Homework 8 in 18.06 Due on Gradescope Sunday May 7 at 11:59 p.m.

To find the SVD $A = U\Sigma V^{\mathrm{T}}$ by hand, here are the steps from page 291.

Find $\boldsymbol{U}$ and $\boldsymbol{\Sigma}$ and $\boldsymbol{V}$ for our original $\boldsymbol{A} = \begin{bmatrix} \mathbf{5} & \mathbf{4} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}$.

With rank 2, this A has two positive singular values σ_1 and σ_2. We will see that σ_1 is larger than $\lambda_{\max} = 5$, and σ_2 is smaller than $\lambda_{\min} = 3$. Begin with $A^{\mathrm{T}}A$ and AA^{T}:

$$\boldsymbol{A^{\mathrm{T}}A} = \begin{bmatrix} \mathbf{25} & \mathbf{20} \\ \mathbf{20} & \mathbf{25} \end{bmatrix} \qquad \boldsymbol{AA^{\mathrm{T}}} = \begin{bmatrix} \mathbf{41} & \mathbf{12} \\ \mathbf{12} & \mathbf{9} \end{bmatrix}$$

Those have the same trace $\lambda_1 + \lambda_2 = 50$ and the same eigenvalues $\lambda_1 = \sigma_1^2 = 45$ and $\lambda_2 = \sigma_2^2 = 5$. The square roots are $\boldsymbol{\sigma_1} = \sqrt{\mathbf{45}} = \mathbf{3}\sqrt{\mathbf{5}}$ and $\boldsymbol{\sigma_2} = \sqrt{\mathbf{5}}$. Then σ_1 times σ_2 equals 15, and this is the determinant of A. The next step is to find V.

The key to V is to find the eigenvectors of $A^{\mathrm{T}}A$ (with eigenvalues 45 and 5):

$$\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \mathbf{45}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \qquad \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}\begin{bmatrix} \mathbf{-1} \\ \mathbf{1} \end{bmatrix} = \mathbf{5}\begin{bmatrix} \mathbf{-1} \\ \mathbf{1} \end{bmatrix}$$

Then $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are those orthogonal eigenvectors rescaled to length 1. Divide by $\sqrt{2}$.

$$\textbf{Right singular vectors} \quad \boldsymbol{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \text{ and } \boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{-1} \\ \mathbf{1} \end{bmatrix} \text{ (as predicted)}$$

The left singular vectors are $\boldsymbol{u}_1 = A\boldsymbol{v}_1/\sigma_1$ and $\boldsymbol{u}_2 = A\boldsymbol{v}_2/\sigma_2$. **Multiply $\boldsymbol{v_1}, \boldsymbol{v_2}$ by $\boldsymbol{A}$:**

$$A\boldsymbol{v}_1 = \frac{3}{\sqrt{2}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \sqrt{45}\frac{\mathbf{1}}{\sqrt{\mathbf{10}}}\begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix} = \sigma_1 \boldsymbol{u}_1$$

$$A\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \sqrt{5}\frac{\mathbf{1}}{\sqrt{\mathbf{10}}}\begin{bmatrix} \mathbf{-3} \\ \mathbf{1} \end{bmatrix} = \sigma_2 \boldsymbol{u}_2$$

The division by $\sqrt{10}$ makes $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ unit vectors. Then $\sigma_1 = \sqrt{45}$ and $\sigma_2 = \sqrt{5}$ as expected. The Singular Value Decomposition of A is $\boldsymbol{U}$ times $\boldsymbol{\Sigma}$ times $\boldsymbol{V^{\mathrm{T}}}$. (**Not $\boldsymbol{V}$**.)

$$\boxed{\boldsymbol{U} = \frac{1}{\sqrt{10}}\begin{bmatrix} \mathbf{1} & \mathbf{-3} \\ \mathbf{3} & \mathbf{1} \end{bmatrix} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sqrt{\mathbf{45}} & \\ & \sqrt{\mathbf{5}} \end{bmatrix} \quad \boldsymbol{V^{\mathrm{T}}} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}}$$

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1. Find the eigenvalues and the singular values of this 2 by 2 matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \quad \text{with} \quad A^{\mathrm{T}}A = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \quad \text{and} \quad AA^{\mathrm{T}} = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}.$$

The eigenvectors $(1, 2)$ and $(1, -2)$ of A are not orthogonal. How do you know the eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ of $A^{\mathrm{T}}A$ will be orthogonal? Notice that $A^{\mathrm{T}}A$ and AA^{T} have the same eigenvalues $\lambda_1 = 25$ and $\lambda_2 = 0$.

This is Problem 7.1.5 on page 295 of ILA6.

2. Find $A^{\mathrm{T}}A$ and AA^{T} and the singular vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{u}_1, \boldsymbol{u}_2$ for A:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{has rank } r = 2. \quad \text{The eigenvalues are } 0, 0, 0.$$

Check the equations $A\boldsymbol{v}_1 = \sigma_1 \boldsymbol{u}_1$ and $A\boldsymbol{v}_2 = \sigma_2 \boldsymbol{u}_2$ and $A = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^{\mathrm{T}}$. If you remove row 3 of A (all zeros), show that σ_1 and σ_2 don't change.

This is Problem 7.1.1 on page 295 of ILA6.

3. If $(A^{\mathrm{T}}A)\boldsymbol{v} = \sigma^2 \boldsymbol{v}$, multiply by A. *Move the parentheses to get* $(AA^{\mathrm{T}})A\boldsymbol{v} = \sigma^2(A\boldsymbol{v})$.

If $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A^{\mathrm{T}}A}$, then ____ is an eigenvector of $\boldsymbol{AA^{\mathrm{T}}}$.

This is Problem 7.1.14 on page 296 of ILA6.

4. If $A = Q$ is an orthogonal matrix, why does every singular value of Q equal 1?

This is Problem 7.1.9 on page 296 of ILA6.

5. (a) Why is the trace of $A^{\mathrm{T}}A$ equal to the sum of all a_{ij}^2?

(b) For every rank-one matrix, why is $\sigma_1^2 =$ sum of all a_{ij}^2?

This is Problem 7.1.16 on page 296 of ILA6.

6. Suppose A_0 holds these 2 measurements of 5 samples:

$$A_0 = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ -1 & 1 & 0 & 1 & -1 \end{bmatrix}$$

Find the average of each row and subtract it to produce the centered matrix A. Compute the sample covariance matrix $S = AA^{\mathrm{T}}/(n-1)$ and find its eigenvalues λ_1 and λ_2. What line through the origin is closest to the 5 samples in columns of A?

This is Problem 7.3.1 about Principal Component Analysis from page 307 of ILA6. The best line is an eigenvector of S and a singular vector of the centered matrix A.