Probability

Probability (first glance):

Probability is a measure of certainty (surety) of an event. It takes a value between ‘0’ and ‘1’. For an impossible event, the probability is zero. For a sure event, the probability is one.

Example: If a regular (fair) coin is tossed, the probability that a ‘head’ occurs is 1/2 = 0.50 (50% chance).

Example: If a fair dice is tossed, the probability that a ‘3’ occurs is 1/6 = 0.167 (16.7% chance).

It should be noted here that in old English, ‘die’ is singular and ‘dice’ is plural. In present-day English, ‘dice’ is both singular and plural.

Deterministic and random experiments:

An experiment that has only one possible outcome is a ‘deterministic’ experiment.

Example: Counting the number of stairs of a particular building is a deterministic experiment. If we do the experiment repeatedly, we will get the same result (if we do not make a mistake).

A ‘random’ experiment has more than one possible outcome.

Example: Tossing a coin, throwing a dice, counting the number of calls received in an hour, etc. are random experiments.

* Probability is associated with random experiments.

Sample space:

The set of all possible outcomes of a random experiment is called the sample space. It is usually denoted by S. It is comparable to the universal set in set theory.

Example: In coin tossing experiment, $S = \{H, T\}$.

Example: In dice throwing experiment, $S = \{1, 2, 3, 4, 5, 6\}$.

Example: In a cricket match, $S = \{\text{win, loss, tie, postponed, cancelled}\}$.

Outcomes (elements) in S should be mutually exclusive (two outcomes cannot occur together). Also, S should be exhaustive (complete, i.e., no possible outcome should be left out).

Event:

Any subset of the sample space is called an event. When a random experiment is going to be conducted, we are interested in the probabilities of different events.

Example: In dice throwing, $A = \{1\}$, $B = \{1, 4, 5\}$, etc. are events. Note that B denotes the event that 1 or 4 or 5 occurs. They cannot happen together!

Example: In cricket match, $C = \{\text{loss}\}$, $D = \{\text{win}, \text{tie}\}$, etc. are events.

Events are sets. So, upper-case letters should be used for notation.

Operations on events:

$A \cup B$ occurs when A or B (or both) occur.

$A \cap B$ occurs when both A and B occur.

A^c occurs when A does not occur.

Classical or mathematical definition of probability:

Let a random experiment have n possible outcomes that are mutually exclusive, exhaustive and equally likely (all the outcomes have same chance). If m of these outcomes are favorable to an event A, then probability of A is given by:

$$P(A) = \frac{m}{n}$$

Note that, we cannot use this formula when all the outcomes are not equally likely, or the total number of possible outcomes, n, is infinite.

Example: Let a fair dice be thrown. Let $A = \{1, 4, 5\}$. Then

$$P(A) = \frac{m}{n} = \frac{3}{6} = 0.5.$$

Example: Consider a cricket match. $S = \{\text{win, loss, tie, postponed, cancelled}\}$. Let $D = \{\text{win, tie}\}$. Then, we should NOT say $P(D) = \frac{2}{5}$. (Why not?)

Empirical or statistical or frequency definition of probability

Empirical means observation-based, i.e., data-based.

Let an experiment be conducted n times, where n is large. If an event A occurs f_A times, then

$$P(A) \approx \frac{f_A}{n}$$

Here, probability is approximately equal to relative frequency. We cannot use this formula when the experiment cannot be repeated under the same conditions. Also,

$$P(A) = \lim_{n \to \infty} \frac{f_A}{n}$$

Example: An unfair dice is tossed 1000 times. "6" occurred 400 times, Then,

$$P(6) \approx \frac{400}{1000} = 0.40$$

* We have not followed set notation to write the event '6'. Some good books have also done that for comfort.

Subjective definition of probability

Sometimes probability is someone's judgement or belief.

Example: Consider the statement: "There is a 90% chance (probability 0.90) that I'll get an 'A' in this course." Here, the probability 0.90 shows the judgement of the 'subject' (the person who made the statement).

We cannot use mathematical definition in the above example, because 'getting A' and 'not getting A' are not equally likely. Also, we cannot use statistical definition here, because we cannot repeat the experiment (taking the course) under the same conditions.

Axioms of probability

Probability follows the following three axioms:

1. $0 \leq P(A) \leq 1$.

2. $P(S) = 1$.

3. When A and B are mutually exclusive events, i.e., $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

Explanation of Axiom 3:

Let a fair die be thrown. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Then, $P(A) = \frac{3}{6}$ and $P(B) = \frac{2}{6}$. Also, $A \cup B = \{1, 2, 3, 4, 5\}$, so that $P(A \cup B) = \frac{5}{6} = P(A) + P(B)$. If A and B are not mutually exclusive, equality will not hold.

Extension of Axiom 3

Axiom 3 can be extended to any number of sets.

If C, D and E are mutually exclusive sets, then

$$
\begin{aligned}
&P(C \cup D \cup E) \\
&= P\big((C \cup D) \cup E\big) \\
&= P(C \cup D) + P(E) \\
&= P(C) + P(D) + P(E)
\end{aligned}
$$

Some important theorems:

Theorem 1: $P(\emptyset) = 0$.

Proof:

$A \cup \emptyset = A$

$\therefore P(A \cup \emptyset) = P(A)$

$\therefore P(A) + P(\emptyset) = P(A)$ [Axiom 3]

$\therefore P(\emptyset) = 0$.

Theorem 2: $P(A^c) = 1 - P(A)$

Proof:

$A \cup A^c = S$

$\therefore P(A \cup A^c) = P(S)$

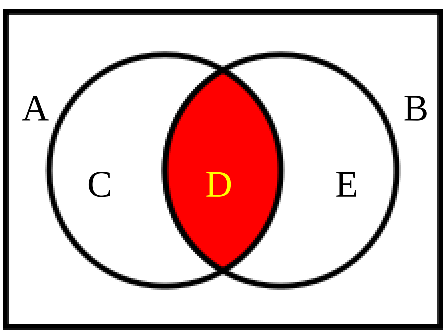
$\therefore P(A) + P(A^c) = 1$ [Axiom 3 for left side]

$\therefore P(A^c) = 1 - P(A)$

Theorem 3: For any two sets A and B, we have

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof:



$A \cup B = C \cup D \cup E$

$\therefore P(A \cup B) = P(C \cup D \cup E)$

$\therefore P(A \cup B) = P(C) + P(D) + P(E)$ [Axiom 3]

$\therefore P(A \cup B)$

$= P(C) + P(D) + P(D) + P(E) - P(D)$

$\therefore P(A \cup B)$

$= P(C \cup D) + P(D \cup E) - P(D)$ [Axiom 3]

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Exercise:

In a community, 25% of the families have cars, 15% have washing machines and 10% have both. A family is selected at random from the community. What is the probability that the family has (i) a car or a washing machine? (ii) neither a car nor a washing machine?

Solution:

$P(C) = 0.25$

$P(W) = 0.15$

$P(C \cap W) = 0.10$

(i) $P(C \cup W) = 0.25 + 0.15 - 0.10 = 0.30$

(ii) $P((C \cup W)') = 1 - P(C \cup W) = 0.70$