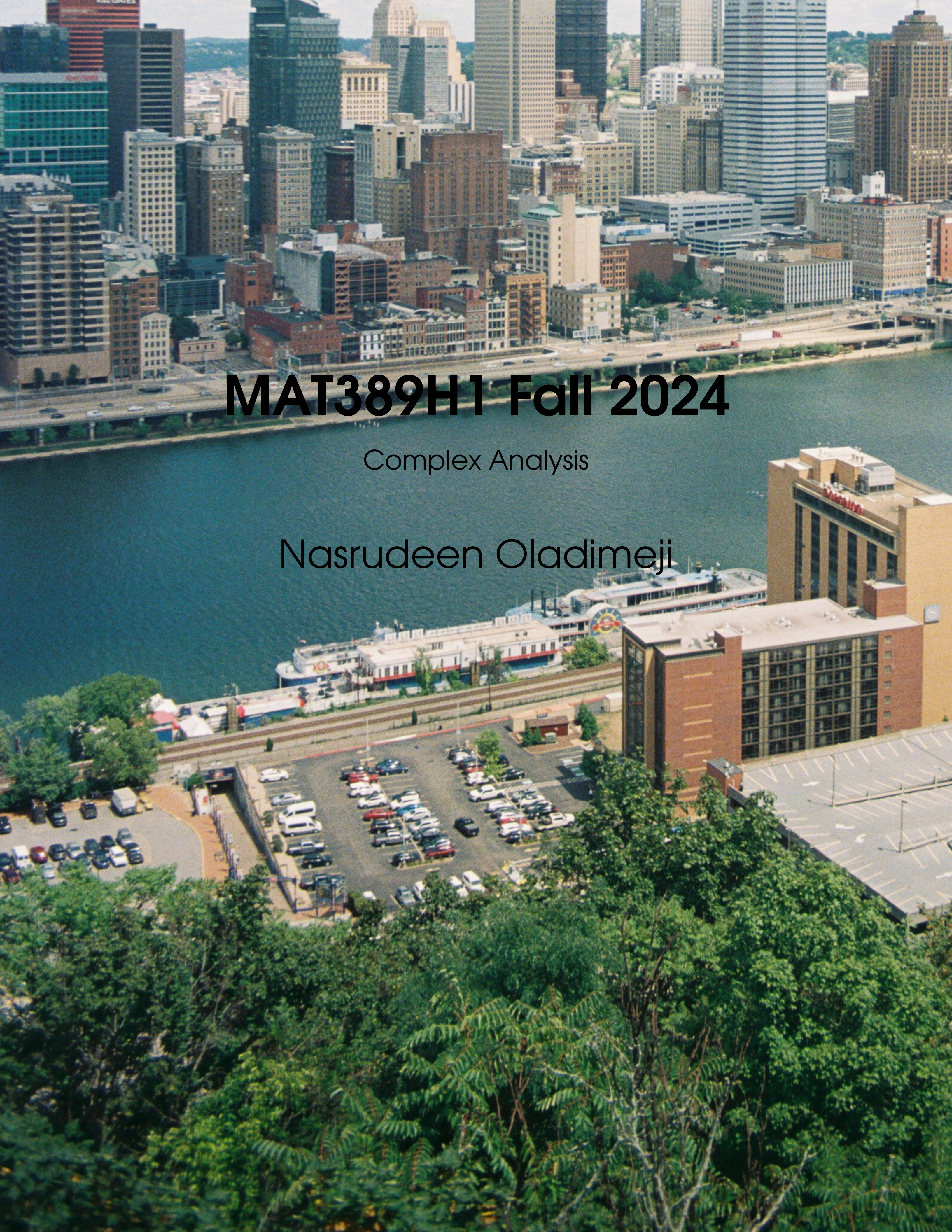

MAT389H1 Fall 2024

Complex Analysis

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I.I Introduction

Definition I.I.I — Complex Numbers. z is a complex number iff $z = a + bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.
The set of complex numbers is denoted by $\mathbb{C}$.

Definition I.I.II — Real and Imaginary Parts. If $z = a + bi$, then $\Re(z) = a \in \mathbb{R}$ and $\Im(z) = b \in \mathbb{R}$. where $\Re(z)$ is the real part of z and $\Im(z)$ is the imaginary part of z.

Definition I.I.III — Modulus. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$. $|z|$ is the modulus of z.

Definition I.I.IV — Conjugate. If $z = a + bi$, then $\bar{z} = a - bi$. $\bar{z}$ is the conjugate of z.

I.II Operations

Definition I.II.I — Addition and Subtraction. If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$.
Similarly $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$.

Definition I.II.II — Multiplication. If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then $z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$.
Note that $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

Definition I.II.III — Inversion. If $z = a + bi$, then $z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$.

Proof. Let's multiply by 1 in the form of the conjugate of z:

$$\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

■

Definition I.II.IV — Division. For $z, w \in \mathbb{C}$, $\frac{w}{z} = w \cdot z^{-1} = \frac{w\bar{z}}{|z|^2}$.

Table I.I: Properties of the Complex Conjugate

Property	Description
Conjugate of the Sum	$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
Conjugate Modulus	$z \cdot \bar{z} = \|z\|^2$
Conjugate of a Conjugate	$\bar{\bar{z}} = z$
Product of Conjugates	$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
Conjugate of a Quotient	$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
Real Part Conjugate	$Re(z) = \frac{z+\bar{z}}{2}$
Imaginary Part Conjugate	$Im(z) = \frac{z-\bar{z}}{2i}$
Real Number Check	$z = \bar{z} \iff z \in \mathbb{R}$
Imaginary Number Check	$z = -\bar{z} \iff z \in \mathbb{I}$
Function Linearity	If $\alpha = f(z)$ then $\overline{\alpha} = \overline{f(z)} = f(\bar{z})$

Table I.II: Properties of the Modulus in Complex Numbers

Property	Description
Positivity	$\|z\| \geq 0$, with equality if and only if $z = 0$
Triangle Inequality	$\|\|z_1\| - \|z_2\|\| \leq \|z_1 \pm z_2\| \leq \|z_1\| + \|z_2\|$
Multiplicative Property	$\|z_1 \cdot z_2\| = \|z_1\| \cdot \|z_2\|$
Division Property	$\left\|\frac{z_1}{z_2}\right\| = \frac{\|z_1\|}{\|z_2\|}$, for $z_2 \neq 0$
Conjugate	$\|z\| = \|\bar{z}\|$
Component Property	$-\|z\| \leq Re(z) \leq \|z\|$ $-\|z\| \leq Im(z) \leq \|z\|$
Cauchy-Schwarz Inequality	$\|z_1 w_1 + \cdots + z_n w_n\|^2 \leq \sum_{j=1}^{n} \|z_j\|^2 \sum_{j=1}^{n} \|w_j\|^2$

Proof. Proof of the Multiplicative Property of the Modulus:

$$\begin{aligned} |z_1 \cdot z_2|^2 &= (z_1 \cdot z_2) \cdot (\overline{z_1} \cdot \overline{z_2}) \\ &= z_1 \cdot \overline{z_1} \cdot z_2 \cdot \overline{z_2} \\ &= |z_1|^2 \cdot |z_2|^2 \end{aligned}$$

■

I.III Polar Representation

A complex number are vectors in $\mathbb{R}^2$, as such, they can be represented by a magnitude and a direction.

