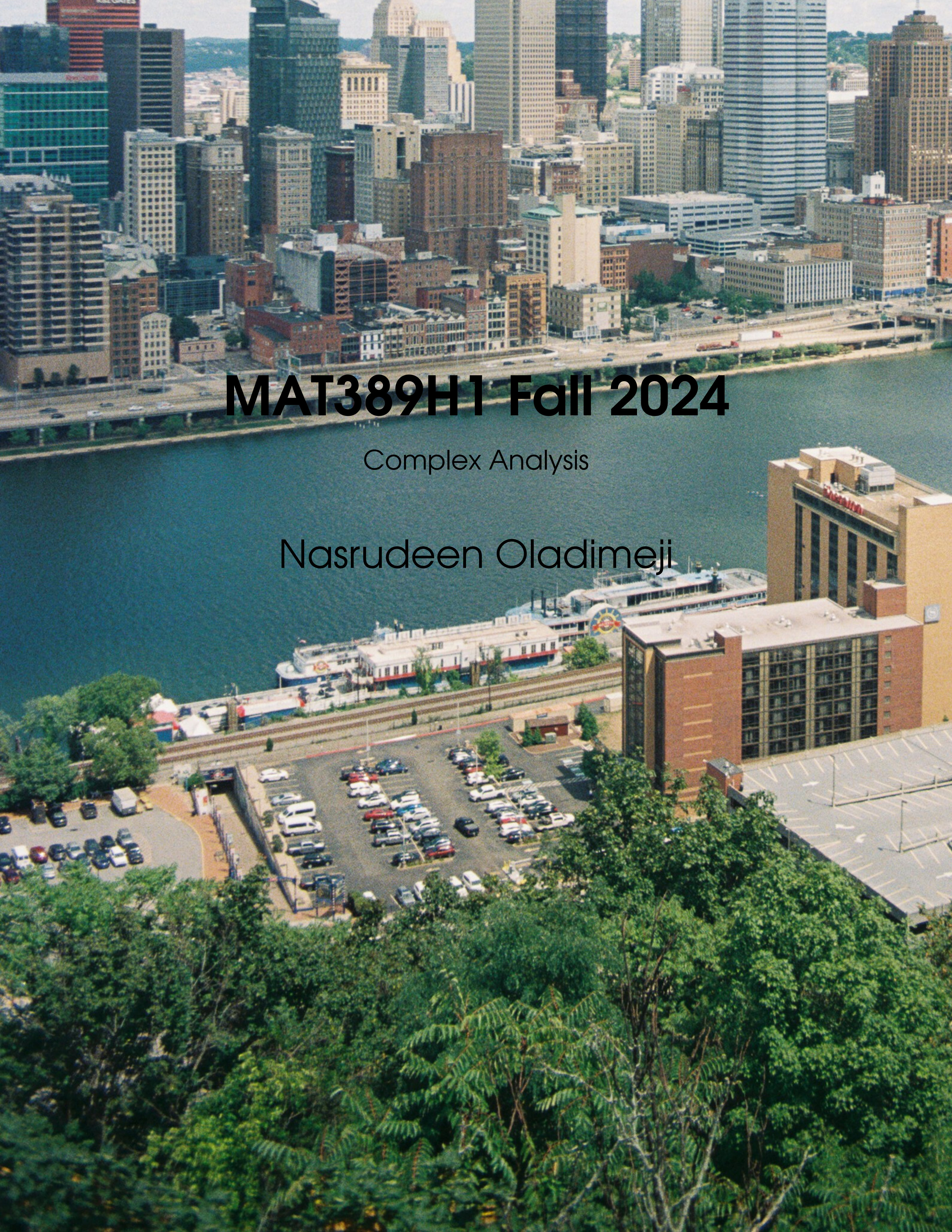

MAT389H1 Fall 2024

Complex Analysis

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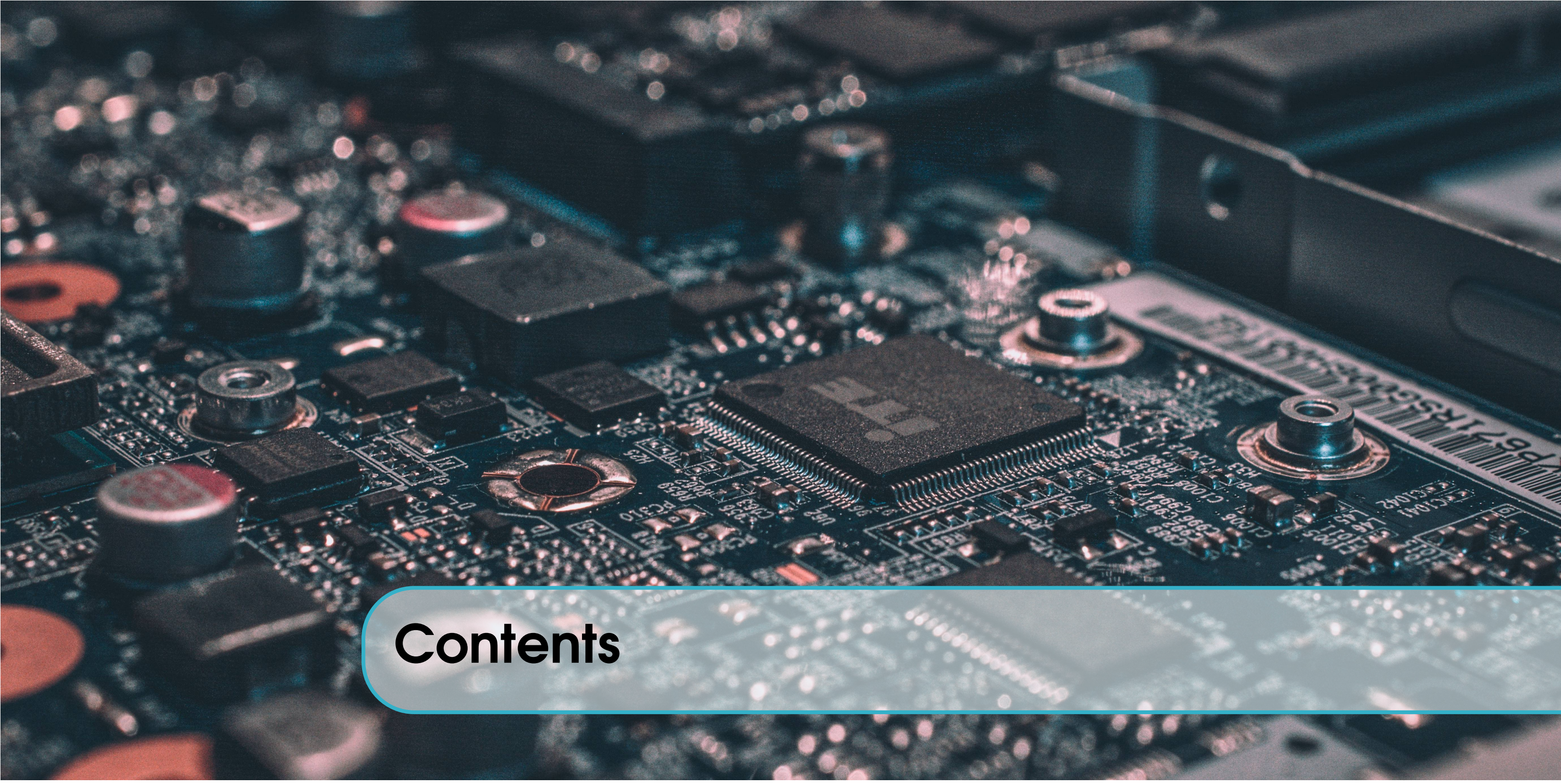

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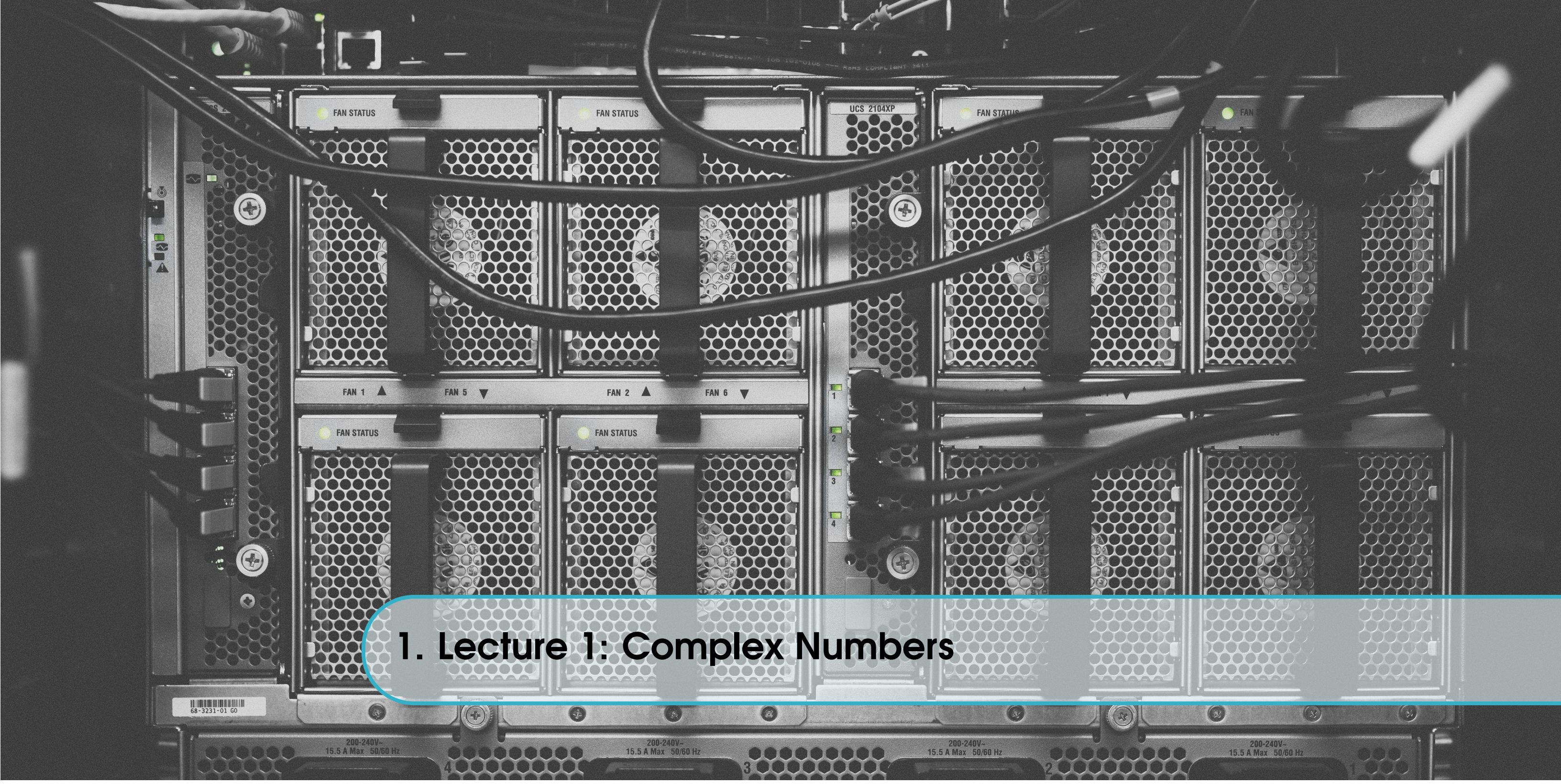


1. Lecture 1: Complex Numbers

1.1 Introduction

Definition 1.1.1 — Complex Numbers. z is a complex number iff $z = a + bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.
The set of complex numbers is denoted by $\mathbb{C}$.

Definition 1.1.2 — Real and Imaginary Parts. If $z = a + bi$, then $\Re(z) = a \in \mathbb{R}$ and $\Im(z) = b \in \mathbb{R}$. where $\Re(z)$ is the real part of z and $\Im(z)$ is the imaginary part of z.

Definition 1.1.3 — Modulus. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$. $|z|$ is the modulus of z.

Definition 1.1.4 — Conjugate. If $z = a + bi$, then $\bar{z} = a - bi$. $\bar{z}$ is the conjugate of z.

1.2 Operations

Definition 1.2.1 — Addition and Subtraction. If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$.
Similarly $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$.

Definition 1.2.2 — Multiplication. If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then $z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$.
Note that $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

Definition 1.2.3 — Inversion. If $z = a + bi$, then $z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$.

Proof. Let's multiply by 1 in the form of the conjugate of z:

$$\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

■

Definition 1.2.4 — Division. For $z, w \in \mathbb{C}$, $\frac{w}{z} = w \cdot z^{-1} = \frac{w\bar{z}}{|z|^2}$.

Table 1.1: Properties of the Complex Conjugate

Property	**Description**
Conjugate of the Sum	$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
Conjugate Modulus	$z \cdot \bar{z} = \|z\|^2$
Conjugate of a Conjugate	$\bar{\bar{z}} = z$
Product of Conjugates	$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
Conjugate of a Quotient	$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
Real Part Conjugate	$Re(z) = \frac{z+\bar{z}}{2}$
Imaginary Part Conjugate	$Im(z) = \frac{z-\bar{z}}{2i}$
Real Number Check	$z = \bar{z} \iff z \in \mathbb{R}$
Imaginary Number Check	$z = -\bar{z} \iff z \in \mathbb{I}$
Function Linearity	If $\alpha = f(z)$ then $\overline{\alpha} = \overline{f(z)} = f(\bar{z})$

Table 1.2: Properties of the Modulus in Complex Numbers

Property	**Description**
Positivity	$\|z\| \geq 0$, with equality if and only if $z = 0$
Triangle Inequality	$\|\|z_1\| - \|z_2\|\| \leq \|z_1 \pm z_2\| \leq \|z_1\| + \|z_2\|$
Multiplicative Property	$\|z_1 \cdot z_2\| = \|z_1\| \cdot \|z_2\|$
Division Property	$\left\|\frac{z_1}{z_2}\right\| = \frac{\|z_1\|}{\|z_2\|}$, for $z_2 \neq 0$
Conjugate	$\|z\| = \|\bar{z}\|$
Component Property	$-\|z\| \leq Re(z) \leq \|z\|$ $-\|z\| \leq Im(z) \leq \|z\|$
Cauchy-Schwarz Inequality	$\|z_1 w_1 + \cdots + z_n w_n\|^2 \leq \sum_{j=1}^{n} \|z_j\|^2 \sum_{j=1}^{n} \|w_j\|^2$

Proof. Proof of the Multiplicative Property of the Modulus:

$$
\begin{aligned}
|z_1 \cdot z_2|^2 &= (z_1 \cdot z_2) \cdot (\overline{z_1} \cdot \overline{z_2}) \\
&= z_1 \cdot \overline{z_1} \cdot z_2 \cdot \overline{z_2} \\
&= |z_1|^2 \cdot |z_2|^2
\end{aligned}
$$

■

1.3 Polar Representation

A complex number are vectors in $\mathbb{R}^2$, as such, they can be represented by a magnitude and a direction.

Definition 1.3.1 — Polar Form.

$$z = r(\cos(\theta) + i \sin(\theta)) \tag{1.1}$$

: $r = |z| \in \mathbb{R}^+$

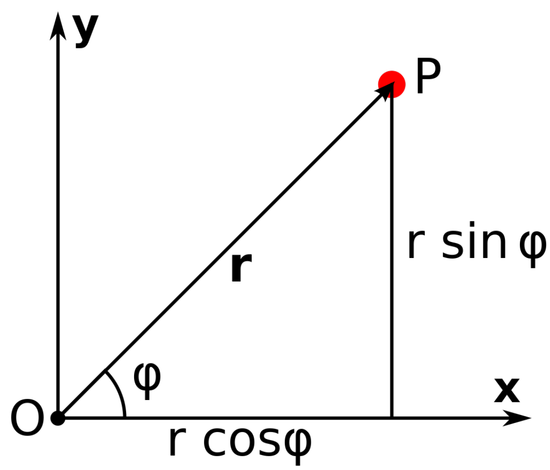


Figure 1.1: Polar Coordinate Components

■ **Example 1.1 — Multiplying Complex Numbers in Polar Form.** Let $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$. Then:

$$z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2))) \tag{1.2}$$

$$= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \tag{1.3}$$

Using the trig addition formula:

$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ and $\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$.

■

Theorem 1.3.1 — De Moivre's Theorem. if $z = r(\cos(\theta) + i\sin(\theta))$

$$z^n = r^n(\cos(\theta n) + i\sin(\theta n)) \tag{1.4}$$

Proof. The following proof will illustrative the steps to inductive reasoning
Case of $n = 1$: $z^n = r^n(\cos(\theta n) + i\sin(\theta n)) = z = r(\cos(\theta) + i\sin(\theta))$
This is true by definition.
Assume that:
$z^{n-1} = r^{n-1}(\cos(\theta(n-1)) + i\sin(\theta(n-1)))$
Then from Equation (1.3) we can verify:

$$zz^{n-1} = rr^{n-1}(\cos(\theta(n-1) + \theta) + i\sin(\theta(n-1) + \theta))$$
$$z^n = r^n(\cos(\theta n) + i\sin(\theta n))$$

■

Definition 1.3.2 — Argument. The argument of a complex number $z = r(\cos(\theta) + i\sin(\theta))$ is any angle, $\arg(z) = \theta$, such that $z = r(\cos(\theta) + i\sin(\theta))$.

From Equation (1.1), we observe that r is unique (because we constrained it to just positive values). θ, however, is not unique.

Definition 1.3.3 — Principle Orientation. We say θ is the principle orientation of z if $\theta \in [-\pi, \pi)$
In this range, θ is unique.

Definition 1.3.4 — Vector Dot Product. The dot product of two complex numbers $z = x + iy$ and $w = s + it$ is defined as:

$$z \cdot w = x \cdot s + y \cdot t = \Re(z\overline{w}) \tag{1.5}$$

$$\cos\theta = \frac{a \cdot b}{|a||b|} \tag{1.6}$$

Corollary 1.3.2 — Perpendicular Vectors. Complex variables z and w are perpendicular if $\Re(z\overline{w}) = 0$.

R [Complex Numbers to Solve Polynomial Equations] Over $\mathbb{C}$, every equation of the form $z^n = a$ has n solutions.

■ **Example 1.2 — Solving $z^n = -1$.** Let $z = r(\cos(\theta) + i\sin(\theta))$. Then:

$$\begin{aligned}
z^n &= r^n(\cos(\theta n) + i\sin(\theta n)) = -1 \\
\implies r^n &= 1 \text{ and } \cos(\theta n) + i\sin(\theta n) = -1 \\
\implies r &= 1 \text{ and } \cos(\theta n) = -1 \text{ and } \sin(\theta n) = 0 \\
\implies \theta n &= \pi + 2\pi k \text{ for } k \in \mathbb{Z} \\
\implies \theta &= \frac{\pi + 2\pi k}{n} \text{ for } k \in \mathbb{Z}
\end{aligned}$$

We can now find the principle solutions for Z

$$\therefore\ \theta_0 = \frac{\pi}{n},\ \theta_1 = \frac{3\pi}{n},\ \ldots,\ \theta_{n-1} = \frac{(2n-1)\pi}{n}$$

■

R Roots of Unity The solutions to $z^n = 1$ are called the nth roots of unity. Plotting these solutions splits the complex plane into n equal parts.

1.4 Subsets of the Plane

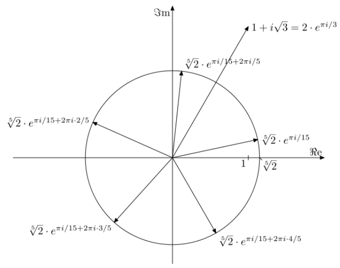


Figure 1.2: Complex Fifth Roots of Unity

Definition 1.4.1 — Open Disc. An open disc of radius R centered at z_0 is the set of all z such that $D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R \subset \mathbb{C}\}$.

Definition 1.4.2 — Interior Point. A point z_0 is an interior point of a set $A \subset \mathbb{C}$ if there exists an open disc centered at z_0 that is contained in A.

$$z_0 \quad \text{is an interior point of } A \text{ if} \quad \exists D_{>0}(z_0) \in A$$

Definition 1.4.3 — Open Set. A set $A \subset \mathbb{C}$ is open if every point in A is an interior point. *I.e. there are no 'hard lines' in the set.*

■ **Example 1.3 — Open Disc.** Show that the disc $D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R\}$ is an open set.

Proof. Let $z_1 \in D$. Then $|z_1 - z_0| < R$. Let $r = R - |z_1 - z_0|$. Then $r > 0$.
Let $z_2 \in D$ be any point in D, such that $|z_2 - z_1| < r$. Then:

$$\begin{aligned} |z_2 - z_0| &\leq |z_2 - z_1| + |z_1 - z_0| \\ &< r + R - r = R \end{aligned}$$

Therefore $z_2 \in D$ and D is open. ■

■

Definition 1.4.4 — Boundary (∂D). The boundary of a set A is the set of all points z such that every open disc centered at z, no matter how small, contains points in A and points not in A. The boundary of A is denoted by ∂A and a boundary point z is denoted by $z \in \partial A$.

$$z_0 \quad \text{is a boundary point of } A \text{ if} \quad \exists z \in D_R(z_0) : z \notin A \forall R > 0$$

Definition 1.4.5 — Closed Set. A set D is closed if it contains all its boundary points.

R A set can be both open and closed ($\mathbb{C}$, $\emptyset$), open and not closed, closed and not open, or neither open nor closed (contains part, but not all of their boundary).

Theorem 1.4.1 — Properties of Open and Closed Sets.

1. D is open iff $\mathbb{C} \setminus D$ is closed.
2. D is closed iff $\mathbb{C} \setminus D$ is open.
3. D is open if and only if it contains none of its boundary points.

1.5 Lines and Circles (Not done in class, Fisher 1.3)

Definition 1.5.1 — Line in the Complex Plane. A line of the form $y = mx + b$ can be formulated as:

$$0 = \Re\{(m + i)z + b\}$$

Such that when the real part of the complex number is zero, the line is satisfied. The general form is:

$$0 = \Re\{az + b\}, \qquad a, b, z \in \mathbb{C} \tag{1.7}$$

$$\text{where } a = A + iB \text{ such that:} \tag{1.8}$$

$$Ax - By + \Re b = 0 \tag{1.9}$$

Note that the imaginary part of b does not affect the line.

Definition 1.5.2 — Simple Circle in the Complex Plane. Circles in the complex plane can be formulated as:

$$|z - z_0| = R \tag{1.10}$$

Where z_0 is the *locus* of the circle and R is the radius.

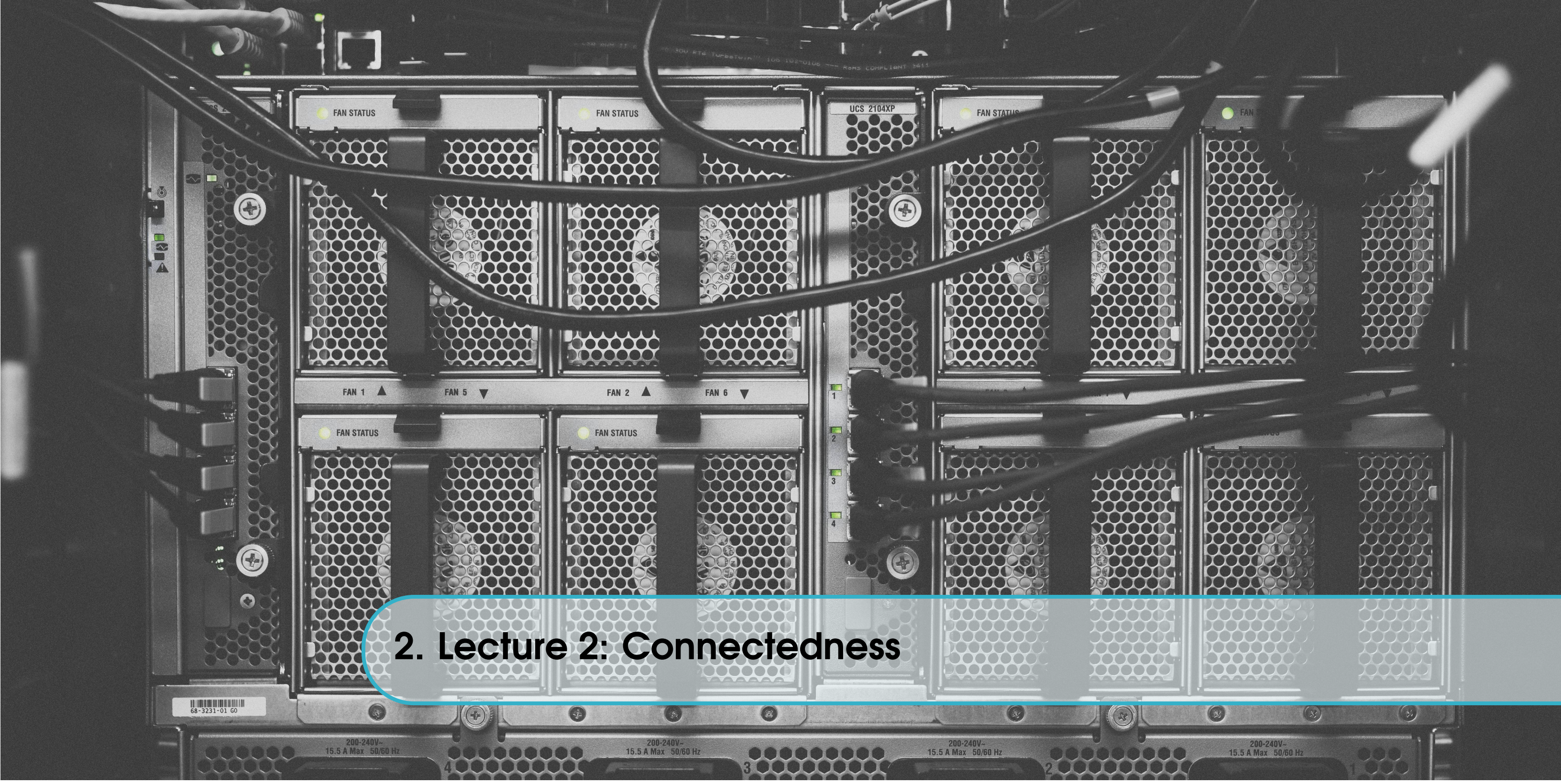
Definition 1.5.3 — Perpendicular Bisector. The perpendicular bisector of the line segment between p and q is the set of all points z such that

$$|z - p| = |z - q|$$

Corollary 1.5.1 — Apollonian Circles. If p and q are distinct complex numbers then a circle can be formulated as:

$$|z - p| = \rho|z - q| \qquad 0 < \rho \in \mathbb{R}, \rho \neq 1 \tag{1.11}$$

$$\to \quad \text{Where} \quad z_0 = \frac{p - \rho^2 q}{1 - \rho^2} \quad \text{and} \quad R = \frac{|p - q|\rho}{1 - \rho^2} \tag{1.12}$$



2. Lecture 2: Connectedness

2.1 Connected Sets

Definition 2.1.1 — Connected Set. An *open* set D is connected if each pair of points $p, q \in D$ can be joined by a polygonal path lying entirely in D. That is:

$$\exists P_2, P_3, \ldots, P_n \in D \quad \text{such that} \quad pP_1, P_1P_2, \ldots, P_nq \in D$$

R The set doesn't *have* to be open, but it is easier to prove connectedness for open sets.

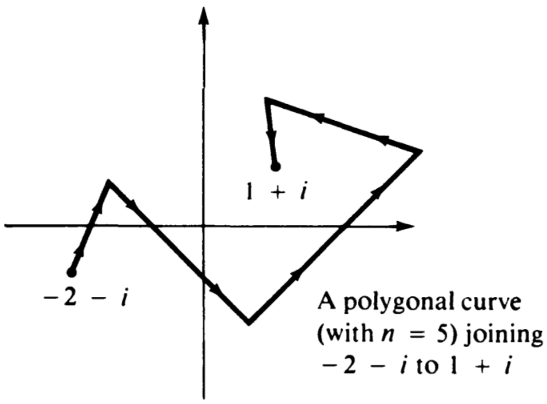


Figure 2.1: Polygonal Path

Definition 2.1.2 — Domain. A domain is a set that's

- Open
- Connected
- Not empty

Definition 2.1.3 — Convex Set. A set D is convex if for each pair of points $p, q \in D$, the line segment pq lies entirely in D.

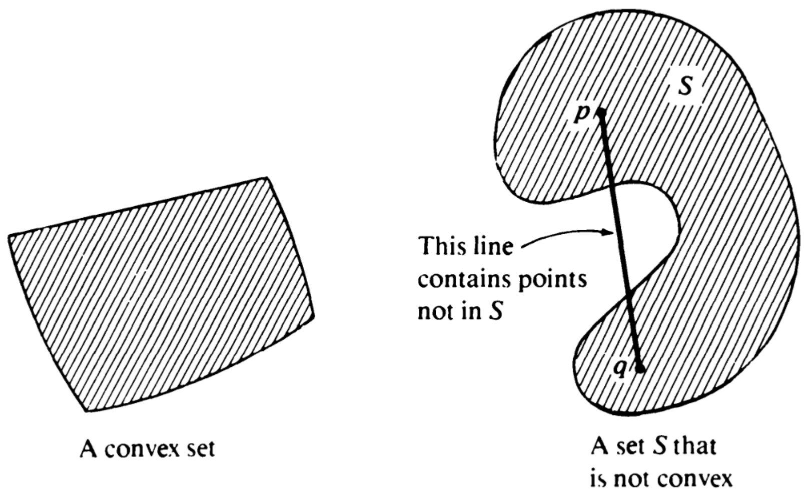


Figure 2.2: Convex Set

Theorem 2.1.1 — Convex $\implies$ Connected. If D is a convex open set, then D is connected.

Definition 2.1.4 — Open Half-plane. A set D is an open half-plane if it is of the form

$$D = \{z \in \mathbb{C} : \Re\{az + b\} > 0\}$$

Each open half-plane is convex and open

Definition 2.1.5 — Closed Half-plane. A set D is a closed half-plane if it is of the form

$$D = \{z \in \mathbb{C} : \Re\{az + b\} \geq 0\}$$

Each closed half-plane is convex and closed

2.2 Point at Infinity

Definition 2.2.1 — Point at Infinity. A set is said to contain the point at infinity if it contains all points z such that $|z| > R$ for some $R > 0$.

■ **Example 2.1** No open Half-plane contains the point at infinity. Even though the set is unbounded, choosing R near the boundary will always give a point outside the set. ■

2.3 Functions and Limits

Definition 2.3.1 — Limit of a Sequence of Complex Numbers.

$$\lim_{n\to\infty} z_n = z \quad \text{or} \quad z_n \to z \quad \iff \quad \forall \varepsilon > 0, \exists N \in \mathbb{N} \tag{2.1}$$

such that $n \geq N \implies |z_n - z| < \varepsilon$ (2.2)

Corollary 2.3.1 — Parts of a Limit. If $z_n = x_n + iy_n$ and $z = x + iy$, then

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

Theorem 2.3.2 — Subsequence. Suppose $\{z_n\}$ converges with limit z. Then every subsequence, $z_{m_n} = f(n)$ also converges to z. Where $1 \leq m_1 < m_2 < \ldots$

Definition 2.3.2 — Limits of Functions.

$$\lim_{z\to z_0} f(z) = w \iff \forall\varepsilon > 0, \exists\delta > 0 \tag{2.3}$$

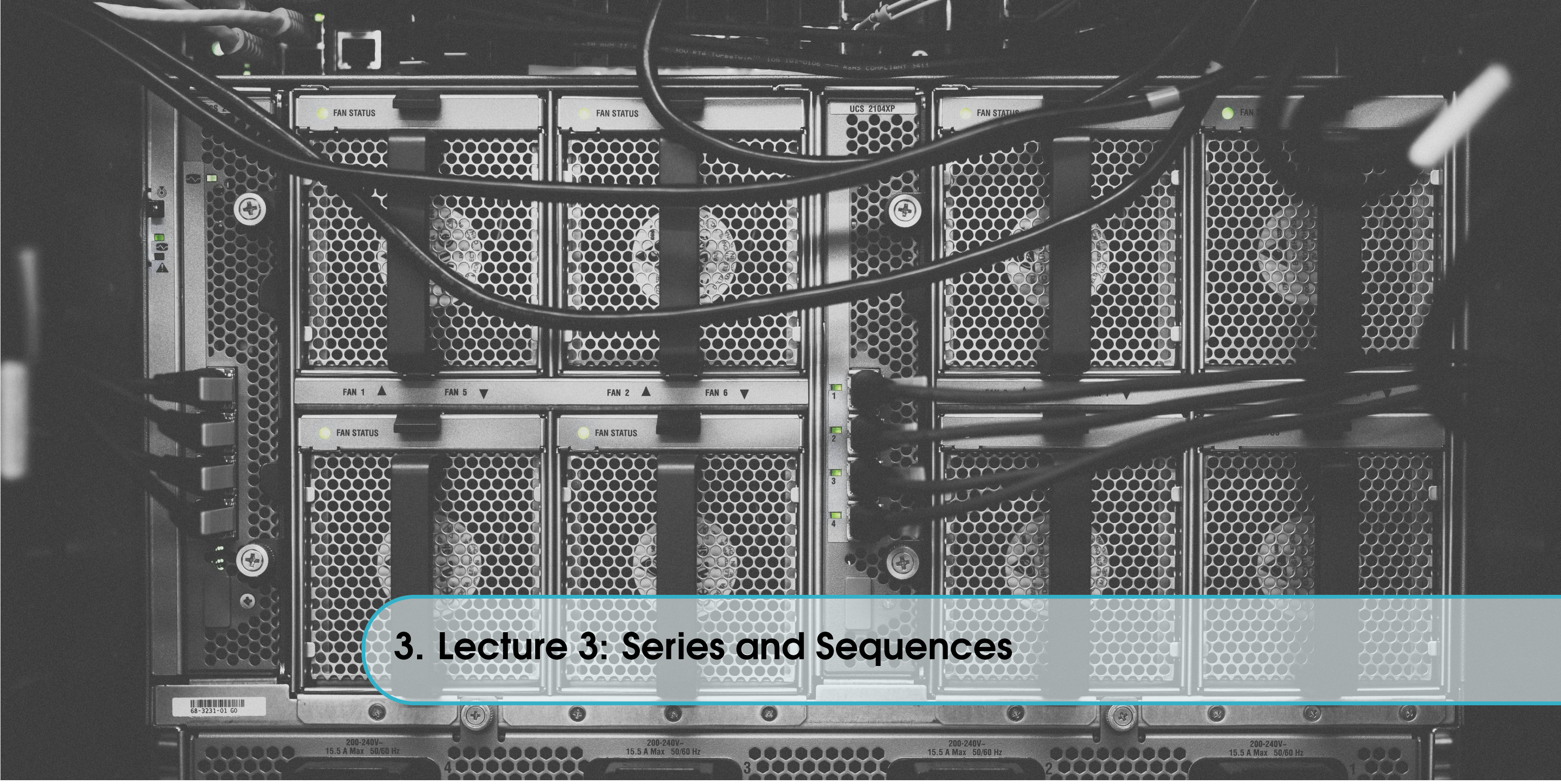
$$\text{such that } 0 < |z - z_0| < \delta \implies |f(z) - w| < \varepsilon \tag{2.4}$$

2.4 Continuity

Definition 2.4.1 — Continuous Function. A function $f(z)$ is continuous at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0)$$

Corollary 2.4.1 — Continuous at Infinity. A function $f(z)$ can be continuous at ∞ if $f(\infty) = \lim_{z\to\infty} f(z) = f(\infty)$. Note, $f(\infty)$ may equal ∞
This is equivalent to saying that $f(1/z)$ is continuous at $z = 0$



3. Lecture 3: Series and Sequences

Definition 3.0.1 — infinite Series. Suppose we have a sequence:

$$z_1, z_2, z_3, \ldots \tag{3.1}$$

We can define the partial sum of the sequence as:

$$S_n = z_1 + z_2 + z_3 + \ldots + z_n \tag{3.2}$$

We say $\sum_{n=1}^{\infty} z_n$ converges and has a sum S if the sequence of partial sums converges to S:

$$\lim_{n\to\infty} S_n = S \tag{3.3}$$

If $\lim_{n\to\infty} S_n$ does not exist, we say the series diverges.

Corollary 3.0.1 — Real and Imaginary Parts of a Series. If $\sum_{n=1}^{\infty} z_n$ converges, then the real and imaginary parts of the series also converge.

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \Re(z_n) + i \sum_{n=1}^{\infty} \Im(z_n) \tag{3.4}$$

3.1 Tests for Convergence

Theorem 3.1.1 If $\sum_{n=1}^{\infty} |z_n|$ converges, then so does $|\sum_{n=1}^{\infty} z_n|$ and:

$$\left|\sum_{n=1}^{\infty} z_n\right| \leq \sum_{n=1}^{\infty} |z_n|$$

Proof. Say $z_n = x_n + iy_n$. Then:

$$\left|\sum_{n=1}^{\infty} x_n\right| \leq \sum_{n=1}^{\infty} |x_n| \leq \sum_{n=1}^{\infty} |z_n|$$

And

$$\left|\sum_{n=1}^{\infty} y_n\right| \leq \sum_{n=1}^{\infty} |y_n| \leq \sum_{n=1}^{\infty} |z_n|$$

So if $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge, then $\sum_{n=1}^{\infty} z_n$ converges. ∎

■ Example 3.1

$$\sum_{j=1}^{\infty} j \left(\frac{1+2i}{3}\right)^j \tag{3.5}$$

We can use the ratio test to determine convergence:

$$\begin{aligned}
\sum_{j=1}^{\infty} |z_j| &= \sum_{j=1}^{\infty} j \left|\frac{1+2i}{3}\right|^j \\
&= \sum_{j=1}^{\infty} j \left(\frac{\sqrt{5}}{3}\right)^j \\
\lim_{j\to\infty} \left|\frac{z_{j+1}}{z_j}\right| &= \lim_{j\to\infty} \frac{(j+1)(\frac{\sqrt{5}}{3})^{j+1}}{j(\frac{\sqrt{5}}{3})^j} \\
&= \lim_{j\to\infty} \frac{j+1}{j}\left(\frac{\sqrt{5}}{3}\right) \\
&= \frac{\sqrt{5}}{3} < 1 \qquad \therefore \text{ The series converges}
\end{aligned}$$

■

3.2 The Exponential Function

Approach 1

Definition 3.2.1 — Exponential Function. If $z = x + iy$, then the exponential function is defined as:

$$e^z = e^x\,(\cos(y) + i\,\sin(y)) \tag{3.6}$$

R [Euler's Formula]

$$e^{i\theta} \triangleq \cos(\theta) + i\,\sin(\theta) \tag{3.7}$$

(3.8)

Test Name	Description	Conditions for Use	Results
Ratio Test	$\lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\|$	Applicable when terms are positive and the limit exists.	Converges if $L < 1$, diverges if $L > 1$, inconclusive if $L = 1$.
Root Test	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|}$	Applicable when terms are positive and the limit exists.	Converges if $L < 1$, diverges if $L > 1$, inconclusive if $L = 1$.
Integral Test	Compares a series to an improper. $\int_1^\infty f(x)\,dx$	Applicable when terms are positive, continuous, and decreasing.	Converges if the integral converges, diverges if the integral diverges.
Comparison Test	Compares a series to a known convergent or divergent series.	Applicable when terms are positive.	Converges if the series being compared to converges.
Limit Comparison Test	Compares the limit of the ratio of terms to a known series. $\lim_{n\to\infty} \frac{a_n}{b_n}$	Applicable when terms are positive and the limit exists.	Converges if the limit is finite and the comparison series converges, diverges otherwise.
Alternating Series Test	$\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$	When dealing with alternating series	Converges if: $a_n > 0$, decreasing, and $\lim_{n\to\infty} a_n = 0$
p-Series Test	Determines convergence based on the exponent in a series of the form $\sum \frac{1}{n^p}$	Applicable for series of the form $\frac{1}{n^p}$.	Converges if $p > 1$, diverges if $p \leq 1$.
Geometric Series Test	Determines convergence for geometric series. $\sum ar^n$	Applicable for series of the form ar^n.	Converges if $\|r\| < 1$, diverges if $\|r\| \geq 1$.
D'Alembert's Ratio Test	Similar to the Ratio Test, but specifically for series with factorial terms.	Applicable when terms involve factorials.	Converges if the ratio is less than 1, diverges if greater than 1.
Cauchy's Condensation Test	Determines convergence by condensing the series. $\sum a_n \sim \sum 2^n a_{2^n}$	Applicable for series with positive, decreasing terms.	Converges if the condensed series converges, diverges if the condensed series diverges.

Property	**Description**
Periodicity	The complex exponential function is periodic with period $2\pi i$, $$e^{z+2\pi i} = e^z$$.
Multiplication	The exponential function satisfies $$e^{z_1+z_2} = e^{z_1}e^{z_2}$$ for any complex numbers z_1 and z_2.
Derivative	The derivative of the exponential function is $$\frac{d}{dz}e^z = e^z$$.
Inverse	The inverse of the exponential function is the complex logarithm, $$\log z$$ such that $$e^{\log z} = z$$ for $z \neq 0$.
Magnitude	The magnitude of the exponential function is $$\lvert e^z \rvert = e^{\Re(z)}$$ where $\Re(z)$ denotes the real part of z.
Argument	The argument of the exponential function is $$\arg(e^z) = \Im(z) \mod 2\pi$$ where $\Im(z)$ denotes the imaginary part of z.
Conjugate	The conjugate of the exponential function is $$\overline{e^z} = e^{\bar{z}}$$.

Table 3.2: Properties of the Complex Exponential Function

Properties of the complex Exponential Function

Approach 2: Taylor Series

Definition 3.2.2 — The Exponential Function. The exponential function can be defined as:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \text{for all } z \in \mathbb{C} \tag{3.9}$$

Claim 3.2.1 — The Taylor Series for the Exponential Function Converges. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$.

Proof. HOMEWORK ■

Problem 3.1 For $\theta \in \mathbb{R}$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \cos(\theta) + i\sin(\theta)$$

3.3 Approach 3: Differential Equations

Definition 3.3.1 — Differential Equation for the Exponential Function. The exponential function satisfies the differential equation:

$$f(z) = \begin{cases} \frac{df}{dz} = f & \text{for all } z \in \mathbb{C} \\ f(0) = 1 \end{cases} \tag{3.10}$$

3.4 The Logarithm Function

Definition 3.4.1 — Logarithm Function. The logarithm function is defined as the inverse of the exponential function:

$$\log z = \log|z| + i\theta \tag{3.11}$$

R There will be many solutions to the logarithm function, as the argument is only defined modulo 2π.

$$\log z = \log|z| + i\,(\arg(z) + 2\pi n) \qquad \text{for } n \in \mathbb{Z}$$

Definition 3.4.2 — Principal Logarithm. The principal branch logarithm is defined as:

$$\mathrm{Log}(z) = \log|z| + i\arg(z) \qquad \text{for } -\pi < \arg(z) \leq \pi$$

Note: We use a capital L to denote the principal logarithm.

Definition 3.4.3 — Fixed θ_0 Logarithm Function. We can fix the argument of the logarithm function by setting θ_0 and letting $D = \{te^{i\theta_0} | t > 0, t \in \mathbb{R}\}$.

We define:

$$\widetilde{\log}_{\theta_0} z = \log|z| + i\,(\widetilde{\arg}(z) + \theta_0) \qquad \text{for } z \in D,\ \widetilde{\arg}(z) \in [0, 2\pi)$$

■ **Example 3.2 — Find the Values of $(-1)^i$.**

$$(-1)^i = e^{i\,\log(-1)} \tag{3.12}$$
$$= e^{i(2n+1)\pi i} \tag{3.13}$$
$$= e^{-2n\pi} \tag{3.14}$$
$$\log(-1) = -(2n+1)\pi i \qquad n \in \mathbb{Z} \tag{3.15}$$
$$(-1)^i = e^{2n+1\pi} \tag{3.16}$$

■

3.5 The Trigonometric Functions

Definition 3.5.1 — Trigonometric Functions. For $z \in \mathbb{C}$ trigonometric functions are defined as:

$$\Re e^{iz} = \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{3.17}$$
$$\Im e^{iz} = \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{3.18}$$
$$\tan(z) = \frac{\sin(z)}{\cos(z)} \tag{3.19}$$
$$\tag{3.20}$$

Lemma 3.5.1

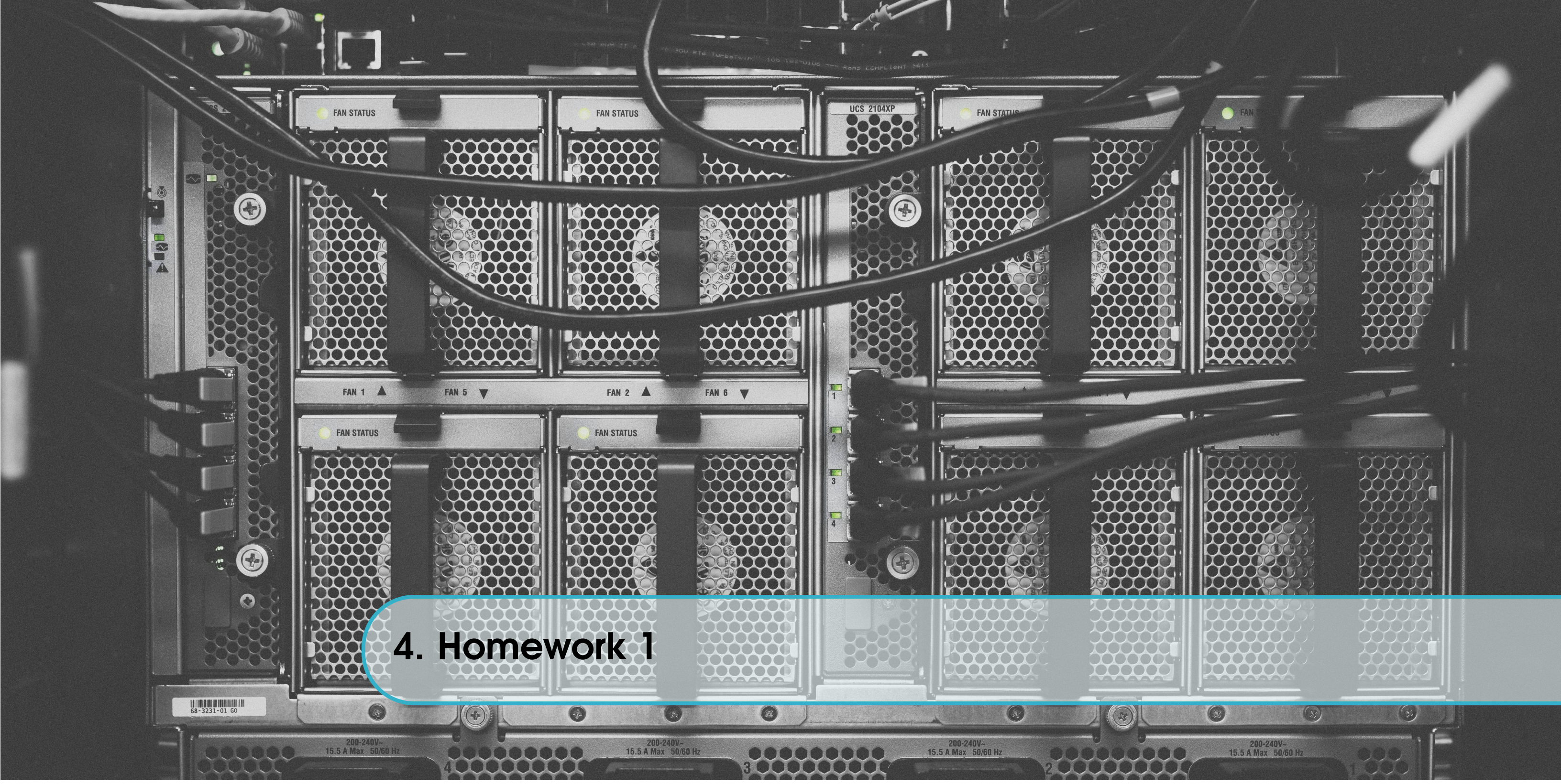
$$\begin{cases} \cos(z+\alpha) = \cos(z) \\ \sin(z+\alpha) = \sin(z) \end{cases} \tag{3.21}$$

iff $\alpha = 2\pi n$ for $n \in \mathbb{Z}$.

Proof.

$$e^{i(z+\alpha)} = e^{iz}e^{i\alpha} \tag{3.22}$$
$$= e^{iz}\,(\cos(\alpha) + i\,\sin(\alpha)) \tag{3.23}$$
$$= e^{iz}\,(\cos(2\pi n) + i\,\sin(2\pi n)) \tag{3.24}$$
$$= e^{iz} \tag{3.25}$$

■



4. Homework 1

■ **Example 4.1 — Fisher, Section 1.2, Problem 2.** Describe the locus of points z satisfying the equation

$$|z - 4| = 4|z|$$

Solution:
Let $z = x + iy$. Then

$$\begin{aligned}
|z-4| &= |x+iy-4| = |x-4+iy| = \sqrt{(x-4)^2+y^2} \\
4|z| &= 4|x+iy| = 4\sqrt{x^2+y^2} \\
\sqrt{(x-4)^2+y^2} &= 4\sqrt{x^2+y^2} \\
(x-4)^2+y^2 &= 16(x^2+y^2) \\
x^2-8x+16+y^2 &= 16x^2+16y^2 \\
15x^2+15y^2+8x-16 &= 0 \\
x^2+y^2+\frac{8}{15}x-\frac{16}{15} &= 0 \\
\Rightarrow \quad & \text{complete the square} \\
x^2+\frac{8}{15}x+(\frac{4}{15})^2-(\frac{4}{15})^2+y^2 &= \frac{16}{15} \\
(x+\frac{4}{15})^2+y^2 &= \frac{16}{15}+(\frac{4}{15})^2 \\
(x+\frac{4}{15})^2+y^2 &= \frac{16}{15}+\frac{16}{225} \\
(x+\frac{4}{15})^2+y^2 &= \frac{256}{225}
\end{aligned}$$

$\therefore$ The locus of points z satisfying the equation $|z - 4| = 4|z|$ is a circle with center $(-\frac{4}{15}, 0)$ and radius $\sqrt{\frac{256}{225}} = \frac{16}{15}$. ■

■ **Example 4.2 — Fisher, Section 1.2, Problem 24.** Find all solutions of the equation

$$(z + 1)^4 = 1 - i$$

Solution:

$$\begin{aligned}
&\text{Convert } 1 - i \text{ to polar form} \\
\arg 1 - i &= \tan^{-1}(-1) = -\frac{\pi}{4} + 2k\pi \quad \text{where } k \in \mathbb{Z} \\
|1 - i| &= \sqrt{1^2 + (-1)^2} = \sqrt{2} \\
1 - i &= \sqrt{2}(cos(-\frac{\pi}{4} + 2k\pi) + isin(-\frac{\pi}{4} + 2k\pi)) \\
&\text{Use De Moivre's Theorem} \\
&\rightarrow z^n = r^n(cos(n\theta) + isin(n\theta)) \\
z + 1 &= 2^{\frac{1}{8}}(\cos(\frac{-\pi}{4} + \frac{2\pi k}{4}) + i\sin(\frac{-\pi}{4} + \frac{2\pi k}{4})) \\
z &= 2^{\frac{1}{8}}(\cos(\frac{-\pi}{4} + \frac{2\pi k}{4}) + i\sin(\frac{-\pi}{4} + \frac{2\pi k}{4})) - 1
\end{aligned}$$

We can now find the solutions by plugging in $k = 0, 1, 2, 3$.

$$\begin{aligned}
\theta_0 &= \frac{-\pi}{4} + \frac{2\pi \cdot 0}{4} = -\frac{\pi}{4} \quad k = 0 \\
\theta_1 &= \frac{-\pi}{4} + \frac{2\pi \cdot 1}{4} = \frac{\pi}{4} \quad k = 1 \\
\theta_2 &= \frac{-\pi}{4} + \frac{2\pi \cdot 2}{4} = \frac{3\pi}{4} \quad k = 2 \\
\theta_3 &= \frac{-\pi}{4} + \frac{2\pi \cdot 3}{4} = \frac{5\pi}{4} \quad k = 3
\end{aligned}$$

So our solutions are:

$$\begin{aligned}
z &= 2^{\frac{1}{8}}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})) - 1 = 2^{\frac{1}{8}}(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) - 1 \\
z &= 2^{\frac{1}{8}}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})) - 1 = 2^{\frac{1}{8}}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) - 1 \\
z &= 2^{\frac{1}{8}}(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})) - 1 = 2^{\frac{1}{8}}(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) - 1 \\
z &= 2^{\frac{1}{8}}(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4})) - 1 = 2^{\frac{1}{8}}(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) - 1
\end{aligned}$$

■

■ **Example 4.3 — Fisher, Section 1.2, Problem 26.** Find all solutions of the equation $z^3 = 8$.
Solution:
First, we convert 8 to polar form.

$$\begin{aligned} 8 &= 8(\cos(0) + i\sin(0)) \\ &= 8(\cos(2\pi k) + i\sin(2\pi k)) \quad \text{where } k \in \mathbb{Z} \end{aligned}$$

Then we use De Moivre's Theorem to find the solutions.

$$\begin{aligned} \rightarrow z^n &= r^n(\cos(n\theta) + i\sin(n\theta)) \\ z &= 2(\cos(\frac{2\pi k}{3}) + i\sin(\frac{2\pi k}{3})) \quad \text{where } k = 0, 1, 2 \end{aligned}$$

So our solutions are:

$$\begin{aligned} z &= 2(\cos(0) + i\sin(0)) = 2(1 + i0) = 2 \\ z &= 2(\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})) = 2(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -1 + i\sqrt{3} \\ z &= 2(\cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3})) = 2(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = -1 - i\sqrt{3} \end{aligned}$$

■

■ **Example 4.4 — Fisher, Section 1.3, Problem 2.** For the following set, describe (i) the interior and the boundary, (ii) state whether the set is open, or closed, or neither open nor closed, (iii) state whether the interior of the set is connected (if it has an interior).

$$A = \{z \in \mathbb{C} : |z| < 1\text{or}|z - 3| \leq 1\}$$

Solution:

1. $A_{int} = \{|z| < 1 \text{ or } |z - 3| < 1\}$
2. $A_{bd} = \{|z| = 1 \text{ or } |z - 3| = 1\}$
3. A is neither open nor closed because $\{|z| = 1\} \notin A$ but $\{|z - 3| = 1\} \in A_{int}$, so A contains only part of its boundary.
4. A_{int} is not connected, because $z_1 = 0, z_2 = 3 \in A_{int}$, but $\nexists P_1P_2 \ldots P_n \in A_{int}$ such that $z_1P_1P_2 \ldots P_nz_2 \in A_{int}$

■

■ **Example 4.5 — Fisher, Section 1.3, Problem 4.** For the following set, describe (i) the interior and the boundary, (ii) state whether the set is open, or closed, or neither open nor closed, (iii) state whether the interior of the set is connected (if it has an interior).

$$A = \{z \in \mathbb{C} : \text{Re}(z^2) = 4\}$$

Solution:
Let $z = x + iy$. Then

$$\begin{aligned} \text{Re}(z^2) &= \text{Re}((x + iy)^2) = \text{Re}(x^2 - y^2 + 2ixy) \\ 4 &= x^2 - y^2 \end{aligned}$$

1. No interior, because $\forall z_0 \in A \quad \nexists D | D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R, R > 0\}$
2. $A_{bd} = \{z \in \mathbb{C} : x^2 - y^2 = 4\}$
3. $A_{bd} = A$ so A is closed.
4. A_{int} is connected because $A_{int} = \emptyset$.

