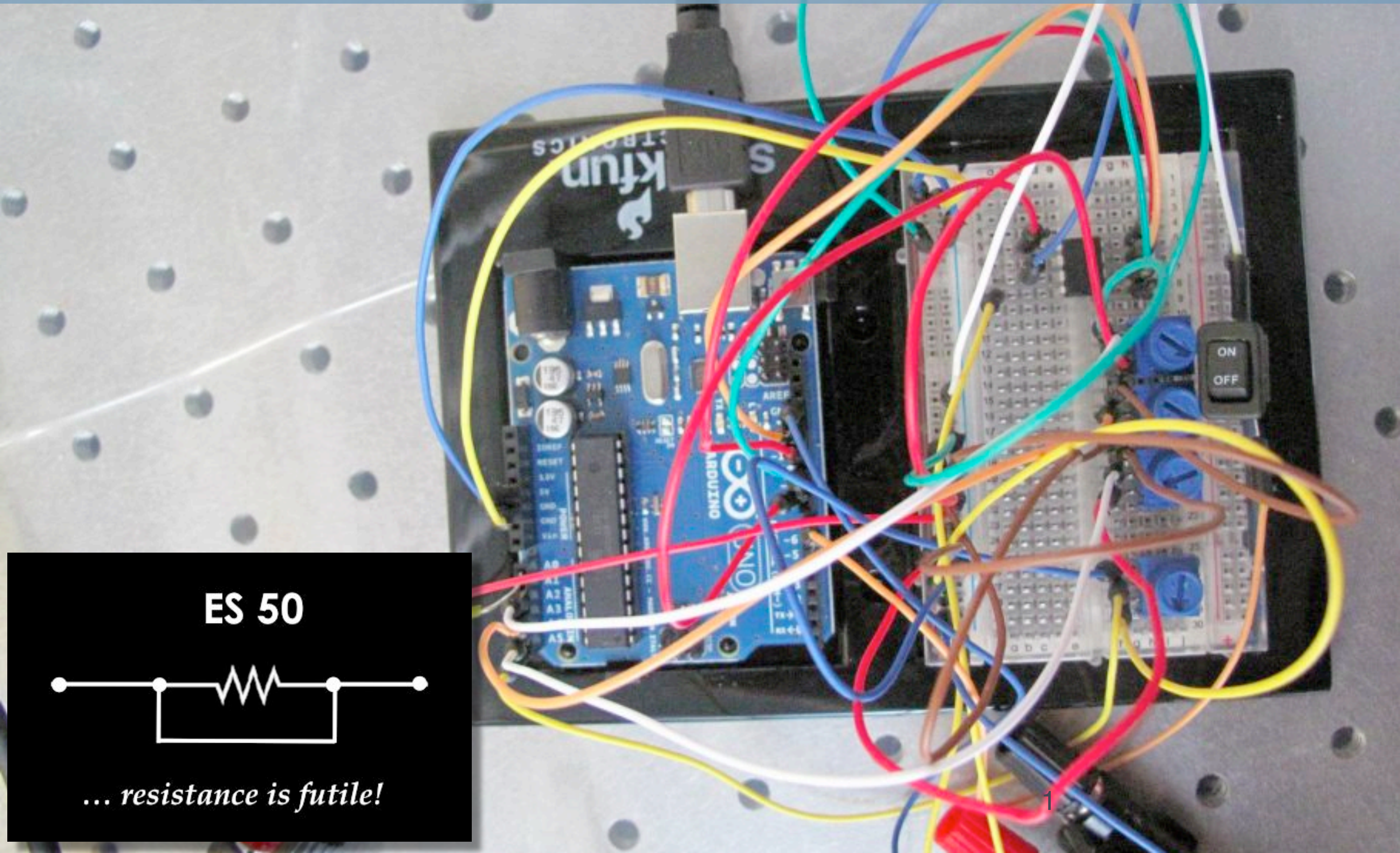
ES 50: Introduction to Electrical Eng.

Explore EE, satisfy a Gen Ed requirement & have fun



Lecture 4: Circuit Analysis

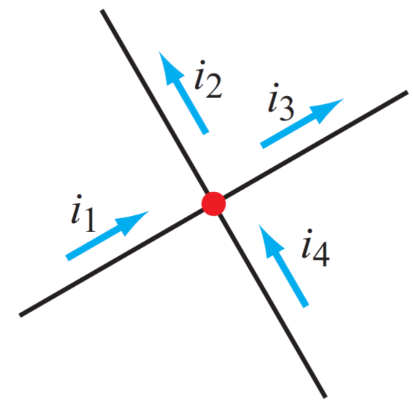
ES 50: Introduction to Electrical Engineering

Announcements

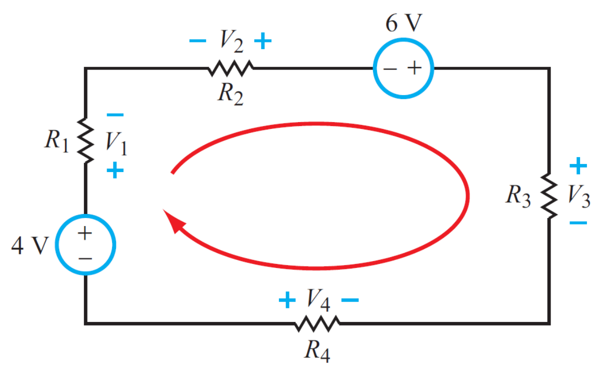
- **Lab sections have begun!**
- **Review / problem solving sessions**
  - **Wed 7-9:30PM in Pierce 301**
  - **Thurs 7-9:30PM in Pierce 301**
- **HW #1 due Saturday, 9PM MD drop boxes**
- **Changing date for Quiz #2 to Wed, Nov 18, 2015**
  - **Go Big Red!**

Last Time:

- Electrical power: P= V*I
- Circuit Definitions (node, loop, branch)
- Kirchhoff's Current Law
  - Sum of current entering node = sum of current exiting node
- Kirchhoff's Voltage Law
  - Sum of voltage drops = sum of voltage rises



$$i_1 + i_4 = i_2 + i_3$$



$$4 + V_2 + 6 + V4 = V_1 + V_3$$

Today we will…

- Revisit example quiz problem
- See more uses for KCL & KVL
- *Parallel and series* connection of resistors and other circuit elements
- Voltage divider and current divider
- Nodal analysis
- Superposition

Summary:
Kirchhoff's Current and Voltage Laws

KCL:
Sum of currents entering/leaving a node is zero
Note: By convention, currents entering a node have "+" sign and currents exiting a node have a "-" sign.

$$\sum_{n=1}^{N} i_n = 0 \qquad \text{(KCL)},$$

KVL:
Sum of voltages around a closed path is zero

$$\sum_{n=1}^{N} v_n = 0 \qquad \text{(KVL)},$$

Sign Convention

- Add up the voltages in a systematic clockwise movement around the loop.
- Assign a positive sign to the voltage across an element if the (+) side of that voltage is encountered first, and assign a negative sign if the (−) side is encountered first.

Simple (though not the most efficient) recipe for solving circuits…

1. Redraw the circuit and simplify it if possible
2. Label polarizations of active elements (sources) – this is already given in most cases
3. Choose current directions through each element arbitrarily
4. Label voltages across each passive element (e.g., resistors) so that current flows from "+" terminal to "-" terminal of the element
5. Use Ohm's Law to establish relationship between current **I** and voltage **V** for each resistor. Use following convention: if **I** is flowing from "+" to "-" terminal through resistor **R**, the voltage **V** is **V=I*R**
6. Use KCL for all extraordinary nodes to establish relationships between currents
7. USE KVLs for all (independent) loops to establish relationships between voltages
8. Solve the system of equations. DONE!

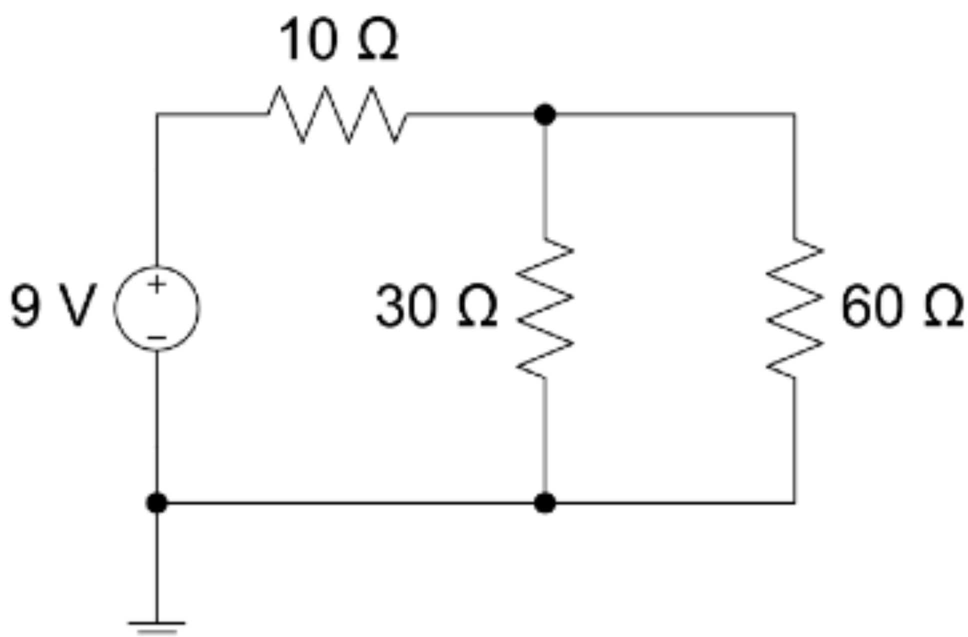
Example: 2012 quiz

Question 1: Easy Circuits (10 points)