Definition I.III.I — Polar Form.

$$z = r(\cos(\theta) + i\sin(\theta)) \tag{I.I}$$

: $r = |z| \in \mathbb{R}^+$

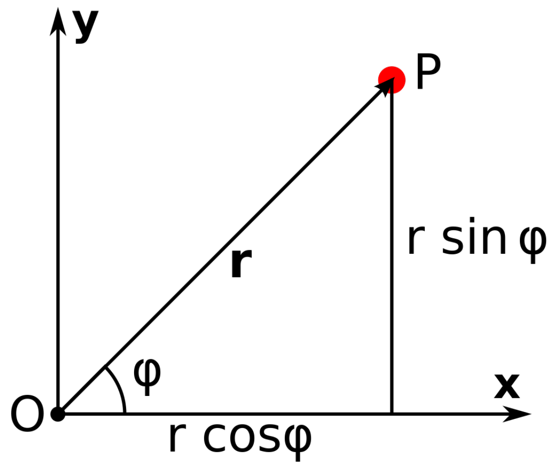

Figure I.1: Polar Coordinate Components

■ **Example I.1 — Multiplying Complex Numbers in Polar Form.** Let $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$. Then:

$$z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2))) \tag{I.II}$$

$$= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \tag{I.III}$$

Using the trig addition formula:

$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ and $\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$.

■

Theorem I.III.I — De Moivre's Theorem. if $z = r(\cos(\theta) + i\sin(\theta))$

$$z^n = r^n(\cos(\theta n) + i\sin(\theta n)) \tag{I.IV}$$

Proof. The following proof will illustrative the steps to inductive reasoning
Case of $n = 1$: $z^n = r^n(\cos(\theta n) + i\sin(\theta n)) = z = r(\cos(\theta) + i\sin(\theta))$
This is true by definition.
Assume that:
$z^{n-1} = r^{n-1}(\cos(\theta(n-1)) + i\sin(\theta(n-1)))$
Then from Equation (**??**) we can verify:

$$zz^{n-1} = rr^{n-1}(\cos(\theta(n-1) + \theta) + i\sin(\theta(n-1) + \theta))$$
$$z^n = r^n(\cos(\theta n) + i\sin(\theta n))$$

■

Definition I.III.II — Argument. The argument of a complex number $z = r(\cos(\theta) + i\sin(\theta))$ is any angle, $\arg(z) = \theta$, such that $z = r(\cos(\theta) + i\sin(\theta))$.

From Equation (**??**), we observe that r is unique (because we constrained it to just positive values). θ, however, is not unique.

Definition I.III.III — Principle Orientation. We say θ is the principle orientation of z if $\theta \in [-\pi, \pi)$
In this range, θ is unique.

Definition I.III.IV — Vector Dot Product. The dot product of two complex numbers $z = x + iy$ and $w = s + it$ is defined as:

$$z \cdot w = x \cdot s + y \cdot t = \Re(z\overline{w}) \tag{I.V}$$

$$\cos\theta = \frac{a \cdot b}{|a||b|} \tag{I.VI}$$

Corollary I.III.II — Perpendicular Vectors. Complex variables z and w are perpendicular if $\Re(z\overline{w}) = 0$.

R [Complex Numbers to Solve Polynomial Equations] Over $\mathbb{C}$, every equation of the form $z^n = a$ has n solutions.

■ **Example I.II — Solving $z^n = -1$.** Let $z = r(\cos(\theta) + i\sin(\theta))$. Then:

$$\begin{aligned}
z^n &= r^n(\cos(\theta n) + i\sin(\theta n)) = -1 \\
\implies r^n &= 1 \text{ and } \cos(\theta n) + i\sin(\theta n) = -1 \\
\implies r &= 1 \text{ and } \cos(\theta n) = -1 \text{ and } \sin(\theta n) = 0 \\
\implies \theta n &= \pi + 2\pi k \text{ for } k \in \mathbb{Z} \\
\implies \theta &= \frac{\pi + 2\pi k}{n} \text{ for } k \in \mathbb{Z}
\end{aligned}$$

We can now find the principle solutions for Z

$$\therefore\ \theta_0 = \frac{\pi}{n},\ \theta_1 = \frac{3\pi}{n}, \ldots, \theta_{n-1} = \frac{(2n-1)\pi}{n}$$

■

R Roots of Unity The solutions to $z^n = 1$ are called the nth roots of unity. Plotting these solutions splits the complex plane into n equal parts.

I.IV Subsets of the Plane

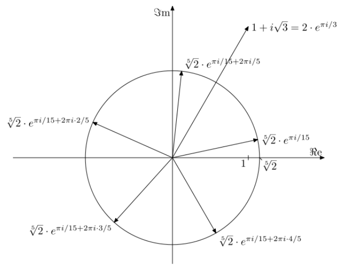

Figure I.2: Complex Fifth Roots of Unity

Definition I.IV.I — Open Disc. An open disc of radius R centered at z_0 is the set of all z such that $D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R \subset \mathbb{C}\}$.

Definition I.IV.II — Interior Point. A point z_0 is an interior point of a set $A \subset \mathbb{C}$ if there exists an open disc centered at z_0 that is contained in A.

$$z_0 \quad \text{is an interior point of } A \text{ if } \quad \exists D_{>0}(z_0) \in A$$

Definition I.IV.III — Open Set. A set $A \subset \mathbb{C}$ is open if every point in A is an interior point. *I.e. there are no 'hard lines' in the set.*

■ **Example I.III — Open Disc.** Show that the disc $D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R\}$ is an open set.

Proof. Let $z_1 \in D$. Then $|z_1 - z_0| < R$. Let $r = R - |z_1 - z_0|$. Then $r > 0$.
Let $z_2 \in D$ be any point in D, such that $|z_2 - z_1| < r$. Then:

$$\begin{aligned} |z_2 - z_0| &\leq |z_2 - z_1| + |z_1 - z_0| \\ &< r + R - r = R \end{aligned}$$

Therefore $z_2 \in D$ and D is open. ■

■

Definition I.IV.IV — Boundary (∂D). The boundary of a set A is the set of all points z such that every open disc centered at z, no matter how small, contains points in A and points not in A. The boundary of A is denoted by ∂A and a boundary point z is denoted by $z \in \partial A$.

$$z_0 \quad \text{is an boundary point of } A \text{ if } \quad \exists z \in D_R(z_0) : z \notin A \forall R > 0$$

Definition I.IV.V — Closed Set. A set D is closed if it contains all its boundary points.

