Different tests:

Equality of two means

Let $X_1, X_2, \cdots, X_n$ be a random sample from $N(\mu_1, \sigma_1^2)$. Let $Y_1, Y_2, \cdots, Y_m$ be a random sample from $N(\mu_2, \sigma_2^2)$ and the two samples are independent. Suppose that all four parameters are unknown. We want to test

H_0: $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$

at level of significance α.

When we can assume that the two population variances are equal: $\sigma_1^2 = \sigma_2^2 = \sigma^2$, that is, the sample variances S_1^2 and S_2^2 are close, we calculate

$$S_P^2 = \frac{(n-1)\, S_1^2 + (m-1)\, S_2^2}{n+m-2}$$

which is a pooled estimator of the common variance σ^2. Then the test statistic

$$T = \frac{\bar{X} - \bar{Y}}{S_P\sqrt{\frac{1}{n} + \frac{1}{m}}}$$

has a t distribution with $n + m - 2$ d.f. under the null hypothesis.

This test is known as **2-sample *t* test** or independent samples t test.

Example

Twenty-two volunteers at a cold research institute caught cold after having been exposed to various cold viruses. Ten of these volunteers were given tablets containing one gram of vitamin C four times a day. The other group (control group) was given placebo tablets. The length of time (days) the cold lasted was recorded for each volunteer. The sample means were 6.45 and 7.13, and the sample standard deviations were 0.58 and 0.78, respectively. Does vitamin C reduce the mean length of time cold lasts?

Solution

Let μ_1 = mean for the control group and μ_2 = mean for vitamin C group.

H_0: $\mu_1 = \mu_2$

H_1: $\mu_1 > \mu_2$

level of significance $\alpha = 0.05$.

$S_P^2 = 0.69$

$T_{\text{obs}} = 1.90$

$t_{0.95,20} = 1.725$

Since $T_{\text{obs}} > t_{0.95,20}$, we reject the null hypothesis and conclude that vitamin C is effective for cold.

Note: For a two-sided test, we reject null if $T_{\text{obs}} < t_{0.025,\text{df}}$ or $T_{\text{obs}} > t_{0.975,\text{df}}$.

When the two population variances are not equal:

In this case, if the sample size is large, due to CLT we assume that

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{m}}}$$

follows standard normal distribution under null hypothesis.

Equality of two means: Paired *t* test

For bivariate data, we collect a pair of observations (x_i, y_i) from the ith individual or object, $i = 1, 2, \cdots, n$. If we want to test the equality of means of X and Y, we calculate $D = X - Y$ so that

$$d_i = x_i - y_i, \quad i = 1, 2, \cdots, n.$$

Testing H_0: $\mu_1 = \mu_2$ is now same as testing H_0: $\mu_D = 0$. The test statistic is

$$T = \frac{\bar{D}}{S_D/\sqrt{n}}$$

which follows t distribution with $n-1$ d.f. under null hypothesis.

Example

An industrial safety program was recently instituted in an industry. The weekly loss in labor-hours due to accidents in 10 similar plants both before and after the program are as follows. Test whether the safety program has been effective.

Before	After
30.5	23.0
18.5	21.0
24.5	22.0
32.0	28.5
16.0	14.5
15.0	15.5
23.5	24.0
25.5	21.0
28.0	23.5
18.0	16.5

Solution: Do it yourself.

Test about variance

Suppose that the data follow normal distribution and we want to test

H_0: $\sigma^2 = \sigma_0^2$

H_1: $\sigma^2 > \sigma_0^2$

at level of significance 0.05.

Here, the test statistic is

$(n-1)\,S^2 \,/\, \sigma_0^2$

which follows χ^2 (chi-square) distribution with $n-1$ degrees of freedom under the null hypothesis. We reject the null hypothesis if $(n-1)\,S^2 \,/\, \sigma_0^2 > \chi^2_{0.95,n-1}$.

Example

A machine that automatically controls the amount of ribbon on a tape has recently been installed. This machine will be judged to be ineffective if the standard deviation

σ of the amount of ribbon on a tape is more than 0.15 cm. If a sample of 20 tapes yields a sample variance of 0.025 cm^2, what is your conclusion?

Solution

$H_0\colon\ \sigma^2 = 0.0225$

$H_1\colon\ \sigma^2 > 0.0225$

$$\frac{(n-1)\,S^2}{\sigma_0^2} = \frac{(20-1)\,0.025}{0.0225} = 21.11$$

$\chi^2_{0.95,19} = 30.14$

We cannot reject the null hypothesis. We cannot say that the machine is ineffective.

Test in Bernoulli population

Let $X_1, X_2, \cdots, X_n \sim \text{Bernoulli}(p)$. We want to test

$H_0\colon\ p = p_0$

$H_1\colon\ p > p_0$

If the sample size is large,

$$Z = \frac{\sum_{i=1}^{n} X_i - np_0}{\sqrt{np_0(1-p_0)}}$$

follows standard normal distribution under null.

Example

A computer chip manufacturer claims that no more than 2 percent of the chips it sends out are defective. An electronics company, impressed with this claim, has purchased a large quantity of such chips. To determine if the manufacturer's claim can be taken literally, the company has decided to test a sample of 300 of these chips. If 11 of these 300 chips are found to be defective, should the manufacturer's claim be rejected?

Solution

$H_0\colon\ p = 0.02$

$H_1\colon\ p > 0.02$

$$Z = \frac{\sum_{i=1}^{n} X_i - np_0}{\sqrt{np_0(1-p_0)}} = \frac{11 - 300 \times 0.02}{\sqrt{300 \times 0.02\ (1-0.02)}} = 2.06$$

$Z_{0.95} = 1.645$

So, the manufacturer's claim can be rejected.

Note:

For tests concerning the mean of Poisson distribution, we can use the fact that, for large λ, a Poisson random variable with parameter λ is approximately normally distributed with mean and variance equal to λ.