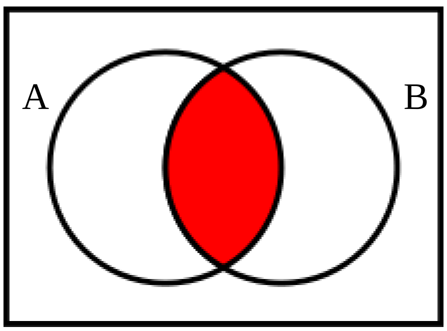
Conditional Probability

Probability calculated under a condition or additional information is called conditional probability.

Notation

$P(B \mid A)$ is the conditional probability that event B occurs given that the event A occurs.



Example (Dice throwing)

$A = \{1, 2, 3\}$ and $B = \{1, 4\}$. We want to calculate $P(B \mid A)$.

If A occurs, then the sample space is reduced to $A = \{1, 2, 3\}$. That is, "4" cannot happen. Only the part of B that is in A may or may not happen. Thus, B happens if $A \cap B = \{1\}$ happens.

$$\therefore P(B \mid A) = \frac{1}{3}$$

Note that

$$P(B \mid A) = \frac{1}{3} = \frac{1/6}{3/6} = \frac{P(A \cap B)}{P(A)}$$

Thus, the formula for conditional probability:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad \text{when } P(A) \neq 0$$

If we cross-multiply, we get the Multiplication Rule:

$$P(A \cap B) = P(A)\, P(B \mid A)$$

Exercise

In a class, 60% of the students are boys. 15% of the students are boys that play cricket. A student is selected at random. If the selected student is a boy, what is the probability that he plays cricket?

Solution

$P(B) = 0.60$

$P(B \cap C) = 0.15$

$$P(C \mid B) = \frac{P(B \cap C)}{P(B)} = \frac{0.15}{0.60} = 0.25$$

Exercise

In a class, 60% of the students are boys. 25% of the boys play cricket. A student is selected at random. What is the probability that the student will be a boy who plays cricket?

Solution

$P(B) = 0.60$

$P(C \mid B) = 0.25$

$P(B \cap C) = P(B)\, P(C \mid B) = 0.60 \times 0.25 = 0.15$

Independence of events

Two events A and B are independent if occurrence of A does not change the probability of B, that is,

$P(B \mid A) = P(B)$.

If A and B are independent, then the multiplication rule reduces to

$P(A \cap B) = P(A)\, P(B)$

If A, B and C are independent, then

$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$

Exercises

1. Two coins are tossed. Let A be the event that the first coin shows head, and B be the event that the second coin shows tail. Are A and B independent? (Intuitively, they are independent. However, we have to show mathematically.)

Solution

$S = \{HH, HT, TH, TT\}$, $A = \{HH, HT\}$, $B = \{HT, TT\}$ and $A \cap B = \{HT\}$.

$P(A) = \frac{2}{4} = 0.50$, $P(B) = \frac{2}{4} = 0.50$ and $P(A \cap B) = \frac{1}{4} = 0.25$.

We see that $0.50 \times 0.50 = 0.25$. That is, $P(A)\ P(B) = P(A \cap B)$.

Therefore, A and B are independent.

* Note that, in the exercise above, the two events A and B are not mutually exclusive, but independent.

2. In a town, 20% of families have cars, 10% of families have washing machines and 5% of families have both. Are “having a car” and “having a washing machine” independent?

Solution

$P(C) = 0.20$, $P(W) = 0.10$ and $P(C \cap W) = 0.05$.

We see that, $0.20 \times 0.10 = 0.02 \neq 0.05$. That is, $P(C)\ P(W) \neq P(C \cap W)$.

Therefore, C and W are not independent.

* Note that, in the exercise above, C and W are not mutually exclusive, but dependent.

3. A dice is tossed. Let $A = \{1\}$ and $B = \{3\}$. Are A and B independent?

Solution:

$P(A) = \frac{1}{6}$.

$P(B) = \frac{1}{6}$.

$A \cap B = \emptyset$

$P(A \cap B) = 0 \neq P(A)\ P(B)$.

Therefore, A and B are dependent (NOT independent).

* Note that, in the exercise above, A and B are mutually exclusive, but dependent!! When '1' occurs, '3' cannot occur, and vice versa. So, they control each other!

Mutually exclusive events vs Independent events

When A and B are mutually exclusive: $P(A \cap B) = 0$.

When A and B are independent: $P(A \cap B) = P(A)\,P(B) \neq 0$ (for events with nonzero probabilities).

Therefore, mutually exclusive events are NOT independent!!

Further explanation: Let A and B be mutually exclusive events. When A occurs, B cannot occur. That is, occurrence of A stops the occurrence of B. Also, occurrence of B stops the occurrence of A. Thus, A and B depend on each other.

Example

Three fair coins are tossed. $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. Each outcome has probability $1/8$. We can use independence to calculate the probabilities.

$$P(HHH) = P(H)\,P(H)\,P(H) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Exercise

An unfair coin (with probability of head 0.7) is tossed three times. Calculate the probabilities of all the possible outcomes.

Solution: Do it yourself

Exercise

A machine works if at least two of the three components A, B and C work. Each component works independently with probability 0.8. What is the probability that the machine works?

Solution: Do it yourself