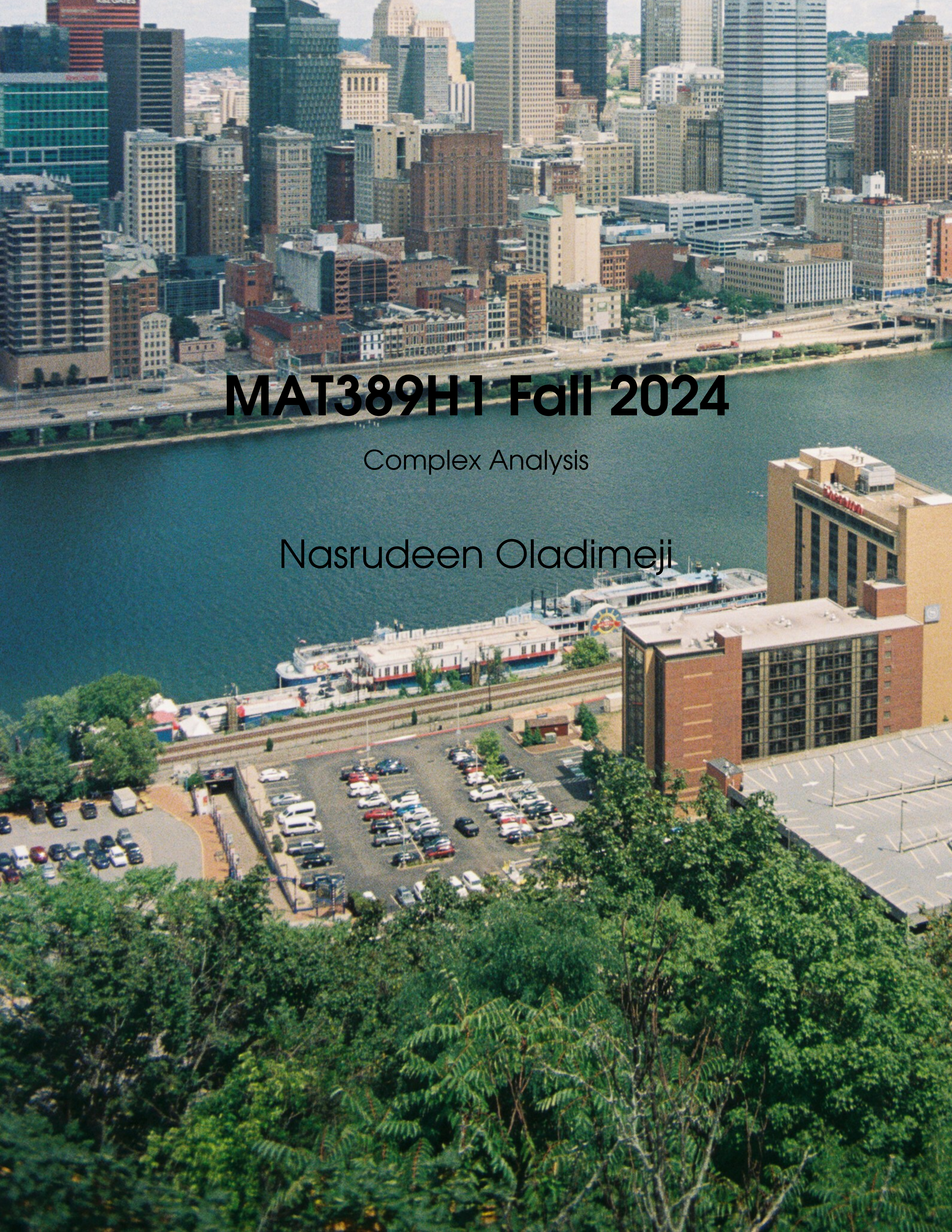

MAT389H1 Fall 2024

Complex Analysis

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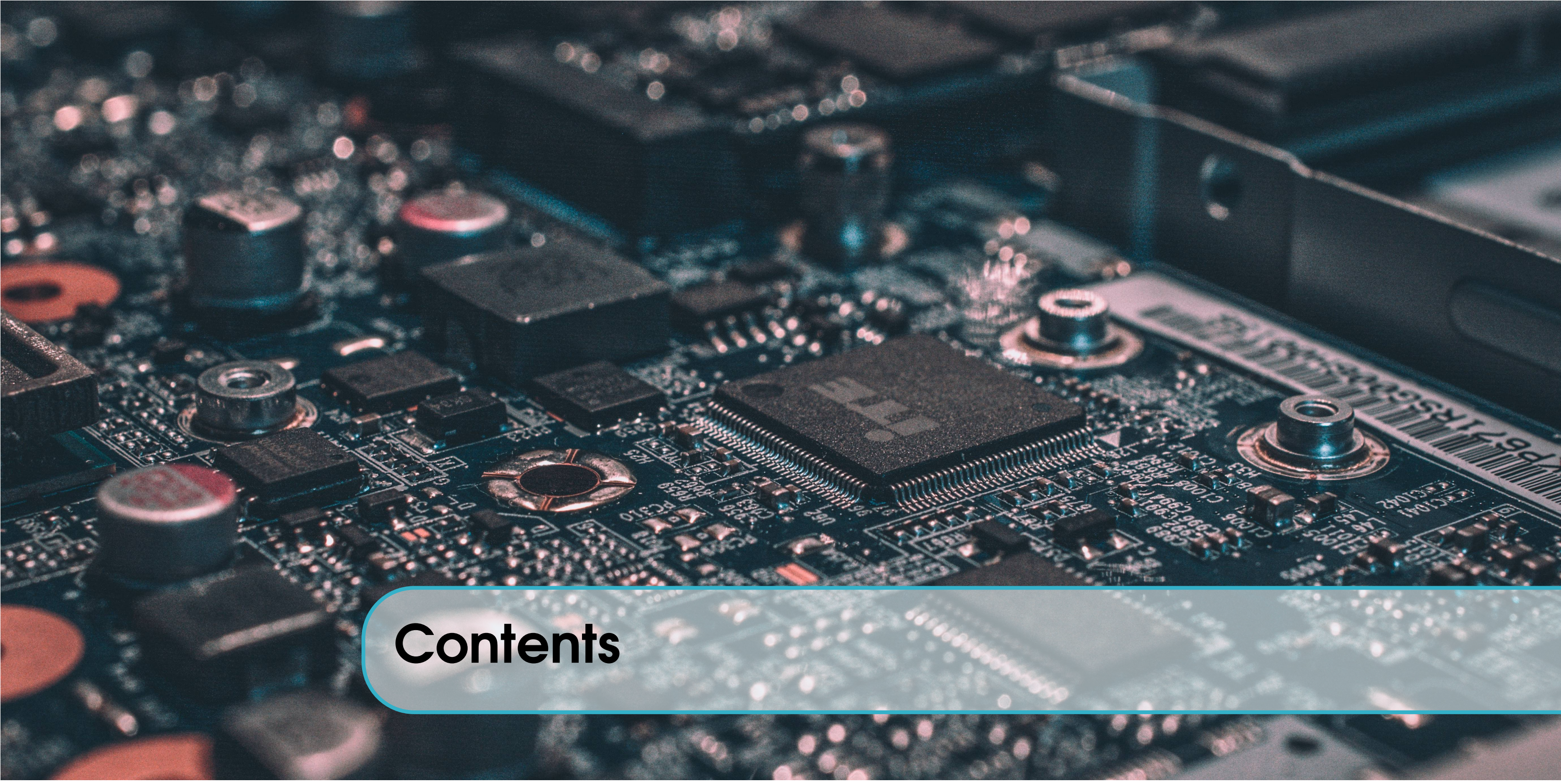
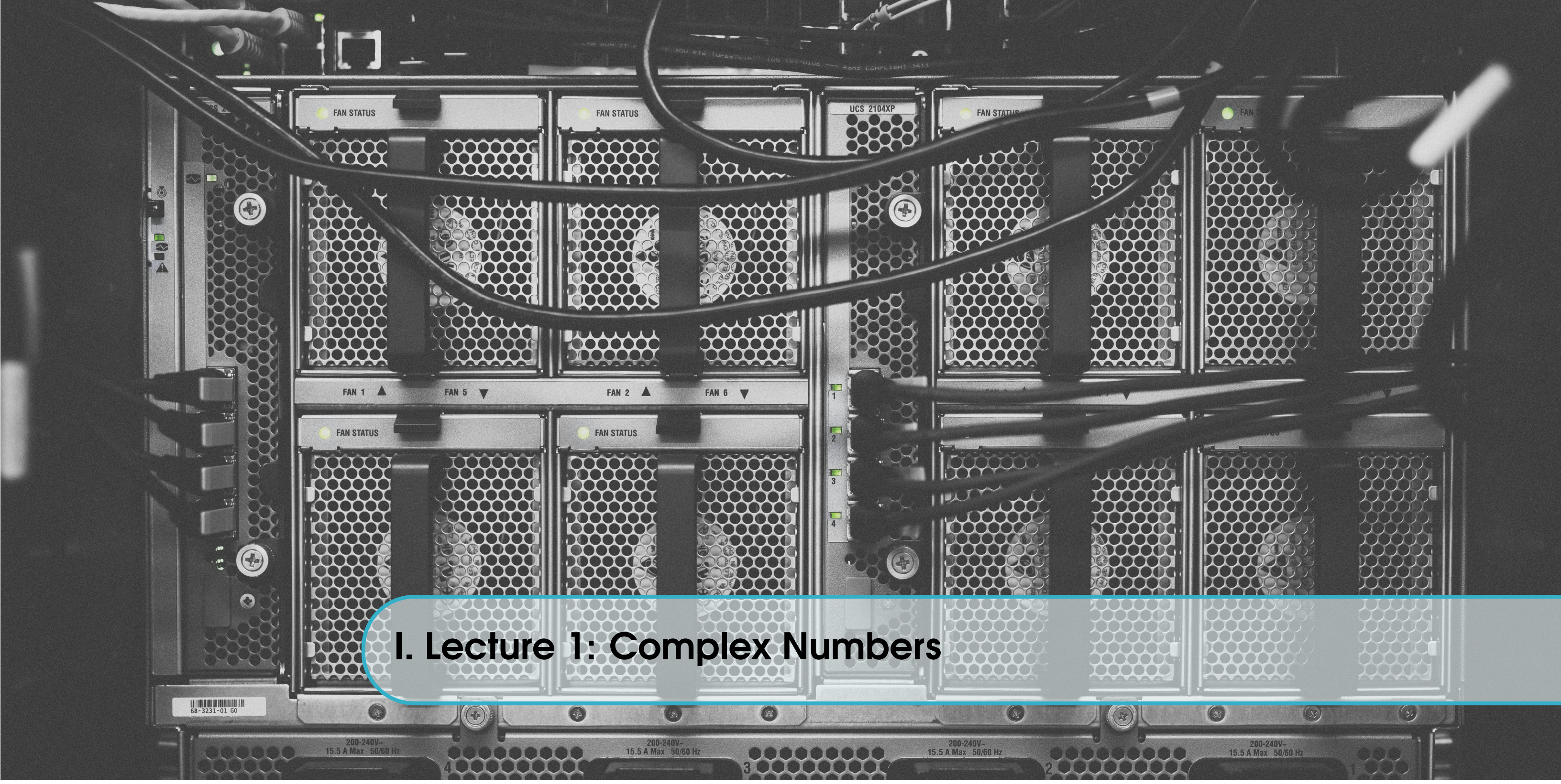


Contents



I. Lecture 1: Complex Numbers

I.I Introduction

Definition I.I.I — Complex Numbers. z is a complex number iff $z = a + bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.
The set of complex numbers is denoted by $\mathbb{C}$.

Definition I.I.II — Real and Imaginary Parts. If $z = a + bi$, then $\Re(z) = a \in \mathbb{R}$ and $\Im(z) = b \in \mathbb{R}$. where $\Re(z)$ is the real part of z and $\Im(z)$ is the imaginary part of z.

Definition I.I.III — Modulus. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$. $|z|$ is the modulus of z.

Definition I.I.IV — Conjugate. If $z = a + bi$, then $\bar{z} = a - bi$. $\bar{z}$ is the conjugate of z.

I.II Operations

Definition I.II.I — Addition and Subtraction. If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$.
Similarly $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$.

Definition I.II.II — Multiplication. If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then $z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$.
Note that $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

Definition I.II.III — Inversion. If $z = a + bi$, then $z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$.

Proof. Let's multiply by 1 in the form of the conjugate of z:

$$\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

■

Definition I.II.IV — Division. For $z, w \in \mathbb{C}$, $\frac{w}{z} = w \cdot z^{-1} = \frac{w\bar{z}}{|z|^2}$.

Table I.I: Properties of the Complex Conjugate

Property	Description
Conjugate of the Sum	$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
Conjugate Modulus	$z \cdot \bar{z} = \|z\|^2$
Conjugate of a Conjugate	$\bar{\bar{z}} = z$
Product of Conjugates	$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
Conjugate of a Quotient	$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
Real Part Conjugate	$Re(z) = \frac{z+\bar{z}}{2}$
Imaginary Part Conjugate	$Im(z) = \frac{z-\bar{z}}{2i}$
Real Number Check	$z = \bar{z} \iff z \in \mathbb{R}$
Imaginary Number Check	$z = -\bar{z} \iff z \in \mathbb{I}$
Function Linearity	If $\alpha = f(z)$ then $\overline{\alpha} = \overline{f(z)} = f(\bar{z})$

Table I.II: Properties of the Modulus in Complex Numbers

Property	Description
Positivity	$\|z\| \geq 0$, with equality if and only if $z = 0$
Triangle Inequality	$\|\|z_1\| - \|z_2\|\| \leq \|z_1 \pm z_2\| \leq \|z_1\| + \|z_2\|$
Multiplicative Property	$\|z_1 \cdot z_2\| = \|z_1\| \cdot \|z_2\|$
Division Property	$\left\|\frac{z_1}{z_2}\right\| = \frac{\|z_1\|}{\|z_2\|}$, for $z_2 \neq 0$
Conjugate	$\|z\| = \|\bar{z}\|$
Component Property	$-\|z\| \leq Re(z) \leq \|z\|$ $-\|z\| \leq Im(z) \leq \|z\|$
Cauchy-Schwarz Inequality	$\|z_1 w_1 + \cdots + z_n w_n\|^2 \leq \sum_{j=1}^{n} \|z_j\|^2 \sum_{j=1}^{n} \|w_j\|^2$

Proof. Proof of the Multiplicative Property of the Modulus:

$$\begin{aligned} |z_1 \cdot z_2|^2 &= (z_1 \cdot z_2) \cdot (\overline{z_1} \cdot \overline{z_2}) \\ &= z_1 \cdot \overline{z_1} \cdot z_2 \cdot \overline{z_2} \\ &= |z_1|^2 \cdot |z_2|^2 \end{aligned}$$

■

I.III Polar Representation

A complex number are vectors in $\mathbb{R}^2$, as such, they can be represented by a magnitude and a direction.

Definition I.III.I — Polar Form.

$$z = r(\cos(\theta) + i\,\sin(\theta)) \tag{I.I}$$

: $r = |z| \in \mathbb{R}^+$

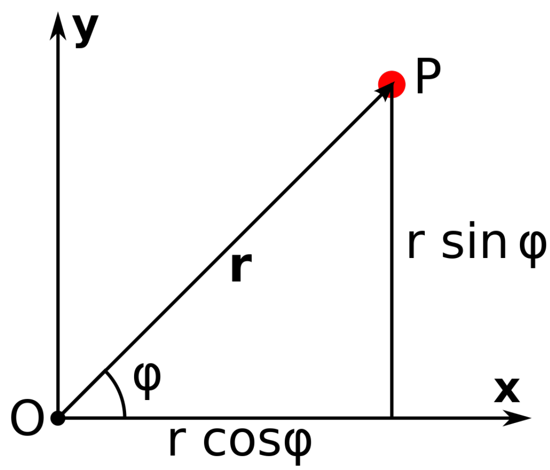


Figure I.1: Polar Coordinate Components

■ **Example I.I — Multiplying Complex Numbers in Polar Form.** Let $z_1 = r_1(\cos(\theta_1) + i\,\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\,\sin(\theta_2))$. Then:

$$z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1)\,\cos(\theta_2) - \sin(\theta_1)\,\sin(\theta_2) + i(\cos(\theta_1)\,\sin(\theta_2) + \sin(\theta_1)\,\cos(\theta_2))) \tag{I.II}$$

$$= r_1 r_2(\cos(\theta_1 + \theta_2) + i\,\sin(\theta_1 + \theta_2)) \tag{I.III}$$

Using the trig addition formula:

$\cos(\alpha+\beta) = \cos(\alpha)\,\cos(\beta) - \sin(\alpha)\,\sin(\beta)$ and $\sin(\alpha+\beta) = \cos(\alpha)\,\sin(\beta) + \sin(\alpha)\,\cos(\beta)$.

■

Theorem I.III.I — De Moivre's Theorem. if $z = r(\cos(\theta) + i\,\sin(\theta))$

$$z^n = r^n(\cos(\theta n) + i\,\sin(\theta n)) \tag{I.IV}$$

Proof. The following proof will illustrative the steps to inductive reasoning
Case of $n = 1$: $z^n = r^n(\cos(\theta n) + i\,\sin(\theta n)) = z = r(\cos(\theta) + i\,\sin(\theta))$
This is true by definition.
Assume that:
$z^{n-1} = r^{n-1}(\cos(\theta(n-1)) + i\,\sin(\theta(n-1)))$
Then from Equation (I.III) we can verify:

$$zz^{n-1} = rr^{n-1}(\cos(\theta(n-1) + \theta) + i\,\sin(\theta(n-1) + \theta))$$
$$z^n = r^n(\cos(\theta n) + i\,\sin(\theta n))$$

■

Definition I.III.II — Argument. The argument of a complex number $z = r(\cos(\theta) + i\sin(\theta))$ is any angle, $\arg(z) = \theta$, such that $z = r(\cos(\theta) + i\sin(\theta))$.

From Equation (I.I), we observe that r is unique (because we constrained it to just positive values). θ, however, is not unique.

Definition I.III.III — Principle Orientation. We say θ is the principle orientation of z if $\theta \in [-\pi, \pi)$
In this range, θ is unique.

Definition I.III.IV — Vector Dot Product. The dot product of two complex numbers $z = x + iy$ and $w = s + it$ is defined as:

$$z \cdot w = x \cdot s + y \cdot t = \Re(z\overline{w}) \tag{I.V}$$

$$\cos\theta = \frac{a \cdot b}{|a||b|} \tag{I.VI}$$

Corollary I.III.II — Perpendicular Vectors. Complex variables z and w are perpendicular if $\Re(z\overline{w}) = 0$.

R [Complex Numbers to Solve Polynomial Equations] Over $\mathbb{C}$, every equation of the form $z^n = a$ has n solutions.

■ **Example I.II — Solving $z^n = -1$.** Let $z = r(\cos(\theta) + i\sin(\theta))$. Then:

$$\begin{aligned}
z^n &= r^n(\cos(\theta n) + i\sin(\theta n)) = -1 \\
\implies r^n &= 1 \text{ and } \cos(\theta n) + i\sin(\theta n) = -1 \\
\implies r &= 1 \text{ and } \cos(\theta n) = -1 \text{ and } \sin(\theta n) = 0 \\
\implies \theta n &= \pi + 2\pi k \text{ for } k \in \mathbb{Z} \\
\implies \theta &= \frac{\pi + 2\pi k}{n} \text{ for } k \in \mathbb{Z}
\end{aligned}$$

We can now find the principle solutions for Z

$$\therefore\ \theta_0 = \frac{\pi}{n},\ \theta_1 = \frac{3\pi}{n}, \ldots, \theta_{n-1} = \frac{(2n-1)\pi}{n}$$

■

R Roots of Unity The solutions to $z^n = 1$ are called the nth roots of unity. Plotting these solutions splits the complex plane into n equal parts.

I.IV Subsets of the Plane

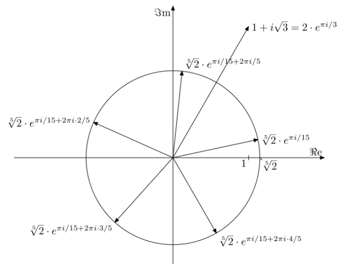


Figure I.2: Complex Fifth Roots of Unity

Definition I.IV.I — Open Disc. An open disc of radius R centered at z_0 is the set of all z such that $D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R \subset \mathbb{C}\}$.

Definition I.IV.II — Interior Point. A point z_0 is an interior point of a set $A \subset \mathbb{C}$ if there exists an open disc centered at z_0 that is contained in A.

$$z_0 \quad \text{is an interior point of } A \text{ if } \quad \exists D_{>0}(z_0) \in A$$

Definition I.IV.III — Open Set. A set $A \subset \mathbb{C}$ is open if every point in A is an interior point. *I.e. there are no 'hard lines' in the set.*

■ **Example I.III — Open Disc.** Show that the disc $D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R\}$ is an open set.

Proof. Let $z_1 \in D$. Then $|z_1 - z_0| < R$. Let $r = R - |z_1 - z_0|$. Then $r > 0$.
Let $z_2 \in D$ be any point in D, such that $|z_2 - z_1| < r$. Then:

$$\begin{aligned} |z_2 - z_0| &\leq |z_2 - z_1| + |z_1 - z_0| \\ &< r + R - r = R \end{aligned}$$

Therefore $z_2 \in D$ and D is open. ■

■

Definition I.IV.IV — Boundary (∂D). The boundary of a set A is the set of all points z such that every open disc centered at z, no matter how small, contains points in A and points not in A. The boundary of A is denoted by ∂A and a boundary point z is denoted by $z \in \partial A$.

$$z_0 \quad \text{is an boundary point of } A \text{ if } \quad \exists z \in D_R(z_0) : z \notin A \forall R > 0$$

Definition I.IV.V — Closed Set. A set D is closed if it contains all its boundary points.