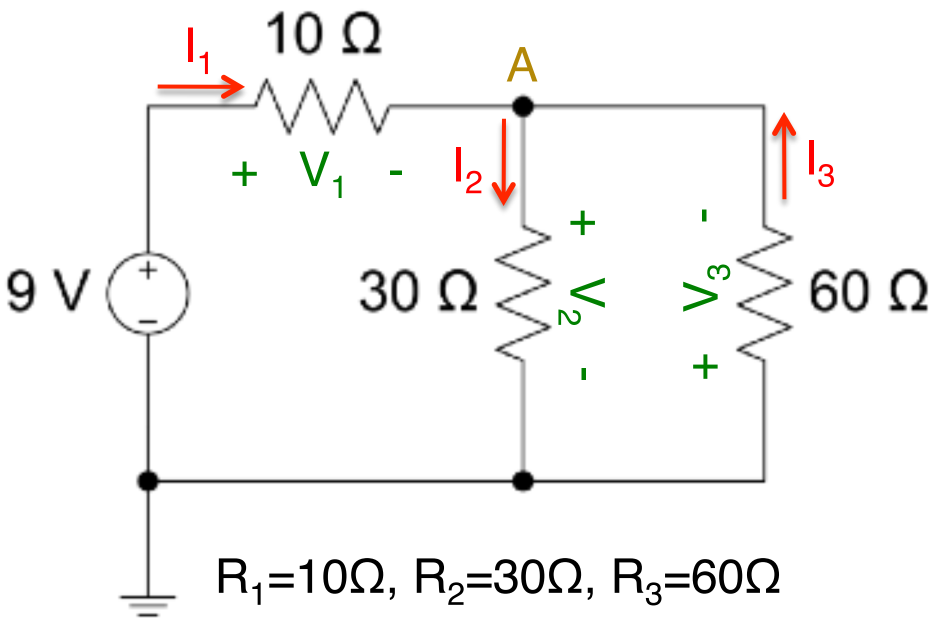
a) **(3 points)** For the circuit below, label all resistor currents and resistor voltages according to reference directions used for passive elements.

b) (**6 points**) Find the currents going through each of the three resistors, with respect to the reference directions you chose in part (a)

c) **(1 points)** What is the total power generated by the source?



Example: 2012 quiz



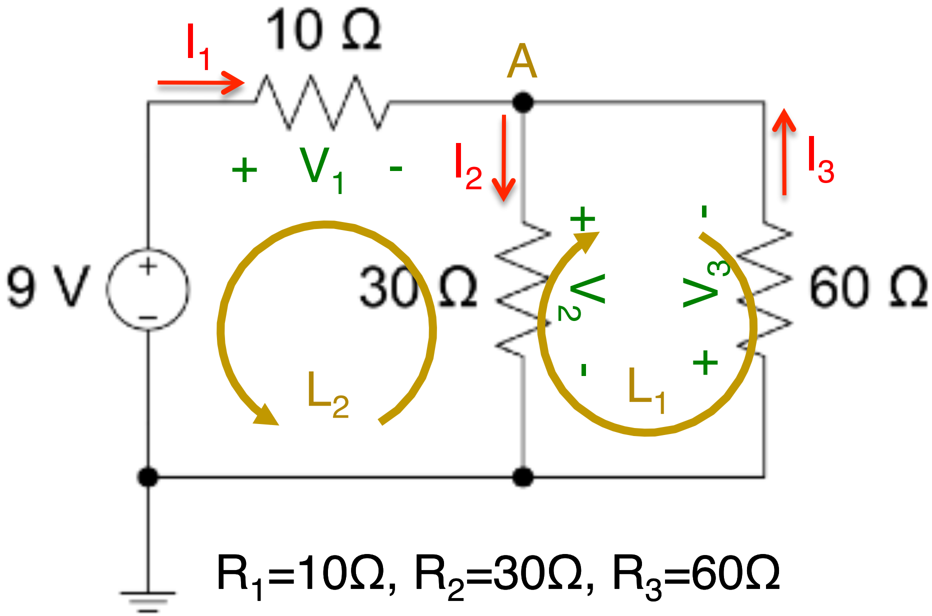
R_1=10Ω, R_2=30Ω, R_3=60Ω

$V_1=I_1*R_1$, $V_2=R_2*I_2$, $V_3=R_3*I_3$ (1)

Using the recipe:

1. Redraw the circuit and simplify it if possible (we will skip for now)
2. Label polarizations of active elements (already done for us)
3. Choose current directions through each element arbitrarily
4. Label voltages across each passive element (e.g. resistors) so that current is flowing from "+" terminal to "-" terminal of the element (done)
5. Use Ohm's law to establish relationship between current **I** and voltage **V** for each resistor

Example: 2012 quiz



$R_1=10\Omega$, $R_2=30\Omega$, $R_3=60\Omega$

$V_1=I_1*R_1$, $V_2=R_2*I_2$, $V_3=R_3*I_3$ (1)

KCL A: $I_1 - I_2 + I_3 = 0$ (2)

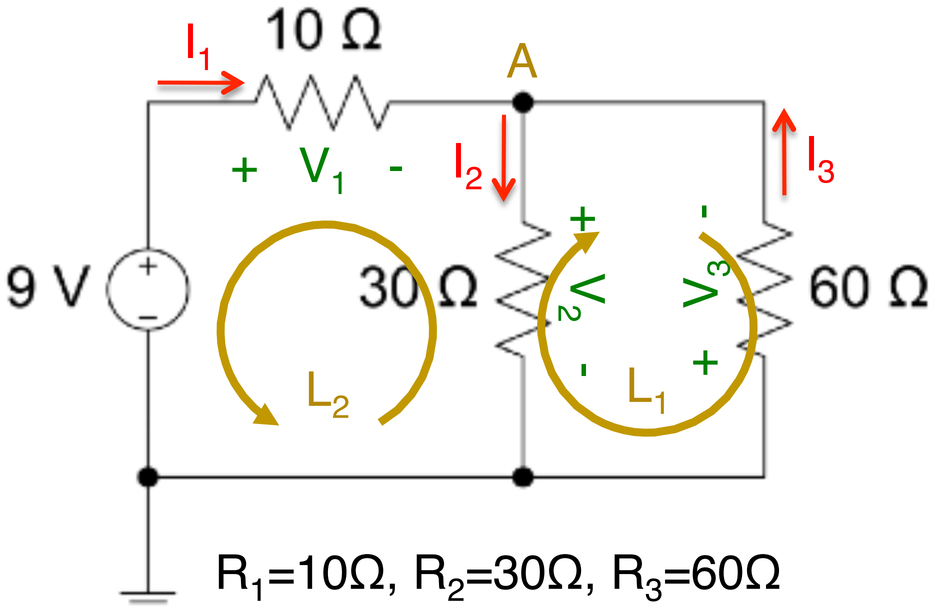
KVL L_1: $-V_2 - V_3 = 0$ (3)

KVL L_2: $9 - V_2 - V_1 = 0$ (4)

Using the recipe:

1. Redraw the circuit and simplify it if possible (we will skip for now)
2. Label polarizations of active elements (already done for us)
3. Choose current directions through each element arbitrarily
4. Label voltages across each passive element (e.g. resistors) so that current is flowing from "+" terminal to "-" terminal of the element (done)
5. Use Ohm's law to establish relationship between current **I** and voltage **V** for each resistor ($V = I*R$)
6. KCL for all nodes (except for one) to establish relationships between currents
7. KVL for all (independent) loops to establish relationships between voltages

Example: 2012 quiz



$R_1=10\Omega$, $R_2=30\Omega$, $R_3=60\Omega$

$V_1=I_1*R_1$, $V_2=R_2*I_2$, $V_3=R_3*I_3$ (1)

KCL A: $I_1 - I_2 + I_3 = 0$ (2)

KVL L_1: $-V_2 - V_3 = 0$ (3)

KVL L_2: $9 - V_2 - V_1 = 0$ (4)

Using the recipe:

8. Solve the equations

Eqs. 1 & 3: $I_2R_2 = - I_3R_3$

→ $I_2 = -R_2/R_3 * I_3$ → $-60/30*I_3$ → $\mathbf{I_2 = -2 * I_3}$

Eq. 2: $I_1 + 2I_3 + I_3 = 0$ → $\mathbf{I_1 = -3 * I_3}$

Eqs. 1 & 4: → $\mathbf{9 = I_1R_1 + I_2R_2}$

Substituting for I_1 and I_2 we get:

$9 = R_1I_1 + R_2I_2 = 10*(-3*I_3) + 30*(-2*I_3)$

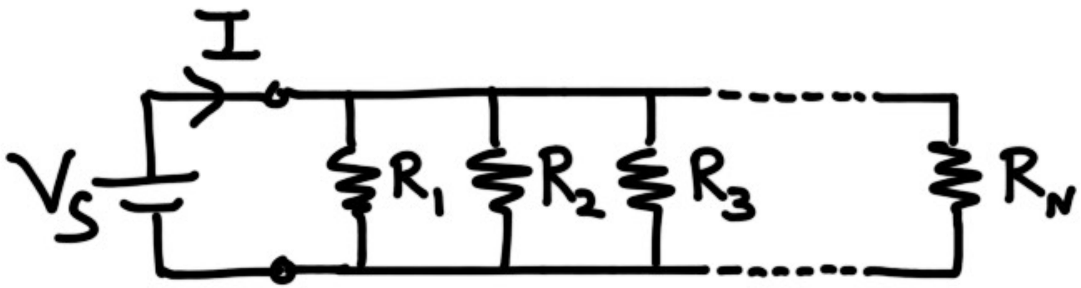
$9 = -90\ I_3$ → $\mathbf{I_3 = -0.1A}$

$\mathbf{I_3}$	**= -0.1A**	$\mathbf{V_3 = -6\ V}$
$\mathbf{I_2 = -2\ I_3}$	**= 0.2 A**	$\mathbf{V_2 = 6\ V}$
$\mathbf{I_1 = -3\ I_3}$	**= 0.3A**	$\mathbf{V_1 = 3\ V}$

Power generated/supplied by the source:

