Homework 6 in 18.06 Due on Gradescope Sunday April 16 at 11:59 p.m.

1. The example at the start of the chapter has powers of this matrix A:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Find the eigenvalues of these matrices. All powers have the same eigenvectors. Show from A how a row exchange can produce different eigenvalues.

This is Problem 6.1.1 on page 226 of ILA6.

2. Find three eigenvectors for this matrix P (projection matrices have $\lambda = 1$ and 0):

$$\textbf{Projection matrix} \qquad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

This is Problem 6.1.12 on page 228 of ILA6.

3. A 3 by 3 matrix B is known to have eigenvalues $0, 1, 2$. This information is enough to find three of these (give the answers where possible):

(a) the rank of B

(b) the determinant of $B^{\mathrm{T}}B$

(c) the eigenvalues of $B^{\mathrm{T}}B$

(d) the eigenvalues of $(B^2 + I)^{-1}$.

This is Problem 6.1.19 on page 229 of ILA6.

4. This matrix is singular with rank one. Find three λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

This is Problem 6.1.24 on page 229 of ILA6.

5. Find the rank and the four eigenvalues of A and C:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

This is Problem 6.1.27 on page 229 of ILA6.

6. Suppose A has eigenvalues $0, 3, 5$ with independent eigenvectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

   (a) Give a basis for the nullspace and a basis for the column space.

   (b) Find a particular solution to $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Find all solutions.

   (c) $A\boldsymbol{x} = \boldsymbol{u}$ has no solution. If it did then ____ would be in the column space.

   This is Problem 6.1.32 on page 230 of ILA6.

7. (a) Factor these two matrices into $A = X\Lambda X^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

   (b) If $A = X\Lambda X^{-1}$ then $A^3 = (\quad)(\quad)(\quad)$ and $A^{-1} = (\quad)(\quad)(\quad)$.

   This is Problem 6.2.1 on page 242 of ILA6.

8. True or false : If the columns of X (eigenvectors of A) are linearly independent, then

   (a) A is invertible (b) A is diagonalizable

   (c) X is invertible (d) X is diagonalizable.

   This is Problem 6.2.4 on page 242 of ILA6.

9. $A^k = X\Lambda^k X^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ____. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

   (Recommended) Find Λ and X to diagonalize A_1 in the above problem. What is the limit of Λ^k as $k \to \infty$? What is the limit of $X\Lambda^k X^{-1}$? In the columns of this limiting matrix you see the ____.

   This is Problem 6.2.15 ***AND*** Problem 6.2.16 on page 243 of ILA6.

10. Show that trace XY = trace YX, by adding the diagonal entries of XY and YX :

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} q & r \\ s & t \end{bmatrix}.$$

   Now choose Y to be ΛX^{-1}. Then $X\Lambda X^{-1}$ has the same trace as $\Lambda X^{-1}X = \Lambda$. This proves that *the trace of A equals the trace of Λ = the sum of the eigenvalues.* $\boldsymbol{AB - BA = I}$ **is impossible** since the left side has trace = ____.

   This is Problem 6.2.21 on page 244 of ILA6.

11. (a) If $A = \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]$ then the determinant of $A - \lambda I$ is $(\lambda - a)(\lambda - d)$. Check the "Cayley-Hamilton Theorem" that $(A - aI)(A - dI) =$ *zero matrix*.

   (b) Test the Cayley-Hamilton Theorem on Fibonacci's $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The theorem predicts that $A^2 - A - I = 0$, since the polynomial $\det(A - \lambda I)$ is $\lambda^2 - \lambda - 1$.

   This is Problem 6.2.29 on page 244 of ILA6.