Homework 7 in 18.06   Due on Gradescope Sunday April 30 at 11:59 p.m.

1. (a) Find a symmetric matrix $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ that has a negative eigenvalue.

(b) How do you know it must have a negative pivot?

(c) How do you know it can't have two negative eigenvalues?

This is Problem 6.3.4 on page 257 of ILA6.

2. Every 2 by 2 symmetric matrix is $\lambda_1 \boldsymbol{x}_1 \boldsymbol{x}_1^{\mathrm{T}} + \lambda_2 \boldsymbol{x}_2 \boldsymbol{x}_2^{\mathrm{T}} = \lambda_1 P_1 + \lambda_2 P_2$.
Explain $P_1 + P_2 = \boldsymbol{x}_1 \boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{x}_2 \boldsymbol{x}_2^{\mathrm{T}} = I$ from columns times rows of Q. Why is $P_1 P_2 = 0$?

This is Problem 6.3.8 on page 257 of ILA6.

3. Show that this A (**symmetric but complex**) has only one line of eigenvectors:

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \text{ is not even diagonalizable: eigenvalues } \lambda = 0, 0.$$

$A^{\mathrm{T}} = A$ is not such a special property for complex matrices. The good property is $\overline{A}^{\mathrm{T}} = A$. Then all λ's are real and the eigenvectors are orthogonal.

This is Problem 6.3.11 on page 258 of ILA6.

4. Write a 2 by 2 *complex* matrix with $\overline{S}^{\mathrm{T}} = S$ (a "Hermitian matrix"). Find λ_1 and λ_2 for your complex matrix. Check that $\overline{\boldsymbol{x}}_1^{\mathrm{T}} \boldsymbol{x}_2 = 0$ (this is complex orthogonality).

This is Problem 6.3.13 on page 258 of ILA6.

5. ***True*** (with reason) ***or false*** (with example).

(a) A matrix with n real eigenvalues and n real eigenvectors is symmetric.

(b) A matrix with n real eigenvalues and n orthonormal eigenvectors is symmetric.

(c) The inverse of an invertible symmetric matrix is symmetric.

(d) The eigenvector matrix Q of a symmetric matrix is symmetric.

(e) The main diagonal of a positive definite matrix is all positive.

This is Problem 6.3.14 on page 258 of ILA6.

6. Which 3 by 3 symmetric matrices S and T produce these quadratics?

$\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 2\left(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3\right)$. Why is S positive definite?

$\boldsymbol{x}^{\mathrm{T}} T \boldsymbol{x} = 2\left(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_1 x_3 - x_2 x_3\right)$. Why is T semidefinite?

This is Problem 6.3.30 on page 260 of ILA6.

7. **Important**! Suppose S is positive definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

(a) What are the eigenvalues of the matrix $\lambda_1 I - S$? Is it positive semidefinite?

(b) How does it follow that $\lambda_1 \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} \geq \boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}$ for every $\boldsymbol{x}$?

(c) Draw this conclusion: **The maximum value of $\boldsymbol{x}^{\mathrm{T}} \boldsymbol{S} \boldsymbol{x} / \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}$ is** ____ .

This is Problem 6.3.48 on page 261 of ILA6.

8. Find the eigenvalues and eigenvectors of the Hermitian matrix $S = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$.

This is Problem 6.4.2 on page 269 of ILA6.

9. If $\overline{Q}^{\mathrm{T}} Q = I$ (unitary matrix = complex orthogonal) and $Q\boldsymbol{x} = \lambda\boldsymbol{x}$, show that $|\lambda| = 1$.
This is Problem 6.4.3 on page 269 of ILA6.

10. Find two λ's and $\boldsymbol{x}$'s so that $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ solves $\dfrac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \boldsymbol{u}$.

What combination $\boldsymbol{u} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ starts from $\boldsymbol{u}(0) = (5, -2)$?
This is Problem 6.5.1 on page 280 of ILA6.

11. (a) If every column of A adds to zero, why is $\lambda = 0$ an eigenvalue ?

(b) With negative diagonal and positive off-diagonal adding to zero, $\boldsymbol{u}' = A\boldsymbol{u}$ will be a "continuous" Markov equation. Find the eigenvalues and eigenvectors, and the *steady state* as $t \to \infty$

$$\text{Solve} \quad \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \boldsymbol{u} \quad \text{with} \quad \boldsymbol{u}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \quad \text{What is } \boldsymbol{u}(\infty)?$$

This is Problem 6.5.3 on page 280 of ILA6.