Homework 3 in 18.06 Due on Gradescope by Sunday midnight, March 5

The first two problems (3.1.3 and 3.1.5, on page 89 of ILA6) are about vector spaces in general. The vectors in those spaces are not necessarily column vectors. In the definition of a *vector space*, vector addition $x + y$ and scalar multiplication cx must obey the following eight rules :

(1) $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$

(2) $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$

(3) There is a unique "zero vector" such that $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{x}$ for all $\boldsymbol{x}$

(4) For each $\boldsymbol{x}$ there is a unique vector $-\boldsymbol{x}$ such that $\boldsymbol{x} + (-\boldsymbol{x}) = \boldsymbol{0}$

(5) 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$

(6) $(c_1c_2)\boldsymbol{x} = c_1(c_2\boldsymbol{x})$ — rules (1) to (4) are about $\boldsymbol{x} + \boldsymbol{y}$

(7) $c(\boldsymbol{x} + \boldsymbol{y}) = c\boldsymbol{x} + c\boldsymbol{y}$ — rules (5) to (6) are about $c\boldsymbol{x}$

(8) $(\boldsymbol{c_1} + \boldsymbol{c_2})\boldsymbol{x} = \boldsymbol{c_1x} + \boldsymbol{c_2x}$. — rules **(7) to (8) connect them**

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1. (a) Which rules are broken if we keep only the positive numbers $x > 0$ in $\mathbf{R}^1$? Every c must be allowed. This half-line is not a subspace.

(b) The positive numbers with $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ redefined to equal the usual xy and x^c *do satisfy the eight rules*. Test rule 7 when $c = 3, x = 2, y = 1$. (Then $\boldsymbol{x} + \boldsymbol{y} = 2$ and $c\boldsymbol{x} = 8$.) Which number acts as the "zero vector" in this space ?

This is Problem 3.1.3 on page 89.

2. M is the space of 2 by 2 matrices.

(a) Describe a subspace of M that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) If a subspace of M does contain A and B, must it contain the identity matrix ?

(c) Describe a subspace of M that contains no nonzero diagonal matrices.

This is Problem 3.1.5 on page 89.

3. The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A*. Give an example where the column spaces of A and AB are not equal.

This is Problem 3.1.21 on page 91.

4. (nullspace of A) Create a 2 by 4 matrix R whose special solutions to $R\boldsymbol{x} = \boldsymbol{0}$ are $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$:

$$\boldsymbol{s}_1 = \begin{bmatrix} -3 \\ \mathbf{1} \\ 0 \\ \mathbf{0} \end{bmatrix} \quad \text{and} \quad \boldsymbol{s}_2 = \begin{bmatrix} -2 \\ \mathbf{0} \\ -6 \\ \mathbf{1} \end{bmatrix}$$

pivot columns 1 and 3
free variables x_2 and x_4
x_2 and x_4 are $1, 0$ and $0, 1$
in the "special solutions"

Describe all 2 by 4 matrices with this nullspace $\mathbf{N}(A)$ spanned by $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.

This is Problem 3.2.3 on page 100.

5. Suppose an m by n matrix has r pivots. The number of special solutions (basis for the nullspace) is ____ by the Counting Theorem. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when $r =$ ____. The column space is all of $\mathbf{R}^m$ when the rank is $r =$ ____.

This is Problem 3.2.12 on page 101.

6. Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose nullspace contains $(1, 1, 2)$.

This is Problem 3.2.17 on page 101.

7. If $AB = 0$ then the column space of B is contained in the ____ of A. Why ?

This is Problem 3.2.21 on page 101.

8. How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

This is Problem 3.2.27 on page 102.

9. $A = C \begin{bmatrix} I & F \end{bmatrix} = C \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$. C has 2 independent columns.

Find the 2 special solutions to $A\boldsymbol{x} = \mathbf{0}$ of the form $(x_1, x_2, 1, 0)$ and $(x_1, x_2, 0, 1)$.

Note : If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution $x = x_p$ then all its solutions have the form $x = x_p + x_n$. Here x_p is the only solution in the row space of A and x_n is in the nullspace of A (so $Ax_n = 0$).

This problem is not in the ILA6 textbook.

Questions 10–13 are about the solution of $A\boldsymbol{x} = \boldsymbol{b}$. Start from the augmented matrix with last column $\boldsymbol{b}$. Eliminate column by column.

10. Write the complete solution as $\boldsymbol{x}_p$ plus any multiple of $\boldsymbol{s}$ in the nullspace :

$$\begin{aligned} x + 3y &= 7 \\ 2x + 6y &= 14 \end{aligned} \qquad \begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned}$$

This is Problem 3.3.3 on page 111.

11. Under what conditions on b_1, b_2, b_3 are these systems solvable ? Include $\boldsymbol{b}$ as a fourth column in elimination. Find all solutions when that solvability condition holds :

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3 \end{aligned} \qquad \begin{aligned} 2x + 2z &= b_1 \\ 4x + 4y &= b_2 \\ 8x + 8y &= b_3 \end{aligned}$$

This is Problem 3.3.5 on page 111.

12. Construct a 2 by 3 system $A\boldsymbol{x} = \boldsymbol{b}$ with particular solution $\boldsymbol{x}_p = (2, 4, 0)$ and homogeneous solution $\boldsymbol{x}_n =$ any multiple of $(1, 1, 1)$.

This is Problem 3.3.10 on page 112.

13. Give examples of matrices A for which the number of solutions to $A\boldsymbol{x} = \boldsymbol{b}$ is

(a) 0 or 1, depending on $\boldsymbol{b}$ (b) ∞, regardless of $\boldsymbol{b}$

(c) 0 or ∞, depending on $\boldsymbol{b}$ (d) 1, regardless of $\boldsymbol{b}$.

This is Problem 3.3.21 on page 113.

14. Find a basis (independent vectors that span the subspace) for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$.

(d) The column space and the nullspace of I $(4 \text{ by } 4)$.

Find a basis (and the dimension) for each of these subspaces of 3 by 3 matrices :

(e) All symmetric matrices $(A^{\mathrm{T}} = A)$.

(f) All skew-symmetric matrices $(A^{\mathrm{T}} = -A)$.

Corresponds to problems 3.4.16 on page 125 and 3.4.27 on page 127.