Independence of two random variables

Discrete Case:

Let X and Y be two discrete random variables with joint distribution $p(x,y)$. X and Y are said to be independent if, for each pair of values x and y,

$P(X = x, Y = y) = P(X = x)\, P(Y = y)$.

That is, X and Y are said to be independent if

$p(x,y) = p_X(x)\, p_Y(y)$

for each pair of values x and y.

Example

Let the joint pmf of X and Y be as follows. Are X and Y independent?

x	y			
	0	1	2	3
0	0.24	0.18	0.12	0.06
1	0.12	0.09	0.06	0.03
2	0.04	0.03	0.02	0.01

Solution

x	y				$p_X(x)$
	0	1	2	3	
0	0.24	0.18	0.12	0.06	0.60
1	0.12	0.09	0.06	0.03	0.30
2	0.04	0.03	0.02	0.01	0.10
$p_Y(y)$	0.40	0.30	0.20	0.10	1.00

$p_X(0)\, p_Y(0) = 0.60 \times 0.40 = 0.24 = p(0,0)$

It is true for each other pair of values of X and Y.

Therefore, X and Y are independent.

Example

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x	y			
	0	1	2	3
0	0.24	0.18	0.12	0.06
1	0.12	0.09	0.07	0.02
2	0.04	0.03	0.01	0.02

Solution

x	y				$p_X(x)$
	0	1	2	3	
0	0.24	0.18	0.12	0.06	0.60
1	0.12	0.09	0.07	0.02	0.30
2	0.04	0.03	0.01	0.02	0.10
$p_Y(y)$	0.40	0.30	0.20	0.10	1.00

$$p_X(1)\,p_Y(2) = 0.30 \times 0.20 = 0.06 \neq p(1,2)$$

Therefore, X and Y are NOT independent.

Continuous Case:

Let X and Y be two continuous random variables. X and Y are independent if

$$f(x,y) = f_X(x)\,f_Y(y)$$

for all x and y.

Example:

The joint pdf of X and Y is given below. Are X and Y independent?

$$f(x,y) = 4xy, \quad 0 < x < 1, \quad 0 < y < 1$$

Solution:

$f_X(x)$

$= \int_0^1 4xy\, dy$

$= 2x, \;\; 0 < x < 1$

$f_Y(y)$

$= \int_0^1 4xy\, dx$

$= 2y, \;\; 0 < y < 1$

$f_X(x)\, f_Y(y) = 2x\; 2y = 4xy = f(x, y)$

Therefore, X and Y are independent.

Example

The joint pdf of X and Y is given below. Are X and Y independent?

$f(x, y) = x + y, \;\; 0 < x < 1, \;\; 0 < y < 1.$

Solution

$f_X(x)$

$= \int_0^1 (x + y)\, dy$

$= x + 0.5, \;\; 0 < x < 1$

$f_Y(y)$

$= \int_0^1 (x + y)\, dx$

$= y + 0.5, \;\; 0 < y < 1$

$f_X(x)\, f_Y(y) = (x + 0.5)(y + 0.5) \neq f(x, y)$

Therefore, X and Y are NOT independent.