Exercise:

Consider the following pmf.

x	1	3	4	9
$p(x)$	0.2	0.5	k	0.1

Determine (a) the value of k (b) $P(X = 3)$ (c) $P(X = 3 \text{ or } 4)$ (d) $P(1.7 < X < 4.1)$ (e) $P(X > 4)$ (f) $P(X \geq 4)$ and (g) $P(X < 1)$.

Solution:

(a) $0.2 + 0.5 + k + 0.1 = 1$

$\therefore k = 0.2$

(b) $P(X = 3) = 0.5$

(c) $P(X = 3 \text{ or } 4) = p(3) + p(4) = 0.7$

(d) $P(0.7 < X < 4.1) = p(1) + p(3) + p(4) = 0.9$

(e) $P(X > 4) = p(9) = 0.1$

(f) $P(X \geq 4) = p(4) + p(9) = 0.3$

(g) $P(X < 1) = 0.$

Exercise:

Check whether the following function is a pdf.

$f(x) = 2x, \;\; 0 < x < 1.$

Solution:

i)

$f(x) \geq 0$ for $0 < x < 1.$

ii)

$$\int_0^1 f(x)\,dx$$

$$= \int_0^1 2x\,dx$$

$$= \left.\frac{2x^2}{2}\right|_0^1$$

$$= 1$$

Therefore, $f(x)$ is a pdf.

Exercise:

Check whether the following function is a pdf.

$f(x) = x - 0.5, \;\; 0 < x < 2.$

Solution:

Do it yourself.

Exercise:

Consider the following pdf.

$f(x) = k(1 - x^2), \;\; -1 < x < 1.$

Determine (a) the value of k (b) $P(X > 0)$ (c) $P(-0.7 < X < 0.7)$ (d) density at $X = 0.5$ (e) $P(X = 0.5)$ (f) $P(X = 0.8)$.

Solution:

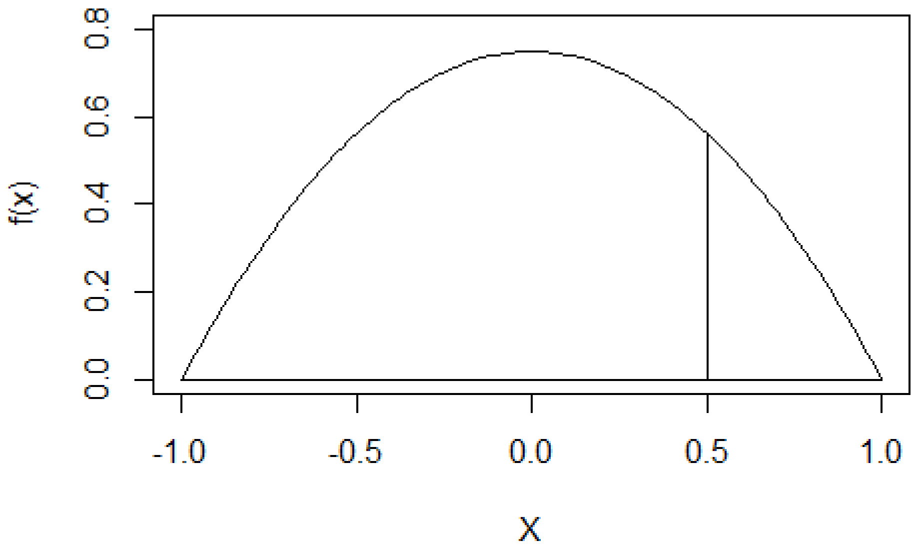
(a)

$$\int_{-1}^1 f(x)\,dx = 1$$

$$\therefore \int_{-1}^1 k(1 - x^2)\,dx = 1$$

$$\therefore k\left(x-\frac{x^3}{3}\right)\Bigg|_{-1}^{1}=1$$

$$\therefore k=\frac{3}{4}=0.75$$



(b)

$P(X>0)$

$=\int_0^1 0.75(1-x^2)\,dx$

$=0.5$

(c)

$P(-0.7<X<0.7)$

$=\int_{-0.7}^{0.7} 0.75(1-x^2)\,dx$

$=0.8785$

(d)

$f(0.5)=0.75(1-0.5^2)=0.5625$

Therefore, density at $X=0.5$ is 0.5625.