R A set can be both open and closed ($\mathbb{C}$, $\emptyset$), open and not closed, closed and not open, or neither open nor closed (contains part, but not all of their boundary).

Theorem I.IV.I — Properties of Open and Closed Sets.
1. D is open iff $\mathbb{C} \setminus D$ is closed.
2. D is closed iff $\mathbb{C} \setminus D$ is open.
3. D is open if and only if it contains none of its boundary points.

I.V Lines and Circles (Not done in class, Fisher 1.3)

Definition I.V.I — Line in the Complex Plane. A line of the form $y = mx + b$ can be formulated as:

$$0 = \Re\{(m + i)z + b\}$$

Such that when the real part of the complex number is zero, the line is satisfied. The general form is:

$$0 = \Re\{az + b\}, \qquad a, b, z \in \mathbb{C} \tag{I.VII}$$

$$\text{where } a = A + iB \text{ such that:} \tag{I.VIII}$$

$$Ax - By + \Re b = 0 \tag{I.IX}$$

Note that the imaginary part of b does not affect the line.

Definition I.V.II — Simple Circle in the Complex Plane. Circles in the complex plane can be formulated as:

$$|z - z_0| = R \tag{I.X}$$

Where z_0 is the *locus* of the circle and R is the radius.

Definition I.V.III — Perpendicular Bisector. The perpendicular bisector of the line segment between p and q is the set of all points z such that

$$|z - p| = |z - q|$$

Corollary I.V.I — Apollonian Circles. If p and q are distinct complex numbers then a circle can be formulated as:

$$|z - p| = \rho|z - q| \qquad 0 < \rho \in \mathbb{R}, \rho \neq 1 \tag{I.XI}$$

$$\rightarrow \quad \text{Where} \quad z_0 = \frac{p - \rho^2 q}{1 - \rho^2} \quad \text{and} \quad R = \frac{|p - q|\rho}{1 - \rho^2} \tag{I.XII}$$

II.I Lecture 2: Connected Sets

Definition II.I.I — Connected Set. An *open* set D is connected if each pair of points $p, q \in D$ can be joined by a polygonal path lying entirely in D. That is:

$$\exists P_2, P_3, \ldots, P_n \in D \quad \text{such that} \quad pP_1, P_1P_2, \ldots, P_nq \in D$$

R The set doesn't *have* to be open, but it is easier to prove connectedness for open sets.

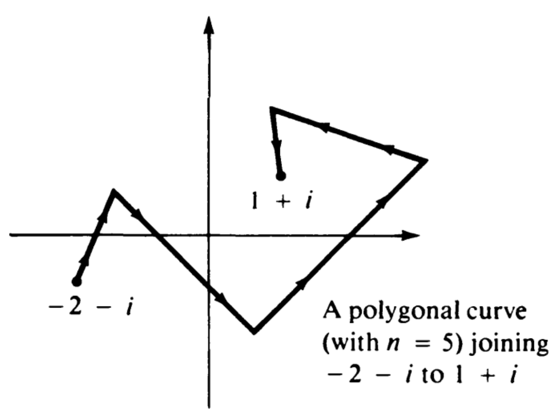

Figure II.1: Polygonal Path

Definition II.I.II — Domain. A domain is a set that's

- Open
- Connected
- Not empty

Definition II.I.III — Convex Set. A set D is convex if for each pair of points $p, q \in D$, the line segment pq lies entirely in D.

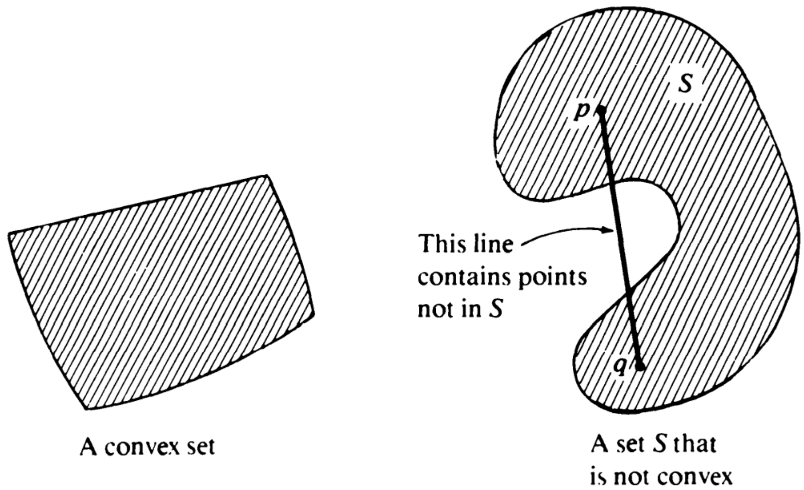

Figure II.2: Convex Set

Theorem II.I.I — Convex $\implies$ Connected. If D is a convex open set, then D is connected.

Definition II.I.IV — Open Half-plane. A set D is an open half-plane if it is of the form

$$D = \{z \in \mathbb{C} : \Re\{az + b\} \geq 0\}$$

Each open half-plane is convex and open

Definition II.I.V — Closed Half-plane. A set D is a closed half-plane if it is of the form

$$D = \{z \in \mathbb{C} : \Re\{az + b\} > 0\}$$

Each closed half-plane is convex and closed

II.II Point at Infinity

Definition II.II.I — Point at Infinity. A set is said to contain the point at infinity if it contains all points z such that $|z| > R$ for some $R > 0$.

■ **Example II.I** No open Half-plane contains the point at infinity. Even though the set is unbounded, choosing R near the boundary will always give a point outside the set. ■

II.III Functions and Limits

Definition II.III.I — Limit of a Sequence of Complex Numbers.

$$\lim_{n\to\infty} z_n = z \quad \text{or} \quad z_n \to z \iff \forall \varepsilon > 0, \exists N \in \mathbb{N} \tag{II.I}$$

$$\text{such that} \quad n \geq N \implies |z_n - z| < \varepsilon \tag{II.II}$$

Corollary II.III.I — Parts of a Limit. If $z_n = x_n + iy_n$ and $z = x + iy$, then

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

Theorem II.III.II — Subsequence. Suppose $\{z_n\}$ converges with limit z. Then every subsequence, $z_{m_n} = f(n)$ also converges to z. Where $1 \leq m_1 < m_2 < \ldots$

Definition II.III.II — Limits of Functions.

$$\lim_{z\to z_0} f(z) = w \iff \forall \varepsilon > 0, \exists \delta > 0 \tag{II.III}$$

$$\text{such that} 0 < |z - z_0| < \delta \implies |f(z) - w| < \varepsilon \tag{II.IV}$$

II.IV Continuity

Definition II.IV.I — Continuous Function. A function $f(z)$ is continuous at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0)$$

Corollary II.IV.I — Continuous at Infinity. A function $f(z)$ can be continuous at ∞ if $f(\infty) = \lim_{z\to\infty} f(z) = f(\infty)$. Note, $f(\infty)$ may equal ∞
This is equivalent to saying that $f(1/z)$ is continuous at $z = 0$

Definition III..I — infinite Series. Suppose we have a sequence:

$$z_1,\ z_2,\ z_3,\ \ldots \tag{III.I}$$

We can define the partial sum of the sequence as:

$$S_n = z_1 + z_2 + z_3 + \ldots + z_n \tag{III.II}$$

We say $\sum_{n=1}^{\infty} z_n$ converges and has a sum S if the sequence of partial sums converges to S:

$$\lim_{n\to\infty} S_n = S \tag{III.III}$$

If $\lim_{n\to\infty} S_n$ does not exist, we say the series diverges.

