Expected value of function of random variable

Discrete Case:

Let X be a discrete random variable with pmf $p(x)$. The expected value or mean of $Y = g(X)$ is obtained as

$$\mu_Y = E(g(X)) = \sum_x g(x)\, p(x)$$

Example

Let X have the following pmf. Obtain the mean of $Y = X^2$.

x	0	1	2
$p(x)$	0.35	0.40	0.25

$$E(X^2) = \sum_x x^2\, p(x)$$

$$= 0^2 \times 0.35 + 1^2 \times 0.40 + 2^2 \times 0.25$$

$$= 1.40$$

Example

Let X have the following pmf. Obtain the variance of $Y = X^2$.

x	-1	0	1	2
$p_X(x)$	0.10	0.50	0.20	0.20

$$E(X^2) = \sum_x x^2\, p(x)$$

$$= (-1)^2 \times 0.10 + 0^2 \times 0.50 + 1^2 \times 0.20 + 2^2 \times 0.20$$

$$= 1.10$$

Continuous Case:

Let X be a continuous random variable with pdf $f(x)$. The expected value or mean of $Y = g(X)$ is obtained as

$$\mu_Y = E(g(X)) = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

Example

Let X have the following pdf. Obtain the mean of $Y = X^2$.

$$f(x) = \frac{1}{9}x^2, \quad 0 < x < 3.$$

$$\mu_Y = E(X^2) = \int_0^3 x^2 \, \frac{1}{9}x^2 \, dx = 5.40$$

Expectation of function of two random variables

Discrete case

Let X and Y be two discrete random variables with joint distribution $p(x, y)$. Then, expected value of $g(X, Y)$ is given by:

$$E\big(g(X, Y)\big) = \sum_x \sum_y g(x, y) \, p(x, y)$$

Example

Let the joint pmf of X and Y be as follows. Calculate $E(XY)$.

x	y			
	0	1	2	3
0	0.20	0.18	0.16	0.06
1	0.12	0.09	0.07	0.02
2	0.04	0.03	0.01	0.02

$$E(XY) = 0 \times 0 \times 0.20 + 0 \times 1 \times 0.18 + \cdots + 2 \times 3 \times 0.02 = 0.59$$

Continuous case

Let X and Y be two continuous random variables with joint distribution $f(x, y)$. Then, expected value of $g(X, Y)$ is given by:

$$E\big(g(X, Y)\big) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \, f(x, y) \, dy \, dx$$

Example

Let the joint pdf of X and Y be as follows. Calculate $E(XY)$.

$f(x, y) = x + y, \;\; 0 < x < 1, \;\; 0 < y < 1$

$$E(XY) = \int_0^1 \int_0^1 xy\,(x + y)\, dy\, dx$$

$$= \int_0^1 \int_0^1 (x^2 y \; + xy^2)\, dy\, dx$$

$$= 0.3333$$

Properties of expectation

1. $E(c) = c$
2. $E(aX + b) = a\,E(X) + b$
3. $E(X + Y) = E(X) + E(Y)$
4. More generally:

   $E(a_0 + a_1 X_1 + a_2 X_2 + \cdots + a_n X_n)$

   $= a_0 + a_1 E(X_1) + a_2 E(X_2) + \cdots + a_n E(X_n)$

Variance

Variance of a random variable X, denoted by σ^2 or $V(X)$, is defined as

$\sigma^2 = V(X) = E((X - E(X))^2) = E(X^2) - (E(X))^2$ (after simplification)

Example (discrete)

Let X have the following pmf. Obtain the variance of X.

x	0	1	2
$p(x)$	0.35	0.40	0.25

$$\mu = \sum_x x\, p(x)$$

$$= 0 \times 0.35 + 1 \times 0.40 + 2 \times 0.25$$

$$= 0.90$$

$$E(X^2) = \sum_x x^2\, p(x)$$

$$= 0^2 \times 0.35 + 1^2 \times 0.40 + 2^2 \times 0.25$$

$$= 1.40$$

$$V(X) = E(X^2) - \mu^2 = 1.40 - 0.90^2 = 0.59$$

Example (continuous)

Let X have the following pdf. Obtain the variance of X.

$$f(x) = \frac{1}{9}x^2, \;\; 0 < x < 3.$$

$$\mu = \int_0^3 x\, \frac{1}{9}x^2\, dx = 2.25$$

$$E(X^2) = \int_0^3 x^2\, \frac{1}{9}x^2\, dx = 5.40$$

$$V(X) = E(X^2) - \mu^2 = 5.40 - 2.25^2 = 0.3375$$

Exercise

Let X have the following pdf. Calculate the variance of X.

$$f(x) = x + 0.5, \;\; 0 < x < 1.$$

Solution

Do it yourself.