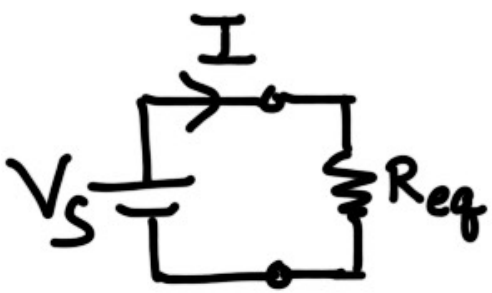
$\mathbf{P = 9V * I_1 = 2.7\ W}$

Multiple Resistors in Parallel



Notice: The total resistance of resistors in parallel is SMALLER than the resistance of each resistor. Why?

$$I = \frac{V_S}{R_1} + \frac{V_S}{R_2} + \frac{V_S}{R_3} + \cdots + \frac{V_S}{R_N} = V_S \cdot \sum_{i=1}^{N} \frac{1}{R_i}$$

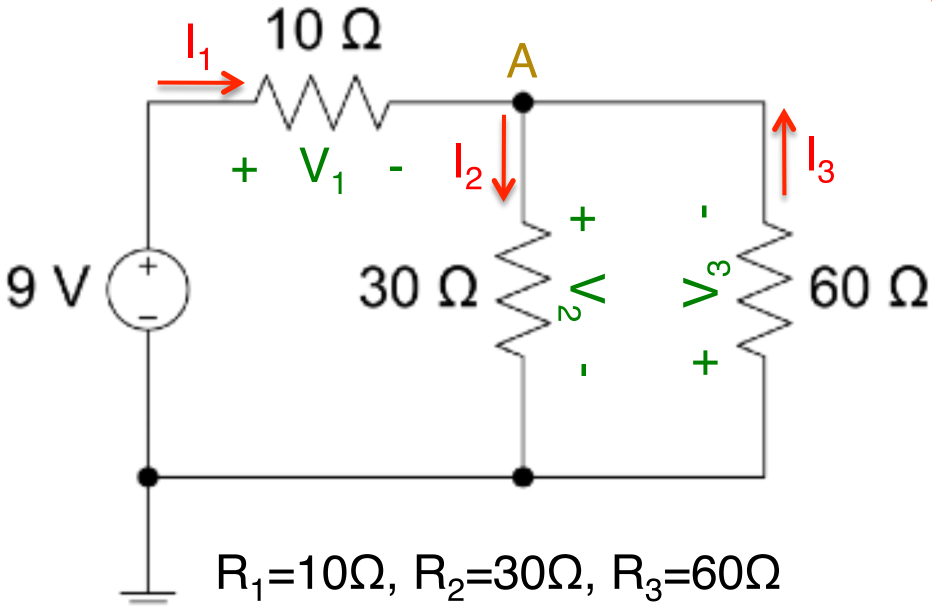


$$\frac{1}{R_{eq}} = \sum_{i=1}^{N} \frac{1}{R_i} \Rightarrow R_{eq} = \frac{1}{\sum_{i=1}^{N} \frac{1}{R_i}}$$

Shorter notation:

$R_{eq} = R_1 \,||\, R_2 \,||\, R_3 \,||\, \ldots \,||\, R_N$

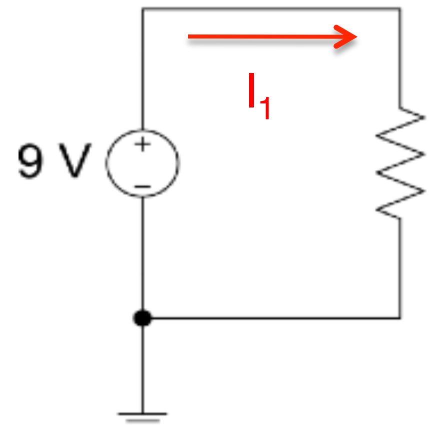
Is there an easier way to solve this?



R_1=10Ω, R_2=30Ω, R_3=60Ω

Recall the recipe…

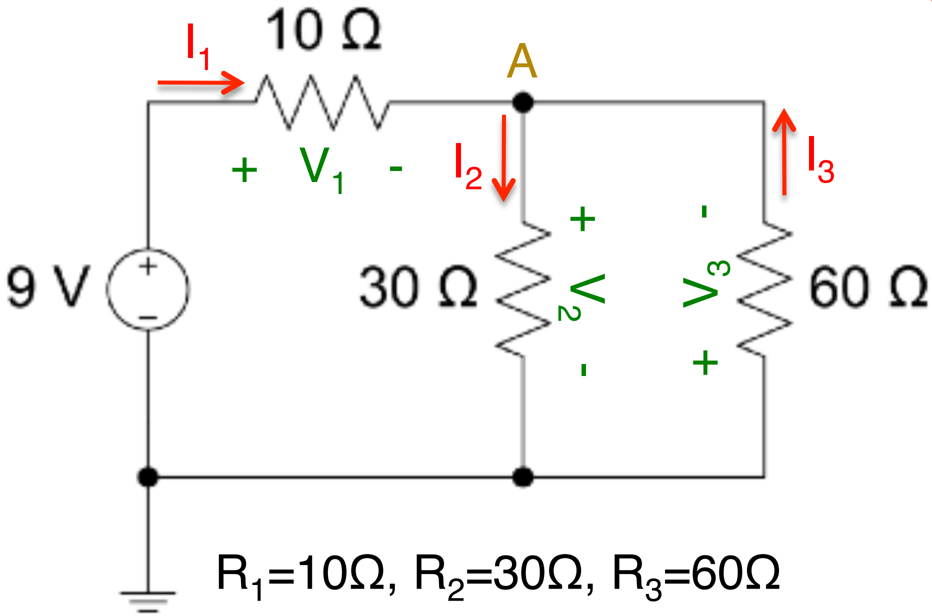
1. Redraw the circuit and simplify if possible



R_{eq} = ?

$$
\begin{aligned}
R_{eq} &= 10 + (30||60) \\
&= 10 + 20 = 30\Omega \\
I_1 &= \frac{9}{R_{eq}} = \frac{9}{30} = 0.3A \\
P_{source} &= 9 \cdot I_1 = 9 \cdot 0.3 = 2.7W
\end{aligned}
$$

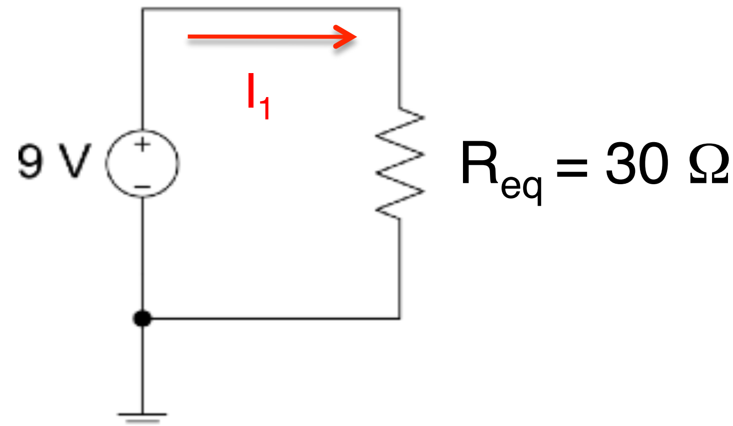
Is there an easier way to solve this?



R_1=10Ω, R_2=30Ω, R_3=60Ω

Recall the recipe…

1. Redraw the circuit and simplify if possible



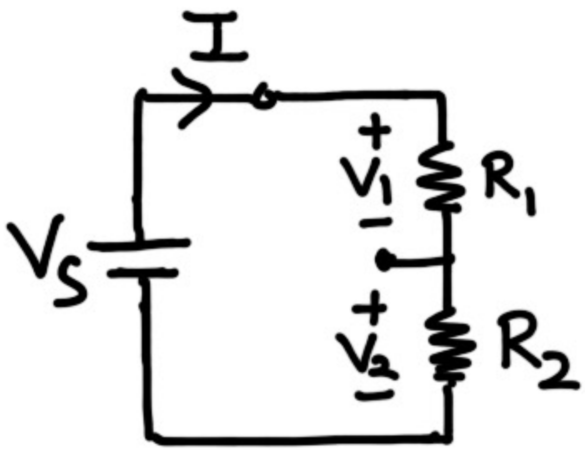
$\mathbf{I_1} = \mathbf{0.3A} \rightarrow V_1 = 3V$

$\mathbf{I_2} = V_2/R_2 = (9 - V_1)/30 = 6/30 = \mathbf{0.2A}$

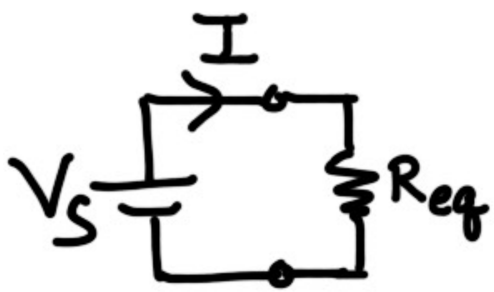
$\mathbf{I_3} = V_3/R_3 = -(9 - V_1)/60 = -6/60 = \mathbf{-0.1A}$

Resistors in series

Definition: Resistors, or any two circuit elements, are connected in SERIES when they share the same CURRENT, i.e., the same current flows through both resistors (circuit elements).