Corollary III..I — Real and Imaginary Parts of a Series. If $\sum_{n=1}^{\infty} z_n$ converges, then the real and imaginary parts of the series also converge.

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \Re(z_n) + i \sum_{n=1}^{\infty} \Im(z_n) \tag{III.IV}$$

III.I Tests for Convergence

Theorem III.I.I If $\sum_{n=1}^{\infty} z_n$ converges, then so does $\sum_{n=1}^{\infty} z_n$ and:

$$\left| \sum_{n=1}^{\infty} z_n \right| \leq \sum_{n=1}^{\infty} |z_n|$$

Proof. Say $z_n = x_n + iy_n$. Then:

$$\left|\sum_{n=1}^{\infty} x_n\right| \leq \sum_{n=1}^{\infty} |x_n| \leq \sum_{n=1}^{\infty} |z_n| \tag{III.V}$$

And (III.VI)

$$\left|\sum_{n=1}^{\infty} y_n\right| \leq \sum_{n=1}^{\infty} |y_n| \leq \sum_{n=1}^{\infty} |z_n| \tag{III.VII}$$

So if $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge, then $\sum_{n=1}^{\infty} z_n$ converges. ■

■ **Example III.I**

$$\sum_{j=1}^{\infty} j \left(\frac{1 + 2i}{3}\right)^j \tag{III.VIII}$$

We can use the ratio test to determine convergence:

$$\sum_{j=1}^{\infty} |z_j| = \sum_{j=1}^{\infty} j \left|\frac{1 + 2i}{3}\right|^j \tag{III.IX}$$

$$= \sum_{j=1}^{\infty} j \left(\frac{\sqrt{5}}{3}\right)^j \tag{III.X}$$

$$\lim_{j\to\infty} \left|\frac{z_{j+1}}{z_j}\right| = \lim_{j\to\infty} \frac{(j + 1)(\frac{\sqrt{5}}{3})^{j+1}}{j(\frac{\sqrt{5}}{3})^j} \tag{III.XI}$$

$$= \lim_{j\to\infty} \frac{j + 1}{j} \left(\frac{\sqrt{5}}{3}\right) \tag{III.XII}$$

$$= \frac{5}{3} < 1 \qquad \therefore \text{ The series converges} \tag{III.XIII}$$

■

III.II The Exponential Function

Approach 1

Definition III.II.I — Exponential Function. If $z = x + iy$, then the exponential function is defined as:

$$e^z = e^x \left(\cos(y) + i \sin(y)\right) \tag{III.XIV}$$

R [Euler's Formula]

$$e^{i\theta} \triangleq \cos(\theta) + i \sin(\theta) \tag{III.XV}$$

(III.XVI)

Test Name	Description	Conditions for Use	Results
Ratio Test	$\lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\|$	Applicable when terms are positive and the limit exists.	Converges if $L < 1$, diverges if $L > 1$, inconclusive if $L = 1$.
Root Test	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|}$	Applicable when terms are positive and the limit exists.	Converges if $L < 1$, diverges if $L > 1$, inconclusive if $L = 1$.
Integral Test	Compares a series to an improper. $\int_1^\infty f(x)\ dx$	Applicable when terms are positive, continuous, and decreasing.	Converges if the integral converges, diverges if the integral diverges.
Comparison Test	Compares a series to a known convergent or divergent series.	Applicable when terms are positive.	Converges if the series being compared to converges.
Limit Comparison Test	Compares the limit of the ratio of terms to a known series. $\lim_{n\to\infty} \frac{a_n}{b_n}$	Applicable when terms are positive and the limit exists.	Converges if the limit is finite and the comparison series converges, diverges otherwise.
Alternating Series Test	$\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$	When dealing with alternating series	Converges if: $a_n > 0$, decreasing, and $\lim_{n\to\infty} a_n = 0$
p-Series Test	Determines convergence based on the exponent in a series of the form $\sum \frac{1}{n^p}$	Applicable for series of the form $\frac{1}{n^p}$.	Converges if $p > 1$, diverges if $p \leq 1$.
Geometric Series Test	Determines convergence for geometric series. $\sum ar^n$	Applicable for series of the form ar^n.	Converges if $\|r\| < 1$, diverges if $\|r\| \geq 1$.
D'Alembert's Ratio Test	Similar to the Ratio Test, but specifically for series with factorial terms.	Applicable when terms involve factorials.	Converges if the ratio is less than 1, diverges if greater than 1.
Cauchy's Condensation Test	Determines convergence by condensing the series. $\sum a_n \sim \sum 2^n a_{2^n}$	Applicable for series with positive, decreasing terms.	Converges if the condensed series converges, diverges if the condensed series diverges.

Property	Description
Periodicity	The complex exponential function is periodic with period $2\pi i$, $$e^{z+2\pi i} = e^z$$.
Multiplication	The exponential function satisfies $$e^{z_1+z_2} = e^{z_1}e^{z_2}$$ for any complex numbers z_1 and z_2.
Derivative	The derivative of the exponential function is $$\frac{d}{dz}e^z = e^z$$.
Inverse	The inverse of the exponential function is the complex logarithm, $$\log z$$, such that $$e^{\log z} = z$$ for $z \neq 0$.
Magnitude	The magnitude of the exponential function is $$\lvert e^z \rvert = e^{\Re(z)}$$, where $\Re(z)$ denotes the real part of z.
Argument	The argument of the exponential function is $$\arg(e^z) = \Im(z) \mod 2\pi$$, where $\Im(z)$ denotes the imaginary part of z.
Conjugate	The conjugate of the exponential function is $$\overline{e^z} = e^{\bar{z}}$$.

Table III.II: Properties of the Complex Exponential Function

Properties of the complex Exponential Function

Approach 2: Taylor Series

Definition III.II.II — The Exponential Function. The exponential function can be defined as:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \text{for all } z \in \mathbb{C} \tag{III.XVII}$$

Claim III.II.I — The Taylor Series for the Exponential Function Converges. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$.