■

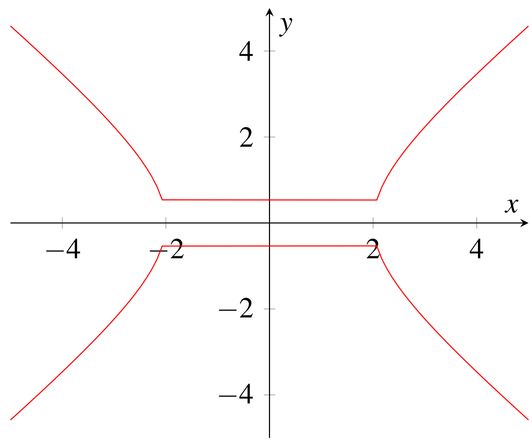


Figure 4.1: Plot of $4 = x^2 - y^2$

■ Example 4.6 — Fisher, Section 1.4, Problem 12. Find

$$\lim_{z \to 2} (z - 2) \log |z - 2|,$$

or explain why it does not exist. **Solution:**
We use L'Hopital's Rule to find the limit.

$$\begin{aligned}
\lim_{z \to 2} (z - 2) \log |z - 2| &= \lim_{z \to 2} \frac{\log |z - 2|}{\frac{1}{z-2}} \\
&= \lim_{z \to 2} \frac{\frac{\partial}{\partial z} \log |z - 2|}{\frac{\partial}{\partial z} \frac{1}{(z-2)}} \\
&= \lim_{z \to 2} \frac{\frac{1}{z-2}}{-\frac{1}{(z-2)^2}} \\
&= \lim_{z \to 2} \frac{1}{\frac{1}{2-z}} \\
&= \lim_{z \to 2} 2 - z \\
&= 0
\end{aligned}$$

■

■ Example 4.7 — Fisher, Section 1.4, Problem 16. Find all the points where the following function is continuous:

$$f(z) = \begin{cases} \frac{z^4 - 1}{z - i}, & z \neq i \\ 4i, & z = i \end{cases}$$

Solution:
First normalize the denominator.

$$\begin{aligned}
f(z) &= \frac{z^4 - 1}{z - i} = \frac{(z^2 + 1)(z + 1)(z - 1)}{z - i} = \frac{(z^2 + 1)(z + 1)(z - 1)}{z - i}\frac{z + i}{z + i} \\
&= \frac{(z^2 + 1)(z + 1)(z - 1)(z + i)}{z^2 + 1} = (z + 1)(z - 1)(z + i), \quad z \neq i
\end{aligned}$$

As this is a polynomial, it is continuous everywhere except at $z = i$. Now we test for continuity at $z = i$.

$$\begin{aligned}
\lim_{z \to i} f(z) &= f(i) \\
\lim_{z \to i}(z + 1)(z - 1)(z + i) &= 4i \\
(i + 1)(i - 1)(i + i) &= 4i \\
4i &= 4i
\end{aligned}$$

So $f(z)$ is continuous everywhere. ∎

∎ **Example 4.8 — Fisher, Section 1.4, Problem 34.** Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{2 + i^n}$$

Solution:
We notice:

$$\begin{aligned}
\frac{1}{2 + i^n} &= \frac{1}{2 + 1} \quad \text{for } n = 0, 4, 8, \ldots \\
\frac{1}{2 + i^n} &= \frac{1}{2 + i} \quad \text{for } n = 1, 5, 9, \ldots \\
\frac{1}{2 + i^n} &= \frac{1}{2 - 1} \quad \text{for } n = 2, 6, 10, \ldots \\
\frac{1}{2 + i^n} &= \frac{1}{2 - i} \quad \text{for } n = 3, 7, 11, \ldots
\end{aligned}$$

Which forms a cycle, so the series diverges. ∎

∎ **Example 4.9 — Fisher, Section 1.4, Problem 36.** Show that each of the following series converges for all z.

1. $$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

2. $$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

3. $$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n + 1)!}$$

Solution:

1. We use the ratio test to show convergence.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{z^{n+1}}{(n+1)!}\frac{n!}{z^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{z}{n+1}\right| \\ &= 0\end{aligned}$$

   So the series converges for all z.
2. We use the ratio test to show convergence.

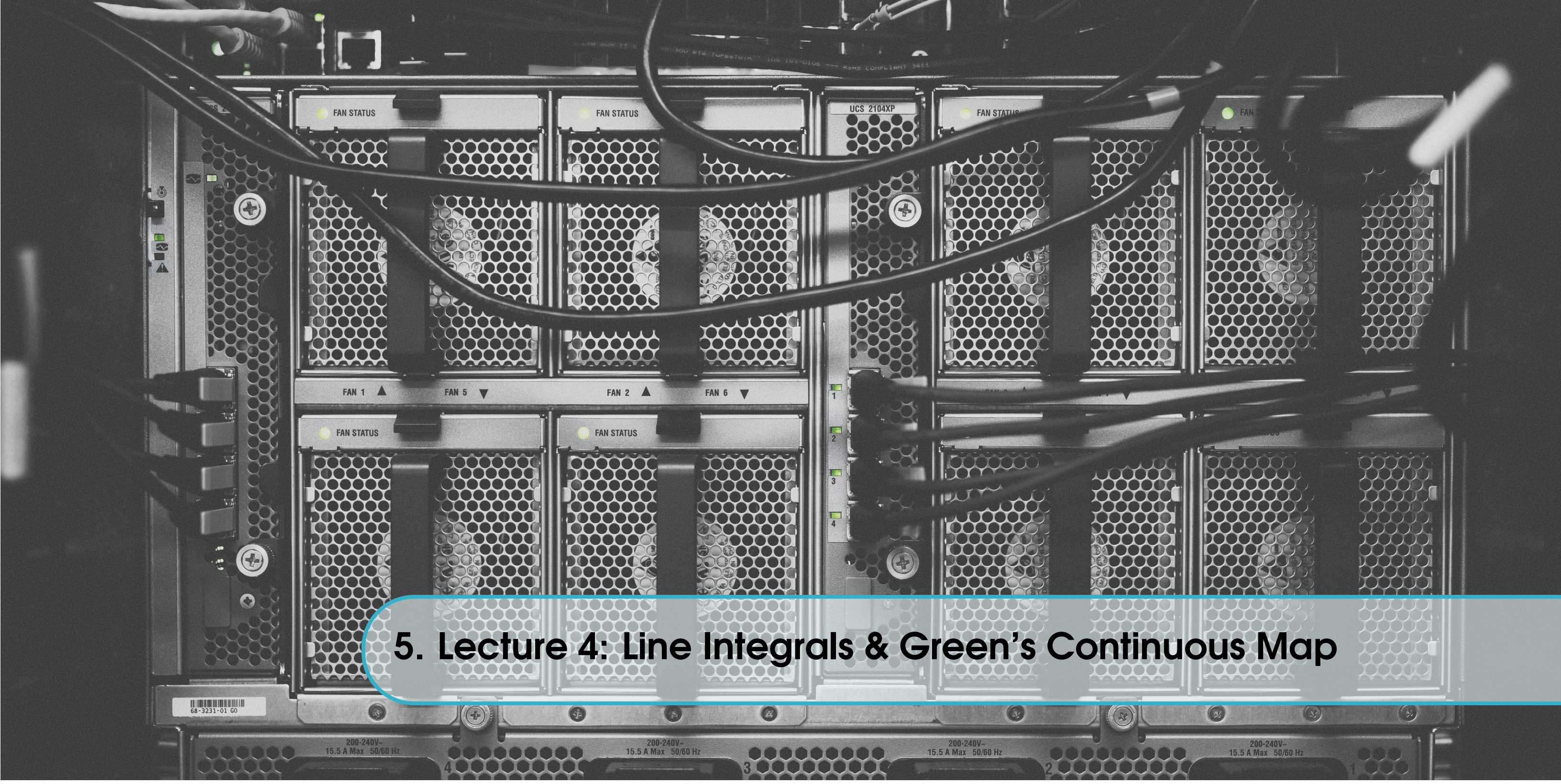
$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{z^{2(n+1)}}{(2(n+1))!}}{(-1)^n\frac{z^{2n}}{(2n)!}}\right| \\ &= \lim_{n\to\infty}\left|\frac{z^2}{(2n+2)(2n+1)}\right| \\ &= 0\end{aligned}$$

   So the series converges for all z.
3. We use the ratio test to show convergence.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{z^{2(n+1)+1}}{(2(n+1)+1)!}\frac{(2n+1)!}{z^{2n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{z^2}{(2n+3)(2n+2)(2n+1)}\right| \\ &= 0\end{aligned}$$

   So the series converges for all z.

∎



5. Lecture 4: Line Integrals & Green's Continuous Map

Theorem 5.0.1 — Parametrized Curves.

$$\gamma(t) = x(t) + iy(t) \quad a \leq t \leq b$$

$\gamma[a, b] \rightarrow \mathbb{C}$ is the image of γ.

■ **Example 5.1 — Circle Parametrization.**

$$\gamma(t) = \cos(t) + i\sin(t) \quad 0 \leq t \leq 2\pi$$
$$\gamma(t) = e^{it} \quad 0 \leq t \leq 2\pi$$

So γ is a circle of radius 1 centered at the origin.

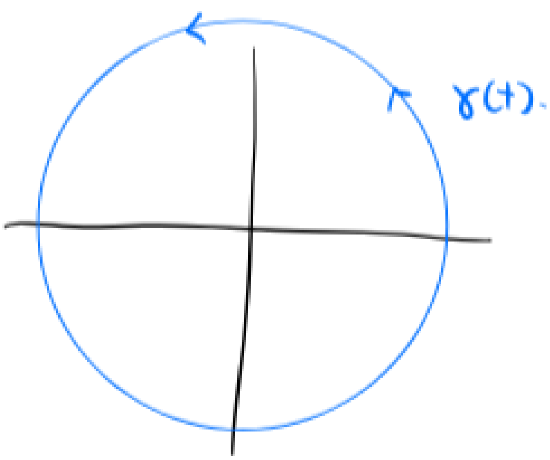


Figure 5.1: Circle Parametrization

■

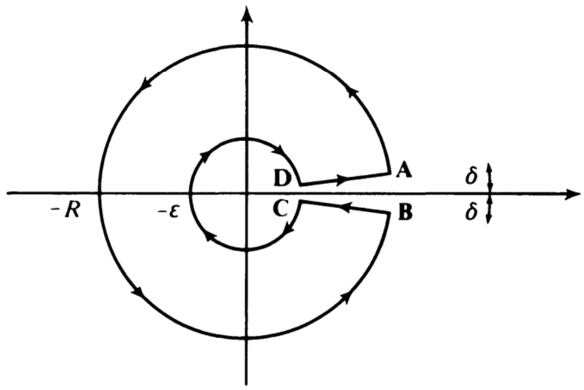


Figure 5.2: Keyhole Parametrization

■ **Example 5.2 — Keyhole Parametrization (Section 1.6, Example 5).** The first part of the keyhole is parametrized by a circle of radius R centered at the origin between δ and $2\pi - \delta$. This is given by the following:

$$\gamma(t) = Re^{i\theta} \quad \delta \leq \theta \leq 2\pi - \delta$$

The inner circle is parametrized by a circle of radius ε centered at the origin between $2\pi - \delta$ and δ. Note that the direction of the parametrization is opposite to the first circle. The lines of the keyhole connects the two circles at $\theta = 2\pi - \delta$ and $\theta = \delta$.
So the keyhole is parametrized by the following:

$$\gamma(t) = \begin{cases} Re^{i\theta} & \delta \leq \theta \leq 2\pi - \delta \\ te^{i(2\pi-\delta)} & R \leq t \leq \varepsilon \\ \varepsilon e^{i\theta} & 2\pi - \delta \leq \theta \leq \delta \\ te^{i\delta} & \varepsilon \leq t \leq R \end{cases}$$

■

Definition 5.0.1 — Simple Curve. A curve γ is **simple** if $\gamma(t_1) = \gamma(t_2) \implies t_1 = t_2$ for $t_1, t_2 \neq a, b$.

Definition 5.0.2 — Closed Curve. A curve γ is **closed** if $\gamma(a) = \gamma(b)$. So if the end point meets the starting point.

R We can *ignore* the parametrization and talk about the curve

$$Image(\gamma) \subset \mathbb{C}$$

as a subset of $\mathbb{C}$.

Definition 5.0.3 — C^1/Smooth Curve. A parametrized curve is C^1 if $\gamma'(t)$ if

$$\gamma'(t) = x'(t) + iy'(t)$$

exists $\forall t \in [a, b]$ and is continuous.

R Here, $\gamma'(a) = x'(a) + iy'(a)$, $\gamma'(b) = x'(b) + iy'(b)$ are the 1-sided derivatives (i.e. they consider the rate of change from the left/right where the function is defined).

Definition 5.0.4 — Piecewise C^1/Smooth Curve.

$$\text{if } \exists a = t_0 < t_1 < \ldots < t_n = b \text{ such that } \gamma|_{[t_i, t_{i+1}]} \text{ is } C^1$$

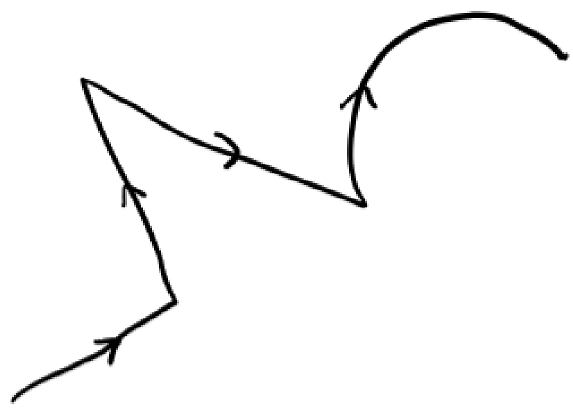


Figure 5.3: Piecewise C^1 Curve

5.1 Line Integrals

Definition 5.1.1 — Line Integral. if $g = u + iv$, $(u, v) \in \mathbb{R}^2$ is a complex-valued function and γ is piecewise C^1, then the line integral of g along γ is

$$\int_\gamma g(z)dz = \sum_{i=0}^{n-1} \int_i^{i+1} g(\gamma(t))\gamma'(t)dt$$

Where

$$\begin{aligned} g(\gamma(t))\gamma'(t) &= ux' - vy' + ivx' + iuy' \\ &= (u(\gamma(t)) + iv(\gamma(t)))(x'(t) + iy'(t)) \end{aligned}$$

is complex multiplication.

■ **Example 5.3 — Line Integral Example.** Compute the line integral of $\int_\gamma z^2 - 3|z| + \Im(z)dz$ where $\gamma(t) = 2e^{it}$, $0 \le t \le \pi/2$.

$$\begin{aligned} \gamma'(t) &= 2ie^{it} \\ g(\gamma(t)) &= \gamma(t)^2 - 3|\gamma(t)| + \Im(\gamma(t)) \\ &= 4e^{2it} - 6 + 2\sin(t) \end{aligned}$$

$$\begin{aligned}
\int_0^{\pi/2} g(\gamma(t))\gamma'(t) &= (4e^{2it} - 6 + 2\sin(t))(2ie^{it}) \\
&= \int_0^{\pi/2} 8ie^{3it} - 12ie^{it} + 4ie^{2it}\sin(t) \\
\Rightarrow \sin(t) &= \frac{e^{it} - e^{-it}}{2i} \\
&= \int_0^{\pi/2} \left(8ie^{3it} - 12ie^{it} + 4ie^{2it}\right)\left(\frac{e^{it} - e^{-it}}{2i}\right) \\
&= \int_0^{\pi/2} 4e^{4it} - 6e^{2it} + 2e^{3it} - 4e^{it} + 2e^{2it} - 3e^{it} + e^{3it} - e^{it} \\
&= \left(\frac{8}{3}e^{3it} - 12e^{it} - ie^{2it} - 2t\right)\Big|_0^{\pi/2} \\
&= \frac{8}{3}e^{3i\pi/2} - 12e^{i\pi/2} - ie^{i\pi} - 2\pi - \left(\frac{8}{3} - 12 - i\right) \\
&= \frac{28}{3} - \pi - \frac{38}{3}i
\end{aligned}$$

■

■ **Example 5.4 — Line Integral Example.** Compute the line integral of $\int_\gamma \cos z dz$ where $\gamma(t)$ is the line segment from $-\pi/2 + i$ to $\pi + i$.

$$\begin{aligned}
\gamma(t) &= -\frac{\pi}{2} + i + t(\pi + i + \frac{\pi}{2} - i) \quad 0 \le t \le 1 \\
&= -\frac{\pi}{2} + i + t(\frac{3\pi}{2})
\end{aligned}$$

$$\gamma'(t) = \frac{3\pi}{2}$$

$$\begin{aligned}
\int_0^1 \cos(\gamma(t))\gamma'(t)dt &= \int_0^1 \cos(-\frac{\pi}{2} + i + t(\frac{3\pi}{2}))\frac{3\pi}{2}dt \\
\to \cos(x + iy) &= \cos(x)\cosh(y) - i\sin(x) \\
&= \int_0^1 \cos(-\frac{\pi}{2} + t(\frac{3\pi}{2}))\cosh(1) - i\sinh(1)\sin(-\frac{\pi}{2} + t(\frac{3\pi}{2}))\frac{3\pi}{2}dt \\
&= \cosh(1)\int_0^1 \cos(-\frac{\pi}{2} + t(\frac{3\pi}{2}))dt - i\sinh(1)\int_0^1 \sin(-\frac{\pi}{2} + t(\frac{3\pi}{2}))\frac{3\pi}{2}dt \\
&= \cosh(1) - i\sinh(1)
\end{aligned}$$

■

Theorem 5.1.1 — Length of a Curve. If γ is a piecewise C^1 curve, then the length of γ is

$$\text{Length}(\gamma) = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |\gamma'(t)|dt$$

So we have

$$|\int_\gamma g| \le \max_{z\in\gamma} |g(z)| \cdot \text{Length}(\gamma)$$

Proof.

$$\begin{aligned}
|\int_\gamma g| &= |\sum_{i=0}^{n-1} \int_i^{i+1} g(\gamma(t))\gamma'(t)dt| \\
&\le \sum_{i=0}^{n-1} \int_i^{i+1} |g(\gamma(t))\gamma'(t)|dt \\
&\le \sum_{i=0}^{n-1} \int_i^{i+1} |g(\gamma(t))||\gamma'(t)|dt \\
&\le \max_{z\in\gamma} |g(z)| \sum_{i=0}^{n-1} \int_i^{i+1} |\gamma'(t)|dt \\
&= \max_{z\in\gamma} |g(z)| \cdot \text{Length}(\gamma)
\end{aligned}$$

■

Theorem 5.1.2 — Green's Theorem. Say $\Omega \subset \mathbb{C}$ such that $\partial\Omega$ is a finite collection of piecewise C^1 closed simple curves. If $g = u + iv$ is C^1 on Ω, and $f = p + iq$ is differentiable in Ω, then ($\Re p$, q have 1^{st} order derivatives). Then

$$\int_{\partial\Omega} fdz = i \int\int_\Omega \left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) dxdy$$

Where $\partial\Omega$ is the boundary of Ω.

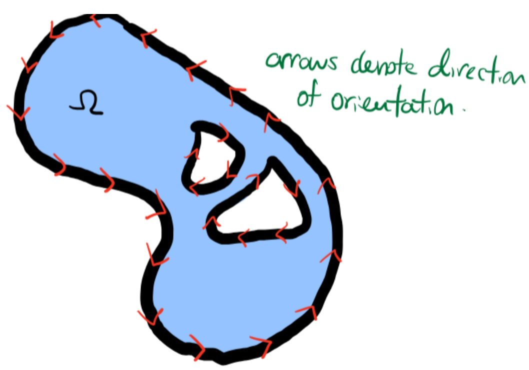


Figure 5.4: Green's Theorem

Corollary 5.1.3 Let's just show this with the real part of f. If $dz = dx + idy$, then

$$\begin{aligned}
\Re(fdz) &= \Re((p + iq)(dx + idy)) \\
&= pdx - qdy
\end{aligned}$$

If f is differentiable in Ω, then the following holds (as will be shown by the Cauchy-Riemann equations next lecture). Note: This is not the derivative.

$$\Re(i\left(\frac{\partial f}{\partial x}+\frac{i\partial f}{\partial y}\right))=\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}$$

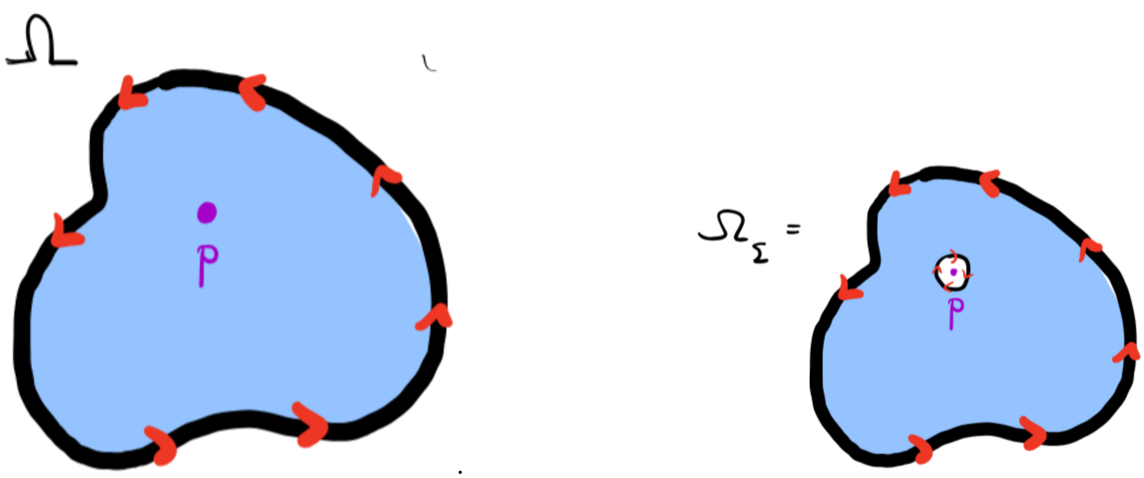
So if we're looking for an integral in the form of $\int_{\partial\Omega} pdx - qdy$, we can use Green's Theorem to convert it to an integral over Ω.

$$\int_{\partial\Omega} pdx - qdy = \int_{\Omega}\left(\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}\right)dxdy$$

R Orient $\partial\Omega$ always on the left (in the counter-clockwise direction outsides, conterclockwise insides) as we walk along $\partial\Omega$ (say $\partial\Omega$ is positively oriented).

Example 5.5 — Very Important Example. Let γ be a simple, closed piecewise C^1 curve. such that $\gamma = \partial\Omega$ for some $\Omega \subset \mathbb{C}$. Then for $p \notin \gamma$,

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-p} = \begin{cases}1 & \text{if } p \in \Omega\\ 0 & \text{if } p \notin \Omega\end{cases}$$



(a) Ω

(b) $\Omega_\varepsilon = \Omega \setminus D_\varepsilon(p)$

Proof. 1) Assume p not in Ω and suppose f is differentiable in Ω such that:

$$f = \frac{1}{z-p}$$

And $\partial\Omega$ is a simple closed curve. So by Green's Theorem:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{1}{z-p} = -\frac{1}{(z-p)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{1}{z - p} = -\frac{i}{(z - p)^2}$$

Then by the Cauchy-Riemann equations:

$$\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} = 0$$

So by Green's Theorem:

$$\int_{\partial\Omega} f dz = 0$$

2) Assume $p \in \Omega$. Green's theorem doesn't apply here, so we need to be a bit more clever.
Let $D_\varepsilon(p)$ be the disk of radius ε centered at p, essentially, we want to remove the point stopping us from applying Green's Theorem. And we make ε small. So by Green's Theorem:

$$0 = \int_{\partial\Omega_\varepsilon} f dz = \int_{\partial\Omega} f dz - \int_{\partial D_\varepsilon(p)} f dz$$

Where $\partial D_\varepsilon(p) = \partial\{|z - p| \leq \varepsilon\}$ is equal to

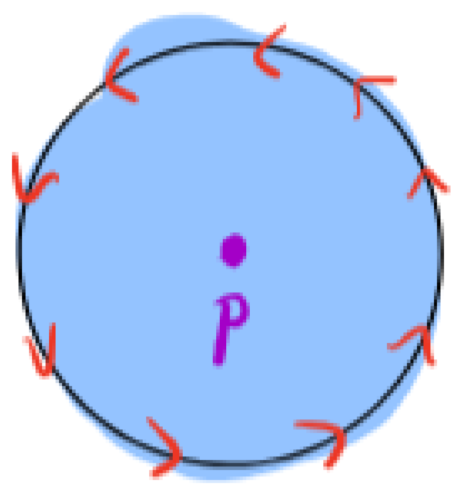


Figure 5.6: Epsilon Disk

$$\int_{\partial\Omega_\varepsilon} \frac{dz}{z - p} = 0$$

$$\int_{\partial\Omega} \frac{dz}{z - p} - \int_{\partial D_\varepsilon(p)} \frac{dz}{z - p} = 0$$

$$\int_{\partial\Omega} \frac{dz}{z - p} = \int_{\partial D_\varepsilon(p)} \frac{dz}{z - p}$$

$$\to \partial D_\varepsilon = p + \varepsilon e^{it} \quad 0 \leq t \leq 2\pi$$

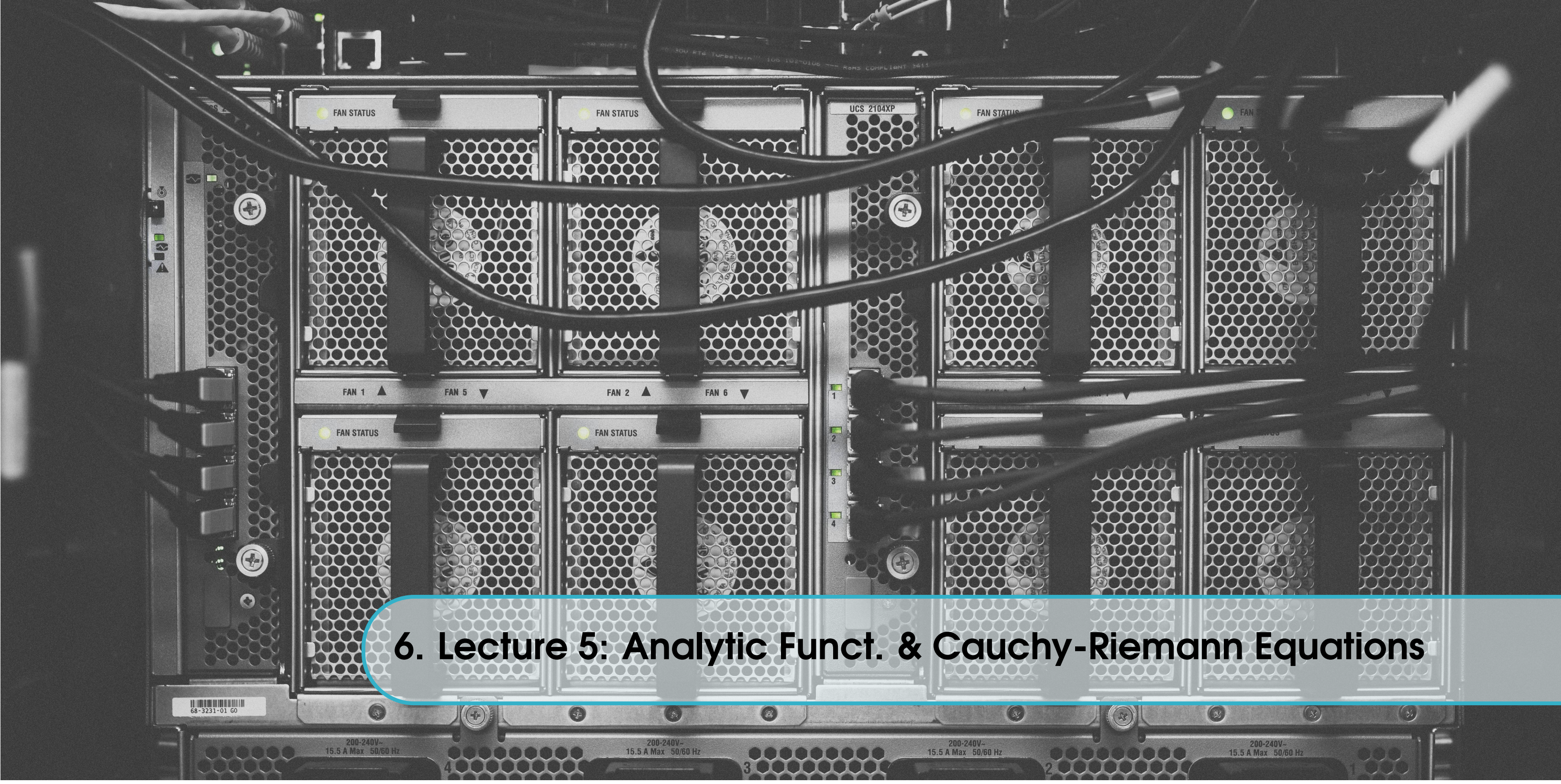
$$\int_{\partial D_\varepsilon(p)} \frac{dz}{z - p} = \int_0^{2\pi} \frac{i\varepsilon e^{it}}{\varepsilon e^{it}} dt = 2\pi i$$
$$\int_{\partial\Omega} \frac{dz}{z - p} = 2\pi i$$

■

■

Theorem 5.1.4 — Complex Partial Derivatives. The previous example gives us the following theorem: Let $f = u + iv$ be differentiable in Ω. Then f is infinitely differentiable in Ω and the partial derivatives of f are given by:

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$



6. Lecture 5: Analytic Funct. & Cauchy-Riemann Equations

6.1 Analytic Functions

Definition 6.1.1 — Complex Differentiability. A complex function $f(z) : D \to \mathbb{C}$, where D is a domain, is **complex differentiable** at $z_0 \in D$ if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists}$$

$$= \lim_{h} \frac{f(z_0 + h) - f(z_0)}{h} \quad h \in \mathbb{C}$$

Definition 6.1.2 — Analytic. A function $f(z)$ is **analytic** on a domain D if $f(z)$ is complex differentiable at every point in D.

Definition 6.1.3 — Entire. A function $f(z)$ is **entire** if $f(z)$ is analytic on $\mathbb{C}$.

■ **Example 6.1 — Prove the Power Rule.**

$$f(z) = z^n \quad n \in \mathbb{Z}$$

f is entire and

$$f'(z) = nz^{n-1}$$

■

Proof.

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{(z+h)^n - z^n}{h}$$

$$= \lim_{h \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^k - z^n}{h}$$

$$= \lim_{h \to 0} \sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^{k-1}$$

$$= \binom{n}{1} z^{n-1}$$
$$= nz^{n-1}$$

■

■ **Example 6.2** Prove that $f(z) = \bar{z}$ is not complex differentiable at any point. ■

Proof. In homework 2... ■

■ **Example 6.3 — Prove the Derivative of the Exponential Function.**

$$f(z) = e^z$$

■

Proof.

$$\begin{aligned}
\lim_{h \to 0} \frac{e^{z+h} - e^z}{h} &= \lim_{h \to 0} \frac{e^z e^h - e^z}{h} \\
&= e^z \lim_{h \to 0} \frac{e^h - 1}{h} \\
&= e^z \lim_{h \to 0} \frac{1 + h + \frac{h^2}{2} + \cdots - 1}{h} \\
&= e^z \lim_{h \to 0} 1 + \frac{h}{2} + \cdots
\end{aligned}$$

■

6.2 Cauchy-Riemann Equations

Lemma 6.2.1 — h can approach from any direction. If $f(z)$ is differentiable then

$$\exists \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f(z) \in \mathbb{C}$$

And yield the same result for any $h \in \mathbb{C}$.

Theorem 6.2.2 — Cauchy-Riemann Equations. If $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z = x + iy$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof. We compute h in two ways:

$$h_1 = is \quad s \in \mathbb{R}$$
$$h_2 = s \in \mathbb{R}$$

Property	**Description**
Linearity	The derivative of a sum is the sum of the derivatives: $$(f+g)'(z) = f'(z) + g'(z)$$ The derivative of a constant multiple is the constant multiple of the derivative: $$(cf)'(z) = cf'(z)$$
Quotient Rule	The derivative of a quotient is given by: $$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$
Chain Rule	The derivative of a composition is given by: $$(f \circ g)'(z) = f'(g(z))g'(z)$$
Exponential Function	The derivative of the exponential function is: $$\frac{d}{dz}e^z = e^z$$
Logarithmic Function	The derivative of the logarithmic function is: $$\frac{d}{dz}\log z = \frac{1}{z}$$
Power Rule	The derivative of a power function is: $$\frac{d}{dz}z^n = nz^{n-1}$$
Trigonometric Functions	The derivatives of the trigonometric functions are: $$\frac{d}{dz}\sin z = \cos z$$ $$\frac{d}{dz}\cos z = -\sin z$$
Hyperbolic Functions	The derivatives of the hyperbolic functions are: $$\frac{d}{dz}\sinh z = \cosh z$$ $$\frac{d}{dz}\cosh z = \sinh z$$

Table 6.1: Properties of Complex Derivatives

$$
\begin{aligned}
&\lim_{h\to 0} \frac{f(z+is)-f(z)}{is} \\
=&lim_{h\to 0}\frac{u(x,y+s)+iv(x,y+s)-u(x,y)-iv(x,y)}{is} \\
=&lim_{h\to 0}\frac{u(x,y+s)-u(x,y)}{is}+\frac{v(x,y+s)-v(x,y)}{s} \\
=&\frac{1}{i}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y})
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{h\to 0} \frac{f(z+s)-f(z)}{s} \\
=&lim_{h\to 0}\frac{u(x+s,y)+iv(x+s,y)-u(x,y)-iv(x,y)}{s} \\
=&lim_{h\to 0}\frac{u(x+s,y)-u(x,y)}{s}+i\frac{v(x+s,y)-v(x,y)}{s} \\
=&\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} &= \frac{1}{i}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}) \\
\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}
$$

■

Theorem 6.2.3 — Harmonic Functions. If $f(z)=u(x,y)+iv(x,y)$ is complex differentiable, then

$$\Delta u=\Delta v=0$$

And u,v are **harmonic functions** and satisfy Cauchy-Riemann equations. Thus they are **harmonic conjugates**. Where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplacian operator.

Proof. Cauchy-Riemann equations give us the partial derivatives of u,v.

$$
\begin{cases}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\end{cases}
$$

$$\text{Take} \quad \frac{\partial}{\partial x}(1) \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}$$

$$\text{Take} \quad \frac{\partial}{\partial y}(2) \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial v}{\partial y} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial x^2}$$

$$\Delta u = 0$$

■

Corollary 6.2.4 If a function $f(z)$ is once complex differentiable, then it is infinitely differentiable and analytic.

Theorem 6.2.5 Let $f = u + iv$ and assume $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are defined and continuous on a disc around z_0. If u, v satisfy the Cauchy-Riemann equations at z_0, then f is complex differentiable at z_0.

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Proof. Using the taylor expansion of $f(z)$

$$\begin{aligned}
\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} &= \lim_{h\to 0} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\
&= \lim_{h\to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i\frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h} \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}
\end{aligned}$$

■

■ **Example 6.4 — Prove the Derivative of the Logarithmic Function.** Let $D \in \mathbb{C}$ be a domain o which there is a single-valued branch of $\log z$. ■

Proof. When $\arctan(y/x) \in (\theta_0, \theta + \pi]$ and $\arctan(y/x)$ is not in D.

$$u = \frac{1}{2}\log(x^2 + y^2) \qquad v = \arctan(y/x)$$

Then

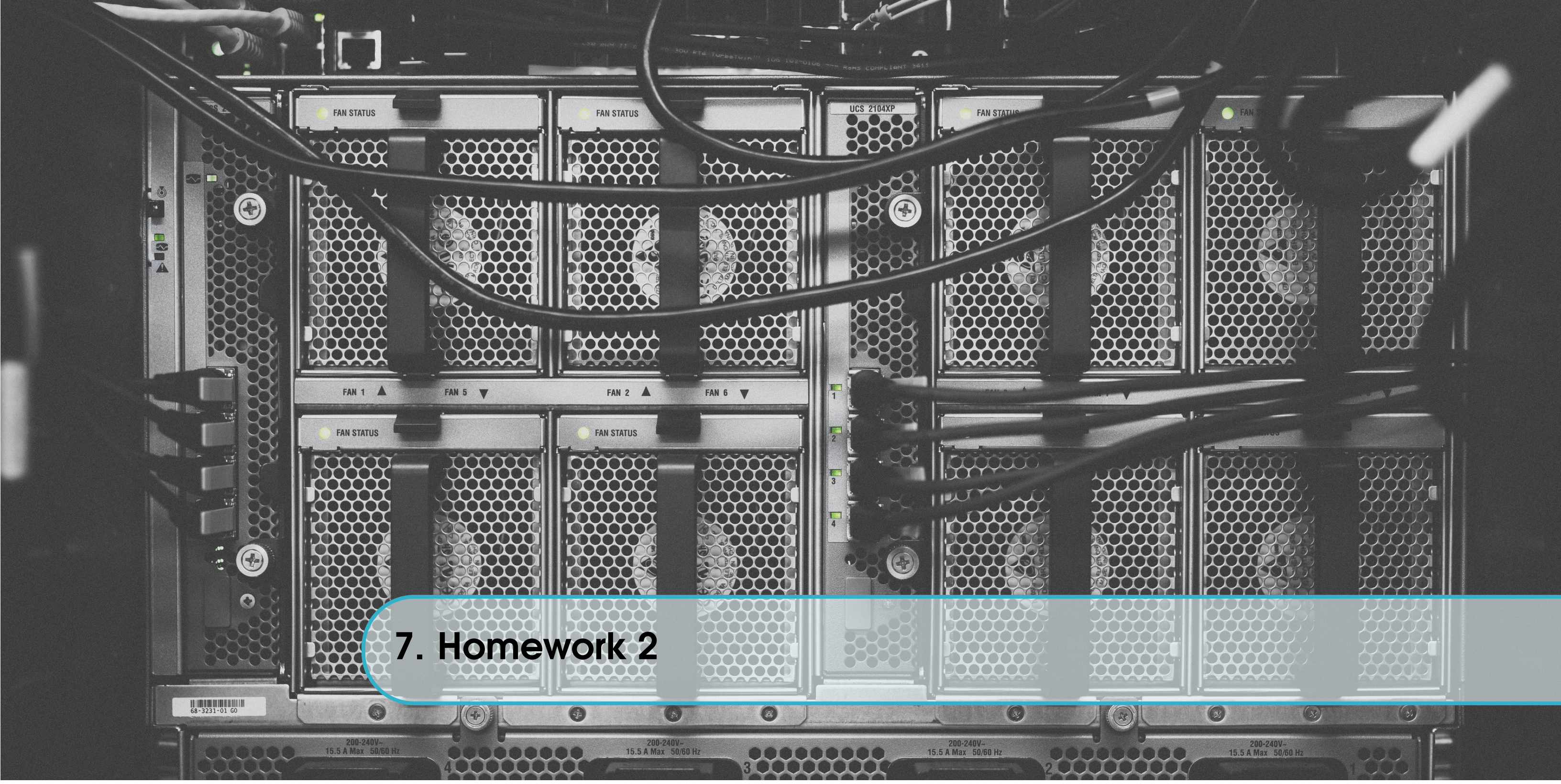
$$\frac{\partial u}{\partial x} = \frac{1}{2(x^2 + y^2)} \cdot 2x = \frac{x}{x^2 + y^2} \tag{6.1}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1 + \frac{y}{x}^2} \times \frac{1}{x} \tag{6.2}$$

$$= \frac{x}{x^2 + y^2} \tag{6.3}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{6.4}$$

INCOMPLETE ■



7. Homework 2

■ **Example 7.1 — Fisher, Section 1.6, Problem 2.** Compute the following line integral:

$$\int_\gamma e^z dz$$

where γ is the line segment from 0 to z_0.

We want some path that approaches z_0 so we parametrize γ as $\gamma(t) = z_0 t$ for $t \in [0, 1]$.

$$\begin{aligned}
\int_\gamma e^z dz &= \int_0^1 e^{z_0 t} z_0 dt \\
&= z_0 \int_0^1 e^{z_0 t} dt \\
&= z_0 \left[\frac{e^{z_0 t}}{z_0} \right]_0^1 \\
&= z_0 \left[\frac{e^{z_0} - e^0}{z_0} \right] \\
&= e^{z_0} - 1
\end{aligned}$$

■

■ **Example 7.2 — Fisher, Section 1.6, Problem 4.** Compute the following line integral:

$$\int_\gamma \frac{1}{z + 4} dz$$

where γ is the circle of radius 1 centered at -4, oriented counterclockwise.

We first recognize that $e^{i\theta} = \cos\theta + i\sin\theta$ represents a point on a unit circle. So we can parametrize γ as $\gamma(t) = -4 + e^{it}$ for $t \in [0, 2\pi]$.

$$\begin{aligned}
\int_\gamma \frac{1}{z + 4} dz &= \int_0^{2\pi} \frac{1}{(-4 + e^{it}) + 4} ie^{it} dt \\
&= \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} dt \\
&= \int_0^{2\pi} i dt \\
&= 2\pi i
\end{aligned}$$

■

■ **Example 7.3 — Fisher, Section 1.6, Problem 10.** Let $f = u + iv$ be a continuous functions and $\gamma(t) = x(t) + iy(t)$ be a piecewise C^1 curve. Show that

$$\text{Re}\left(\int_\gamma f(z)dz\right) = \int_\gamma udx - vdy$$

and,

$$\text{Im}\left(\int_\gamma f(z)dz\right) = \int_\gamma vdx + udy$$

where $dx = x'(t)dt$ and $dy = y'(t)dt$.

We know that $f(z) = u + iv$ and $dz = dx + idy$. So we can write the integral as:

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_\gamma (u + iv)(dx + idy) \\
&= \int_\gamma udx + i \int_\gamma vdx + i \int_\gamma udy - \int_\gamma vdy
\end{aligned}$$

Taking the real part of the integral, we get:

$$\text{Re}\left(\int_\gamma f(z)dz\right) = \int_\gamma udx - \int_\gamma vdy$$

Taking the imaginary part of the integral, we get:

$$\text{Im}\left(\int_\gamma f(z)dz\right) = \int_\gamma vdx + \int_\gamma udy$$

■

■ **Example 7.4 — Fisher, Section 1.6, Problem 16.** Let γ be a piecewise C^1, simple closed curve. Let z_0 be a point which does not lie on γ. Show that

$$\int_\gamma \frac{dz}{(z - z_0)^m} = 0 \quad \text{for } m = 2, 3, 4, \ldots$$

Let's see if we can apply Cauchy's Integral Theorem, which says that if f is analytic on a simply connected domain D and γ is a simple closed curve in D, then $\int_\gamma f(z)dz \;=\; 0$.
Say $\exists D | z_0 \;\notin\; D$ and D is simply connected. Say also that γ is a simple closed curve in D. Then we can write $f(z) \;=\; \frac{1}{(z-z_0)^m}$ for $m \;=\; 2,\, 3,\, 4,\, \ldots$. Then $f(z)$ is analytic on D and γ is a simple closed curve in D. So by Cauchy's Integral Theorem, $\int_\gamma f(z)dz \;=\; 0 \quad \forall m \;=\; 2,\, 3,\, 4,\, \ldots$. ■

■ **Example 7.5 — Fisher, Section 2.1, Problem 4.** Find the derivative of the function $f(z) \;=\; (\cos(z^2))^3$.

We can write $f(z) \;=\; (\cos(z^2))^3 \;=\; \cos^3(z^2)$. So we can apply the chain rule to get:

$$\begin{aligned} f'(z) \;&=\; 3\,\cos^2(z^2)\,\left(-\,\sin(z^2)\right)\,2z \\ &=\; -6z\,\cos^2(z^2)\,\sin(z^2) \end{aligned}$$

■

■ **Example 7.6 — Fisher, Section 2.1, Problem 6.** Find the derivative of the function $(\mathrm{Log}(z))^3$ on the plane minus the negative reals.

Say $w \;=\; \mathrm{Log}(z) \;=\; \ln|z| \,+\, i\,\arg(z) \quad -\,\pi \;<\; \arg(z) \;<\; \pi$. We can write $f(z) \;=\; (\mathrm{Log}(z))^3 \;=\; (w)^3$. So:

$$\begin{aligned} \frac{dw}{dz} \;&=\; \frac{d}{dz}(\ln|z| \,+\, i\,\arg(z)) \\ \frac{dw}{dz} \;&=\; \frac{1}{z} \end{aligned}$$

So we can apply the chain rule to get:

$$\begin{aligned} f'(z) \;&=\; 3(w)^2\frac{dw}{dz} \\ &=\; 3(\mathrm{Log}(z))^2\frac{1}{z} \end{aligned}$$

■

■ **Example 7.7 — Fisher, Section 2.1, Problem 14.** Let $P(z) \;=\; A(z \,-\, z_1)\,\cdots\,(z \,-\, z_n)$, where $A,\, z_1,\, \ldots,\, z_n$ are complex numbers. Show that

$$\frac{P'(z)}{P(z)} \;=\; \sum_{j=1}^{n} \frac{1}{z \,-\, z_j}$$

for any $z \;\neq\; z_1,\, \ldots,\, z_n$.

We can write $P(z) \;=\; A(z \,-\, z_1)\,\cdots\,(z \,-\, z_n)$. So we can apply the product rule to get:

$$
\begin{aligned}
P'(z) &= A\left(\prod_{j=1}^{n}(z-z_j)\right)' \\
&= A\sum_{j=1}^{n}\left(\prod_{k\neq j}(z-z_k)\right)
\end{aligned}
$$

Because we know $d(z-z_n) = dz$. So we can write:

$$
\begin{aligned}
\frac{P'(z)}{P(z)} &= \frac{A\ \sum_{j=1}^{n}\ \left(\prod_{k\neq j}(z-z_k)\right)}{A\ \prod_{j=1}^{n}(z-z_j)} \\
&= \sum_{j=1}^{n}\frac{\prod_{k\neq j}(z-z_k)}{\prod_{j=1}^{n}(z-z_j)} \\
&= \sum_{j=1}^{n}\frac{1}{z-z_j}
\end{aligned}
$$

■

■ **Example 7.8 — Fisher, Section 2.1, Problem 18.** Show that $f(z) = \bar{z}$ is not analytic on any domain

---

We can write $f(z) = \bar{z} = x - iy$. So we can write $u(x, y) = x$ and $v(x, y) = -y$. We can apply the Cauchy-Riemann equations to get:

$$
\begin{aligned}
u_x &= 1 = -v_y \\
u_y &= 0 = v_x
\end{aligned}
$$

So the Cauchy-Riemann equations are not satisfied. So $f(z) = \bar{z}$ is not analytic on any domain.

■

■ **Example 7.9 — Fisher, Section 2.1, Problem 20.** Let $f = u + iv$ and suppose that f is analytic. In each of the following, find v, given u:
1. $u = x^2 - y^2$
2. $u = \frac{x}{x^2+y^2}$

Firstly, we remind ourselves of the Cauchy-Riemann equations:

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}
$$

1. Say $u = x^2 - y^2$. We can apply the Cauchy-Riemann equations to get:

$$
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}
$$

$$
\begin{aligned}
&= 2x \\
\int \partial v &= 2x \int \partial y \\
v(x,\, y) &= 2xy + h(x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\
&= -(-2y) \\
&= 2y \\
\int \partial v &= 2y \int \partial x \\
v(x,\, y) &= 2xy + h(y)
\end{aligned}
$$

So $h(x) = h(y) = 0$. Therefore $v(x,\, y) = 2xy + c$. Where c is a constant in $\mathbb{C}$.

2. Say $u = \frac{x}{x^2+y^2}$. We can apply the Cauchy-Riemann equations to get:

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
&= \frac{y^2 - x^2}{(x^2 + y^2)^2} \\
\int \partial v &= \int \frac{y^2 - x^2}{(x^2 + y^2)^2} \partial y \\
v(x,\, y) &= -\frac{y}{x^2 + y^2} + h(x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\
&= \frac{2xy}{(x^2 + y^2)^2} \\
\int \partial v &= \int \frac{2xy}{(x^2 + y^2)^2} \partial x \\
v(x,\, y) &= -\frac{y}{x^2 + y^2} + h(y)
\end{aligned}
$$

So $h(x) = h(y) = 0$. Therefore $v(x,\, y) = -\frac{y}{x^2+y^2} + c$. Where c is a constant in $\mathbb{C}$.

■

■ **Example 7.10 — Fisher, Section 2.1, Problem 26.** Suppose that γ is a piecewise C^1 simple closed curve and that u is a continuous function on γ. Let D be a domain disjoint from γ, and define a function h on D by the rule

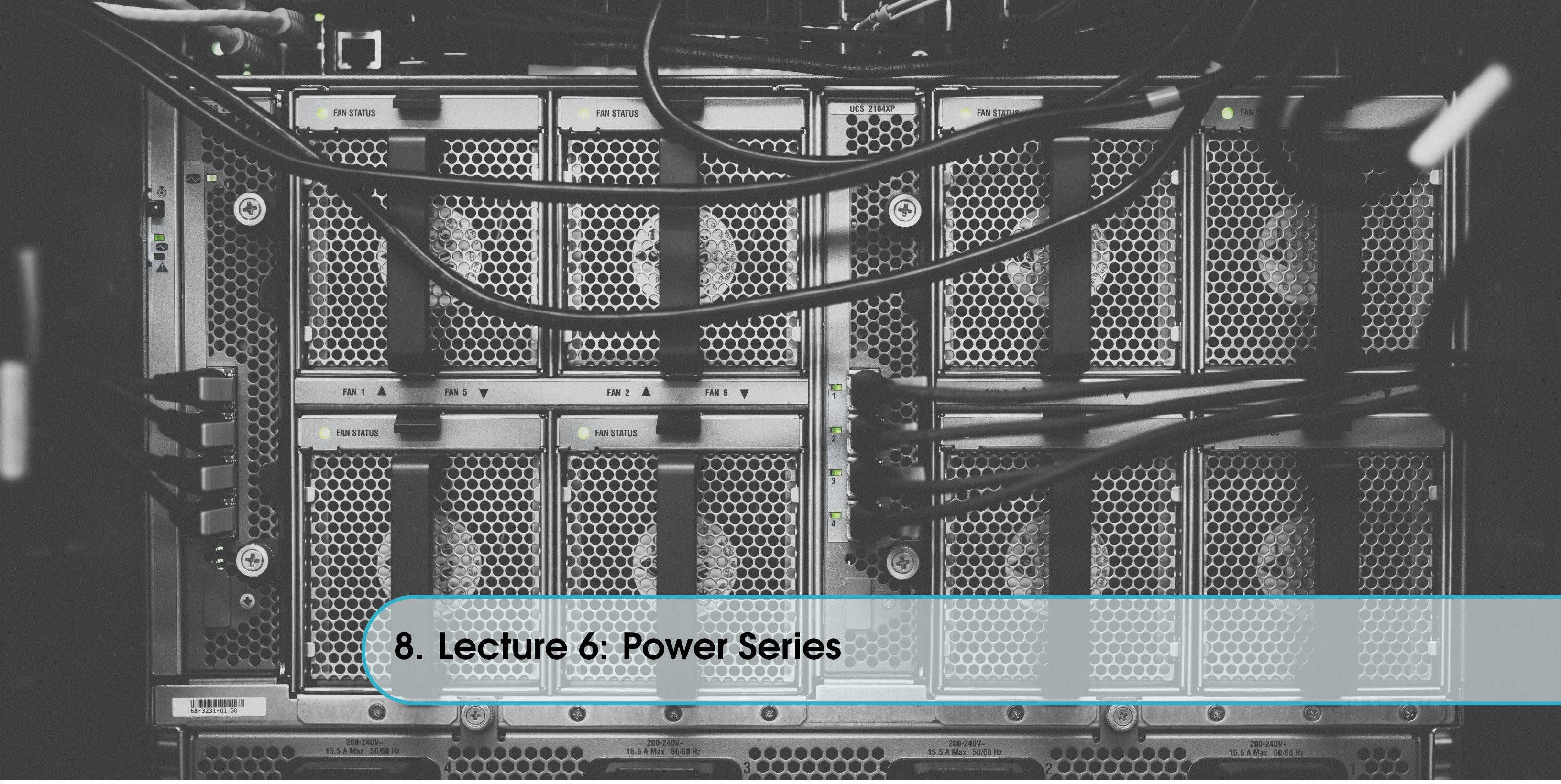
$$h(z) = \int_\gamma \frac{u(\xi)}{\xi - z} d\xi$$

Show that h is analytic in D.

