1. $\frac{1}{2}e^{\mathtt{j}\,\pi} = -\frac{1}{2}+0\,\mathtt{j}$, $\frac{1}{2}e^{-\mathtt{j}\,\pi} = -\frac{1}{2}+0\,\mathtt{j}$, $e^{\mathtt{j}\,\pi/2} = 0+\mathtt{j}$, $e^{-\mathtt{j}\,\pi/2} = 0-\mathtt{j}$, $e^{\mathtt{j}\,5\pi/2} = e^{\mathtt{j}\,\pi/2} = 0+\mathtt{j}$, $\sqrt{2}e^{\mathtt{j}\,\pi/4} = 1+\mathtt{j}$, $\sqrt{2}e^{\mathtt{j}\,9\pi/4} = \sqrt{2}e^{\mathtt{j}\,\pi/4} = 1+\mathtt{j}$.

2. $5 = 5e^{\mathtt{j}\,0}$, $-2 = 2e^{\mathtt{j}\,\pi}$, $-3\,\mathtt{j} = 3e^{-\mathtt{j}\,\pi/2}$, $1+\mathtt{j} = \sqrt{2}e^{\mathtt{j}\,\pi/4}$, $(1-\mathtt{j})^2 = 1-2\,\mathtt{j}+\mathtt{j}^2 = -2\,\mathtt{j} = 2e^{-\mathtt{j}\,\pi/2}$, $\mathtt{j}(1-\mathtt{j}) = \mathtt{j}-\mathtt{j}^2 = 1+\mathtt{j} = \sqrt{2}e^{\mathtt{j}\,\pi/4}$, $\frac{1+\mathtt{j}}{1-\mathtt{j}} = \frac{(1+\mathtt{j})(1+\mathtt{j})}{(1-\mathtt{j})(1+\mathtt{j})} = \frac{1+2\,\mathtt{j}+\mathtt{j}^2}{1+1} = \mathtt{j} = e^{\mathtt{j}\,\pi/2}$.

3. (a) The function f function is not injective since $f(-1) = f(1)$, so "distinctness" is not preserved. It is not surjective either, since $f^{-1}(-1) = \varnothing$. We have $\mathrm{im}_f(\mathbb{R}) = [0,\infty) = \{x \in \mathbb{R} : x \geq 0\}$, i.e., the range of f is the set of nonnegative real numbers.

   (b) The function g is not injective since $g(-1) = g(1)$; however, g *is* surjective since $g^{-1}(y) = \{\pm\sqrt{y}\}$, which is not empty for any $y \in [0,\infty)$. As in the previous question, $\mathrm{im}_g(\mathbb{R}) = [0,\infty)$.

   This problem shows that associated with any nonsurjective function is a surjective function obtained from the original function by redefining its codomain to agree with its range. The two functions f and g are considered *different*, since their codomains are not the same.

   (c) The complex exponential function is not injective, since, for example, $\exp(0) = \exp(2\pi\mathbf{j}) = 1$. It is not surjective since $\exp(z) \neq 0$ for any $z \in \mathbb{C}$. The range of the complex exponential function is $\mathbb{C} \setminus \{0\}$, i.e., the entire complex plane excluding the origin.

   (d) The function L is injective, since if $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$ we must have $x_1+x_2 = y_1+y_2$ and $x_1-x_2 = y_1-y_2$. By adding these equations we deduce that $2x_1 = 2y_1$, i.e., $x_1 = y_1$. By subtracting these equations we deduce that $2x_2 = 2y_2$, i.e., $x_2 = y_2$. Thus L never maps distinct elements of the domain to the same element of the codomain. The function L is surjective since every element $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ of the codomain has the singleton set $\left\{\frac{1}{2}\begin{bmatrix} y_1+y_2 \\ y_1-y_2 \end{bmatrix}\right\}$ as its inverse image under mapping by L. Being both injective and surjective, we can conclude that L is bijective. The inverse function is given as $L^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \overset{L^{-1}}{\mapsto} \frac{1}{2}\begin{bmatrix} y_1+y_2 \\ y_1-y_2 \end{bmatrix}$. The range of L is $\mathbb{R}^2$.

4. Let $A = \{a_1, \ldots, a_{|A|}\}$. Consider defining a function $f : A \to B$. We have $|B|$ distinct choices for $f(a_1)$, we have $|B|$ distinct choices for $f(a_2)$, and so on, up to and including $|B|$ distinct choices for $f(a_{|A|})$. These choices can be combined freely, so in total we will have $|B| \cdot |B| \cdot \cdots \cdot |B| = |B|^{|A|}$ distinct functions from A to B. Stated more succinctly, $|B^A| = |B|^{|A|}$.

5. (a) $x[n-3]$ has support $\{1, 2, \ldots, 7\}$, (b) $x[n+4]$ has support $\{-6, -5, \ldots, 0\}$, (c) $x[-n]$ has support $\{-4, -3, \ldots, 2\}$, (d) $x[-n+2]$ has support $\{-2, -1, \ldots, 4\}$, (e) $x[-n-2]$ has support $\{-6, -5, \ldots, 0\}$.

6. $x(t)$ is guaranteed to be zero for values of t outside of its support, i.e., for all $t < 3$. Thus (a) $x(1-t)$ is guaranteed to be zero for all $t > -2$, (b) $x(1-t) + x(2-t)$ is guaranteed to be zero for all $t > -1$, (c) $x(1-t)x(2-t)$ is guaranteed to be zero for all $t > -2$, (d) $x(3t)$ is guaranteed to be zero for all $t < 1$, (e) $x(t/3)$ is guaranteed to be zero for all $t < 9$.

7. (a) Note that
$$S = 1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1},$$
while
$$\alpha S = \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^N.$$
It follows that $S - \alpha S = 1 - \alpha^N$. Thus $S(1-\alpha) = 1 - \alpha^N$ for any value of α. If $\alpha \neq 1$, we can divide both sides by $(1-\alpha)$ to obtain $S = \frac{1-\alpha^N}{1-\alpha}$. If $\alpha = 1$, we find $S = 1 + 1 + \cdots + 1 = N$.

   (b) When $|\alpha| < 1$, $\lim_{N\to\infty} \frac{1-\alpha^N}{1-\alpha} = \frac{1}{1-\alpha}$, since $\lim_{N\to\infty} \alpha^N = 0$.

   (c) For any integer k, assuming $|\alpha| < 1$,
$$\sum_{n=k}^{\infty} \alpha^n = \alpha^k \sum_{n=0}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha}.$$