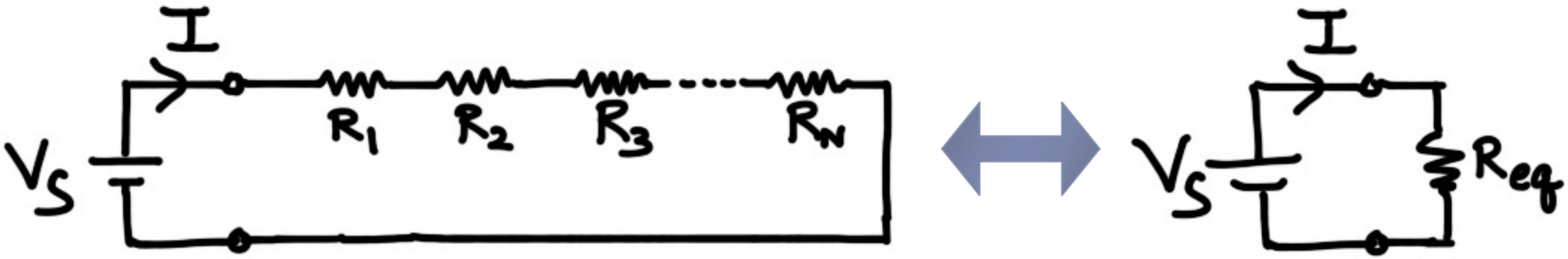
$$
\begin{aligned}
V_1 &= I \cdot R_1 \\
V_2 &= I \cdot R_2 \\
V_S &= V_1 + V_2 \\
V_S &= I \cdot (R_1 + R_2) \\
I &= \frac{V_S}{(R_1 + R_2)}
\end{aligned}
$$



$$I = \frac{V_S}{R_{eq}}$$

$$\boxed{R_{eq} = R_1 + R_2}$$

Multiple resistors in series

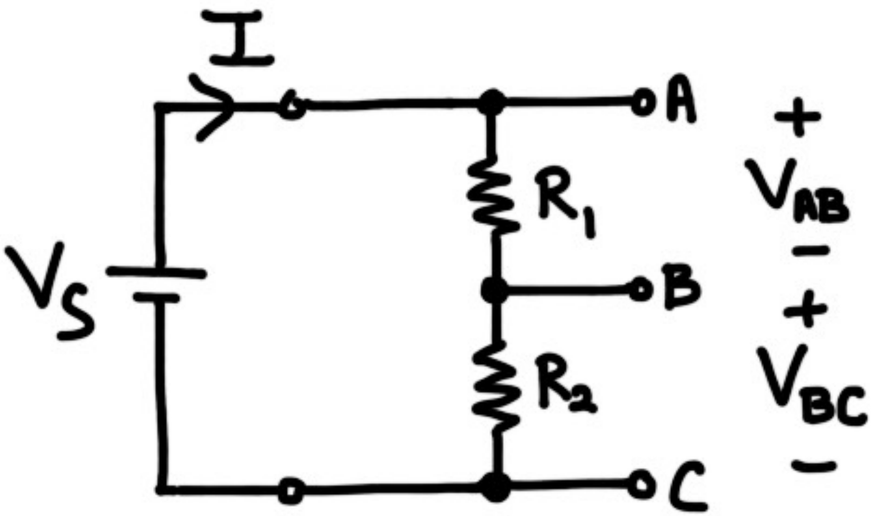


$$R_{eq} = R_1 + R_2 + R_3 + \cdots + R_N$$

$$\boxed{R_{eq} = \sum_{i=1}^{N} R_i}$$

Notice: The total resistance of resistors in series is LARGER than resistance of each resistor.

Example: Voltage divider (a useful circuit)

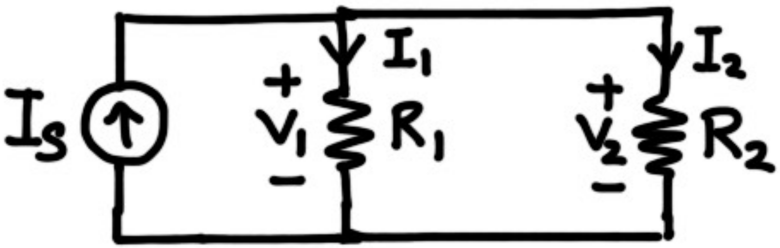


$$V_S = I \cdot (R_1 + R_2)$$
$$V_{BC} = I \cdot R_2$$

$$I = \frac{V_S}{R_1 + R_2}$$
$$V_{BC} = \frac{V_S}{R_1 + R_2} \cdot R_2 = \boxed{V_S \cdot \frac{R_2}{R_1 + R_2}}$$

You will do this in Lab 2!

Example: Current divider (useful circuit!)



This circuit divides the current, I_s, between two resistor branches so that the smaller resistor gets more current

KCL: $I_S - I_1 - I_2 = 0 \Rightarrow I_S = I_1 + I_2$

Ohm's law: $V_1 = I_1 R_1$ **and** $V_2 = I_2 R_2$

KVL: $V_2 - V_1 = 0 \Rightarrow V_2 = V_1$

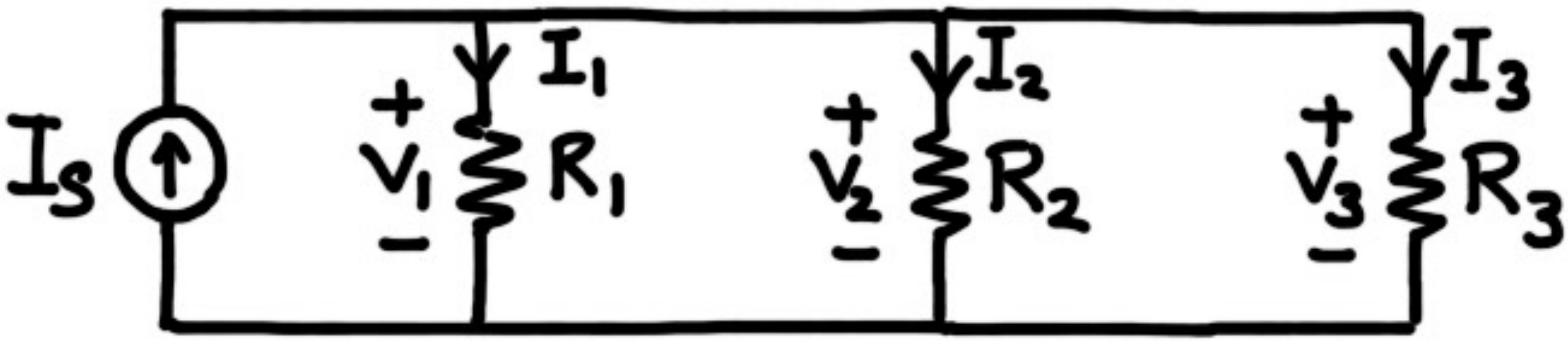
KVL + Ohm's law: $I_1 R_1 = I_2 R_2 \Rightarrow I_1 = \frac{R_2}{R_1} I_2$

$$\boxed{\Rightarrow I_2 = I_S \frac{R_1}{R_1 + R_2}}$$

Using KCL: $I_S = \frac{R_2}{R_1} I_2 + I_2 = I_2 \left(1 + \frac{R_2}{R1}\right) \Rightarrow I_1 = I_S - I_2$

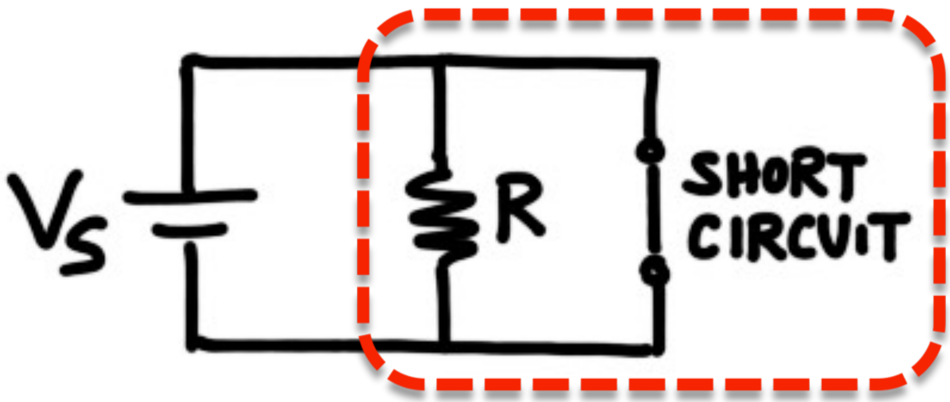
$$\boxed{\Rightarrow I_1 = I_S \frac{R_2}{R_1 + R_2}}$$

Now, your turn. I'll give you 5mins…



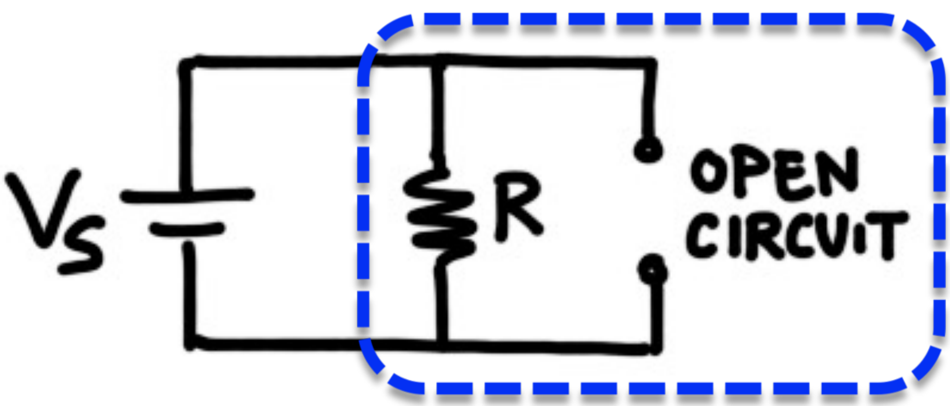
- What is the expression for I_1?
  - (1) $I_1 = (R_2 + R_3) / (R_1 + R_2 + R_3)$
  - (2) $I_1 = R_1 / (R_2R_3 + R_1R_3 + R_1R_2)$
  - (3) $I_1 = R_2R_3 / (R_1 + R_2 + R_3)$
  - **(4) None of the above** $\leftarrow I_1 = R_2R_3/(R_2R_3+R_1R_3+R_1R_2)*I_S$
- Hint: What is R_{eq}?