R A set can be both open and closed ($\mathbb{C}$, $\emptyset$), open and not closed, closed and not open, or neither open nor closed (contains part, but not all of their boundary).

Theorem I.IV.I — Properties of Open and Closed Sets.
1. D is open iff $\mathbb{C} \setminus D$ is closed.
2. D is closed iff $\mathbb{C} \setminus D$ is open.
3. D is open if and only if it contains none of its boundary points.

I.V Lines and Circles (Not done in class, Fisher 1.3)

Definition I.V.I — Line in the Complex Plane. A line of the form $y = mx + b$ can be formulated as:

$$0 = \Re\{(m + i)z + b\}$$

Such that when the real part of the complex number is zero, the line is satisfied. The general form is:

$$0 = \Re\{az + b\}, \qquad a, b, z \in \mathbb{C} \tag{I.VII}$$

$$\text{where } a = A + iB \text{ such that:} \tag{I.VIII}$$

$$Ax - By + \Re b = 0 \tag{I.IX}$$

Note that the imaginary part of b does not affect the line.

Definition I.V.II — Simple Circle in the Complex Plane. Circles in the complex plane can be formulated as:

$$|z - z_0| = R \tag{I.X}$$

Where z_0 is the *locus* of the circle and R is the radius.

Definition I.V.III — Perpendicular Bisector. The perpendicular bisector of the line segment between p and q is the set of all points z such that

$$|z - p| = |z - q|$$

Corollary I.V.I — Apollonian Circles. If p and q are distinct complex numbers then a circle can be formulated as:

$$|z - p| = \rho|z - q| \qquad 0 < \rho \in \mathbb{R}, \rho \neq 1 \tag{I.XI}$$

$$\rightarrow \quad \text{Where} \quad z_0 = \frac{p - \rho^2 q}{1 - \rho^2} \quad \text{and} \quad R = \frac{|p - q|\rho}{1 - \rho^2} \tag{I.XII}$$

II. Connectedness

II.I Lecture 2: Connected Sets

Definition II.I.I — Connected Set. An *open* set D is connected if each pair of points $p, q \in D$ can be joined by a polygonal path lying entirely in D. That is:

$$\exists P_2, P_3, \ldots, P_n \in D \quad \text{such that} \quad pP_1, P_1P_2, \ldots, P_nq \in D$$

R The set doesn't *have* to be open, but it is easier to prove connectedness for open sets.

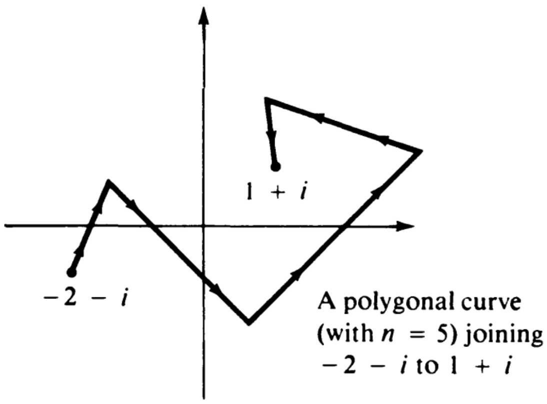


Figure II.1: Polygonal Path

Definition II.I.II — Domain. A domain is a set that's
- Open
- Connected
- Not empty

Definition II.I.III — Convex Set. A set D is convex if for each pair of points $p, q \in D$, the line segment pq lies entirely in D.

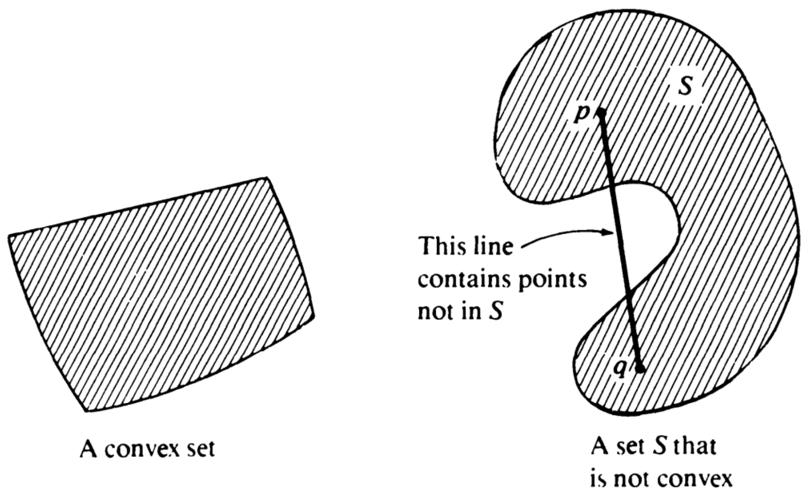


Figure II.2: Convex Set

Theorem II.I.I — Convex $\implies$ Connected. If D is a convex open set, then D is connected.

Definition II.I.IV — Open Half-plane. A set D is an open half-plane if it is of the form

$$D = \{z \in \mathbb{C} : \Re\{az + b\} \geq 0\}$$

Each open half-plane is convex and open

Definition II.I.V — Closed Half-plane. A set D is a closed half-plane if it is of the form

$$D = \{z \in \mathbb{C} : \Re\{az + b\} > 0\}$$

Each closed half-plane is convex and closed

II.II Point at Infinity

Definition II.II.I — Point at Infinity. A set is said to contain the point at infinity if it contains all points z such that $|z| > R$ for some $R > 0$.

■ **Example II.I** No open Half-plane contains the point at infinity. Even though the set is unbounded, choosing R near the boundary will always give a point outside the set. ■

II.III Functions and Limits

Definition II.III.I — Limit of a Sequence of Complex Numbers.

$$\lim_{n\to\infty} z_n = z \quad \text{or} \quad z_n \to z \iff \forall \varepsilon > 0, \exists N \in \mathbb{N} \tag{II.I}$$

$$\text{such that} \quad n \geq N \implies |z_n - z| < \varepsilon \tag{II.II}$$

Corollary II.III.I — Parts of a Limit. If $z_n = x_n + iy_n$ and $z = x + iy$, then

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

Theorem II.III.II — Subsequence. Suppose $\{z_n\}$ converges with limit z. Then every subsequence, $z_{m_n} = f(n)$ also converges to z. Where $1 \leq m_1 < m_2 < \ldots$

Definition II.III.II — Limits of Functions.

$$\lim_{z\to z_0} f(z) = w \iff \forall\varepsilon > 0, \exists\delta > 0 \tag{II.III}$$

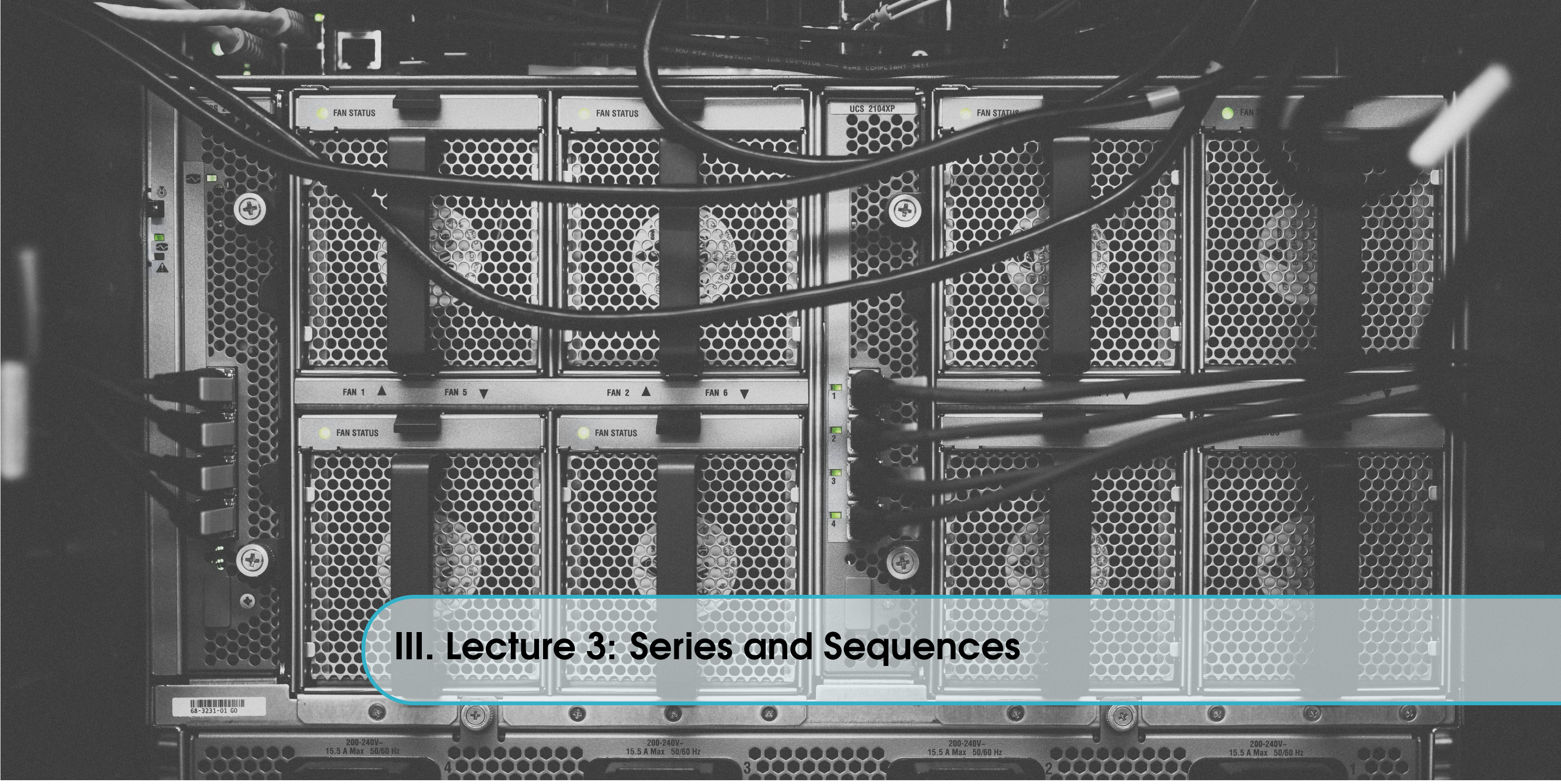
$$\text{such that} 0 < |z - z_0| < \delta \implies |f(z) - w| < \varepsilon \tag{II.IV}$$

II.IV Continuity

Definition II.IV.I — Continuous Function. A function $f(z)$ is continuous at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0)$$

Corollary II.IV.I — Continuous at Infinity. A function $f(z)$ can be continuous at ∞ if $f(\infty) = \lim_{z\to\infty} f(z) = f(\infty)$. Note, $f(\infty)$ may equal ∞
This is equivalent to saying that $f(1/z)$ is continuous at $z = 0$



III. Lecture 3: Series and Sequences

Definition III..I — infinite Series. Suppose we have a sequence:

$$z_1, z_2, z_3, \ldots \tag{III.I}$$

We can define the partial sum of the sequence as:

$$S_n = z_1 + z_2 + z_3 + \ldots + z_n \tag{III.II}$$

We say $\sum_{n=1}^{\infty} z_n$ converges and has a sum S if the sequence of partial sums converges to S:

$$\lim_{n\to\infty} S_n = S \tag{III.III}$$

If $\lim_{n\to\infty} S_n$ does not exist, we say the series diverges.

Corollary III..I — Real and Imaginary Parts of a Series. If $\sum_{n=1}^{\infty} z_n$ converges, then the real and imaginary parts of the series also converge.

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \Re(z_n) + i \sum_{n=1}^{\infty} \Im(z_n) \tag{III.IV}$$

III.I Tests for Convergence

Theorem III.I.I If $\sum_{n=1}^{\infty} z_n$ converges, then so does $\sum_{n=1}^{\infty} z_n$ and:

$$\left|\sum_{n=1}^{\infty} z_n\right| \leq \sum_{n=1}^{\infty} |z_n|$$

Proof. Say $z_n = x_n + iy_n$. Then:

$$\left|\sum_{n=1}^{\infty} x_n\right| \leq \sum_{n=1}^{\infty} |x_n| \leq \sum_{n=1}^{\infty} |z_n| \tag{III.V}$$

And (III.VI)

$$\left|\sum_{n=1}^{\infty} y_n\right| \leq \sum_{n=1}^{\infty} |y_n| \leq \sum_{n=1}^{\infty} |z_n| \tag{III.VII}$$

So if $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge, then $\sum_{n=1}^{\infty} z_n$ converges. ∎

Example III.I

$$\sum_{j=1}^{\infty} j \left(\frac{1 + 2i}{3}\right)^j \tag{III.VIII}$$

We can use the ratio test to determine convergence:

$$\sum_{j=1}^{\infty} |z_j| = \sum_{j=1}^{\infty} j \left|\frac{1 + 2i}{3}\right|^j \tag{III.IX}$$

$$= \sum_{j=1}^{\infty} j \left(\frac{\sqrt{5}}{3}\right)^j \tag{III.X}$$

$$\lim_{j\to\infty} \left|\frac{z_{j+1}}{z_j}\right| = \lim_{j\to\infty} \frac{(j + 1)(\frac{\sqrt{5}}{3})^{j+1}}{j(\frac{\sqrt{5}}{3})^j} \tag{III.XI}$$

$$= \lim_{j\to\infty} \frac{j + 1}{j} \left(\frac{\sqrt{5}}{3}\right) \tag{III.XII}$$

$$= \frac{5}{3} < 1 \qquad \therefore \text{ The series converges} \tag{III.XIII}$$

∎

III.II The Exponential Function

Approach 1

Definition III.II.I — Exponential Function. If $z = x + iy$, then the exponential function is defined as:

$$e^z = e^x \left(\cos(y) + i \sin(y)\right) \tag{III.XIV}$$

R [Euler's Formula]

$$e^{i\theta} \triangleq \cos(\theta) + i \sin(\theta) \tag{III.XV}$$

(III.XVI)

Test Name	Description	Conditions for Use	Results
Ratio Test	$$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert$$	Applicable when terms are positive and the limit exists.	Converges if $L < 1$, diverges if $L > 1$, inconclusive if $L = 1$.
Root Test	$$\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert}$$	Applicable when terms are positive and the limit exists.	Converges if $L < 1$, diverges if $L > 1$, inconclusive if $L = 1$.
Integral Test	Compares a series to an improper. $$\int_1^\infty f(x)\ dx$$	Applicable when terms are positive, continuous, and decreasing.	Converges if the integral converges, diverges if the integral diverges.
Comparison Test	Compares a series to a known convergent or divergent series.	Applicable when terms are positive.	Converges if the series being compared to converges.
Limit Comparison Test	Compares the limit of the ratio of terms to a known series. $$\lim_{n\to\infty}\frac{a_n}{b_n}$$	Applicable when terms are positive and the limit exists.	Converges if the limit is finite and the comparison series converges, diverges otherwise.
Alternating Series Test	$$\sum(-1)^n a_n \quad \text{or} \quad \sum(-1)^{n+1}a_n$$	When dealing with alternating series	Converges if: $a_n > 0$, decreasing, and $\lim_{n\to\infty} a_n = 0$
p-Series Test	Determines convergence based on the exponent in a series of the form $$\sum\frac{1}{n^p}$$	Applicable for series of the form $\frac{1}{n^p}$.	Converges if $p > 1$, diverges if $p \leq 1$.
Geometric Series Test	Determines convergence for geometric series. $$\sum ar^n$$	Applicable for series of the form ar^n.	Converges if $\lvert r\rvert < 1$, diverges if $\lvert r\rvert \geq 1$.
D'Alembert's Ratio Test	Similar to the Ratio Test, but specifically for series with factorial terms.	Applicable when terms involve factorials.	Converges if the ratio is less than 1, diverges if greater than 1.
Cauchy's Condensation Test	Determines convergence by condensing the series. $$\sum a_n \sim \sum 2^n a_{2^n}$$	Applicable for series with positive, decreasing terms.	Converges if the condensed series converges, diverges if the condensed series diverges.

Property	Description
Periodicity	The complex exponential function is periodic with period $2\pi i$, $$e^{z+2\pi i} = e^z$$.
Multiplication	The exponential function satisfies $$e^{z_1+z_2} = e^{z_1}e^{z_2}$$ for any complex numbers z_1 and z_2.
Derivative	The derivative of the exponential function is $$\frac{d}{dz}e^z = e^z$$.
Inverse	The inverse of the exponential function is the complex logarithm, $$\log z$$, such that $$e^{\log z} = z$$ for $z \neq 0$.
Magnitude	The magnitude of the exponential function is $$\lvert e^z \rvert = e^{\Re(z)}$$, where $\Re(z)$ denotes the real part of z.
Argument	The argument of the exponential function is $$\arg(e^z) = \Im(z) \mod 2\pi$$, where $\Im(z)$ denotes the imaginary part of z.
Conjugate	The conjugate of the exponential function is $$\overline{e^z} = e^{\bar{z}}$$.

Table III.II: Properties of the Complex Exponential Function

Properties of the complex Exponential Function

Approach 2: Taylor Series

Definition III.II.II — The Exponential Function. The exponential function can be defined as:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \text{for all } z \in \mathbb{C} \tag{III.XVII}$$

Claim III.II.I — The Taylor Series for the Exponential Function Converges. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$.

Proof. HOMEWORK ■

Problem I For $\theta \in \mathbb{R}$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \cos(\theta) + i\sin(\theta)$$

III.III Approach 3: Differential Equations

Definition III.III.I — Differential Equation for the Exponential Function. The exponential function satisfies the differential equation:

$$f(z) = \begin{cases} \frac{df}{dz} = f & \text{for all } z \in \mathbb{C} \\ f(0) = 1 \end{cases} \tag{III.XVIII}$$

III.IV The Logarithm Function

Definition III.IV.I — Logarithm Function. The logarithm function is defined as the inverse of the exponential function:

$$\log z = \log|z| + i\theta \tag{III.XIX}$$

R There will be many solutions to the logarithm function, as the argument is only defined modulo 2π.

$$\log z = \log|z| + i(\arg(z) + 2\pi n) \qquad \text{for } n \in \mathbb{Z}$$

Definition III.IV.II — Principal Logarithm. The principal branch logarithm is defined as:

$$\mathrm{Log}(z) = \log|z| + i\arg(z) \qquad \text{for } -\pi < \arg(z) \leq \pi$$

Note: We use a capital L to denote the principal logarithm.

Definition III.IV.III — Fixed θ_0 Logarithm Function. We can fix the argument of the logarithm function by setting θ_0 and letting $D = \{te^{i\theta_0} | t > 0, t \in \mathbb{R}\}$.

We define:

$$\widetilde{\log}_{\theta_0} z = \log|z| + i\,(\widetilde{\arg}(z) + \theta_0) \qquad \text{for } z \in D,\ \arg(z) \in [\theta_0, \theta_0 + 2\pi)$$

■ **Example III.II — Find the Values of $(-1)^i$.**

$$(-1)^i = e^{i\log(-1)} \tag{III.XX}$$
$$= e^{i(2n+1)\pi i} \tag{III.XXI}$$
$$= e^{-2n\pi} \tag{III.XXII}$$
$$\log(-1) = -(2n+1)\pi i \qquad n \in \mathbb{Z} \tag{III.XXIII}$$
$$(-1)^i = e^{2n+1}\pi \tag{III.XXIV}$$

■

III.V The Trigonometric Functions

Definition III.V.I — Trigonometric Functions. For $z \in mathbbC$ trigonometric functions are defined as:

$$\Re e^{iz} = \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{III.XXV}$$
$$\Im e^{iz} = \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{III.XXVI}$$
$$\tan(z) = \frac{\sin(z)}{\cos(z)} \tag{III.XXVII}$$
$$\tag{III.XXVIII}$$

Lemma III.V.I

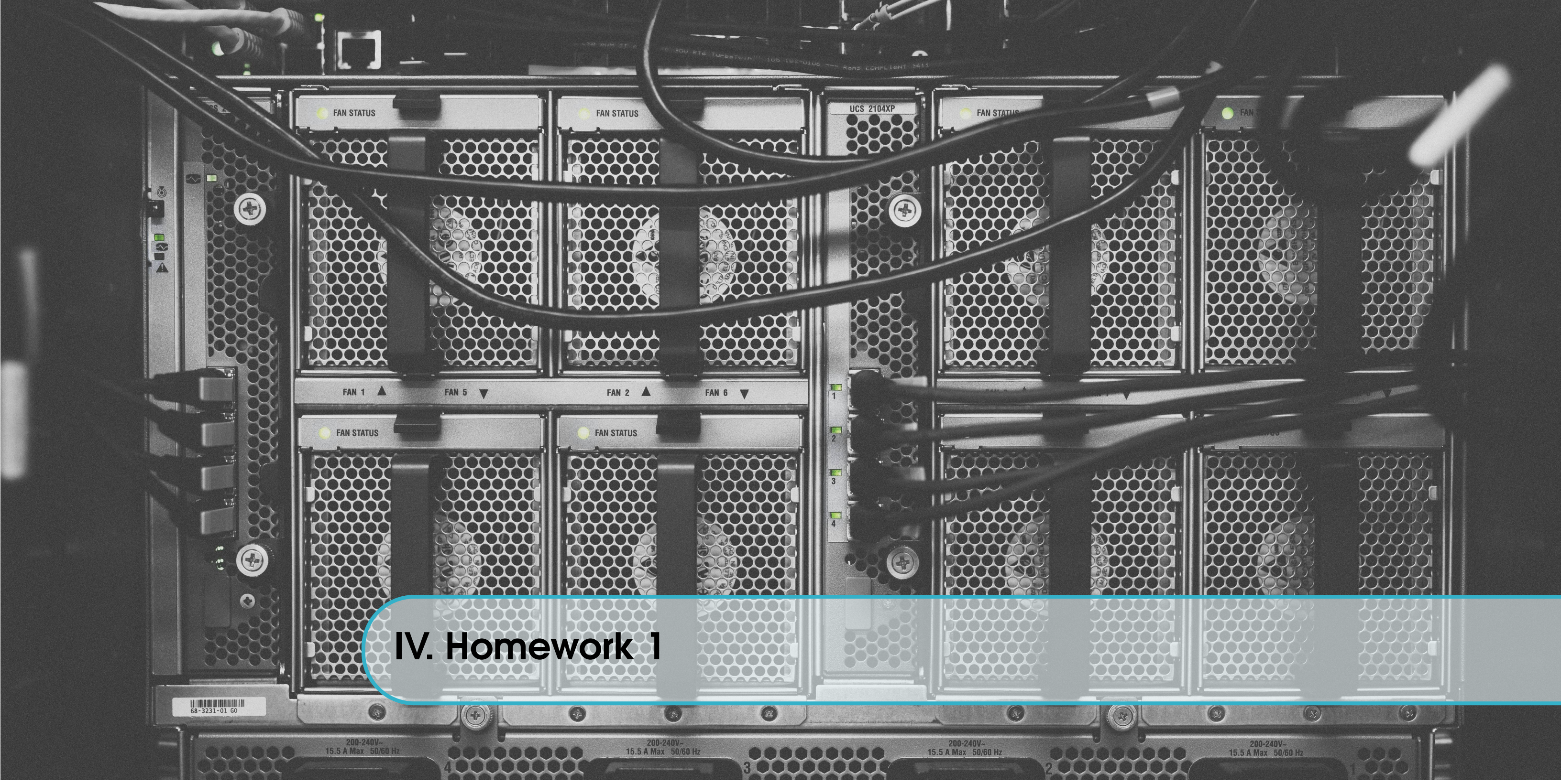
$$\begin{cases} \cos(z+\alpha) = \cos(z) \\ \sin(z+\alpha) = \sin(z) \end{cases} \tag{III.XXIX}$$

iff $\alpha = 2\pi n$ for $n \in \mathbb{Z}$.