Proof. HOMEWORK ■

Problem I For $\theta \in \mathbb{R}$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \cos(\theta) + i\sin(\theta)$$

III.III Approach 3: Differential Equations

Definition III.III.I — Differential Equation for the Exponential Function. The exponential function satisfies the differential equation:

$$f(z) = \begin{cases} \frac{df}{dz} = f & \text{for all } z \in \mathbb{C} \\ f(0) = 1 \end{cases} \tag{III.XVIII}$$

III.IV The Logarithm Function

Definition III.IV.I — Logarithm Function. The logarithm function is defined as the inverse of the exponential function:

$$\log z = \log |z| + i\theta \tag{III.XIX}$$

R There will be many solutions to the logarithm function, as the argument is only defined modulo 2π.

$$\log z = \log |z| + i\,(\arg(z) + 2\pi n) \qquad \text{for } n \in \mathbb{Z}$$

Definition III.IV.II — Principal Logarithm. The principal branch logarithm is defined as:

$$\mathrm{Log}(z) = \log |z| + i\arg(z) \qquad \text{for } -\pi < \arg(z) \leq \pi$$

Note: We use a capital L to denote the principal logarithm.

Definition III.IV.III — Fixed θ_0 Logarithm Function. We can fix the argument of the logarithm function by setting θ_0 and letting $D = \{te^{i\theta_0} | t > 0, t \in \mathbb{R}\}$.

We define:

$$\widetilde{\log}_{\theta_0} z = \log|z| + i\,(\widetilde{\arg}(z) + \theta_0) \qquad \text{for } z \in D, \arg(z) \in [\theta_0, \theta_0 + 2\pi)$$

■ **Example III.II — Find the Values of $(-1)^i$.**

$$(-1)^i = e^{i\,\log(-1)} \tag{III.XX}$$
$$= e^{i(2n+1)\pi i} \tag{III.XXI}$$
$$= e^{-2n\pi} \tag{III.XXII}$$
$$\log(-1) = -(2n+1)\pi i \qquad n \in \mathbb{Z} \tag{III.XXIII}$$
$$(-1)^i = e^{2n+1}\pi \tag{III.XXIV}$$

■

III.V The Trigonometric Functions

Definition III.V.I — Trigonometric Functions. For $z \in mathbbC$ trigonometric functions are defined as:

$$\Re e^{iz} = \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{III.XXV}$$
$$\Im e^{iz} = \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{III.XXVI}$$
$$\tan(z) = \frac{\sin(z)}{\cos(z)} \tag{III.XXVII}$$
$$\tag{III.XXVIII}$$

Lemma III.V.I

$$\begin{cases} \cos(z+\alpha) = \cos(z) \\ \sin(z+\alpha) = \sin(z) \end{cases} \tag{III.XXIX}$$

iff $\alpha = 2\pi n$ for $n \in \mathbb{Z}$.

Proof.

$$e^{i(z+\alpha)} = e^{iz}e^{i\alpha} \tag{III.XXX}$$
$$= e^{iz}\,(\cos(\alpha) + i\,\sin(\alpha)) \tag{III.XXXI}$$
$$= e^{iz}\,(\cos(2\pi n) + i\,\sin(2\pi n)) \tag{III.XXXII}$$
$$= e^{iz} \tag{III.XXXIII}$$

■

■ **Example IV.I — Fisher, Section 1.2, Problem 2.** Describe the locus of points z satisfying the equation

$$|z - 4| = 4|z|$$

Solution:
Let $z = x + iy$. Then

$$\begin{aligned}
|z - 4| &= |x + iy - 4| = |x - 4 + iy| = \sqrt{(x-4)^2 + y^2} \\
4|z| &= 4|x + iy| = 4\sqrt{x^2 + y^2} \\
\sqrt{(x-4)^2 + y^2} &= 4\sqrt{x^2 + y^2} \\
(x-4)^2 + y^2 &= 16(x^2 + y^2) \\
x^2 - 8x + 16 + y^2 &= 16x^2 + 16y^2 \\
15x^2 + 15y^2 + 8x - 16 &= 0 \\
x^2 + y^2 + \frac{8}{15}x - \frac{16}{15} &= 0
\end{aligned}$$

$\Rightarrow$ complete the square

$$x^2 + \frac{8}{15}x + (\frac{4}{15})^2 - (\frac{4}{15})^2 + y^2 = \frac{16}{15}$$

$$\begin{aligned}
(x + \frac{4}{15})^2 + y^2 &= \frac{16}{15} + (\frac{4}{15})^2 \\
(x + \frac{4}{15})^2 + y^2 &= \frac{16}{15} + \frac{16}{225} \\
(x + \frac{4}{15})^2 + y^2 &= \frac{256}{225}
\end{aligned}$$

$\therefore$ The locus of points z satisfying the equation $|z - 4| = 4|z|$ is a circle with center $(-\frac{4}{15}, 0)$ and radius $\sqrt{\frac{256}{225}} = \frac{16}{15}$. ■

■ **Example IV.II — Fisher, Section 1.2, Problem 24.** Find all solutions of the equation

$$(z + 1)^4 = 1 - i$$

Solution:

$$\begin{aligned}
&\text{Convert } 1 - i \text{ to polar form}\\
\arg 1 - i &= \tan^{-1}(-1) = -\frac{\pi}{4} + 2k\pi \quad \text{where } k \in \mathbb{Z}\\
|1 - i| &= \sqrt{1^2 + (-1)^2} = \sqrt{2}\\
1 - i &= \sqrt{2}(cos(-\frac{\pi}{4} + 2k\pi) + isin(-\frac{\pi}{4} + 2k\pi))\\
&\text{Use De Moivre's Theorem}\\
&\rightarrow z^n = r^n(cos(n\theta) + isin(n\theta))\\
z + 1 &= 2^{\frac{1}{8}}(\cos(\frac{-\pi}{4} + \frac{2\pi k}{4}) + i\sin(\frac{-\pi}{4} + \frac{2\pi k}{4}))\\
z &= 2^{\frac{1}{8}}(\cos(\frac{-\pi}{4} + \frac{2\pi k}{4}) + i\sin(\frac{-\pi}{4} + \frac{2\pi k}{4})) - 1
\end{aligned}$$

We can now find the solutions by plugging in $k = 0, 1, 2, 3$.

$$\begin{aligned}
\theta_0 &= \frac{-\pi}{4} + \frac{2\pi \cdot 0}{4} = -\frac{\pi}{4} \quad k = 0\\
\theta_1 &= \frac{-\pi}{4} + \frac{2\pi \cdot 1}{4} = \frac{\pi}{4} \quad k = 1\\
\theta_2 &= \frac{-\pi}{4} + \frac{2\pi \cdot 2}{4} = \frac{3\pi}{4} \quad k = 2\\
\theta_3 &= \frac{-\pi}{4} + \frac{2\pi \cdot 3}{4} = \frac{5\pi}{4} \quad k = 3
\end{aligned}$$