Example: Find equivalent resistance, R_{eq}



$R_{eq} = R \,||\, 0 = 0$

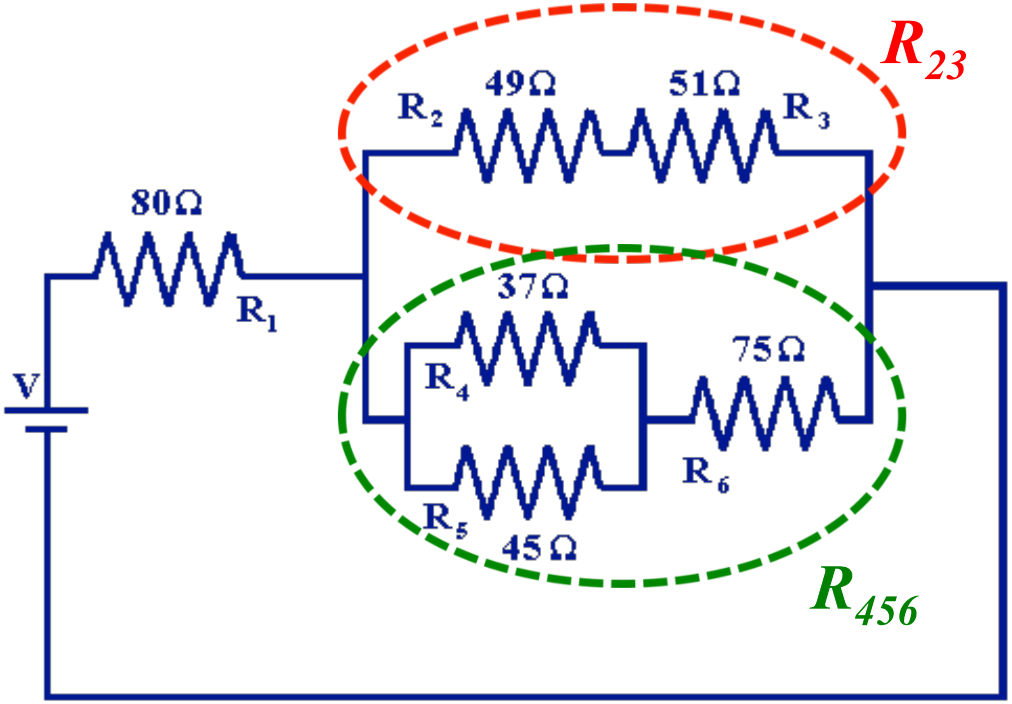
- R is shorted and all current flows through the *short circuit* since it has zero resistance



$R_{eq} = R \,||\, \infty = R$

- All current flows through R since no current can flow through the *open circuit*

A more “complicated” resistive network



$$R_{23} = R_2 + R_3 = 100\Omega$$

$$\begin{aligned} R_{456} &= R_4 || R_5 + R_6 \\ &= \frac{1}{1/37 + 1/45} + 75 \\ &\approx 95\Omega \end{aligned}$$

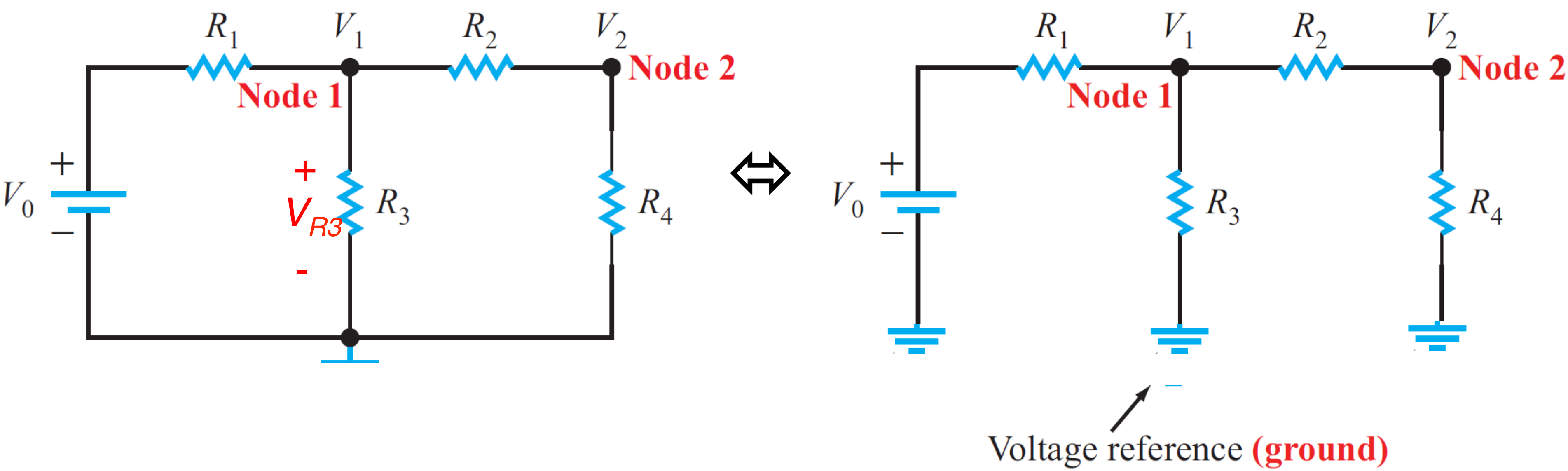
What is the equivalent resistance of this network?
How much resistance does the *V* source see?

$$\begin{aligned} R_{eq} &= R_1 + R_{23} || R_{456} \\ &= 80 + (100||95) \\ &\approx 129\Omega \end{aligned}$$

Find the *equivalent* resistance by identifying what is in series and what is in parallel…

Reference/Ground (GND)

- Choose reference point for potential and assign it potential of 0 Volts. This point is called ground. (Note: this ground can be *virtual* and does not have to be *physically* grounded.)
- Now potentials of all nodes in the circuit are expressed relative to this ground terminal!

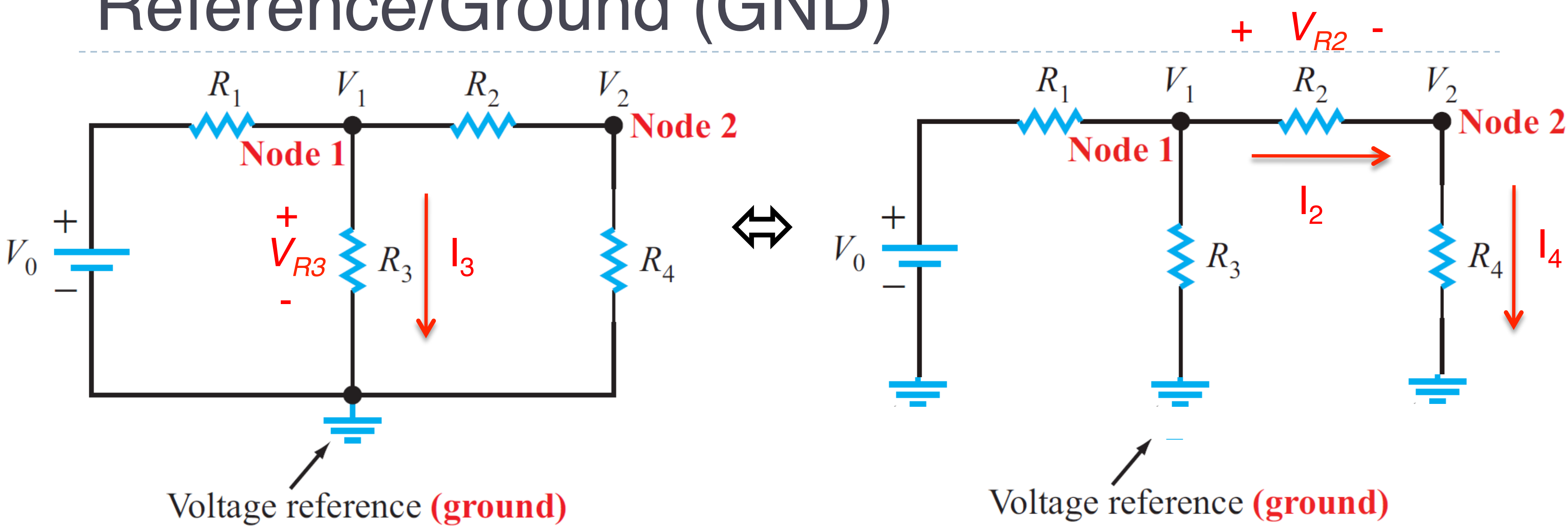


Remember that **voltage** between any two nodes is defined as **difference of potentials** in the two nodes (potentials are related to potential energy….). Therefore, voltage **across** resistor R_3 is the **potential difference** between Node 1 and GND:

$$V_{R3} = V_1 - V_{GND} = V_1 - 0 = V_1 \quad => \quad \mathbf{V_1 = V_{R3}}$$

Reference ground (GND) node allows us to use terms like "voltage of node 1", where we imply "voltage difference between node 1 and reference GND node".
(strictly speaking, we should be saying "potential of node 1 with respect to GND is….")