Proof.

$$e^{i(z+\alpha)} = e^{iz}e^{i\alpha} \tag{III.XXX}$$
$$= e^{iz}\,(\cos(\alpha) + i\sin(\alpha)) \tag{III.XXXI}$$
$$= e^{iz}\,(\cos(2\pi n) + i\sin(2\pi n)) \tag{III.XXXII}$$
$$= e^{iz} \tag{III.XXXIII}$$

■



IV. Homework 1

■ **Example IV.I — Fisher, Section 1.2, Problem 2.** Describe the locus of points z satisfying the equation

$$|z - 4| = 4|z|$$

Solution:
Let $z = x + iy$. Then

$$\begin{aligned}
|z - 4| &= |x + iy - 4| = |x - 4 + iy| = \sqrt{(x-4)^2 + y^2} \\
4|z| &= 4|x + iy| = 4\sqrt{x^2 + y^2} \\
\sqrt{(x-4)^2 + y^2} &= 4\sqrt{x^2 + y^2} \\
(x-4)^2 + y^2 &= 16(x^2 + y^2) \\
x^2 - 8x + 16 + y^2 &= 16x^2 + 16y^2 \\
15x^2 + 15y^2 + 8x - 16 &= 0 \\
x^2 + y^2 + \frac{8}{15}x - \frac{16}{15} &= 0
\end{aligned}$$

$\Rightarrow$ complete the square

$$x^2 + \frac{8}{15}x + (\frac{4}{15})^2 - (\frac{4}{15})^2 + y^2 = \frac{16}{15}$$

$$\begin{aligned}
(x + \frac{4}{15})^2 + y^2 &= \frac{16}{15} + (\frac{4}{15})^2 \\
(x + \frac{4}{15})^2 + y^2 &= \frac{16}{15} + \frac{16}{225} \\
(x + \frac{4}{15})^2 + y^2 &= \frac{256}{225}
\end{aligned}$$

$\therefore$ The locus of points z satisfying the equation $|z - 4| = 4|z|$ is a circle with center $(-\frac{4}{15}, 0)$ and radius $\sqrt{\frac{256}{225}} = \frac{16}{15}$. ■

■ **Example IV.II — Fisher, Section 1.2, Problem 24.** Find all solutions of the equation

$$(z + 1)^4 = 1 - i$$

Solution:

$$\begin{aligned}
&\text{Convert } 1 - i \text{ to polar form} \\
\arg 1 - i &= \tan^{-1}(-1) = -\frac{\pi}{4} + 2k\pi \quad \text{where } k \in \mathbb{Z} \\
|1 - i| &= \sqrt{1^2 + (-1)^2} = \sqrt{2} \\
1 - i &= \sqrt{2}(cos(-\frac{\pi}{4} + 2k\pi) + isin(-\frac{\pi}{4} + 2k\pi)) \\
&\text{Use De Moivre's Theorem} \\
&\rightarrow z^n = r^n(cos(n\theta) + isin(n\theta)) \\
z + 1 &= 2^{\frac{1}{8}}(\cos(\frac{-\pi}{4} + \frac{2\pi k}{4}) + i\,\sin(\frac{-\pi}{4} + \frac{2\pi k}{4})) \\
z &= 2^{\frac{1}{8}}(\cos(\frac{-\pi}{4} + \frac{2\pi k}{4}) + i\,\sin(\frac{-\pi}{4} + \frac{2\pi k}{4})) - 1
\end{aligned}$$

We can now find the solutions by plugging in $k = 0, 1, 2, 3$.

$$\begin{aligned}
\theta_0 &= \frac{-\pi}{4} + \frac{2\pi \cdot 0}{4} = -\frac{\pi}{4} \quad k = 0 \\
\theta_1 &= \frac{-\pi}{4} + \frac{2\pi \cdot 1}{4} = \frac{\pi}{4} \quad k = 1 \\
\theta_2 &= \frac{-\pi}{4} + \frac{2\pi \cdot 2}{4} = \frac{3\pi}{4} \quad k = 2 \\
\theta_3 &= \frac{-\pi}{4} + \frac{2\pi \cdot 3}{4} = \frac{5\pi}{4} \quad k = 3
\end{aligned}$$

So our solutions are:

$$\begin{aligned}
z &= 2^{\frac{1}{8}}(\cos(-\frac{\pi}{4}) + i\,\sin(-\frac{\pi}{4})) - 1 = 2^{\frac{1}{8}}(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) - 1 \\
z &= 2^{\frac{1}{8}}(\cos(\frac{\pi}{4}) + i\,\sin(\frac{\pi}{4})) - 1 = 2^{\frac{1}{8}}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) - 1 \\
z &= 2^{\frac{1}{8}}(\cos(\frac{3\pi}{4}) + i\,\sin(\frac{3\pi}{4})) - 1 = 2^{\frac{1}{8}}(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) - 1 \\
z &= 2^{\frac{1}{8}}(\cos(\frac{5\pi}{4}) + i\,\sin(\frac{5\pi}{4})) - 1 = 2^{\frac{1}{8}}(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) - 1
\end{aligned}$$

■

■ **Example IV.III — Fisher, Section 1.2, Problem 26.** Find all solutions of the equation $z^3 = 8$.
Solution:
First, we convert 8 to polar form.

$$\begin{aligned} 8 &= 8(\cos(0) + i\sin(0)) \\ &= 8(\cos(2\pi k) + i\sin(2\pi k)) \quad \text{where } k \in \mathbb{Z} \end{aligned}$$

Then we use De Moivre's Theorem to find the solutions.

$$\begin{aligned} \rightarrow z^n &= r^n(\cos(n\theta) + i\sin(n\theta)) \\ z &= 2(\cos(\frac{2\pi k}{3}) + i\sin(\frac{2\pi k}{3})) \quad \text{where } k = 0, 1, 2 \end{aligned}$$

So our solutions are:

$$\begin{aligned} z &= 2(\cos(0) + i\sin(0)) = 2(1 + i0) = 2 \\ z &= 2(\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})) = 2(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -1 + i\sqrt{3} \\ z &= 2(\cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3})) = 2(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = -1 - i\sqrt{3} \end{aligned}$$

■

■ **Example IV.IV — Fisher, Section 1.3, Problem 2.** For the following set, describe (i) the interior and the boundary, (ii) state whether the set is open, or closed, or neither open nor closed, (iii) state whether the interior of the set is connected (if it has an interior).

$$A = \{z \in \mathbb{C} : |z| < 1 \text{or} |z - 3| \leq 1\}$$

Solution:

1. $A_{int} = \{|z| < 1 \text{ or } |z - 3| < 1\}$
2. $A_{bd} = \{|z| = 1 \text{ or } |z - 3| = 1\}$
3. A is neither open nor closed because $\{|z| = 1\} \notin A$ but $\{|z - 3| = 1\} \in A_{int}$, so A contains only part of its boundary.
4. A_{int} is not connected, because $z_1 = 0$, $z_2 = 3 \in A_{int}$, but $\nexists P_1 P_2 \ldots P_n \in A_{int}$ such that $z_1 P_1 P_2 \ldots P_n z_2 \in A_{int}$

■

■ **Example IV.V — Fisher, Section 1.3, Problem 4.** For the following set, describe (i) the interior and the boundary, (ii) state whether the set is open, or closed, or neither open nor closed, (iii) state whether the interior of the set is connected (if it has an interior).

$$A = \{z \in \mathbb{C} : \text{Re}(z^2) = 4\}$$

Solution:
Let $z = x + iy$. Then

$$\begin{aligned} \text{Re}(z^2) &= \text{Re}((x + iy)^2) = \text{Re}(x^2 - y^2 + 2ixy) \\ 4 &= x^2 - y^2 \end{aligned}$$

1. No interior, because $\forall z_0 \in A \quad \nexists D | D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R, R > 0\}$
2. $A_{bd} = \{z \in \mathbb{C} : x^2 - y^2 = 4\}$
3. $A_{bd} = A$ so A is closed.
4. A_{int} is connected because $A_{int} = \emptyset$.

■

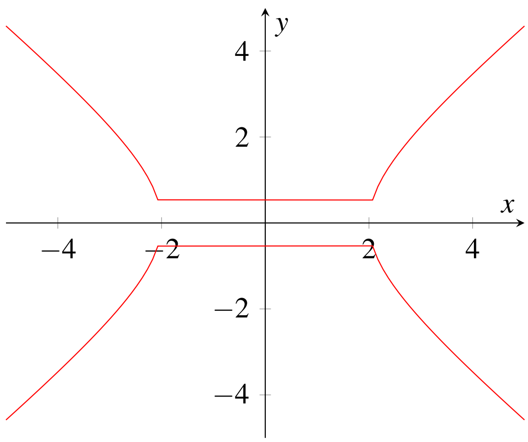


Figure IV.1: Plot of $4 = x^2 - y^2$

■ **Example IV.VI — Fisher, Section 1.4, Problem 12.** Find

$$\lim_{z\to 2}(z - 2) \log |z - 2|,$$

or explain why it does not exist. **Solution:**
We use L'Hopital's Rule to find the limit.

$$\begin{aligned}
\lim_{z\to 2}(z - 2) \log |z - 2| &= \lim_{z\to 2} \frac{\log |z - 2|}{\frac{1}{z-2}} \\
&= \lim_{z\to 2} \frac{\frac{\partial}{\partial z} \log |z - 2|}{\frac{\partial}{\partial z} \frac{1}{(z-2)}} \\
&= \lim_{z\to 2} \frac{\frac{1}{z-2}}{-\frac{1}{(z-2)^2}} \\
&= \lim_{z\to 2} \frac{1}{\frac{1}{2-z}} \\
&= \lim_{z\to 2} 2 - z \\
&= 0
\end{aligned}$$

■

■ **Example IV.VII — Fisher, Section 1.4, Problem 16.** Find all the points where the following function is continuous:

$$f(z) = \begin{cases} \frac{z^4-1}{z-i}, & z \neq i \\ 4i, & z = i \end{cases}$$

Solution:
First normalize the denominator.

$$\begin{aligned} f(z) &= \frac{z^4 - 1}{z - i} = \frac{(z^2 + 1)(z + 1)(z - 1)}{z - i} = \frac{(z^2 + 1)(z + 1)(z - 1)}{z - i}\frac{z + i}{z + i} \\ &= \frac{(z^2 + 1)(z + 1)(z - 1)(z + i)}{z^2 + 1} = (z + 1)(z - 1)(z + i), \quad z \neq i \end{aligned}$$

As this is a polynomial, it is continuous everywhere except at $z = i$. Now we test for continuity at $z = i$.

$$\begin{aligned} \lim_{z \to i} f(z) &= f(i) \\ \lim_{z \to i}(z + 1)(z - 1)(z + i) &= 4i \\ (i + 1)(i - 1)(i + i) &= 4i \\ 4i &= 4i \end{aligned}$$

So $f(z)$ is continuous everywhere. ■

■ **Example IV.VIII — Fisher, Section 1.4, Problem 34.** Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{2 + i^n}$$

Solution:
We notice:

$$\begin{aligned} \frac{1}{2 + i^n} &= \frac{1}{2 + 1} \quad \text{for } n = 0, 4, 8, \ldots \\ \frac{1}{2 + i^n} &= \frac{1}{2 + i} \quad \text{for } n = 1, 5, 9, \ldots \\ \frac{1}{2 + i^n} &= \frac{1}{2 - 1} \quad \text{for } n = 2, 6, 10, \ldots \\ \frac{1}{2 + i^n} &= \frac{1}{2 - i} \quad \text{for } n = 3, 7, 11, \ldots \end{aligned}$$

Which forms a cycle, so the series diverges. ■

■ **Example IV.IX — Fisher, Section 1.4, Problem 36.** Show that each of the following series converges for all z.

1. $$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

2. $$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

3. $$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n + 1)!}$$

Solution:

1. We use the ratio test to show convergence.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{z^{n+1}}{(n+1)!}\frac{n!}{z^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{z}{n+1}\right| \\
&= 0
\end{aligned}$$

   So the series converges for all z.
2. We use the ratio test to show convergence.

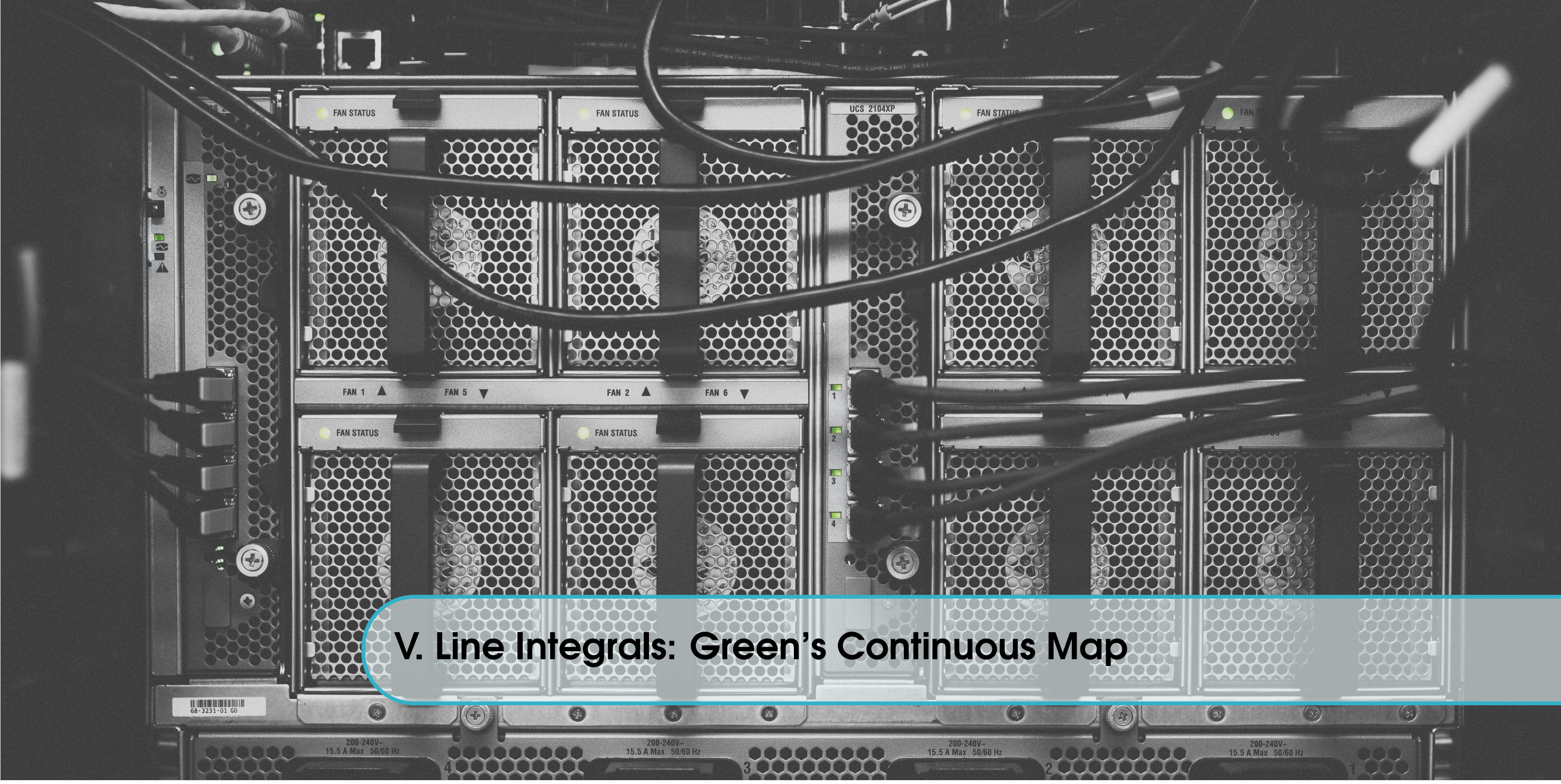
$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{z^{2(n+1)}}{(2(n+1))!}}{(-1)^n\frac{z^{2n}}{(2n)!}}\right| \\
&= \lim_{n\to\infty}\left|\frac{z^2}{(2n+2)(2n+1)}\right| \\
&= 0
\end{aligned}$$

   So the series converges for all z.
3. We use the ratio test to show convergence.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{z^{2(n+1)+1}}{(2(n+1)+1)!}\frac{(2n+1)!}{z^{2n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{z^2}{(2n+3)(2n+2)(2n+1)}\right| \\
&= 0
\end{aligned}$$

   So the series converges for all z.

■



V. Line Integrals: Green's Continuous Map

Theorem V..I — Parametrized Curves.

$$\gamma(t) = x(t) + iy(t) \quad a \le t \le b$$

$\gamma[a, b] \to \mathbb{C}$ is the image of γ.

Definition V..I — Simple Curve. A curve γ is **simple** if $\gamma(t_1) = \gamma(t_2) \implies t_1 = t_2$ for $t_1, t_2 \neq a, b$.

Definition V..II — Closed Curve. A curve γ is **closed** if $\gamma(a) = \gamma(b)$. So if the end point meets the starting point.

R We can *ignore* the parametrization and talk about the curve

$$Image(\gamma) \subset \mathbb{C}$$

as a subset of $\mathbb{C}$.

Definition V..III — C^1/Smooth Curve. A parametrized curve is C^1 if $\gamma'(t)$ if

$$\gamma'(t) = x'(t) + iy'(t)$$

exists $\forall t \in [a, b]$ and is continuous.

R Here, $\gamma'(a)$, $\gamma'(b)$ are the 1-sided derivatives.

Definition V..IV — Piecewise C^1/Smooth Curve.

$$\text{if } \exists a = t_0 < t_1 < \ldots < t_n = b \text{ such that } \gamma|_{[t_i, t_{i+1}]} \text{ is } C^1$$

V.I Line Integrals

Definition V.I.I — Line Integral. if $g = u + iv$, $(u, v) \in \mathbb{R}^2$ is a complex-valued function and γ is piecewise C^1, then the line integral of g along γ is

$$\int_\gamma g(z)dz = \sum_{i=0}^{n-1} \int_i^{i+1} g(\gamma(t))\gamma'(t)dt$$

Where

$$\begin{aligned} g(\gamma(t))\gamma'(t) &= ux' - vy' + ivx' + iuy' \\ &= (u(\gamma(t)) + iv(\gamma(t)))(x'(t) + iy'(t)) \end{aligned}$$

is complex multiplication

Theorem V.I.I — Length of a Curve. If γ is a piecewise C^1 curve, then the length of γ is

$$\text{Length}(\gamma) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\gamma'(t)|dt$$

So we have

$$|\int_\gamma g| \leq \max_{z\in\gamma} |g(z)| \cdot \text{Length}(\gamma)$$

Theorem V.I.II — Green's Theorem. Say $\Omega \subset \mathbb{C}$ such that $\partial\Omega$ is a finite collection of piecewise C^1 closed simple curves. If $g = u + iv$ is C^1 on Ω, then
if $= p + iq$ is differentiable in Ω, then ($\Re p$, q have 1^{st} order derivatives). Then

$$\int_{\partial\Omega} f = i \int_\Omega \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

Where $\partial\Omega$ is the boundary of Ω.