So our solutions are:

$$\begin{aligned}
z &= 2^{\frac{1}{8}}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})) - 1 = 2^{\frac{1}{8}}(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) - 1\\
z &= 2^{\frac{1}{8}}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})) - 1 = 2^{\frac{1}{8}}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) - 1\\
z &= 2^{\frac{1}{8}}(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})) - 1 = 2^{\frac{1}{8}}(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) - 1\\
z &= 2^{\frac{1}{8}}(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4})) - 1 = 2^{\frac{1}{8}}(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) - 1
\end{aligned}$$

■

■ **Example IV.III — Fisher, Section 1.2, Problem 26.** Find all solutions of the equation $z^3 = 8$.
Solution:
First, we convert 8 to polar form.

$$\begin{aligned} 8 &= 8(\cos(0) + i\sin(0)) \\ &= 8(\cos(2\pi k) + i\sin(2\pi k)) \quad \text{where } k \in \mathbb{Z} \end{aligned}$$

Then we use De Moivre's Theorem to find the solutions.

$$\begin{aligned} \rightarrow z^n &= r^n(\cos(n\theta) + i\sin(n\theta)) \\ z &= 2(\cos(\frac{2\pi k}{3}) + i\sin(\frac{2\pi k}{3})) \quad \text{where } k = 0, 1, 2 \end{aligned}$$

So our solutions are:

$$\begin{aligned} z &= 2(\cos(0) + i\sin(0)) = 2(1 + i0) = 2 \\ z &= 2(\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})) = 2(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -1 + i\sqrt{3} \\ z &= 2(\cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3})) = 2(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = -1 - i\sqrt{3} \end{aligned}$$

■

■ **Example IV.IV — Fisher, Section 1.3, Problem 2.** For the following set, describe (i) the interior and the boundary, (ii) state whether the set is open, or closed, or neither open nor closed, (iii) state whether the interior of the set is connected (if it has an interior).

$$A = \{z \in \mathbb{C} : |z| < 1\text{or}|z-3| \leq 1\}$$

Solution:

1. $A_{int} = \{|z| < 1 \text{ or } |z-3| < 1\}$
2. $A_{bd} = \{|z| = 1 \text{ or } |z-3| = 1\}$
3. A is neither open nor closed because $\{|z| = 1\} \notin A$ but $\{|z-3| = 1\} \in A_{int}$, so A contains only part of its boundary.
4. A_{int} is not connected, because $z_1 = 0$, $z_2 = 3 \in A_{int}$, but $\nexists P_1P_2 \ldots P_n \in A_{int}$ such that $z_1P_1P_2 \ldots P_nz_2 \in A_{int}$

■

■ **Example IV.V — Fisher, Section 1.3, Problem 4.** For the following set, describe (i) the interior and the boundary, (ii) state whether the set is open, or closed, or neither open nor closed, (iii) state whether the interior of the set is connected (if it has an interior).

$$A = \{z \in \mathbb{C} : \text{Re}(z^2) = 4\}$$

Solution:
Let $z = x + iy$. Then

$$\begin{aligned} \text{Re}(z^2) &= \text{Re}((x+iy)^2) = \text{Re}(x^2 - y^2 + 2ixy) \\ 4 &= x^2 - y^2 \end{aligned}$$

1. No interior, because $\forall z_0 \in A \quad \nexists D | D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R, R > 0\}$
2. $A_{bd} = \{z \in \mathbb{C} : x^2 - y^2 = 4\}$
3. $A_{bd} = A$ so A is closed.
4. A_{int} is connected because $A_{int} = \emptyset$.

■

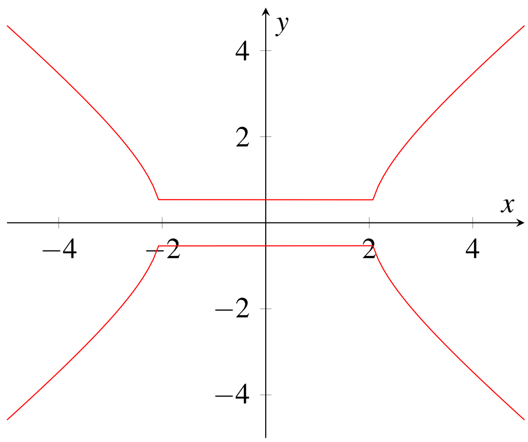

Figure IV.1: Plot of $4 = x^2 - y^2$

■ **Example IV.VI — Fisher, Section 1.4, Problem 12.** Find

$$\lim_{z\to 2}(z - 2)\log|z - 2|,$$

or explain why it does not exist. **Solution:**
We use L'Hopital's Rule to find the limit.

$$\begin{aligned}
\lim_{z\to 2}(z - 2)\log|z - 2| &= \lim_{z\to 2}\frac{\log|z - 2|}{\frac{1}{z-2}} \\
&= \lim_{z\to 2}\frac{\frac{\partial}{\partial z}\log|z - 2|}{\frac{\partial}{\partial z}\frac{1}{(z-2)}} \\
&= \lim_{z\to 2}\frac{\frac{1}{z-2}}{-\frac{1}{(z-2)^2}} \\
&= \lim_{z\to 2}\frac{1}{\frac{1}{2-z}} \\
&= \lim_{z\to 2} 2 - z \\
&= 0
\end{aligned}$$

■

■ **Example IV.VII — Fisher, Section 1.4, Problem 16.** Find all the points where the following function is continuous:

$$f(z) = \begin{cases} \frac{z^4-1}{z-i}, & z \neq i \\ 4i, & z = i \end{cases}$$

Solution:
First normalize the denominator.

$$\begin{aligned} f(z) &= \frac{z^4 - 1}{z - i} = \frac{(z^2 + 1)(z + 1)(z - 1)}{z - i} = \frac{(z^2 + 1)(z + 1)(z - 1)}{z - i}\frac{z + i}{z + i} \\ &= \frac{(z^2 + 1)(z + 1)(z - 1)(z + i)}{z^2 + 1} = (z + 1)(z - 1)(z + i), \quad z \neq i \end{aligned}$$

As this is a polynomial, it is continuous everywhere except at $z = i$. Now we test for continuity at $z = i$.

$$\begin{aligned} \lim_{z \to i} f(z) &= f(i) \\ \lim_{z \to i} (z + 1)(z - 1)(z + i) &= 4i \\ (i + 1)(i - 1)(i + i) &= 4i \\ 4i &= 4i \end{aligned}$$

So $f(z)$ is continuous everywhere. ∎

∎ **Example IV.VIII — Fisher, Section 1.4, Problem 34.** Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{2 + i^n}$$

Solution:
We notice:

$$\begin{aligned} \frac{1}{2 + i^n} &= \frac{1}{2 + 1} \quad \text{for } n = 0, 4, 8, \ldots \\ \frac{1}{2 + i^n} &= \frac{1}{2 + i} \quad \text{for } n = 1, 5, 9, \ldots \\ \frac{1}{2 + i^n} &= \frac{1}{2 - 1} \quad \text{for } n = 2, 6, 10, \ldots \\ \frac{1}{2 + i^n} &= \frac{1}{2 - i} \quad \text{for } n = 3, 7, 11, \ldots \end{aligned}$$

Which forms a cycle, so the series diverges. ∎

∎ **Example IV.IX — Fisher, Section 1.4, Problem 36.** Show that each of the following series converges for all z.