Reference/Ground (GND)



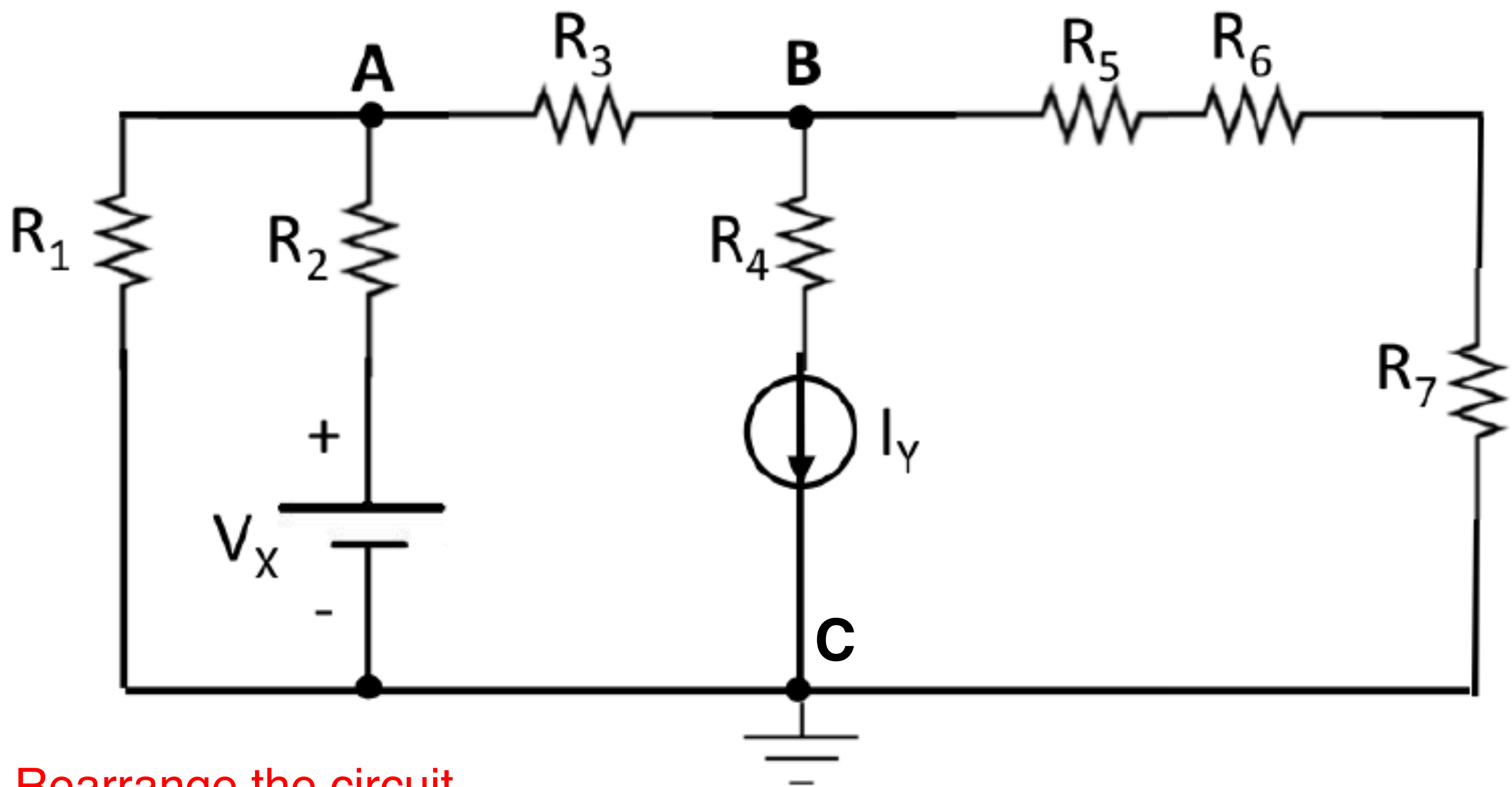
Allows us to write: $I_3 = V_{R3} / R_3 = (V_1 - V_{GND}) / R_3 = (V_1 - 0) / R_3 =$ $\mathbf{V_1 / R_3}$

Similarly, we can now express currents of each resistor in terms of voltages (potentials) of nodes that a resistor is attached to:

- $I_2 = V_{R2} / R_2 = (V_1 - V_2) / R_2$
- $I_4 = V_{R4} / R_4 = (V_2 - V_{GND}) / R_2 = V_2 / R_2$
- Also notice that $I_2 = I_4$ ➔ $(V_1 - V_2) / R_2 = V_2 / R_4$

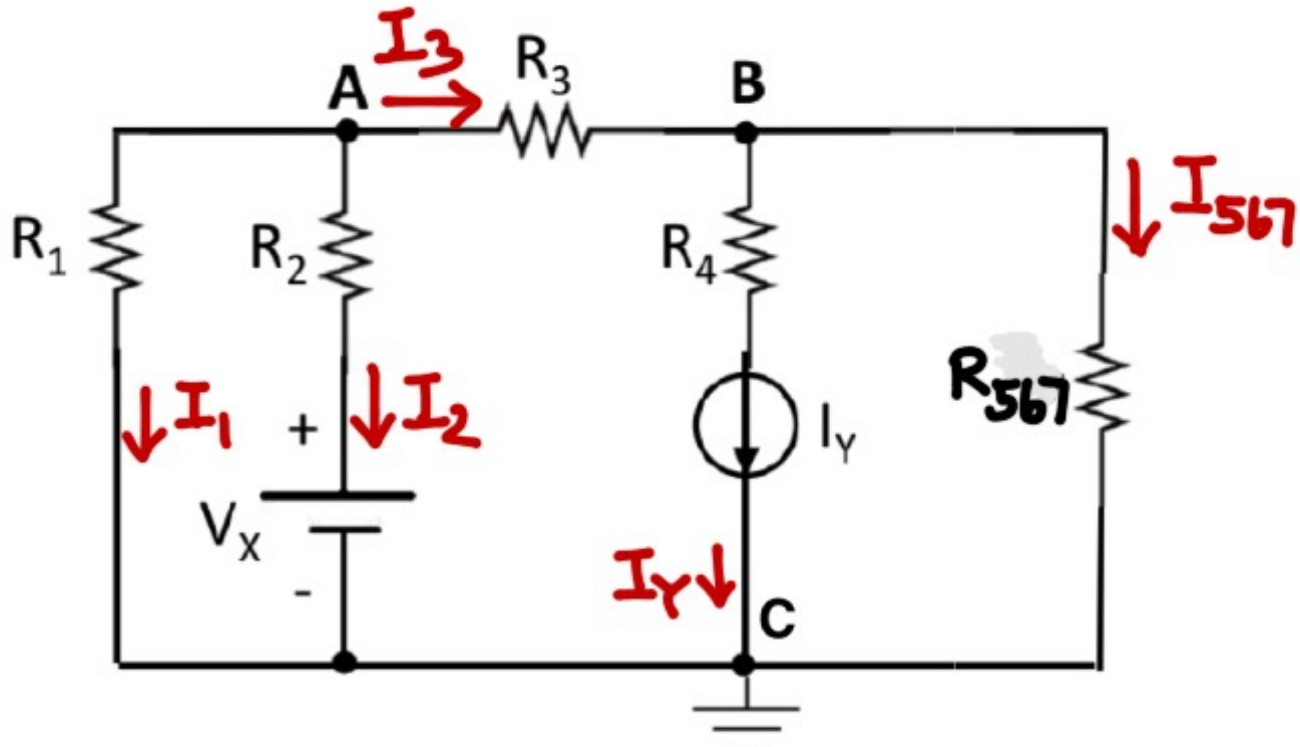
Leads to a systematic method for circuit analysis, the "Node-Voltage method."

What about this circuit?

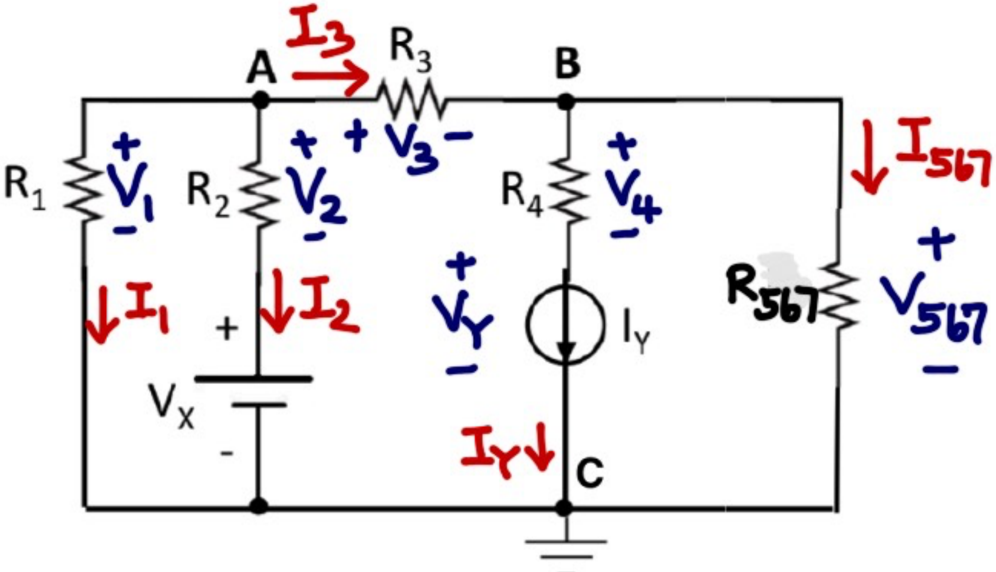


1. Rearrange the circuit

What about this circuit?



1. Rearranging the circuit → $\mathbf{R_{567} = R_5 + R_6 + R_7}$
2. Label polarization of active elements (done)
3. Choose current directions through each branch arbitrarily. For simplicity, use one label for identical currents:
   (a) I_2 flows through R_2 and V_x (b) I_Y must also flow through R_4



KVL:
$$I_1 R_1 = V_X + I_2 R_2$$
$$V_X + I_2 R_2 = I_3 R_3 + I_{567} R_{567}$$

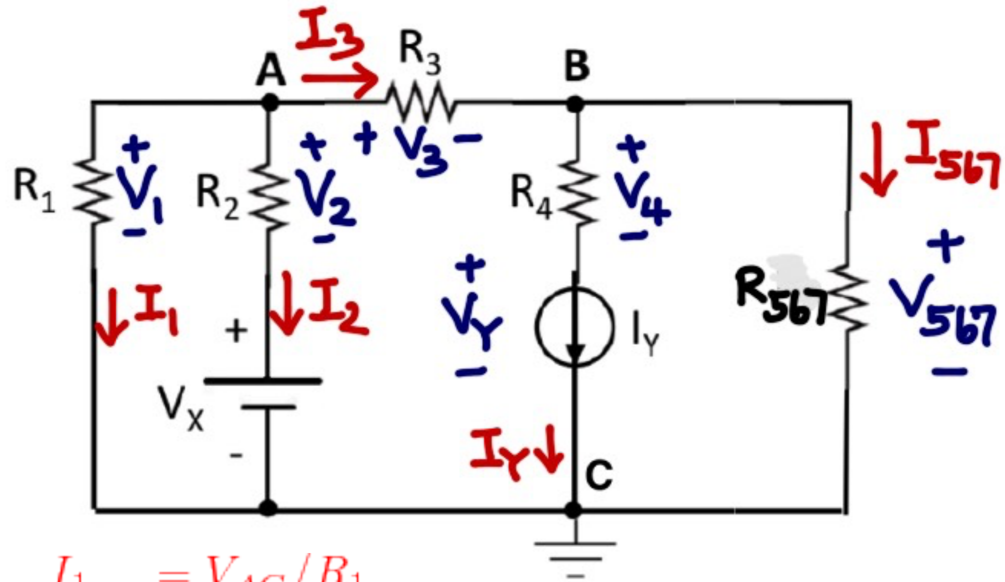
KCL:
$$-I_1 - I_2 - I_3 = 0$$
$$I_3 - I_Y - I_{567} = 0$$

4 Unknowns: I_1, I_2, I_3, I_{567}
4 Equations ➔ Solve

4. Label voltages across each passive element (e.g., resistors) so that current is flowing from "+" terminal to "-" terminal of the element;
5. Use Ohm's Law to establish relationship between current ***I*** and voltage ***V*** for each ***R***. We can skip this and simply write I*R, with proper sign, for every resistor voltage when we apply KVL.
6. **NEXT: Apply KVL and KCL to solve for the unknowns…**

IS THERE AN EASIER, MORE SYSTEMATIC WAY?