Corollary V.I.III If $dz = dx + idy$, then

$$\begin{aligned} \Re(fdz) &= \Re(f)dx - \Im(f)dy \\ &= pdx - qdy \end{aligned}$$

$$\Re(i\left(\frac{\partial f}{\partial x} + \frac{i\partial f}{\partial y}\right)) = -\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}$$

So

$$\int_{\partial\Omega} pdx + qdy = \left(\int_{\Omega}\left(\frac{\partial q}{\partial x} + \frac{\partial p}{\partial y}\right) dxdy\right)$$

R Orient $\partial\Omega$ always on the left (in the counter-clockwise direction outsides, conterclockwise insides) as we walk along $\partial\Omega$ (say $\partial\Omega$ is positively oriented).

■ **Example V.I — Very Important Example.** Let γ be a simple, closed piecewise C^1 curve. such that $\gamma = \partial\Omega$ for some $\Omega \subset \mathbb{C}$. Then for $p \notin \Omega$,

$$\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-p} = \begin{cases} 1 & \text{if } p \in \Omega \\ 0 & \text{if } p \notin \Omega \end{cases}$$

Proof. 1) Assume p not in Ω, then $\frac{1}{z-p}$ is differentiable in Ω and $\partial\Omega$ is a simple closed curve. So by Green's Theorem,

$$\int_{\partial\Omega}\frac{dz}{z-p} = i\int_{\Omega}\left(\frac{\partial}{\partial x}\frac{1}{z-p} - i\frac{\partial}{\partial y}\frac{1}{z-p}\right) dxdy = 0$$

■

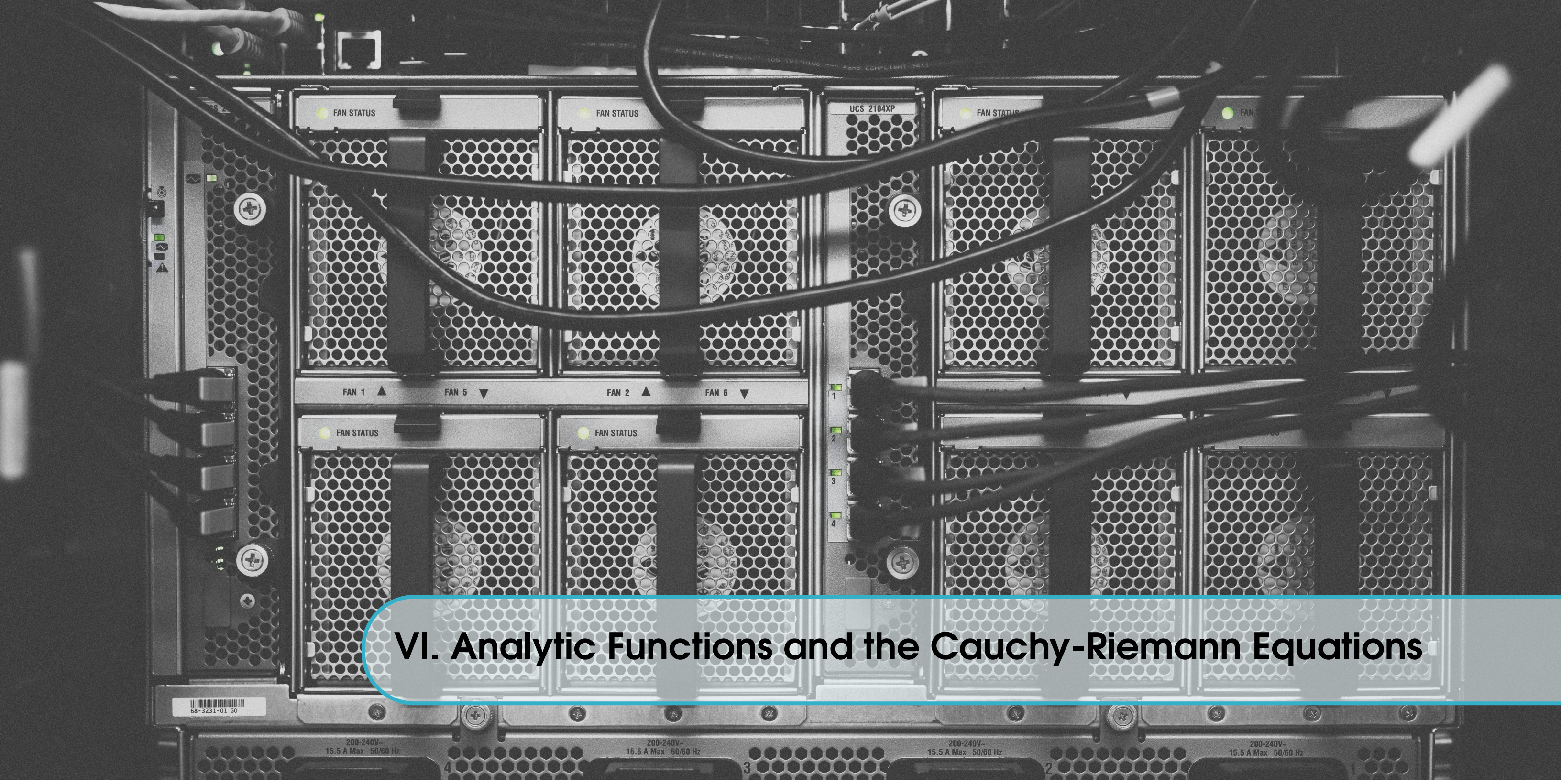
Let $D_\varepsilon(p)$ be the disk of radius ε centered at p, essentially, we want to remove the point stopping us from applying Green's Theorem.

$$\Omega_\varepsilon = \Omega \setminus D_\varepsilon(p)$$

If ε is sufficiently small, Ω_ε is still a domain. So by Green's Theorem,

$$\begin{aligned}
\int_{\partial\Omega_\varepsilon}\frac{dz}{z-p} &= 0 \\
\int_{\partial\Omega}\frac{dz}{z-p} - \int_{\partial D_\varepsilon(p)}\frac{dz}{z-p} &= 0 \\
\int_{\partial\Omega}\frac{dz}{z-p} &= \int_{\partial D_\varepsilon(p)}\frac{dz}{z-p} \\
\to \partial D_\varepsilon &= p + \varepsilon e^{it} \quad 0 \le t \le 2\pi \\
\int_{\partial D_\varepsilon(p)}\frac{dz}{z-p} &= \int_0^{2\pi}\frac{i\varepsilon e^{it}}{\varepsilon e^{it}}dt = 2\pi i \\
\int_{\partial\Omega}\frac{dz}{z-p} &= 2\pi i
\end{aligned}$$

■



VI. Analytic Functions and the Cauchy-Riemann Equations

VI.I Analytic Functions

Definition VI.I.I — Complex Differentiability. A complex function $f(z) : D \to \mathbb{C}$, where D is a domain, is **complex differentiable** at $z_0 \in D$ if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists}$$
$$= \lim_{h} \frac{f(z_0 + h) - f(z_0)}{h} \quad h \in \mathbb{C}$$

Definition VI.I.II — Analytic. A function $f(z)$ is **analytic** on a domain D if $f(z)$ is complex differentiable at every point in D.

Definition VI.I.III — Entire. A function $f(z)$ is **entire** if $f(z)$ is analytic on $\mathbb{C}$.

■ **Example VI.I — Prove the Power Rule.**

$$f(z) = z^n \quad n \in \mathbb{Z}$$

f is entire and

$$f'(z) = nz^{n-1}$$

■

Proof.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{(z+h)^n - z^n}{h} \\
&= \lim_{h \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^k - z^n}{h} \\
&= \lim_{h \to 0} \sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^{k-1}
\end{aligned}$$

$$= \binom{n}{1} z^{n-1}$$
$$= nz^{n-1}$$

■

■ **Example VI.II** Prove that $f(z) = \bar{z}$ is not complex differentiable at any point. ■

Proof. In homework 2... ■

■ **Example VI.III — Prove the Derivative of the Exponential Function.**

$$f(z) = e^z$$

■

Proof.

$$\begin{aligned}
\lim_{h\to 0} \frac{e^{z+h} - e^z}{h} &= \lim_{h\to 0} \frac{e^z e^h - e^z}{h} \\
&= e^z \lim_{h\to 0} \frac{e^h - 1}{h} \\
&= e^z \lim_{h\to 0} \frac{1 + h + \frac{h^2}{2} + \cdots - 1}{h} \\
&= e^z \lim_{h\to 0} 1 + \frac{h}{2} + \cdots
\end{aligned}$$

■

VI.II Cauchy-Riemann Equations

Lemma VI.II.I — h can approach from any direction. If $f(z)$ is differentiable then

$$\exists \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = f(z) \in \mathbb{C}$$

And yield the same result for any $h \in \mathbb{C}$.

Theorem VI.II.II — Cauchy-Riemann Equations. If $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z = x + iy$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof. We compute h in two ways:

$$h_1 = is \quad s \in \mathbb{R}$$
$$h_2 = s \in \mathbb{R}$$

<table>
<tr><th>Property</th><th>Description</th></tr>
<tr><td>Linearity</td><td>The derivative of a sum is the sum of the derivatives:
$$(f + g)'(z) = f'(z) + g'(z)$$
The derivative of a constant multiple is the constant multiple of the derivative:
$$(cf)'(z) = cf'(z)$$</td></tr>
<tr><td>Product Rule</td><td>The derivative of a product is given by:
$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$</td></tr>
<tr><td>Quotient Rule</td><td>The derivative of a quotient is given by:
$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$</td></tr>
<tr><td>Chain Rule</td><td>The derivative of a composition is given by:
$$(f \circ g)'(z) = f'(g(z))g'(z)$$</td></tr>
<tr><td>Exponential Function</td><td>The derivative of the exponential function is:
$$\frac{d}{dz}e^z = e^z$$</td></tr>
<tr><td>Logarithmic Function</td><td>The derivative of the logarithmic function is:
$$\frac{d}{dz}\log z = \frac{1}{z}$$</td></tr>
<tr><td>Power Rule</td><td>The derivative of a power function is:
$$\frac{d}{dz}z^n = nz^{n-1}$$</td></tr>
<tr><td>Trigonometric Functions</td><td>The derivatives of the trigonometric functions are:
$$\frac{d}{dz}\sin z = \cos z$$
$$\frac{d}{dz}\cos z = -\sin z$$</td></tr>
<tr><td>Hyperbolic Functions</td><td>The derivatives of the hyperbolic functions are:
$$\frac{d}{dz}\sinh z = \cosh z$$
$$\frac{d}{dz}$$</td></tr>
</table>

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(z+is)-f(z)}{is}\\
=&lim_{h\to 0}\frac{u(x,y+s)+iv(x,y+s)-u(x,y)-iv(x,y)}{is}\\
=&lim_{h\to 0}\frac{u(x,y+s)-u(x,y)}{is}+\frac{v(x,y+s)-v(x,y)}{s}\\
=&\frac{1}{i}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y})
\end{aligned}$$

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(z+s)-f(z)}{s}\\
=&lim_{h\to 0}\frac{u(x+s,y)+iv(x+s,y)-u(x,y)-iv(x,y)}{s}\\
=&lim_{h\to 0}\frac{u(x+s,y)-u(x,y)}{s}+i\frac{v(x+s,y)-v(x,y)}{s}\\
=&\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}
\end{aligned}$$

So

$$\begin{aligned}
\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} &= \frac{1}{i}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y})\\
\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial y}\\
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}$$

■

Theorem VI.II.III — Harmonic Functions. If $f(z) = u(x,y)+iv(x,y)$ is complex differentiable, then

$$\Delta u = \Delta v = 0$$

And u, v are **harmonic functions** and satisfy Cauchy-Riemann equations. Thus they are **harmonic conjugates**. Where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator.

Corollary VI.II.IV If a function $f(z)$ is once complex differentiable, then it is infinitely differentiable and analytic.

Proof. Cauchy-Riemann equations give us the partial derivatives of u, v.

$$\begin{cases}\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\end{cases}$$

$$\text{Take } \frac{\partial}{\partial x}(1) \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}$$
$$\text{Take } \frac{\partial}{\partial y}(2) \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial v}{\partial y} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial x^2}$$
$$\Delta u = 0$$

■

Theorem VI.II.V Let $f = u + iv$ and assume u, v, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ are defined and continuous on a disc around z_0. If u, v satisfy the Cauchy-Riemann equations at z_0, then f is complex differentiable at z_0.

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Proof. Using the taylor expansion of $f(z)$

$$\begin{aligned}
\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} &= \lim_{h\to 0} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\
&= \lim_{h\to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i\frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h} \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}
\end{aligned}$$

■

■ **Example VI.IV — Prove the Derivative of the Logarithmic Function.** Let $D \in \mathbb{C}$ be a domain o which there is a single-valued branch of $\log z$. ■

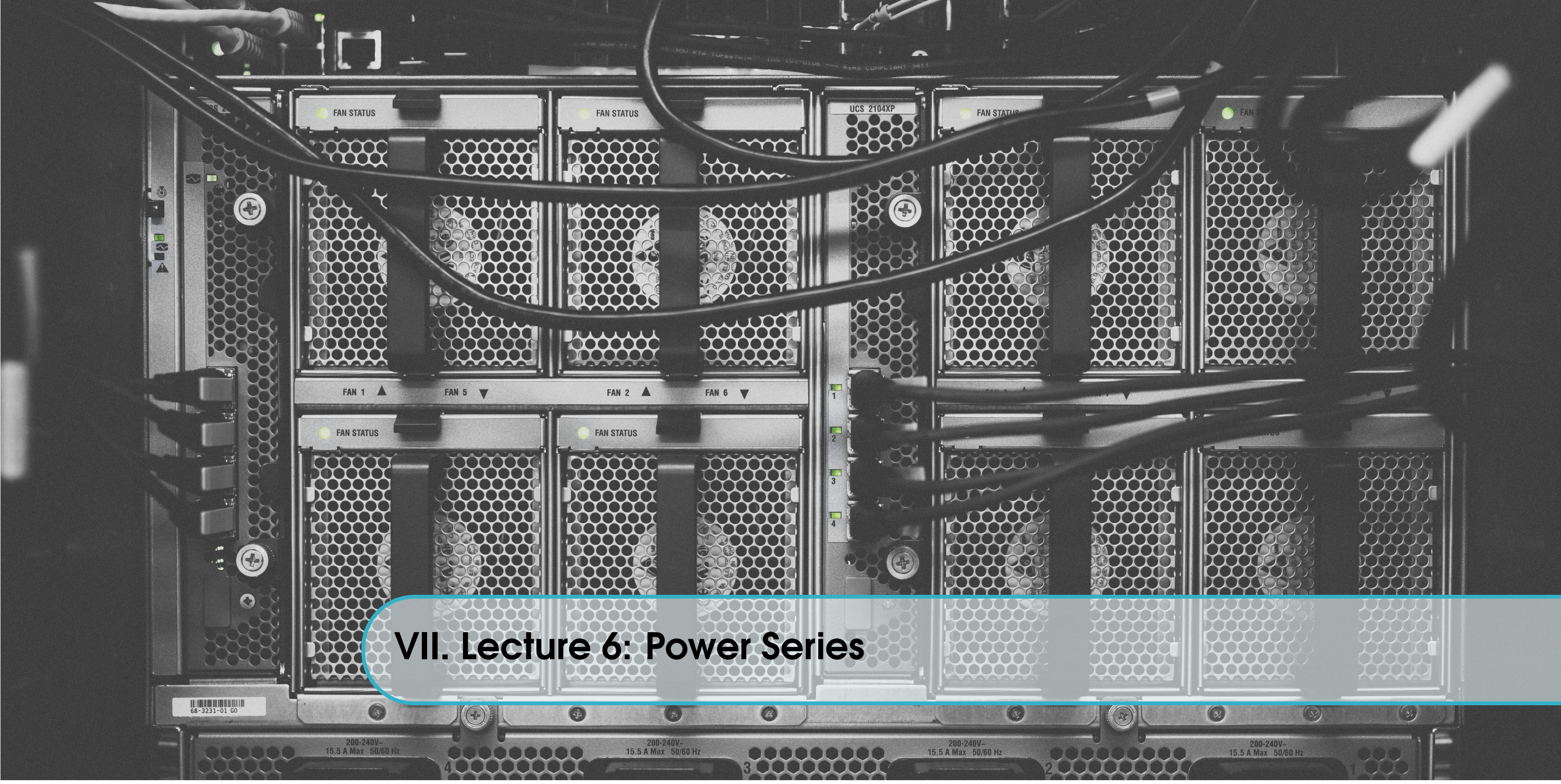
Proof. When $\arctan(y/x) \in (\theta_0, \theta + \pi]$ and $\arctan(y/x)$ is not in D.

$$u = \frac{1}{2}\log(x^2 + y^2) \quad v = \arctan(y/x)$$

Then

$$\frac{\partial u}{\partial x} = \frac{1}{2(x^2 + y^2)} \cdot 2x = \frac{x}{x^2 + y^2} \tag{VI.I}$$
$$\frac{\partial v}{\partial y} = \frac{1}{1 + \frac{y}{x}^2} \times \frac{1}{x} \tag{VI.II}$$
$$= \frac{x}{x^2 + y^2} \tag{VI.III}$$
$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{VI.IV}$$

INCOMPLETE ■



VII. Lecture 6: Power Series

VII.I Introduction

Definition VII.I.I — Power Series. A power series in z is a n infinite series of the form:

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

where a_n are complex numbers, z is a complex variable, and $z_0 \in \mathbb{C}$ is the centre.

Theorem VII.I.I — Absolute Convergence of Power Series. Suppose $\exists z_1 \neq z_0 | \sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges. Then for each $z \in \mathbb{C} : |z - z_0| < |z_1 - z_0|$, the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges **absolutely**.

Proof. Since $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges, we know:

$$\lim_{n\to\infty} a_n(z_1 - z_0)^n = 0 \tag{VII.I}$$

so: $|a_n||z_1 - z_0|^n \leq M \forall n$ as $n \to \infty$.
by VII.I, $\exists N \in \mathbb{N} | n \geq N \implies |a_n||z_1 - z_0|^n \leq 1$.
So

$$\begin{aligned} |a_n||z - z_0|^n &= \frac{|a_n||z_1 - z_0}{|z_1 - z_0|}|z - z_0|^n \\ &\leq M \end{aligned}$$

Let $\rho = \frac{|z-z_0|}{|z_1-z_0|} < 1$. So

$$\sum_{n=0} \infty |a_n||z - z_0| \leq M \sum_{n=0}^{\infty} \rho^n$$

Since $\rho < 1$, the geometric series $\sum_{n=0}^{\infty} \rho^n$ converges. Hence so does $\sum_{n=0}^{\infty} |a_n||z - z_0|^n$.
Absolute Convergence ■

Definition VII.I.II — Absolute Convergence. A series $\sum_{n=0}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges.

Definition VII.I.III — Radius of Convergence. The radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is the number R such that the series converges absolutely for $|z - z_0| < R$ and diverges for $|z - z_0| > R$.
There are three possibilities:
1. Converges only for $z = z_0$ (i.e. $R = 0$)
2. Converges for all $z \in \mathbb{C}$ (i.e. $R = \infty$)
3. Converges for some $z \neq z_0$ (i.e. $0 < R < \infty$)

Lemma VII.I.II Assume $z', z'' \in \mathbb{C}$ are points such that:
- $\sum_{n=0}^{\infty} a_n(z' - z_0)^n$ converges
- $\sum_{n=0}^{\infty} a_n(z'' - z_0)^n$ diverges

There is a unique $R > 0$ such that:
- If $|z - z_0| < R$, then $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges
- if $|z - z_0| > R$, then $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ diverges

R is called the radius of convergence.

R The behaviour o on the circle of convergence $|z - z_0| = R$ can be complicated! The series may converge or diverge, or both (depending on where on the circle you are).