Note:

$f(0.5)$ is the height at $X = 0.5$. Height does not represent probability.

Area (integration) gives probability.

(e)

$P(X = 0.5)$

$$= \int_{0.5}^{0.5} 0.75(1 - x^2)\, dx$$

$$= 0.75\left(x - \frac{x^3}{3}\right)\Bigg|_{0.5}^{0.5}$$

$= 0$

Note:

For a continuous variable, probability of a single point is zero.

$P(X = a) = 0$ for any real number a.

(f)

$P(X = 0.8) = 0$ (since X is continuous)

Note:

$f(x)$ is NOT probability. It gives density at $X = x$. In fact, $f(x)$ can be greater than one. Consider the pdf below (from Exercise 1).

$f(x) = 2x, \quad 0 < x < 1.$

Here, $f(0.7) = 2 \times 0.7 = 1.4$ which is greater than one.

Cumulative distribution function
(or distribution function or cdf)

The cdf of a discrete or continuous random variable X, denoted by F, is defined as:

$F(a) = P(X \leq a)$

Discrete cases:

$$F(a) = \sum_{x \leq a} p(x)$$

Example:

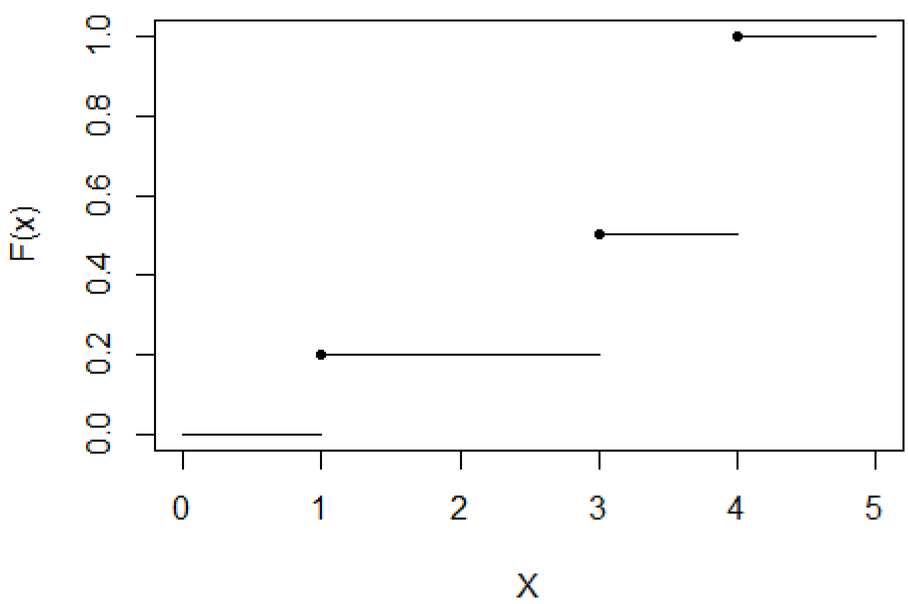
x	1	3	4
$p(x)$	0.2	0.3	0.5

Determine (a) the cdf (b) $P(X \leq 3)$ (c) $P(X \leq 3.7)$.

Solution:

(a)

$$F(a) = \begin{cases} 0, & a < 1 \\ 0.2, & 1 \leq a < 3 \\ 0.5, & 3 \leq a < 4 \\ 1, & a \geq 4 \end{cases}$$



(b) $P(X \leq 3) = F(3) = 0.5$

(c) $P(X \leq 3.7) = F(3.7) = 0.5$

Exercise:

x	0	1	3	5
$p(x)$	0.2	0.5	0.2	0.1

Determine (a) the cdf (b) $P(X \leq 2)$ (c) $P(X \leq 4.7)$.

Solution:

Do it yourself.