We can write $h(z) = \int_\gamma \frac{u(\xi)}{\xi - z} d\xi$. We can apply the Cauchy Integral Formula to get:

$$\begin{aligned} \frac{d}{dz} h(z) &= \frac{d}{dz} \int_\gamma \frac{u(\xi)}{\xi - z} d\xi \\ &= \int_\gamma \frac{\partial}{\partial z} \left(\frac{u(\xi)}{(\xi - z)^2} \right) d\xi \\ \therefore\ h'(z) &= \int_\gamma \frac{u(\xi)}{(\xi - z)^2} d\xi \end{aligned}$$

Since the function $\frac{u(\xi)}{(\xi - z)^2}$ is continuous with respect to z in D and the curve γ is piecewise C^1, the integral defines a smooth function of z in D .Therefore, $h'(z)$ exists for all $z \in D$,and $h(z)$ is differentiable.
Moreover, the existence and continuity of $h'(z)$ imply that $h(z)$ is analytic in D ,as $h(z)$ is differentiable and its derivative is continuous in D. ∎



8. Lecture 6: Power Series

8.1 Introduction

Definition 8.1.1 — Power Series. A power series in z is a n infinite series of the form:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where a_n are complex numbers, z is a complex variable, and $z_0 \in \mathbb{C}$ is the centre.

Theorem 8.1.1 — Absolute Convergence of Power Series. Suppose $\exists z_1 \neq z_0 | \sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges. Then for each $z \in \mathbb{C} : |z - z_0| < |z_1 - z_0|$, the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges **absolutely**.

Proof. Since $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges, we know:

$$\lim_{n \to \infty} a_n (z_1 - z_0)^n = 0 \tag{8.1}$$

so: $|a_n||z_1 - z_0|^n \leq M \forall n$ as $n \to \infty$.
by 8.1, $\exists N \in \mathbb{N} | n \geq N \implies |a_n||z_1 - z_0|^n \leq 1$.
So

$$\begin{aligned} |a_n||z - z_0|^n &= \frac{|a_n||z_1 - z_0}{|z_1 - z_0|}|z - z_0|^n \\ &\leq M \end{aligned}$$

Let $\rho = \frac{|z - z_0|}{|z_1 - z_0|} < 1$. So

$$\sum_{n=0} \infty |a_n||z - z_0| \leq M \sum_{n=0}^{\infty} \rho^n$$

Since $\rho < 1$, the geometric series $\sum_{n=0}^{\infty} \rho^n$ converges. Hence so does $\sum_{n=0}^{\infty} |a_n||z - z_0|^n$.
Absolute Convergence ■

Definition 8.1.2 — Absolute Convergence. A series $\sum_{n=0}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges.

Definition 8.1.3 — Radius of Convergence. The radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is the number R such that the series converges absolutely for $|z - z_0| < R$ and diverges for $|z - z_0| > R$.
There are three possibilities:

1. Converges only for $z = z_0$ (i.e. $R = 0$)
2. Converges for all $z \in \mathbb{C}$ (i.e. $R = \infty$)
3. Converges for some $z \neq z_0$ (i.e. $0 < R < \infty$)

Lemma 8.1.2 Assume $z', z'' \in \mathbb{C}$ are points such that:

- $\sum_{n=0}^{\infty} a_n(z' - z_0)^n$ converges
- $\sum_{n=0}^{\infty} a_n(z'' - z_0)^n$ diverges

There is a unique $R > 0$ such that:

- If $|z - z_0| < R$, then $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges
- if $|z - z_0| > R$, then $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ diverges

R is called the radius of convergence.

R The behaviour o on the circle of convergence $|z - z_0| = R$ can be complicated! The series may converge or diverge, or both (depending on where on the circle you are).

8.2 Computing the Radius of Convergence

Theorem 8.2.1 Suppose $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is a power series with radius of convergence $0 < R \leq \infty$. Then:

1. $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ exists, then $R = \frac{1}{L}$

Proof. Assume

$$\exists \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$

Then:

$$\lim_{n\to\infty} \left|\frac{(z - z_0)^{n+1} a_{n+1}}{(z - z_0)^n a_n}\right| = |z - z_0| L$$

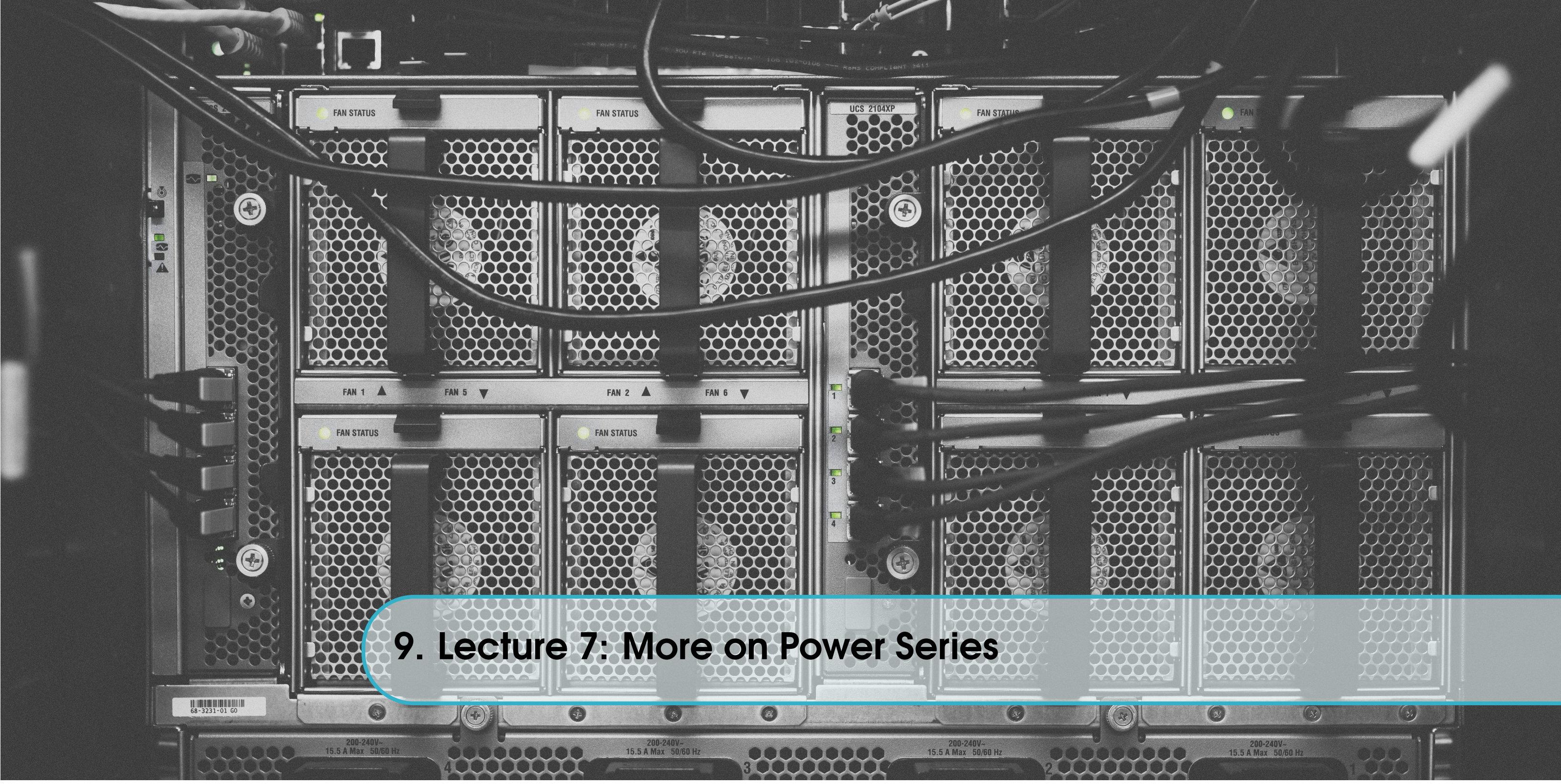
if $|z - z_0|L < \frac{1}{L}$, then $|z - z_0|L < 1$ and the series converges by the ratio test, and if $|z - z_0|L > \frac{1}{L}$, then the series diverges by the ratio test.

Similarly, if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ exists, then:

$$\lim_{n\to\infty} \sqrt[n]{|a_n(z - z_0)^n|} = |z - z_0| L$$

If $|z - z_0|L < 1 \to R = |z - z_0| < \frac{1}{L}$, then the series converges by the root test, and if $|z - z_0|L > 1$, then the series diverges by the root test. ■

R If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ does not exist, then $R = 0$



9. Lecture 7: More on Power Series

Theorem 9.0.1 — Differentiation of Power Series. Consider a power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \qquad \text{with } 0 < R \leq \infty$$

Then: $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is ***analytic*** in the disc $\{|z - z_0| < R\}$ and

$$\frac{d}{dz}f(z) = \sum_{n=1}^{\infty} na_n(z - z_0)^{n-1}$$

■ **Example 9.1 — Proof of Convergence of Power Series Differentiation.** Show that

$$\frac{d}{dz}f(z) = \sum_{n=1}^{\infty} na_n(z - z_0)^{n-1}$$

converges for $|z - z_0| < R$.

Proof. Let $|z - z_0| < R$. and let $r < s < R$. then, $\exists N | \forall n \geq N$ we have.

$$nr^{n-1} \leq S^n$$

Since $\frac{r}{s} < 1$, by the ratio test:

$$\lim_{n\to\infty} n(\frac{r}{s})^{n-1} = 0$$

Thus:

$$\begin{aligned}\sum_{n=1}^{\infty} n|a_n|r^{n-1} &\leq \sum_{n=1}^{N} n|a_n|r^{n-1} + \sum_{n=N}^{\infty} n|a_n|s^n \\ &\leq \sum_{n=1}^{N} n|a_n|r^{n-1} + \sum_{n=1}^{\infty} n|a_n|s^n\end{aligned}$$

Of course $\sum_{n=1}^{\infty} |a_n|s^n$ converges by the ratio test since $s < R$. Hence $\sum_{n=1}^{\infty} n|a_n|r^{n-1}$ converges. ■

■

Theorem 9.0.2 — Infinite Differentiation of Power Series. if $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence $0 < R \leq \infty$, then in the disc $\{|z - z_0| < R\}$, $f(z)$ is infinitely differentiable and

$$f^{(m)}(z) = \sum_{n=m}^{\infty} n(n-1)\ldots(n-(m-1))a_n(z-z_0)^{n-m} \qquad k = 1, 2, \ldots$$

9.1 Cauchy's Theorem

Definition 9.1.1 A domain D is *simply-connected* if, whenever γ is a closed curve in D, the inside of γ is also a subset of D.
Informally, a domain is simply connected if it has no holes.

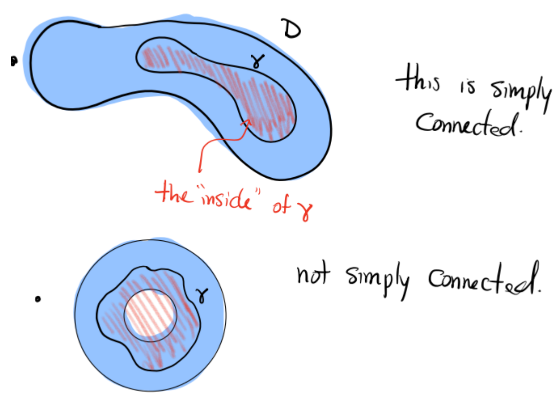


Figure 9.1: Simply Connected Domain

Theorem 9.1.1 — Cauchy's Theorem. Suppose f is analytic on a domain D, and let γ be a C^1 simple closed curve in D such that the inside of $\gamma = \Omega \subset D$. Then:

$$\oint_\gamma f(z)dz = 0$$

Proof. By Green's Theorem:

$$\oint_\gamma f(z)dz = i\iint_\Omega \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

But if $f = u + iv$ is analytic, so:

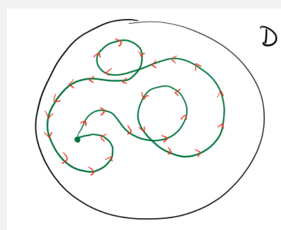
$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$
\begin{aligned}
&= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \\
&= -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \\
&= -i\frac{\partial f}{\partial y}
\end{aligned}
$$

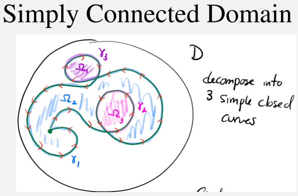
So $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$ and the result follows. ■

Theorem 9.1.2 — More General Cauchy's Theorem. if D is a simply connected domain and γ is any closed, piecewise C^1 curve in D, then, if f is analytic on D:

$$\oint_\gamma f(z)dz = 0$$



(a) Simply Connected Domain in D



(b) Decomposition of γ

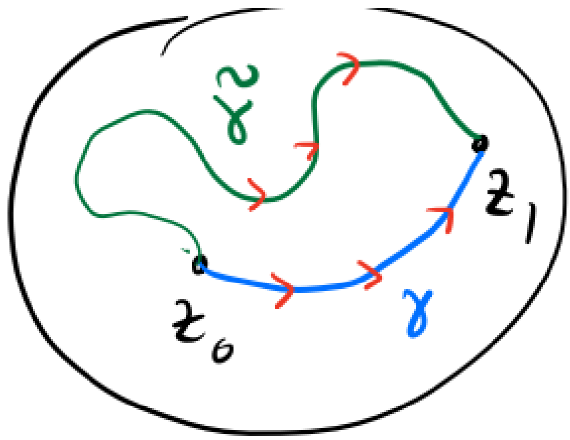
$$\oint_\gamma f(z)dz = \oint_{\gamma_1} f(z)dz + \oint_{\gamma_2} f(z)dz + \oint_{\gamma_3} f(z)dz = 0$$

Theorem 9.1.3 — Differentiability of Analytic Functions. If D is a simply connected domain and f is analytic on D, then there is an analytic function F on D such that $F' = f$.

Lemma 9.1.4 For $F(z) = \int_\gamma f(z)dz$, F is independent of the path γ.

Proof. Let's say γ_1 and γ_2 are two paths from z_0 to z_1, and $\Gamma = \gamma_1 - \gamma_2$. Then:

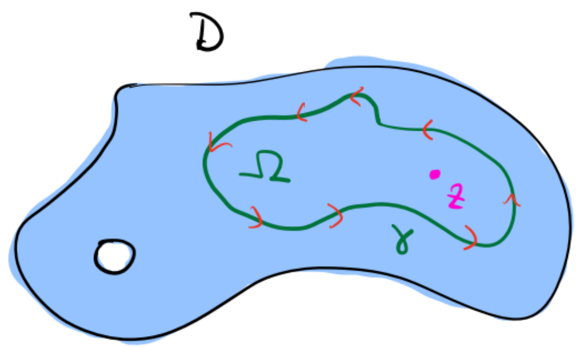
$$
\begin{aligned}
\oint_\Gamma f(z)dz &= \oint_{\gamma_1} f(z)dz - \oint_{\gamma_2} f(z)dz \\
0 &= F_1 - F_2 \\
F_1 &= F_2
\end{aligned}
$$



■

Theorem 9.1.5 — Cauchy's Integral Formula. Suppose f is analytic on a domain D, γ is piecewise C^1, positively oriented, simple closed curve in D such that: inside$(\gamma) = \Omega \subset D$. Then:

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}d\xi \qquad \forall z \in \Omega$$



Proof. Let $g(\xi) = \frac{f(\xi)}{\xi - z}$ be an analytic function in $\Omega/\{z\}$.
Let $D_\varepsilon(z) = \{z \in \mathbb{C} | |z - z_0| < \varepsilon\}$ be a disc of radius ε centered at z_0.
We choose ε to be small such that $D_\varepsilon(z) \subset \Omega$.
By Cauchy's Theorem:

$$\oint_\gamma \frac{f(\xi)}{\xi - z}d\xi = \oint_{\partial D_\varepsilon(z)} \frac{f(\xi)}{\xi - z}d\xi$$

Note: $\partial D_\varepsilon(z)$ is the boundary of $D_\varepsilon(z)$.
Now we parametrize $\partial D_\varepsilon(z)$ by $\partial D_\varepsilon = z + \varepsilon e^{i\theta}, 0 \leq \theta \leq 2\pi$, Then:

$$\begin{aligned}\oint_{\partial D_\varepsilon(z)} \frac{f(\xi)}{\xi - z}d\xi &= \int_0^{2\pi} \frac{f(z + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\varepsilon i e^{i\theta} d\theta \\ &= i\int_0^{2\pi} f(z + \varepsilon e^{i\theta}) d\theta\end{aligned}$$

as $\varepsilon \to 0$, $f(z + \varepsilon e^{i\theta}) \to f(z)$ (by continuity of f).
Hence:

$$\lim_{\varepsilon \to 0} i \int_0^{2\pi} f(z + \varepsilon e^{i\theta}) d\theta = 2\pi i f(z)$$

Thus

$$\boxed{\oint_\gamma \frac{f(\xi)}{\xi - z} d\xi = 2\pi i f(z)}$$

■

9.2 Applications of Cauchy's Integral Formula

■ **Example 9.2** Compute:

$$\int_0^{2\pi} \frac{d\theta}{2 + \sin\theta}$$

Idea: Write this as an integral for an analytic function over the circle $z = e^{i\theta} \quad 0 \le \theta \le 2\pi$.
If $|z| = 1$, then

$$\begin{aligned} \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \\ &= \frac{z - z^{-1}}{2i} \end{aligned}$$

$$\therefore\ d\theta = \frac{dz}{iz}$$

So, integral is:

$$\begin{aligned} \int_{\gamma(\theta)=e^{i\theta}} \frac{1}{2 + \frac{z - z^{-1}}{2i}} \frac{dz}{iz} &= \int_\gamma \frac{2dz}{4iz + (z^2 - 1)} \\ \to z^2 - 4iz - 1 &= \left[z - \left(\frac{-4i + \sqrt{-16 + 4}}{2}\right)\right]\left[z - \left(\frac{-4i - \sqrt{-16 + 4}}{2}\right)\right] \\ &= (z - i(\sqrt{3} - 2))(z + (\sqrt{3} + 2)i) \end{aligned}$$

Since $|\sqrt{3} - 2| < 1$, $|\sqrt{3} + 2| > 1$, we can apply the cauchy integral formula:

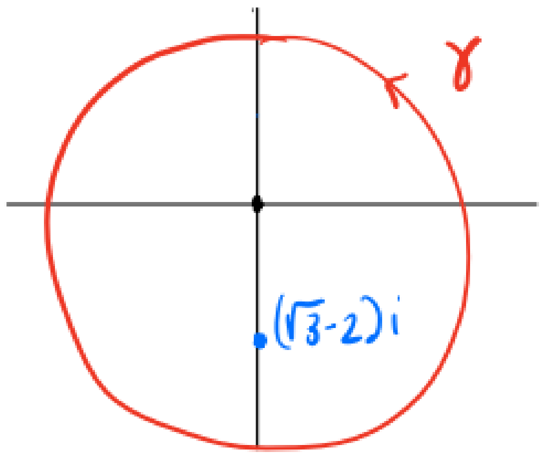
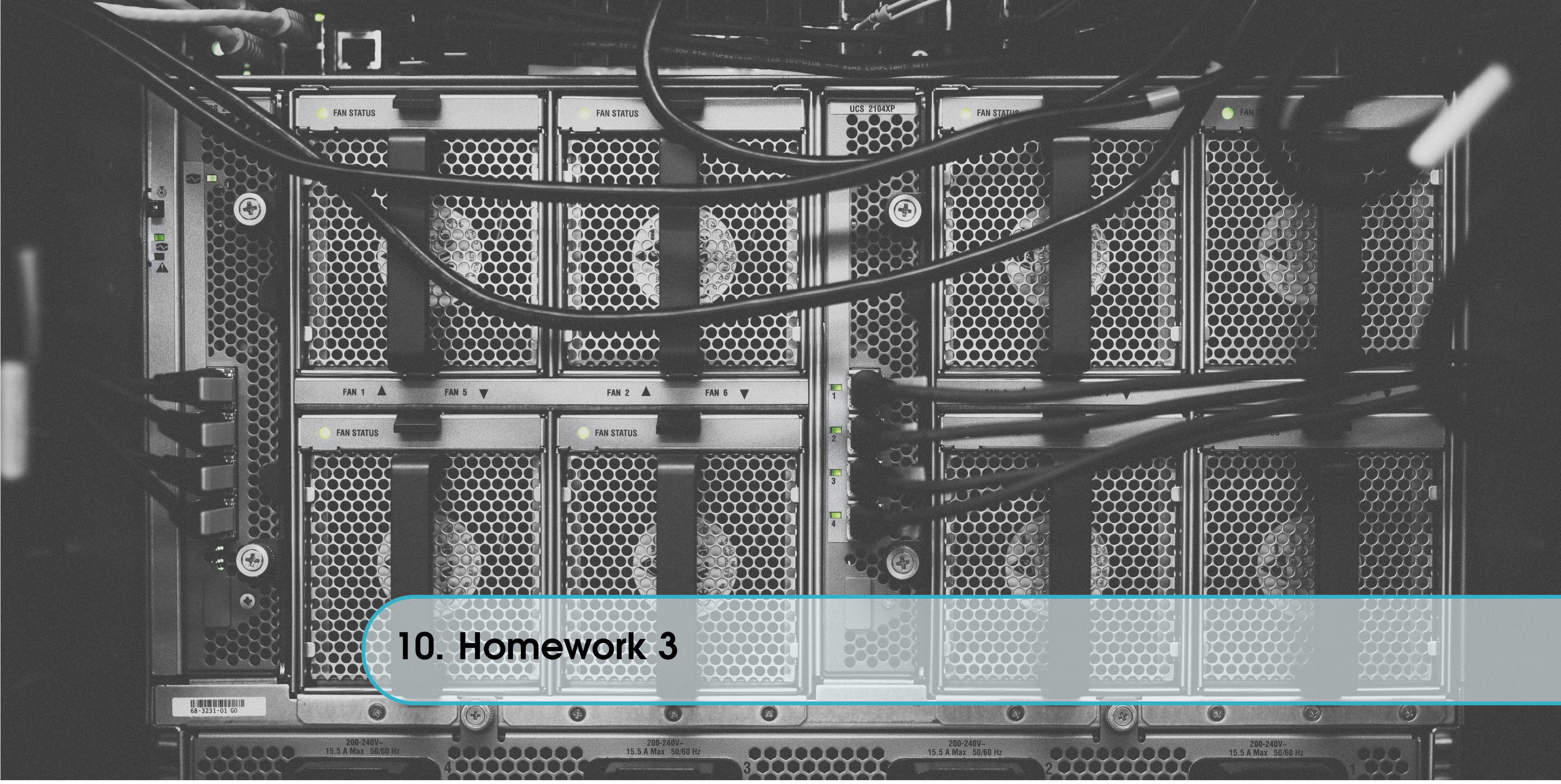


Figure 9.3: Circle of Radius 1

$$\begin{aligned}
\int_\gamma \frac{1}{(z - i(\sqrt{3} - 2))} \cdot \frac{2dz}{(z + i(\sqrt{3} + 2))} &= 2\pi i \left(\frac{2}{(\sqrt{(3)} - 2)i + (\sqrt{3} + 2)i} \right) \\
&= \frac{4\pi}{2\sqrt{3}} \\
&= \frac{2\pi}{\sqrt{3}}
\end{aligned}$$

■



10. Homework 3

■ **Example 10.1 — Fisher, Section 2.2, Problem 2.** Find the radius of convergence for the series

$$\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}(z - 2)^k$$

Let $a_k = \frac{(k!)^2}{(2k)!}$, then

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(k+1)!^2}{(2k+2)!}\cdot\frac{(2k)!}{(k!)^2}\right| \\
&= \lim_{k\to\infty}\left|\frac{(k+1)^2}{(2k+2)(2k+1)}\right| \\
&= \lim_{k\to\infty}\left|\frac{k^2+2k+1}{4k^2+6k+2}\right| \\
&= \lim_{k\to\infty}\left|\frac{1+2/k+1/k^2}{4+6/k+2/k^2}\right| \\
&= \left|\frac{1}{4}\right| = \frac{1}{4}
\end{aligned}$$

Therefore, the radius of convergence is $R = \frac{1}{1/4} = 4$.

■

■ **Example 10.2 — Fisher, Section 2.2, Problem 4.** Find the radius of convergence for the series

$$\sum_{k=0}^{\infty}(-1)^k z^{2k}$$

We can relate this to a geometric series with $a = 1$ and $r = -z^2$. The limit of the ratio of consecutive terms is

$$L = \frac{1}{1 - (-z^2)} = \frac{1}{1 + z^2}$$

Provided that $|r| = |-z^2| < 1$. Therefore, the radius of convergence is $R = 1$.

■

■ **Example 10.3 — Fisher, Section 2.2, Problem 22.** (a) If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence $R > 0$,and if $f(z) = 0$ for all z such that $|z - z_0| < r \leq R$, show that all the coefficients are zero (ie. $a_0 = a_1 = a_2 = \cdots = 0$).
(b) if $F(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and $G(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$ are convergent and equal on some disc $|z - z_0| < r$,show that
$a_n = b_n$ for all n.

(a) if $f(z)$ is analytic in a domain D, $z_0 \in D$ and $\{|z - z_0| < R\} \subseteq D$, then $f(z)$ hasa convergent power series expansion about z_0 given by:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{10.1}$$

Where:

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz \tag{10.2}$$

for any simple, closed, positively oriented curve γ in D containing z_0 and $\gamma = |z - z_0| = R$.
This means that functions that are equal on a disc of radius r must have the same power series expansion. Furthermore, $a_n = 0$ being a valid power series expansion for $f(z)$ means it's the only power series expansion for $f(z)$.
(b) If $F(z) = G(z)$ on a disc of radius r, using the previous theorem, $F(z)$ and $G(z)$ must have the same power series expansion. Therefore, $a_n = b_n$ for all n. ■

■ **Example 10.4** (Fisher, Section 2.3, Problem 2) Evaluate the following integral:

$$\int_{|z|=2} \frac{e^z}{z(z - 3)} dz$$

We can use Cauchy's Integral Formula to evaluate this integral. Let $f(z) = \frac{e^z}{z-3}$ and $z_0 = 0$. Then, the integral evaluates to:

$$\int_{|z|=2} \frac{e^z}{z(z - 3)} dz = 2\pi i f(0) = 2\pi i \frac{e^0}{-3} = -\frac{2\pi i}{3}$$

■

■ **Example 10.5 — Fisher, Section 2.3, Problem 4.** Evaluate the following integral:

$$\int_{|z|=1} \frac{\sin(z)}{z} dz$$

We can use Cauchy's Integral Formula to evaluate this integral. Let $f(z) = \sin(z)$ and $z_0 = 0$. Then, the integral evaluates to:

$$\int_{|z|=1} \frac{\sin(z)}{z} dz = 2\pi i f(0) = 2\pi i \sin(0) = 0$$

■

■ **Example 10.6 — Fisher, Section 2.3, Problem 8.** Evaluate the following definite trigonometric integral. (Hint: it may be useful to review the technique of Examples 6 and 7 of Section 2.3).

$$\int_0^{\pi} \frac{1}{1 + \sin^2 \theta} d\theta$$

We know that:

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

So we can parametrize using the relation, $z = e^{i\theta}$:

$$\begin{aligned}
\sin^2 \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \cdot \frac{e^{i\theta} - e^{-i\theta}}{2i} \\
&= \frac{(e^{i\theta})^2 - 2 + (e^{i\theta})^-2}{-4} \\
&= \frac{z^2 - 2 + z^{-2}}{-4}
\end{aligned}$$

And the differential becomes:

$$d\theta = \frac{dz}{iz}$$

So the integral becomes:

$$\begin{aligned}
\int_0^{\pi} \frac{1}{1 + \sin^2 \theta} d\theta &= \int_{|z|=1} \frac{1}{1 + \frac{z^2 - 2 + z^{-2}}{-4}} \frac{dz}{iz} \\
&= \int_{|z|=1} \frac{-4iz}{z^4 - 6z^2 + 1} dz
\end{aligned}$$

We can find the roots using the quadratic formula ($z^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 1$, $b = -6$, $c = 1$)

$$\begin{aligned}
z^2 &= \frac{-6 \pm \sqrt{36 - 4}}{2} = \frac{-6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2} \\
z &= \pm\sqrt{3 \pm 2\sqrt{2}}
\end{aligned}$$

Now let's rewrite the integral in terms of z:

$$\int_{|z|=1} \frac{-4iz}{z^4 - 6z^2 + 1}dz =$$
$$\int_{|z|=1} \frac{-4iz}{(z - \sqrt{3 + 2\sqrt{2}})(z + \sqrt{3 + 2\sqrt{2}})(z - \sqrt{3 - 2\sqrt{2}})(z + \sqrt{3 - 2\sqrt{2}})}dz$$

We choose $|(z - \sqrt{3 - 2\sqrt{2}})| < 1$ as the contour of integration and we can write:

$$f(z) = \frac{-4iz}{(z + \sqrt{3 - 2\sqrt{2}})(z - \sqrt{3 + 2\sqrt{2}})(z + \sqrt{3 + 2\sqrt{2}})}$$

So the integral evaluates to:

$$\int_{|z|=1} \frac{-4iz}{z^4 - 6z^2 + 1}dz = \int_{|z|=1} \frac{\frac{-4iz}{(z+\sqrt{3-2\sqrt{2}})(z-\sqrt{3+2\sqrt{2}})(z+\sqrt{3+2\sqrt{2}})}}{(z - \sqrt{3 - 2\sqrt{2}})}dz$$

Applying Cauchy's Integral Formula, we get:

$$\begin{aligned}
\int_{|z|=1} \frac{-4iz}{z^4 - 6z^2 + 1}dz &= 2\pi i f(\sqrt{3 - 2\sqrt{2}}) \\
&= 2\pi i \frac{-4i\sqrt{3 - 2\sqrt{2}}}{2\sqrt{3 - 2\sqrt{2}}(\sqrt{3 - 2\sqrt{2}} - \sqrt{3 + 2\sqrt{2}})(\sqrt{3 - 2\sqrt{2}} + \sqrt{3 + 2\sqrt{2}})} \\
&= 2\pi i(\frac{i}{2\sqrt{2}}) \\
&= -\frac{\pi}{\sqrt{2}}
\end{aligned}$$

■

■ **Example 10.7 — Fisher, Section 2.3, Problem 10.** Evaluate the following integral.(Hint: It may be useful to review the technique of Example 10 of Section 2.3)

$$\int_\gamma (z + z^{-1})dz$$

Where γ is any curve contained in the region $\{\mathrm{Im}(z) > 0\}$ which joints $-4 + i$ to $6 + 2i$.

We know that the derivative of $F(z) = \frac{z^2}{2} + \ln(z)$ is $f(z) = z + z^{-1}$. But this is valid only where $\ln(z)$ is analytic, which is the case for $\{\mathrm{Im}(z) > 0\}$. So the integral evaluates to:

$$\begin{aligned}
\int_\gamma (z + z^{-1})dz = \int_\gamma f(z)dz = \int_\gamma F'(z)dz \\
&= F(\text{end}) - F(\text{start}) \\
&= F(6 + 2i) - F(-4 + i) \\
&= \frac{(6 + 2i)^2}{2} + \ln(6 + 2i) - \frac{(-4 + i)^2}{2} - \ln(-4 + i) \\
&= 8.5 + 16i + \ln(6 + 2i) - \ln(-4 + i)
\end{aligned}$$

■

■ **Example 10.8 — Fisher, Section 2.3, Problem 14.** (a)(5 points) Suppose f is analytic on the disc $|z - z_0| < R$.Show that for any $r < R$ we have

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt$$

(b) (5 points) Using part (a) and the triangle inequality, conclude that

$$|f(z_0)| \leq \max_{0\leq t\leq 2\pi} |f(z_0 + re^{it})|$$

(c)(5 points) Conclude from (b) that $|f|$ cannot have a strict local maximum within the domain of analyticity of f

(a) The Cauchy Integral Formula states that for any $r < R$:

$$f(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z - z_0}dz$$

We can parametrize the integral using:

$$\begin{aligned} z &= z_0 + re^{it} \quad \text{where} \quad 0 \leq t \leq 2\pi \\ dz &= ire^{it}dt \end{aligned}$$

So the integral becomes:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt \end{aligned}$$

As required. (b) using what we found in part (a), we can write:

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt$$

Thus:

$$\begin{aligned} \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt\right| &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|dt \\ \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|dt &\leq \frac{1}{2\pi}\int_0^{2\pi} \max_{0\leq t\leq 2\pi} |f(z_0 + re^{it})|dt \end{aligned}$$

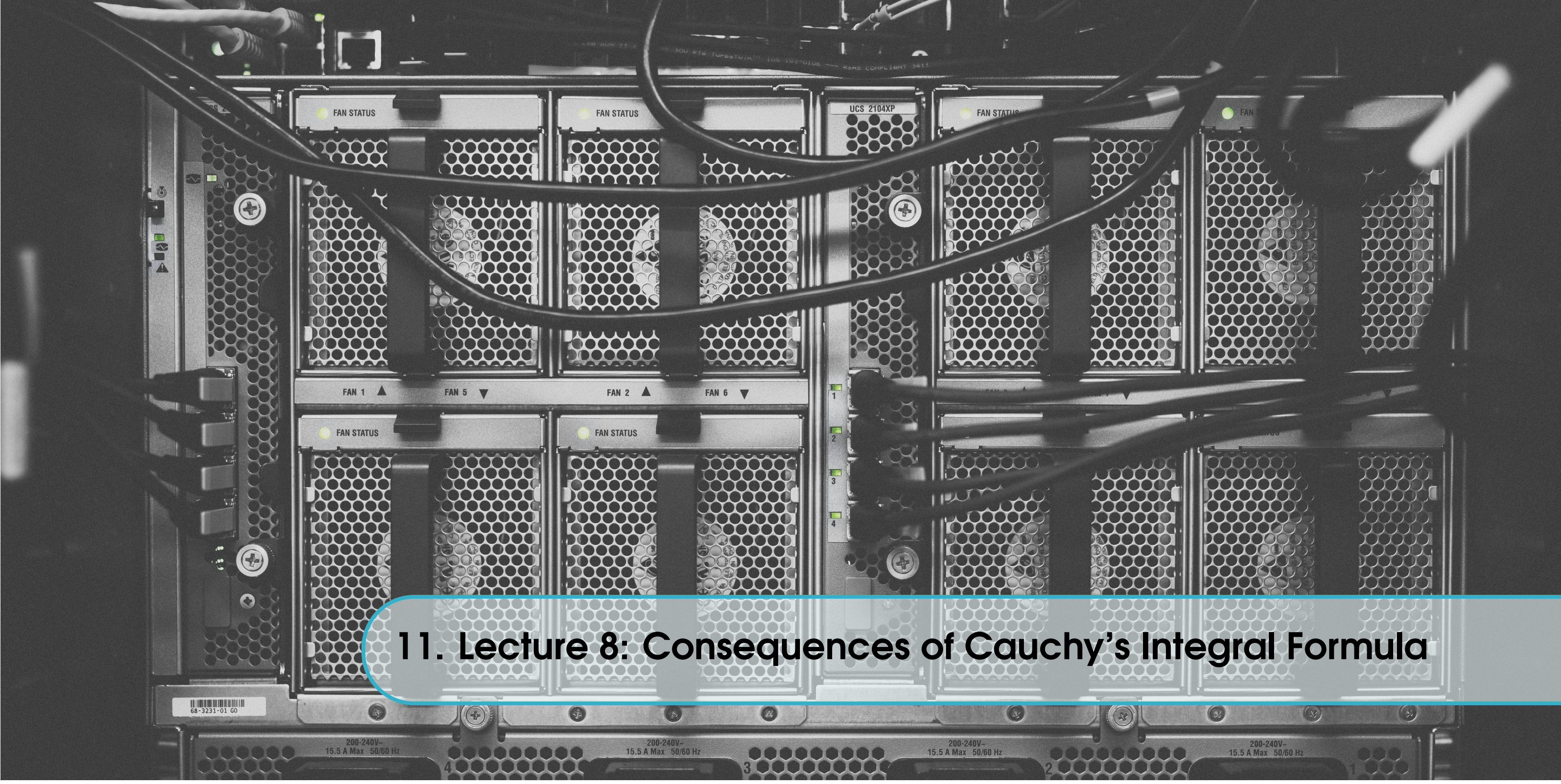
Since the maximum is constant, we can take it out of the integral

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \max_{0\leq t\leq 2\pi} |f(z_0 + re^{it})|dt &= \max_{0\leq t\leq 2\pi} |f(z_0 + re^{it})|\frac{1}{2\pi}\int_0^{2\pi} dt \\ &= \max_{0\leq t\leq 2\pi} |f(z_0 + re^{it})| \end{aligned}$$

Therefore I can write:

$$|f(z_0)| \leq \max_{0\leq t\leq 2\pi} |f(z_0 + re^{it})|$$

(c) If $|f|$ has a strict local maximum at z_0, then $|f(z_0)| > |f(z_0 + re^{it})|$ for some $r < R$. But this contradicts the inequality we found in part (b). Therefore, $|f|$ cannot have a strict local maximum within the domain of analyticity of f. ■



11. Lecture 8: Consequences of Cauchy's Integral Formula

Theorem 11.0.1 if $f(z)$ is analytic in a domain D, $z_0 \in D$ and $\{|z - z_0| < R\} \subseteq D$, then $f(z)$ hasa convergent power series expansion about z_0 given by:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{11.1}$$

Where:

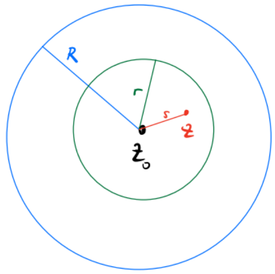
$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz \tag{11.2}$$

for any simple, closed, positively oriented curve γ in D containing z_0 and $\gamma = |z - z_0| = R$.

Proof. Let $\Delta = \{|z - z_0| < R\}$. if $z \in \Delta$, $|z - z_0| = s < R$, then by Cauchy's Integral Formula:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{11.3}$$

Where $\gamma = \{|z - z_0| = r\}$, positively oriented, $r > s$.



Now we write:

$$\begin{aligned}\xi - z &= \xi - z_0 - (z - z_0)\\ &= (\xi - z_0)\left(1 - \frac{z - z_0}{\xi - z_0}\right)\end{aligned}$$

Since $\frac{z-z_0}{\xi-z_0} = \frac{s}{r} < 1$, we can write (for $\xi \in \gamma$):

$$\frac{1}{1 - \frac{z-z_0}{\xi-z_0}} = \sum_{k=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^k \qquad \text{Geometric Series} \tag{11.4}$$

So, *for any $N \in \mathbb{N}$ we have*:

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z_0}\sum_{k=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^k d\xi\\ &= \sum_{k=0}^{N}(z - z_0)^k\left(\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z_0)^{k+1}}d\xi\right)\\ &+ \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z_0}\left[\sum_{k=N+1}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^k\right] d\xi\end{aligned}$$

Since $\sum_{k=0}^{\infty}\frac{|z-z_0|^k}{|\xi-z_0|^k}$ converges, if $\varepsilon > 0$
We can choose L such that $\forall N > L$ we have:

$$\sum_{k=N+1}^{\infty}\frac{|z - z_0|^k}{|\xi - z_0|^k} < \varepsilon \tag{11.5}$$

Since f analytic, there is a constant M such that $\max |f(\xi)| \leq M$ for $\xi \in \gamma$.
By definition: $|\xi - z_0| = r$ on γ, thus, by the triangle inequality:

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z_0}\left[\sum_{k=N+1}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^k\right] d\xi \geq \frac{\text{length}(\gamma)}{r}M\varepsilon = 2\pi M\varepsilon$$

Thus, $\forall\varepsilon > 0$, $\exists L|\forall N > L$ we have:

$$\left|f(z) - \sum_{k=0}^{N} a_k(z - z_0)^k\right| < 2\pi\varepsilon \tag{11.6}$$

Where $a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi-z_0)^{k+1}}d\xi$. So: $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$. as desired.

■

Corollary 11.0.2 If $f(z)$ is analytic in a domain D, then so is $f'(z)$. In particular, if f is analytic, then it is infinitely differentiable.

Proof. Every f has a power series expansion about z_0, and every term in the series is analytic. Therefore, the series converges to an analytic function. ■

Corollary 11.0.3 — Unique Analytic Continuation. If $f(z)$ is analytic in a domain D, and $f(z) = 0$ for all $z \in \Delta \subseteq D$, then $f(z) = 0$ for all $z \in D$. In the above setting we have:

$$\frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z_0)^{k+1}}d\xi \tag{11.7}$$

In particular, if f is analytic in a domain D, and, for some $z_0 \in D$, $f^{(k)}(z_0) = 0$ for all $k \in \mathbb{N}$, then $f(z) = 0 \forall z \in D$.

11.1 Comparison with Real Functions

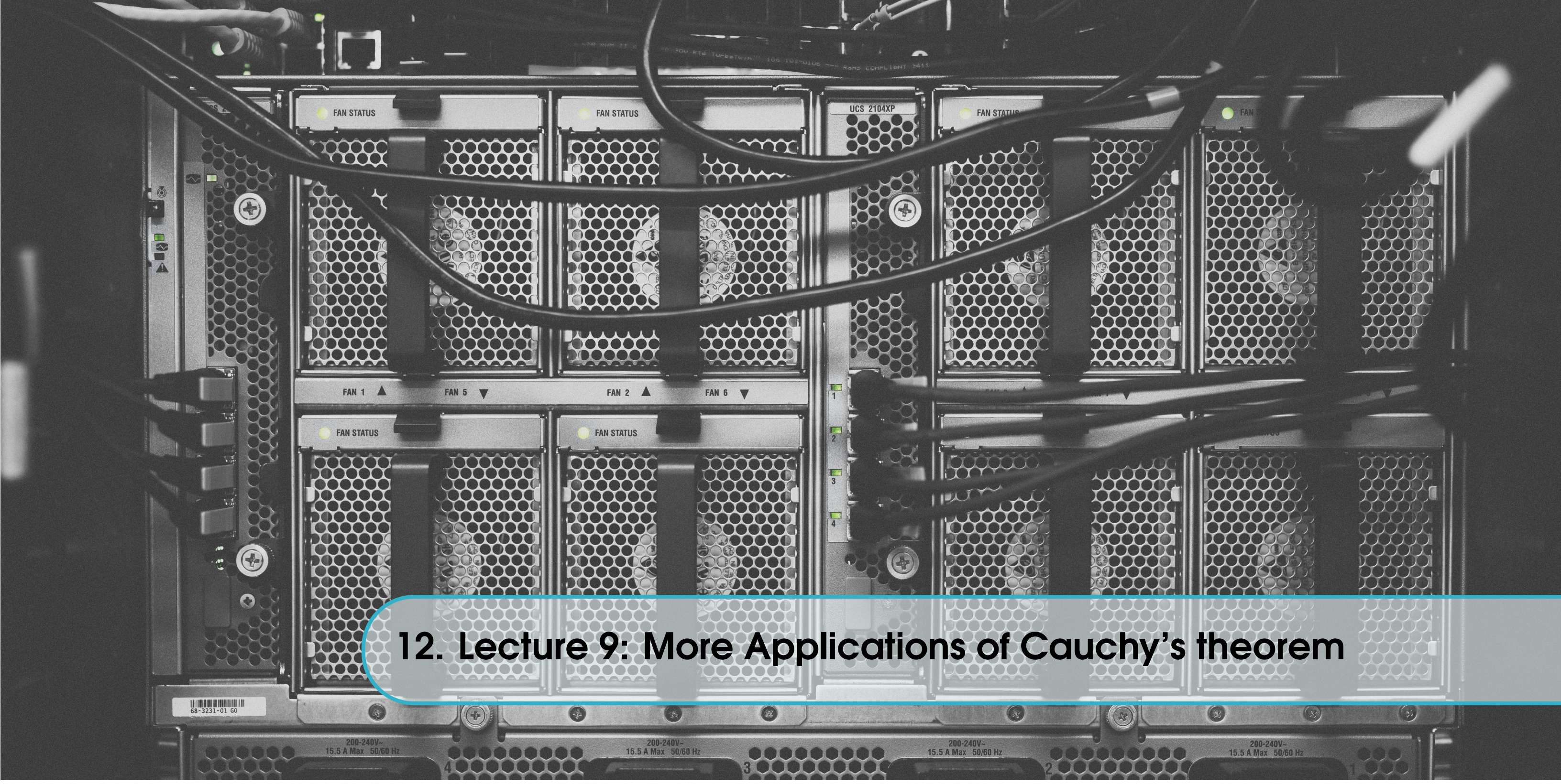
■ **Example 11.1** To get a sense for the difference between real and complex functions, consider the following **real function**:

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{11.8}$$

f is infinitely differentiable, and $\forall k \in \mathbb{N}$, $f^{(k)}(0) = 0$.
So, the Taylor series of f at $0 \in \mathbb{R}$ is:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = 0 \tag{11.9}$$

Thus, f is infinitely differentiable, but ***not*** equal to a power series in any neighborhood of $0 \in \mathbb{R}$. ■



12. Lecture 9: More Applications of Cauchy's theorem

12.1 The order of a zero of a function

Lemma 12.1.1 — Order of a zero of a function. Suppose f is analytic in a disc D, f is not identically zero, and $f(z_0) = 0$. Then for some z_0 in D, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Let $m \geq 1$ be the smallest $n \in \mathbb{Z}_{\geq 0}$ such that $a_n \neq 0$. That is:

$$f(z) = a_m (z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots$$

Say: f has a zero of order m at z_0. Equivalently:

$$f(z) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

Yet $f^{(m)}(z_0) \neq 0$.
Then $g(z) = \frac{f(z)}{(z-z_0)^m}$ is analytic in D and $g(z_0) \neq 0$. Since

$$g(z) = a_m + a_{m+1}(z - z_0) + \cdots$$

Converges because:

$$(z - z_0)^m \left[a_{m+1} + a_{m+2}(z - z_0) + \cdots\right] = f(z) - a_m (z - z_0)^m$$

Converges in D.

Proof. This proof will demonstrate the conclusion of equation **??**.
Let $f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots$
Notice that $(z - z_0) = 0$ at $z = z_0$.
Thus:

$$f'(z) = m a_m (z - z_0)^{m-1} + (m + 1) a_{m+1} (z - z_0)^m + \cdots = 0$$

$$f''(z) = m(m-1)a_m(z-z_0)^{m-2} + (m+1)ma_{m+1}(z-z_0)^{m-1} + \cdots = 0$$
$$\vdots$$
$$f^{(m-1)}(z) = m!a_m(z-z_0) + m!a_{m+1}(z-z_0)^2 + \cdots = 0$$
$$f^{(m)}(z) = m!a_m(z-z_0)^0 + m!a_{m+1}(z-z_0)^1 + \cdots \neq 0$$

■

12.2 A partial converse to the Cauchy integral formula

Theorem 12.2.1 — Morera's Theorem. if f is continuous in a domain D and

$$\int_\gamma f(z)\,dz = 0$$

for every triangle γ where $\gamma \in D$, and inside$(\gamma) \subseteq D$, f is analytic in D.

Proof.

$$\Omega = \{|z-z_0| < r\} \quad r > 0 \quad r \text{ is small}$$

such that $\Omega \in D$

for $z \in \Omega$ we define

$$F(z) = \int_\gamma f(\zeta)\,d\zeta,$$

where the integral is taken along a radial curve

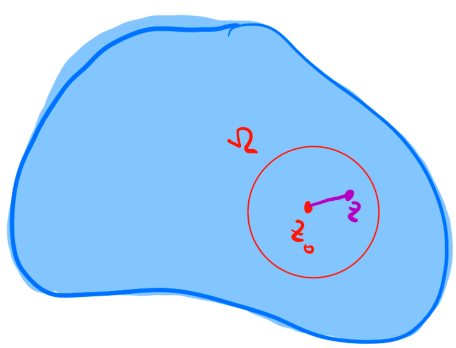


Figure 12.1: Radial curve

<u>Goal:</u> F is analytic and $F'(z) = f(z)$ (then it follows that f is also analytic).

$$F(z+h) - F(z) = \int_z^{z+h} f(\zeta)\,d\zeta$$

Since

$$\int_{z_0}^{z} f(\zeta)\,d\zeta + \int_z^{z+h} f(\zeta)\,d\zeta - \int_{z_0}^{z+h} f(\zeta)\,d\zeta = 0$$

by assumption

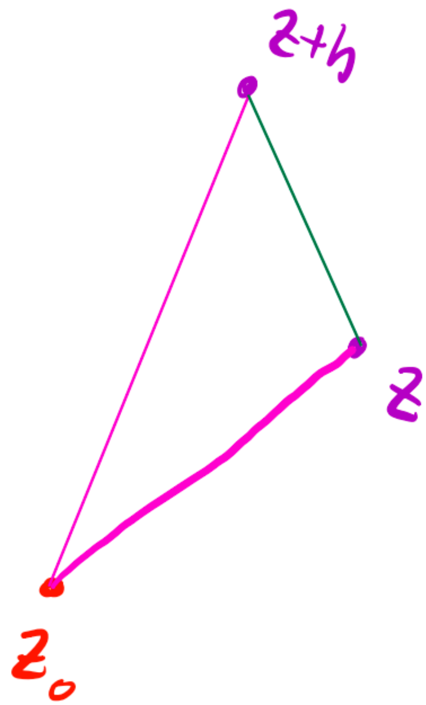


Figure 12.2: Curve

Thus:

$$\frac{F(z+h)-F(z)}{h}-f(z)=\frac{1}{h}\int_z^{z+h} f(\zeta)-f(z)d\zeta$$

$\rightarrow$ This is because

$$\begin{aligned}&\int_z^{z+h} f(z)d\zeta\\ =&f(z)\int_z^{z+h} d\zeta\\ =&f(z)\cdot h\end{aligned}$$

By continuity of f, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(w)-f(z)| < \varepsilon$ if $|w-z| < \delta$. Then if $|h| < \delta$, we have:

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| \leq \frac{1}{|h|}\varepsilon|h| = \varepsilon$$

Since ε is arbitrary, we have:

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=f(z)$$

Thus: F is analytic and $F'(z)=f(z)$, as desired. ∎

12.3 Applications

Theorem 12.3.1 — Liouville's Theorem. If F is entire and $|F(z)| \leq M$ then F is constant.

Proof.

$$
\begin{aligned}
g(z) &= \frac{F(z) - F(0)}{z} \quad \text{is entire since} \\
F(z) &= F(0) + \sum_{n=1}^{\infty} a_n z^n \\
\text{so } g(z) &= \sum_{n=1}^{\infty} a_n z^{n-1} = \sum_{n=0}^{\infty} a_{n+1} z^n
\end{aligned}
$$

Now

$$
|g(Re^{i\theta})| \leq \frac{|F(Re^{i\theta})| + |F(0)|}{R} \leq \frac{2M}{R}
$$

Using Cauchy's theorem, we have:

$$
\begin{aligned}
g(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{g(z_0 + Re^{i\theta})}{Re^{i\theta}} iRe^{i\theta} \, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} g(Re^{i\theta}) \, d\theta
\end{aligned}
$$

if $R >> |z_0|$, then $|z_0 + Re^{i\theta}| \geq R - |z_0|$

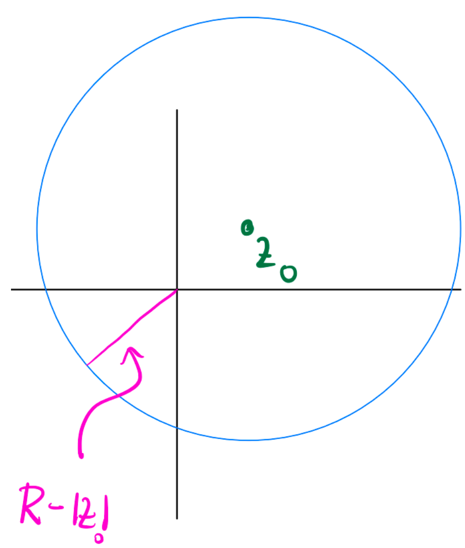


Figure 12.3: Circle

so

$$
|g(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |g(z_0 + Re^{i\theta})| \, d\theta \leq \frac{2M}{R - |z_0|}
$$

But $R >> |z_0|$ is arbitrary, so taking $R \to \infty$ yields $g(z_0) = 0$ for all z_0 and $F(z_0) = F(0)$. ■

12.4 Analytic Logarithms

Lemma 12.4.1 — The logarithmic derivative. Let D be a simply connected domain.
Suppose f is analytic in D and $f \neq 0$ anywhere in D.
Then $\frac{f'}{f}$ is analytic in D and hence so let's define:

$$h'(z) = \frac{f'(z)}{f(z)}$$

Using the fundamental theorem of calculus, we have:

$$h(z) = \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta$$

Cauchy's theorem implies that $h(z)$ is analytic in D.
Morera's theorem ensures that the integral can be taken over any path from z_0 to z in D and thus, $h(z)$ is path independent.
So:

$$\begin{aligned}\left[e^{-h(z)} f(z)\right]' &= -e^{-h(z)} h'(z) f(z) + e^{-h(z)} f'(z) \\ &= -e^{-h(z)} \frac{f'(z)}{f(z)} f(z) + e^{-h(z)} f'(z) \\ &= 0\end{aligned}$$

So $e^{-h(z)} f(z) = f(z_0)$ is a constant. Or:

$$\begin{aligned} e^{-h(z)} f(z) &= f(z_0) \\ f(z) &= f(z_0) e^{h(z)}\end{aligned}$$

Thus $g(z) = h(z) + \mathrm{Log}(f(z_0))$ satisfies:

$$\begin{cases} e^{g(z)} = f(z) \\ g(z) \quad \text{is analytic in } D \end{cases}$$

12.5 Isolated Singularities

Definition 12.5.1 — Isolated Singularities. An analytic function has an isolated singularity at z_0 if it is analytic in a punctured disc $\{0 < |z - z_0| < r\}$ for some $r > 0$.