1. $$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

2. $$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

3. $$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n + 1)!}$$

Solution:

1. We use the ratio test to show convergence.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{z^{n+1}}{(n+1)!}\frac{n!}{z^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{z}{n+1}\right| \\ &= 0\end{aligned}$$

   So the series converges for all z.
2. We use the ratio test to show convergence.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{z^{2(n+1)}}{(2(n+1))!}}{(-1)^n\frac{z^{2n}}{(2n)!}}\right| \\ &= \lim_{n\to\infty}\left|\frac{z^2}{(2n+2)(2n+1)}\right| \\ &= 0\end{aligned}$$

   So the series converges for all z.
3. We use the ratio test to show convergence.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{z^{2(n+1)+1}}{(2(n+1)+1)!}\frac{(2n+1)!}{z^{2n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{z^2}{(2n+3)(2n+2)(2n+1)}\right| \\ &= 0\end{aligned}$$

   So the series converges for all z.

■

Theorem V..I — Parametrized Curves.

$$\gamma(t) = x(t) + iy(t) \quad a \le t \le b$$

$\gamma[a, b] \to \mathbb{C}$ is the image of γ.

Definition V..I — Simple Curve. A curve γ is **simple** if $\gamma(t_1) = \gamma(t_2) \implies t_1 = t_2$ for $t_1, t_2 \neq a, b$.

Definition V..II — Closed Curve. A curve γ is **closed** if $\gamma(a) = \gamma(b)$. So if the end point meets the starting point.

R We can *ignore* the parametrization and talk about the curve

$$Image(\gamma) \subset \mathbb{C}$$

as a subset of $\mathbb{C}$.

Definition V..III — C^1/Smooth Curve. A parametrized curve is C^1 if $\gamma'(t)$ if

$$\gamma'(t) = x'(t) + iy'(t)$$

exists $\forall t \in [a, b]$ and is continuous.

R Here, $\gamma'(a), \gamma'(b)$ are the 1-sided derivatives.

Definition V..IV — Piecewise C^1/Smooth Curve.

$$\text{if } \exists a = t_0 < t_1 < \ldots < t_n = b \text{ such that } \gamma|_{[t_i,t_{i+1}]} \text{ is } C^1$$

V.I Line Integrals

Definition V.I.I — Line Integral. if $g = u + iv$, $(u, v) \in \mathbb{R}^2$ is a complex-valued function and γ is piecewise C^1, then the line integral of g along γ is

$$\int_\gamma g(z)dz = \sum_{i=0}^{n-1} \int_i^{i+1} g(\gamma(t))\gamma'(t)dt$$

Where

$$\begin{aligned} g(\gamma(t))\gamma'(t) &= ux' - vy' + ivx' + iuy' \\ &= (u(\gamma(t)) + iv(\gamma(t)))(x'(t) + iy'(t)) \end{aligned}$$

is complex multiplication

Theorem V.I.I — Length of a Curve. If γ is a piecewise C^1 curve, then the length of γ is

$$\text{Length}(\gamma) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\gamma'(t)|dt$$

So we have

$$|\int_\gamma g| \leq \max_{z\in\gamma} |g(z)| \cdot \text{Length}(\gamma)$$

Theorem V.I.II — Green's Theorem. Say $\Omega \subset \mathbb{C}$ such that $\partial\Omega$ is a finite collection of piecewise C^1 closed simple curves. If $g = u + iv$ is C^1 on Ω, then
if $= p + iq$ is differentiable in Ω, then ($\Re p$, q have 1^{st} order derivatives). Then

$$\int_{\partial\Omega} f = i \int_\Omega \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

Where $\partial\Omega$ is the boundary of Ω.

Corollary V.I.III If $dz = dx + idy$, then

$$\begin{aligned} \Re(fdz) &= \Re(f)dx - \Im(f)dy \\ &= pdx - qdy \end{aligned}$$

$$\Re(i\left(\frac{\partial f}{\partial x} + \frac{i\partial f}{\partial y}\right)) = -\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}$$

So

$$\int_{\partial\Omega} pdx + qdy = \left(\int_{\Omega} \left(\frac{\partial q}{\partial x} + \frac{\partial p}{\partial y}\right) dxdy\right)$$

R Orient $\partial\Omega$ always on the left (in the counter-clockwise direction outsides, conterclockwise insides) as we walk along $\partial\Omega$ (say $\partial\Omega$ is positively oriented).

Example V.I — Very Important Example. Let γ be a simple, closed piecewise C^1 curve. such that $\gamma = \partial\Omega$ for some $\Omega \subset \mathbb{C}$. Then for $p \notin \Omega$,

$$\frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z - p} = \begin{cases} 1 & \text{if } p \in \Omega \\ 0 & \text{if } p \notin \Omega \end{cases}$$

Proof. 1) Assume p not in Ω, then $\frac{1}{z-p}$ is differentiable in Ω and $\partial\Omega$ is a simple closed curve. So by Green's Theorem,

$$\int_{\partial\Omega} \frac{dz}{z - p} = i\int_{\Omega} \left(\frac{\partial}{\partial x}\frac{1}{z - p} - i\frac{\partial}{\partial y}\frac{1}{z - p}\right) dxdy = 0$$

∎

Let $D_\varepsilon(p)$ be the disk of radius ε centered at p, essentially, we want to remove the point stopping us from applying Green's Theorem.

$$\Omega_\varepsilon = \Omega \setminus D_\varepsilon(p)$$

If ε is sufficiently small, Ω_ε is still a domain. So by Green's Theorem,

$$\begin{aligned}
\int_{\partial\Omega_\varepsilon} \frac{dz}{z - p} &= 0 \\
\int_{\partial\Omega} \frac{dz}{z - p} - \int_{\partial D_\varepsilon(p)} \frac{dz}{z - p} &= 0 \\
\int_{\partial\Omega} \frac{dz}{z - p} &= \int_{\partial D_\varepsilon(p)} \frac{dz}{z - p} \\
\rightarrow \partial D_\varepsilon &= p + \varepsilon e^{it} \quad 0 \le t \le 2\pi \\
\int_{\partial D_\varepsilon(p)} \frac{dz}{z - p} &= \int_0^{2\pi} \frac{i\varepsilon e^{it}}{\varepsilon e^{it}} dt = 2\pi i \\
\int_{\partial\Omega} \frac{dz}{z - p} &= 2\pi i
\end{aligned}$$

■

VI.I Analytic Functions

Definition VI.I.I — Complex Differentiability. A complex function $f(z) : D \to \mathbb{C}$, where D is a domain, is **complex differentiable** at $z_0 \in D$ if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists}$$

$$= \lim_{h} \frac{f(z_0 + h) - f(z_0)}{h} \quad h \in \mathbb{C}$$

Definition VI.I.II — Analytic. A function $f(z)$ is **analytic** on a domain D if $f(z)$ is complex differentiable at every point in D.

