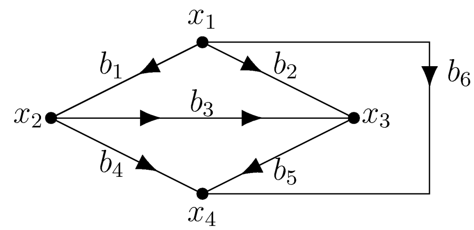
18.06 Homework 4 Due Sunday March 12 on Gradescope at $11:59$ pm

1. Find the **incidence matrix** and its rank and one vector in each subspace for this complete graph—all six edges included.

   This is Problem 3.5.31 on page 140 of ILA6.



2. **If $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ then $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$.** Reason: $A\boldsymbol{x}$ is in the nullspace of A^{T} and also in the ____ of A and those spaces are ____. *Conclusion*: $A\boldsymbol{x} = \mathbf{0}$ and therefore $A^{\mathrm{T}}A$ *has the same nullspace as* A. *This key fact will be repeated when we need it.*

   This is Problem 4.1.9 on page 149 of ILA6.

3. Compute the projection matrices $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto the lines through $\boldsymbol{a}_1 = (-1, 2, 2)$ and $\boldsymbol{a}_2 = (2, 2, -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is.

   This is Problem 4.2.5 on page 159 of ILA6.

4. (Quick and Recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\boldsymbol{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

   This is Problem 4.2.14 on page 160 of ILA6.

5. What linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ is closest to $\boldsymbol{b} = (2, 1, 1)$?

   This is Problem 4.2.17 on page 161 of ILA6.

6. To find the projection matrix onto the plane $x - y - 2z = 0$, choose two vectors in that plane and make them the columns of A. The plane will be the column space of A! Then compute $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

   OR

   To find the projection matrix P onto the same plane $x-y-2z = 0$, write down a vector $\boldsymbol{e}$ that is perpendicular to that plane. Compute the projection $Q = \boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}/\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}$ and then $P = I - Q$.

   This is Problem 4.2.20 ***OR*** Problem 4.2.21 on page 161 of ILA6.

7. The first three Chebyshev polynomials are given by

   $T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1$

   (a) Show that $\{T_0, T_1, T_2\}$ is a basis for $\mathbb{P}_2$, the space of polynomials of degree ≤ 2 (more generally, the space of polynomials of degree $\leq n$ is denoted by $\mathbb{P}_n$).

   (b) Check that differentiation defines a linear transformation $T_D : \mathbb{P}_2 \to \mathbb{P}_1$ and write down the matrix of each linear transformation in the Chebyshev basis. Similarly, check that integration is a linear transformation $T_S = \mathbb{P}_1 \to \mathbb{P}_2$.

   (c) Let D and S be the differentiation and integration matrices from part (b). Compute the matrix products DS and SD. Interpret the results using calculus: choose a suitable polynomial in $\mathbb{P}_2$, differentiate it, and then integrate it.

   (d) Write down bases for the null spaces and column spaces of D and S. Provide the corresponding polynomials. Can you interpret your results about D and S in light of what you know about differentiation and integration from calculus?

   This problem is not in the ILA6 textbook.