■ **Example 12.1** $f(z) = \frac{z^2 - z_0^2}{z - z_0}$
In this case $|f(z)|$ is bounded as $z \to z_0$.
in fact, $f(z) = z + z_0$ $(z \neq z_0)$ and f can be extended to an analytic function at z_0.
Thus: z_0 is a **removable** singularity. ■

■ **Example 12.2** $f(z) = \frac{1}{(z-z_0)^4}$

$|f(z)| = \frac{1}{|z-z_0|^4} \to +\infty$ as $z \to z_0$.
This is an example of a pole. ■

Example 12.3 $f(z) = e^{\frac{1}{z-z_0}}$

$z_0 = 0$ for simplicity.
$|f(z)| = e^{\frac{1}{2z}+\frac{1}{2\bar{z}}} = e^{\frac{x}{x^2+y^2}}$

1. If $y = 0$ and $x \to 0$ from $x > 0$, then $|f(z)| \to +\infty$.
2. If $y = 0$ and $x \to 0$ from $x < 0$, then $|f(z)| \to 0$.
3. If $x = 0$ and $y \to 0$ then $|f(z)| \to 1$, this is an **essential** singularity.

■

12.6 Removable Singularities

Example 12.4 — Removable Singularity. Suppose $|f(z)|$ is bounded near z_0.
Let

$$g(z) = \begin{cases} (z-z_0)^2 f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

Then, $g(z)$ is analytic on $\{|z-z_0| < r\}$ for some $r > 0$.

Since

$$\frac{g(z)-g(z_0)}{z-z_0} = (z-z_0)f(z)$$

and

$$\lim_{z \to z_0} (z-z_0)f(z) = 0$$

so $g'(z_0) = 0$ and thus:

$$g(z) = a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$

So:

$$f(z) = \frac{g(z)}{(z-z_0)^2} = a_2 + a_3(z-z_0) + \cdots$$

If we set $f(z_0) = a_2$, then f is analytic on $\{|z-z_0| < r\}$ and, we have **removed** the singularity.
■

Definition 12.6.1 z_0 is a removable singularity of f is bounded in a neighborhood of z_0.

12.7 Poles

R Recall: if f is analytic on $\{0 < |z-z_0| < R\}$ and $\lim_{z \to z_0} f(z) = \infty$, then z_0 is a pole of f.

Lemma 12.7.1 — Poles. Choose $r < R$ small enough that $|f(z)| > 1$ on $\{0 < |z - z_0| < r\}$.
Then: $g(z) = \frac{1}{f(z)}$ is analytic on $\{0 < |z - z_0| < r\}$ and $g(z) \to 0$ as $z \to z_0$.
Thus, g has a removable singularity and $g(z_0) = 0$.
z_0 is a zero of order $m \geq 1$

$$g(z) = (z - z_0)^m h(z)$$

where $h(z)$ is analytic and $h(z_0) \neq 0$ on $\{|z - z_0| < r\}$.
Then

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z - z_0)^m} \cdot \frac{1}{h(z)} = \frac{H(z)}{(z - z_0)^m}$$

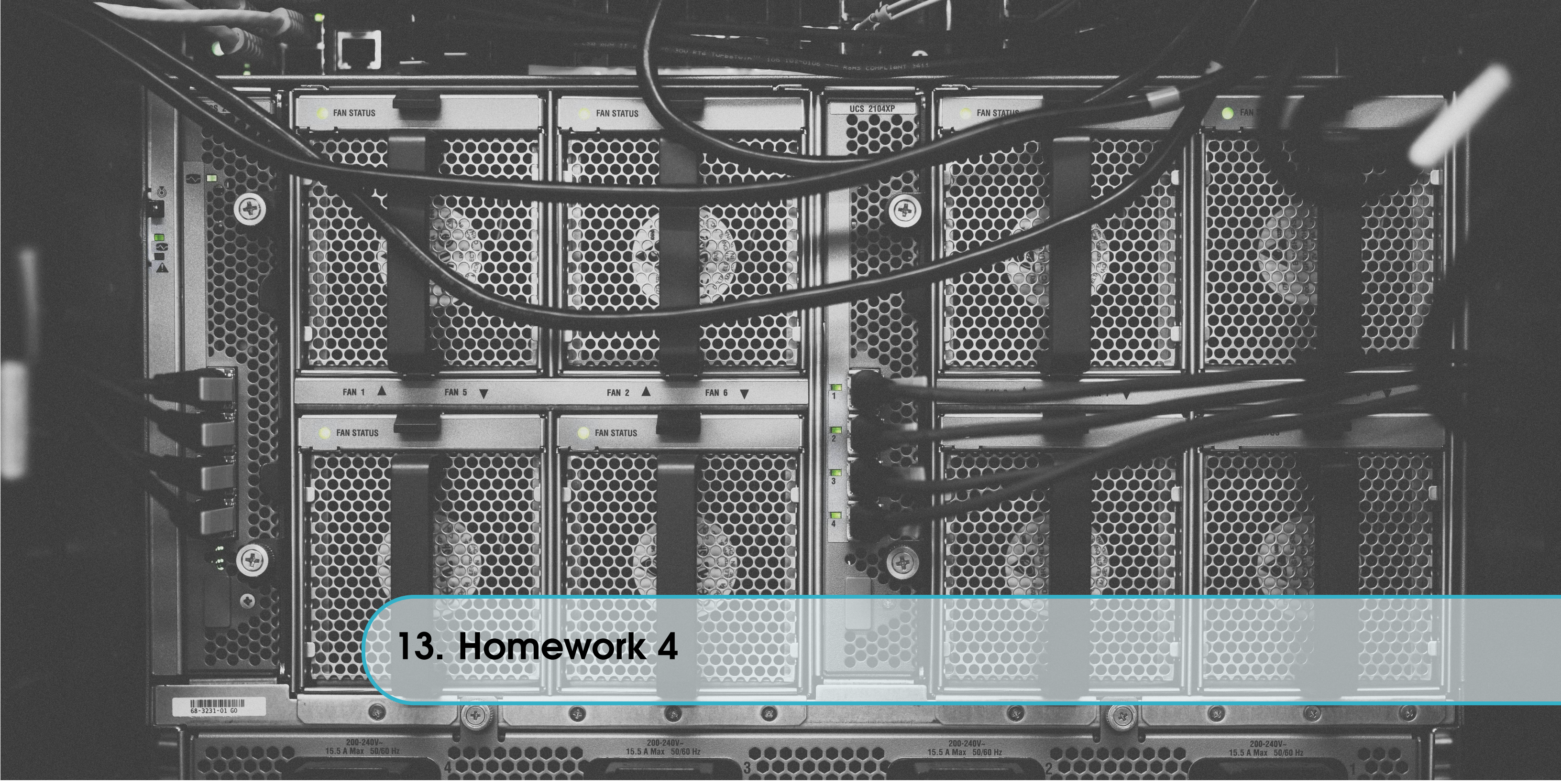
Where $H(z)$ is analytic on $\{|z - z_0| < r\}$ and $H(z_0) \neq 0$.

Definition 12.7.1 if $f(z) = \frac{H(z)}{(z-z_0)^m}$, where $H(z)$ is analytic on $\{|z - z_0| < r\}$ and $H(z_0) \neq 0$, then we say $f(z)$ has a pole of order m at z_0.

12.8 Essential Singularities

R Recall $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $z = 0$. The behavior near essential singularities is wild.

Proposition 12.8.1 if f is analytic on $\{0 < |z - z_0| < r\}$, with an essential singularity at z_0, then for any $w \in \mathbb{C}$, and any $\delta > 0$, $\exists z_\delta$ such that $0 < |z - z_0| < r$ and $|f(z) - w| < \delta$.



13. Homework 4

■ **Example 13.1 — Fisher, Section 2.4, Problem 10.** find the power series expansion of $f(z) = e^z$ about the point $z_0 = \pi i$. What is the largest disc on which this series is valid?

We know that a taylor series is given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{13.1}$$

for $f(z) = e^z$, we know that $f^{(n)}(z) = e^z$ for all $n \in \mathbb{N}$. Therefore, the taylor series expansion of $f(z)$ about $z_0 = \pi i$ is

$$e^z = \sum_{n=0}^{\infty} \frac{e^{\pi i}}{n!}(z - \pi i)^n \tag{13.2}$$

The largest disk on which this series is valid can be found by using the ratio test. The ratio test states that a series converges if the limit of the ratio of the $(n + 1)$th term to the nth term is less than 1. That is

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \tag{13.3}$$

for the series $\sum a_n$. In this case, we have

$$\lim_{n\to\infty} \left| \frac{e^{\pi i}}{(n + 1)!}(z - \pi i)^{n+1} \cdot \frac{n!}{e^{\pi i}}(z - \pi i)^n \right| \tag{13.4}$$

Simplifying, we get

$$\lim_{n\to\infty}\left|\frac{z - \pi i}{n + 1}\right| = 0 \tag{13.5}$$

Therefore, $R = \frac{1}{L} = \infty$. This means that the series converges for all $z \in \mathbb{C}$. ■

■ **Example 13.2 — Fisher, Section 2.4, Problem 12.** Find the power series expansion of $f(z) = \frac{z^2}{1-z}$ about $z_0 = 0$. What is the largest disc on which this series is valid?

We know that $\frac{1}{1-z}$ has a taylor series expansion about $z_0 = 0$ given by

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n$$

So, the taylor series expansion of $f(z)$ about $z_0 = 0$ is

$$\begin{aligned} f(z) &= z^2 \sum_{n=0}^{\infty} z^n \\ &= \sum_{n=0}^{\infty} z^{n+2} && = \sum_{n=2}^{\infty} z^n \end{aligned}$$

The radius of convergence of this series is inherited from the radius of convergence of the series for $\frac{1}{1-z}$, which is $R = 1$. ■

■ **Example 13.3 — Fisher, Section 2.4, Problem 18.** Using the (consequence of) the Cauchy Integral Formula, prove the Cauchy estimates:

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{r^n} \max_{\{|z-z_0|=r\}} |f(z)|, \quad n = 0, 1, 2, \ldots,$$

whenever f is analytic on a domain containing the set $\{|z - z_0| < r\}$.

If $f(z)$ is analytic on a domain containing the set $\{|z - z_0| < r\}$, then there exists a power series expansion of $f(z)$ about z_0 given by

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{13.6}$$

We know that the nth derivative of $f(z)$ is given by

$$\begin{aligned} f^{(n)}(z) &= \sum_{k=n}^{\infty} a_k \frac{k!}{(k - n)!}(z - z_0)^{k-n} \\ f^{(n)}(z_0) &= a_n n! \end{aligned}$$

The coefficient a_n are given by the following line integral where γ is a simple, closed, positively oriented curve in $\{|z - z_0| < r\}$:

$$\begin{aligned}
a_n &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}dz \\
a_n n! &= \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}dz \\
&\le \frac{n!}{2\pi}\max_{\{|z-z_0|=r\}}\left|\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}dz\right| \\
&\le \frac{n!}{2\pi}2\pi r\max_{\{|z-z_0|=r\}}|\frac{f(z)}{r^{n+1}}| \\
&\le n!\max_{\{|z-z_0|=r\}}\frac{|f(z)|}{r^n} \\
|f^{(n)}(z_0)| &\le \frac{n!}{r^n}\max_{\{|z-z_0|=r\}}|f(z)|
\end{aligned}$$

As required. ■

■ **Example 13.4 — Fisher, Section 2.4, Problem 20.** Suppose that $f(z)$ is an entire function and $\Re f(z) \le c$ for all z. Show that f is constant. (Hint: Consider $e^{f(z)}$.)

if $f(z)$ is an entire function, and $\Re f(z) \le c$ for all z, then:

$$\begin{aligned}
\Re f(z) &\le c \\
e^{\Re f(z)} &\le e^c \\
g(z) = |e^{f(z)}| &\le e^c
\end{aligned}$$

So by Liouville's theorem, $g(z) = C$ is constant and by extension:

$$\begin{aligned}
e^{f(z)} &= C \\
f(z) &= \log C
\end{aligned}$$

$f(z)$ is also constant. ■

■ **Example 13.5 — Fisher, Section 2.4, Problem 21.** Suppose that $f(z)$ is an entire function and that there are constants $A, R_0 > 0$ and $m \in \mathbb{Z}_{>0}$ so that

$$|f(z)| \le A|z|^m \quad \text{for all } |z| \ge R_0.$$

Show that f is a polynomial of degree at most m. (Hint: Use Problem 3, above, for $n > m$, and let $r \to \infty$.)

The cauchy estimates state that for an entire function $f(z)$ with a power series expansion about z_0 given by

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

we have, for $z_0 = 0$:

$$
\begin{aligned}
|f^{(n)}(z_0)| &\leq \frac{n!}{r^n} \max_{\{|z-z_0|=r\}} |f(z)| \\
|f^{(n)}(z_0)| &\leq \frac{n!}{r^n} \max_{\{|z|=r\}} A|z|^m \\
|f^{(n)}(z_0)| &\leq n!Ar^{m-n}
\end{aligned}
$$

If $f(z)$ is an entire function, then it must be defined for $r \to \infty$.

$$
\lim_{r\to\infty} |f^{(n)}(z_0)| \leq \lim_{r\to\infty} n!Ar^{m-n} = \begin{cases} 0 & \text{if } n > m \\ n!A & \text{if } n = m \\ \infty & \text{if } n < m \end{cases}
$$

Because the limit of the n^{th} derivative of $f(z)$ as $r \to \infty$ is 0 for $n > m$, this implies that the power series expansion of $f(z)$ is a polynomial of degree at most m. ■

■ **Example 13.6 — Fisher, Section 2.4, Problem 22.** Let D be a simply connected domain and f an analytic function on D that has no zeroes in D. Let $\gamma \neq 0$ be a non-zero complex number. Show that there is an analytic function g on D such that $f = g^{\gamma}$.

$$
\begin{aligned}
f(z) &= g^{\gamma} \\
\log f(z) &= \gamma \log g \\
\frac{\log f(z)}{\gamma} &= \log g \\
g(z) &= e^{\frac{\log f(z)}{\gamma}}
\end{aligned}
$$

Because $f(z)$ has no zeroes in D, we can define a branch of the logarithm, $\log f(z)$, such that it is analytic in D. And since $\gamma \neq 0$, $\frac{\log f(z)}{\gamma}$ is also analytic in D. Therefore, $g(z)$ is analytic in D. ■

■ **Example 13.7 — Fisher, Section 2.4, Problem 23.** Locate the isolated singularities of $f(z) = \frac{z^2}{\sin(z)}$. Identify each singularity as a removable singularity, a pole, or an essential singularity. If the singularity is removable, give the value of the function at the point.

We know $\sin(z)$ has zeros at $z = n\pi$ for $n \in \mathbb{Z}$. Therefore, $f(z)$ has singularities at $z = n\pi$ for $n \in \mathbb{Z}$. Near $z = n\pi$, $\sin(n\pi) \approx (-1)^n(z - n\pi)$. We can classify these singularities by examining the limit of $f(z)$ as $z \to n\pi$.

$$
\begin{aligned}
f(z) &= \frac{z^2}{\sin(z)} \\
&= \frac{z^2}{(-1)^n(z - n\pi)} \\
&= \frac{z^2}{(-1)^n(z - n\pi)}
\end{aligned}
$$

at $z = 0$, $f(z)$ has a removable singularity, because

$$\begin{aligned} f(z) &= \frac{z^2}{(-1)^n(z - n\pi)} \\ &= \frac{z^2}{(-1)^0(z - 0)} \\ &= z \end{aligned}$$

at $z = n\pi$, $f(z)$ has a pole of order 1, because

$$\begin{aligned} f(z) &= \frac{z^2}{(-1)^n(z - n\pi)} \\ &= \frac{z^2}{(-1)^n(z - n\pi)} \end{aligned}$$

■

■ **Example 13.8 — Fisher, Section 2.5, Problem 4.** Locate the isolated singularities of $f(z) = \pi \cot(\pi z)$. Identify each singularity as a removable singularity, a pole, or an essential singularity. If the singularity is removable, give the value of the function at the point.

$$\begin{aligned} f(z) &= \pi \cot(\pi z) \\ &= \pi \frac{\cos(\pi z)}{\sin(\pi z)} \\ &\approx \pi \frac{(-1)^n}{\pi(z - n)} \end{aligned}$$

The above approximation is valid near $z = n$ for $n \in \mathbb{Z}$. We can classify these singularities by examining the limit of $f(z)$ as $z \to n$.

$$\begin{aligned} f(z) &= \pi \frac{(-1)^n}{\pi(z - n)} \\ &= \frac{(-1)^n}{z - n} \\ \lim_{z \to n} f(z) &= \frac{(-1)^n}{0} = \infty \quad \text{On both sides of } n \end{aligned}$$

Therefore, $f(z)$ has poles of order 1 at $z = n$ for $n \in \mathbb{Z}$. ■

■ **Example 13.9 — Fisher, Section 2.5, Problem 6.** Locate the isolated singularities of $f(z) = \frac{e^z - 1}{e^{2z} - 1}$. Identify each singularity as a removable singularity, a pole, or an essential singularity. If the singularity is removable, give the value of the function at the point.

We can start by finding the zeros of the denominator, $e^{2z} - 1 = 0$:

$$\begin{aligned} e^{2z} - 1 &= 0 \\ e^{2z} &= 1 \\ 2z &= 2\pi in \quad \text{for } n \in \mathbb{Z} \\ z &= \pi in \quad \text{for } n \in \mathbb{Z} \end{aligned}$$

To examine the singularities of $f(z)$, we can simplify the function:

$$\begin{aligned} f(z) &= \frac{e^z - 1}{e^{2z} - 1} \\ &= \frac{e^z - 1}{(e^z - 1)(e^z + 1)} \\ &= \frac{1}{e^z + 1} \\ f(z_0) &= \frac{1}{e^{z_0} + 1} \\ &= \frac{1}{e^{\pi in} + 1} \\ &= \frac{1}{1 + 1} = \frac{1}{2} \end{aligned}$$

Therefore, $f(z)$ has removable singularities at $z = \pi in$ for $n \in \mathbb{Z}$ equal to $\frac{1}{2}$. ■

■ **Example 13.10 — Fisher, Section 2.5, Problem 16.** Suppose f and g are analytic in $|z - z_0| < R$ and f has a zero of order $m \geq 1$ at z_0, while g has a zero of order $l \leq m$ at z_0. Show that $\frac{f}{g}$ has a removable singularity at z_0.

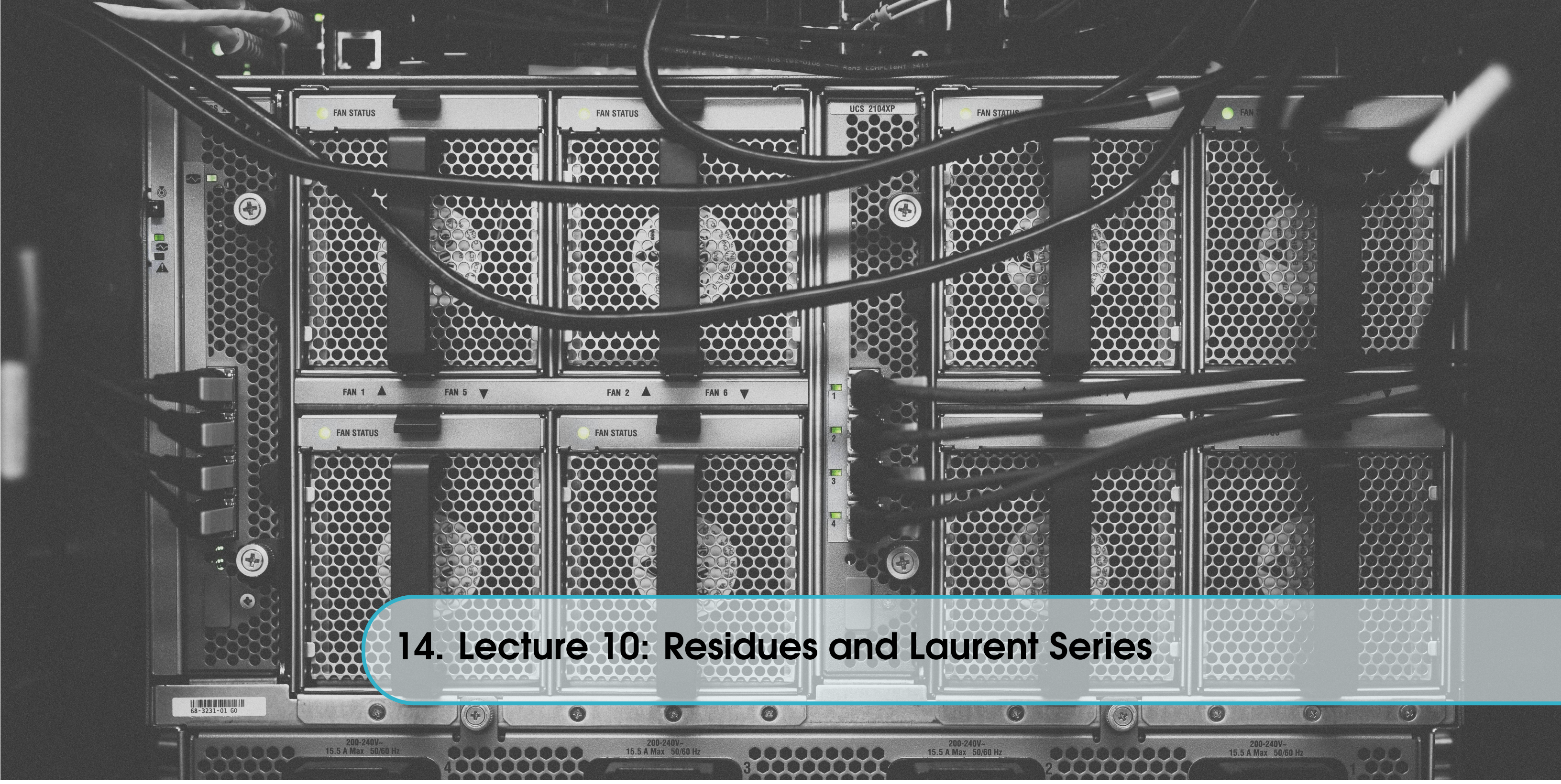
Since $f(z)$ and $g(z)$ are analytic in $|z - z_0| < R$, we can write their zeroes around z_0 as:

$$\begin{aligned} f(z) &= (z - z_0)^m f_1(z) \\ g(z) &= (z - z_0)^l g_1(z) \end{aligned}$$

where $f_1(z)$ and $g_1(z)$ are analytic at z_0 and $f_1(z_0) \neq 0$ and $g_1(z_0) \neq 0$. We can then write $\frac{f}{g}$ as:

$$\begin{aligned} \frac{f}{g} &= \frac{(z - z_0)^m f_1(z)}{(z - z_0)^l g_1(z)} \\ &= (z - z_0)^{m-l} \frac{f_1(z)}{g_1(z)} \\ \frac{f}{g}(z_0) &= \begin{cases} 0 & \text{if } m > l \\ \frac{f_1(z_0)}{g_1(z_0)} & \text{if } m = l \end{cases} \end{aligned}$$

Therefore, $\frac{f}{g}$ has a removable singularity at z_0. ■



14. Lecture 10: Residues and Laurent Series

Definition 14.0.1 — Residue. Suppose f is analytic on $0 < |z - z_0| < r$, if $0 < s < r$ Define the ***residue of*** f ***at*** z_0 to be

$$\text{Res}(f;\, z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=s} f(z)dz$$

Note: $\text{Res}(f;\, z_0)$ does not depend on s!

Theorem 14.0.1 — Green's Theorem. Suppose f is analytic on $0 < |z - z_0| < r$, then

$$\int_{|z-z_0|=s_1} f(\xi)d\xi = \int_{|z-z_0|=s_2} f(\xi)d\xi$$

14.1 Computing Residues with Power Series

Theorem 14.1.1 Suppose $f(z)$ has a pole of order $m \geq 1$ at z_0, then

$$f(z) = \frac{H(z)}{(z - z_0)^m} \tag{14.1}$$

Where H is analytic at z_0 and $H(z_0) \neq 0$ on $\{|z - z_0| < r\}$ and $H(z_0) \neq 0$. Expand $H(z)$ in a Laurent series about z_0:

$$H(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$$

Then

$$\boxed{f(z) = \frac{c_0}{(z-z_0)^m} + \frac{c_1}{(z-z_0)^{m-1}} + \cdots + \sum_{k=m}^{\infty} c_k (z-z_0)^{k-m}} \tag{14.2}$$

So we say:

$$\text{Res}(f; z_0) = c_{m-1}$$

i.e.the coefficent of $\frac{1}{(z-z_0)}$ in the Laurent series of f about z_0 (equation 14.2).

■ **Example 14.1** Find the residue of:

$$f(z) = \frac{e^z - 1}{z^2} \quad \text{at} \quad z_0 = 0$$

Solution: We know that:

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$\frac{e^z - 1}{z^2} = \frac{1}{z^2} + \frac{z}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{n-2}}{n!}$$

So the residue is $c_{m-1} = \frac{1}{1!} = 1$. ■

■ **Example 14.2** Find the residue of:

$$f(z) = \frac{z^2 + 3z - 1}{z + 2} \quad \text{at} \quad z_0 = -2$$

Solution: Let $w = z + 2$, then $z = w - 2$ and $dz = dw$. So

$$\begin{aligned} f(w) &= \frac{(w-2)^2 + 3(w-2) - 1}{w} = \frac{w^2 - 4w + 4 + 3w - 6 - 1}{w} = \frac{w^2 - w - 3}{w} \\ &= \frac{w^2}{w} - \frac{w}{w} - \frac{3}{w} \\ &= w - 1 - \frac{3}{w} \\ &= \frac{-3}{z+2} - 1 + z + 2 \end{aligned}$$

So the residue is $c_{m-1} = -3$. ■

14.2 Laurent series

Definition 14.2.1 — Laurent Series. Suppose f is analytic on $0 \leq r < |z - z_0| < R$, then f has a ***Laurent series*** about z_0.

Theorem 14.2.1 — Laurent Series. if f is analytic on $0 \leq r < |z - z_0| < R$, then we can write $f(z) = f_1(z) + f_2(z)$ where:

1. $f_1(z)$ is analytic on $\{|z - z_0| < R\}$ and has a power series expansion about z_0.
2. $f_2(z)$ is analytic on $\{|z - z_0| < R\}$ and has a power series expansion about ∞.

And:

1. $f_1(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^n$
2. $f_2(z) = \sum_{k=1}^{\infty} b_k(z - z_0)^{-n}$

So we can write:

$$\boxed{f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^n} \quad \text{Laurent Series} \tag{14.3}$$

Where $a_k = b_k$ if $k < 0$. And the expression converges on $r < |z - z_0| < R$.

Proof. Take $z : r < |z - z_0| < R$, and r_1, R_1 such that:

$$r < r_1 < |z - z_0| < R_1 < R$$

Then f is analytic on $\{|z - z_0| < R_1\}$. As we can see from figure 14.1, C_1 is a simple, closed, positively oriented curve in $\{|z - z_0| < R_1\}$, so we can apply Cauchy's theorem to get:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \int_{|\xi - z_0| = R_1} \frac{f(\xi)}{\xi - z_0} dz - \frac{1}{2\pi i} \int_{|\xi - z_0| = r_1} \frac{f(\xi)}{\xi - z_0} d\xi \end{aligned}$$

$$\Gamma = \{|\xi - z_0| = R_1\} \quad \text{and} \quad \gamma = \{|\xi - z_0| = r_1\}$$

Now we use a trick!

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 - (z - z_0)} = \frac{1}{\xi - z_0} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}}$$

On $\Gamma = \frac{z-z_0}{\xi - z_0} < 1$ so:

$$\frac{1}{\xi - z} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\xi - z_0)^{k+1}}$$

And on *gamma* we have:

$$\frac{1}{\xi - z} = -\sum_{k=1}^{\infty} \frac{(z - z_0)^k}{(\xi - z_0)^{k+1}}$$

Thus:

$$f(z) = \sum_{k=0}^{\infty}(z - z_0)^k \frac{1}{2\pi i}\int_{\Gamma=|\xi - z_0|=R_1} \frac{f(\xi)}{(\xi - z_0)^{k+1}}d\xi$$
$$- \sum_{k=0}^{\infty}(z - z_0)^{-k} \frac{1}{2\pi i}\int_{\gamma=|\xi - z_0|=r_1} \frac{f(\xi)}{(\xi - z_0)^{k}}d\xi$$

■

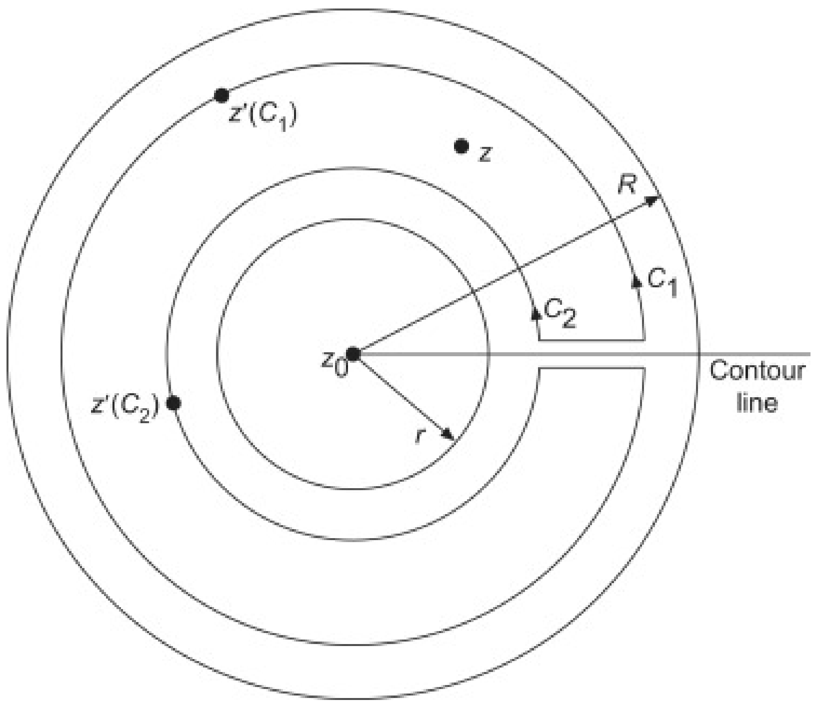


Figure 14.1: Laurent Series

■ **Example 14.3**

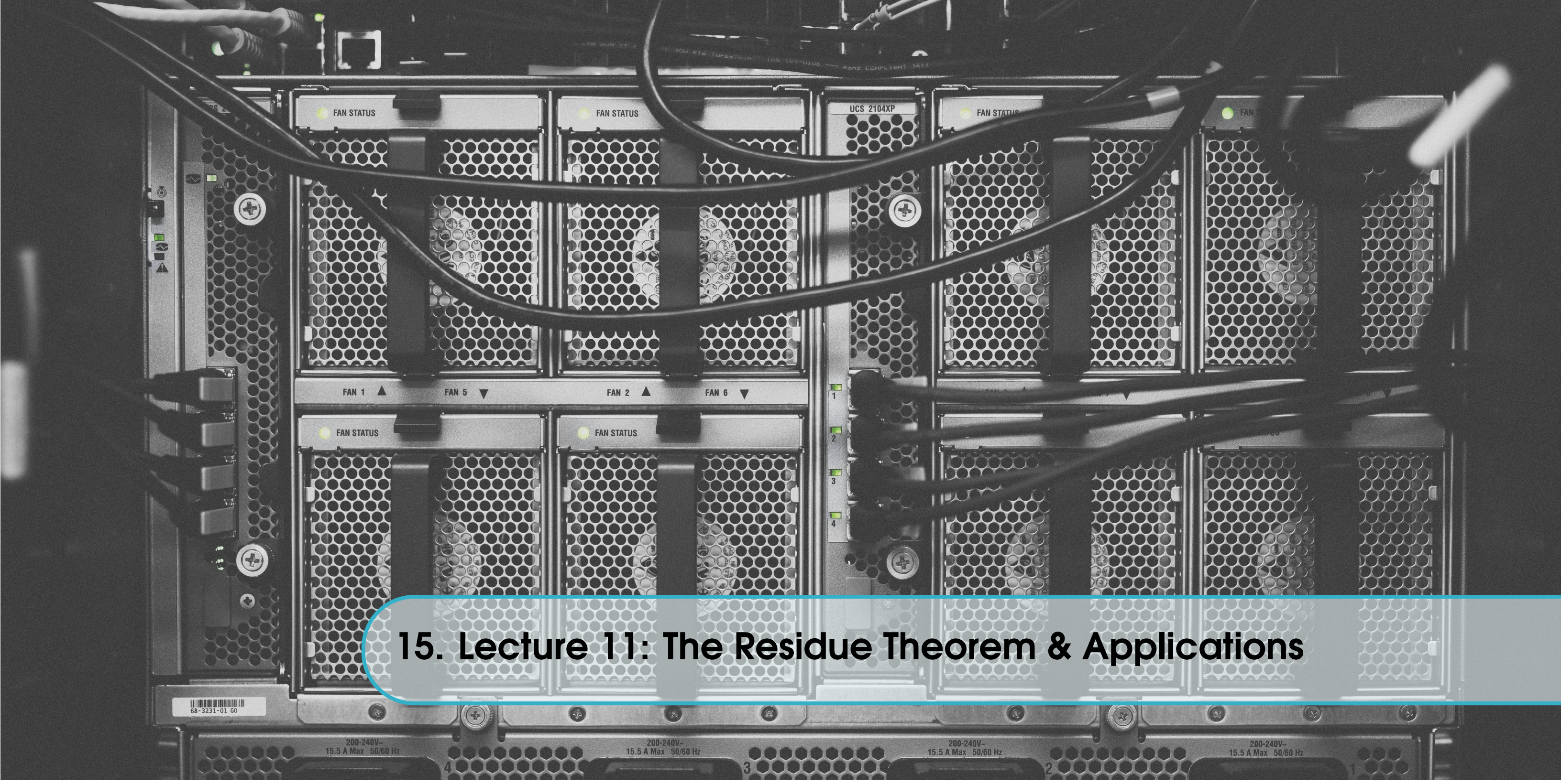
1. $f(z) = \frac{H(z)}{(z-z_0)^m}$ where $H(z)$ is analytic at z_0 and $H(z_0) \neq 0$.
2. $f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$

■

14.3 Common Series Expansions

Function	**Series Expansion**	**Radius of Convergence**
$\frac{1}{1-z}$	$\sum_{n=0}^{\infty} z^n$	$\|z\| < 1$
e^z	$\sum_{n=0}^{\infty} \frac{z^n}{n!}$	All $z \in \mathbb{C}$
$\sin(z)$	$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$	All $z \in \mathbb{C}$
$\cos(z)$	$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$	All $z \in \mathbb{C}$
$\ln(1+z)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$	$\|z\| < 1$
$(1+z)^\alpha$	$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$	$\|z\| < 1$
$\sinh(z)$	$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$	All $z \in \mathbb{C}$
$\cosh(z)$	$\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$	All $z \in \mathbb{C}$
$\frac{1}{(1-z)^k}$	$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} z^n$	$\|z\| < 1$

Table 14.1: Common Series Expansions



15. Lecture 11: The Residue Theorem & Applications

Theorem 15.0.1 — Residue Theorem. Suppose f is analytic on a simply connected domain D, except for a finite number of isolated singularities at $z_1, z_2, \cdots, z_n \in D$. Let γ be a piecewise C^1, positively oriented, simple closed curve in D which does not pass through any of the singularities. Then:

$$\boxed{\oint_\gamma f(z)dz = 2\pi i \sum_{z_k \text{ inside } \gamma} \text{Res}(f, z_k)} \tag{15.1}$$

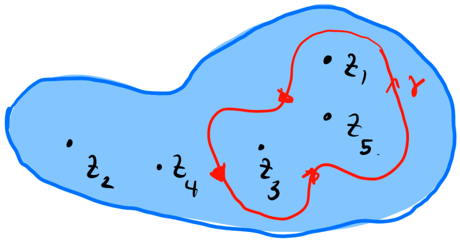


Figure 15.1: Residue Theorem

Proof. We'll use Green's Theorem with $\gamma = \partial\Omega$ where Ω is the region enclosed by γ.

$$\int_{\partial\Omega} f dz = i \int\int_\Omega \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) dxdy$$

We know that the integral of an analytic function over a closed curve is zero, so the only contributions to the integral are from the singularities inside Ω. Let $\gamma_1, \gamma_2, \cdots, \gamma_n$ be disjoint, positively oriented, circles around singularities $z_1, z_2, \cdots, z_n$ respectively. We apply Green's theorem to get:

$$\oint_\gamma f(z)dz = \sum_{k:z_k \text{ inside } \gamma} \oint_{\gamma_k} f(z)dz$$

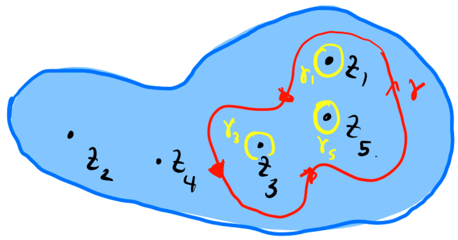


Figure 15.2: circles around $z_1, z_2, \cdots, z_n$

To compute $\oint_{\gamma_k} f(z)dz$, we can write $f(z)$ as a Laurent series about z_k:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_k)^n = \frac{\text{Res}(f, z_k)}{z - z_k} + \text{regular terms} \tag{15.2}$$

Now, we can compute the integral:

$$\oint_{\gamma_k} f(z)dz = \oint_{\gamma_k} \frac{\text{Res}(f, z_k)}{z - z_k}dz + \oint_{\gamma_k} \text{regular terms } dz$$

$\oint_{\gamma_k}$ regular terms dz is zero because when $n > -1$ the integrand is analytic and thus the integral of an analytic function over a closed curve is zero. When $n < -1$, the integrand is oscillatory and the integral is zero. So we are left with:

$$\oint_{\gamma_k} f(z)dz = \oint_{\gamma_k} \frac{\text{Res}(f, z_k)}{z - z_k}dz$$

Which resembles the Cauchy Integral Formula.

$$2\pi i f(z_0) = \oint_{\gamma} \frac{f(z)}{z - z_0}dz$$

So we can write:

$$\begin{aligned}\oint_{\gamma_k} f(z)dz &= \oint_{\gamma_k} \frac{\text{Res}(f, z_k)}{z - z_k}dz \\ &= 2\pi i \text{Res}(f, z_k)\end{aligned}$$

Summing over all z_k inside γ:

$$\begin{aligned}\oint_{\gamma} f(z)dz &= \sum_{k:z_k \text{ inside } \gamma} \oint_{\gamma_k} f(z)dz \\ &= 2\pi i \sum_{z_k \text{ inside } \gamma} \text{Res}(f, z_k)\end{aligned}$$

■

■ **Example 15.1** Compute:

$$\int_{-\infty}^{\infty} \frac{x^2}{((1 + x^2)(4 + x^2))} dx$$

Step 1: Change to a complex integral:

$$P(z) = z^2, \quad Q(z) = (1 + z^2)(4 + z^2)$$

Step 2: Choose a contour:

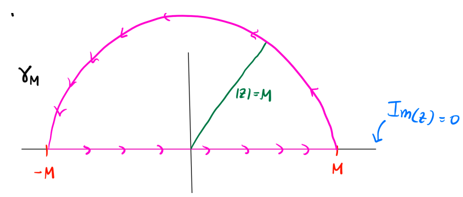


Figure 15.3: Contour

Suppose M is very large

1. $\int_{\gamma_M} \frac{P(z)}{Q(z)} dz$ can be computed by the residue formula.
2. On the other hand:

$$\int_{\gamma_M} \frac{P(z)}{Q(z)} dz = \underbrace{\int_{-M}^{M} \frac{x^2}{(1 + x^2)(4 + x^2)} dx}_{\text{The integral we want}} + \int_0^{\pi} \frac{P(Me^{i\theta})}{Q(Me^{i\theta})} iMe^{i\theta} d\theta$$

<u>Now</u>:

$$\begin{aligned} |P(Me^{i\theta})| &\leq M^2 \\ |Q(Me^{i\theta})| &= |(Me^{i\theta})^2 + 1||(Me^{i\theta})^2 + 4| \\ &\geq \frac{1}{10} M^4 \qquad \text{for a very large } M \end{aligned}$$

<u>So</u>

$$\left| \int_0^{\pi} \frac{P(Me^{i\theta})}{Q(Me^{i\theta})} iMe^{i\theta} d\theta \right| \leq 10\pi \frac{M^3}{M^4} \to 0 \quad \text{as } M \to \infty$$

<u>**Thus**</u>

$$\int_{-\infty}^{\infty} \frac{x^2}{(1 + x^2)(4 + x^2)} dx = \lim_{M \to \infty} \int_{\gamma_M} \frac{P(z)}{Q(z)} dz$$

Which can be computed using the residue formula.
$Q(z)$ has zeroes at $z = \pm i, \pm 2i$. only $+i, +2i$ are inside γ_M for large M. So:

$$\boxed{z = i} \quad \to \quad \frac{z^2}{(z + i)(z - i)(z^2 + 4)} = \frac{1}{z - i} \left[\frac{z^2}{(z + i)(z^2 + 4)} \right]$$

$$\text{So } \mathrm{Res}(f,\, i) \;=\; \frac{i^2}{(i\,+\,i)(i^2\,+\,4)} \;=\; \frac{-1}{6i}$$

$$\boxed{z\;=\;2i} \quad\rightarrow\quad \frac{z^2}{(z\,+\,i)(z\,-\,i)(z^2\,+\,4)} \;=\; \frac{1}{z\,-\,2i}\left[\frac{z^2}{(z\,+\,2i)(z^2\,+\,1)}\right]$$

$$\text{So } \mathrm{Res}(f,\, 2i) \;=\; \frac{-4}{4i(-4\,+\,1)} \;=\; \frac{1}{3i}$$

Thus, for a large M:

$$\begin{aligned}\int_{\gamma_M} \frac{P(z)}{Q(z)}dz \;&=\; 2\pi\left(\frac{-1}{6i}\,+\,\frac{1}{3i}\right)\\ &=\; \frac{\pi}{3}\end{aligned}$$

■

Theorem 15.0.2 — Solving Residue Problems. What do we need?
Polynomials $P(z)$, $Q(z)$ such that:

1. $Q(z)$ has no zeroes on the line $\Re(z)\;=\;0$.
   Why?
   - To apply the Residue Theorem to the contour shown in figure 15.3.
2. $\deg(Q)\;\geq\;\deg(P)\;+\;2$.
   Why?
   - To ensure that the integral over the arc goes to zero.
   - $\int_0^{\pi} \frac{P(Me^{i\theta})}{Q(Me^{i\theta})} iMe^{i\theta}d\theta \;\rightarrow\; 0$ as $M\;\rightarrow\;\infty$.

Proposition 15.0.3 P, Q polynomials that are real-valued on $\Im(z)\;=\;0$ and $\deg(Q)\;\geq\;\deg(P)+2$, then if $Q(x)\;\neq\;0$ for all $x\;\in\;\mathbb{R}$, then:

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx \;=\; 2\pi i \sum_{z_j\in U} \mathrm{Res}\left(\frac{P(z)}{Q(z)},\, z_j\right) \tag{15.3}$$

Where $U\;=\;\{z_j\;:\;\Im(z_j)\;>\;0\}$.

Proof. Take the contour γ_M as shown in figure 15.3, for a large M, the residue theorem applies and all zeroes of $Q\;\in\;\{z\;:\;\Im(z)\;>\;0\}$ are inside γ_M.
Claim: for M large:

- $|Q(Me^{i\theta})|\;\geq\;c_1M^{\deg(Q)}$ for some $c_1\;>\;0$.
- $|P(Me^{i\theta})|\;\leq\;c_2M^{\deg(P)}$ for some $c_2\;>\;0$.

This can be proved using the triangle inequality:

$$PERFORMTHEINEQUALITY$$

Since $\deg(P)\;+\;1\;-\;\deg(Q)\;\leq\;-1$ by assumption.
Now apply the residue theorem to γ_M:

$$\int_{\gamma_M} \frac{P(z)}{Q(z)}dz \;=\; 2\pi i \sum_{z_j\in U} \mathrm{Res}\left(\frac{P(z)}{Q(z)},\, z_j\right)$$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{z_j \in U} \text{Res}\left(\frac{P(z)}{Q(z)}, z_j\right)$$

■

15.1 Integrals Involving Trigonometric Functions

Proposition 15.1.1 Suppose R is a rational function (ratio of two polynomials) that is real on $\Im(z) = 0$. We can integrate $(x)\sin(x)$ or $R(x)\cos(x)$ over $(-\infty, \infty)$ by integrating

$$\Re R(z)e^{iz}, \quad \Im R(z)e^{iz}$$

Respectively over the contour shown in figure 15.3.

■ **Example 15.2** Compute:

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + \alpha^2} dx \quad \text{for } \alpha > 0$$

Step 1: Change to a complex integral and replace $\cos(x)$ with $\Re e^{iz}$

$$P(z) = e^{iz}$$
$$Q(z) = z^2 + \alpha^2$$

Step 2: $\deg(Q) \geq \deg(P) + 2$.

$$\lim_{z\to\infty} P(z) = \lim_{z\to\infty} e^{iz} = 1 \tag{15.4}$$
$$\lim_{z\to\infty} Q(z) = \lim_{z\to\infty} z^2 + \alpha^2 = \infty \therefore \deg(Q) \geq \deg(P) + 2 \tag{15.5}$$

Step 3: Choose a contour as shown in figure 15.3. Because we satisfy the conditions of proposition 15.0.3, we can apply it to compute the integral.

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + \alpha^2} dx = \Re\left(\lim_{M\to\infty} \oint_{\gamma_M} \frac{e^{iz}}{z^2 + \alpha^2} dz\right) \tag{15.6}$$
$$= \Re\left(2\pi i \sum_{z_j \in U} \text{Res}\left(\frac{e^{iz}}{z^2 + \alpha^2}, z_j\right)\right) \tag{15.7}$$

Step 4: Compute the residues: $z^2 + \alpha^2 = 0 \implies z = \pm i\alpha$ and only $i\alpha$ is in γ_M for large M.

$$\text{Res}\left(\frac{e^{iz}}{z^2 + \alpha^2}, i\alpha\right) = \lim_{z\to i\alpha} (z - i\alpha) \frac{e^{iz}}{z^2 + \alpha^2} \tag{15.8}$$
$$= \frac{e^{i(i\alpha)}}{2i\alpha} = \frac{e^{-\alpha}}{2i\alpha} \tag{15.9}$$

Step 5: Compute the integral:

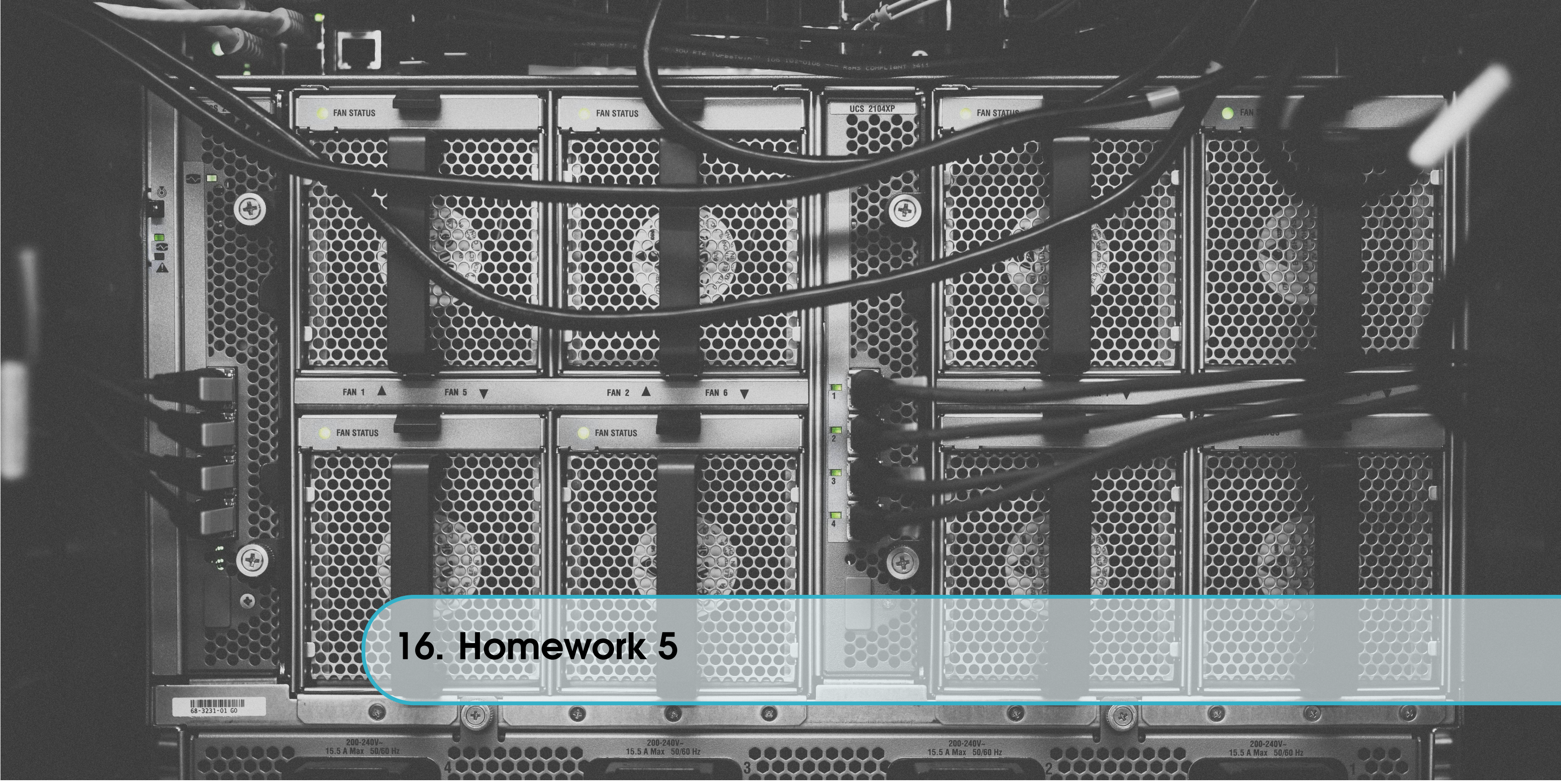
$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + \alpha^2} dx = \Re\left(2\pi i \sum_{z_j \in U} \text{Res}\left(\frac{e^{iz}}{z^2 + \alpha^2}, z_j\right)\right) \tag{15.10}$$

$$= \Re\left(2\pi i \frac{e^{-\alpha}}{2i\alpha}\right) \tag{15.11}$$

$$= \frac{\pi e^{-\alpha}}{\alpha} \tag{15.12}$$

■

15.1.1 There is another example in the notes... But I'm tired....



16. Homework 5

■ **Example 16.1** Solve the integral:

$$f(z) = \frac{z^2}{1 - z^2}, \quad z_0 = 1$$

Solution:

$$\begin{aligned} 2\pi i \text{Res}(f, z_0) &= \int_\gamma f(z)dz \\ &= \int_\gamma \frac{z^2}{(z - 1)(z + 1)}dz \\ &= \int_\gamma \frac{-\frac{z^2}{z+1}}{z - 1}dz \end{aligned}$$

Say, $g(z) = -\frac{z^2}{z+1}$, then using Cauchy's Integral Formula:

$$\begin{aligned} 2\pi i \text{Res}(f, z_0) &= \int_\gamma \frac{-\frac{z^2}{z+1}}{z - 1}dz \\ &= -2\pi i g(1) = -\pi i \\ &= 2\pi i \left(-\frac{1}{2}\right) = -\pi i \\ &\therefore \text{ Res}(f, z_0) = -\frac{\pi i}{2} \end{aligned}$$

■

■ **Example 16.2** Solve the integral:

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 5x^2 + 4}dx$$

Solution:
First, we factor the denominator:

$$\begin{aligned} x^4 + 5x^2 + 4 &= (x^2 + 4)(x^2 + 1) \\ &= (x + 2i)(x - 2i)(x + i)(x - i) \end{aligned}$$

We only care about two residues here, because the other two are not in the upper half plane. So we have:

$$\begin{aligned} \int_{\gamma_R} \frac{z^2}{z^4 + 5z^2 + 4} dz &= \frac{1}{2\pi i}(\mathrm{Res}(f, i) + \mathrm{Res}(f, 2i)) \\ &= 2\pi(\frac{1}{6} - \frac{2i}{6}) = \frac{-\pi}{3} + \frac{2\pi}{3} = \frac{\pi}{3} \end{aligned}$$

Next, let's find the residues at i and $2i$:

$$\begin{aligned} \mathrm{Res}(f, i) &= \frac{1}{2\pi i} \int_\gamma f(z)dz = \frac{1}{2\pi i} \int_\gamma \frac{z^2}{(z + 2i)(z - 2i)(z + i)(z - i)} dz \\ &= \frac{1}{2\pi i} \int_\gamma \frac{\frac{z^2}{(z+2i)(z-2i)(z+2i)}}{z - i} dz \\ &= \frac{1}{2\pi i} 2\pi g(i) = \frac{i^2}{(i + 2i)(i - 2i)(i + 2i)} = -\frac{1}{6} \end{aligned}$$

$$\begin{aligned} \mathrm{Res}(f, 2i) &= \frac{1}{2\pi i} \int_\gamma f(z)dz = \frac{1}{2\pi i} \int_\gamma \frac{z^2}{(z + 2i)(z - 2i)(z + i)(z - i)} dz \\ &= \frac{1}{2\pi i} \int_\gamma \frac{\frac{z^2}{(z+2i)(z-i)(z+2i)}}{z - 2i} dz \\ &= g(2i) = \frac{(2 - i)^2}{(2i + 2i)(2i - i)(2i + i)} = \frac{4}{12i} \end{aligned}$$

$$\begin{aligned} P(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \\ \lim_{|z| \to \infty} \left| \frac{P(z)}{z^n} \right| &= |a_n| \end{aligned}$$