Definition VI.I.III — Entire. A function $f(z)$ is **entire** if $f(z)$ is analytic on $\mathbb{C}$.

■ **Example VI.I — Prove the Power Rule.**

$$f(z) = z^n \quad n \in \mathbb{Z}$$

f is entire and

$$f'(z) = nz^{n-1}$$

■

Proof.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{(z+h)^n - z^n}{h} \\
&= \lim_{h \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^k - z^n}{h} \\
&= \lim_{h \to 0} \sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^{k-1}
\end{aligned}$$

$$= \binom{n}{1} z^{n-1}$$
$$= nz^{n-1}$$

■

■ **Example VI.II** Prove that $f(z) = \bar{z}$ is not complex differentiable at any point. ■

Proof. In homework 2... ■

■ **Example VI.III — Prove the Derivative of the Exponential Function.**

$$f(z) = e^z$$

■

Proof.

$$\begin{aligned}
\lim_{h \to 0} \frac{e^{z+h} - e^z}{h} &= \lim_{h \to 0} \frac{e^z e^h - e^z}{h} \\
&= e^z \lim_{h \to 0} \frac{e^h - 1}{h} \\
&= e^z \lim_{h \to 0} \frac{1 + h + \frac{h^2}{2} + \cdots - 1}{h} \\
&= e^z \lim_{h \to 0} 1 + \frac{h}{2} + \cdots
\end{aligned}$$

■

VI.II Cauchy-Riemann Equations

Lemma VI.II.I — h can approach from any direction. If $f(z)$ is differentiable then

$$\exists \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = f(z) \in \mathbb{C}$$

And yield the same result for any $h \in \mathbb{C}$.

Theorem VI.II.II — Cauchy-Riemann Equations. If $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z = x + iy$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof. We compute h in two ways:

$$h_1 = is \quad s \in \mathbb{R}$$
$$h_2 = s \in \mathbb{R}$$

Property	**Description**
Linearity	The derivative of a sum is the sum of the derivatives: $(f + g)'(z) = f'(z) + g'(z)$ The derivative of a constant multiple is the constant multiple of the derivative: $(cf)'(z) = cf'(z)$
Product Rule	The derivative of a product is given by: $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$
Quotient Rule	The derivative of a quotient is given by: $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$
Chain Rule	The derivative of a composition is given by: $(f \circ g)'(z) = f'(g(z))g'(z)$
Exponential Function	The derivative of the exponential function is: $\frac{d}{dz}e^z = e^z$
Logarithmic Function	The derivative of the logarithmic function is: $\frac{d}{dz} \log z = \frac{1}{z}$
Power Rule	The derivative of a power function is: $\frac{d}{dz}z^n = nz^{n-1}$
Trigonometric Functions	The derivatives of the trigonometric functions are: $\frac{d}{dz} \sin z = \cos z$ $\frac{d}{dz} \cos z = - \sin z$
Hyperbolic Functions	The derivatives of the hyperbolic functions are: $\frac{d}{dz} \sinh z = \cosh z$ d

$$
\begin{aligned}
&\lim_{h\to 0} \frac{f(z+is)-f(z)}{is} \\
=&lim_{h\to 0}\frac{u(x,y+s)+iv(x,y+s)-u(x,y)-iv(x,y)}{is} \\
=&lim_{h\to 0}\frac{u(x,y+s)-u(x,y)}{is}+\frac{v(x,y+s)-v(x,y)}{s} \\
=&\frac{1}{i}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y})
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{h\to 0} \frac{f(z+s)-f(z)}{s} \\
=&lim_{h\to 0}\frac{u(x+s,y)+iv(x+s,y)-u(x,y)-iv(x,y)}{s} \\
=&lim_{h\to 0}\frac{u(x+s,y)-u(x,y)}{s}+i\frac{v(x+s,y)-v(x,y)}{s} \\
=&\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} &= \frac{1}{i}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}) \\
\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}
$$

■

Theorem VI.II.III — Harmonic Functions. If $f(z) = u(x,y)+iv(x,y)$ is complex differentiable, then

$$\Delta u = \Delta v = 0$$

And u, v are **harmonic functions** and satisfy Cauchy-Riemann equations. Thus they are **harmonic conjugates**. Where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator.

Corollary VI.II.IV If a function $f(z)$ is once complex differentiable, then it is infinitely differentiable and analytic.

Proof. Cauchy-Riemann equations give us the partial derivatives of u, v.

$$
\begin{cases}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\end{cases}
$$

$$\begin{aligned}
\text{Take } \frac{\partial}{\partial x}(1) \quad \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} \\
\text{Take } \frac{\partial}{\partial y}(2) \quad \frac{\partial^2 u}{\partial y^2} &= -\frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial v}{\partial y} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial x^2} \\
\Delta u &= 0
\end{aligned}$$

■

Theorem VI.II.V Let $f = u + iv$ and assume u, v, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ are defined and continuous on a disc around z_0. If u, v satisfy the Cauchy-Riemann equations at z_0, then f is complex differentiable at z_0.

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Proof. Using the taylor expansion of $f(z)$

$$\begin{aligned}
\lim_{h\to 0}\frac{f(z_0+h) - f(z_0)}{h} &= \lim_{h\to 0}\frac{u(x_0+h, y_0) + iv(x_0+h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\
&= \lim_{h\to 0}\frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} + i\frac{v(x_0+h, y_0) - v(x_0, y_0)}{h} \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}
\end{aligned}$$

■

■ **Example VI.IV — Prove the Derivative of the Logarithmic Function.** Let $D \in \mathbb{C}$ be a domain o which there is a single-valued branch of $\log z$. ■

Proof. When $\arctan(y/x) \in (\theta_0, \theta + \pi]$ and $\arctan(y/x)$ is not in D.

$$u = \frac{1}{2}\log(x^2 + y^2) \quad v = \arctan(y/x)$$

Then

$$\frac{\partial u}{\partial x} = \frac{1}{2(x^2+y^2)} \cdot 2x = \frac{x}{x^2+y^2} \tag{VI.I}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\frac{y}{x}^2} \times \frac{1}{x} \tag{VI.II}$$

$$= \frac{x}{x^2+y^2} \tag{VI.III}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{VI.IV}$$

INCOMPLETE ■