Covariance

Covariance of two random variables X and Y, denoted by σ_{XY} or $\text{Cov}(X, Y)$, is defined as

$$\sigma_{XY} = \text{Cov}(X, Y) = E\big((X - E(X))(Y - E(Y))\big)$$

$$= E(XY) - E(X)E(Y) \text{ (after simplification)}$$

When X and Y are independent, covariance is zero. However, when covariance is zero, X and Y may or may not be independent.

Example (discrete)

Let the joint pmf of X and Y be as follows. Calculate $\text{Cov}(X, Y)$.

x	y				$p_X(x)$
	0	1	2	3	
0	0.20	0.18	0.16	0.06	0.60
1	0.12	0.09	0.07	0.02	0.30
2	0.04	0.03	0.01	0.02	0.10
$p_Y(y)$	0.36	0.30	0.24	0.10	1.00

$E(XY) = 0 \times 0 \times 0.20 + 0 \times 1 \times 0.18 + \cdots + 2 \times 3 \times 0.02 = 0.59$

$\mu_X = 0 \times 0.60 + 1 \times 0.30 + 2 \times 0.10 = 0.50$

$\mu_Y = 0 \times 0.36 + 1 \times 0.30 + 2 \times 0.24 + 3 \times 0.10 = 1.08$

$\text{Cov}(X, Y) = 0.59 - 0.50 \times 1.08 = 0.05$

Example (Continuous)

Let the joint pdf of X and Y be as follows. Calculate $\text{Cov}(X, Y)$.

$f(x, y) = x + y, \;\; 0 < x < 1, \;\; 0 < y < 1.$

Solution:

$$E(XY) = \int_0^1 \int_0^1 xy\,(x + y)\,dy\,dx$$

$$= \int_0^1 \int_0^1 (x^2 y + xy^2)\,dy\,dx$$

$$= 0.3333$$

$f_X(x)$

$$= \int_0^1 (x + y)\,dy$$

$$= x + 0.5, \;\; 0 < x < 1$$

$$\mu_X = \int_0^1 x\,(x + 0.5)\,dx = 0.5833$$

Similarly, $\mu_Y = 0.5833$

$\text{Cov}(X, Y) = 0.3333 - 0.5833 \times 0.5833 = -0.0069$

Properties of variance

1. $V(c) = 0$
2. $V(aX + b) = a^2 V(X)$
3. $V(X + Y) = V(X) + V(Y) + 2\,\text{Cov}(X, Y)$

   $V(X - Y) = V(X) + V(Y) - 2\,\text{Cov}(X, Y)$
4. When X and Y are independent

   $V(X + Y) = V(X) + V(Y)$

   $V(X - Y) = V(X) + V(Y)$

Moments of a random variable

The rth raw moment of a random variable X is defined as

$\mu_r' = E(X^r)$

Note that $\mu_1' = E(X) = \mu$ is the mean.

The rth central moment of a random variable X is defined as

$\mu_r' = E((X - \mu)^r)$

Note that $\mu_2' = E((X - \mu)^2)$ is the variance.

Moment generating function

The moment generating function (mgf) of a random variable X, denoted by $M_X(t)$, is defined as

$M_X(t) = E(e^{tX})$,

where t is an auxiliary variable. If we expand the mgf, we get

$$M_X(t)$$

$$= E(e^{tX})$$

$$= E\left(1 + \frac{t}{1!}X + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \cdots\right)$$

$$= 1 + \frac{t}{1!}\,\mu_1' + \frac{t^2}{2!}\,\mu_2' + \frac{t^3}{3!}\,\mu_3' + \cdots$$

That is, the rth raw moment is the coefficient of $\frac{t^r}{r!}$ in the expansion of the moment generating function. Therefore, the rth raw moment is obtained by differentiating the mgf r times with respect to t and then setting t to zero. For example,

$$\mu_1' = \frac{d}{dt}M_X(t)\bigg|_{t=0}$$

$$\mu_2' = \frac{d^2}{dt^2}M_X(t)\bigg|_{t=0}$$

and so on.