R is large enough so that $|f(z)| < \varepsilon$ for $|z| > R$, so we have:

$$\begin{aligned} \frac{1}{2}|a_n| &\leq \left| \frac{P(z)}{z^n} \right| \leq 2|a_n| \\ \text{for} \quad z &= \alpha_R(z), \quad |z^n| = R^n \end{aligned}$$

$$P(z) = z^2$$

$$
\begin{aligned}
Q(z) &= z^4 + 5z^2 + 4 \\
\left|\frac{P(z)}{Q(z)}\right| &\leq \frac{2|1|R^2}{\frac{1}{2}|1|R^4} \\
&= 4R^{-2}
\end{aligned}
$$

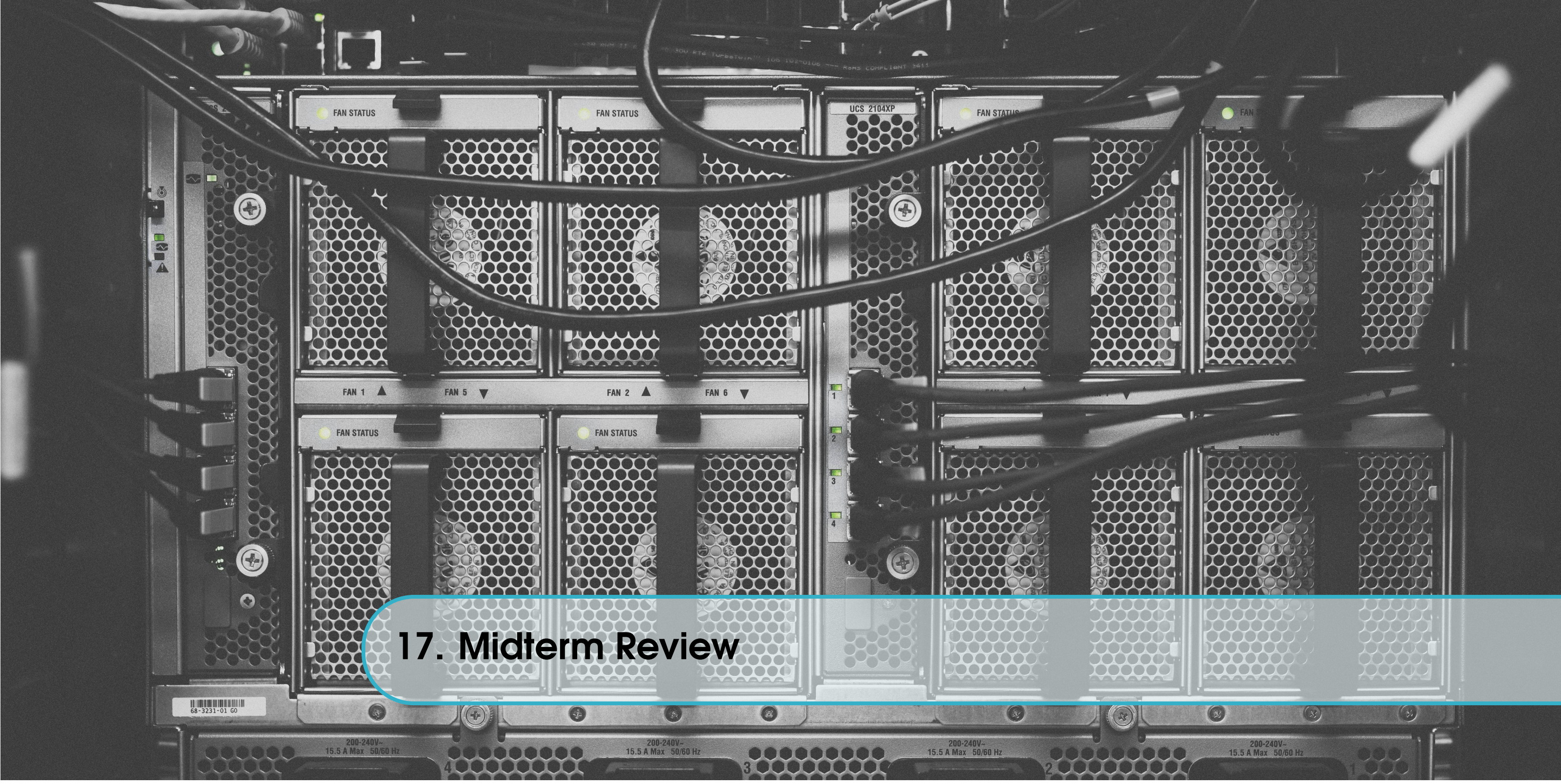
$$
\begin{aligned}
\left|\int_{\alpha_R} f(z)dz\right| &\leq \int_{\alpha_R} \left|\frac{P(z)}{Q(z)}\right| dz \\
&\leq \int_{\alpha_R} 4R^{-2}dz \\
&= 4R^{-2} \int_{\alpha_R} dz \\
&= 4R^{-2}\pi R \\
&= \frac{4\pi}{R} \to 0 \quad \text{as} \quad R \to \infty
\end{aligned}
$$

∎

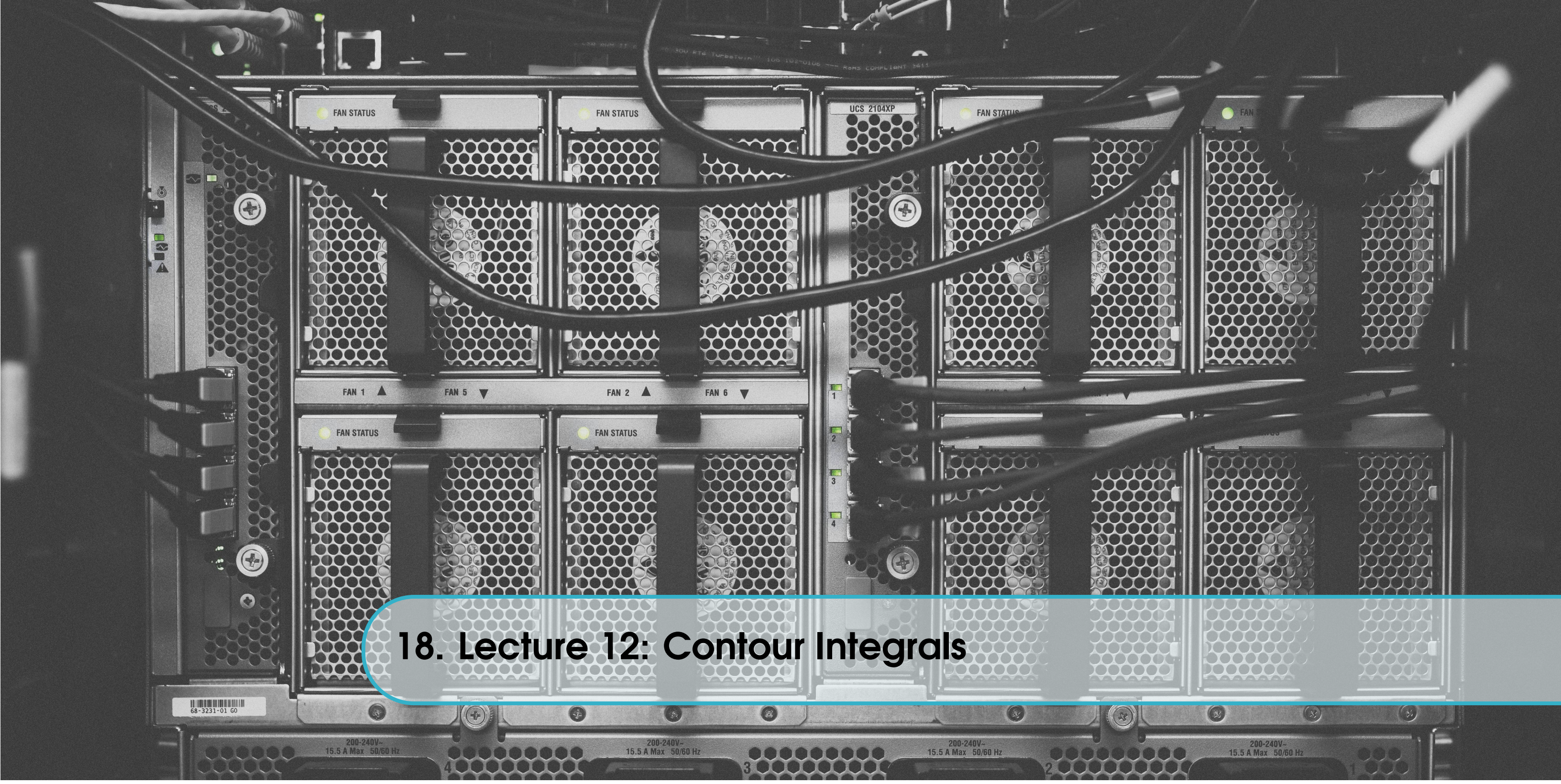
Proposition 16.0.1 In general if $f = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials and deg $Q \geq$ deg $P + 2$, then:

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx \\
I &= 2\pi i \sum_{\mathrm{Im} z > 0} \mathrm{Res}(f, z)
\end{aligned}
$$

ensure that $Q \neq 0$



17. Midterm Review



18. Lecture 12: Contour Integrals

■ **Example 18.1** Compute:

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx$$

step 1: Replace with a complex function:

$$\begin{aligned}
2\sin^2 x &= 1 - \cos 2x \\
\int_0^\infty \frac{\sin^2 x}{x^2} dx &= \frac{1}{2}\int_{-\infty}^\infty \frac{\sin^2 x}{x^2} dx \\
&= \frac{1}{4}\int_{-\infty}^\infty \frac{1 - \cos 2x}{x^2} dx \\
&= \frac{1}{4}\int_{-\infty}^\infty \frac{1 - \cos 2z}{z^2} dz \\
&= \Re\left(\frac{1}{4}\int_{-\infty}^\infty \frac{1 - e^{2iz}}{z^2} dz\right)
\end{aligned}$$

step 2: Choose the right contour: Previously, we used a semi-circle in the upper half plane with the condition that there are no zeroes on the line $Im(z) = 0$. This time, there is a zero at $z = 0$. So, we need to use a contour that excludes this zero. We can use a semi-circle in the upper half plane with a small semi-circle around the origin removed (half-keyhole contour) as seen in the figure below.

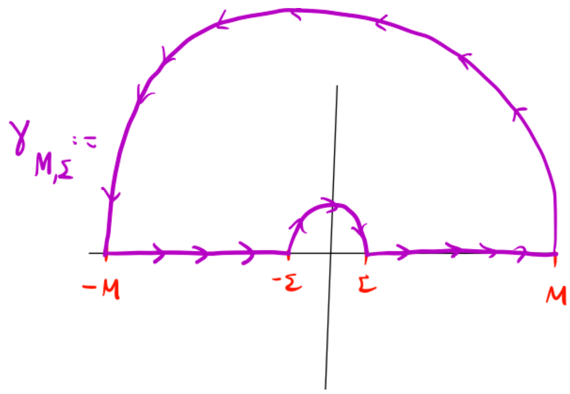


Figure 18.1: Half-keyhole contour

step 3: If there are singularities in the contour, find their residues: There are no singularities in the contour, so:

$$\int_{\gamma_{M,\varepsilon}} f(z)dz = 0 \tag{18.1}$$

step 4: Evaluate the consequences of the contour: By Cauchy's Integral Formula, we know that the integral around a closed loop is zero for a function that is analytic in the region enclosed by the loop. Say $f(z) = \frac{1-e^{2iz}}{z^2}$, then we can write:

$$\begin{aligned} 0 = \int_{\gamma_{M,\varepsilon}} f(z)dz &= \int_{\{z=Me^{i\theta},\theta\in[0,\pi]\}} f(z)dz && (I) \\ &+ \int_{z=\varepsilon e^{i\theta},\theta\in[0,\pi]} f(z)dz && (II) \\ &+ \int_{-M}^{-\varepsilon} f(z)dz + \int_{\varepsilon}^{M} f(z)dz && (III) \end{aligned}$$

We know from last lecture that $(I) = 0$ as $M \to \infty$ and as $M \to \infty$, $\varepsilon \to 0$, $(III) \to \int_{-\infty}^{\infty} f(z)dz$ (the integral we want to compute). So, we can write:

$$\begin{aligned} \int_{-M}^{-\varepsilon} f(z)dz + \int_{\varepsilon}^{M} f(z)dz &= -\int_{z=\varepsilon e^{i\theta},\theta\in[0,\pi]} f(z)dz \\ (III) &= -(II) \end{aligned}$$

We compute (II) explicitly, let $z = \varepsilon e^{i\theta}$:

$$\begin{aligned} (II) &= \int_{z=\varepsilon e^{i\theta},\theta\in[0,\pi]} f(z)dz \\ &= \int_{\pi}^{0} f(\varepsilon e^{i\theta})d(\varepsilon e^{i\theta}) \\ &= \int_{\pi}^{0} f(\varepsilon e^{i\theta})i\varepsilon e^{i\theta}d\theta \end{aligned}$$

$$= \int_{\pi}^{0} \frac{1 - e^{2i\varepsilon e^{i\theta}}}{(\varepsilon e^{i\theta})^2} i\varepsilon e^{i\theta} d\theta$$
$$= \int_{\pi}^{0} \frac{1 - e^{2i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} i d\theta$$

Now let's use the Taylor expansion of $e^{2i\varepsilon e^{i\theta}}$:

$$\rightarrow e^{2i\varepsilon e^{i\theta}} = 1 + 2i\varepsilon e^{i\theta} - \frac{4(\varepsilon e^{i\theta})^2}{2} + \frac{8i(\varepsilon e^{i\theta})^3}{6} \dots$$
$$\rightarrow 1 - e^{2i\varepsilon e^{i\theta}} = -2i\varepsilon e^{i\theta} + \frac{4(\varepsilon e^{i\theta})^2}{2} - \frac{8i(\varepsilon e^{i\theta})^3}{6} \dots$$
$$\rightarrow \frac{1 - e^{2i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} = -2i + 2\varepsilon e^{i\theta} - \frac{4\varepsilon e^{i\theta}}{2} + \frac{8i(\varepsilon e^{i\theta})^2}{6} \dots$$
$$\lim_{\varepsilon \to 0} \frac{1 - e^{2i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} = -2i$$
$$\therefore (II) = \int_{\pi}^{0} -2i^2 d\theta = 2\pi$$

So, we have:

$$\int_{-\infty}^{\infty} \frac{1 - e^{2iz}}{z^2} dz = \Re\left(\frac{1}{4}\int_{-\infty}^{\infty} \frac{1 - e^{2iz}}{z^2} dz\right) \tag{18.2}$$
$$= \Re\left(\frac{1}{4} \cdot 2\pi\right) = \frac{\pi}{2} \tag{18.3}$$

■

18.1 Integrals Involving the Log or Fractional Powers

■ **Example 18.2** Compute:

$$\int_0^{\infty} \frac{\log x}{(1 + x^2)^2} dx$$

step 1: Replace with a complex function:

$$\int_0^{\infty} \frac{\log x}{(1 + x^2)^2} dx = \int_0^{\infty} \frac{\log z}{(1 + z^2)^2} dz$$

Because we're in the complex plane now, we must define a branch cut for the logarithm. We can choose the negative real axis as the branch cut, so $\Im \log z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$.
step 2: Choose the right contour: We want to use a contour that excludes the branch cut and the part of the real axis where the integrand is not defined ($z = 0$). We can use a semi-circle in the upper half plane with a small semi-circle around the origin removed (half-keyhole contour) as seen in figure 18.1.
step 3: If there are singularities in the contour, find their residues: There are singularities where $1 + z^2 = 0 \rightarrow z = \pm i$. We can see that the singularity at $z = i$ is enclosed by the contour, so we need to find the residue at $z = i$, which is a pole of order 2. We can write:

$$\begin{aligned}
\operatorname{Res}(f, z_k) &= \frac{1}{(m-1)!} \lim_{z \to z_k} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_k)^m f(z)\right) && (18.4)\\
\operatorname{Res}(f, i) &= \frac{1}{1!} \lim_{z \to i} \frac{d}{dz} \left((z - i)^2 \frac{\log z}{(z+i)^2 (z-i)^2}\right) && (18.5)\\
&= \lim_{z \to i} \frac{d}{dz} \left(\frac{\log z}{(z+i)^2}\right) && (18.6)\\
&= \lim_{z \to i} \frac{1}{z} \cdot \frac{1}{(z+i)^2} - \frac{2 \log z}{(z+i)^3} && (18.7)\\
&= \frac{1}{i} \cdot \frac{1}{(2i)^2} - \frac{2 \log i}{(2i)^3} && (18.8)\\
&= \frac{1}{-4i} + \frac{\log i}{4i} && (18.9)\\
&= \frac{1}{-4i} + \frac{(\log 1 + i\pi/2)}{4i} && (18.10)\\
&= \frac{1}{-4i} + \frac{i\pi/2}{4i} && (18.11)\\
&= \frac{\pi/2 + i}{4} && (18.12)
\end{aligned}$$

Therefore we can state:

$$\begin{aligned}
\int_{\gamma_{M,\varepsilon}} f(z)dz &= 2\pi i \cdot \frac{\pi/2 + i}{4} && (18.13)\\
&= \frac{\pi^2}{4} i - \frac{\pi}{2} && (18.14)
\end{aligned}$$

step 4: Find the consequences of the contour:
We can write:

$$\begin{aligned}
0 = \int_{\gamma_{M,\varepsilon}} f(z)dz &= \int_{\{z = Me^{i\theta}, \theta \in [0,\pi]\}} f(z)dz && (I)\\
&+ \int_{z = \varepsilon e^{i\theta}, \theta \in [0,\pi]} f(z)dz && (II)\\
&+ \int_{-M}^{-\varepsilon} f(z)dz + \int_{\varepsilon}^{M} f(z)dz && (III)
\end{aligned}$$

We can see that:

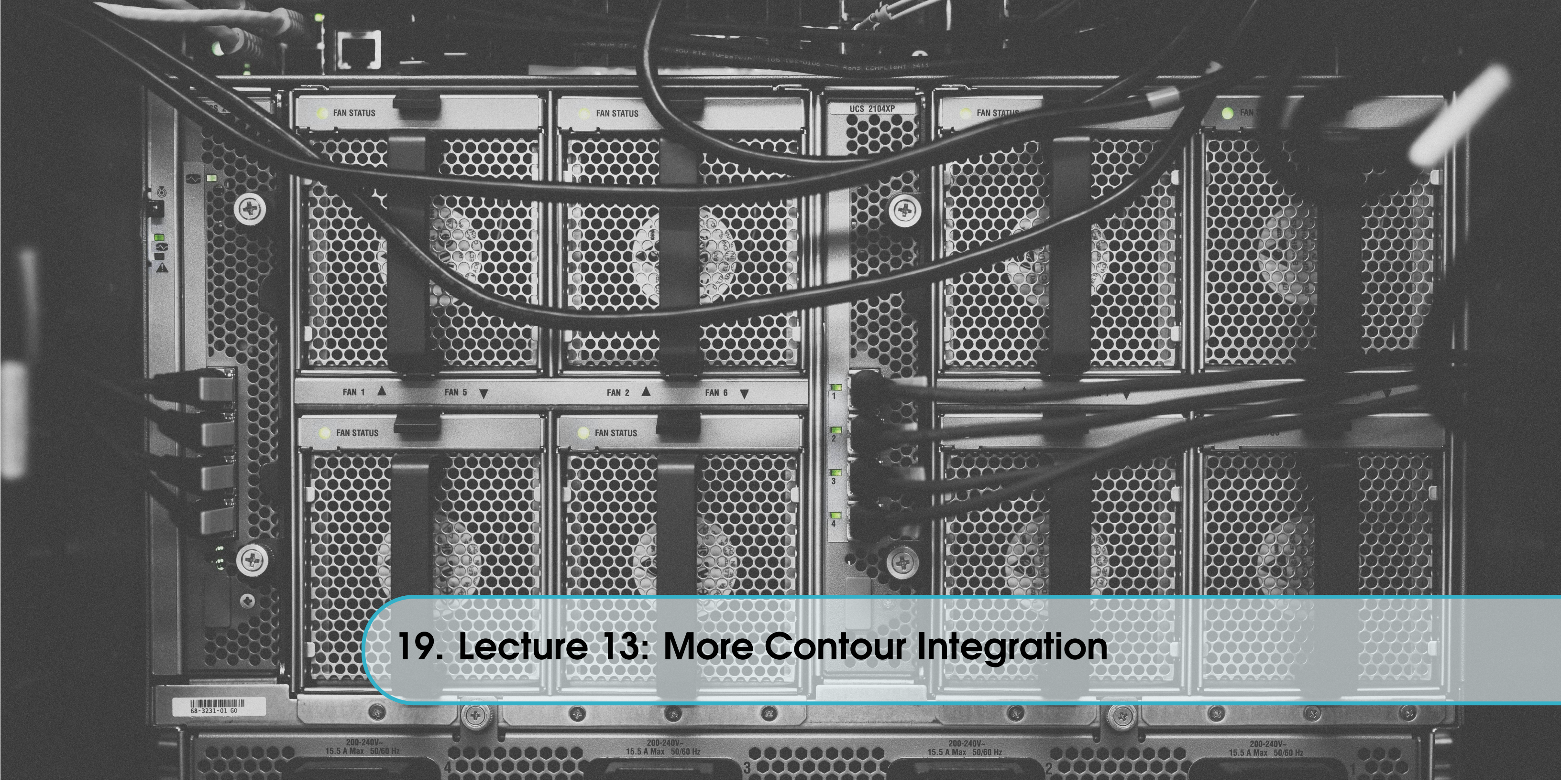
$$\begin{aligned}
(I) &= 0 \text{ as } M \to \infty && (18.15)\\
(II) &= \lim_{\varepsilon \to 0} \int_0^{\infty} f(\varepsilon e^{i\theta}) \varepsilon i e^{i\theta} d\theta && (18.16)\\
&= \lim_{\varepsilon \to 0} \int_0^{\infty} \frac{\log \varepsilon + i\theta}{(1 + \varepsilon^2 e^{2i\theta})^2} \varepsilon i e^{i\theta} d\theta = 0 && (18.17)\\
(III) &= \int_{-M}^{-\varepsilon} f(z)dz + \int_{\varepsilon}^{M} f(z)dz && (18.18)
\end{aligned}$$

$$\frac{\pi^2}{4}i - \frac{\pi}{2} = \int_{-M}^{-\varepsilon} \frac{\log|z| + i\pi}{(1 + z^2)^2} dz + \int_{\varepsilon}^{M} \frac{\log|z|}{(1 + z^2)^2} dz \tag{18.19}$$

$$\frac{\pi^2}{4}i - \frac{\pi}{2} = i\pi \int_{-M}^{-\varepsilon} \frac{1}{(1 + z^2)^2} dz + 2 \underbrace{\int_{\varepsilon}^{M} \frac{\log|z|}{(1 + z^2)^2} dz}_{\text{Integral we want}} \tag{18.20}$$

$$\int_0^{\infty} \frac{\log x}{(1 + x^2)^2} dx = -\frac{\pi}{4} \tag{18.21}$$

■



19. Lecture 13: More Contour Integration

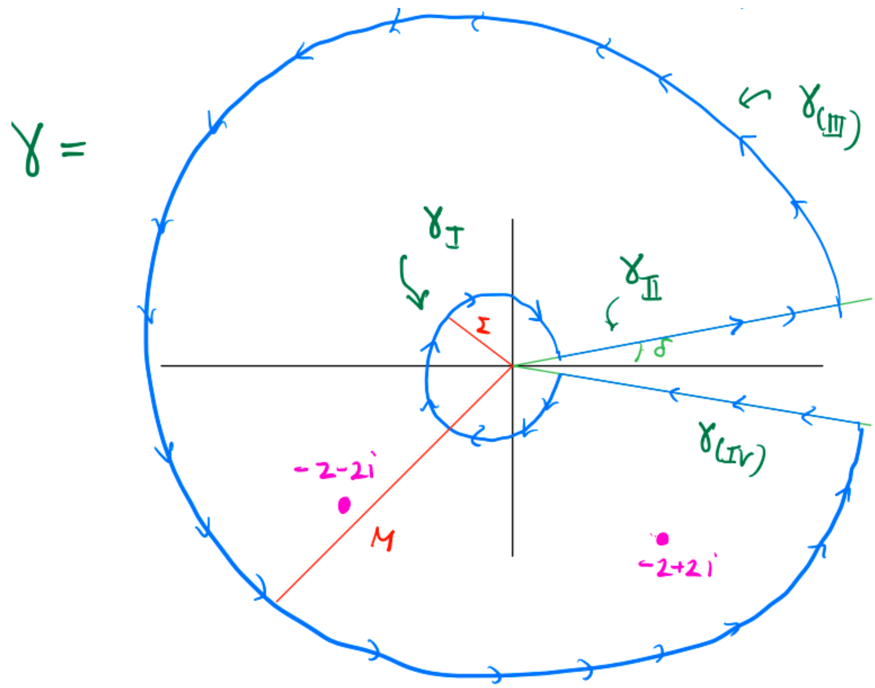
■ **Example 19.1** Find the value of:

$$\int_0^\infty \frac{x^{1/3}}{x^2 + 4x + 8} dx$$

step 1: Replace with a complex function: Let $f(z) = \frac{z^{1/3}}{z^2+4z+8}$, then:

$$\begin{aligned} \int_0^\infty \frac{x^{1/3}}{x^2 + 4x + 8} dx &= \int_{-\infty}^\infty \frac{z^{1/3}}{z^2 + 4z + 8} dz \\ &= \int_{-\infty}^\infty \frac{e^{\frac{1}{3} \log z}}{z^2 + 4z + 8} dz \end{aligned}$$

step 2: Choose the right contour: **NOTE:** $\int_{-\infty}^0 \frac{|z|^{1/3}}{z^2+4z+8} dz \neq \int_0^\infty \frac{z^{1/3}}{z^2+4z+8} dz$ so, $\int_{-\infty}^\infty \frac{z^{1/3}}{z^2+4z+8} \neq 2\int_0^\infty \frac{x^{1/3}}{x^2+4x+8} dx$ because $z^2 + 4x + 8$ is not even. (There's an absolute value in the numerator because we'd be able to evaluate the imaginary part of the logarithm explicitly) Therefore a half-keyhole contour is not appropriate. We can use a full keyhole contour as seen in the figure below.



Step 2.5: Choose the branch cut: We need to choose a branch cut over which $z^{1/3}$ is analytic. We can choose the positive real axis so that $\arg(z) \in [0, 2\pi)$.
step 3: If there are singularities in the contour, find their residues: There are singularities at $z = -2 \pm 2i$. They are both enclosed by the contour, so we need to find the residues at both points. We can write:

$$
\begin{aligned}
\rightarrow f(z) &= \frac{z^{1/3}}{(z+2-2i)(z+2+2i)} \\
\operatorname{Res}(f, z_k) &= \lim_{z \to z_k} (z - z_k) f(z) \\
\operatorname{Res}(f, -2+2i) &= \lim_{z \to -2+2i} (z-(-2+2i)) \frac{z^{1/3}}{(z+2-2i)(z+2+2i)} \\
&= \lim_{z \to -2+2i} \frac{z^{1/3}}{(z+2-2i)} \\
&= \frac{(-2+2i)^{1/3}}{(-2+2i+2-2i)} \\
&= \frac{2^{1/3} e^{i\pi/4}}{4} = \frac{e^{i\pi/4}}{2^{2/3}} \\
\operatorname{Res}(f, -2-2i) &= \lim_{z \to -2-2i} (z-(-2-2i)) \frac{z^{1/3}}{(z+2-2i)(z+2+2i)} \\
&= \lim_{z \to -2-2i} \frac{z^{1/3}}{(z+2+2i)} \\
&= \frac{(-2-2i)^{1/3}}{(-2-2i+2+2i)}
\end{aligned}
$$

$$= \frac{2^{1/3}e^{-i\pi/4}}{4} = \frac{e^{-i\pi/4}}{2^{2/3}}$$
$$\therefore \ \text{Res}(f, -2 \pm 2i) = \frac{e^{\pm i\pi/4}}{2^{2/3}}$$

We can compute the integral:

$$i\pi/4 \int_{\gamma_{M,\varepsilon,\delta}} f(z)dz = 2\pi i \left(\frac{e^{i\pi/4}}{2^{2/3}} + \frac{e^{-i\pi/4}}{2^{2/3}} \right)$$

step 4: Find the consequences of the contour: Recall Example 5.2 From chapter 5, lecture 4 for the parametrization of the keyhole contour:

$$\gamma(t) = \begin{cases} Me^{i\theta} & \delta \leq \theta \leq 2\pi - \delta \\ te^{i(2\pi-\delta)} & M \leq t \leq \varepsilon \\ \varepsilon e^{i\theta} & 2\pi - \delta \leq \theta \leq \delta \\ te^{i\delta} & \varepsilon \leq t \leq M \end{cases}$$

From the residue theorem and the contour in the figure above, we can write:

$$i\pi/4 \int_{\gamma_{M,\varepsilon,\delta}} f(z)dz = \int_{\gamma_I} f(z)dz + \int_{\gamma_{II}} f(z)dz + \int_{\gamma_{III}} f(z)dz + \int_{\gamma_{IV}} f(z)dz$$

By previous arguments, as $M \to \infty$ and $\varepsilon \to 0$, we can write:

$$\int_{\gamma_I} f(z)dz = 0$$
$$\int_{\gamma_{III}} f(z)dz = 0$$

The other two integrals can be computed as:

$$\begin{aligned} \int_{\gamma_{II}} f(z)dz &= \int_{\varepsilon}^{M} f(re^{i\delta})rie^{i\delta}dr \\ &= \int_{\varepsilon}^{M} \frac{r^{1/3}e^{i\delta/3}}{(re^{i\delta} + 2 - 2i)(re^{i\delta} + 2 + 2i)} rie^{i\delta}dr \\ &\text{As } M \to \infty,\ \varepsilon \to 0,\ \delta \to 0 \\ &= \int_{0}^{\infty} \frac{r^{1/3}}{(r + 2 - 2i)(r + 2 + 2i)} ridr \end{aligned}$$

On the other hand:

$$\int_{\gamma_{IV}} f(z)dz = \int_{M}^{\varepsilon} f(re^{i\delta})rie^{i\delta}dr$$

$$= \int_M^{\varepsilon} \frac{r^{1/3}e^{i(2\pi-\delta)/3}}{(re^{i(2\pi-\delta)} + 2 - 2i)(re^{i(2\pi-\delta)} + 2 + 2i)} rie^{i(2\pi-\delta)}dr$$

As $M \to \infty,\ \varepsilon \to 0,\ \delta \to 0$

$$= \int_{\infty}^{0} \frac{r^{1/3}e^{i2\pi/3}}{(r + 2 + 2i)(r + 2 - 2i)} ridr$$

$$= e^{i2\pi/3} \int_0^{\infty} \frac{r^{1/3}}{(r + 2 - 2i)(r + 2 + 2i)} ridr$$

Therefore:

$$\left[\int_0^{\infty} \frac{r^{1/3}}{(r + 2 - 2i)(r + 2 + 2i)} ridr\right] \left[1 - e^{i2\pi/3}\right] = i\pi/4 \int_{\gamma_{M,\varepsilon,\delta}} f(z)dz$$

$$\int_0^{\infty} \frac{r^{1/3}}{(r + 2 - 2i)(r + 2 + 2i)} ridr = \frac{2\pi i}{1 - e^{i2\pi/3}} \left(\frac{e^{i\pi/4}}{2^{2/3}} + \frac{e^{-i\pi/4}}{2^{2/3}}\right)$$

■

Proposition 19.0.1 — Keyhole Contour Integral. Use a keyhole contour to evaluate real integrals with logarithms or fractional powers where the integrand is not even.

$$\left[\int_0^{\infty} f(x)dx\right] (1 - e^{i\theta}) = \int_{\gamma_{M,\varepsilon,\delta}} f(z)dz = 2\pi i \sum \text{Res}(f, z_k)$$

Where $\gamma_{M,\varepsilon,\delta}$ is the sum of the residues of the singularities enclosed by the contour. $\theta = i \arg z \times$ fractional power of z for a fractional power of z.

19.1 Analytic Functions as Mappings (Chap. 3)

19.1.1 3.1 - The Zeroes of an Analytic Function

Lemma 19.1.1 Recall that if $f(z)$ is not identically 0 on a domain D and $f(z_0) = 0$ for some $z_0 \in D$, then $f(z)$ has a pole at z_0 of order m if $f(z) = (z - z_0)^m g(z)$ where $g(z)$ is analytic and $g(z_0) \neq 0$.

Proposition 19.1.2 — The Identity Theorem. If $f(z)$ is analytic on a domain D and there exists a sequence of points $\{z_n\}$ in D such that $f(z_n) = 0$ and $z_n \to z_0 \in D$, then $f(z) = 0$ for all $z \in D$. **Alternatively:** If f is analytic on D, not identically zero, and $f(z_0) = 0$ for some $z_0 \in D$, then $\exists\delta > 0 | z_0$ is the only zero of $f \in \{z \in D || z - z_0| < \delta\}$.

Proof. Suppose f is analytic, $f(z_0) =$ and $\exists z_n \in \{|z_0 - z| < \frac{1}{n}\}$ such that $f(z_n) = 0$ this would mean:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

$$f(z_0) = a_0(z_0 - z_0)^0 + a_1(z_0 - z_0)^1 + \ldots = 0 \qquad = a_0 \times 0^0$$

$$0 = a_0 \times 1$$

If $a_0, a_1, \ldots a_{n-1}$ are all 0 then:

$$\begin{aligned} g(z) &= \frac{f(z)}{(z - z_0)^n} = a_n + a_{n+1}(z - z_0) + \ldots = 0 \\ g(z_n) &= \frac{f(z_n)}{(z_n - z_0)^n} = \frac{0}{(z_n - z_0)^n} = 0 \\ a_n &= 0 \end{aligned}$$

■

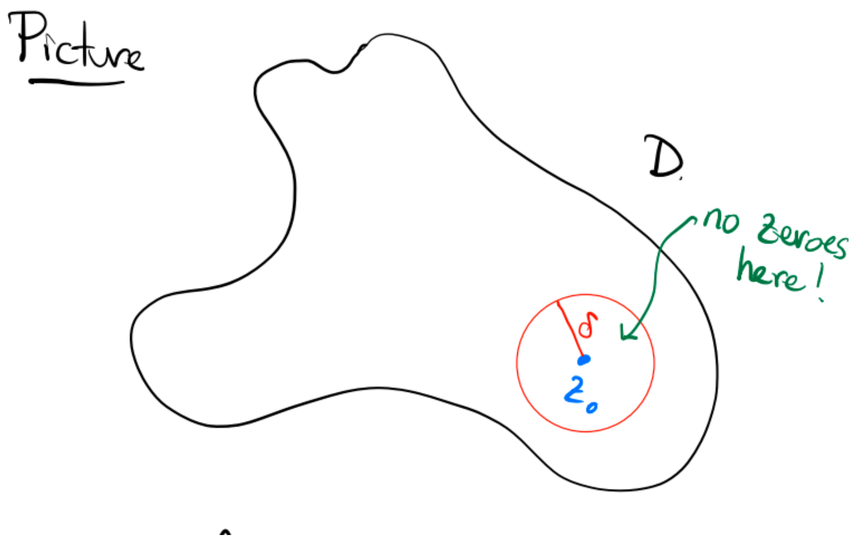


Figure 19.1: δ where no other zeroes exist

Proposition 19.1.3 If D is a bounded domain (i.e. $D \subset \{|z| < M\}$ for some M) then f has only finitely many zeroes. Thus, we can count them. This can be done using the residue theorem.

Proof. Zeroes are isolated in D so they must be at least some δ apart. If D then there is only finite space for zeroes and thus, there are only finitely many zeroes. ■

Theorem 19.1.4 — Argument Principle. Suppose h is analytic in a domain D except for a finite number of isolated poles. Let γ be a piecewise C^1, positively oriented, simple closed curve in D, which does not pass through any pole or zero of h, and such that inside$(\gamma) \subset D$. Then:

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}dz = \text{number of zeroes of } h \text{ inside } \gamma - \text{number of poles of } h \text{ inside } \gamma$$

$$\frac{1}{2\pi}\left\{\begin{matrix}\text{change in } \arg h(z) \\ \text{as } z \text{ traverses } \gamma\end{matrix}\right\} = \left\{\begin{matrix}\text{no. of zeros of } h \\ \text{inside } \gamma\end{matrix}\right\} - \left\{\begin{matrix}\text{no. of poles of } h \\ \text{inside } \gamma\end{matrix}\right\}.$$

Where all zeroes and poles are counted with their multiplicities.

■ **Example 19.2** The function z^k has k zeroes inside the unit circle and k poles outside of it. ■

Proof. $\frac{h'(z)}{h(z)}$ is analytic except at zeroes or poles of h. If h has a zero of order k at z_0, then $h = (z - z_0)^k g(z)$ $g(z)$ analytic $g(z_0) \neq 0$. so:

$$\begin{aligned}\frac{h'}{h} &= \frac{k(z - z_0)^{k-1} g(z) + (z - z_0)^k g'(z)}{(z - z_0)^k g(z)} \\ &= \frac{k}{z - z_0} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{analytic near } z_0}\end{aligned}$$

Thus:

$$\operatorname{Res}(h'/h, z_0) = k = \text{order of zero at } z_0$$

Now if h has a pole order k at z_0 then $h = \frac{H(z)}{(z-z_0)^k}$ $H(z)$ analytic $H(z_0) \neq 0$. so:

$$\begin{aligned}\frac{h'}{h} &= \left[\frac{H'(z)}{(z - z_0)^k} - \frac{kH(z)}{(z - z_0)^{k+1}}\right] \frac{(z - z_0)^k}{H(z)} \\ &= -\frac{k}{z - z_0} + \underbrace{\frac{H'(z)}{H(z)}}_{\text{analytic near } z_0}\end{aligned}$$

Thus:

$$\operatorname{Res}(h'/h, z_0) = -k = \text{order of pole at } z_0$$

We can conclude that:

$$\begin{aligned}\frac{1}{2\pi i} \int_\gamma \frac{h'(z)}{h(z)} dz &= \sum_{z_j \text{ zeroes } h} \text{order of zeroes at } z_j - \sum_{z_k \text{ poles } h} \text{order of poles at } z_k \\ &= \text{number of zeroes of } h \text{ inside } \gamma - \text{number of poles of } h \text{ inside } \gamma\end{aligned}$$

■

■ **Example 19.3** Consider $h(z) = z^k$, and the curve $\gamma = e^{i\theta}$ then:

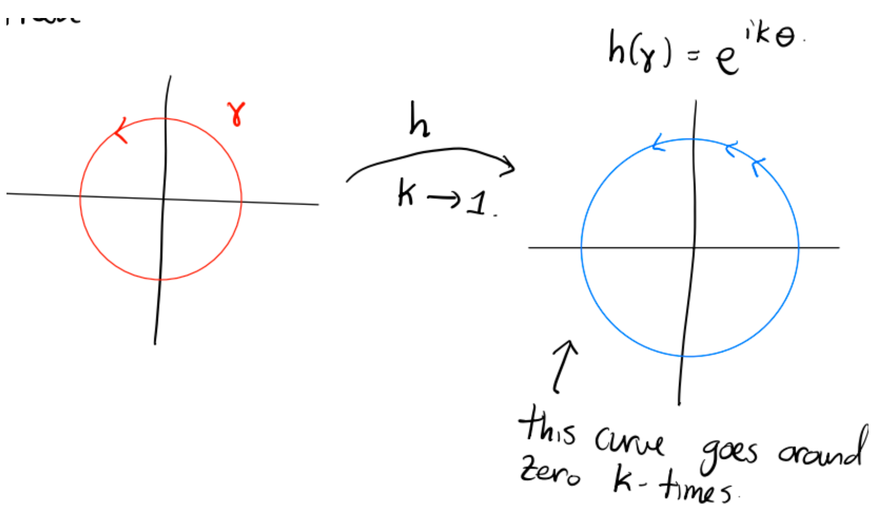


Figure 19.2: The curve γ

Step 1: find $h'(z)/h(z)dz$:

$$\begin{aligned}\frac{h'(z)}{h(z)} &= d\ \log\ h(z) \text{ By definition}\\ &= \underbrace{d\ \log\ |h|}_{\text{change in } |h(z)|} + \underbrace{id\theta}_{\text{change in } \arg\ h(z)}\\ &= dr + id\theta\end{aligned}$$

So:

$$\begin{aligned}\int_\gamma \frac{h'(z)}{h(z)}dz &= \int_{h(\gamma)} dr + id\theta\\ &= \log \frac{|h(\gamma(2\pi))|}{|h(\gamma(0))|} + i(\arg\ h(\gamma(2\pi)))k\\ &= 0_i 2\pi k\end{aligned}$$

Similarly if $h(z) = \frac{1}{z^k}$ then:

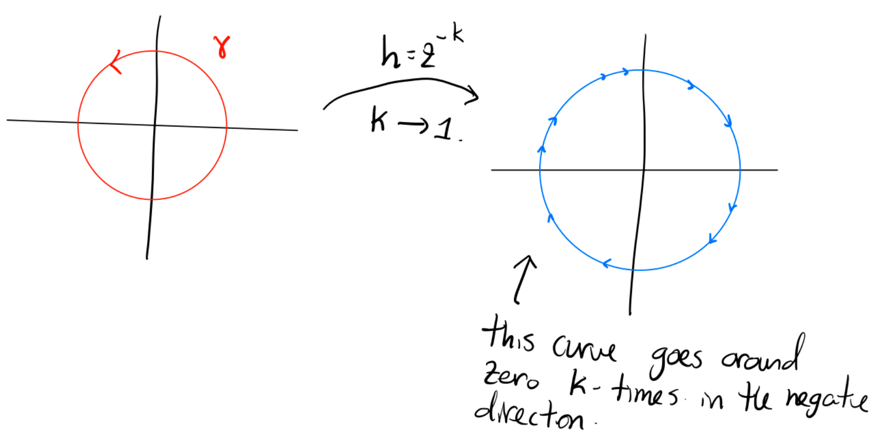


Figure 19.3: The reverse curve γ

■

■ **Example 19.4** Find the number of zeroes of $f(z) = z^3 - 2z^2 + 4$ in the first quadrant

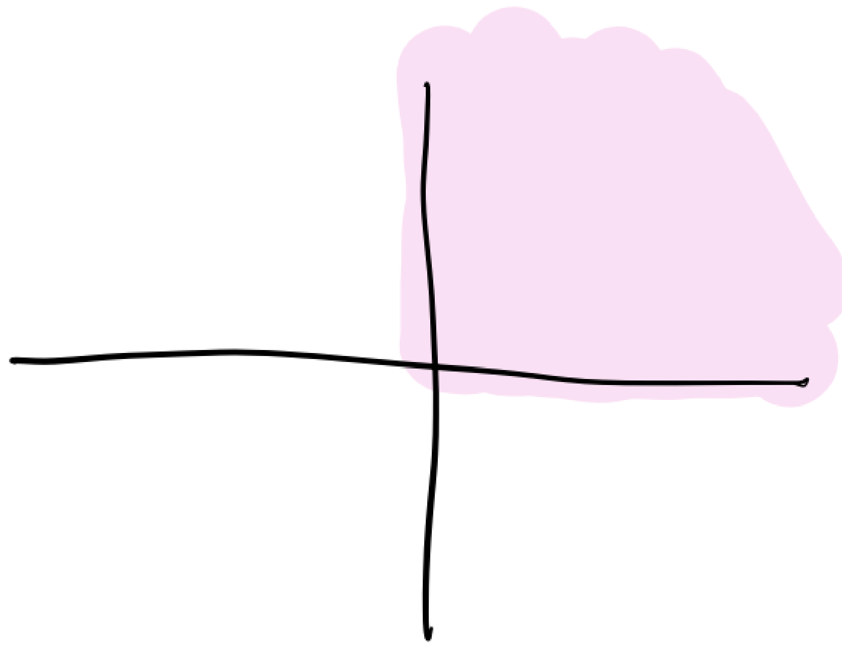


Figure 19.4: The first quadrant

Solution: Step 1: Consider the contour **Step 2:** Find Δ arg $f(\gamma_k) \forall k$

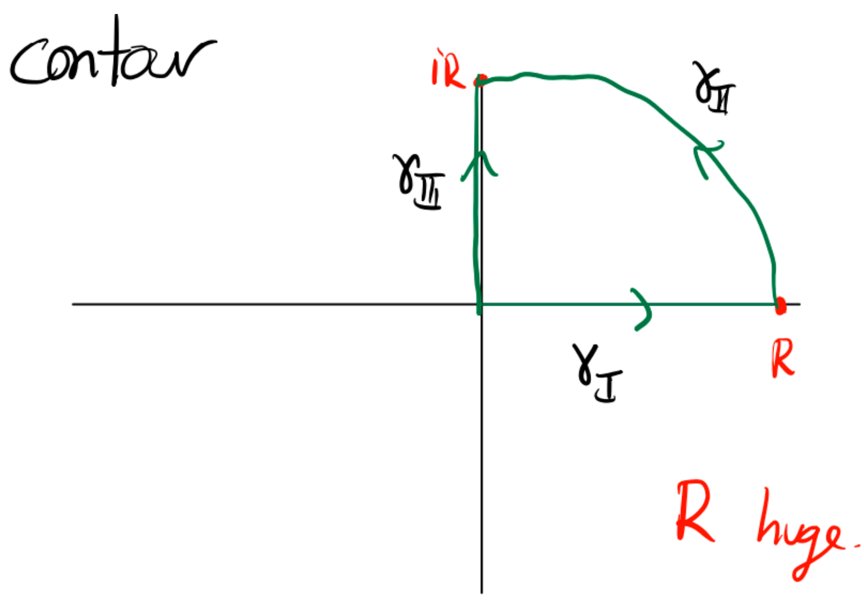


Figure 19.5: The contour

$$\gamma_I = x \quad t \in [0, R] \quad \rightarrow \quad f(\gamma_I) = t^3 - 2t^2 + 4$$

This is a real function, so as long as $f(\gamma_I) \geq 0$ from $t = 0$ to $t = R$, then Δ arg $f(\gamma_I) = 0$. We know this function will have a minimum at $f'(t)$

$$\begin{aligned}
f' &= 3t^2 - 4t = x(3x-4) \\
\to f(0) &= 4 \\
\to f(\frac{4}{3}) &= t^3 - 2t^2 + 4 = \frac{64}{27} - \frac{32}{9} + 4 = \frac{64}{27} - \frac{96}{27} + \frac{108}{27} = \frac{76}{27} > 0
\end{aligned}$$

So Δ arg $f(\gamma_I) = 0$.

$$\begin{aligned}
\gamma_{II} &= Re^{i\theta} \quad \theta \in [0, \pi/2] \\
f(\gamma_{II}) &= R^3e^{3i\theta} - 2R^2e^{2i\theta} + 4 \\
&= R^3e^{3i\theta}\left(1 - \frac{2}{R}e^{i\theta} + \frac{4}{R^3}e^{-3i\theta}\right) \\
&\text{As } R \to \infty \\
&= R^3e^{3i\theta}
\end{aligned}$$

R is the magnitude of the function, complex exponential function gives the argument of the function.

$$e^{ig(\theta)} \to g(\theta) = \arg f(\gamma_{II}) = 3\theta$$

Because $\theta \in [0, \pi/2]$ then Δ arg $f(\gamma_{II}) = 3\pi/2$.

$$\begin{aligned}
\gamma_{III} &= iy \quad \to \quad t \in [0, R] \\
f(\gamma_{III}) &= iy^3 + 2y^2 + 4
\end{aligned}$$

We know the argument of a function is given by $\tan^{-1}(\frac{\Im(f)}{\Re(f)})$

$$\arg f(\gamma_{III})_1 = 0 \quad \text{because } f(\gamma_{III}(0)) = 4$$

Because the function begins on the real axis when $t = 0$, the argument is 0.

$$\begin{aligned}
\arg f(\gamma_{III})_2 &= \lim_{y=R\to\infty} \tan^{-1}\left(\frac{y^3}{2y^2+4}\right) \\
&= tan^{-1}(\text{inf}) = \frac{\pi}{2} \\
\therefore \Delta \arg f(\gamma_{III}) &= \frac{\pi}{2}
\end{aligned}$$

So all together:

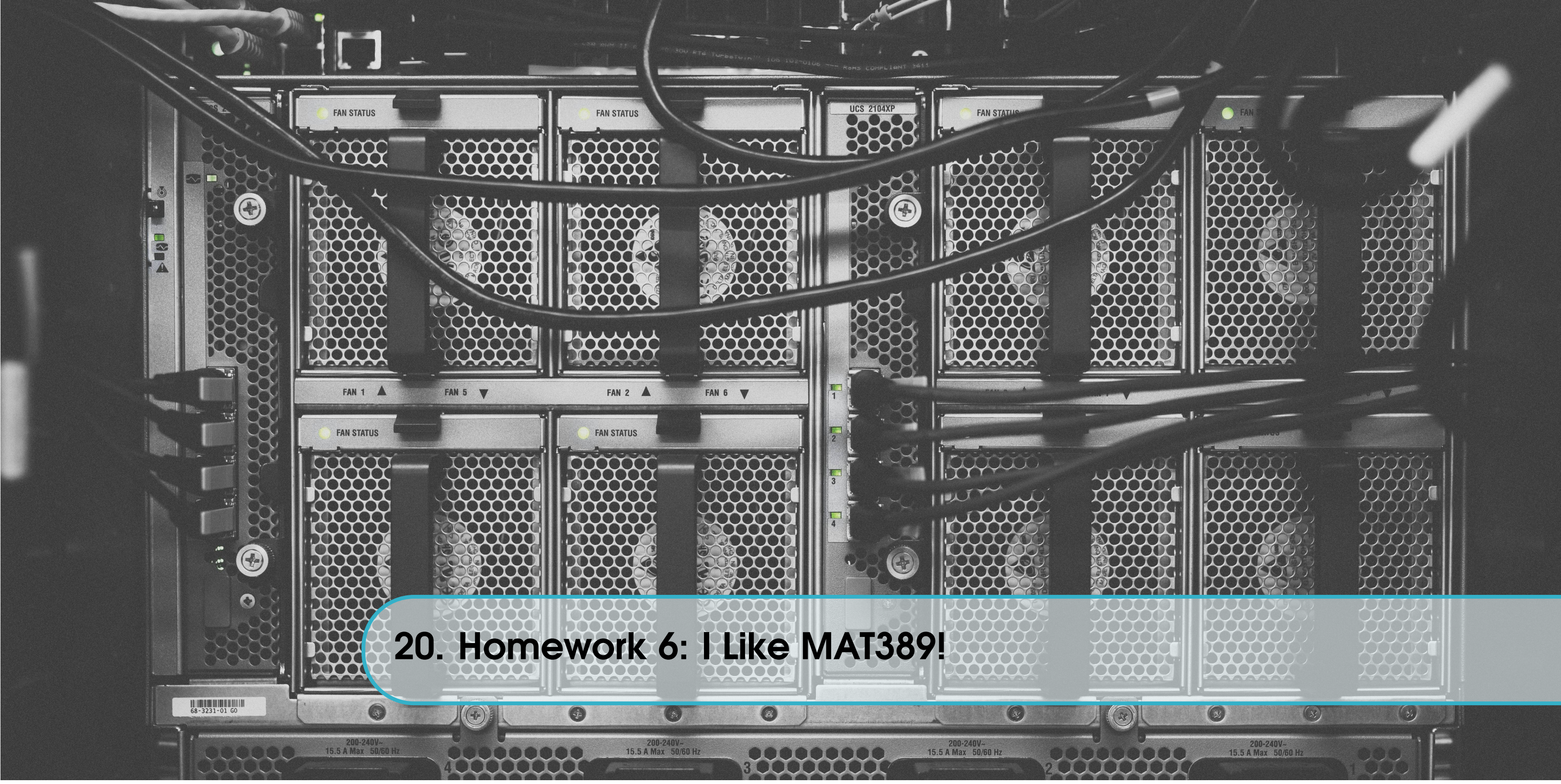
$$\begin{aligned}
\Delta \arg f(\gamma) &= \Delta \arg f(\gamma_I) + \Delta \arg f(\gamma_{II}) + \Delta \arg f(\gamma_{III}) \\
&= 0 + \frac{3\pi}{2} + \frac{\pi}{2} = 2\pi
\end{aligned}$$

We notice that f has no poles in the first quadrant, so by the argument principle:

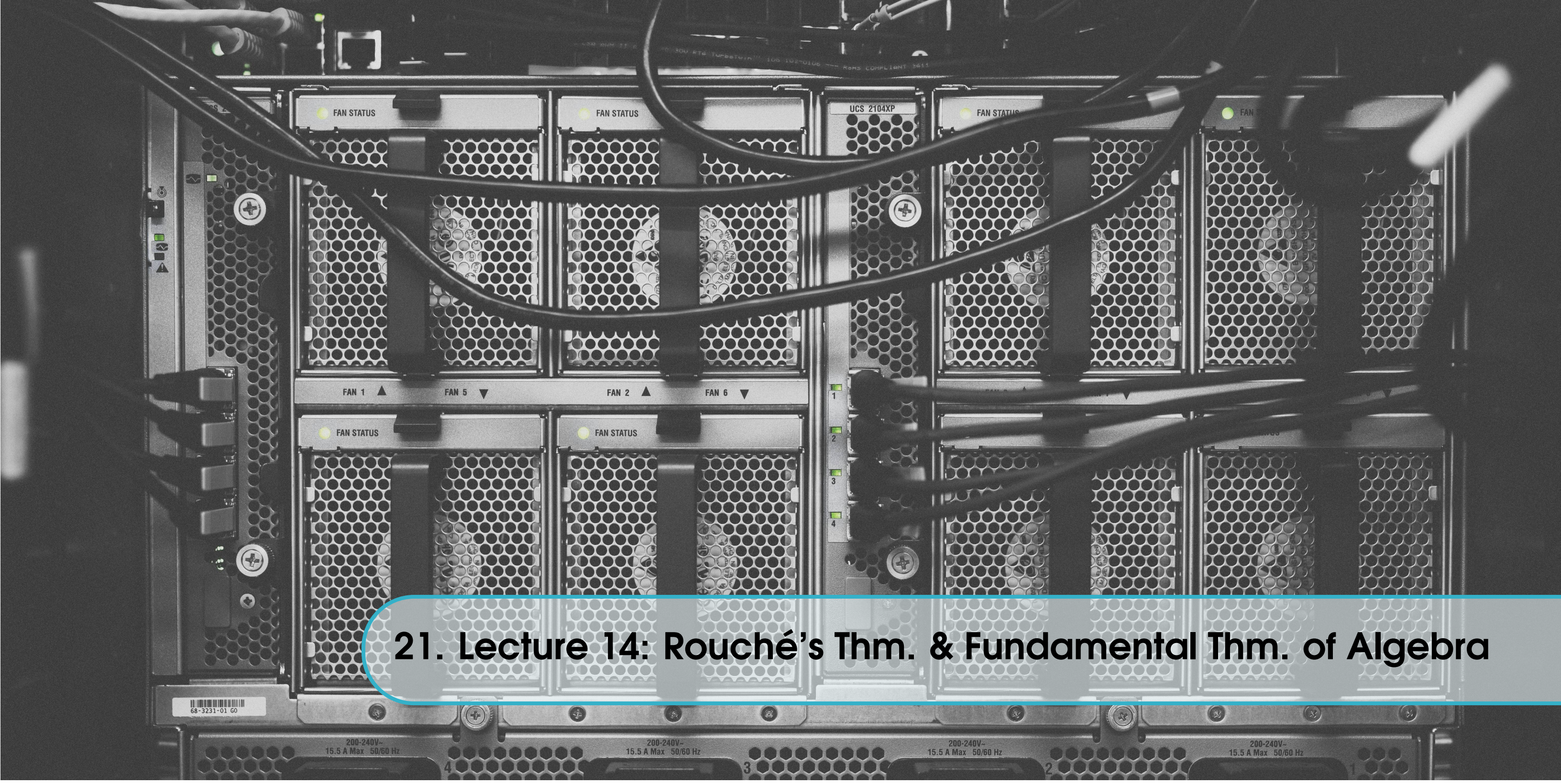
$$\frac{1}{2\pi}\begin{Bmatrix} \text{change in } \arg h(z) \\ \text{as } z \text{ traverses } \gamma \end{Bmatrix} + \begin{Bmatrix} \text{no. of poles of } h \\ \text{inside } \gamma \end{Bmatrix} = \begin{Bmatrix} \text{no. of zeros of } h \\ \text{inside } \gamma \end{Bmatrix}$$

$$
\begin{aligned}
\frac{1}{2\pi}\Delta \arg f(\gamma) &= \frac{1}{2\pi}\left(\frac{3\pi}{2} + \frac{\pi}{2}\right) \\
1 &= \text{no. of zeroes of } f \text{ in the first quadrant}
\end{aligned}
$$

■



20. Homework 6: I Like MAT389!



21. Lecture 14: Rouché's Thm. & Fundamental Thm. of Algebra

21.1 Last Time: Counting zeroes and poles of analytic functions

Proposition 21.1.1 We say:

- h analytic in D, except finite number of poles $z_1, \ldots, z_n$.
- γ piecewise C^1 closed curve, positively oriented, inside$(\gamma) \subset D$ avoiding zeroes and poles of h.