VII.II Computing the Radius of Convergence

Theorem VII.II.I Suppose $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is a power series with radius of convergence $0 < R \leq \infty$. Then:
1. $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ exists, then $R = \frac{1}{L}$

Proof. Assume

$$\exists \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$

Then:

$$\lim_{n\to\infty} \left|\frac{(z - z_0)^{n+1} a_{n+1}}{(z - z_0)^n a_n}\right| = |z - z_0|L$$

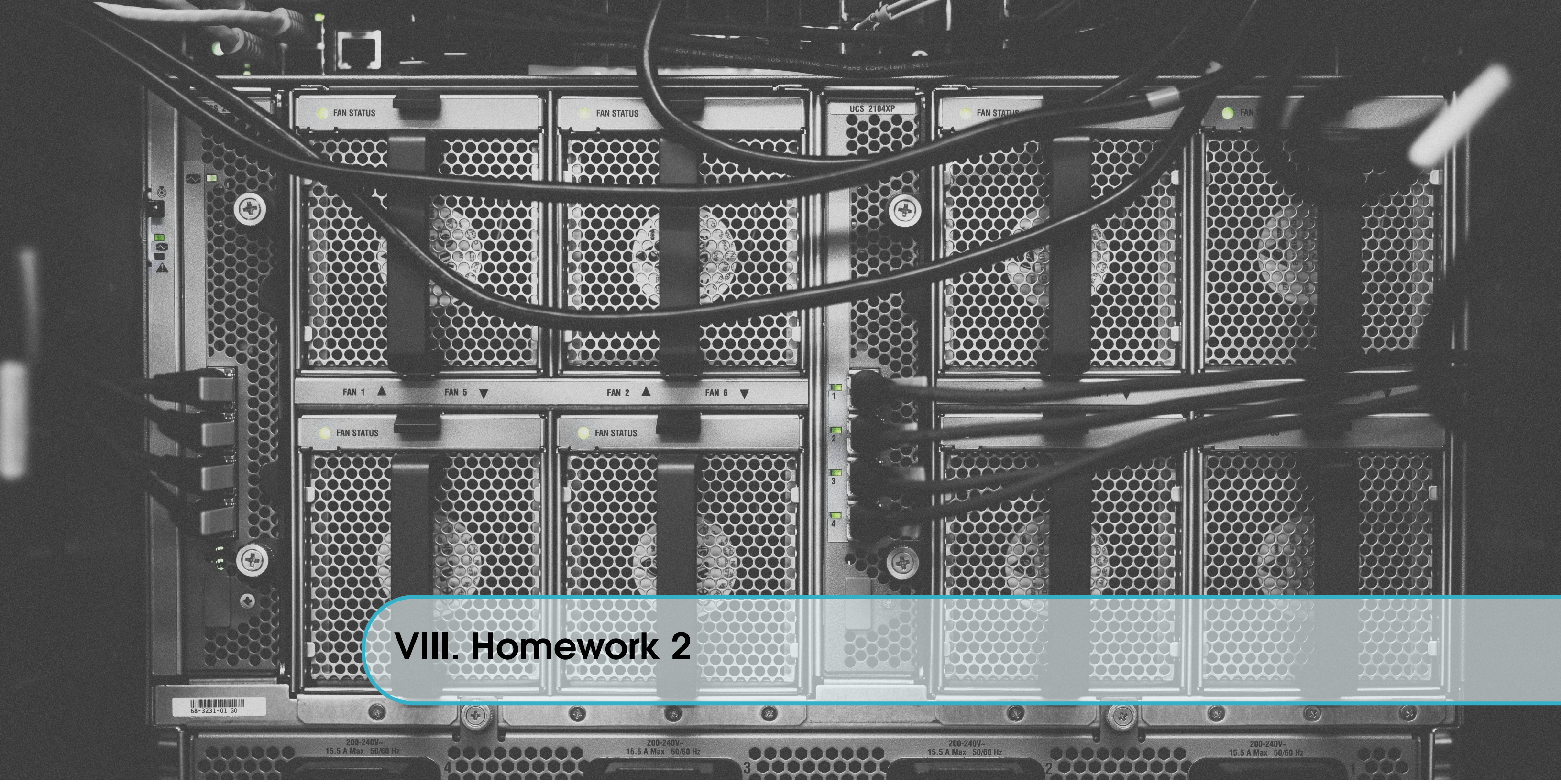
if $|z - z_0|L < \frac{1}{L}$, then $|z - z_0|L < 1$ and the series converges by the ratio test, and if $|z - z_0|L > \frac{1}{L}$, then the series diverges by the ratio test.

Similarly, if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ exists, then:

$$\lim_{n\to\infty} \sqrt[n]{|a_n(z - z_0)^n|} = |z - z_0|L$$

If $|z - z_0|L < 1 \to R = |z - z_0| < \frac{1}{L}$, then the series converges by the root test, and if $|z - z_0|L > 1$, then the series diverges by the root test. ■

R If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ does not exist, then $R = 0$



VIII. Homework 2

■ **Example VIII.I — Fisher, Section 1.6, Problem 2.** Compute the following line integral:

$$\int_\gamma e^z dz$$

where γ is the line segment from 0 to z_0.

We want some path that approaches z_0 so we parametrize γ as $\gamma(t) = z_0 t$ for $t \in [0, 1]$.

$$\begin{aligned}
\int_\gamma e^z dz &= \int_0^1 e^{z_0 t} z_0 dt \\
&= z_0 \int_0^1 e^{z_0 t} dt \\
&= z_0 \left[\frac{e^{z_0 t}}{z_0} \right]_0^1 \\
&= z_0 \left[\frac{e^{z_0} - e^0}{z_0} \right] \\
&= e^{z_0} - 1
\end{aligned}$$

■

■ **Example VIII.II — Fisher, Section 1.6, Problem 4.** Compute the following line integral:

$$\int_\gamma \frac{1}{z + 4} dz$$

where γ is the circle of radius 1 centered at -4, oriented counterclockwise.

We first recognize that $e^{i\theta} = \cos\theta + i \sin\theta$ represents a point on a unit circle. So we can parametrize γ as $\gamma(t) = -4 + e^{it}$ for $t \in [0, 2\pi]$.

$$\begin{aligned}
\int_\gamma \frac{1}{z+4} dz &= \int_0^{2\pi} \frac{1}{(-4+e^{it})+4} ie^{it} dt \\
&= \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} dt \\
&= \int_0^{2\pi} idt \\
&= 2\pi i
\end{aligned}$$

■

■ **Example VIII.III — Fisher, Section 1.6, Problem 10.** Let $f = u + iv$ be a continuous functions and $\gamma(t) = x(t) + iy(t)$ be a piecewise C^1 curve. Show that

$$\text{Re}\left(\int_\gamma f(z)dz\right) = \int_\gamma udx - vdy$$

and,

$$\text{Im}\left(\int_\gamma f(z)dz\right) = \int_\gamma vdx + udy$$

where $dx = x'(t)dt$ and $dy = y'(t)dt$.

We know that $f(z) = u + iv$ and $dz = dx + idy$. So we can write the integral as:

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_\gamma (u+iv)(dx+idy) \\
&= \int_\gamma udx + i\int_\gamma vdx + i\int_\gamma udy - \int_\gamma vdy
\end{aligned}$$

Taking the real part of the integral, we get:

$$\text{Re}\left(\int_\gamma f(z)dz\right) = \int_\gamma udx - \int_\gamma vdy$$

Taking the imaginary part of the integral, we get:

$$\text{Im}\left(\int_\gamma f(z)dz\right) = \int_\gamma vdx + \int_\gamma udy$$

■

■ **Example VIII.IV — Fisher, Section 1.6, Problem 16.** Let γ be a piecewise C^1, simple closed curve. Let z_0 be a point which does not lie on γ. Show that

$$\int_\gamma \frac{dz}{(z-z_0)^m} = 0 \quad \text{for } m = 2, 3, 4, \ldots$$

Let's see if we can apply Cauchy's Integral Theorem, which says that if f is analytic on a simply connected domain D and γ is a simple closed curve in D, then $\int_\gamma f(z)dz \;=\; 0$.
Say $\exists D|z_0 \;\notin\; D$ and D is simply connected. Say also that γ is a simple closed curve in D. Then we can write $f(z) \;=\; \frac{1}{(z-z_0)^m}$ for $m \;=\; 2,\, 3,\, 4,\, \ldots$. Then $f(z)$ is analytic on D and γ is a simple closed curve in D. So by Cauchy's Integral Theorem, $\int_\gamma f(z)dz \;=\; 0 \quad \forall m \;=\; 2,\, 3,\, 4,\, \ldots$. ■

■ **Example VIII.V — Fisher, Section 2.1, Problem 4.** Find the derivative of the function $f(z) \;=\; (\cos(z^2))^3$.

We can write $f(z) \;=\; (\cos(z^2))^3 \;=\; \cos^3(z^2)$. So we can apply the chain rule to get:

$$\begin{aligned} f'(z) \;&=\; 3\,\cos^2(z^2)\,\big(-\,\sin(z^2)\big)\,2z \\ &=\; -6z\,\cos^2(z^2)\,\sin(z^2) \end{aligned}$$

■

■ **Example VIII.VI — Fisher, Section 2.1, Problem 6.** Find the derivative of the function $(\mathrm{Log}(z))^3$ on the plane minus the negative reals.

Say $w \;=\; \mathrm{Log}(z) \;=\; \ln|z| \;+\; i\,\arg(z) \quad -\pi \;<\; \arg(z) \;<\; \pi$. We can write $f(z) \;=\; (\mathrm{Log}(z))^3 \;=\; (w)^3$. So:

$$\begin{aligned} \frac{dw}{dz} \;&=\; \frac{d}{dz}(\ln|z| \;+\; i\,\arg(z)) \\ \frac{dw}{dz} \;&=\; \frac{1}{z} \end{aligned}$$

So we can apply the chain rule to get:

$$\begin{aligned} f'(z) \;&=\; 3(w)^2\frac{dw}{dz} \\ &=\; 3(\mathrm{Log}(z))^2\frac{1}{z} \end{aligned}$$

■

■ **Example VIII.VII — Fisher, Section 2.1, Problem 14.** Let $P(z) \;=\; A(z \,-\, z_1)\cdots(z \,-\, z_n)$, where $A,\, z_1,\, \ldots,\, z_n$ are complex numbers. Show that

$$\frac{P'(z)}{P(z)} \;=\; \sum_{j=1}^{n}\frac{1}{z \,-\, z_j}$$

for any $z \;\neq\; z_1,\, \ldots,\, z_n$.

We can write $P(z) \;=\; A(z \,-\, z_1)\cdots(z \,-\, z_n)$. So we can apply the product rule to get:

$$\begin{aligned}
P'(z) &= A\left(\prod_{j=1}^{n}(z - z_j)\right)' \\
&= A\sum_{j=1}^{n}\left(\prod_{k\neq j}(z - z_k)\right)
\end{aligned}$$

Because we know $d(z - z_n) = dz$. So we can write:

$$\begin{aligned}
\frac{P'(z)}{P(z)} &= \frac{A\sum_{j=1}^{n}\left(\prod_{k\neq j}(z - z_k)\right)}{A\prod_{j=1}^{n}(z - z_j)} \\
&= \sum_{j=1}^{n}\frac{\prod_{k\neq j}(z - z_k)}{\prod_{j=1}^{n}(z - z_j)} \\
&= \sum_{j=1}^{n}\frac{1}{z - z_j}
\end{aligned}$$

■

■ **Example VIII.VIII — Fisher, Section 2.1, Problem 18.** Show that $f(z) = \bar{z}$ is not analytic on any domain

---

We can write $f(z) = \bar{z} = x - iy$. So we can write $u(x, y) = x$ and $v(x, y) = -y$. We can apply the Cauchy-Riemann equations to get:

$$\begin{aligned}
u_x &= 1 = -v_y \\
u_y &= 0 = v_x
\end{aligned}$$

So the Cauchy-Riemann equations are not satisfied. So $f(z) = \bar{z}$ is not analytic on any domain.

■

■ **Example VIII.IX — Fisher, Section 2.1, Problem 20.** Let $f = u + iv$ and suppose that f is analytic. In each of the following, find v, given u:

1. $u = x^2 - y^2$
2. $u = \frac{x}{x^2+y^2}$

Firstly, we remind ourselves of the Cauchy-Riemann equations:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}$$

1. Say $u = x^2 - y^2$. We can apply the Cauchy-Riemann equations to get:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\begin{aligned} &= 2x \\ \int \partial v &= 2x \int \partial y \\ v(x, y) &= 2xy + h(x) \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ &= -(-2y) \\ &= 2y \\ \int \partial v &= 2y \int \partial x \\ v(x, y) &= 2xy + h(y) \end{aligned}$$

So $h(x) = h(y) = 0$. Therefore $v(x, y) = 2xy + c$. Where c is a constant in $\mathbb{C}$.

2. Say $u = \frac{x}{x^2+y^2}$. We can apply the Cauchy-Riemann equations to get:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \\ \int \partial v &= \int \frac{y^2 - x^2}{(x^2 + y^2)^2} \partial y \\ v(x, y) &= -\frac{y}{x^2 + y^2} + h(x) \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ &= \frac{2xy}{(x^2 + y^2)^2} \\ \int \partial v &= \int \frac{2xy}{(x^2 + y^2)^2} \partial x \\ v(x, y) &= -\frac{y}{x^2 + y^2} + h(y) \end{aligned}$$

So $h(x) = h(y) = 0$. Therefore $v(x, y) = -\frac{y}{x^2+y^2} + c$. Where c is a constant in $\mathbb{C}$.

■

■ **Example VIII.X — Fisher, Section 2.1, Problem 26.** Suppose that γ is a piecewise C^1 simple closed curve and that u is a continuous function on γ. Let D be a domain disjoint from γ, and define a function h on D by the rule

$$h(z) \;=\; \int_{\gamma} \frac{u(\xi)}{\xi \;-\; z} d\xi$$

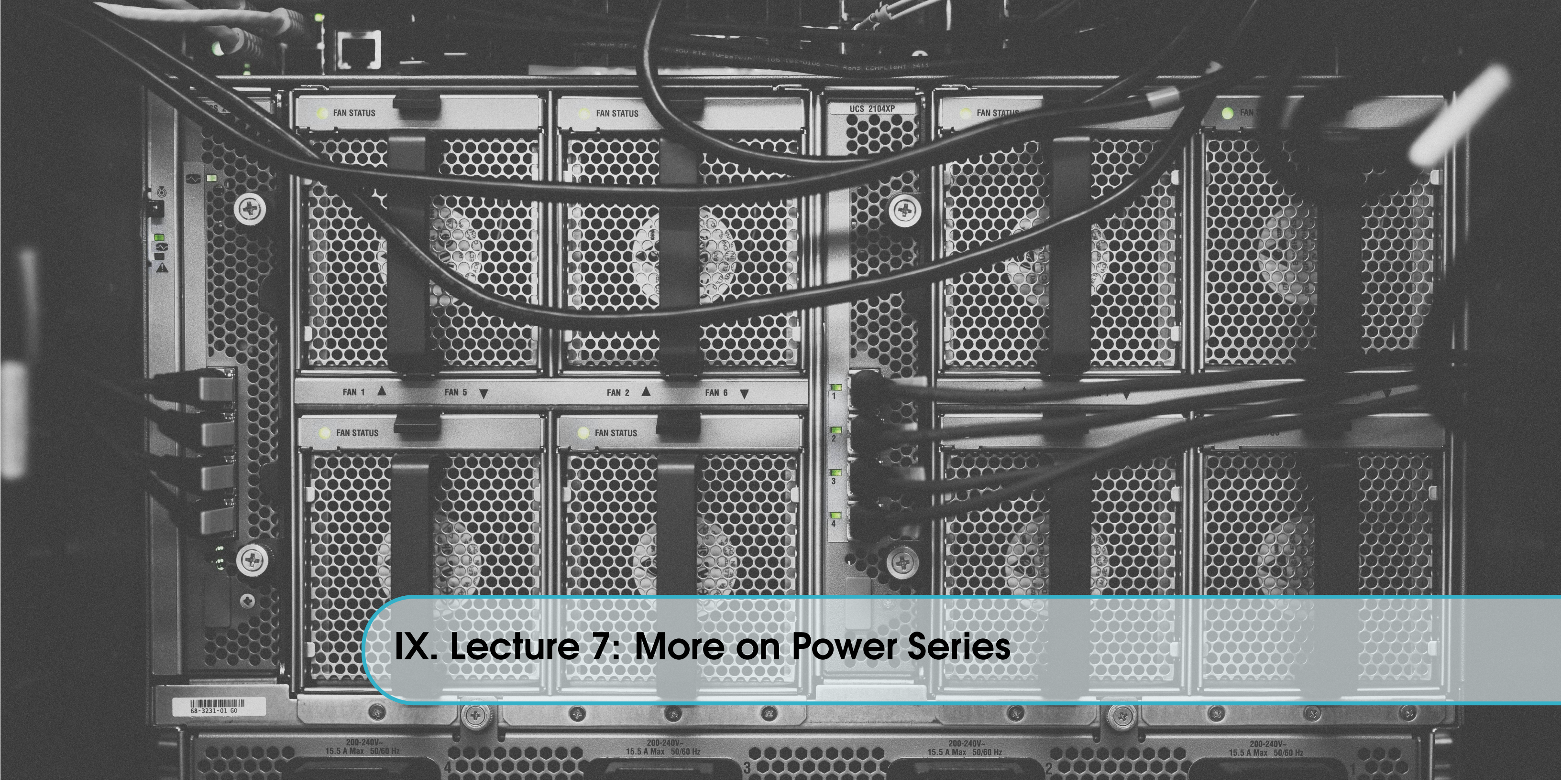
Show that h is analytic in D.

We can write $h(z) \;=\; \int_{\gamma} \frac{u(\xi)}{\xi - z} d\xi$. We can apply the Cauchy Integral Formula to get:

$$\begin{aligned} \frac{d}{dz} h(z) \;&=\; \frac{d}{dz} \int_{\gamma} \frac{u(\xi)}{\xi \;-\; z} d\xi \\ &=\; \int_{\gamma} \frac{\partial}{\partial z} \left(\frac{u(\xi)}{(\xi \;-\; z)^2} \right) d\xi \\ \therefore\; h'(z) \;&=\; \int_{\gamma} \frac{u(\xi)}{(\xi \;-\; z)^2} d\xi \end{aligned}$$

Since the function $\frac{u(\xi)}{(\xi - z)^2}$ is continuous with respect to z in D and the curve γ is piecewise C^1, the integral defines a smooth function of z in D .Therefore, $h'(z)$ exists for all $z \;\in\; D$,and $h(z)$ is differentiable.

Moreover, the existence and continuity of $h'(z)$ imply that $h(z)$ is analytic in D ,as $h(z)$ is differentiable and its derivative is continuous in D. ■



IX. Lecture 7: More on Power Series

Theorem IX..I — Differentiation of Power Series. Consider a power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \qquad \text{with } 0 < R \leq \infty$$

Then: $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is ***analytic*** in the disc $\{|z - z_0| < R\}$ and

$$\frac{d}{dz} f(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

■ **Example IX.I — Proof of Convergence of Power Series Differentiation.** Show that

$$\frac{d}{dz} f(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

converges for $|z - z_0| < R$.

Proof. Let $|z - z_0| < R$. and let $r < s < R$. then, $\exists N | \forall n \geq N$ we have.

$$n r^{n-1} \leq S^n$$

Since $\frac{r}{s} < 1$, by the ratio test:

$$\lim_{n \to \infty} n \left(\frac{r}{s}\right)^{n-1} = 0$$

Thus:

$$\begin{aligned}
\sum_{n=1}^{\infty} n|a_n|r^{n-1} &\leq \sum_{n=1}^{N} n|a_n|r^{n-1} + \sum_{n=N}^{\infty} n|a_n|s^n \\
&\leq \sum_{n=1}^{N} n|a_n|r^{n-1} + \sum_{n=1}^{\infty} n|a_n|s^n
\end{aligned}$$

Of course $\sum_{n=1}^{\infty} |a_n|s^n$ converges by the ratio test since $s < R$. Hence $\sum_{n=1}^{\infty} n|a_n|r^{n-1}$ converges. ■

▪

Theorem IX..II — Infinite Differentiation of Power Series. if $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence $0 < R \leq \infty$, then in the disc $\{|z - z_0| < R\}$, $f(z)$ is infinitely differentiable and

$$f^{(m)}(z) = \sum_{n=m}^{\infty} n(n-1)\dots(n-(m-1))a_n(z-z_0)^{n-m} \qquad k = 1, 2, \dots$$

IX.I Cauchy's Theorem

Definition IX.I.I A domain D is *simply-connected* if, whenever γ is a closed curve in D, the inside of γ is also a subset of D.
Informally, a domain is simply connected if it has no holes.

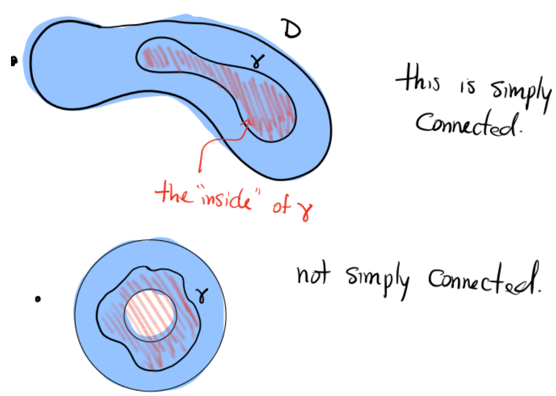


Figure IX.1: Simply Connected Domain

Theorem IX.I.I — Cauchy's Theorem. Suppose f is analytic on a domain D, and let γ be a C^1 simple closed curve in D such that the inside of $\gamma = \Omega \subset D$. Then:

$$\oint_{\gamma} f(z)dz = 0$$

Proof. By Green's Theorem:

$$\oint_{\gamma} f(z)dz = i \iint_{\Omega} \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

But if $f = u + iv$ is analytic, so:

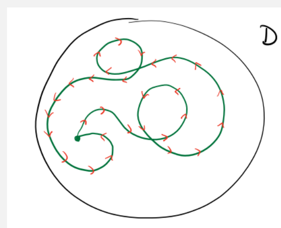
$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$
\begin{aligned}
&= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \\
&= -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \\
&= -i\frac{\partial f}{\partial y}
\end{aligned}
$$

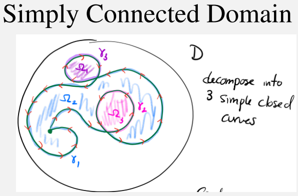
So $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$ and the result follows. ■

Theorem IX.I.II — More General Cauchy's Theorem. if D is a simply connected domain and γ is any closed, piecewise C^1 curve in D, then, if f is analytic on D:

$$\oint_\gamma f(z)dz = 0$$



(a) Simply Connected Domain in D



(b) Decomposition of γ

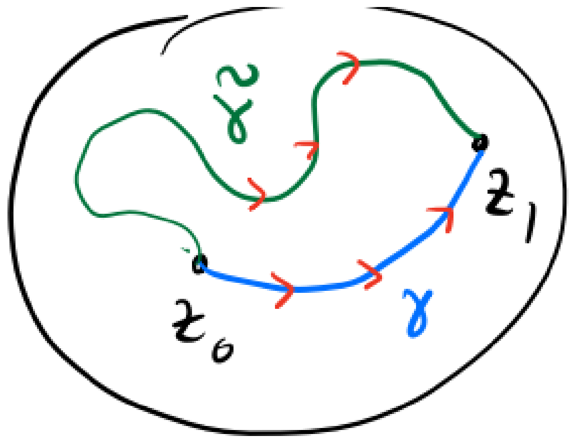
$$\oint_\gamma f(z)dz = \oint_{\gamma_1} f(z)dz + \oint_{\gamma_2} f(z)dz + \oint_{\gamma_3} f(z)dz = 0$$

Theorem IX.I.III — Differentiability of Analytic Functions. If D is a simply connected domain and f is analytic on D, then there is an analytic function F on D such that $F' = f$.

