 terms in $\det A$?

This is Problem 5.2.3 on page 209 of ILA6.

8. *Quick proof of Cramer's rule.* The determinant is a linear function of column 1. It is zero if two columns are equal. When $\boldsymbol{b} = A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3$ goes into the first column of A, we have the matrix B_1 and Cramer's Rule $x_1 = \det B_1 / \det A$:

$$|\boldsymbol{b} \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3| = |x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3| = x_1|\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3| = x_1 \det A.$$

What steps lead to the middle equation?

This is Problem 5.2.8 on page 209 of ILA6.

9. (prize for the max determinant) If a 3 by 3 matrix has entries $1, 2, 3, 4, \ldots, 9$, what is the maximum determinant ? I would use a computer to decide. This problem does not seem easy.

This is Problem 5.2.11 on page 210 of ILA6.