Another approach…



$$V_{AC} = I_1 R_1 \Rightarrow I_1 = V_{AC}/R_1$$

$$V_{AC} = V_2 + V_X = I_2 R_2 + V_X \Rightarrow I_2 = (V_{AC} - V_X)/R_2$$

$$V_{AB} = I_3 R_3 \Rightarrow I_3 = V_{AB}/R_3 = (V_{AC} - V_{BC})/R_3$$

$$V_{BC} = I_{567} R_{567} \Rightarrow I_{567} = V_{BC}/R_{567}$$

$$I_1 + I_2 + I_3 = 0 \Rightarrow \frac{V_{AC}}{R_1} + \frac{V_{AC} - V_X}{R_2} + \frac{V_{AC} - V_{BC}}{R_3} = 0$$

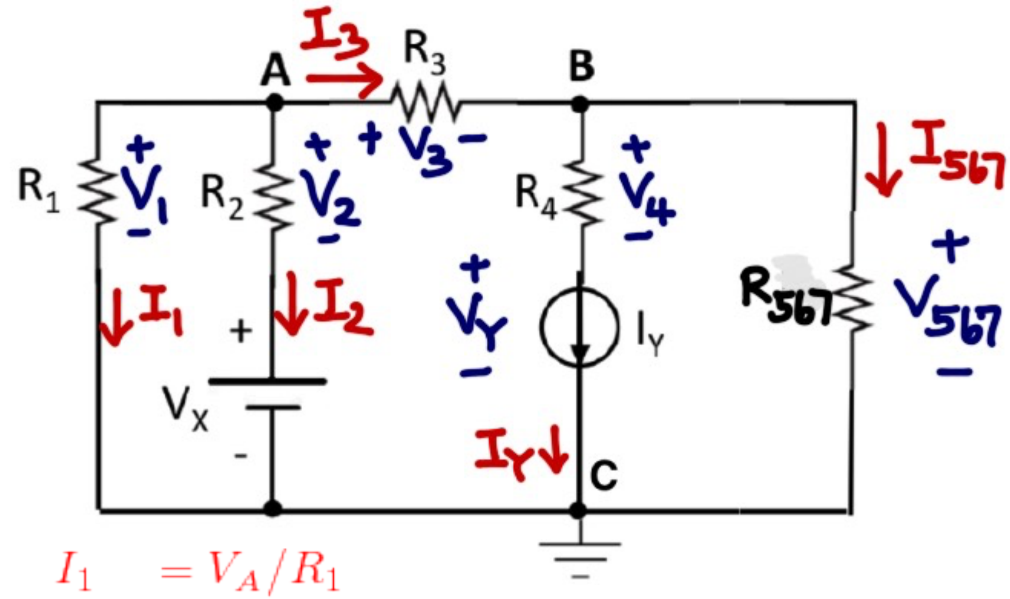
$$-I_3 + I_Y + I_{567} = 0 \Rightarrow -\frac{V_{AC} - V_{BC}}{R_3} + I_Y + \frac{V_{BC}}{R_{567}} = 0$$

$$\Rightarrow V_{AC}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) - \frac{V_{BC}}{R_3} = \frac{V_X}{R_3}$$

$$\Rightarrow \frac{V_{AC}}{R_3} - V_{BC}\left(\frac{1}{R_3} + \frac{1}{R_{567}}\right) = I_Y$$

- Unknown **nodal voltages**
  V_{AC} and V_{BC}
- Known **currents**
  V_X/R_3 and I_Y

Redo recognizing C = 0V



$$
\begin{aligned}
V_A &= I_1 R_1 & \Rightarrow\quad I_1 &= V_A / R_1 \\
V_A &= V_2 + V_X = I_2 R_2 + V_X & \Rightarrow\quad I_2 &= (V_A - V_X) / R_2 \\
V_{AB} &= V_A - V_B = I_3 R_3 & \Rightarrow\quad I_3 &= (V_A - V_B) / R_3 \\
V_B &= I_{567} R_{567} & \Rightarrow\quad I_{567} &= V_B / R_{567}
\end{aligned}
$$

$$I_1 + I_2 + I_3 = 0 \quad \Rightarrow \quad \frac{V_A}{R_1} + \frac{V_A - V_X}{R_2} + \frac{V_A - V_B}{R_3} = 0$$

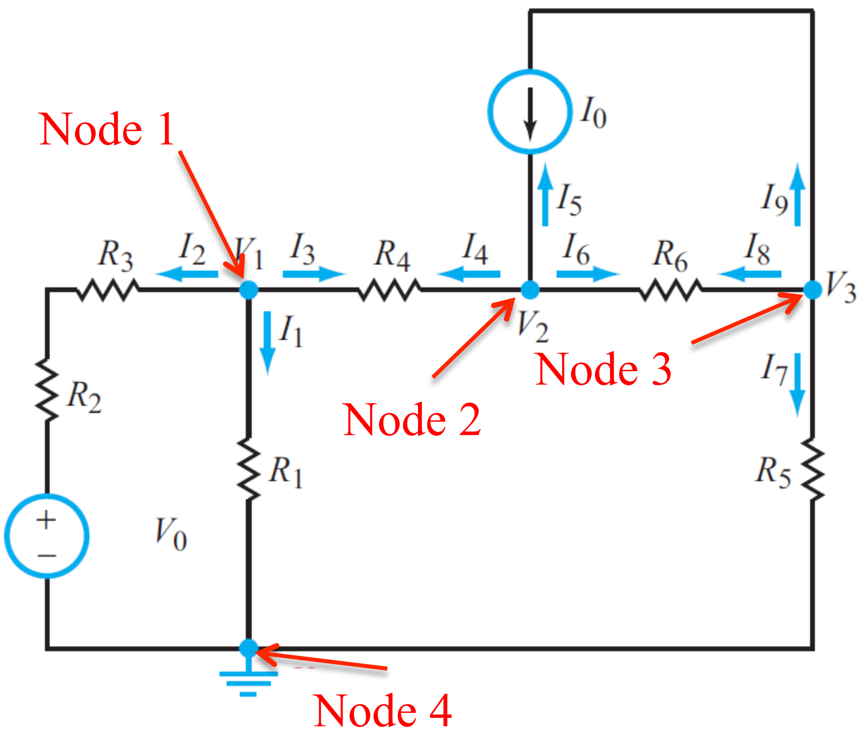
$$-I_3 + I_Y + I_{567} = 0 \quad \Rightarrow \quad -\frac{V_A - V_B}{R_3} + I_Y + \frac{V_B}{R_{567}} = 0$$

$$\Rightarrow \quad V_A \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) - \frac{V_B}{R_3} = \frac{V_X}{R_3}$$

$$\Rightarrow \quad \frac{V_A}{R_3} - V_B \left(\frac{1}{R_3} + \frac{1}{R_{567}} \right) = I_Y$$

- Unknown **nodal voltages**
  V_A and V_B
- Known **currents**
  V_X/R_3 and I_Y

Node-voltage method



Solution Procedure: Node Voltage

Step 1: Identify all extraordinary nodes, select one of them as a reference node (ground), and then assign node voltages to the remaining $(n_{ex} - 1)$ extraordinary nodes.

Step 2: At each of the $(n_{ex} - 1)$ extraordinary nodes, apply the form of KCL requiring the sum of all currents *leaving* a node to be zero.

Step 3: Solve the $(n_{ex} - 1)$ independent simultaneous equations to determine the unknown node voltages.

Node 2:

$$I_4 + I_5 + I_6 = 0$$

$$\frac{V_2 - V_1}{R_4} - I_0 + \frac{V_2 - V_3}{R_6} = 0$$

Node 1:

$$I_1 + I_2 + I_3 = 0$$

$$\frac{V_1}{R_1} + \frac{V_1 - V_0}{R_2 + R_3} + \frac{V_1 - V_2}{R_4} = 0$$

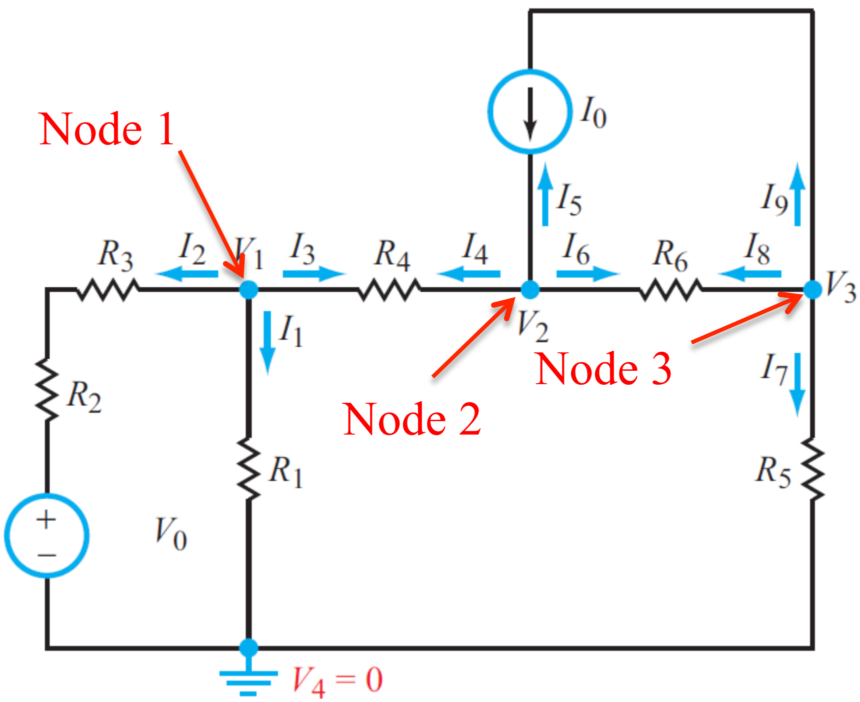
Node 3:

$$I_7 + I_8 + I_9 = 0$$

$$\frac{V_3}{R_5} + \frac{V_3 - V_2}{R_6} + I_0 = 0$$

How many unknowns are there?