Then

$$(A) \quad \frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\,dz = \#\text{ of zeroes of } h \text{ inside } \gamma - \#\text{ of poles of } h \text{ inside } \gamma.$$

(B) Argument Principle

$$\frac{1}{2\pi i}\left\{\text{Total change in } \operatorname{Arg}(h(z))_\gamma \text{ as } z \text{ traverses}\right\}$$
$$= \#\text{ of zeroes of } h \text{ inside } \gamma - \#\text{ of poles of } h \text{ inside } \gamma.$$

Theorem 21.1.2 — Rouche's Theorem. Suppose $f,\ g$ are analytic on D, γ a curve in D (piecewise C^1, simple, closed).
if $|f(z) + g(z)| < |f(z)| \quad \forall z \in \gamma$ then f and g have the same number of zeroes inside(γ) (counting multiplicities).

Proof. Neither f nor g have zeroes on γ.
Suppose $f,\ g$ have zeroes of multiplicity k at z_0. Then (by definition of zero multiplicity):

$$\tilde{f} = \frac{f(z)}{(z - z_0)^k}, \quad \tilde{g} = \frac{g(z)}{(z - z_0)^k}$$

are analytic on D (even at z_0). And $f(\tilde{z_0}) \neq 0 \neq g(\tilde{z_0})$, and

$$|\tilde{f}(z) + \tilde{g}(z)| < |\tilde{f}(z)| \quad \forall z \in \gamma$$

So the number of zeroes of $\tilde{f}$ (or $\tilde{g}$) inside γ is

$$\#\text{zeroes}(f) = \#\text{zeroes}((z - z_0)^k) + \#\text{zeroes}(\tilde{f}) = k + \#\text{zeroes}(\tilde{f}) \tag{21.1}$$

$$\#\text{zeroes}(f) - k = \#\text{zeroes}(\tilde{f}) \tag{21.2}$$

$$\#\text{zeroes}(g) - k = \#\text{zeroes}(\tilde{g}) \tag{21.3}$$

So the number of zeroes of $\tilde{f}$ (or $\tilde{g}$) inside γ is the number of zeroes of f minus k So it suffices to assume f, g have no common zeroes with γ.
Also, if $\tilde{f}(z_0) = 0 = \tilde{g}(z_0)$ then $|f(z_0) + g(z_0)| < |f(z_0)|$ is impossible anyways.

■

■ **Example 21.1** Consider $\frac{g}{f}$ Which has zeroes at zeroes of g and poles at zeroes of f.
and: $|\frac{g}{f} + 1| = |\frac{g+f}{f}| < 1$ on γ.
Consider the curve $\frac{g}{f}(\gamma)$ So, total change in argument of $\frac{g}{f}$ on γ is zero.

$\frac{g}{f}(\gamma)$ lies in $\{z \mid \mathrm{Re}(z) < 0\}$.

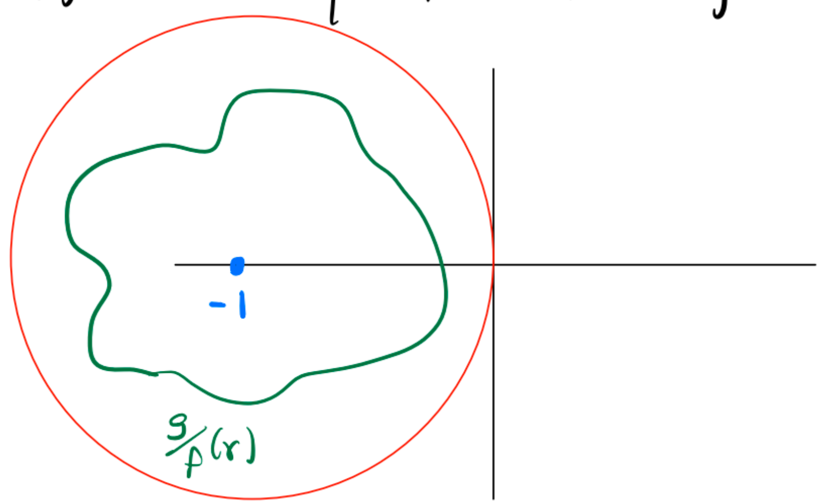


Figure 21.1: Curve $\frac{g}{f}(\gamma)$

So by the Argument Principle:

$$\#\text{zeroes}(\frac{g}{f}) = \#\text{poles}(\frac{g}{f}) = \#\text{zeroes}(f) = \#\text{poles}(g)$$

■

■ **Example 21.2** Show that all zeroes of the polynomial $p(z) = 3z^3 - 2z^2 + 2iz - 8$ lie in the annulus $1 < |z| < 2$.

■

Proof. **We apply Rouche's Theorem: Step 1:** Show that $p(z)$ has no zeroes inside $|z| = 1$.
Consider the claim:

$$|p(z) + 8| < |8| \quad \forall z \in |z| = 1$$

Let's show that it's true:

$$\begin{aligned}|p(z)+8| &= |3z^3-2z^2+2iz| \\ &\leq 3|z|^3+2|z|^2+2|z| \\ &= 3+2+2=7<8\end{aligned}$$

And therefore, the number of zeroes of p in $|z| \leq 1$ is equal to the number of zeroes of 8, which is 0.

Step 2: We expect an order 3 polynomial to have 3 zeroes, because there are none inside $|z|=1$ we expect all 3 to be in $1<|z|<2$.

Consider:

$$\begin{aligned}f(z) &= -3z^3 \\ |p(z)+f(z)| &< |f(z)| \text{ on } |z|=2 \\ |p(z)+f(z)| &= |-2z^2+2iz-8| \\ &\leq 2|z|^2+2|z|+8=2(4)+2(2)+8=20<3\cdot 8=24=|f(z)|\end{aligned}$$

So $p(z)$ has 3 zeroes in $1<|z|<2$ by Rouche's Theorem.

■

Theorem 21.1.3 — The Fundamental Theorem of Algebra. If you have a polynomial $p(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$ with $a_n\neq 0$ and $a_n, z\in\mathbb{C}$ then $p(z)$ has exactly n zeroes in $\mathbb{C}$ (counting multiplicities).

Proof. Let's compare $p(z)$ with z^n on $|z|=R$ for R large enough.

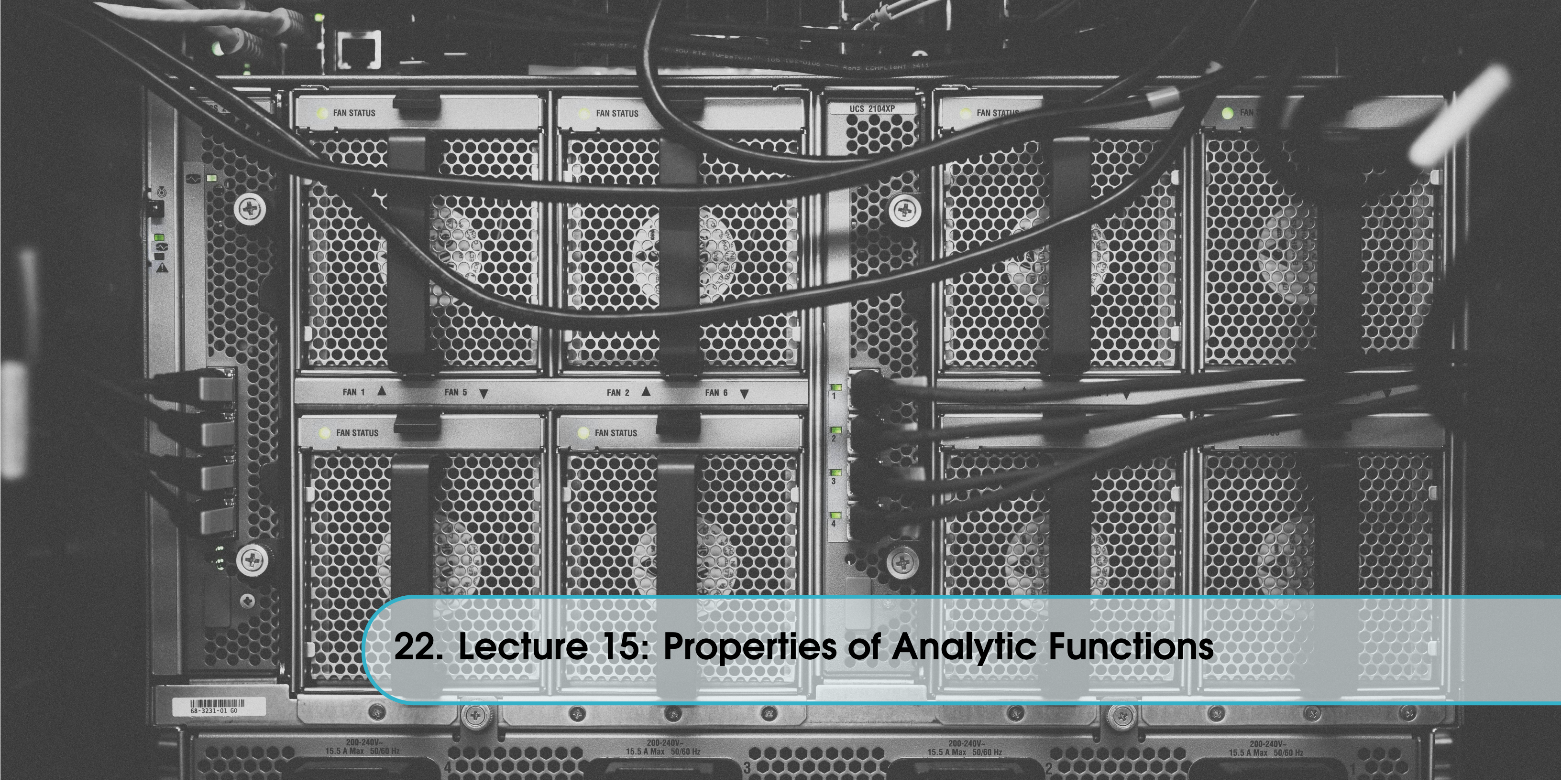
$$\begin{aligned}|p(z)-z^n| &\leq n\max_{0\leq i\leq n-1}|a_i||z|^i \\ &= n\max_{|z|=R}^{n-1}|a_i|R^i \leq R^n=|z|^n\end{aligned}$$

So:

$$|p(z)-z^n|<|z^n| \quad \forall |z|=R \quad \text{for } R \text{ large enough}$$

Rouche's Theorem implies that $p(z)$ and z^n have the same number of zeroes in $|z|<R$ for R large enough.

■



22. Lecture 15: Properties of Analytic Functions

22.1 Maximum Modulus & Mean Value

Theorem 22.1.1 — The Open Mapping Theorem. if f is analytic on a Domain D, then either:

- f is constant
- $f(D) \subseteq D$ is open

Proof. Recall that if $\Re(f) = 0$, then f is constant Suppose f is analytic, but not constant.
Let $w_0 = f(z_0)$
Since f is not constant, we choose $r > 0$ (small), such that $f(z) - w_0$ has no zero in the set of $0 < |z - z_0| < r$
Why?: Since zeroes of analytic functions are isolated
Let $0 < \delta = \min_{z \in |z - z_0| = r} |f(z) - w_0| > 0$
if $|w - w_0| < \delta$, then on $|z - z_0| = r$:

$$|(f(z) - w) - (f(z) - w_0)| = |w - w_0| < \delta \leq |f(z) - w_0|$$

By Rouche's Theorem, $f(z) - w$ has the same number of zeroes in $|z - z_0| \leq r$ as $f(z) - w_0$ (in particular, at least 1)

$$\Rightarrow w \in f(D) \text{ so } \forall w \mid |w - w_0| < \delta, \exists z \in D \mid f(z) = w$$
$$\{|w - w_0| < \delta \in f(D)\}$$

So $f(D)$ is open ∎

Definition 22.1.1 — Mapping. A function $f : D \to \mathbb{C}$ is said to be $m \to 1$ at $z_0 \in D$ if in the neighborhood around z_0 there are exactly m points in the domain z that map to the same point $w = f(z_0)$ in the codomain.

Corollary 22.1.2 if f is a non-constant analytic function on D and $f(z) - f(z_0)$ has a zero of order m at z_0, then $f(z)$ is $m \to 1$ map near z_0, in particular, if $f'(z_0) = 0$, then f is not $1 \to 1$ in any disc around z_0. Furthermore, any disc around z_0 contains m points $z \neq z_0$ such that $f(z) = f(z_0)$

■ **Example 22.1** if $f(z) = z^2$, has a zero of order 2 at 0
if $z \neq 0$, then $f(z) = f(-z)$, so f is $2 \to 1$ ■

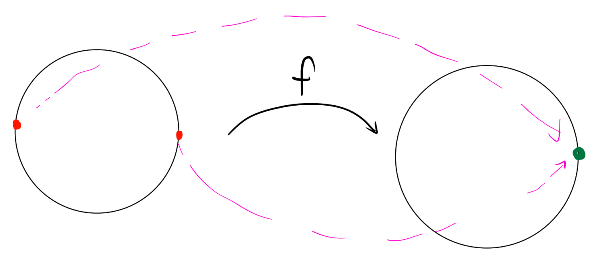


Figure 22.1: Mapping of $f(z) = z^2$

Corollary 22.1.3 — Maximum Modulus Principle. if f is a non-constant analytic function on the domain D, then $|f|$ has no local maximum.

R This may seem counterintuitive, but this is unique to complex functions. It is possible for a real differentiable function, f, to be non-constant and $|f|$ to have a local maximum.

Proof. Suppose $|f|$ has a local max at z_0, i.e.:

$$|f(z)| \leq |f(z_0)| \quad \forall |z - z_0| \leq r$$

For some $r > 0$, then $f(z_0)$ lies on the boundary of the set $\{f(z) \mid |z - z_0| < r\}$ because the maximum modulus in an open set is on the boundary, which is a contradiction to the open mapping theorem which states that $f(D)$ is open. ■

Theorem 22.1.4 — Schwartz Lemma. Suppose f is analytic in a disc:

$$\{|z| < 1\},\ f(0) = 0 \text{ and } |f(z)| \leq 1 \forall |z| < 1 \tag{22.1}$$

$$\tag{22.2}$$

<u>Then</u> $|f(z)| \leq |z|\ \forall\ |z| < 1$, <u>and</u>

$$|f(\tilde{z})| = |\tilde{z}| \text{ for some } z \neq 0, \text{ if and only if} \tag{22.3}$$

$$f(z) = \lambda z \text{ for } \lambda \in \mathbb{C},\ |\lambda| = 1 \tag{22.4}$$

Proof. Say:

$$g(z) = \frac{f(z)}{z}, \quad |z| < 1$$

Since $f(0) = 0$ we know $g(z) = \frac{f(z)}{z}$ is analytic for $|z| < 1$
And on $|z| = r \quad (r < 1)$ we can write:

$$|g(z)| \leq \frac{|f(z)|}{r} \leq \frac{1}{r} \tag{22.5}$$

So, by max modulus principle:

$$|g(z)| < \frac{1}{r} \text{ on } \{|z| \leq r\}$$

Because $\{g(z) \;:\; |z| = r \to 1\}$ needs to be open.
Taking $r \to 1$, conclude

$$|g(z)| \leq \frac{|f(z)|}{|z|} \leq 1 \to f(z) \leq |z|$$

We say $f(z) \leq |z|$ because $z \leq 1$, so we don't worry about openness.
if $|f(z_0)| = |z_0|$, for some
Then $|g(z)| = 1$, so $g(z)$ has an interior maximum. This means $g(z)$ is constant, so $f(z) = \lambda z$ for $|\lambda| = 1$ ■

22.2 Mean-Value Property

Theorem 22.2.1 Suppose $f = u + iv$ is analytic on $\{|z - z_0| \leq r\}$. Then, for any $s \leq r$, the average value of u and v on the circle $|z - z_0| = s$ is given by:

$$\left.\begin{aligned} u(z_0) &= \tfrac{1}{2\pi} \textstyle\int_0^{2\pi} u(z_0 + se^{i\theta})\, d\theta \\ v(z_0) &= \tfrac{1}{2\pi} \textstyle\int_0^{2\pi} v(z_0 + se^{i\theta})\, d\theta \end{aligned}\right\} = \text{Average value of } u \text{ or } v \text{ on the circle } |z - z_0| = s$$

Proof. **Cauchy's Integral Formula** with $\gamma(\theta) = z_0 + se^{i\theta}$:

$$f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)}{z - z_0} dz$$

$$\to z = z_0 + se^{i\theta} \quad dz = ise^{i\theta} d\theta$$

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + se^{i\theta})}{se^{i\theta}} ise^{i\theta} d\theta$$

Now we take the real and imaginary parts of the above equation to get the average value of u and v on the circle $|z - z_0| = s$.

■

22.3 Fractional Linear Transformations

Definition 22.3.1 — Fractional Linear Transformation. A FLT is a rational function of the form:

$$T(z) = \frac{az + b}{cz + d} \quad a, b, c, d \in \mathbb{C}, ad - bc \neq 0$$

R Note: if $ad = bc$, then:

$$T'(z) = \frac{a(cz + d) - (az + b)c}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2} = 0$$

So $T(z)$ is constant, and we're not interested in constant functions

Claim 22.3.1 T is $1 \to 1$
If $T(z_1) = T(z_2) \implies z_1 = z_2$.

Proof.

$$\begin{aligned} \frac{az_1 + b}{cz_1 + d} &= \frac{az_2 + b}{cz_2 + d} \\ \iff \quad acz_1^2 + adz_1 + bcz_1 + bd &= acz_2^2 + adz_2 + bcz_2 + bd \\ (ad - bc)z_1 &= (ad - bc)z_2 \\ z_1 &= z_2 \end{aligned}$$

■

Claim 22.3.2 — T has an inverse.

$$T^{-1}(z) = \frac{dz - b}{-cz + a}$$

Which is also a FLT

Proof.

$$\begin{aligned} T(z) &= w \\ &= \frac{az + b}{cz + d} \\ \to z &= \frac{dw - b}{-cw + a} \\ \to T^{-1}(w) &= \frac{dw - b}{-cw + a} = z \end{aligned}$$

■

Claim 22.3.3 — Composition of FLT. if T_1, T_2 are FLT, then $T_1 \circ T_2$ is also a FLT

Proof.

$$T_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$$

$$\begin{aligned}
T_2(z) &= \frac{a_2z + b_2}{c_2z + d_2} \\
T_1(T_2(z)) &= \frac{a_1(\frac{a_2z+b_2}{c_2z+d_2}) + b_1}{c_1(\frac{a_2z+b_2}{c_2z+d_2}) + d_1} \\
&= \frac{(a_1a_2z + a_1b_2 + b_1c_2z + b_1d_2)}{(c_1a_2z + c_1b_2 + d_1c_2z + d_1d_2)} \\
&= \frac{(a_1a_2z + b_1b_2)}{(c_1a_2z + d_1b_2)} \\
&= \frac{a_1a_2z + b_1b_2}{c_1a_2z + d_1b_2}
\end{aligned}$$

■

22.3.1 Matrix Representation of FLT

Theorem 22.3.4 Given $T(z) = \frac{az+b}{cz+d}$, we can represent T as a matrix:

$$A_T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Where A_T is a 2×2 matrix with determinant $ad - bc \neq 0$. So, A_T is invertible

R Also:

$$A_{T_1 \circ T_2} = A_{T_1}A_{T_2}$$

Note: if $\lambda \in \mathbb{C}$, $\lambda \neq 0$, then A and λA give rise to the same FLT.

22.3.2 Point at Infinity

Definition 22.3.2 Recall that we can add the point at ∞ to $\mathbb{C}$ to get the 2 sphere $S^2 = \mathbb{C} \cup \{\infty\}$, $\{x^2 + y^2 + z^2 = 1\}$

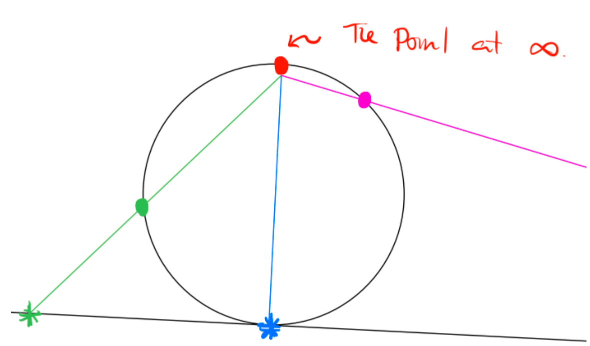


Figure 22.2: The Riemann Sphere

R An FLT should be be thought of in terms of how it moves points in S^2

Theorem 22.3.5 Consider $T(z) = \frac{az+b}{cz+d}$, $(c \neq 0)$
- when $z = \frac{-d}{c}$, $T(z) = \infty$
- $\lim_{z\to\infty} T(z) = \frac{a+\frac{b}{z}}{c+\frac{d}{z}} = \frac{a}{c}$

■ **Example 22.2** If $a, c = 1$, Then $(d \neq 0)$:
- $T(0) = \frac{b}{d}$
- $T(\infty) = \frac{1}{d}$
- $T(\frac{-d}{c}) = \infty$

■

22.3.3 Fixed Points

Definition 22.3.3 — Fixed Points. A fixed point of $T(z) = z$ is a point z_0 such that $T(z_0) = z_0$

Lemma 22.3.6 An FLT has either $\leq$ 2 fixed points or ∞ fixed points (it's the identity function)

Proof.

$$T(z) = \frac{az+b}{cz+d} = z$$

So:

$$cz^2 + (d-a)z - b = 0$$

So, z has at most 2 solutions, unless $c = 0$, in which case $T(z) = z$ for all z ■

Proposition 22.3.7 A consequence of the above lemma is that if we're given:

$$\{z_1, z_2, z_3\} \mathbb{C} \cup \{\infty\} = S^2 \quad \text{distinct}$$
$$\{w_1, w_2, w_3\} \subset \mathbb{C} \cup \{\infty\} \quad \text{distinct}$$

Then, there exists a unique FLT T such that $T(z_i) = w_i, \quad i = 1, 2, 3$

Proof. Consider:

$$L(z) = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}$$

Then:
- $L(z_1) = 0$
- $L(z_2) = 1$
- $L(z_3) = \infty$

Similarly

$$S(w) = \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1}$$

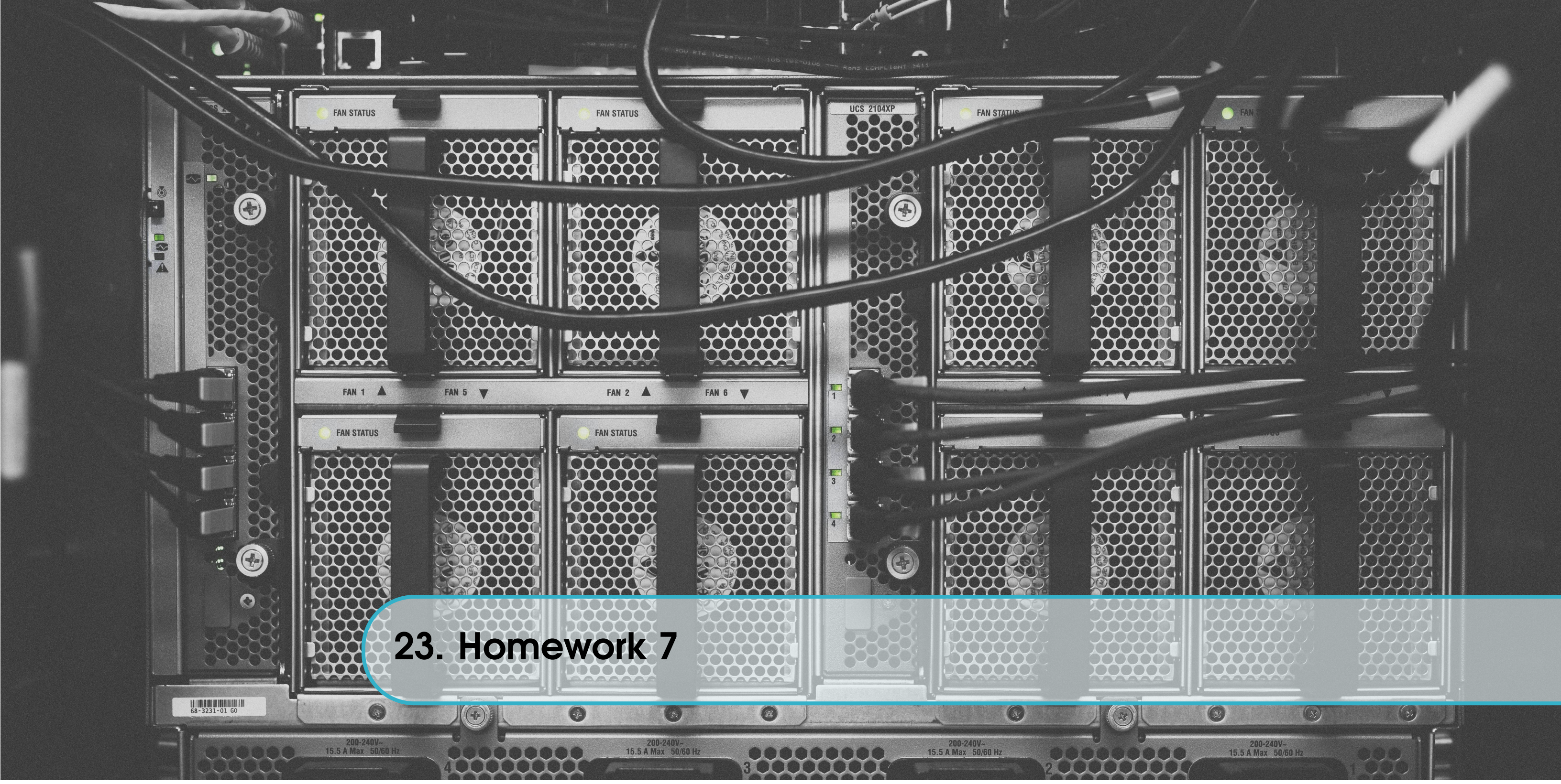
Then:

- $S(w_1) = 0$
- $S(w_2) = 1$
- $S(w_3) = \infty$

So, $S^{-1} \circ L$ is the FLT we're looking for. T is unique since if $\tilde{T}$ maps $z_1 \to w_1$, then $T^{-1} \circ \tilde{T}(f)$ fixes z_1, z_2, z_3, so it must be the identity function.

$$\tilde{T} = T$$

■



23. Homework 7

■ **Example 23.1 — Fisher, Section 2.5, Problem 10.** Find the first four terms of the Laurent series of $f(z) = \frac{z}{(\sin(z))^2}$ around $z_0 = 0$.That is, find an expansion of the form

$$f(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + O(z^3)$$

For $a_{-1}, a_0, a_1, a_2 \in \mathbb{C}$.

We know that:

$$\begin{aligned}
\sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \\
(\sin(z))^2 &= z^2 - \frac{2z^4}{3!} + \frac{2z^6}{5!} - \frac{2z^8}{7!} + \cdots
\end{aligned}$$

Substituting that back into the original function, we get:

$$\begin{aligned}
f(z) &= \frac{z}{(\sin(z))^2} \\
&= \frac{z}{z^2 - \frac{2z^4}{3!} + \frac{2z^6}{5!} - \frac{2z^8}{7!} + \cdots} \\
&= \frac{1}{z - \frac{2z^3}{3!} + \frac{2z^5}{5!} - \frac{2z^7}{7!} + \cdots} \\
&= \frac{1}{z} \cdot \frac{1}{1 - \frac{2z^2}{3!} + \frac{2z^4}{5!} - \frac{2z^6}{7!} + \cdots} \\
&= \frac{1}{z} \cdot \left(1 + \frac{2z^2}{3!} + \frac{2z^4}{5!} + \frac{2z^6}{7!} + \cdots\right) \\
&= \frac{1}{z} + \frac{2z}{3!} + \frac{2z^3}{5!} + \frac{2z^5}{7!} + \cdots \\
&= \frac{1}{z} + \frac{2}{3!}z + \frac{2}{5!}z^3 + \frac{2}{7!}z^5 + \cdots
\end{aligned}$$

$$= \frac{1}{z} + 0 + \frac{1}{3}z + 0 \cdot z^2 + O(z^3)$$

Therefore, the first four terms of the Laurent series of $f(z) = \frac{z}{(\sin(z))^2}$ around $z_0 = 0$ are:

$$a_{-1} = 1 \quad a_0 = 0 \quad a_1 = \frac{1}{3} \quad a_2 = 0$$

■

■ **Example 23.2 — Fisher, Section 2.5, Problem 12.** Find the first four terms of the Laurent series of $f(z) = \frac{1}{e^z - 1}$ around $z_0 = 0$. That is,

$$f(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + O(z^3)$$

for $a_{-1}, a_0, a_1, a_2 \in \mathbb{C}$.

We know that:

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \\ e^z - 1 &= z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \end{aligned}$$

Substituting that back into the original function, we get:

$$\begin{aligned} f(z) &= \frac{1}{e^z - 1} \\ &= \frac{1}{z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots} \\ &= \frac{1}{z} \cdot \frac{1}{1 + \frac{z}{2} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots} \end{aligned}$$

We can use the geometric series formula to simplify the above expression, we know that:

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots$$

Therefore, we can rewrite the denominator as:

$$\begin{aligned} \frac{1}{1 + \frac{z}{2} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots} &= \frac{1}{1 - \left(-\frac{z}{2} - \frac{z^2}{3!} - \frac{z^3}{4!} - \cdots\right)} \\ &\approx \frac{1}{1 - (-\frac{z}{2})} \quad \text{for small } z \\ &\rightarrow w = -\frac{z}{2} \quad \text{so} \\ &= 1 - \frac{z}{2} + (-\frac{z}{2})^2 + (-\frac{z}{2})^3 + \cdots \end{aligned}$$

$$= 1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots$$

Substituting that back into the original function, we get:

$$\begin{aligned} f(z) &= \frac{1}{z} \cdot \left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right) \\ &= \frac{1}{z} - \frac{1}{2} + \frac{z}{4} - \frac{z^2}{8} + \cdots \end{aligned}$$

Therefore the first four terms of the Laurent series of $f(z) = \frac{1}{e^z - 1}$ around $z_0 = 0$ are:

$$a_{-1} = 1 \quad a_0 = -\frac{1}{2} \quad a_1 = \frac{1}{4} \quad a_2 = -\frac{1}{8}$$

■

■ **Example 23.3 — Fisher, Section 2.6, Problem 9.** Compute the integral

$$\int_0^{2\pi} \frac{d\theta}{(2 - \sin(\theta))^2}$$

From the theorem in lecture 10, we know that residues of a function $f(z) = \frac{H(z)}{(z - z_0)^m}$ at a pole z_0 are given by: $\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{H(z)}{(z - z_0)^m} = c_{m-1}$, where c_{m-1} is the coefficient of the $(z - z_0)^{m-1}$ term in the Laurent series of $f(z)$ around z_0. Therefore, we can find the residue of the integrand at $z = 0$ by finding the coefficient of the z^{-1} term in the Laurent series of the integrand around $z = 0$. We know that $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, so:

$$\begin{aligned} (2 - \frac{e^{i\theta} - e^{-i\theta}}{2i})^2 &= (\frac{4i - e^{i\theta} + e^{-i\theta}}{2i})^2 \\ &\to z = e^{i\theta} \\ &= (\frac{4i - z + \frac{1}{z}}{2i})^2 \\ &= -\frac{-16 - 4iz + \frac{4i}{z} - 4iz + z^2 - 1 + \frac{4i}{z} - 1 + \frac{1}{z^2}}{4} \\ &= -\frac{-18 - 8iz + z^2 + \frac{1}{z^2} + \frac{8i}{z}}{4} \\ &= \frac{9}{2} + 2iz - \frac{z^2}{4} - \frac{1}{4z^2} - \frac{2i}{z} \end{aligned}$$

We can also change the infinitesimal $d\theta$ to dz:

$$\begin{aligned} dz &= ie^{i\theta} d\theta \\ d\theta &= \frac{dz}{iz} \end{aligned}$$

Therefore, the integral becomes:

$$\int_0^{2\pi} \frac{d\theta}{(2 - \sin(\theta))^2} = \oint_{|z|=1} \frac{dz}{iz(\frac{9}{2} + 2iz - \frac{z^2}{4} - \frac{1}{4z^2} - \frac{2i}{z})}$$

Now we can use the residue theorem to evaluate the integral. Now we want to find where the integrand is singular. We can do this by setting the denominator to zero and solving for z:

$$4z^2\left(\frac{9}{2} + 2iz - \frac{z^2}{4} - \frac{1}{4z^2} - \frac{2i}{z}\right) = 0$$
$$18z^2 + 8iz^3 - z^4 - 1 - 8iz = 0$$

We assume that the solution is of the form $(z^2 + Az + B)(z^2 + Cz + D)$, so we can expand the above equation to get:
$z^4 + (A + C)z^3 + (AC + B + D)z^2 + (AD + BC)z + BD = z^4 - 8iz^3 - 18z^2 + 8iz + 1 = 0$
This gives the system of equations:

$$\begin{aligned} A + C &= -8i \\ AC + B + D &= -18 \\ AD + BC &= 8i \\ BD &= 1 \end{aligned}$$

Let's guess that $B = D = -1$ Because this would satisfy the last equation and make the first and third equations equivalent.

$$\begin{aligned} A + C &= -8i \\ AC &= -16 \\ -A - C &= 8i \end{aligned}$$

By inspection we can see that $A = C = -4i$, this allows us to factor the denominator as:

$$(z^2 - 4iz - 1)^2 = 0$$

We can use the quadratic formula to find the roots of the above equation:

$$\begin{aligned} z &= \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \\ z &= \frac{-4i \pm \sqrt{(-4i)^2 - 4(1)(-1)}}{2(1)} \\ z &= \frac{-4i \pm \sqrt{-16 + 4}}{2} \\ z &= \frac{-4i \pm \sqrt{-12}}{2} \\ z &= i(2 \pm \sqrt{3}) \end{aligned}$$

Our contour is the unit circle where $|z| = 1$, so we're only considering residues within the unit circle.

$$|z_1| = |i(2 + \sqrt{3})| = 2 + \sqrt{3} > 1$$

$$|z_2| = |i(2 - \sqrt{3})| = 2 - \sqrt{3} < 1$$

Therefore, the only pole within the unit circle is $z_2 = i(2 - \sqrt{3})$, and it's a double pole. We can find the residue at z_2 by finding the coefficient of the z^{-1} term in the Laurent series of the integrand around z_2.

$$\begin{aligned} f(z) &= \frac{1}{iz(\frac{9}{2} + 2iz - \frac{z^2}{4} - \frac{1}{4z^2} - \frac{2i}{z})} \\ &= \frac{4zi}{(z - i(2 + \sqrt{3}))^2(z - i(2 - \sqrt{3}))^2} \end{aligned}$$

Using the residue theorem that says:

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$$

$$\oint_{|z|=1} \frac{4zi}{(z - i(2 + \sqrt{3}))^2(z - i(2 - \sqrt{3}))^2} = 2\pi i \mathrm{Res}(f, z_2)$$

We can find the residue at z_2, at $m = 2$

$$\begin{aligned} \mathrm{Res}(f, z_k) &= \frac{1}{(m - 1)!} \lim_{z \to z_k} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_k)^m f(z)\right) \\ &= \frac{1}{1!} \lim_{z \to z_2} \frac{d}{dz} \left((z - z_2)^2 f(z)\right) \\ &= \lim_{z \to z_2} \frac{d}{dz} \left(\frac{4zi(z - z_2)^2}{(z - z_1)^2(z - z_2)^2}\right) \\ &= \lim_{z \to z_2} \frac{d}{dz} \left(\frac{4zi}{(z - z_1)^2}\right) \\ &= \lim_{z \to z_2} \frac{4(z - z_1) - 8i}{(z - z_1)^3} \\ &= \frac{4(i(2 - \sqrt{3}) - i(2 + \sqrt{3})) - 8i}{(i(2 - \sqrt{3}) - i(2 + \sqrt{3}))^3} \\ &= \frac{4(-2\sqrt{3}i) - 8i}{(-2\sqrt{3}i)^3} \\ &= \frac{-8\sqrt{3}i - 8i}{8 \cdot 3i\sqrt{3}} \\ &= \frac{-\sqrt{3} - 1}{3\sqrt{3}} \end{aligned}$$

Therefore, the integral is:

$$\int_0^{2\pi} \frac{d\theta}{(2 - \sin(\theta))^2} = \oint_{|z|=1} \frac{dz}{iz(\frac{9}{2} + 2iz - \frac{z^2}{4} - \frac{1}{4z^2} - \frac{2i}{z})}$$

$$
\begin{aligned}
&= 2\pi i \text{Res}(f, z_2) \\
&= 2\pi i \left(= \frac{-\sqrt{3} - 1}{3\sqrt{3}}\right) \\
&= \frac{-2\pi}{3\sqrt{3}} i
\end{aligned}
$$

So in summary, the steps to solve the integral are:

- Recognize that integrals of the form $\int f(\sin(\theta), \cos(\theta))d\theta$ can be solved by converting to complex form.
- Convert the integrand to complex form by using the identity $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.
- Perform a change of variables to convert the integral from $d\theta$ to dz by using $z = e^{i\theta}, \ d\theta = \frac{dz}{iz}$.
- Also remember to change the limits of integration to $|z| = 1$.
- Next we realize that it's possible that within our contour, the integrand has singularities. So we must use either the residue theorem or the Cauchy integral formula to evaluate the integral.
- In order to use the residue theorem, or Cauchy Integral Theorem, we must find the singularities of the integrand. We do this by setting the denominator to zero and solving for z.
- We find that the integrand has a double pole at $z = i(2 - \sqrt{3})$.
- Because there's one pole within the unit circle, we could use the cauchy integral formula to evaluate the integral. However, we chose to use the residue theorem.
- There are multiple ways to solve for a residue, note that m is the order of the pole.
  - We can use the formula $\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{H(z)}{(z - z_0)^m} = c_{m-1}$, where c_{m-1} is the coefficient of the $(z - z_0)^{m-1}$ term in the Laurent series of $f(z)$ around z_0.
  - We can also use the formula $\text{Res}(f, z_k) = \frac{1}{(m-1)!} \lim_{z \to z_k} \frac{d^{m-1}}{dz^{m-1}}((z - z_k)^m f(z))$
  - or we can just expand the integrand into a Laurent series **around the singularity (not zero)**, then find the coefficient of the z^{m-1} term
- After finding the residue, we can use the residue theorem to evaluate the integral: $\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$.

■

■ **Example 23.4 — Fisher, Section 2.6, Problem 10.** Compute the integral

$$\int_0^{2\pi} \frac{d\theta}{(1 + \beta \cos(\theta))^2} \quad \text{for } -1 < \beta < 1$$

We can convert the denominator to a complex form by using the identity $\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$, so:

$$
\begin{aligned}
(1 + \beta \cos(\theta))^2 &= (1 + \frac{\beta}{2}(e^{i\theta} + e^{-i\theta}))^2 \\
&\to z = e^{i\theta} \quad dz = ie^{i\theta} d\theta \to d\theta = \frac{dz}{iz} \\
&= (1 + \frac{\beta}{2}(z + \frac{1}{z}))^2 \\
\frac{d\theta}{(1 + \beta \cos(\theta))^2} &= \frac{dz}{iz(1 + \frac{\beta}{2}(z + \frac{1}{z}))^2}
\end{aligned}
$$

The limits of integration can be converted as well:

$$\begin{aligned}\theta = 0 &\rightarrow z = e^{i0} = 1\\ \theta = 2\pi &\rightarrow z = e^{i2\pi} = 1\end{aligned}$$

The path that the integral is taken over is the unit circle, so $|z| = 1$. We can now substitute the limits of integration and the integrand into the integral:

$$\int_0^{2\pi} \frac{d\theta}{(1 + \beta\cos(\theta))^2} = \oint_{|z|=1} \frac{dz}{iz(1 + \frac{\beta}{2}(z + \frac{1}{z}))^2}$$

We can now find the singularities of the integrand by setting the denominator to zero and solving for z:

$$\begin{aligned}
iz(1 + \frac{\beta}{2}(z + \frac{1}{z}))^2 &= 0\\
(1 + \frac{\beta}{2}(z + \frac{1}{z}))^2 &= 0\\
(\beta(z + \frac{1}{z}) + \frac{\beta^2}{4}(z + \frac{1}{z})^2 + 1) &= 0\\
\beta(z + z^{-1}) + \frac{\beta^2}{4}(z^2 + 2 + z^{-2}) + 1 &= 0\\
z^2\left(\beta z + \beta z^{-1} + \frac{\beta^2}{4}z^2 + \frac{\beta^2}{2} + \frac{\beta^2}{4}z^{-2} + 1\right) &= 0\\
\beta z^3 + \beta z + \frac{\beta^2}{4}z^4 + \frac{\beta^2}{2}z^2 + \frac{\beta^2}{4} + z^2 &= 0\\
\frac{\beta^2}{4}z^4 + \beta z^3 + \frac{\beta^2}{2}z^2 + z^2 + \beta z + \frac{\beta^2}{4} &= 0\\
\frac{\beta^2}{4}z^4 + \beta z^3 + (\frac{\beta^2}{2} + 1)z^2 + \beta z + \frac{\beta^2}{4} &= 0
\end{aligned}$$

We guess that the solution is symmetric, that is, it can be decomposed into two identical quadratics, i.e. $(Az^2 + Bz + C)^2$:
$A^2z^4 + 2ABz^3 + (2CA + B^2)z^2 + 2BCz + C^2 = \frac{\beta^2}{4}z^4 + \beta z^3 + (\frac{\beta^2}{2} + 1)z^2 + \beta z + \frac{\beta^2}{4}$

This gives the system of equations:

$$\begin{aligned}
A^2 &= \frac{\beta^2}{4}\\
2AB &= \beta\\
(2CA + B^2) &= (\frac{\beta^2}{2} + 1)\\
2BC &= \beta
\end{aligned}$$

$$C^2 = \frac{\beta^2}{4}$$

Right away we know $C = \pm\frac{\beta}{2}$, and so by inspection we can see that:

$$A = C = \frac{\beta}{2}$$
$$B = 1$$

Therefore, the denominator can be factored as:

$$\left(\frac{\beta}{2}z^2 + z + \frac{\beta}{2}\right)^2 = 0$$

We can use the quadratic formula to find the roots of the above equation:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$z = \frac{-1 \pm \sqrt{1 - 4(\frac{\beta}{2})(\frac{\beta}{2})}}{2(\frac{\beta}{2})}$$
$$z = \frac{-1 \pm \sqrt{1 - \beta^2}}{\beta}$$
$$z_1 = \frac{-1 + \sqrt{1 - \beta^2}}{\beta}$$
$$z_2 = \frac{-1 - \sqrt{1 - \beta^2}}{\beta}$$

Multiplying the simplifying factors back in $(\frac{iz}{z^2})$ gets us the factorization:

$$\frac{i}{z}(z - z_1)^2(z - z_2)^2$$

Now we find which poles are within the unit circle:

$$|z_1| \rightarrow 0 \leq \left|\frac{-1 + \sqrt{1 - \beta^2}}{\beta}\right| \leq 1$$
$$|z_2| \rightarrow 1 \leq \left|\frac{-1 - \sqrt{1 - \beta^2}}{\beta}\right| \leq 2$$

Therefore, the only pole within the unit circle is $z_1 = \frac{-1+\sqrt{1-\beta^2}}{\beta}$, and it's a double pole. We can now use the residue theorem to evaluate the integral. It says:

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$$

$$\oint_{|z|=1} \frac{z}{i(z - z_1)^2(z - z_2)^2} = 2\pi i \text{Res}(f, z_1)$$

Using the formula for the residue of a double pole:

$$\begin{aligned}
\text{Res}(f, z_k) &= \frac{1}{(m-1)!} \lim_{z \to z_k} \frac{d^{m-1}}{dz^{m-1}}((z - z_k)^m f(z)) \\
\text{Res}(f, z_1) &= \frac{1}{1!} \lim_{z \to z_1} \frac{d}{dz} \left((z - z_1)^2 f(z)\right) \\
\text{Res}(f, z_1) &= \lim_{z \to z_1} \frac{d}{dz} \left(\frac{z(z - z_1)^2}{i(z - z_1)^2(z - z_2)^2}\right) \\
\text{Res}(f, z_1) &= \lim_{z \to z_1} \frac{d}{dz} \left(\frac{z}{i(z - z_2)^2}\right) \\
\text{Res}(f, z_1) &= \lim_{z \to z_1} \left(\frac{2i - i(z - z_2)}{(z - z_2)^3}\right) \\
\to z_1 - z_2 &= \frac{-1 + \sqrt{1 - \beta^2}}{\beta} - \frac{-1 - \sqrt{1 - \beta^2}}{\beta} = \frac{2\sqrt{1 - \beta^2}}{\beta} \\
\text{Res}(f, z_1) &= \frac{2i - i(\frac{2\sqrt{1-\beta^2}}{\beta})}{(\frac{2\sqrt{1-\beta^2}}{\beta})^3} \\
\text{Res}(f, z_1) &= \frac{i\beta^3}{4\sqrt{1 - \beta^2}^3} - \frac{i\beta^2}{4(1 - \beta^2)}
\end{aligned}$$

Therefore, the integral is:

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{(1 + \beta \cos(\theta))^2} &= \oint_{|z|=1} \frac{dz}{iz(1 + \frac{\beta}{2}(z + \frac{1}{z}))^2} \\
&= 2\pi i \text{Res}(f, z_1) \\
&= 2\pi i \left(\frac{i\beta^3}{4\sqrt{1 - \beta^2}^3} - \frac{i\beta^2}{4(1 - \beta^2)}\right) \\
&= \frac{\pi}{2} \left(-\frac{\beta^3}{\sqrt{1 - \beta^2}^3} + \frac{\beta^2}{1 - \beta^2}\right)
\end{aligned}$$

■

■ **Example 23.5 — Fisher, Section 2.6, Problem 13.** Compute the following integral

$$\int_0^{\infty} \frac{x^\alpha}{x^2 + 3x + 2} dx \quad \text{for } 0 < \alpha < 1$$

We can convert the integral into a complex integral by realizing that $x = z$ and $x^\alpha = e^{\alpha \log(x)} = e^{\alpha \log(z)} = z^\alpha, \ dx = dz$.

$$\int_0^{\infty} \frac{x^\alpha}{x^2 + 3x + 2} dx = \int_0^{\infty} \frac{e^{\alpha \log(z)}}{z^2 + 3z + 2} dz$$

Because we've introduced a logarithm into the integral, we know we must choose a keyhole contour to evaluate the integral. Because the integral is evaluated over the positive real axis, we can choose the branch cut to be the positive real axis. This is because we know for a keyhole contour that:

$$\begin{aligned}\oint_{\text{Keyhole}=C} f(z)dz &= (\text{Path above the branch cut}) - (\text{Path below the branch cut}) \\ &+ (\text{Large arc}) - (\text{Small arc})\end{aligned}$$

And as $R \to \infty$ and $R \to 0$, the integrals over the large and small arcs go to zero. So:

$$\begin{aligned}\oint_{\text{Keyhole}=C} f(z)dz &= (\text{Path above the branch cut}) - (\text{Path below the branch cut}) \\ &= (1 - e^{2\pi i\alpha}) \int_0^\infty f(z)dz\end{aligned}$$

We can now convert the integrand into a complex form by using the identity $\log(z) = \log|z| + i\arg(z)$:

$$\begin{aligned}\frac{e^{\alpha \log(z)}}{z^2 + 3z + 2} &= \frac{e^{\alpha(\log|z| + i\arg(z))}}{z^2 + 3z + 2} \\ &= \frac{e^{\alpha \log|z|} e^{i\alpha \arg(z)}}{z^2 + 3z + 2} \\ &= \frac{z^\alpha e^{i\alpha \arg(z)}}{z^2 + 3z + 2}\end{aligned}$$

Now let's proceed with using the residue theorem.
We can factor the denominator as:

$$z^2 + 3z + 2 = (z + 2)(z + 1)$$

This gives us the single poles at $z = -2$ and $z = -1$. We can now find the residues at these poles by using the formula, where $m = 1$:

$$\begin{aligned}\text{Res}(f, z_k) &= \frac{1}{(m-1)!} \lim_{z\to z_k} \frac{d^{m-1}}{dz^{m-1}}((z - z_k)^m f(z)) \\ \text{Res}(f, -2) &= \lim_{z\to -2} \frac{z^\alpha e^{i\alpha \arg(z)}}{(z + 1)} \\ &= \frac{(-2)^\alpha e^{i\alpha \arg(-2)}}{(-2 + 1)} \\ &= -(-2)^\alpha e^{i\alpha\pi} \\ \to u^\alpha &= e^{\alpha \log(u)} \\ &= -e^{\alpha \log(-2)} e^{i\alpha\pi} \\ &= -e^{\alpha(\log(2) + i\arg(-2))} e^{i\alpha\pi} \\ &= -e^{\alpha(\log(2) + i\pi)} e^{i\alpha\pi} \\ &= -2^\alpha e^{2i\alpha\pi}\end{aligned}$$

$$
\begin{aligned}
\operatorname{Res}(f,\ -1) &= \lim_{z\to -1} \frac{z^{\alpha} e^{i\alpha\ \arg(z)}}{(z\ +\ 2)} \\
&= \frac{(-1)^{\alpha} e^{i\alpha\ \arg(-1)}}{(-1\ +\ 2)} \\
&= (-1)^{\alpha} e^{i\alpha\ \arg(-1)} \\
&= (-1)^{\alpha} e^{i\alpha\pi} \\
&= e^{\alpha\ \log(-1)} e^{i\alpha\pi} \\
&= e^{\alpha(\log(1)+i\ \arg(-1))} e^{i\alpha\pi} \\
&= e^{\alpha(0+i\pi)} e^{i\alpha\pi} \\
&= e^{i\alpha\pi} e^{i\alpha\pi} \\
&= e^{2i\alpha\pi}
\end{aligned}
$$

Therefore, the integral is:

$$
\begin{aligned}
\int_0^{\infty} \frac{x^{\alpha}}{x^2\ +\ 3x\ +\ 2} dx &= \frac{1}{1\ -\ e^{2\pi i\alpha}} \oint_{\text{Keyhole}=C} f(z) dz \\
&= \frac{2\pi i}{1\ -\ e^{2\pi i\alpha}} (\operatorname{Res}(f,\ -2)\ +\ \operatorname{Res}(f,\ -1)) \\
&= \frac{2\pi i}{1\ -\ e^{2\pi i\alpha}} (-2^{\alpha} e^{2i\alpha\pi}\ +\ e^{2i\alpha\pi})
\end{aligned}
$$

So in summary, the steps to solve the integral are:

1. Recognize that the integral is over the positive real axis, and that the integrand has a logarithm in it.
2. Because of the logarithm, we must choose a keyhole contour to evaluate the integral, and since the integral is over the positive real axis, we can choose the branch cut to be the positive real axis because $\oint_{\text{Keyhole}=C} f(z)dz\ =\ (\text{Path above the branch cut}) - (\text{Path below the branch cut})\ =\ (1\ -\ e^{2\pi i\alpha}) \int_0^{\infty} f(z)dz$ as $R\ \to\ \infty$ and $R\ \to\ 0$.
3. Convert the integrand into a complex form by using the identity $\log(z)\ =\ \log|z|\ +\ i\ \arg(z)$.
4. Find the singularities of the integrand by setting the denominator to zero and solving for z.
5. Find the residues at the poles by using the formula $\operatorname{Res}(f,\ z_k)\ =\ \frac{1}{(m-1)!} \lim_{z\to z_k} \frac{d^{m-1}}{dz^{m-1}}((z\ -\ z_k)^m f(z))$.
6. Use the residue theorem to evaluate the integral: $\oint_{\text{Keyhole}=C} f(z)dz\ =\ (\text{Path above the branch cut}) - (\text{Path below the branch cut})\ =\ (1\ -\ e^{2\pi i\alpha}) \int_0^{\infty} f(z)dz$.
7. After finding the residues, we can use the formula $\int_0^{\infty} \frac{x^{\alpha}}{x^2+3x+2} dx\ =\ \frac{2\pi i}{1-e^{2\pi i\alpha}} (\operatorname{Res}(f,\ -2)\ +\ \operatorname{Res}(f,\ -1))$ to evaluate the integral.