Exercise:

$$F(a) = \begin{cases} 0, & a < 0 \\ 0.3, & 0 \leq a < 2 \\ 0.7, & 2 \leq a < 3 \\ 1, & a \geq 3 \end{cases}$$

Determine the pmf.

Solution:

x	0	2	3
$p(x)$	0.3	0.4	0.3

Note:

For discrete cases:

From pmf to cdf: addition.

From cdf to pmf: subtraction

Continuous cases:

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)\, dx$$

Example:

Consider the following pdf.

$f(x) = 0.75(1 - x^2), \qquad -1 < x < 1.$

Determine (a) the cdf (b) $P(X \leq -0.5)$ (c) $P(X < 0.4)$ (d) $P(X > 0.7)$ (e) $P(0.3 < X < 0.6)$

Solution:

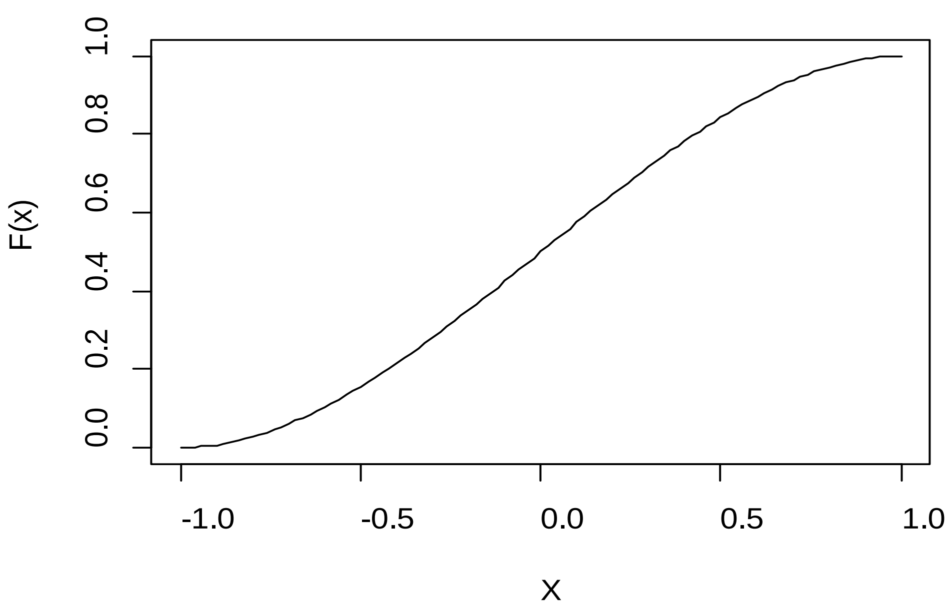
(a)

$$F(a) = \int_{-\infty}^{a} f(x)\, dx$$

$$= \int_{-1}^{a} 0.75(1 - x^2)\, dx$$

$$= (3a - a^3 + 2)/4$$

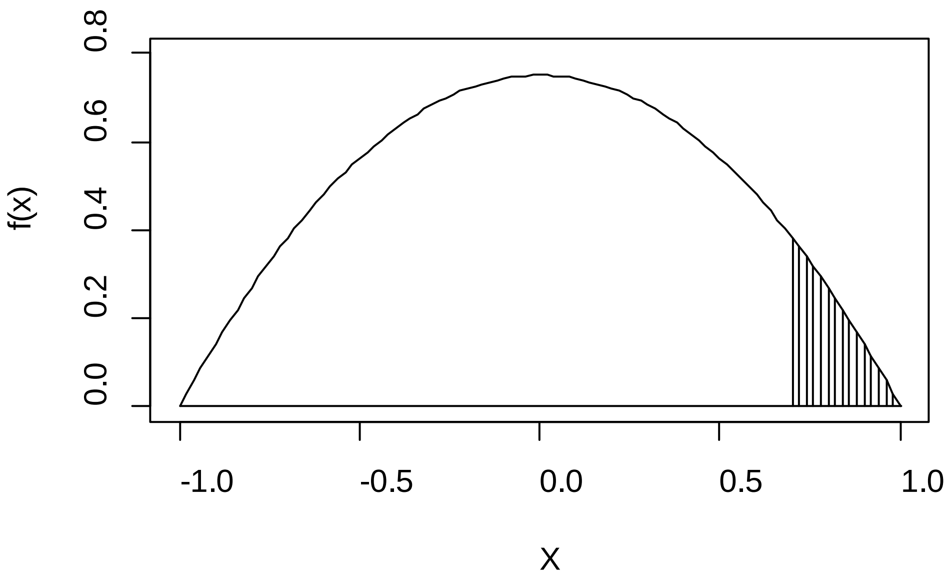
$$F(a) = \begin{cases} 0, & x \leq -1 \\ (3a - a^3 + 2)/4, & -1 < x < 1 \\ 1, & x \geq 1 \end{cases}$$