Node-voltage method



Node 1: $\dfrac{V_1}{R_1} + \dfrac{V_1 - V_0}{R_2 + R_3} + \dfrac{V_1 - V_2}{R_4} = 0$

Node 2: $\dfrac{V_2 - V_1}{R_4} - I_0 + \dfrac{V_2 - V_3}{R_6} = 0$

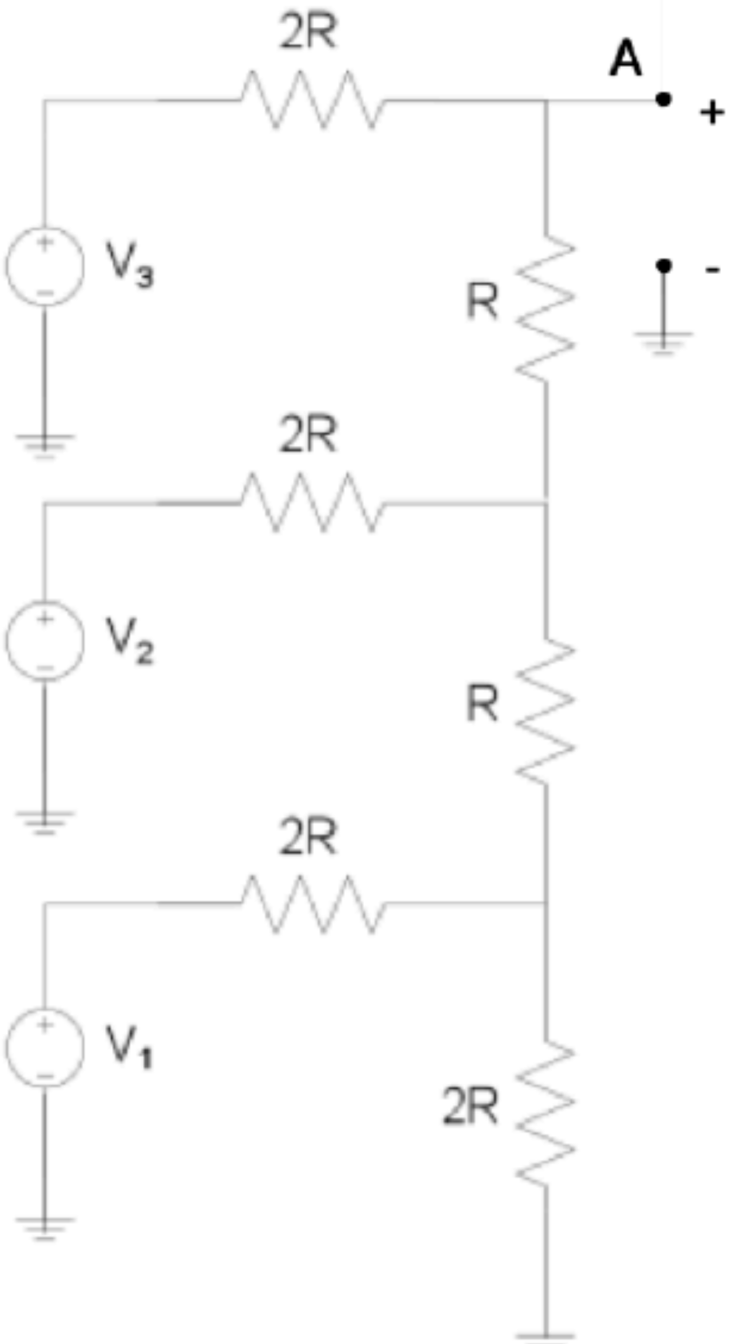
Node 3: $\dfrac{V_3}{R_5} + \dfrac{V_3 - V_2}{R_6} + I_0 = 0$

$$\begin{aligned}
\left(\frac{1}{R_1} + \frac{1}{R_2 + R_3} + \frac{1}{R_4}\right) V_1 - \frac{1}{R_4} V_2 + 0 \cdot V_3 &= \frac{V_0}{R_2 + R_3} \\
-\frac{1}{R_4} V_1 + \left(\frac{1}{R_4} + \frac{1}{R_6}\right) V_2 - \frac{1}{R_6} V_3 &= I_0 \\
0 \cdot V_1 - \frac{1}{R_6} V_2 + \left(\frac{1}{R_5} + \frac{1}{R_6}\right) V_3 &= -I_0
\end{aligned}$$

Can solve with matrix math…

Write matrix and solve w/ a computer

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2+R_3} + \frac{1}{R_4} & \frac{1}{R_4} & 0 \\ -\frac{1}{R_4} & \frac{1}{R_4} + \frac{1}{R_6} & \frac{1}{R_6} \\ 0 & -\frac{1}{R_6} & \frac{1}{R_5} + \frac{1}{R_6} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{V_0}{R_2+R_3} \\ I_0 \\ -I_0 \end{bmatrix}$$



What do we do with this?

There are multiple voltage sources!

Superposition

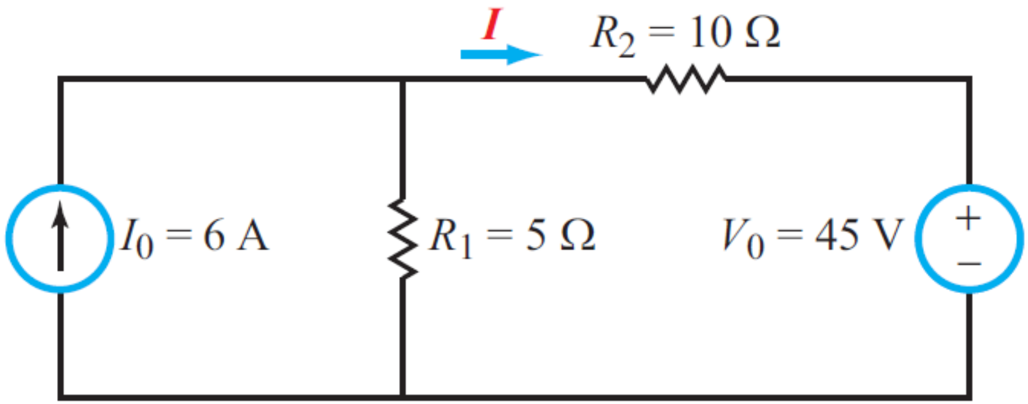
If a circuit contains more than one independent source, the voltage (or current) response of any element in the circuit is equal to the algebraic sum of the individual responses associated with the individual independent sources, as if each had been acting alone.

Superposition trades off the examination of several simpler circuits in place of one complex circuit.

Superposition Procedure:

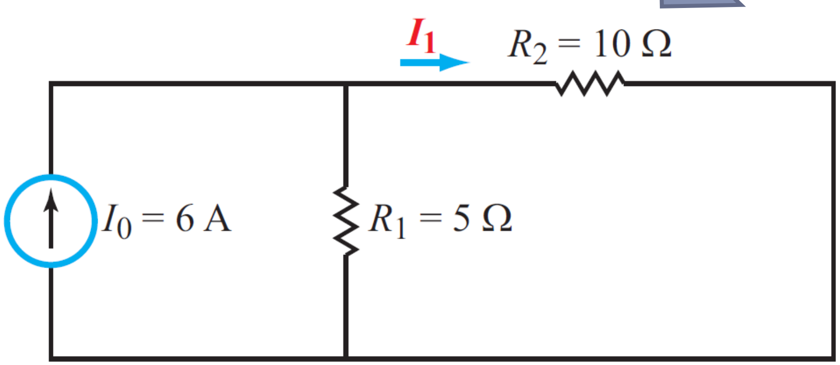
1. Set all independent sources to zero (replace voltage source with short circuits and current sources with open circuits), except for source 1.
2. Solve circuit in response to source 1.
3. Repeat (1) and (2) for source 2 through *n*.
4. Sum up the responses to sources 1 through *n*.

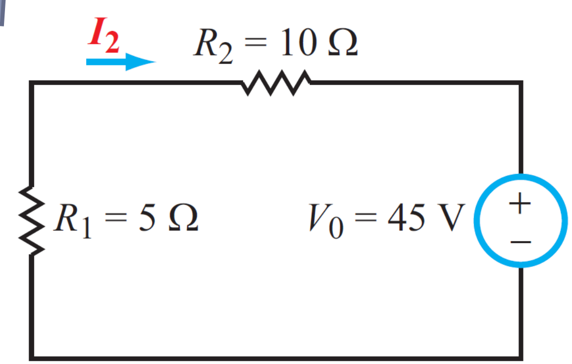
Example: Superposition



Contribution from I_0 alone

Contribution from V_0 alone





$I_1 = 2$ A

$$\boxed{I = I_1 + I_2 = 2 - 3 = -1 \text{ A}}$$

$I_2 = -3$ A