Mathematical Expectation

Discrete Case:

Let X be a discrete random variable with pmf $p(x)$. The mathematical expectation or expected value or population mean of X, denoted by $E(X)$ or μ, is defined as

$$\mu = E(X) = \sum_x x\,p(x)$$

Example

Let X have the following pmf.

x	0	1	4
$p(x)$	0.25	0.50	0.25

$$\mu = \sum_x x\,p(x)$$

$$= 0 \times 0.25 + 1 \times 0.50 + 4 \times 0.25$$

$$= 1.5$$

Comparison with sample mean

Sample mean is defined as

$$\bar{x} = \frac{1}{n}\sum_i x_i\,f_i = \sum_i x_i \frac{f_i}{n}$$

Population mean is defined as

$$\mu = \sum_x x\,p(x)$$

That is, for sample mean each value is multiplied by relative frequency, while for population mean each value is multiplied by probability.

Example: population mean vs sample mean

Consider the following population (pmf):

x	1	2	7
$p(x)$	0.25	0.50	0.25

Here, the population mean is

$$\mu = \sum_x x\, p(x)$$

$$= 1 \times 0.25 + 2 \times 0.50 + 7 \times 0.25$$

$$= 3.00$$

Let us take a sample of size $n = 1000$ from this population and construct the following frequency table.

X	Frequency	Relative Frequency
1	240	0.24
2	510	0.51
7	250	0.25

The sample mean is

$$\bar{x} = \sum_i x_i \frac{f_i}{n}$$

$$= 1 \times 0.24 + 2 \times 0.51 + 7 \times 0.25$$

$$= 3.01$$

Continuous Case:

Let X be a continuous random variable with pdf $p(x)$. The mathematical expectation or expected value or population mean of X, denoted by $E(X)$ or μ, is defined as

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

Example

Let X have the following pdf.

$$f(x) = \frac{1}{9}x^2, \;\; 0 < x < 3.$$

$$\mu = E(X) = \int_0^3 x\, \frac{1}{9}x^2\, dx = 2.25$$

Example: population mean vs sample mean

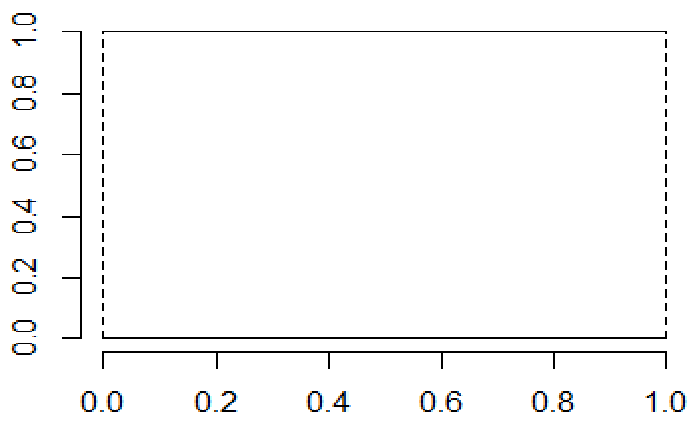
Let us consider the population (pdf):

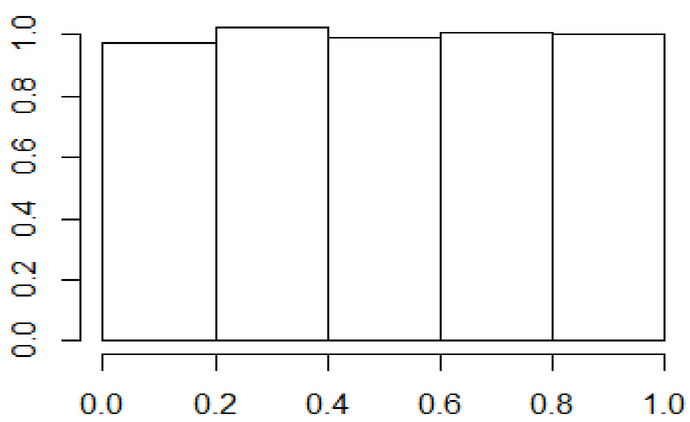
$f(x) = 1, \;\; 0 < x < 1$

Let us take a sample of size $n = 1000$ from this population and construct the following frequency table.

X	Frequency	Relative Frequency	Density
0.0 – 0.2	196	0.196	0.98
0.2 – 0.4	204	0.204	1.02
0.4 – 0.6	199	0.199	1.00
0.6 – 0.8	201	0.201	1.01
0.8 – 1.0	200	0.200	1.00

Let us compare the pdf (first plot below) and histogram (second plot below):





The population mean is

$$\mu = E(X) = \int_0^1 x\, 1\, dx = 0.50$$

The sample mean is

$$\bar{x} = \sum_i m_i \frac{f_i}{n}$$

$$= 0.1 \times 0.196 + 0.3 \times 0.204 + 0.5 \times 0.199 + 0.7 \times 0.201 + 0.9 \times 0.200$$

$$= 0.501$$

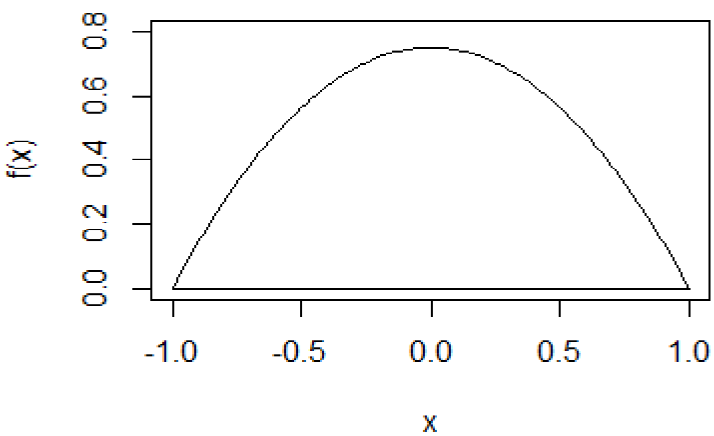
Example

Let X have the following pdf.

$$f(x) = 0.75\,(1 - x^2), \quad -1 < x < 1.$$

$$\mu = E(X) = \int_{-1}^{1} x\, 0.75\,(1 - x^2)\, dx = 0$$

This is a symmetric distribution. So, mean is exactly in the middle.



Exercise

Let X have the following pdf. Calculate the mean of X.

$$f(x) = x + 0.5, \quad 0 < x < 1.$$

Solution

Do it yourself.