Lemma IX.I.IV For $F(z) = \int_\gamma f(z)dz$, F is independent of the path γ.

Proof. Let's say γ_1 and γ_2 are two paths from z_0 to z_1, and $\Gamma = \gamma_1 - \gamma_2$. Then:

$$
\begin{aligned}
\oint_\Gamma f(z)dz &= \oint_{\gamma_1} f(z)dz - \oint_{\gamma_2} f(z)dz \\
0 &= F_1 - F_2 \\
F_1 &= F_2
\end{aligned}
$$



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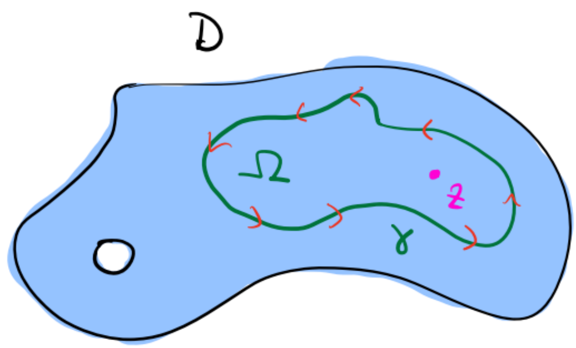
Theorem IX.I.V — Cauchy's Integral Formula. Suppose f is analytic on a domain D, γ is piecewise C^1, positively oriented, simple closed curve in D such that: inside$(\gamma) = \Omega \subset D$. Then:

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}d\xi \qquad \forall z \in \Omega$$



Proof. Let $g(\xi) = \frac{f(\xi)}{\xi - z}$ be an analytic function in $\Omega/\{z\}$.
Let $D_\varepsilon(z) = \{z \in \mathbb{C}||z - z_0| < \varepsilon\}$ be a disc of radius ε centered at z_0.
We choose ε to be small such that $D_\varepsilon(z) \subset \Omega$.
By Cauchy's Theorem:

$$\oint_\gamma \frac{f(\xi)}{\xi - z}d\xi = \oint_{\partial D_\varepsilon(z)} \frac{f(\xi)}{\xi - z}d\xi$$

Note: $\partial D_\varepsilon(z)$ is the boundary of $D_\varepsilon(z)$.
Now we parametrize $\partial D_\varepsilon(z)$ by $\partial D_\varepsilon = z + \varepsilon e^{i\theta}, 0 \leq \theta \leq 2\pi$, Then:

$$\begin{aligned}\oint_{\partial D_\varepsilon(z)} \frac{f(\xi)}{\xi - z}d\xi &= \int_0^{2\pi} \frac{f(z + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\varepsilon i e^{i\theta}d\theta \\ &= i\int_0^{2\pi} f(z + \varepsilon e^{i\theta})d\theta\end{aligned}$$

as $\varepsilon \to 0$, $f(z + \varepsilon e^{i\theta}) \to f(z)$ (by continuity of f).
Hence:

$$\lim_{\varepsilon \to 0} i \int_0^{2\pi} f(z + \varepsilon e^{i\theta}) d\theta = 2\pi i f(z)$$

Thus

$$\boxed{\oint_\gamma \frac{f(\xi)}{\xi - z} d\xi = 2\pi i f(z)}$$

■

IX.II Applications of Cauchy's Integral Formula

■ **Example IX.II** Compute:

$$\int_0^{2\pi} \frac{d\theta}{2 + \sin\theta}$$

Idea: Write this as an integral for an analytic function over the circle $z = e^{i\theta} \quad 0 \leq \theta \leq 2\pi$.
If $|z| = 1$, then

$$\begin{aligned} \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \\ &= \frac{z - z^{-1}}{2i} \end{aligned}$$

$$\therefore\ d\theta = \frac{dz}{iz}$$

So, integral is:

$$\begin{aligned} \int_{\gamma(\theta) = e^{i\theta}} \frac{1}{2 + \frac{z - z^{-1}}{2i}} \frac{dz}{iz} &= \int_\gamma \frac{2dz}{4iz + (z^2 - 1)} \\ \to z^2 - 4iz - 1 &= \left[z - \left(\frac{-4i + \sqrt{-16 + 4}}{2} \right) \right] \left[z - \left(\frac{-4i - \sqrt{-16 + 4}}{2} \right) \right] \\ &= (z - i(\sqrt{3} - 2))(z + (\sqrt{3} + 2)i) \end{aligned}$$

Since $|\sqrt{3} - 2| < 1$, $|\sqrt{3} + 2| > 1$, we can apply the cauchy integral formula:

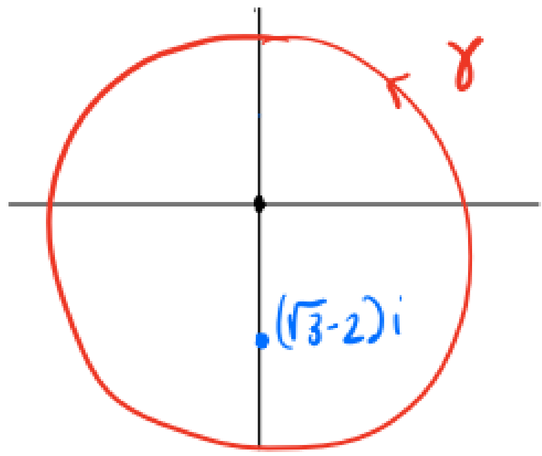
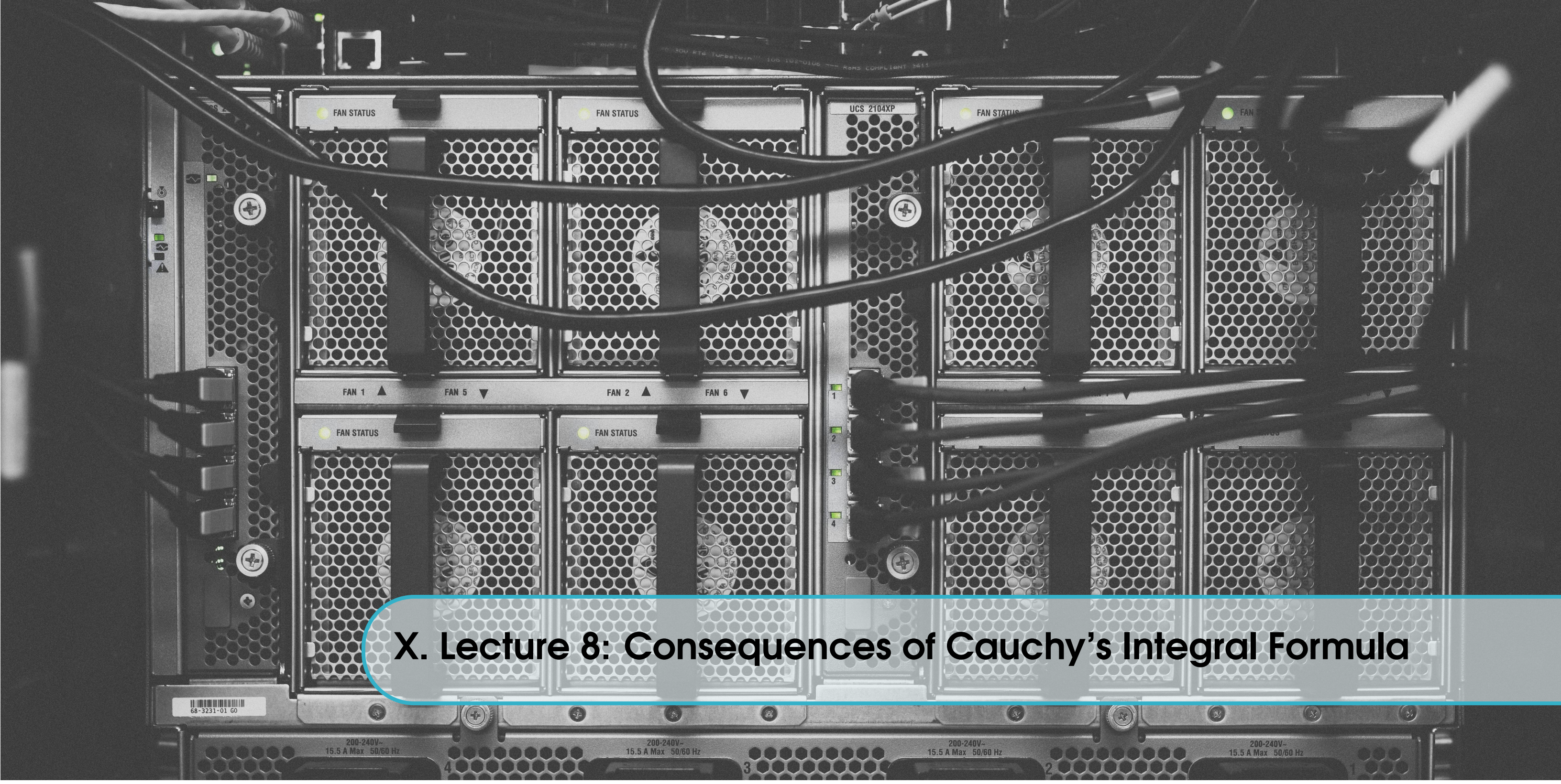


Figure IX.3: Circle of Radius 1

$$\begin{aligned}\int_\gamma \frac{1}{(z - i(\sqrt{3} - 2))} \cdot \frac{2dz}{(z + i(\sqrt{3} + 2))} &= 2\pi i \left(\frac{2}{(\sqrt{(3)} - 2)i + (\sqrt{3} + 2)i} \right) \\ &= \frac{4\pi}{2\sqrt{3}} \\ &= \frac{2\pi}{\sqrt{3}}\end{aligned}$$

■



X. Lecture 8: Consequences of Cauchy's Integral Formula

Theorem X..I if $f(z)$ is analytic in a domain D, $z_0 \in D$ and $\{|z - z_0| < R\} \subseteq D$, then $f(z)$ hasa convergent power series expansion about z_0 given by:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{X.I}$$

Where:

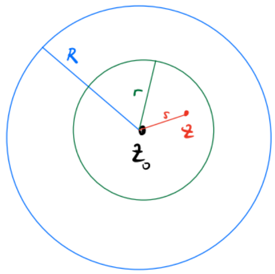
$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz \tag{X.II}$$

for any simple, closed, positively oriented curve γ in D containing z_0 and $\gamma = |z - z_0| = R$.

Proof. Let $\Delta = \{|z - z_0| < R\}$. if $z \in \Delta$, $|z - z_0| = s < R$, then by Cauchy's Integral Formula:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{X.III}$$

Where $\gamma = \{|z - z_0| = r\}$, positively oriented, $r > s$.



Now we write:

$$\begin{aligned}\xi - z &= \xi - z_0 - (z - z_0)\\ &= (\xi - z_0)\left(1 - \frac{z - z_0}{\xi - z_0}\right)\end{aligned}$$

Since $\frac{z-z_0}{\xi-z_0} = \frac{s}{r} < 1$, we can write (for $\xi \in \gamma$):

$$\frac{1}{1 - \frac{z-z_0}{\xi-z_0}} = \sum_{k=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^k \qquad \text{Geometric Series} \tag{X.IV}$$

So, *for any* $N \in \mathbb{N}$ *we have*:

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)}{\xi - z_0}\sum_{k=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^k d\xi\\ &= \sum_{k=0}^{N}(z - z_0)^k\left(\frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)}{(\xi - z_0)^{k+1}}d\xi\right)\\ &+ \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)}{\xi - z_0}\left[\sum_{k=N+1}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^k\right]d\xi\end{aligned}$$

Since $\sum_{k=0}^{\infty}\frac{|z-z_0|^k}{|\xi-z_0|^k}$ converges, if $\varepsilon > 0$
We can choose L such that $\forall N > L$ we have:

$$\sum_{k=N+1}^{\infty}\frac{|z - z_0|^k}{|\xi - z_0|^k} < \varepsilon \tag{X.V}$$

Since f analytic, there is a constant M such that max $|f(\xi)| \leq M$ for $\xi \in \gamma$.
By definition: $|\xi - z_0| = r$ on γ, thus, by the triangle inequality:

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)}{\xi - z_0}\left[\sum_{k=N+1}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^k\right]d\xi \geq \frac{\text{length}(\gamma)}{r}M\varepsilon = 2\pi M\varepsilon$$

Thus, $\forall\varepsilon > 0$, $\exists L|\forall N > L$ we have:

$$\left|f(z) - \sum_{k=0}^{N}a_k(z - z_0)^k\right| < 2\pi\varepsilon \tag{X.VI}$$

Where $a_k = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)}{(\xi-z_0)^{k+1}}d\xi$. So: $f(z) = \sum_{k=0}^{\infty}a_k(z - z_0)^k$. as desired.

■

Corollary X..II If $f(z)$ is analytic in a domain D, then so is $f'(z)$. In particular, if f is analytic, then it is infinitely differentiable.

Proof. Every f has a power series expansion about z_0, and every term in the series is analytic. Therefore, the series converges to an analytic function. ■

Corollary X..III — Unique Analytic Continuation. If $f(z)$ is analytic in a domain D, and $f(z) = 0$ for all $z \in \Delta \subseteq D$, then $f(z) = 0$ for all $z \in D$. In the above setting we have:

$$\frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi \tag{X.VII}$$

In particular, if f is analytic in a domain D, and, for some $z_0 \in D$, $f^{(k)}(z_0) = 0$ for all $k \in \mathbb{N}$, then $f(z) = 0 \forall z \in D$.

X.I Comparison with Real Functions

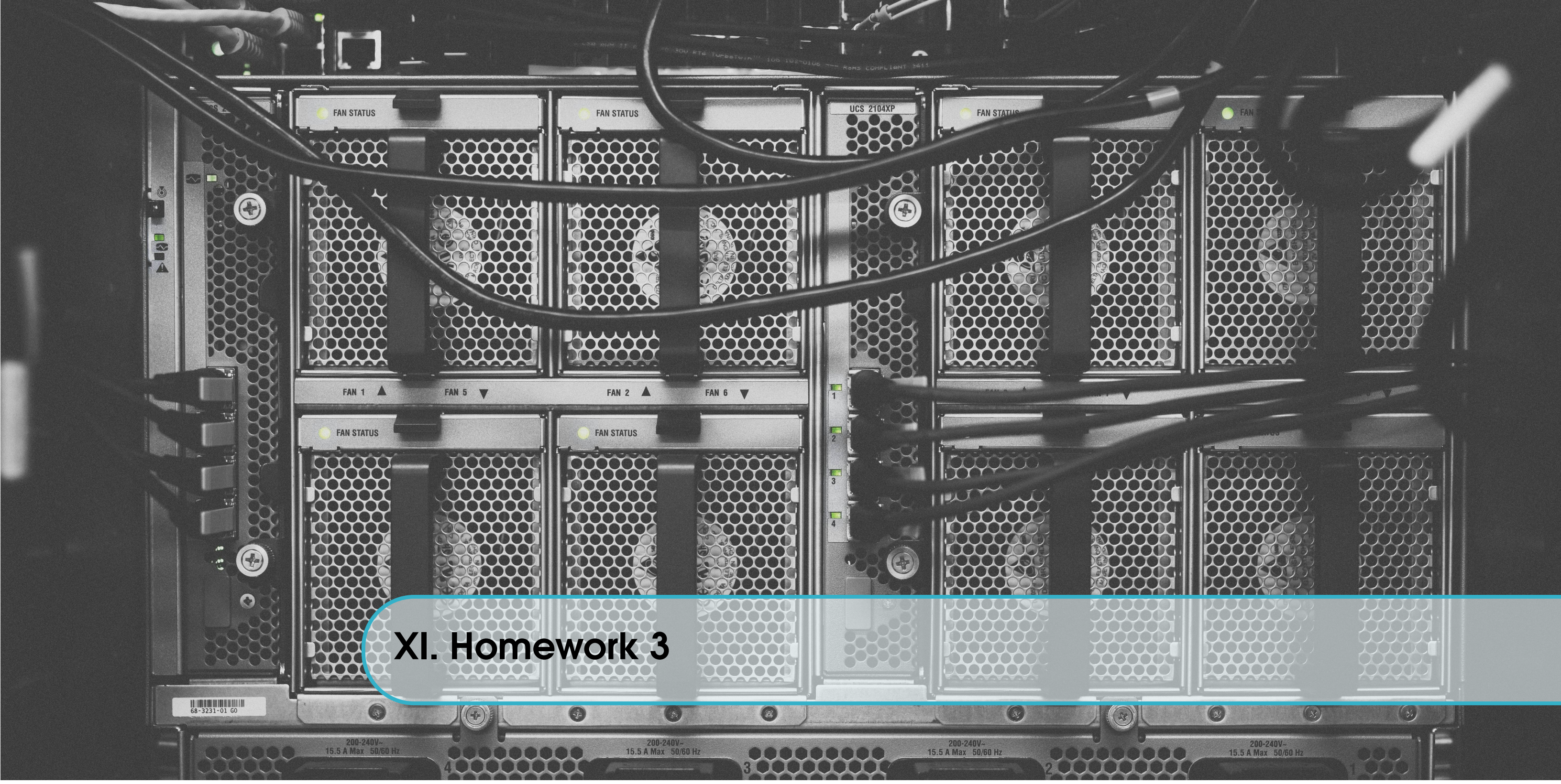
■ **Example X.I** To get a sense for the difference between real and complex functions, consider the following **real function**:

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{X.VIII}$$

f is infinitely differentiable, and $\forall k \in \mathbb{N}$, $f^{(k)}(0) = 0$.
So, the Taylor series of f at $0 \in \mathbb{R}$ is:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0 \tag{X.IX}$$

Thus, f is infinitely differentiable, but ***not*** equal to a power series in any neighborhood of $0 \in \mathbb{R}$. ■



XI. Homework 3

■ **Example XI.I — Fisher, Section 2.2, Problem 2.** Find the radius of convergence for the series

$$\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}(z - 2)^k$$

Let $a_k = \frac{(k!)^2}{(2k)!}$, then

$$\begin{aligned}
\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty} \left|\frac{(k+1)!^2}{(2k+2)!} \cdot \frac{(2k)!}{(k!)^2}\right| \\
&= \lim_{k\to\infty} \left|\frac{(k+1)^2}{(2k+2)(2k+1)}\right| \\
&= \lim_{k\to\infty} \left|\frac{k^2 + 2k + 1}{4k^2 + 6k + 2}\right| \\
&= \lim_{k\to\infty} \left|\frac{1 + 2/k + 1/k^2}{4 + 6/k + 2/k^2}\right| \\
&= \left|\frac{1}{4}\right| = \frac{1}{4}
\end{aligned}$$

Therefore, the radius of convergence is $R = \frac{1}{1/4} = 4$.

■

■ **Example XI.II — Fisher, Section 2.2, Problem 4.** Find the radius of convergence for the series

$$\sum_{k=0}^{\infty} (-1)^k z^{2k}$$

We can relate this to a geometric series with $a = 1$ and $r = -z^2$. The limit of the ratio of consecutive terms is

$$L = \frac{1}{1 - (-z^2)} = \frac{1}{1 + z^2}$$

Provided that $|r| = |-z^2| < 1$. Therefore, the radius of convergence is $R = 1$.

■

■ **Example XI.III — Fisher, Section 2.2, Problem 22.** (a) If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence $R > 0$,and if $f(z) = 0$ for all z such that $|z - z_0| < r \leq R$, show that all the coefficients are zero (ie. $a_0 = a_1 = a_2 = \cdots = 0$).
(b) if $F(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and $G(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$ are convergent and equal on some disc $|z - z_0| < r$,show that
$a_n = b_n$ for all n.