■

■ **Example 23.6 — Fisher, Section 2.6 Problem 14.** Compute the following integral

$$
\int_0^{\infty} \frac{\sqrt{x}}{x^2\ +\ 2x\ +\ 5} dx
$$

We can convert the integral into a complex integral by realizing that $x\ =\ z$ and $dx\ =\ dz$.

$$
\int_0^{\infty} \frac{\sqrt{x}}{x^2\ +\ 2x\ +\ 5} dx\ =\ \int_0^{\infty} \frac{\sqrt{z}}{z^2\ +\ 2z\ +\ 5} dz
$$

Now we can find the singularities of the integrand by the quadratic formula:

$$
\begin{aligned}
z^2 + 2z + 5 &= 0 \\
z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
z &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\
z &= \frac{-2 \pm \sqrt{-16}}{2} \\
z &= \frac{-2 \pm 4i}{2} \\
z &= -1 \pm 2i \\
z_1 &= -1 + 2i \\
z_2 &= -1 - 2i \\
\therefore\ z^2 + 2z + 5 &= (z + 1 - 2i)(z + 1 + 2i)
\end{aligned}
$$

We can now find the residues at these poles by using the formula, where $m = 1$:

$$
\begin{aligned}
\operatorname{Res}(f, z_k) &= \frac{1}{(m-1)!} \lim_{z \to z_k} \frac{d^{m-1}}{dz^{m-1}}((z - z_k)^m f(z)) \\
\operatorname{Res}(f, -1 + 2i) &= \lim_{z \to -1+2i} \frac{z^{\frac{1}{2}}}{(z + 1 + 2i)} \\
&= \frac{(-1 + 2i)^{\frac{1}{2}}}{(-1 + 2i + 1 + 2i)} \\
&\to (-1 + 2i)^{\frac{1}{2}} \quad \text{De Moivre's Theorem} \\
&\to z^n = r^n(\cos(\theta n) + i \sin(\theta n)) \\
&\to (-1 + 2i)^{\frac{1}{2}} = (\sqrt{5}(\cos(\pi - \tan^{-1} \frac{2}{1}) + i \sin(\pi - \tan^{-1} \frac{2}{1})))^{\frac{1}{2}} \\
&\to (-1 + 2i)^{\frac{1}{2}} \approx 5^{\frac{1}{4}}(\cos(\frac{2\pi}{3 \times 2}) + i \sin(\frac{2\pi}{3 \times 2})) \\
&\to (-1 + 2i)^{\frac{1}{2}} \approx 5^{\frac{1}{4}}(\cos(\frac{\pi}{3}) + i \sin(\frac{\pi}{3})) \\
&\to (-1 + 2i)^{\frac{1}{2}} \approx 5^{\frac{1}{4}}(\frac{1}{2} + i\frac{\sqrt{3}}{2}) \\
&\to (-1 + 2i)^{\frac{1}{2}} \approx \frac{5^{\frac{1}{4}}}{2} + i\frac{5^{\frac{1}{4}}\sqrt{3}}{2} \\
&= \frac{\frac{5^{\frac{1}{4}}}{2} + i\frac{5^{\frac{1}{4}}\sqrt{3}}{2}}{4i} \\
&= -i\frac{5^{\frac{1}{4}}}{8} - \frac{5^{\frac{1}{4}}\sqrt{3}}{8}
\end{aligned}
$$

We can also find the residue at z_2:

$$\begin{aligned}
\text{Res}(f, -1 - 2i) &= \lim_{z \to -1-2i} \frac{z^{\frac{1}{2}}}{(z + 1 - 2i)} \\
&= \frac{(-1 - 2i)^{\frac{1}{2}}}{(-1 - 2i + 1 - 2i)} \\
&\to (-1 - 2i)^{\frac{1}{2}} \quad \text{De Moivre's Theorem} \\
&\to (-1 - 2i)^{\frac{1}{2}} = (\sqrt{5}(\cos(\pi + \tan^{-1}\frac{2}{1}) + i\sin(\pi + \tan^{-1}\frac{2}{1})))^{\frac{1}{2}} \\
&\to (-1 - 2i)^{\frac{1}{2}} \approx 5^{\frac{1}{4}}(\cos(\frac{4\pi}{3 \times 2}) + i\sin(\frac{4\pi}{3 \times 2})) \\
&\to (-1 - 2i)^{\frac{1}{2}} \approx 5^{\frac{1}{4}}(\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})) \\
&\to (-1 - 2i)^{\frac{1}{2}} \approx 5^{\frac{1}{4}}(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) \\
&\to (-1 - 2i)^{\frac{1}{2}} \approx -\frac{5^{\frac{1}{4}}}{2} + i\frac{5^{\frac{1}{4}}\sqrt{3}}{2} \\
&= \frac{-\frac{5^{\frac{1}{4}}}{2} + i\frac{5^{\frac{1}{4}}\sqrt{3}}{2}}{4i} \\
&= i\frac{5^{\frac{1}{4}}}{8} - \frac{5^{\frac{1}{4}}\sqrt{3}}{8}
\end{aligned}$$

Therefore, the integral is:

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + 2x + 5}dx = \frac{1}{1 - e^{2\pi i \frac{1}{2}}} \oint_{\text{Keyhole}=C} f(z)dz \tag{23.1}$$

$$= \frac{2\pi i}{1 - (-1)}(\text{Res}(f, -1 + 2i) + \text{Res}(f, -1 - 2i)) \tag{23.2}$$

$$= -\pi i(-i\frac{5^{\frac{1}{4}}}{8} - \frac{5^{\frac{1}{4}}\sqrt{3}}{8} + i\frac{5^{\frac{1}{4}}}{8} - \frac{5^{\frac{1}{4}}\sqrt{3}}{8}) \tag{23.3}$$

$$= \frac{\pi i}{4}5^{\frac{1}{4}}\sqrt{3} \tag{23.4}$$

■

■ **Example 23.7 — Fisher, Section 3.1, Problem 1.** Determine the number of zeroes of $f(z) = z^2 - z + 1$ in the first quadrant (i.e. where $\Re(z) > 0, \Im(z) > 0$).

We can use the quadratic formula to find the zeroes of the function:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{23.5}$$

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} \tag{23.6}$$

$$= \frac{1 \pm \sqrt{-3}}{2}z_1 = \frac{1}{2} + i\sqrt{3}2 \tag{23.7}$$

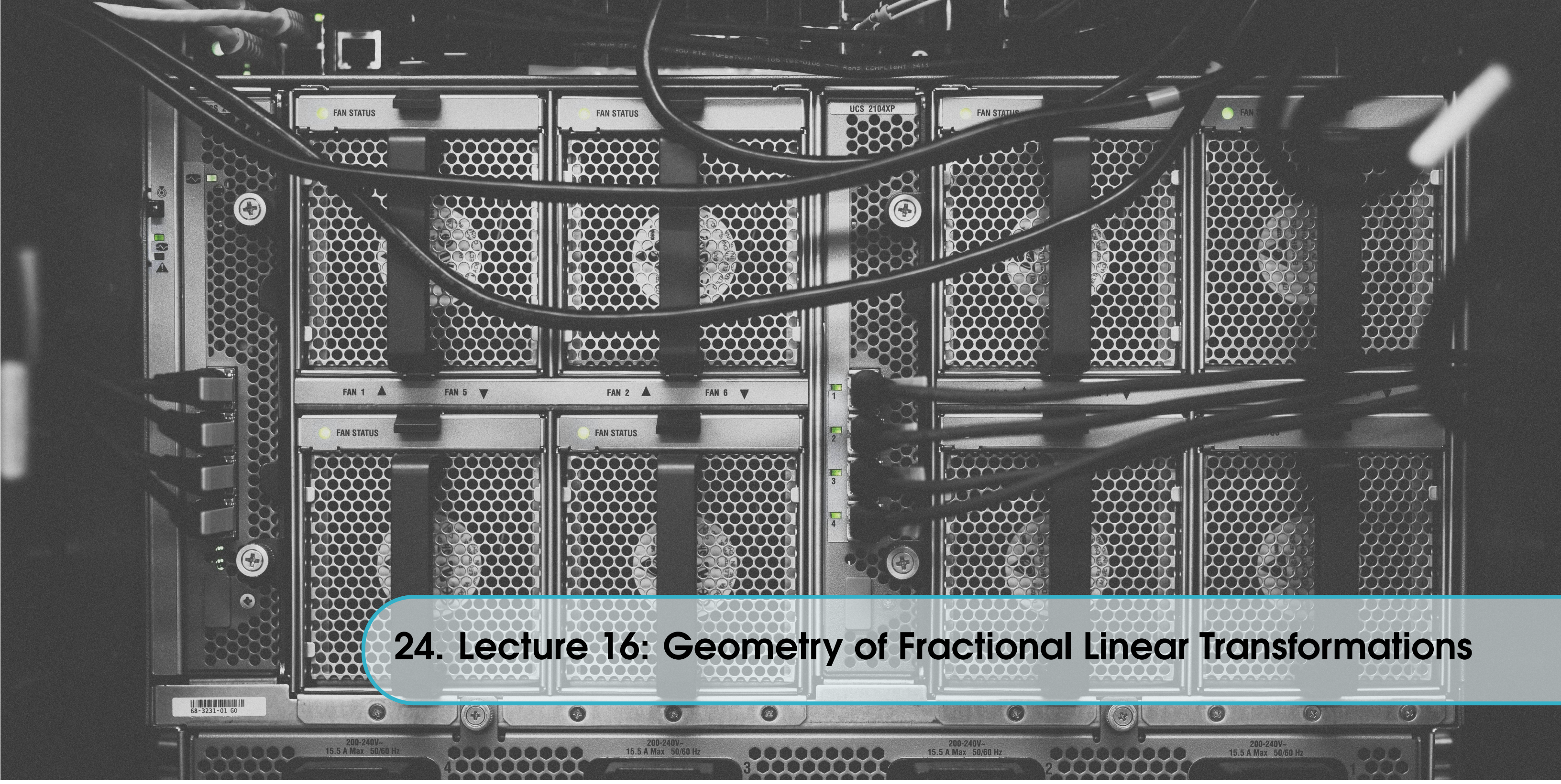
$$z_2 = \frac{1}{2} - i\sqrt{3}2 \tag{23.8}$$

$$\tag{23.9}$$

Only z_1 is in the first quadrant, so the answer is 1. ■

■ **Example 23.8 — Fisher, Section 3.1, Problem 3.** Determine the number of zeroes of $f(z) = z^3 - 3z + 6$ in the first quadrant (i.e. where $\Re(z) > 0$, $\Im(z) > 0$).

We ca ■



24. Lecture 16: Geometry of Fractional Linear Transformations

24.1 Lines & Circles

■ **Example 24.1** $T(z) = \frac{1}{z}$
If $f(z) = |z - z_0|^2 - r^2 = 0$, then we can compute $f(T(z))$:

$$
\begin{aligned}
|z|^2 z\bar{z} - z\overline{z_0} - z_0\bar{z} + |z_0|^2 &= r^2 \\
|z|^2 - 2\Re(z\overline{z_0}) + |z_0|^2 &= r^2 \\
1 - 2\Re(z\overline{z_0}\frac{z}{|z|^2}) &= \frac{r^2 - |z_0|^2}{|z|^2} \\
\text{Then } w &= \frac{1}{z} \text{ satisfies} \\
|w|^2(r^2 - |z_0|^2) &= 1 - 2\Re(\frac{|w|^2}{w}\overline{z_0}) = 1 - 2\Re(\overline{wz_0}) \\
\to \Re(\overline{wz_0}) &= \Re(\overline{z_0}w)
\end{aligned}
$$

Which is just another circle, unless $z_0 = r$.

$$
\begin{aligned}
\text{If } r &= |z_0|, \text{ then} \\
\Re(wz_0) &= \frac{1}{2} \text{ A line}
\end{aligned}
$$

Therefore we see that $T(z) = \frac{1}{z}$ maps circles to lines if and only if $z = 0$ lies on the circle since $T(T(z)) = z$, we also see that T takes lines to circles. ■

Proposition 24.1.1 — Möbius Transformation (Lines & Circles). Given a circle: $f(z) = |z - z_0|^2 - r^2 = 0$, then $T(z) = \frac{1}{z-a}$ maps the circle to a line if and only if the edge of the circle passes through a so that $|a - z_0| = r$.
The equation of the line is $\Re(wz_0) = \frac{1}{2}$ where $w = \frac{1}{z-a}$.

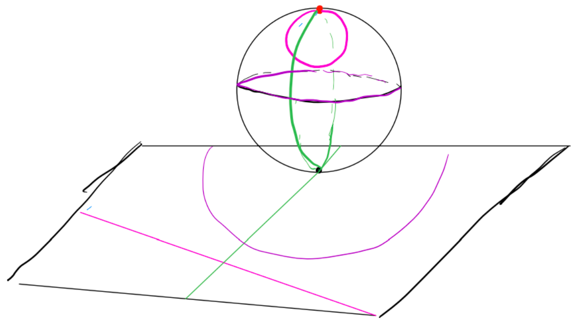


Figure 24.1: Circle through ∞

Lemma 24.1.2 If T is a fractional linear transformation, then either T is linear or, these exist T_1, T_3 are linear (i.e. $T_1(z) = a_1 z + b_1$, $T_3(z) = a_3 z + b_3$) such that if we take $T_2(z) = \frac{1}{z}$, then:

$$T = T_3 \circ T_2 \circ T_1$$

Proof. $T = \frac{az+b}{cz+d}$, T is linear if and only if $c = 0$, so we assume $c \neq 0$.

$$T = \frac{1}{c}\left(\frac{bc - ad}{cz + d} + a\right)$$

Then we define:

$$T_3(z) = \frac{1}{c}(bc - ad)z + a$$
$$T_1(z) = cz + d$$

■

Corollary 24.1.3 A fractional linear transformation takes:

$$\begin{Bmatrix} \text{Circles or} \\ \text{Lines} \end{Bmatrix} \xrightarrow{T} \begin{Bmatrix} \text{Circles or} \\ \text{Lines} \end{Bmatrix}$$

Proof. If T is linear, then T takes lines to lines and circles to circles. If T is not linear, then $T = T_3 \circ T_2 \circ T_1$ where T_1, T_3 are linear and $T_2(z) = \frac{1}{z}$ takes lines to circles/lines and circles to lines/circles. ■

24.2 Conformal Mapping

Proposition 24.2.1 Suppose $z_1(t)$ and $z_2(t)$ are curves in $\mathbb{C}$, (say for $t \in [-1, 1]$) and $z_1(0) = z_2(0) = z_0$. The tangent vector of the curves $z_j(t) = x_j(t) + iy_j(t), \quad j = 1, 2$ is given by:

$$z_j'(t) = x_j'(t) + iy_j'(t)$$

The angle between z_1, z_2 at $(t = 0, z_0)$ is given by:

$$\theta = \arg(z_2'(0)) - \arg(z_1'(0))$$

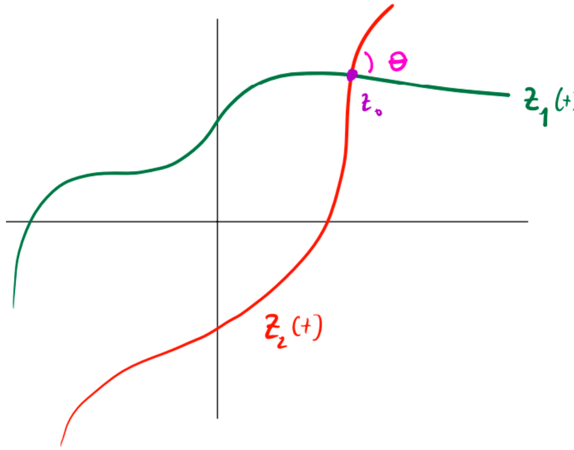


Figure 24.2: Angle between curves

Now suppose f is an analytic function defined in $|z - z_0| < r$ such that $f'(z_0) \neq 0$. Then:

$w_1(t) = f(z_1(t)), \quad w_2(t) = f(z_2(t))$ are curves in $\mathbb{C}$

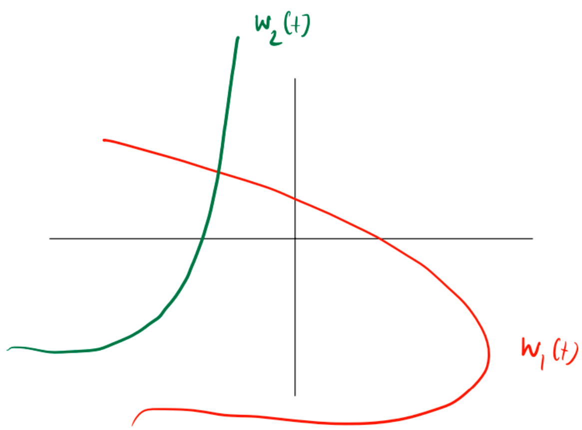


Figure 24.3: Curves under f

The tangent vector to $w_j(t)$ at $t = 0$ is given by:

$$w_j'(0) = f'(z_0)z_j'(0)$$

And the angle between some w_i, w_j is:

$$\arg(w_j'(0)) = \arg(f'(z_0)) + \arg(z_j'(0)) \quad j = 1, 2 \tag{24.1}$$

so

$$\arg(w_2'(0)) - \arg(w_1'(0)) = \arg(z_2'(0)) - \arg(z_1'(0)) \tag{24.2}$$

(This is because if $u = u_1 \cdot u_2$, $\arg(u) = \arg(u_1) + \arg(u_2)$) also

$$|w_j'(0)| = |f'(z_0)||z_j'(0)| \quad j = 1, 2 \tag{24.3}$$

Definition 24.2.1 — Conformal Map. A conformal map $\phi : \mathbb{C} \to \mathbb{C}$ is a map such that

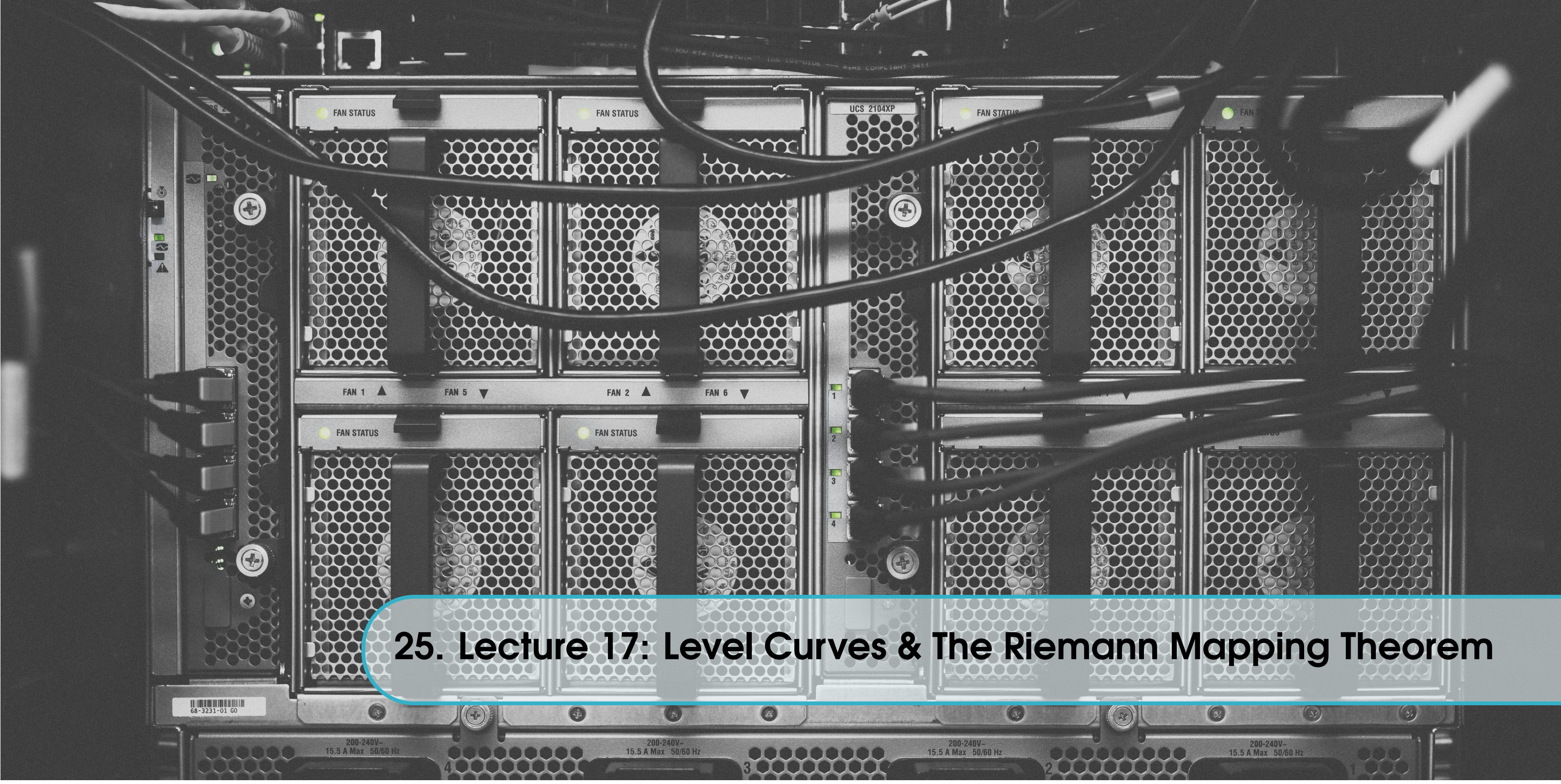
1. if $\gamma_1(t)$, $\gamma_2(t)$ are two curves such that $\gamma_1(0) = \gamma_2(0)$ then
   Angle between $\gamma_1'(0)$ and $\gamma_2'(0)$ = Angle between $\phi(\gamma_1)'(0)$ and $\phi(\gamma_2)'(0)$

2. The lengths of tangent vectors are scaled by a positive constant
   $|\gamma_j'(0)|c = |\phi(\gamma_j)'(0)|$

Essentially, a conformal map preserves angles, but not necessarily lengths.

Theorem 24.2.2 If f is analytic near z_0, and $f'(z_0) \neq 0$, then f is a conformal map near z_0.

■ **Example 24.2 — Map Projections.** The mercator projection is conformal, so it preserves compass angles and hence very useful for navigation. ■



25. Lecture 17: Level Curves & The Riemann Mapping Theorem

25.1 Level Curves

Claim 25.1.1 — Orthogonal Level Curves. Suppose $f = u + iv$ is an analytic function and some $f'(z_0) \neq 0$. Then the level curves of u and v centered around z_0 intersect orthogonally at $z_0 = x_0 + iy_0$.

$$\gamma_1 = \{z : u(z) = u(z_0)\} \text{ the set of points in } \mathbb{C} \text{ where real part of } f \text{ is constant}$$
$$\gamma_2 = \{z : v(z) = v(z_0)\} \text{ the set of points in } \mathbb{C} \text{ where imaginary part of } f \text{ is constant}$$

This is because at z_0 is the only point shared between γ_1 and γ_2.

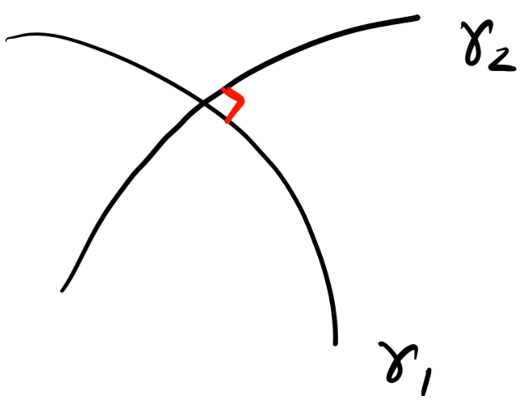


Figure 25.1: Level Curves

Proof. f is a conformal map and γ_1, γ_2 are curves in $\mathbb{C}$ such that $\gamma_1(0) = \gamma_2(0) = z_0$. So the angle between γ_1, γ_2 at z_0 is the same as the angle between $f(\gamma_1), f(\gamma_2)$ at $f(z_0)$.

- $f(\gamma_1) = \{w \in \mathbb{C} : \Re(w) = u(z_0)\}$

$\to w = f(z)$ for each $z \in \gamma_1$

We know for each $z \in \gamma_1$ that $u(z) = u(z_0)$
Because $f^{-1}(w) = z$, we have $u(f^{-1}(w)) = u(z) = u(z_0) = \Re(w)$

- $f(\gamma_2) = \{w \in \mathbb{C} : \Im(w) = v(z_0)\}$

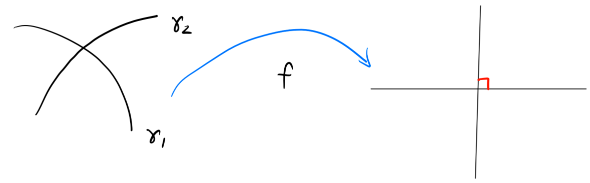


Figure 25.2: Level Curves under f

Due to the fact that *nablau* $\cdot \nabla v = 0$ at every point, this means $f(\gamma_1) = u(z_0)$ and $f(\gamma_2) = v(z_0)$ intersect orthogonally at $f(z_0)$, and thus γ_1, γ_2 intersect orthogonally at z_0. ■

Example 25.1 (i) $f(z) = e^z = \underbrace{e^x(\cos\ y)}_{u} + i\ \underbrace{e^x(\sin\ y)}_{v}$

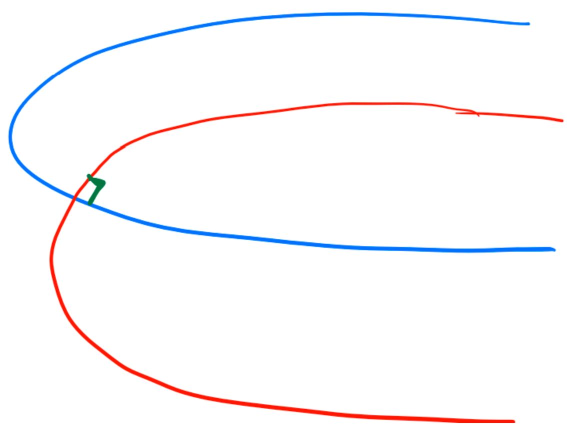


Figure 25.3: Level Curves of e^z

Let's consider the level curves of u and v.

$$\gamma_1 = \{z \in \mathbb{C} : u(z) = u(z_0)\}$$
$$e^x(\cos\ y) = C_1 \quad \text{for some constant } C_1 \in \mathbb{R}$$

If $C_1 = 0$ then:

$$e^x(\cos\ y) = 0 \implies \cos\ y = 0 \implies y = \frac{\pi}{2} + n\pi$$

If $C_1 \neq 0$ then:

$$e^x(\cos\ y) = C_1 \implies e^x = \frac{C_1}{\cos\ y} \quad \text{for } y \neq \frac{\pi}{2} + n\pi$$
$$x = \log\left(\frac{C_1}{\cos\ y}\right)$$

So γ_1 is a set of $z = (x, y)$ such that $x = \log\left(\frac{C_1}{\cos y}\right)$ for $y \neq \frac{\pi}{2} + n\pi$.
Similarly, γ_2 is a set of $z = (x, y)$ such that $x = \log\left(\frac{C_2}{\sin y}\right)$ for $y \neq n\pi$.
Because $f'(z_0) = e^{z_0} \neq 0$, we can show that the level curves of u and v intersect orthogonally at any point z_0.

$$\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x \cos y \\ -e^x \sin y \end{pmatrix}$$

$$\nabla v = \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x \sin y \\ e^x \cos y \end{pmatrix}$$

Then we can use the dot product, which is 0 iff the vectors are orthogonal.

$$\nabla u \cdot \nabla v = e^x \cos(y) e^x \sin(y) - e^x \cos(y) e^x \sin(y) = 0$$

■

■ **Example 25.2** (ii) $\text{Log} z = \underbrace{\log |z|}_{u} + i \underbrace{\text{Arg}(z)}_{v}$

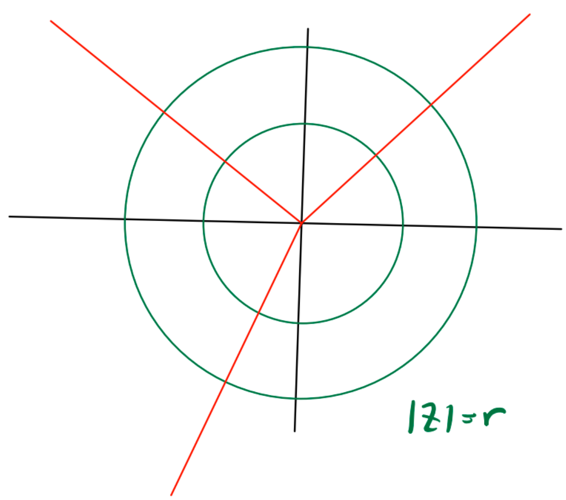


Figure 25.4: Level Curves of Log$z$

■

25.2 The Riemann Mapping Theorem

Theorem 25.2.1 — Riemann Mapping Theorem. Suppose $D \subset \mathbb{C}$ is a simply connected domain such that $D \neq \mathbb{C}$. Let $p \in D$ be a point in D. Then there is a one-to-one analytic function ϕ such that:

$$\phi : D \to \{w \in \mathbb{C} : |w| < 1\}$$

and $\phi(p) = 0$. ϕ is uniquely determined if we require $\phi'(p) > \mathbb{R}_{>0}$.

Basically, we can map any simply connected domain to the unit disk.

Corollary 25.2.2 If D_1, D_2 are simply connected domains $D_1 \neq \mathbb{C} \neq D_2$, then $\exists\phi$ that's one-to-one and analytic such that:

$$\phi(D_1) = D_2$$

Proof. $D_1 \xrightarrow{\phi_1} \{w \in \mathbb{C} : |w| < 1\}$ and $D_2 \xrightarrow{\phi_2} \{w \in \mathbb{C} : |w| < 1\}$.
$\phi_2^{-1} \circ \phi_1$ is the desired map. ■

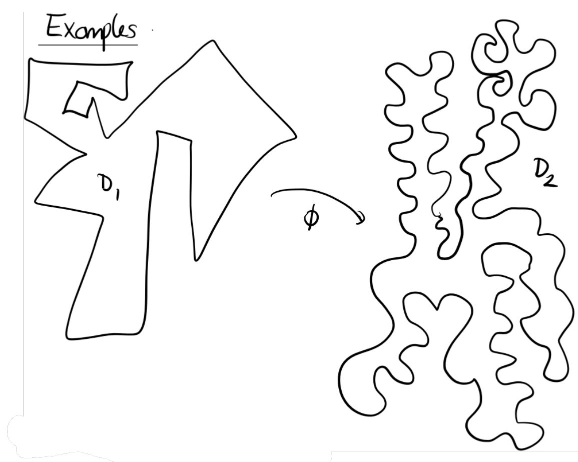


Figure 25.5: Riemann Mapping Theorem

R This only proves the existence! In general, it is not constructive!

25.3 Constructing Conformal Maps

■ **Example 25.3** Mapping the unit disk to the upper half plane.

$$f = i\left(\frac{1+z}{1-z}\right)$$

■

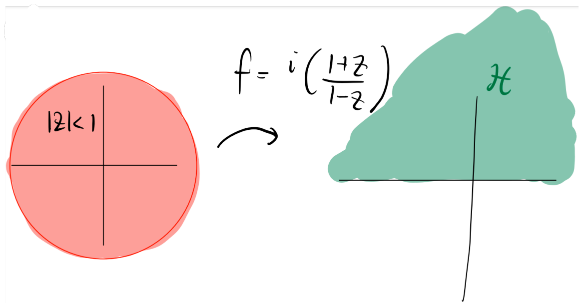


Figure 25.6: Conformal Map

■ **Example 25.4** Mapping the upper half plane to the upper half plane plus a fraction of the lower half plane.

$h(z) = z^p \quad 0 < p < 2$
Defined on $\mathbb{H}$

■

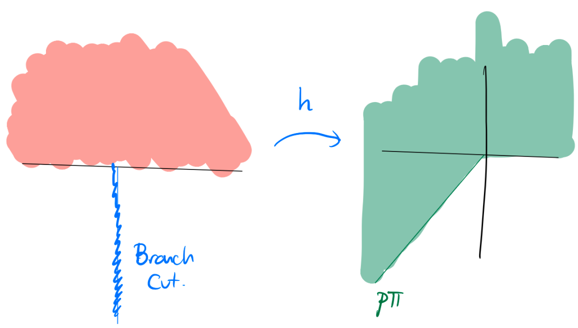


Figure 25.7: Conformal Map

25.4 Schwarz-Christoffel Symbols

Definition 25.4.1 This is a technique for constructing conformal maps from the upper half-plan to a polygon.

$$\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\} \to \text{Polygon}$$

Proposition 25.4.1 — Conformal Map from Real Numbers to Edges. Consider the function $f(z) = A(z - x_0)^\beta + B$ where x_0, β are constants in $\mathbb{R}$, $0 < \beta < 2$ and A, B are constants in $\mathbb{C}$. Argument chosen to lie in $(-\frac{\pi}{2}, \frac{3\pi}{2})$.

Our objective is to show that f maps real numbers to edges with a specific angle.

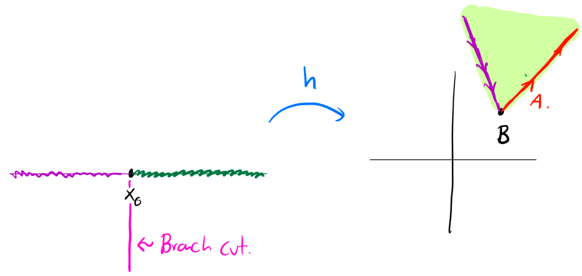


Figure 25.8: Conformal Map f

We can do this by find the difference in the tangents just before and after the point B.
Let's restrict z to be real, so $z = x \in \mathbb{R}$.
You can think of $f(x)$ now as a function that starts at B and moves in the direction of A. This is true when $x > x_0$ and $\beta = 1$.

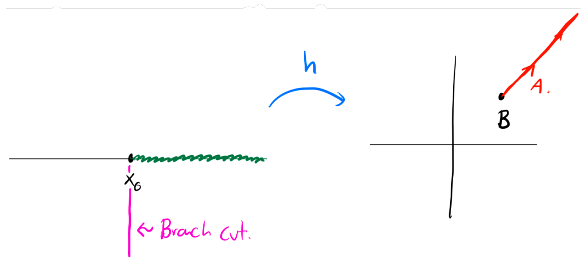


Figure 25.9: Conformal Map f

Now let's write $f(x)$ in a more explicit form:

$$\begin{aligned} f(x) &= A(x - x_0)^{\beta} + B \\ &= Ae^{\beta \log(x - x_0)} + B \\ &= Ae^{\beta \log |x - x_0| + i\beta \operatorname{Arg}(x - x_0)} + B \\ &= A|x - x_0|^{\beta} e^{i\beta \operatorname{Arg}(x - x_0)} + B \end{aligned}$$

We can see now that β will cause some rotation when it's not an integer.
We can think of $\operatorname{Arg}(f(z_0))$ as the angle pointing from the origin to $f(z_0)$.
We can think of $\operatorname{Arg}(f'(z_0))$ as the angle pointing from $f(z_0)$ to $f(z_0^+)$, essentially the direction of the tangent vector.
With this in mind, we can find $f'(x_0^+) = B$, which the direction of the tangent vector at right after B.

$$\begin{aligned} f'(x) &= A\beta(x - x_0)^{\beta - 1} \\ &= A\beta e^{(\beta - 1) \log |x - x_0| + i(\beta - 1)\operatorname{Arg}(x - x_0)} \\ &= A\beta|x - x_0|^{\beta - 1} e^{i(\beta - 1)\operatorname{Arg}(x - x_0)} \\ f'(x_0^+) &= A\beta|x_0^+ - x_0|^{\beta - 1} e^{i(\beta - 1)\operatorname{Arg}(x_0^+ - x_0)} \\ \operatorname{Arg}(f'(x_0^+)) &= \operatorname{Arg}(A) + (\beta - 1)\operatorname{Arg}(x_0^+ - x_0) \end{aligned}$$

$$= \text{Arg}(A)$$

The $(\beta - 1)\text{Arg}(x_0^+ - x_0) = 0$ because $x_0^+ - x_0$ is real and thus has an argument of 0. Now we can find the tangent vector at x_0^-.

$$\begin{aligned}
f'(x_0^-) &= A\beta|x_0^- - x_0|^{\beta-1}e^{i(\beta-1)\text{Arg}(x_0^- - x_0)} \\
\text{Arg}(f'(x_0^-)) &= \text{Arg}(A) + (\beta - 1)\text{Arg}(x_0^- - x_0) \\
&= \text{Arg}(A) + (\beta - 1)\pi
\end{aligned}$$

Since $x_0^- - x_0$ is real and negative, it has an argument of π.
Now we can find the difference in the angles of the tangent vectors at x_0^- and x_0^+.

$$\begin{aligned}
\text{Arg}(f'(x_0^+)) - \text{Arg}(f'(x_0^-)) &= \text{Arg}(A) - \text{Arg}(A) - (\beta - 1)\pi \\
&= (\beta - 1)\pi
\end{aligned}$$

This means we can now map the real numbers to edges with a specific angle. We can also now see why $\beta < 2$ is required, it's because the difference between the tangents of the edges must be between $(-\pi, \pi)$.

Proposition 25.4.2 — A Generalization of the Previous Proposition. Consider the analytic function on $\mathbb{H}$ satisfying:

$$\begin{aligned}
f'(z) &= A(z - x_1)^{\alpha_1}(z - x_2)^{\alpha_2} \ldots (z - x_n)^{\alpha_n} \\
&\quad \text{where } x_1 < x_2 < \ldots < x_n \in \mathbb{R} \text{ and } \alpha_1, \alpha_2, \ldots, \alpha_n \in (-1, 1) \\
\text{Arg} &\in (-\frac{\pi}{2}, \frac{3\pi}{2})
\end{aligned}$$

You can think of α_i as $\beta - 1$ in the previous proposition.

■ **Example 25.5** Suppose $x \in \mathbb{R}$ and $x_{j-1} < x_j < x_{j+1} < \ldots < x_n$. Then:

$$\begin{aligned}
textArgf'(x_j) &= \text{Arg}A + \pi\alpha_{j+1} + \ldots + \pi\alpha_n \\
\text{Arg}f'(x_{j+1}) &= \text{Arg}A + \pi\alpha_{j+2} + \ldots + \pi\alpha_n \\
\therefore \text{Arg}f'(x_{j+1}) - \text{Arg}f'(x_j) &= \pi(\alpha_{j+1})
\end{aligned}$$

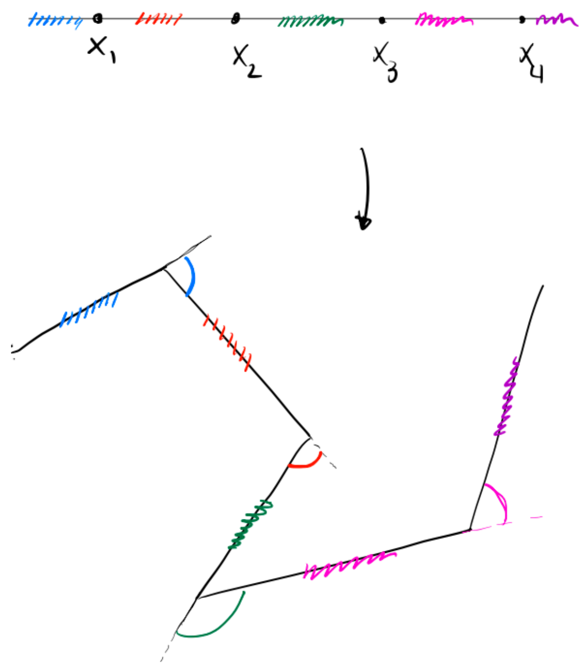


Figure 25.10: Conformal Map f

■

Proposition 25.4.3 — Generalizing to Polygons. Let P be a polygon with $N + 1$ sides, and vertices $w_0, w_1, \ldots, w_N$. Arranged in a counter-clockwise fashion.

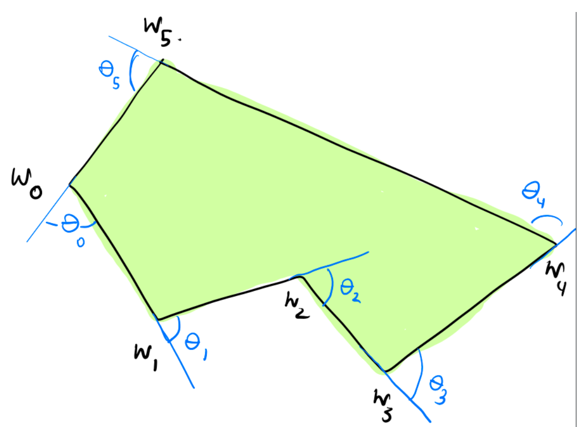


Figure 25.11: Polygon P

Let $\theta_0, \theta_1, \ldots, \theta_N$ be the exterior angles of the polygon.
Then we can say $\theta_0, \theta_1, \ldots, \theta_N \in (-\pi, \pi)$ and because P is closed: $\theta_0 + \theta_1 + \ldots + \theta_N = 2\pi$.

We know that $\alpha_j \in (-1, 1)$ so let's define $\alpha_j = -\frac{\theta_j}{\pi}$. Then:

$$\sum_{j=0}^{N} \alpha_j = -2$$

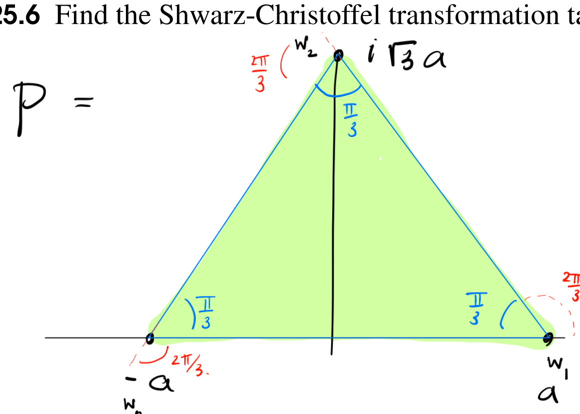
Theorem 25.4.4 There exists $A \in \mathbb{C}$ and real numbers $x_1 < x_2 < \ldots < x_N$ such that the Analytic function $f(z)$ on $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ is defined by:

$$f'(z) = A(z - x_1)^{\alpha_1}(z - x_2)^{\alpha_2} \ldots (z - x_N)^{\alpha_N}$$

Which gives a 1 to 1 analytic map from $\mathbb{H}$ to the polygon P.

- For $j = 1, 2, \ldots, N \quad f(z_j) = w_j$. Essentially, the points $z_j|\Im(z_j) > 0$ are mapped to the inside of the polygon.
- $f(x) \quad x \in \mathbb{R}$ maps to the edges of the polygon.
- $\lim_{x\to\pm\infty} f(x) = w_0 \quad x \in \mathbb{R}$. Essentially, the points positive and negative real infinity are mapped to the left and right of the first vertex w_0 of the polygon.

■ **Example 25.6** Find the Shwarz-Christoffel transformation taking $\mathbb{H}$ to the triangle with vertices



$-a$, a, $i\sqrt{3}a$. The triangle has 3 sides, so $N = 2$. We choose the points $z_1 = 0$, $z_2 = 1$ and $z = \pm\infty$ to be mapped to the vertices of the triangle.

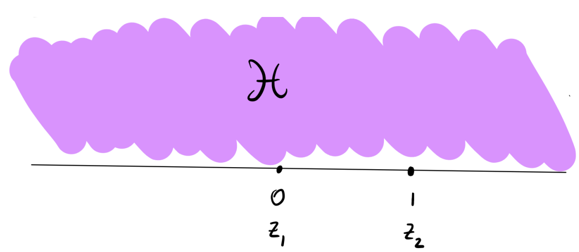


Figure 25.12: Upper Half Plane with points z_1, z_2

We know that:

$$f'(z) = A(z - x_1)^{\alpha_1}(z - x_2)^{\alpha_2}$$
$$f'(z) = A(z - 0)^{\alpha_1}(z - 1)^{\alpha_2}$$

Now we can find $\alpha_1,\ \alpha_2$.

$$\begin{aligned}
\alpha_1 &= -\frac{\theta_1}{\pi} = -\frac{2\pi}{3\pi} = -\frac{2}{3} \\
\alpha_2 &= -\frac{\theta_2}{\pi} = -\frac{2\pi}{3\pi} = -\frac{2}{3}
\end{aligned}$$

Now we can find A. Remember that any segment of the real line, as long as you don't cross the pre-image of a vertex, will have the same derivative argument as its corresponding image. So if we take the line segment $z = x \in \mathbb{R}|x < 0$ we know:

$$\begin{aligned}
\operatorname{Arg}(z)' = 0 &= \operatorname{Arg}(f'(z)) \\
&= \operatorname{Arg}(A(z - x_1)^{\alpha_1}(z - x_2)^{\alpha_2}) \\
&= \operatorname{Arg}(A) + \alpha_1 \operatorname{Arg}(z - x_1) + \alpha_2 \operatorname{Arg}(z - x_2)
\end{aligned}$$

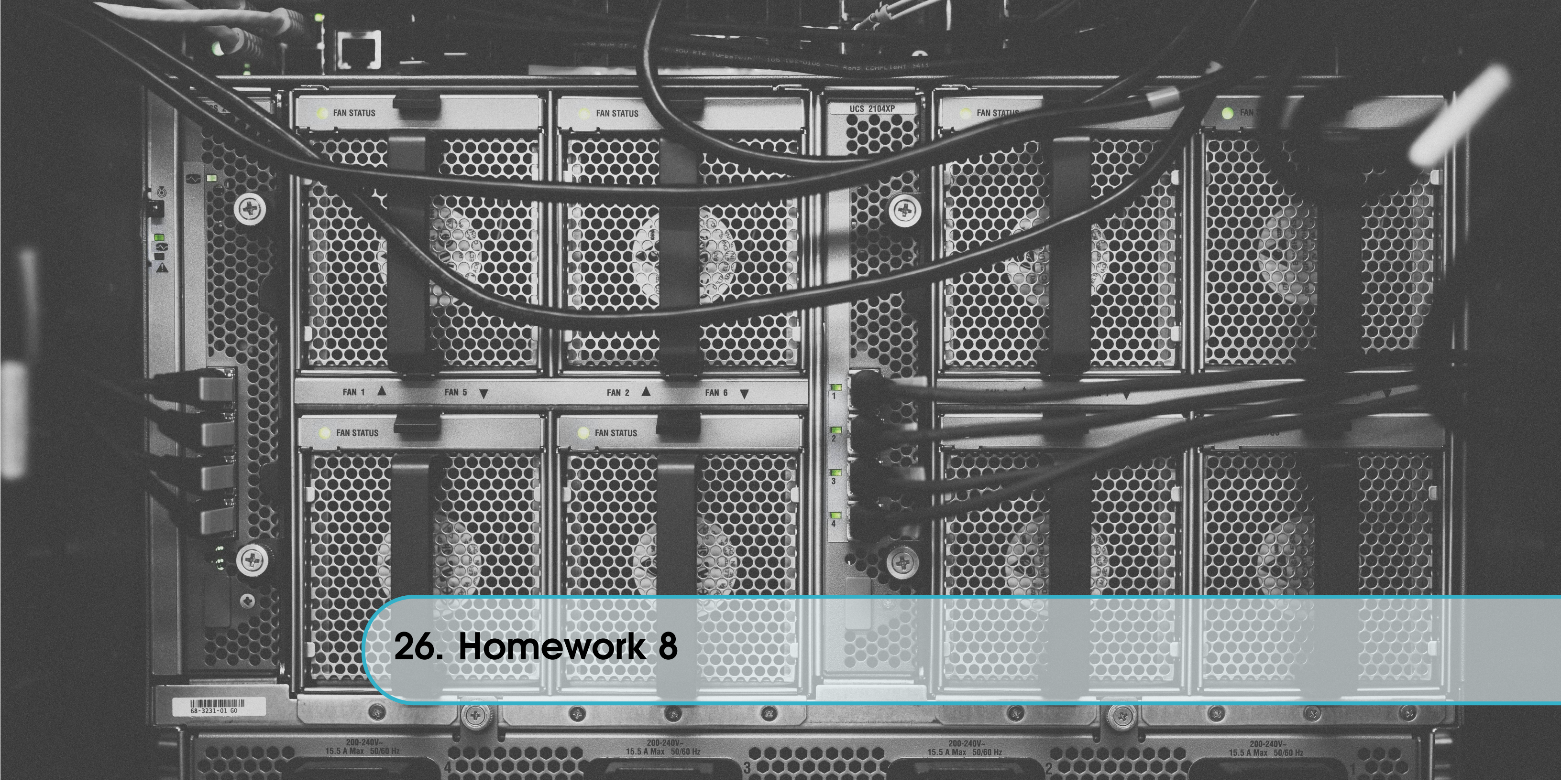
Because $(z - x_1)$ and $(z - x_2)$ are real and negative, their arguments are π. So:

$$\begin{aligned}
0 &= \operatorname{Arg}(A) + \frac{2}{3}\pi + \frac{2}{3}\pi \\
\operatorname{Arg}(A) &= \frac{4}{3}\pi
\end{aligned}$$

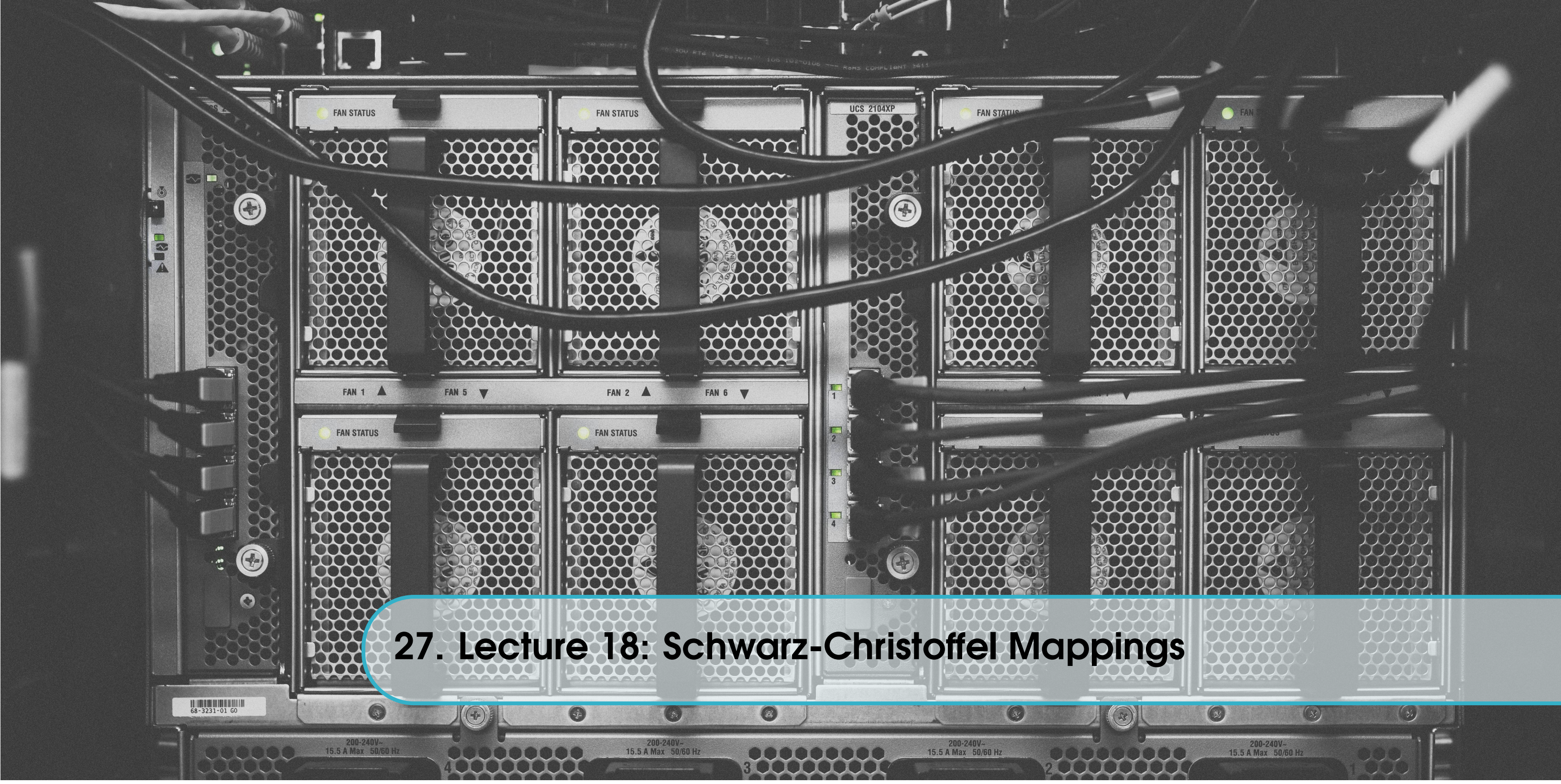
We can integrate to find $f(z)$.

$$\begin{aligned}
f(z) &= A\int_1^z t^{-\frac{2}{3}}(t - 1)^{-\frac{2}{3}}dt + B \\
f(1) &= A\int_1^1 t^{-\frac{2}{3}}(t - 1)^{-\frac{2}{3}}dt + B \\
&= B = i\sqrt{3}a \\
f(\infty) &= A\int_1^\infty t^{-\frac{2}{3}}(t - 1)^{-\frac{2}{3}}dt + i\sqrt{3}a = -a \\
A &= -a\frac{1 + i\sqrt{3}}{\int_1^\infty t^{-\frac{2}{3}}(t - 1)^{-\frac{2}{3}}dt}
\end{aligned}$$

■



26. Homework 8



27. Lecture 18: Schwarz-Christoffel Mappings

■ **Example 27.1** Find the Schwarz-Christoffel mapping from $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ to

$$w_1 = 0 \qquad w_0 = \sigma_0 + i\tau_0 \qquad \sigma_0 < 0,\ \tau_0 > 0$$

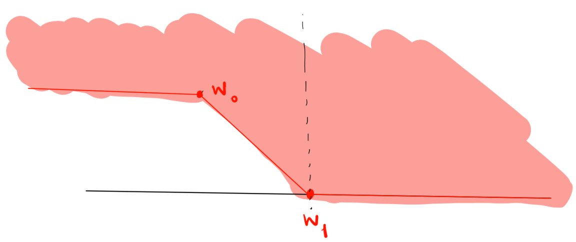


Figure 27.1: Graph

$$\theta_0 = \arctan(\frac{\tau_0}{\sigma_0}) \in (-\pi, 0), \quad \theta_1 = -\theta_0 \tag{27.1}$$

We choose the points $x_0 = -1$, $x_1 = 1$ as the pre-image vertices. Now we write:

$$f'(z) = A(z+1)^{-\frac{\theta_0}{\pi}}(z-1)^{\frac{\theta_0}{\pi}} \tag{27.2}$$

This gives us our general form of $f'(z)$, which can't be integrated easily. So let's take a specific angle where $\sigma_0 = 0 \rightarrow \theta_0 = -\frac{\pi}{2}$.