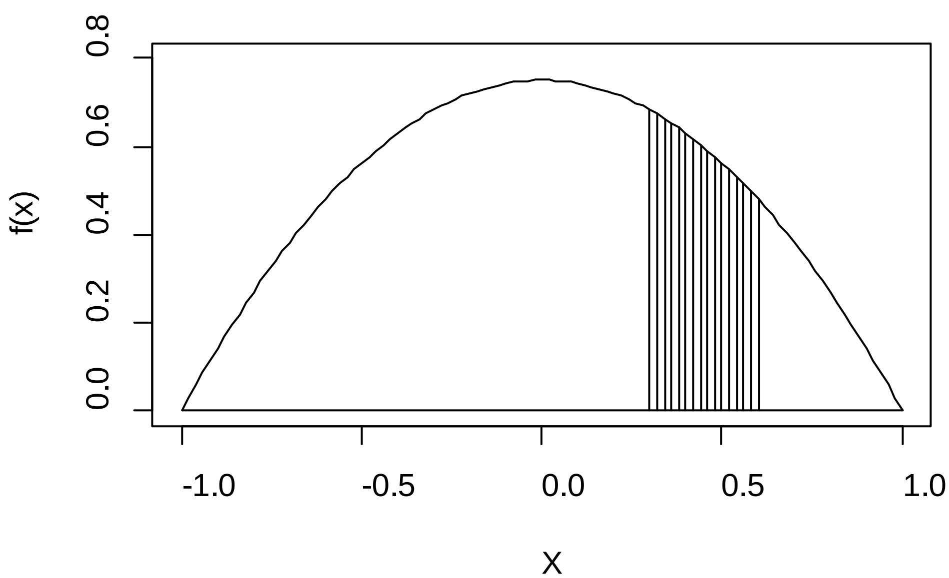
(b) $P(X \leq -0.5) = F(-0.5) = 0.1563$

(c) $P(X < 0.4) = F(0.4) = 0.784$

(d) $P(X > 0.7) = 1 - P(X \leq 0.7) = 1 - F(0.7) = 1 - 0.9393 = 0.0607$



(e) $P(0.3 < X < 0.6) = F(0.6) - F(0.3) = 0.896 - 0.718 = 0.178$



Example:

Consider the following pdf.

$f(x) = 2x, \;\; 0 < x < 1.$

Determine (a) the cdf (b) $P(X \leq 0.5)$ (c) $P(X < 0.7)$ (d) $P(X < 1.9)$ (e) $P(X < -0.4)$ (f) $P(X > 0.6)$ (g) $P(0.5 < X < 0.7)$.

Solution:

Do it yourself.

Exercise:

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x}{7}, & 0 < x < 7 \\ 1, & x \geq 7 \end{cases}$$

Determine the pdf.

Solution:

For $0 < x < 7$,

$$f(x) = \frac{d}{dx} F(x) = \frac{1}{7}$$

Thus, we have

$$f(x) = \begin{cases} \frac{1}{7}, & 0 < x < 7 \\ 0, & \text{otherwise} \end{cases}$$

Note:

For continuous cases:

From pdf to cdf: integration.

From cdf to pdf: differentiation

Properties of cdf:

- $F(-\infty) = 0$
- $F(\infty) = 1$
- If $a < b$, then $F(a) \leq F(b)$. That is, $F(x)$ is non-decreasing.