(a) if $f(z)$ is analytic in a domain D, $z_0 \in D$ and $\{|z - z_0| < R\} \subseteq D$, then $f(z)$ hasa convergent power series expansion about z_0 given by:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{XI.I}$$

Where:

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz \tag{XI.II}$$

for any simple, closed, positively oriented curve γ in D containing z_0 and $\gamma = |z - z_0| = R$.
This means that functions that are equal on a disc of radius r must have the same power series expansion. Furthermore, $a_n = 0$ being a valid power series expansion for $f(z)$ means it's the only power series expansion for $f(z)$.
(b) If $F(z) = G(z)$ on a disc of radius r, using the previous theorem, $F(z)$ and $G(z)$ must have the same power series expansion. Therefore, $a_n = b_n$ for all n. ■

■ **Example XI.IV** (Fisher, Section 2.3, Problem 2) Evaluate the following integral:

$$\int_{|z|=2} \frac{e^z}{z(z - 3)} dz$$

We can use Cauchy's Integral Formula to evaluate this integral. Let $f(z) = \frac{e^z}{z-3}$ and $z_0 = 0$. Then, the integral evaluates to:

$$\int_{|z|=2} \frac{e^z}{z(z - 3)} dz = 2\pi i f(0) = 2\pi i \frac{e^0}{-3} = -\frac{2\pi i}{3}$$

■

■ **Example XI.V — Fisher, Section 2.3, Problem 4.** Evaluate the following integral:

$$\int_{|z|=1} \frac{\sin(z)}{z} dz$$

We can use Cauchy's Integral Formula to evaluate this integral. Let $f(z) = \sin(z)$ and $z_0 = 0$. Then, the integral evaluates to:

$$\int_{|z|=1} \frac{\sin(z)}{z} dz = 2\pi i f(0) = 2\pi i \sin(0) = 0$$

■

■ **Example XI.VI — Fisher, Section 2.3, Problem 8.** Evaluate the following definite trigonometric integral. (Hint: it may be useful to review the technique of Examples 6 and 7 of Section 2.3).

$$\int_0^{\pi} \frac{1}{1 + \sin^2 \theta} d\theta$$

We know that:

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

So we can parametrize using the relation, $z = e^{i\theta}$:

$$\begin{aligned} \sin^2 \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \cdot \frac{e^{i\theta} - e^{-i\theta}}{2i} \\ &= \frac{(e^{i\theta})^2 - 2 + (e^{i\theta})^-2}{-4} \\ &= \frac{z^2 - 2 + z^{-2}}{-4} \end{aligned}$$

And the differential becomes:

$$d\theta = \frac{dz}{iz}$$

So the integral becomes:

$$\begin{aligned} \int_0^{\pi} \frac{1}{1 + \sin^2 \theta} d\theta &= \int_{|z|=1} \frac{1}{1 + \frac{z^2 - 2 + z^{-2}}{-4}} \frac{dz}{iz} \\ &= \int_{|z|=1} \frac{-4iz}{z^4 - 6z^2 + 1} dz \end{aligned}$$

We can find the roots using the quadratic formula ($z^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 1$, $b = -6$, $c = 1$)

$$\begin{aligned} z^2 &= \frac{-6 \pm \sqrt{36 - 4}}{2} = \frac{-6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2} \\ z &= \pm\sqrt{3 \pm 2\sqrt{2}} \end{aligned}$$

Now let's rewrite the integral in terms of z:

$$\int_{|z|=1} \frac{-4iz}{z^4 - 6z^2 + 1} dz =$$
$$\int_{|z|=1} \frac{-4iz}{(z - \sqrt{3 + 2\sqrt{2}})(z + \sqrt{3 + 2\sqrt{2}})(z - \sqrt{3 - 2\sqrt{2}})(z + \sqrt{3 - 2\sqrt{2}})} dz$$

We choose $|(z - \sqrt{3 - 2\sqrt{2}})| < 1$ as the contour of integration and we can write:

$$f(z) = \frac{-4iz}{(z + \sqrt{3 - 2\sqrt{2}})(z - \sqrt{3 + 2\sqrt{2}})(z + \sqrt{3 + 2\sqrt{2}})}$$

So the integral evaluates to:

$$\int_{|z|=1} \frac{-4iz}{z^4 - 6z^2 + 1} dz = \int_{|z|=1} \frac{\frac{-4iz}{(z+\sqrt{3-2\sqrt{2}})(z-\sqrt{3+2\sqrt{2}})(z+\sqrt{3+2\sqrt{2}})}}{(z - \sqrt{3 - 2\sqrt{2}})} dz$$

Applying Cauchy's Integral Formula, we get:

$$\int_{|z|=1} \frac{-4iz}{z^4 - 6z^2 + 1} dz = 2\pi i f(\sqrt{3 - 2\sqrt{2}})$$
$$= 2\pi i \frac{-4i\sqrt{3 - 2\sqrt{2}}}{2\sqrt{3 - 2\sqrt{2}}(\sqrt{3 - 2\sqrt{2}} - \sqrt{3 + 2\sqrt{2}})(\sqrt{3 - 2\sqrt{2}} + \sqrt{3 + 2\sqrt{2}})}$$
$$= 2\pi i (\frac{i}{2\sqrt{2}})$$
$$= -\frac{\pi}{\sqrt{2}}$$

■

■ **Example XI.VII — Fisher, Section 2.3, Problem 10.** Evaluate the following integral.(Hint: It may be useful to review the technique of Example 10 of Section 2.3)

$$\int_\gamma (z + z^{-1}) dz$$

Where γ is any curve contained in the region $\{\mathrm{Im}(z) > 0\}$ which joints $-4 + i$ to $6 + 2i$.

We know that the derivative of $F(z) = \frac{z^2}{2} + \ln(z)$ is $f(z) = z + z^{-1}$. But this is valid only where $\ln(z)$ is analytic, which is the case for $\{\mathrm{Im}(z) > 0\}$. So the integral evaluates to:

$$\begin{aligned}
\int_\gamma (z + z^{-1}) dz &= \int_\gamma f(z) dz = \int_\gamma F'(z) dz \\
&= F(\text{end}) - F(\text{start}) \\
&= F(6 + 2i) - F(-4 + i) \\
&= \frac{(6 + 2i)^2}{2} + \ln(6 + 2i) - \frac{(-4 + i)^2}{2} - \ln(-4 + i) \\
&= 8.5 + 16i + \ln(6 + 2i) - \ln(-4 + i)
\end{aligned}$$

■

■ **Example XI.VIII — Fisher, Section 2.3, Problem 14.** (a)(5 points) Suppose f is analytic on the disc $|z - z_0| < R$.Show that for any $r < R$ we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})dt$$

(b) (5 points) Using part (a) and the triangle inequality, conclude that

$$|f(z_0)| \leq \max_{0 \leq t \leq 2\pi} |f(z_0 + re^{it})|$$

(c)(5 points) Conclude from (b) that $|f|$ cannot have a strict local maximum within the domain of analyticity of f

(a) The Cauchy Integral Formula states that for any $r < R$:

$$f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z - z_0} dz$$

We can parametrize the integral using:

$$\begin{aligned} z &= z_0 + re^{it} \quad \text{where} \quad 0 \leq t \leq 2\pi \\ dz &= ire^{it}dt \end{aligned}$$

So the integral becomes:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})dt \end{aligned}$$

As required. (b) using what we found in part (a), we can write:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})dt$$

Thus:

$$\left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|dt$$

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|dt \leq \frac{1}{2\pi} \int_0^{2\pi} \max_{0 \leq t \leq 2\pi} |f(z_0 + re^{it})|dt$$

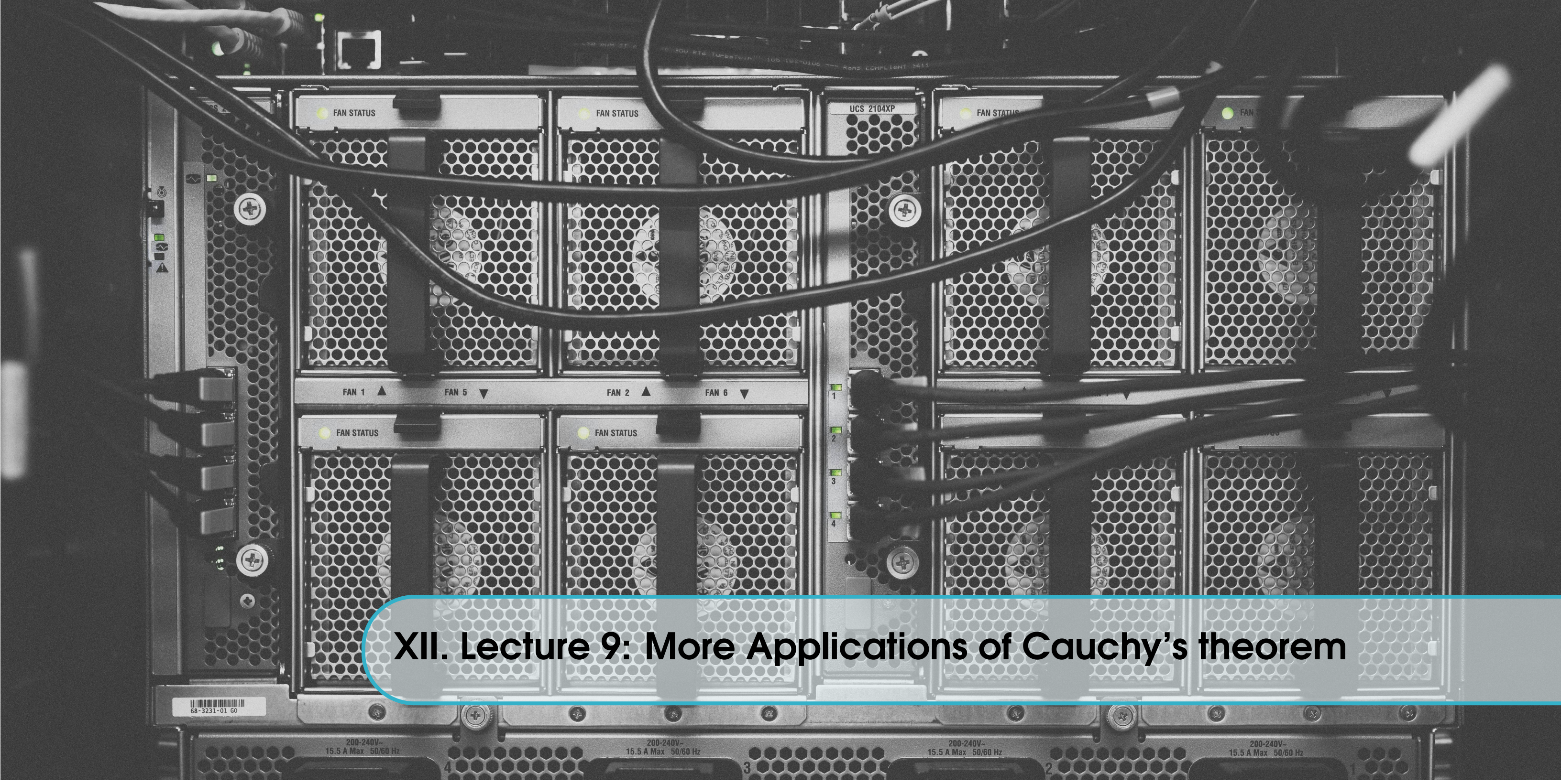
Since the maximum is constant, we can take it out of the integral

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \max_{0 \leq t \leq 2\pi} |f(z_0 + re^{it})|dt &= \max_{0 \leq t \leq 2\pi} |f(z_0 + re^{it})| \frac{1}{2\pi} \int_0^{2\pi} dt \\ &= \max_{0 \leq t \leq 2\pi} |f(z_0 + re^{it})| \end{aligned}$$

Therefore I can write:

$$|f(z_0)| \leq \max_{0 \leq t \leq 2\pi} |f(z_0 + re^{it})|$$

(c) If $|f|$ has a strict local maximum at z_0, then $|f(z_0)| > |f(z_0 + re^{it})|$ for some $r < R$. But this contradicts the inequality we found in part (b). Therefore, $|f|$ cannot have a strict local maximum within the domain of analyticity of f. ∎



XII. Lecture 9: More Applications of Cauchy's theorem

XII.I The order of a zero of a function

Lemma XII.I.I — Order of a zero of a function. Suppose f is analytic in a disc D, f is not identically zero, and $f(z_0) = 0$. Then for some z_0 in D, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

Let $m \geq 1$ be the smallest $n \in \mathbb{Z}_{\geq 0}$ such that $a_n \neq 0$. That is:

$$f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots$$

Say: f has a zero of order m at z_0. Equivalently:

$$f(z) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

Yet $f^{(m)}(z_0) \neq 0$.
Then $g(z) = \frac{f(z)}{(z-z_0)^m}$ is analytic in D and $g(z_0) \neq 0$. Since

$$g(z) = a_m + a_{m+1}(z - z_0) + \cdots$$

Converges because:

$$(z - z_0)^m \left[a_{m+1} + a_{m+2}(z - z_0) + \cdots\right] = f(z) - a_m(z - z_0)^m$$

Converges in D.

Proof. This proof will demonstrate the conclusion of equation **??**.
Let $f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots$
Notice that $(z - z_0) = 0$ at $z = z_0$.
Thus:

$$f'(z) = ma_m(z - z_0)^{m-1} + (m + 1)a_{m+1}(z - z_0)^m + \cdots = 0$$

$$\begin{aligned}
f''(z) &= m(m-1)a_m(z-z_0)^{m-2} + (m+1)ma_{m+1}(z-z_0)^{m-1} + \cdots = 0 \\
&\vdots \\
f^{(m-1)}(z) &= m!a_m(z-z_0) + m!a_{m+1}(z-z_0)^2 + \cdots = 0 \\
f^{(m)}(z) &= m!a_m(z-z_0)^0 + m!a_{m+1}(z-z_0)^1 + \cdots \neq 0
\end{aligned}$$

■

XII.II A partial converse to the Cauchy integral formula

Theorem XII.II.I — Morera's Theorem. if f is continuous in a domain D and

$$\int_\gamma f(z)\, dz = 0$$

for every triangle γ where $\gamma \in D$, and inside$(\gamma) \subseteq D$, f is analytic in D.

Proof.

$$\Omega = \{|z - z_0| < r\} \quad r > 0 \quad r \text{ is small}$$

such that $\Omega \in D$

for $z \in \Omega$ we define

$$F(z) = \int_\gamma f(\zeta)\, d\zeta,$$

where the integral is taken along a radial curve

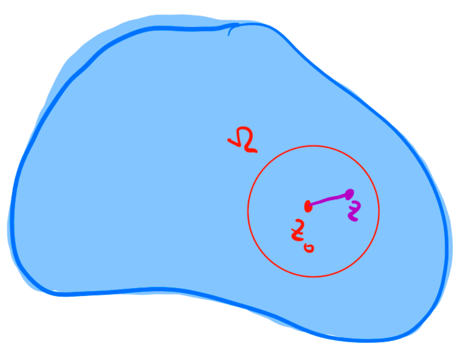


Figure XII.1: Radial curve

<u>**Goal:**</u> F is analytic and F$'(z) = f(z)$ (then it follows that f is also analytic).

$$F(z+h) - F(z) = \int_z^{z+h} f(\zeta)\, d\zeta$$

Since

$$\int_{z_0}^{z} f(\zeta)\, d\zeta + \int_z^{z+h} f(\zeta)\, d\zeta - \int_{z_0}^{z+h} f(\zeta)\, d\zeta = 0$$

by assumption

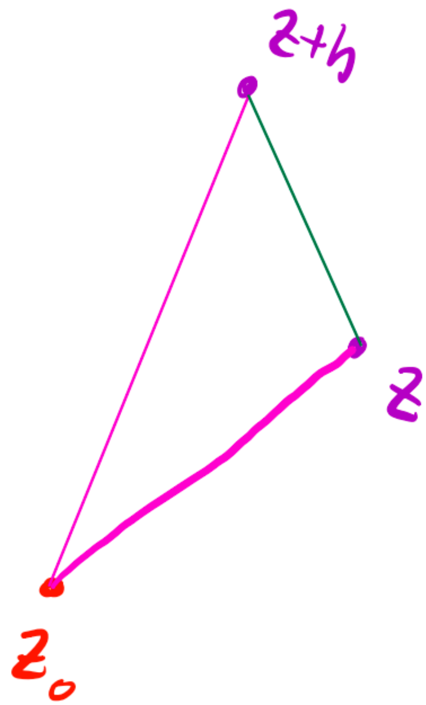


Figure XII.2: Curve

Thus:

$$\frac{F(z+h)-F(z)}{h}-f(z)=\frac{1}{h}\int_z^{z+h}f(\zeta)-f(z)d\zeta$$

$\rightarrow$ This is because

$$\begin{aligned}&\int_z^{z+h}f(z)d\zeta\\=&f(z)\int_z^{z+h}d\zeta\\=&f(z)\cdot h\end{aligned}$$

By continuity of f, for any $\varepsilon>0$, there exists $\delta>0$ such that $|f(w)-f(z)|<\varepsilon$ if $|w-z|<\delta$. Then if $|h|<\delta$, we have:

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right|\leq\frac{1}{|h|}\varepsilon|h|=\varepsilon$$

Since ε is arbitrary, we have:

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=f(z)$$

Thus: F is analytic and $F'(z)=f(z)$, as desired. ■

XII.III Applications

Theorem XII.III.I — Liouville's Theorem. If F is entire and $|F(z)| \leq M$ then F is constant.

Proof.

$$g(z) = \frac{F(z) - F(0)}{z} \quad \text{is entire since}$$

$$F(z) = F(0) + \sum_{n=1}^{\infty} a_n z^n$$

$$\text{so } g(z) = \sum_{n=1}^{\infty} a_n z^{n-1} = \sum_{n=0}^{\infty} a_{n+1} z^n$$

Now

$$|g(Re^{i\theta})| \leq \frac{|F(Re^{i\theta})| + |F(0)|}{R} \leq \frac{2M}{R}$$

Using Cauchy's theorem, we have:

$$\begin{aligned} g(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{g(z_0 + Re^{i\theta})}{Re^{i\theta}} iRe^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(Re^{i\theta})\, d\theta \end{aligned}$$

if $R >> |z_0|$, then $|z_0 + Re^{i\theta}| \geq R - |z_0|$

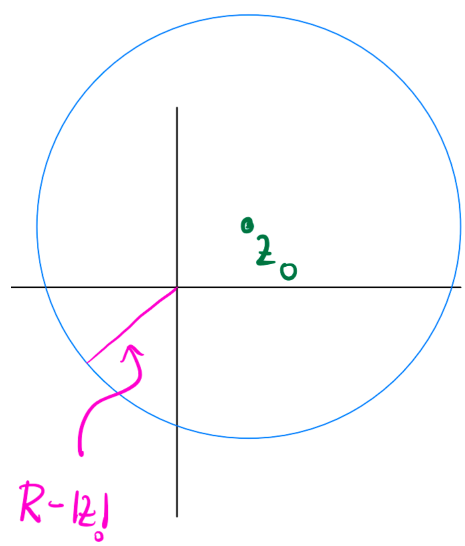


Figure XII.3: Circle

so

$$|g(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |g(z_0 + Re^{i\theta})|\, d\theta \leq \frac{2M}{R - |z_0|}$$

But $R >> |z_0|$ is arbitrary, so taking $R \to \infty$ yields $g(z_0) = 0$ for all z_0 and $F(z_0) = F(0)$. ∎

XII.IV Analytic Logarithms

Lemma XII.IV.I — The logarithmic derivative. Let D be a simply connected domain.
Suppose f is analytic in D and $f \neq 0$ anywhere in D.
Then $\frac{f'}{f}$ is analytic in D and hence so let's define:

$$h'(z) = \frac{f'(z)}{f(z)}$$

Using the fundamental theorem of calculus, we have:

$$h(z) = \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta$$

Cauchy's theorem implies that $h(z)$ is analytic in D.
Morera's theorem ensures that the integral can be taken over any path from z_0 to z in D and thus, $h(z)$ is path independent.
So:

$$\begin{aligned}\left[e^{-h(z)} f(z)\right]' &= -e^{-h(z)} h'(z) f(z) + e^{-h(z)} f'(z) \\ &= -e^{-h(z)} \frac{f'(z)}{f(z)} f(z) + e^{-h(z)} f'(z) \\ &= 0\end{aligned}$$

So $e^{-h(z)} f(z) = f(z_0)$ is a constant. Or:

$$\begin{aligned} e^{-h(z)} f(z) &= f(z_0) \\ f(z) &= f(z_0) e^{h(z)}\end{aligned}$$

Thus $g(z) = h(z) + \mathrm{Log}(f(z_0))$ satisfies:

$$\begin{cases} e^{g(z)} = f(z) \\ g(z) \quad \text{is analytic in } D\end{cases}$$

XII.V Isolated Singularities

Definition XII.V.I — Isolated Singularities. An analytic function has an isolated singularity at z_0 if it is analytic in a punctured disc $\{0 < |z - z_0| < r\}$ for some $r > 0$.

■ **Example XII.I** $f(z) = \frac{z^2 - z_0^2}{z - z_0}$
In this case $|f(z)|$ is bounded as $z \to z_0$.
in fact, $f(z) = z + z_0$ $(z \neq z_0)$ and f can be extended to an analytic function at z_0.
Thus: z_0 is a **removable** singularity. ■

■ **Example XII.II** $f(z) = \frac{1}{(z-z_0)^4}$

$|f(z)| = \frac{1}{|z-z_0|^4} \to +\infty$ as $z \to z_0$.
This is an example of a pole. ■

■ **Example XII.III** $f(z) = e^{\frac{1}{z-z_0}}$

$z_0 = 0$ for simplicity.
$|f(z)| = e^{\frac{1}{2z}+\frac{1}{2\bar{z}}} = e^{\frac{x}{x^2+y^2}}$

1. If $y = 0$ and $x \to 0$ from $x > 0$, then $|f(z)| \to +\infty$.
2. If $y = 0$ and $x \to 0$ from $x < 0$, then $|f(z)| \to 0$.
3. If $x = 0$ and $y \to 0$ then $|f(z)| \to 1$, this is an **essential** singularity.

■

XII.VI Removable Singularities

■ **Example XII.IV — Removable Singularity.** Suppose $|f(z)|$ is bounded near z_0.
Let

$$g(z) = \begin{cases} (z-z_0)^2 f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

Then, $g(z)$ is analytic on $\{|z-z_0| < r\}$ for some $r > 0$.

Since

$$\frac{g(z) - g(z_0)}{z - z_0} = (z-z_0)f(z)$$

and

$$\lim_{z\to z_0}(z-z_0)f(z) = 0$$

so $g'(z_0) = 0$ and thus:

$$g(z) = a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$

So:

$$f(z) = \frac{g(z)}{(z-z_0)^2} = a_2 + a_3(z-z_0) + \cdots$$

If we set $f(z_0) = a_2$, then f is analytic on $\{|z-z_0| < r\}$ and, we have **removed** the singularity.
■

Definition XII.VI.I z_0 is a removable singularity of f is bounded in a neighborhood of z_0.

XII.VII Poles

R Recall: if f is analytic on $\{0 < |z-z_0| < R\}$ and $\lim_{z\to z_0} f(z) = \infty$, then z_0 is a pole of f.

Lemma XII.VII.I — Poles. Choose $r < R$ small enough that $|f(z)| > 1$ on $\{0 < |z - z_0| < r\}$.
Then: $g(z) = \frac{1}{f(z)}$ is analytic on $\{0 < |z - z_0| < r\}$ and $g(z) \to 0$ as $z \to z_0$.
Thus, g has a removable singularity and $g(z_0) = 0$.
z_0 is a zero of order $m \geq 1$

$$g(z) = (z - z_0)^m h(z)$$

where $h(z)$ is analytic and $h(z_0) \neq 0$ on $\{|z - z_0| < r\}$.
Then

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z - z_0)^m} \cdot \frac{1}{h(z)} = \frac{H(z)}{(z - z_0)^m}$$

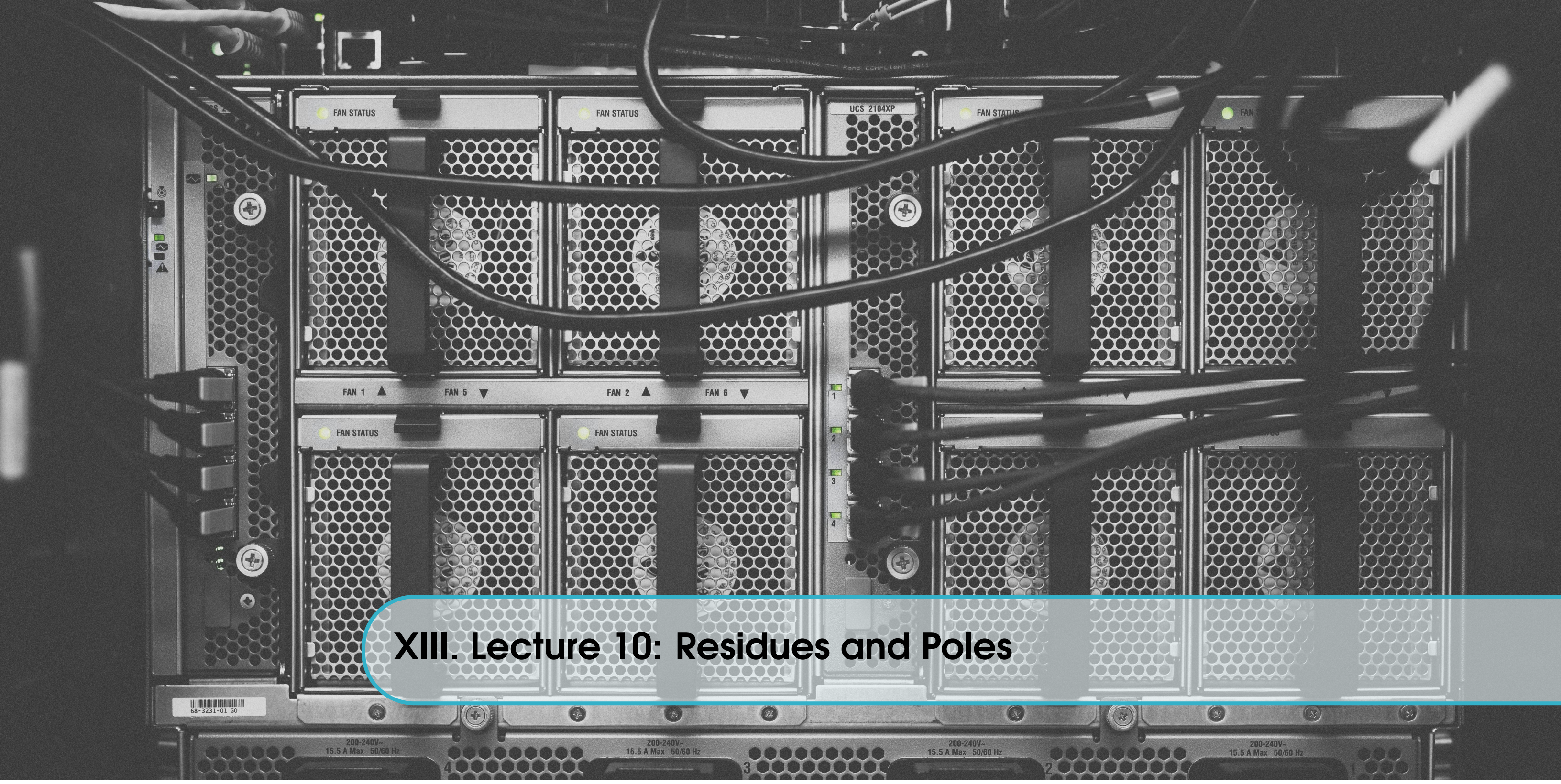
Where $H(z)$ is analytic on $\{|z - z_0| < r\}$ and $H(z_0) \neq 0$.

Definition XII.VII.I if $f(z) = \frac{H(z)}{(z-z_0)^m}$, where $H(z)$ is analytic on $\{|z - z_0| < r\}$ and $H(z_0) \neq 0$, then we say $f(z)$ has a pole of order m at z_0.

XII.VIII Essential Singularities

R Recall $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $z = 0$. The behaviour near essential singularities is wild.

Proposition XII.VIII.I if f is analytic on $\{0 < |z - z_0| < r\}$, with an essential singularity at z_0, then for any $w \in \mathbb{C}$, and any $\delta > 0$, $\exists z_\delta$ such that $0 < |z - z_0| < r$ and $|f(z) - w| < \delta$.



XIII. Lecture 10: Residues and Poles

Definition XIII..I — Residue. Suppose f is analytic on $0 < |z - z_0| < r$, if $0 < s < r$ Define the ***residue of*** f ***at*** z_0 to be

$$\text{Res}(f; z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=s} f(z)dz$$

R *Note:* $\text{Res}(f; z_0)$ does not depend on s!

Theorem XIII..I — Green's Theorem. Suppose f is analytic on $0 < |z - z_0| < r$, then

$$\int_{|z-z_0|=s_1} f(\xi)d\xi = \int_{|z-z_0|=s_2} f(\xi)d\xi$$

XIII.I Computing Residues with Power Series

Theorem XIII.I.I Suppose $f(z)$ has a pole of order $m \geq 1$ at z_0, then

$$f(z) = \frac{H(z)}{(z - z_0)^m} \tag{XIII.I}$$

Where H is analytic at z_0 and $H(z_0) \neq 0$ on $\{|z - z_0| < r\}$ and $H(z_0) \neq 0$. Expand $H(z)$ in a Laurent series about z_0:

$$H(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Then

$$\boxed{f(z) = \frac{c_0}{(z - z_0)^m} + \frac{c_1}{(z - z_0)^{m-1}} + \cdots + \sum_{k=m}^{\infty} c_k (z - z_0)^{k-m}} \quad \text{(XIII.II)}$$

So we say:

$$\text{Res}(f; z_0) = c_{m-1}$$

i.e.the coefficent of $\frac{1}{(z-z_0)}$ in the Laurent series of f about z_0 (equation XIII.II).

■ **Example XIII.I** Find the residue of:

$$f(z) = \frac{e^z - 1}{z^2} \quad \text{at} \quad z_0 = 0$$

Solution: We know that:

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\frac{e^z - 1}{z^2} = \frac{1}{z^2} + \frac{z}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{n-2}}{n!}$$

So the residue is $c_{-1} = \frac{1}{1!} = 1$. ■

■ **Example XIII.II** Find the residue of:

$$f(z) = \frac{z^2 + 3z - 1}{z + 2} \quad \text{at} \quad z_0 = -2$$

Solution: Let $w = z + 2$, then $z = w - 2$ and $dz = dw$. So

$$\begin{aligned} f(w) &= \frac{(w-2)^2 + 3(w-2) - 1}{w} = \frac{w^2 - 4w + 4 + 3w - 6 - 1}{w} = \frac{w^2 - w - 3}{w} \\ &= \frac{w^2}{w} - \frac{w}{w} - \frac{3}{w} \\ &= w - 1 - \frac{3}{w} \\ &= \frac{-3}{z+2} - 1 + z + 2 \end{aligned}$$

So the residue is $c_{-1} = -3$. ■

XIII.II Laurent series

Definition XIII.II.I — Laurent Series. Suppose f is analytic on $0 \le r < |z - z_0| < R$, then f has a ***Laurent series*** about z_0.

Theorem XIII.II.I — Laurent Series. if f is analytic on $0 \leq r < |z - z_0| < R$, then we can write $f(z) = f_1(z) + f_2(z)$ where:

1. $f_1(z)$ is analytic on $\{|z - z_0| < R\}$ and has a power series expansion about z_0.
2. $f_2(z)$ is analytic on $\{|z - z_0| < R\}$ and has a power series expansion about ∞.

And:

1. $f_1(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^n$
2. $f_2(z) = \sum_{k=1}^{\infty} b_k(z - z_0)^{-n}$

So we can write:

$$\boxed{f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^n} \quad \text{Laurent Series} \tag{XIII.III}$$

Where $a_k = b_k$ if $k < 0$. And the expression converges on $r < |z - z_0| < R$.

Proof. Take $z : r < |z - z_0| < R$, and r_1, R_1 such that:

$$r < r_1 < |z - z_0| < R_1 < R$$

Then f is analytic on $\{|z - z_0| < R_1\}$. As we can see from figure XIII.1, C_1 is a simple, closed, positively oriented curve in $\{|z - z_0| < R_1\}$, so we can apply Cauchy's theorem to get:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \int_{|\xi - z_0| = R_1} \frac{f(\xi)}{\xi - z_0} dz - \frac{1}{2\pi i} \int_{|\xi - z_0| = r_1} \frac{f(\xi)}{\xi - z_0} d\xi \end{aligned}$$

$$\Gamma = \{|\xi - z_0| = R_1\} \quad \text{and} \quad \gamma = \{|\xi - z_0| = r_1\}$$

Now we use a trick!

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 - (z - z_0)} = \frac{1}{\xi - z_0} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}}$$

On $\Gamma = \frac{z - z_0}{\xi - z_0} < 1$ so:

$$\frac{1}{\xi - z} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\xi - z_0)^{k+1}}$$

And on *gamma* we have:

$$\frac{1}{\xi - z} = -\sum_{k=1}^{\infty} \frac{(z - z_0)^k}{(\xi - z_0)^{k+1}}$$

Thus:

$$f(z) = \sum_{k=0}^{\infty}(z - z_0)^k \frac{1}{2\pi i}\int_{\Gamma=|\xi - z_0|=R_1} \frac{f(\xi)}{(\xi - z_0)^{k+1}}d\xi$$
$$- \sum_{k=0}^{\infty}(z - z_0)^{-k}\frac{1}{2\pi i}\int_{\gamma=|\xi - z_0|=r_1} \frac{f(\xi)}{(\xi - z_0)^{k}}d\xi$$

■

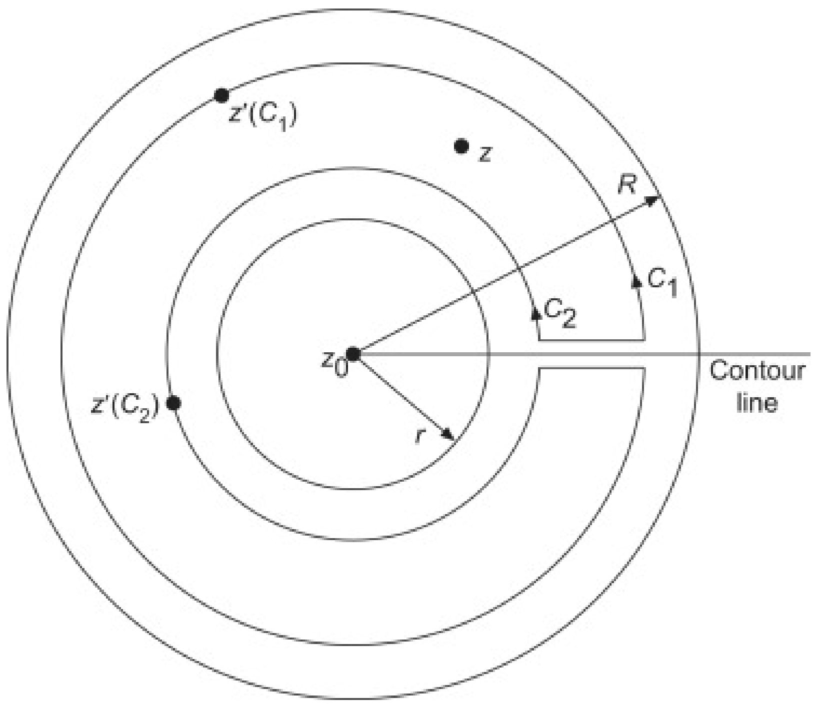
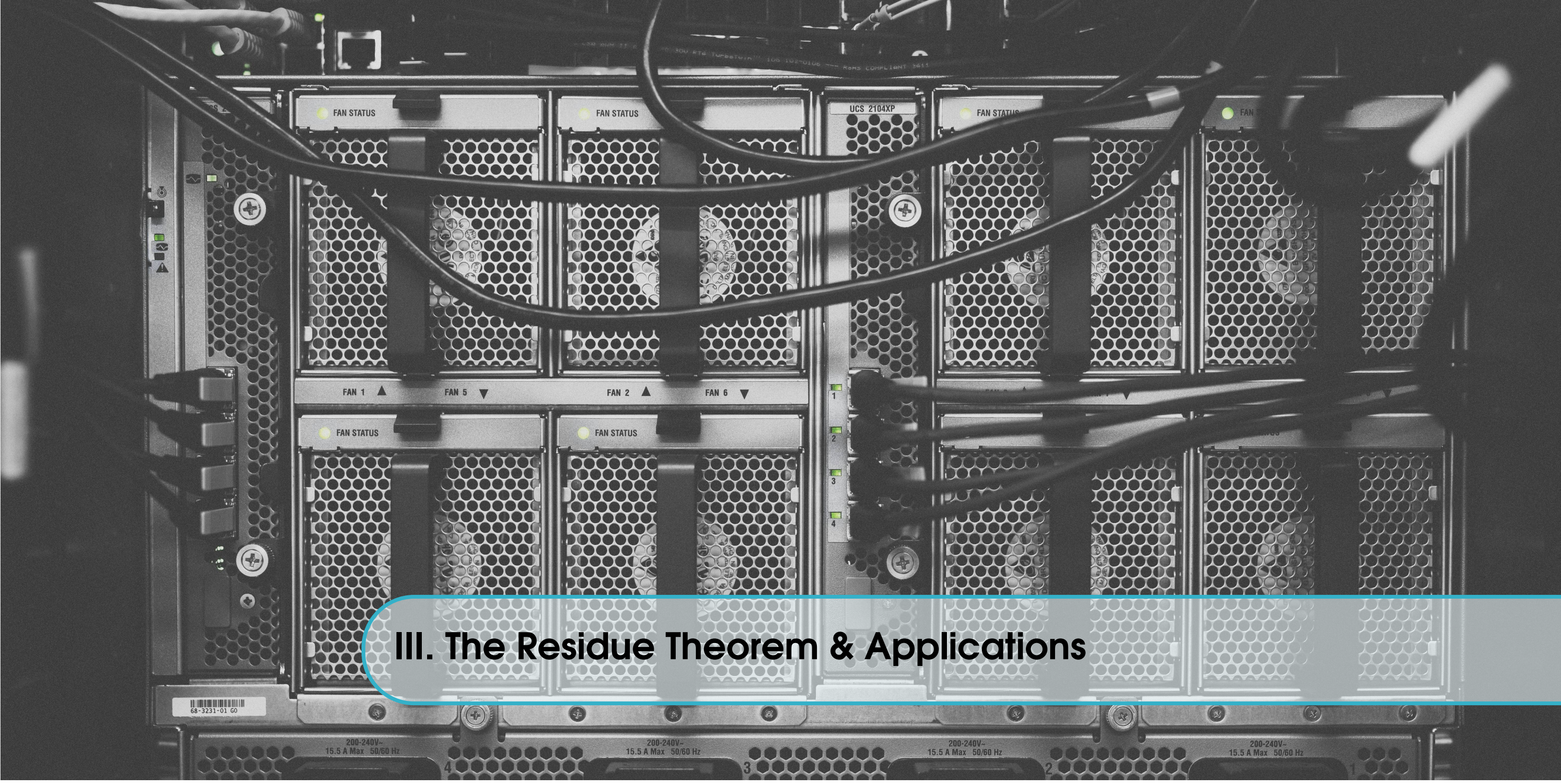


Figure XIII.1: Laurent Series

■ **Example XIII.III** 1. $f(z) = \frac{H(z)}{(z-z_0)^m}$
2. $f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$

■



III. The Residue Theorem & Applications

Theorem III..I — Residue Theorem. Suppoes f is analytic on a simply connected domain D, except for a finite number of isolated singularities at $z_1, z_2, \cdots, z_n \in D$. Let γ be a piecewise C^1, positively oriented, simple closed curve in D which does not pass through any of the singularities. Then:

$$\boxed{\oint_\gamma f(z)dz = 2\pi i \sum_{z_k \text{ inside } \gamma} \operatorname{Res}(f, z_k)} \tag{III.I}$$

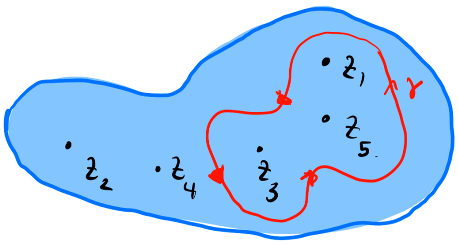


Figure III.1: Residue Theorem

Proof. Let $\gamma_1, \gamma_2, \cdots, \gamma_n$ be disjoin, positively oriented, circles around $z_1, z_2, \cdots, z_n$ respectively. We apply Green's theorem to get:

$$\oint_\gamma f(z)dz = \sum_{k:z_k \text{ inside } \gamma} \oint_{\gamma_k} f(z)dz$$

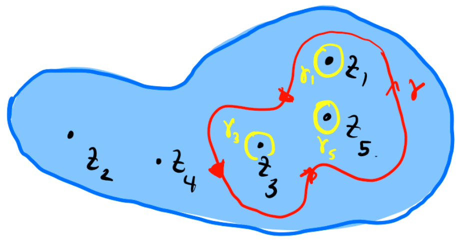


Figure III.2: circles around $z_1, z_2, \cdots, z_n$

But now by the Cauchy Integral Formula

$$\oint_{\gamma_k} f(z)dz = 2\pi i \mathrm{Res}(f, z_k)$$

■

■ **Example III.1** Compute:

$$\int_{-\infty}^{\infty} \frac{x}{((1 + x^2)(4 + x^2))} dx$$

Step 1: Change to a complex integral:

$$P(z) = z^2, \quad Q(z) = (1 + z^2)(4 + z^2)$$

Step 2: Choose a contour:

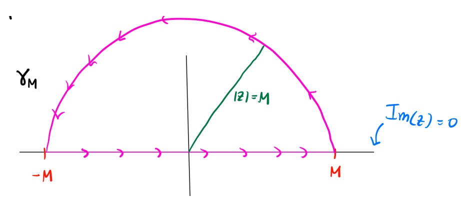


Figure III.3: Contour

Suppose M is very large

1. $\int_{\gamma_M} \frac{P(z)}{Q(z)} dz$ can be computed by the residue formula.
2. On the other hand:

$$\int_{\gamma_M} \frac{P(z)}{Q(z)} dz = \underbrace{\int_{-M} M \frac{x^2}{(1 + x^2)(4 + x^2)} dx}_{\text{The integral we want}} + \int_0 \pi \frac{P(Me^{i\theta})}{Q(Me^{i\theta})} iMe^{i\theta} d\theta$$

<u>Now</u>:

$$|P(Me^{i\theta})| \leq M^2$$

$$|Q(Me^{i\theta})| = |(Me^{i\theta})^2 + 1||(Me^{i\theta})^2 + 4| \qquad \geq \frac{1}{10}M^4 \qquad \text{for a very large } M$$

<u>So</u>

$$\left| \int_0^{\pi} \frac{P(Me^{i\theta})}{Q(Me^{i\theta})} iMe^{i\theta} d\theta \right| \le 10\pi \frac{M^3}{M^4} \to 0 \quad \text{as } M \to \infty$$

<u>Thus</u>

$$\int_{-\infty}^{\infty} \frac{x^2}{(1 + x^2)(4 + x^2)} dx = \lim_{M \to \infty} \int_{\gamma_M} \frac{P(z)}{Q(z)} dz$$

Which can be computed using the residue formula.
$Q(z)$ has zeroes at $z = \pm i, \pm 2i$. only $+i, +2i$ are inside γ_M for large M. So:

$$\to \boxed{z = i} \quad \frac{z^2}{(z + i)(z - i)(z^2 + 4)} = \frac{1}{z - i} \left[\frac{z^2}{(z + i)(z^2 + 4)} \right]$$

$$\text{Res}(f, i) = \frac{i^2}{(i + i)(i^2 + 4)} = \frac{-1}{6i}$$

$$\to \boxed{z = 2i} \quad \frac{z^2}{(z + i)(z - i)(z^2 + 4)} = \frac{1}{z - 2i} \left[\frac{z^2}{(z + i)(z - i)} \right]$$

∎