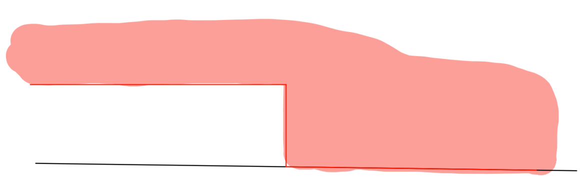


Figure 27.2: Graph

$$f'(z) = A\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}} \tag{27.3}$$

$$f'(z) = A\left(\frac{z+1}{z-1}\frac{z+1}{z+1}\right)^{\frac{1}{2}} \tag{27.4}$$

$$f'(z) = A\frac{z+1}{(z^2-1)^{\frac{1}{2}}} \tag{27.5}$$

$$f(z) = A\int \frac{z+1}{(z^2-1)^{\frac{1}{2}}}dz + B \tag{27.6}$$

$$f(z) = A\left(\int \frac{z}{(z^2-1)^{\frac{1}{2}}}dz + \int \frac{1}{(z^2-1)^{\frac{1}{2}}}dz\right) + B \tag{27.7}$$

$$f(z) = A\left((z^2-1)^{\frac{1}{2}} + \mathrm{Log}(z + (z^2-1)^{\frac{1}{2}})\right) + B \tag{27.8}$$

$$\tag{27.9}$$

Now we can find use the pre-image vertices to find A and B. First we map $x_0 = -1$ to $w_0 = i\tau_0$:

$$\begin{aligned} f(-1) = w_0 = i\tau_0 &= A\left((1-1)^{\frac{1}{2}} + \mathrm{Log}(-1 + (1-1)^{\frac{1}{2}})\right) + B && (27.10)\\ &= A(\mathrm{Log}(-1)) + B && (27.11)\\ &= A(i\pi) + B && (27.12)\end{aligned}$$

$$\begin{aligned} f(1) = w_1 = 0 &= A\left((1-1)^{\frac{1}{2}} + \mathrm{Log}(1 + (1-1)^{\frac{1}{2}})\right) + B\\ &= A(\mathrm{Log}(1)) + B\\ &= B\end{aligned}$$

This gives us $B = 0$ and $A = \frac{\tau_0}{\pi}$. So our final mapping is: ■

■ **Example 27.2** Find the Schwarz-Christoffel mapping from $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ to

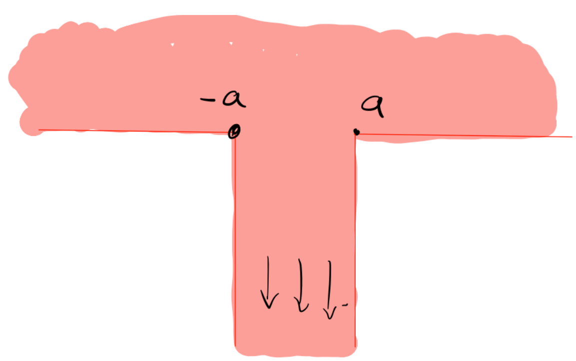


Figure 27.3: Graph

$$a > 0, a \in \mathbb{R} \tag{27.13}$$

Idea: Let's approximate the valley with a triangle extending infinitely downwards

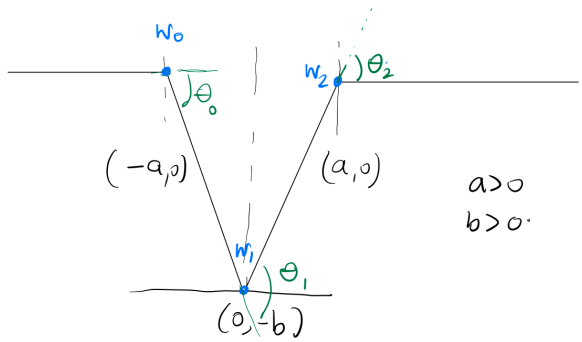


Figure 27.4: Graph

We take the limit as $b \to \infty$. Then we can approximate the angles as:

$$\theta_0 = \arctan(\frac{-b}{a}) \approx \frac{-\pi}{2} \tag{27.14}$$

$$\theta_2 = -\arctan(\frac{b}{a}) \approx \frac{-\pi}{2} \tag{27.15}$$

$$\theta_1 = 0 \quad \text{triangles can only have angles that sum to } \pi \tag{27.16}$$

We choose the points $x_0 = -1$, $x_1 = 1$ as the pre-image vertices. Now we write:

$$f'(z) = A(z+1)^{-\frac{-\theta_0}{\pi}} z^{\frac{-\theta_1}{\pi}} (z-1)^{\frac{-\theta_2}{\pi}} \tag{27.17}$$

$$f'(z) = A(z+1)^{\frac{1}{2}} z^{-1} (z-1)^{\frac{1}{2}} \tag{27.18}$$

$$\tag{27.19}$$

Thus:

$$f'(z) = A\left(\frac{z^2-1}{z^2}\right)^{\frac{1}{2}} \tag{27.20}$$

$$= A\sqrt{1-\frac{1}{z^2}} \tag{27.21}$$

$$f(z) = A\int\sqrt{1-\frac{1}{z^2}}dz + B \qquad = A\left(\sqrt{z^2-1} + \arcsin(\frac{1}{z})\right) + B \tag{27.22}$$

Don't worry about how the integration was done, if you're asked to do it on the exam, you're fucked. Now we can find use the pre-image vertices to find A and B. First we map $x_0 = -1$ to $w_0 = -a$:

$$f(-1) = w_0 = -a = A\left(\sqrt{1-1} + \arcsin(-1)\right) + B \tag{27.23}$$

$$-a = A\left(\arcsin(-1)\right) + B \tag{27.24}$$

$$-a = A\left(-\frac{\pi}{2}\right) + B \tag{27.25}$$

Now we map $x_1 = 1$ to $w_1 = a$:

$$f(1) = w_1 = a = A\left(\sqrt{1-1} + \arcsin(1)\right) + B \tag{27.26}$$

$$a = A\,(\arcsin(1)) + B \tag{27.27}$$

$$a = A\left(\frac{\pi}{2}\right) + B \tag{27.28}$$

This gives us $B = 0$ and $A = \frac{2a}{\pi}$. So our final mapping is:

$$f(z) = \frac{2a}{\pi}\left(\sqrt{z^2 - 1} + \arcsin(\frac{1}{z})\right) \tag{27.29}$$

■

■ **Example 27.3** Find the Schwarz-Christoffel mapping from $\mathbb{H}/\{it \in \mathbb{C} : 0 \leq t \leq a\}$ to $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$

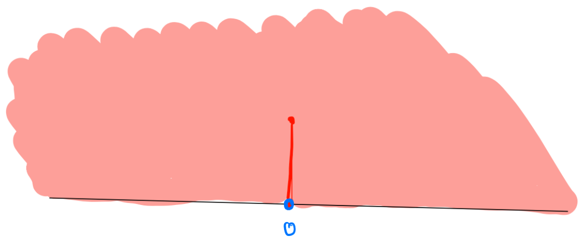


Figure 27.5: Image Mapping

We can observe the image vertices to be $w_0 = -\varepsilon$, $w_1 = ia$, $w_2 = \varepsilon$ where $\varepsilon \to 0$. First, let's view this as the limit of the following domain:

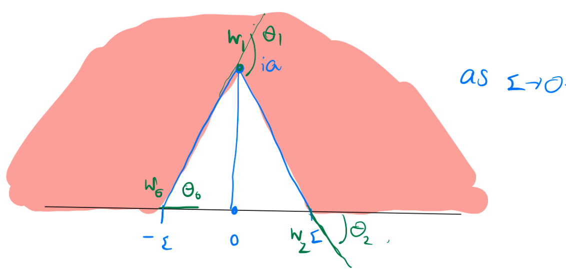


Figure 27.6: Domain as the limit of a triangle

Then we can say that the angles are:

$$\theta_0 = \frac{\pi}{2} \tag{27.30}$$

$$\theta_1 = -\pi \tag{27.31}$$

$$\theta_2 = \frac{\pi}{2} \tag{27.32}$$

We choose the points $x_0 = -1$, $x_1 = 0$, $x_2 = 1$ as the pre-image vertices. Now we write:

$$f'(z) = A(z + 1)^{\frac{-1}{2}} z(z - 1)^{\frac{-1}{2}} \tag{27.33}$$

$$= A\frac{z}{\sqrt{z^2 - 1}} \tag{27.34}$$

$$f(z) = A\int \frac{z}{\sqrt{z^2 - 1}}dz + B \tag{27.35}$$
$$= A(\sqrt{z^2 - 1}) + B \tag{27.36}$$

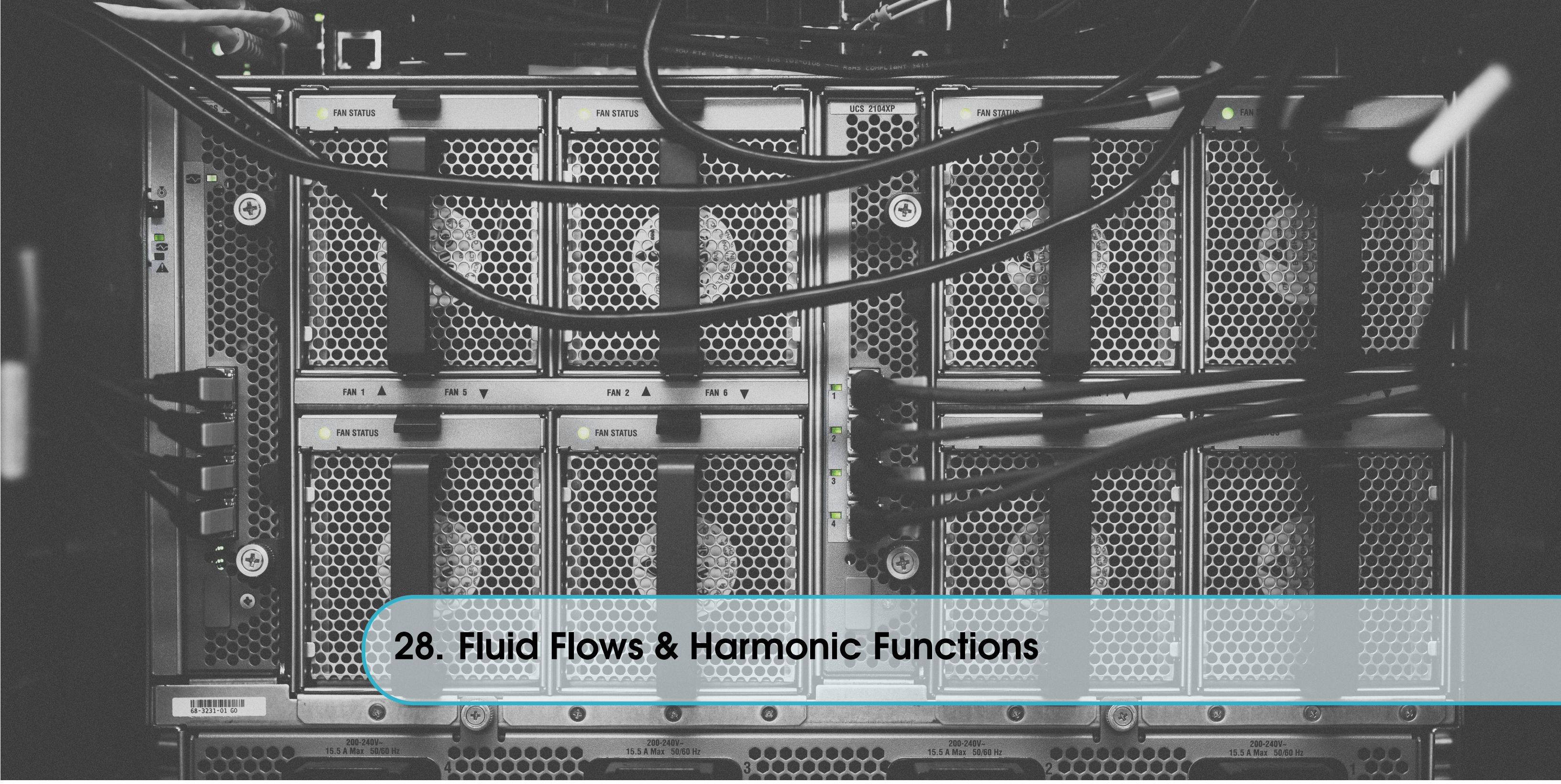
Now we can find use the pre-image vertices to find A and B. First we map $x_0 = -1$ to $w_0 = 0$:

$$f(-1) = w_0 = 0 = A(\sqrt{1 - 1}) + B \tag{27.37}$$
$$0 = B \tag{27.38}$$
$$f(0) = w_1 = ia = A(\sqrt{0 - 1}) \tag{27.39}$$
$$A = a \tag{27.40}$$
$$\tag{27.41}$$

This gives us $B = 0$ and $A = a$. So our final mapping is:

$$f(z) = a\sqrt{z^2 - 1} \tag{27.42}$$

■



28. Fluid Flows & Harmonic Functions

Proposition 28.0.1 Suppose $D \subseteq \mathbb{C}(= \mathbb{R}^2)$ and the following kind of flow fluid flowing through D.

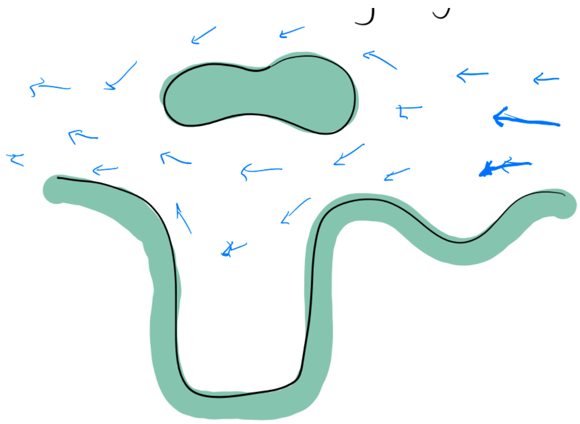


Figure 28.1: Fluid Flow through D

We assume:

(i) The flow is in a **steady state** (velocity doesn't change with time).
The direction of the flow at $(x, y) \in D$ is given by a vector field $\mathbf{v}(x, y) = (u(x, y), v(x, y))$.

(ii) the flow is **irrotational**, imagine a small paddle wheel at (x, y), it doesn't rotate.

(a) Irrotational Flow

(b) Rotational Flow

Mathematically: $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}, \quad \forall (x, y) \in D.$
You can think of $\frac{\partial v}{\partial x}$ as the change in vertical velocity as you move in the horizontal direction. Essentially, the amount of x rotation (counter clockwise). Similarly, $\frac{\partial u}{\partial y}$ is the amount of y rotation (clockwise). Therefore, if there is a horizontal rotation, there must be a vertical rotation to maintain irrotationality.

(iii) The flow is **sourceless/sinkless**, no fluid is created or destroyed (incompressible).
Mathematically: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \forall (x, y) \in D.$
You can think of $\frac{\partial u}{\partial x}$ as the change in horizontal velocity as you move in the horizontal direction. Essentially, the amount of x stretch/compression. Similarly, $\frac{\partial v}{\partial y}$ is the amount of y stretch/compression. Therefore, if there is a horizontal stretch, there must be a vertical stretch to maintain the same volume.

R For a function $f(z)$ that is analytic in D, and the independent variable $z = x + iy$ where $x, y \in \mathbb{R}$, we can define $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ as follows:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Corollary 28.0.2 — Cauchy-Riemann Equations for Flows. If $\mathbf{v}(x, y) = (u(x, y), v(x, y))$ is a sourcelss irrotational flow then:
(i) $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$
(ii) $\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$
(iii) $\overline{f} = u - iv$ is analytic in D by the Cauchy-Riemann equations.
Furthermore, there exists an analytic function G on D such that $G' = \overline{f}$ on D.

Proof. Why must there exist such a G?
Because sources and irrotational flows imply $\int_\gamma \overline{f} dz = 0 \quad \forall$ closed curves γ and vice-versa. ∎

Now then, by (the proof of) Morera's Theorem

$$\begin{aligned}
G &= \phi + i\psi \\
\frac{dG}{dz} &= \frac{1}{2}\left[\frac{\partial G}{\partial x} - i\frac{\partial G}{\partial y}\right] \\
&= \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right) - i\left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right)\right] \\
&\to \text{Now we can separate the real and imaginary parts} \\
&= \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) + i\left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right)\right] \\
&\to \text{We know } \overline{f} = G' \text{ is analytic} \\
&\to \text{Then } G \text{ must be analytic and follow the Cauchy-Riemann equations}
\end{aligned}$$

$$
\begin{aligned}
&\rightarrow \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} = 2\frac{\partial \phi}{\partial x} \\
&\rightarrow \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} = 2\frac{\partial \psi}{\partial x}
\end{aligned}
$$

$$
G' = \overline{f} = \underbrace{\frac{\partial \phi}{\partial x}}_{u} + i \underbrace{\frac{\partial \psi}{\partial x}}_{-v}
$$

$$
G' = \overline{f} = \underbrace{\frac{\partial \psi}{\partial y}}_{u} - i \underbrace{\frac{\partial \phi}{\partial y}}_{v}
$$

So:

$$
\nabla \phi = (u, v) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)
$$

$$
\nabla \psi = (-v, u) = \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right)
$$

Thus:

(i) The level sets of $\{\phi = c\}$ are orthogonal to the flow

(ii) The level sets of $\{\psi = c\}$ are flow lines.

ϕ is called the potential function ψ is called the stream function.

R

- The sets $\{\phi(x, y) = c\}$ are sets with equal potential energy at different (x, y). Hence ϕ is called the potential function.
- The sets $\{\psi(x, y) = c\}$ are streamlines of the flow . Hence ψ is called the stream function.

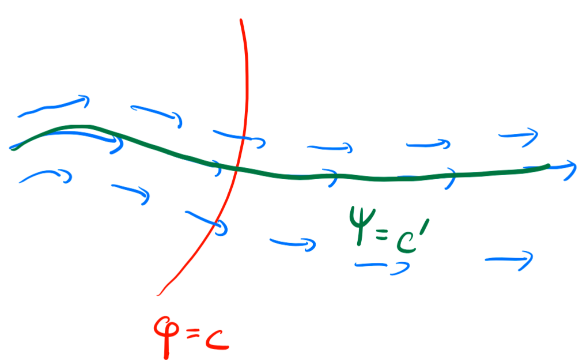


Figure 28.3: Potential and Stream Functions

Example 28.1 Find the flow lines of a sourceless irrotational flow past a wall of height a (assuming take height is $>> a$).

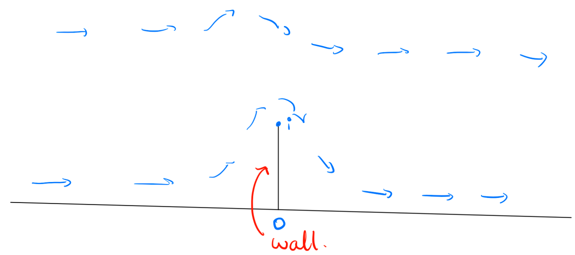


Figure 28.4: Fluid Flow past a wall

Solution: Since the fluid can't flow through the wall, the flow lines must be orthogonal ($\nabla\phi = (u, v) = (0, v)$) to the wall on $x = 0, \quad 0 \le y \le a$. We can solve this with conformal mapping. Our idea is we'll have a domain with a flow from left to right. We'll take the flow lines of that flow and map it to our domain with the wall. We'll then have the flow lines around the wall.
Recall that $f(z) = a(z^2 - 1)^{1/2}$ defines a conformal map from $\mathbb{H}/\{it \in \mathbb{C} : 0 \le t \le a\}$ to the upper half plane, sending $f(-1) = f(1) = 0$. Let's consider the right moving flow in the pre-image $\mathbb{H}$:

$$\begin{aligned} (\tilde{u}, \tilde{v}) &= (1, 0) \\ \tilde{G} &= z = x + iy \end{aligned}$$

Whose flow lines are given by $\{\Im(z) = y\}$ because there's only one component of velocity (there are no vertical flow lines), so we observe the flow at different values of y.
The Image of these flow lines is under f will be the flow lines around the wall ($a = 1$).

$$\begin{aligned} f(x + iy) &= (z^2 - 1)^{\frac{1}{2}} = \sqrt{z - 1}\sqrt{z + 1} \\ &= \sqrt{(x + iy) - 1}\sqrt{(x + iy) + 1} \\ &= ((x - 1) + y)^{1/2}((x + 1) + y)^{1/2} \\ &= \pm|(x - 1) + iy|^{1/2} e^{\frac{i}{2}\arg(x+iy-1)} \times \pm|(x + 1) + iy|^{1/2} e^{\frac{i}{2}\arg(x+iy+1)} \\ &\to \arg(x + iy - 1) = \cos^{-1}\left(\frac{\Re(x + iy - 1)}{|x + iy - 1|}\right) = \cos^{-1}\left(\frac{x - 1}{\sqrt{(x - 1)^2 + y^2}}\right) \\ &= ((x - 1)^2 + y^2)^{1/4} e^{\frac{i}{2}\cos^{-1}(\frac{x-1}{\sqrt{(x-1)^2+y^2}})}((x + 1)^2 + y^2)^{1/4} e^{\frac{i}{2}\cos^{-1}(\frac{x+1}{\sqrt{(x+1)^2+y^2}})} \end{aligned}$$

Note: the root is multi-valued, because a conformal map is definite (and it doesn't make sense for a fluid to have two different directions in the same place), we have to choose a value. We choose the positive value to keep us in the upper-half plane.

$$\text{x-component of the flow} =$$

$$((x - 1)^2 + y^2)^{1/4}((x + 1)^2 + y^2)^{1/4}\cos(\frac{1}{2}\left(\cos^{-1}(\frac{x - 1}{\sqrt{(x - 1)^2 + y^2}}) + \cos^{-1}(\frac{x + 1}{\sqrt{(x + 1)^2 + y^2)}}\right))$$

$$\text{y-component of the flow} =$$

$$((x - 1)^2 + y^2)^{1/4}((x + 1)^2 + y^2)^{1/4}\sin(\frac{1}{2}\left(\cos^{-1}(\frac{x - 1}{\sqrt{(x - 1)^2 + y^2}}) + \cos^{-1}(\frac{x + 1}{\sqrt{(x + 1)^2 + y^2)}}\right))$$

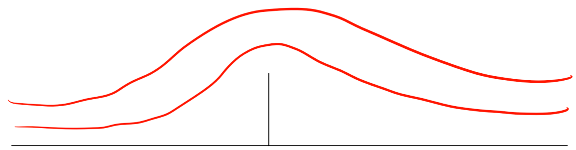


Figure 28.5: Flow Lines around the wall

■

■ **Example 28.2** Find the flow lines of a sourceless irrotational flow past an infinitely deep trench.

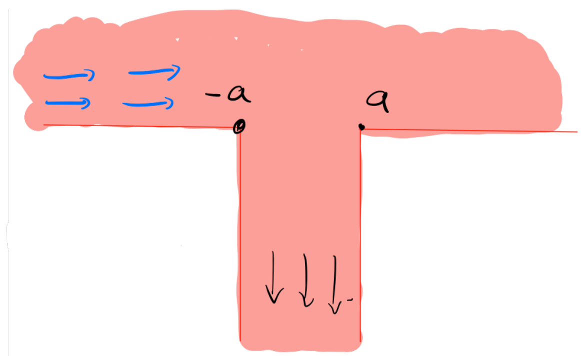


Figure 28.6: Fluid Flow past a trench

Solution: Recall that:

$$f(z) = \frac{2}{\pi}((z^2 - 1)^{1/2} + \arcsin(\frac{1}{z}))$$

Defines a conformal map from $\mathbb{H} \to$ the an infinitely deep trench from $-a$ to a.
Consider the right moving flow

$$(\tilde{u}, \tilde{v}) = (1, 0)$$

The flow lines through the trench are the image of the flow lines $\{\Im(z) = y\}$ under f.

$$\begin{aligned} f(x+iy) &= \sqrt{(x+iy)-1}\sqrt{(x+iy)+1} \\ &= \frac{2}{\pi}[((x-1)^2+y^2)^{1/4}e^{\frac{i}{2}\cos^{-1}(\frac{x-1}{\sqrt{(x-1)^2+y^2}})}((x+1)^2+y^2)^{1/4}e^{\frac{i}{2}\cos^{-1}(\frac{x+1}{\sqrt{(x+1)^2+y^2})}} \\ &+ \arcsin(\frac{1}{x+iy})] \end{aligned}$$

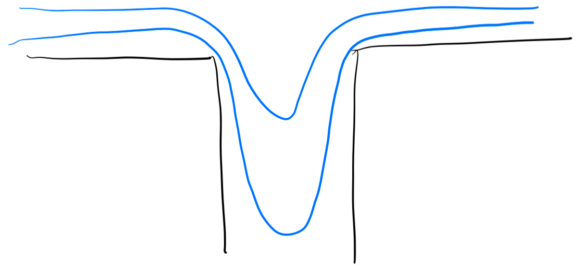


Figure 28.7: Flow Lines through the trench

■

■ **Example 28.3 — Fluid Flow Past a Solid Body.** Lets Assume the flow is (locally) sourceless and irrotational.

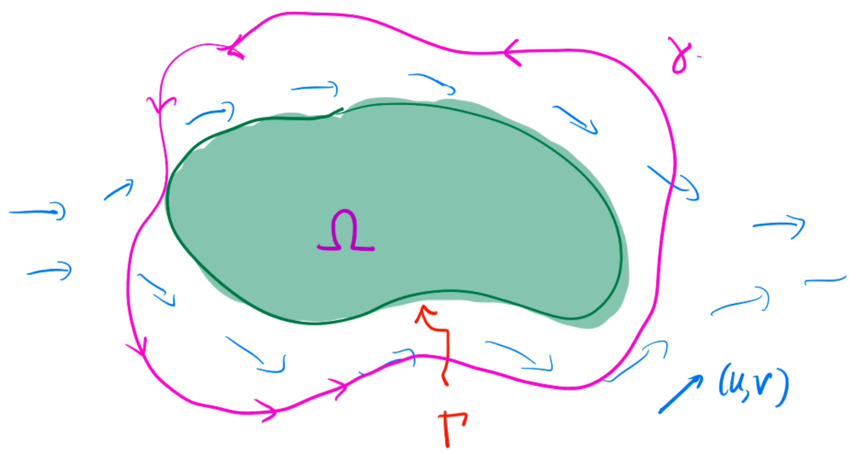


Figure 28.8: Fluid Flow past a solid body

- Γ is the simple closed curve, piecewise C^1 boundary around body.
- Ω is the region inside the body.
- γ is some simple, closed, piecewise C^1, positively oriented curve such that $\Omega \subseteq \text{inside}(\gamma)$.
- $\overline{f} = u - iv$ is analytic outside Ω

We call $\tau = \frac{\gamma'}{|\gamma'|}$ the tangent unit vector to γ, C the circulation of the flow around Ω, and $ds = |\gamma'|dt$ the arc length element of γ.

$$
\begin{aligned}
C &= \int_\gamma f(\gamma') \cdot \tau \, ds \\
&= \int_\gamma (u + iv) \cdot \frac{\gamma'}{|\gamma'|} \, |\gamma'| \, dt \\
&= \int_\gamma \Re((u + iv) \cdot (\Re(\gamma') + i\Im(\gamma'))) \, dt \quad \text{Since the dot product is always real} \\
&= \Re(\int_\gamma (u\Re(\gamma') - v\Im(\gamma')) \, dt)
\end{aligned}
$$

$$
\begin{aligned}
&= \Re\left(\int_\gamma \overline{f}\gamma'\ dt\right) \\
&= \Re\left(\int_\gamma \overline{f}(\gamma')\gamma' dt\right) \quad \text{if } z = \gamma(t) \\
&= \Re\left(\int_\gamma \overline{f} dz\right)
\end{aligned}
$$

Note: C does not depend on γ by Cauchy's Theorem. Since $\overline{f}$ is analytic in Ω. Since the flow goes around Ω, and hence, there's no flow come out through the surface of the body:

$$(u, v) \cdot \vec{\eta} = 0 \quad \text{on } \partial\Omega$$

Where $\vec{\eta} = (\cos\theta, \sin\theta)$ (where $\theta = \arg(\vec{\eta})$) is the outward unit normal to $\partial\Omega$. So if you have a vertical wall, $\cos\theta = 0$ and thus $u\cos\theta = 0$ for all vertical flows, which makes sense since the flow is parallel to the wall. Similar for horizontal walls.
So let's take the parametrization $\Gamma(t)$, $\quad ds = |\Gamma'(t)|dt$ of $\partial\Omega$.

$$0 = \int_{\partial\Omega=\Gamma} f(\Gamma(t)) \cdot \vec{\eta} \quad \Gamma'(t)\ dt \qquad = \int_{\partial\Omega=\Gamma} (u + iv) \cdot (\cos\theta + i\sin\theta)$$

$$
\begin{aligned}
0 &= \int_{\partial\Omega=\Gamma} f \cdot \vec{\eta}\ dt \\
&= \int_{\partial\Omega=\Gamma} (u + iv) \cdot (\cos\theta + i\sin\theta) \qquad = \int_\Gamma \Re((u + iv) \cdot (-\tau_y + i\tau_x))\ ds \\
&= \int_\Gamma \Re((u + iv) \cdot (-\Im(\tau) + i\Re(\tau)))\ ds \\
&= \int_\Gamma -u\Im(\tau) - v\Re(\tau)\ ds \\
&= \Im\left(\int_\Gamma \overline{f}(z)\ dz\right) = \Im\left(\int_\Gamma \overline{f}(z)\ dz\right)
\end{aligned}
$$

Assume the flow is uniform far from Ω.

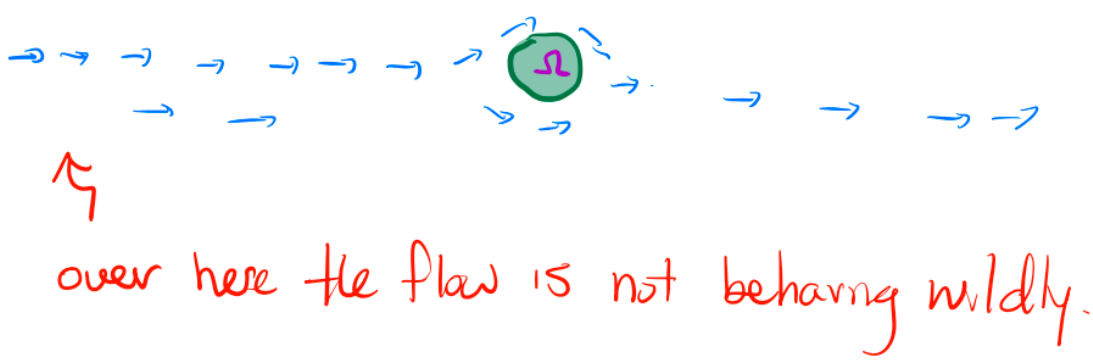


Figure 28.9: Uniform Flow

$$\lim_{|z|\to\infty} \overline{f}(z) = a \in \mathbb{C}$$

Thus, $\overline{f}$ has a removable singularity at ∞. i.e. If we set $w = \frac{1}{z}$ then $h(w) = \overline{f}(\frac{1}{w})$ is analytic in $\{0 < |w| < \varepsilon\}$ and bounded as $w \to 0$.
Since $\overline{f}$ is analytic and has a removable singularity at ∞, it has a power series expansion at ∞.

$$\overline{f} = \sum_{k=0}^{\infty} b_k (\frac{1}{z})^k$$

is valid for $|z|$ large.

$$\begin{aligned}
\lim_{|z|\to\infty} \overline{f} &= \lim_{|z|\to\infty} \left(b_0 + \frac{b_1}{z} + \sum_{k=2}^{\infty} b_k (\frac{1}{z})^k \right) \\
b_0 = a &= \lim_{|z|\to\infty} f(z)
\end{aligned}$$

Cauchy's Integral Formula can be used to find coefficients in a power series expansion.

$$\begin{aligned}
b_k &= \frac{1}{2\pi i} \int_{|z|=R} \frac{\overline{f}(z)}{z^{k-1}} \, dz \\
&= \frac{1}{2\pi i} \int_{|z|=R} \overline{f}(z) \, dz \\
b_1 &= \frac{C}{2\pi i} \quad \text{where} \quad C = \text{circulation of the flow around } \Omega
\end{aligned}$$

This gives us the final result:

$$\overline{f}(z) = a + \frac{C}{2\pi i z} + \sum_{k=2}^{\infty} b_k (\frac{1}{z})^k$$

Our Goal: To fix C, to get some $\overline{f}$ such that $\overline{f} \cdot \vec{\eta}|_{\Gamma} = 0$ on $\partial\Omega$.

■

Proposition 28.0.3 — Kutta-Joukowski Theorem. Assume Ω has density ρ.
We can compute: The total vertical force, V, and horizontal force H on Ω due to the flow.

$$\boxed{V + iH = -\rho C a}$$

If $a \in R$, then the only force is vertical, the lift force.

Definition 28.0.1 — Unit Tangent & Normal Vectors. Say we have some body Ω with boundary parametrized by $\Gamma(t)$. The unit tangent vector is:

$$\tau = \frac{\Gamma'}{|\Gamma'|} \tag{28.1}$$

The unit normal vector is a clockwise $90°$ rotation, which can be achieved by multiplying by $-i$, you can remember this by remembering how the vector $(1, 0)$ rotates after multiplying by i or $-i$.

$$\eta = -i\tau = \frac{-i\Gamma'}{|\Gamma'|} \tag{28.2}$$

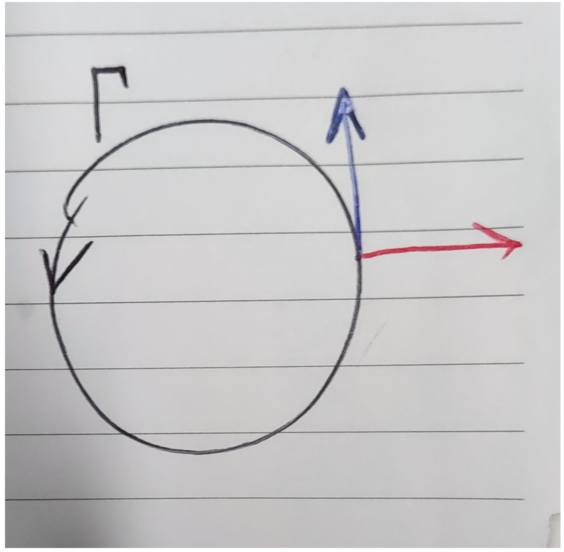


Figure 28.10: Unit Tangent and Normal Vectors

■ **Example 28.4 — Flow Past a Disk.**

$$\Omega = \{|z| < 1\}$$
$$\lim_{|z|\to\infty} \overline{f} = (1, 0)$$

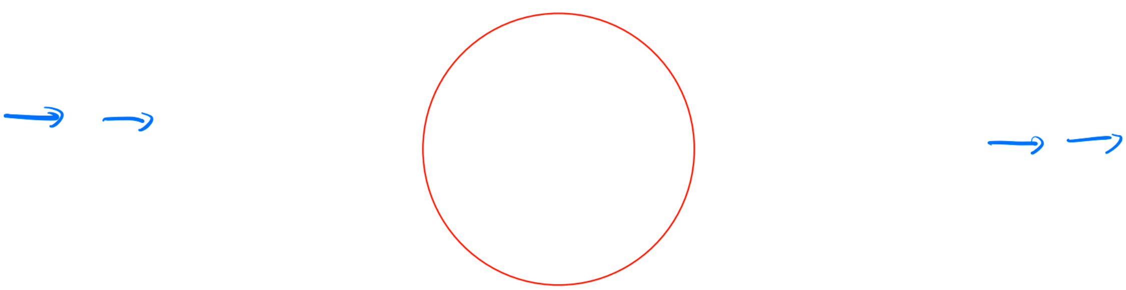


Figure 28.11: Fluid Flow past a disk

Therefore we want

$$\overline{f} = a + \frac{C}{2\pi iz} + \sum_{k=2}^{\infty} b_k(\frac{1}{z})^k$$
$$\overline{f} = 1 + \frac{C}{2\pi iz} + \sum_{k=2}^{\infty} b_k(\frac{1}{z})^k$$

Such that on $\partial\Gamma, \Gamma(t) = e^{it}, \quad 0 \leq t \leq 2\pi$ which will parametrize a unit circle. First we find $\vec{\eta}$:

$$\vec{\eta} = -i\tau \tag{28.3}$$

$$= -i\frac{\Gamma'}{|\Gamma'|} \tag{28.4}$$

$$= -i\frac{ie^{it}}{|ie^{it}|} \tag{28.5}$$

$$= e^{it} \tag{28.6}$$

Now we want $\overline{f} \cdot \vec{\eta} = 0$ on $\partial\Omega$.

$$\begin{aligned}
0 &= f(\Gamma) \cdot \eta \\
&= (u, v) \cdot (\cos\theta, \sin\theta) \\
&= u\cos\theta + v\sin\theta \\
&= \Re((u - iv)(\cos\theta + i\sin\theta)) \\
&\to e^{i\theta} = \cos\theta + i\sin\theta \\
&= \Re((u - iv)e^{i\theta}) = \Re(\overline{f}(\Gamma)e^{i\theta})
\end{aligned}$$

This way we can use what we know about $\overline{f}$

$$\Re\left[(1 + \frac{C}{2\pi i}e^{-i\theta} + \sum_{k=2}^{\infty} b_k e^{-ik\theta})e^{i\theta}\right] = 0$$

$$\Re\left[\frac{C}{2\pi i} + e^{i\theta} + b_2 e^{-i\theta} + \sum_{k=3}^{\infty} b_k e^{i(1-k)\theta}\right] = 0$$

In order to make $\Re(\frac{C}{2\pi i}) = 0$, we must have C be real so the whole term is imaginary.
We can have $b_2 e^{-i\theta}$ cancel out $e^{i\theta}$:

$$\Re(e^{i\theta} + b_2 e^{-i\theta}) = \Re(\cos\theta + i\sin\theta + b_2(\cos(-\theta) + i\sin(-\theta))) \tag{28.7}$$

$$= \cos\theta + b_2\cos(-\theta) \tag{28.8}$$

$$= \cos\theta + b_2\cos\theta = 0 \quad \text{if } b_2 = -1 \tag{28.9}$$

Lastly, we can set $b_k = 0$ for $k \geq 3$ to make the sum term zero.
In summary:

$$C \in \mathbb{R}, \quad b_2 = -1, \quad b_k = 0 \quad \forall k \geq 3$$

Therefore:

$$\overline{f} = 1 - \frac{C}{2\pi iz} - \frac{1}{z^2}$$

We can extend this analysis to other domains using conformal maps (e.g. conformal map ϕ)
A common example is the Joukowski airfoil:

$$\Omega = |z - z_0| < R$$

$$\phi(z) = (z + \frac{R^2}{z})$$

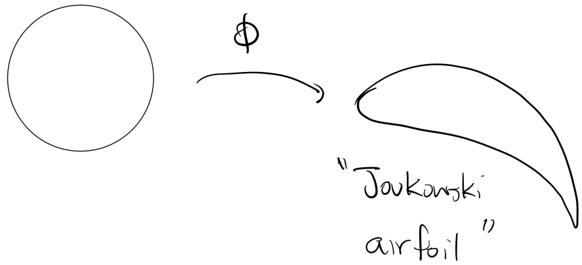
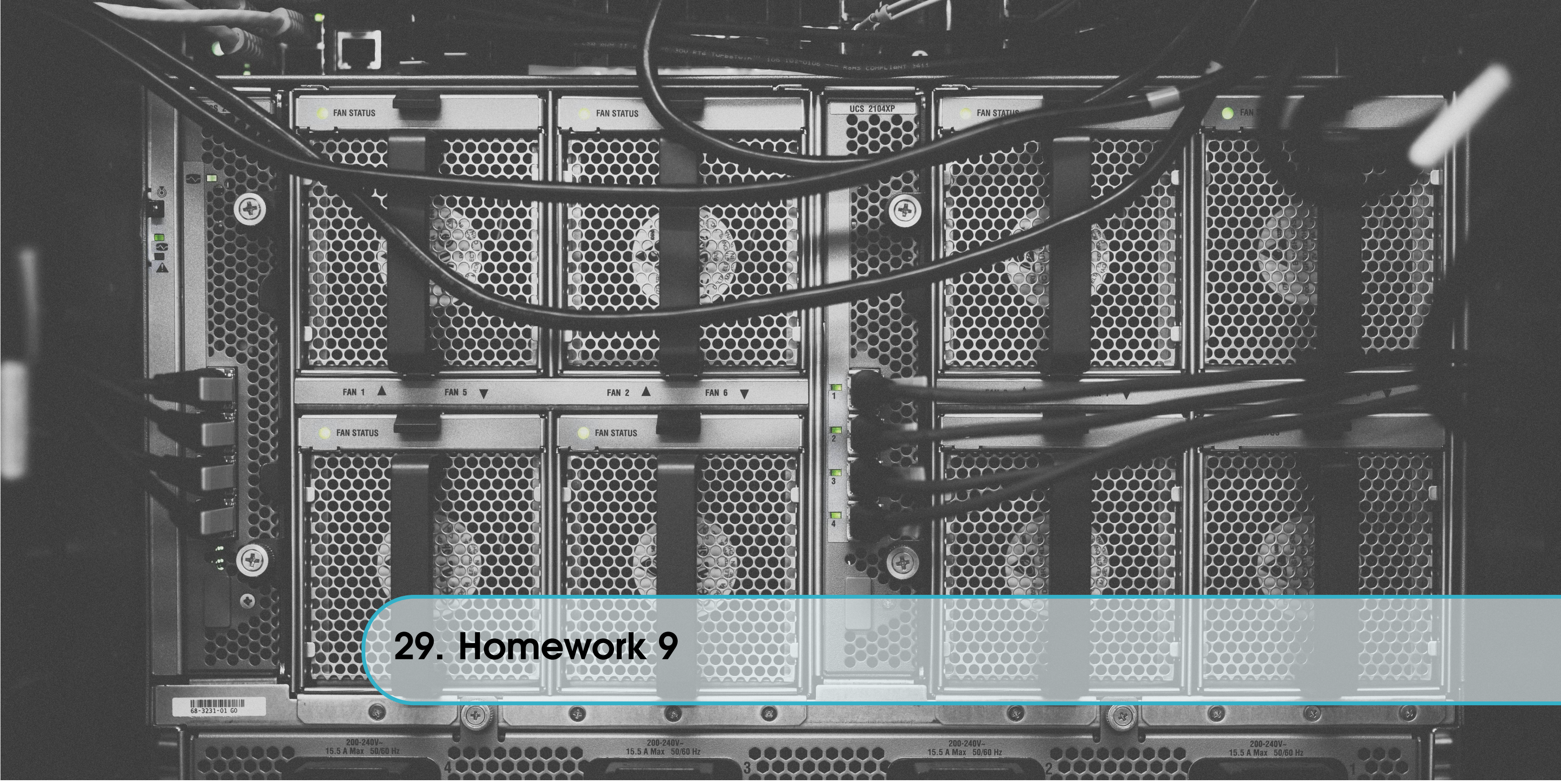


Figure 28.12: Conformal Map of a Disk

- The circulation is determined by z_0, R.
- The lift force is determined by $\Im(z_0)$.

∎



29. Homework 9

■ **Example 29.1 — Fisher, Section 3.5, Problem 2.** Find a conformal map from $D = \{z \in \mathbb{C} : |\mathrm{Arg}(z)| < \alpha\}$ for $0 < \alpha \leq \pi$, to the upper half-plane $\mathscr{H} = \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$.

The wedge D is symmetric about the real axis, with boundaries given by:

$$\mathrm{Arg}(z) = \pm\alpha.$$

We will use the function:

$$f(z) = \left(z^{\frac{\pi}{2\alpha}}\right) e^{i\frac{\pi}{2}}.$$

Express z in polar form:

$$\begin{aligned} z &= re^{i\theta}, \quad \text{so} \\ f(z) &= \left(r^{\frac{\pi}{2\alpha}} e^{i\left(\frac{\pi}{2\alpha}\theta\right)}\right) e^{i\frac{\pi}{2}} \\ &= r^{\frac{\pi}{2\alpha}} e^{i\left(\frac{\pi}{2\alpha}\theta + \frac{\pi}{2}\right)}. \end{aligned}$$

When $\theta = -\alpha$ (the boundaries of D):

$$\mathrm{Arg}(f(z)) = \frac{\pi}{2\alpha}(-\alpha) + \frac{\pi}{2} = -\frac{\pi}{2} + \frac{\pi}{2} = 0.$$

When $\theta = \alpha$:

$$\mathrm{Arg}(f(z)) = \frac{\pi}{2\alpha}(\alpha) + \frac{\pi}{2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

This maps the boundaries of D to the real axis ($\mathrm{Im}(w) = 0$).
For $|\ \mathrm{Arg}(z)| < \alpha$ (the interior of D), we have:

$$-\alpha < \theta < \alpha \implies -\frac{\pi}{2} + \frac{\pi}{2} < \mathrm{Arg}(f(z)) < \frac{\pi}{2} + \frac{\pi}{2} \implies 0 < \mathrm{Arg}(f(z)) < \pi.$$

This places the image of the interior of D in the upper half-plane $(\mathrm{Im}(w) > 0)$.
Thus, the function $f(z) = \left(z^{\frac{\pi}{2\alpha}}\right) e^{i\frac{\pi}{2}}$ is a conformal map from D to $\mathscr{H}$.
This map transforms the wedge D with angle 2α into the upper half-plane, preserving the conformal property.

■

■ **Example 29.2 — Fisher, Section 3.5, Problem 3.** Find a conformal map from $D = \{x + iy \in \mathbb{C} : |y - 1| < 2\}$ to the upper half-plane $\mathscr{H} = \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$.

1. Understand the Domain D

The domain is given by:

$$D = \{x + iy \in \mathbb{C} : |y - 1| < 2\}.$$

This inequality can be rewritten as:

$$-2 < y - 1 < 2 \implies -1 < y < 3.$$

Therefore, D is the horizontal strip in the complex plane where $y \in (-1, 3)$.
2. Shift the Strip Vertically

To simplify the mapping, we can shift the strip so that it starts at zero. Let's define:

$$z' = z + i.$$

This transformation shifts the imaginary part:

$$y' = y + 1.$$

Now, the strip becomes:

$$y' \in (-1 + 1, 3 + 1) \implies y' \in (0, 4).$$

So the new domain is $y' \in (0, 4)$.
3. Apply the Exponential Function

The exponential function maps horizontal strips to sectors or the entire complex plane minus a ray. We can use the function:

$$w = e^{\frac{\pi}{4}z'}.$$

Substituting back $z' = z + i$, we get:

$$w = e^{\frac{\pi}{4}(z+i)}.$$

The function:

$$f(z) = e^{\frac{\pi}{4}(z+i)}$$

is a conformal map from the domain D to the upper half-plane $\mathscr{H}$. ■

■ **Example 29.3** (Fisher, Section 3.5, Problem 10)
Find a Schwarz-Christoffel transformation from the upper half-plane $\mathscr{H}$ to $D = \{z \in \mathbb{C} : 0 < \text{Arg}(z) < \frac{4\pi}{3}\}$.

To find the Schwarz-Christoffel transformation from the upper half-plane $\mathscr{H}$ to the sector D, we'll proceed step by step:

1. Understand the Target Domain D

The domain D is the sector defined by:

$$D = \left\{z \in \mathbb{C} : 0 < \text{Arg}(z) < \frac{4\pi}{3}\right\}.$$

This is a sector with an opening angle of $\frac{4\pi}{3}$, bounded by two rays from the origin at angles 0 and $\frac{4\pi}{3}$.

2. Consider the Mapping Function

We can consider a mapping of the form:

$$w = f(z) = z^{\lambda},$$

where λ is a positive real number to be determined. This power function maps the upper half-plane onto a sector in the complex plane.

3. Determine the Appropriate Exponent λ

In the upper half-plane $\mathscr{H}$, z has an argument θ in the range:

$$0 < \text{Arg}(z) < \pi.$$

Under the mapping $w = z^{\lambda}$, the argument of w becomes:

$$\text{Arg}(w) = \lambda \text{ Arg}(z).$$

We want the image of $\mathscr{H}$ under $f(z)$ to be the sector D, so we set:

$$0 < \text{Arg}(w) = \lambda \text{ Arg}(z) < \lambda\pi = \frac{4\pi}{3}.$$

Solving for λ:

$$\lambda\pi = \frac{4\pi}{3} \implies \lambda = \frac{4}{3}.$$

4. Define the Mapping Function

With $\lambda = \frac{4}{3}$, the mapping becomes:

$$w = f(z) = z^{\frac{4}{3}}.$$

6. Addressing the Branch Cut

The function $w = z^{\frac{4}{3}}$ is multi-valued, so we need to specify the branch of the logarithm used. We choose the principal branch, where the argument of z satisfies $0 < \operatorname{Arg}(z) < \pi$, corresponding to the upper half-plane.
The Schwarz-Christoffel transformation from the upper half-plane $\mathscr{H}$ to the sector D is:

$$f(z) = z^{\frac{4}{3}}.$$

■

■ **Example 29.4** (a) (5 points) Find a conformal map from the infinite strip $\{z = x + iy : 0 < y < \pi\}$ to the upper half-plane $\mathscr{H}$.
(b) (5 points) Using your solution to part (a), and the solution to Q4, find a conformal map from $\{z = x + iy : 0 < y < \pi\}$ to the semi-infinite strip $\{\sigma + i\tau : 0 < \sigma < 1, \text{ and } \tau > 0\}$.

Part (a)

To find a conformal map from the infinite strip $\{z = x + iy : 0 < y < \pi\}$ to the upper half-plane $\mathscr{H}$, we can use the exponential function:

$$w = f(z) = e^z.$$

Verification of the Mapping

Let $z = x + iy$, then:

$$\begin{aligned} w &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x(\cos y + i \sin y). \end{aligned}$$

Since $0 < y < \pi$, we have $\sin y > 0$. Therefore, the imaginary part of w is:

$$\operatorname{Im}(w) = e^x \sin y > 0.$$

This means that w lies in the upper half-plane $\mathscr{H}$. The mapping $w = e^z$ is conformal because the exponential function is analytic everywhere and its derivative is never zero. **Part (b)**

Using the solution from part (a) and the solution to Q4, we aim to find a conformal map from $\{z = x + iy : 0 < y < \pi\}$ to the semi-infinite strip $\{\sigma + i\tau : 0 < \sigma < 1, \tau > 0\}$.
Step 1: Map the Strip to the Upper Half-Plane

From part (a), we have the mapping:

$$w = e^z,$$

which maps the strip $\{z = x + iy : 0 < y < \pi\}$ to the upper half-plane $\mathscr{H}$.

Step 2: Map the Upper Half-Plane to the Semi-Infinite Strip

We need a conformal map from $\mathscr{H}$ to the semi-infinite strip $\{\sigma + i\tau : 0 < \sigma < 1, \tau > 0\}$. One such map, inspired by the solution to Q4, is:

$$s = \frac{1}{\pi} \ln w.$$

Verification of the Mapping

Let $w = u + iv$, where $v > 0$ (since $w \in \mathscr{H}$). Then:

$$\begin{aligned} s &= \frac{1}{\pi} \ln w \\ &= \frac{1}{\pi}(\ln |w| + i \operatorname{Arg}(w)). \end{aligned}$$

Since w is in the upper half-plane, $\operatorname{Arg}(w) \in (0, \pi)$. Therefore:

$$\operatorname{Im}(s) = \frac{1}{\pi} \operatorname{Arg}(w) \in (0, 1) .$$

However, the real part $\operatorname{Re}(s) = \frac{1}{\pi} \ln |w|$ varies over $\mathbb{R}$.
Step 3: Map the Infinite Strip to the Semi-Infinite Strip

To restrict the real part to $(0, 1)$, we can use the transformation:

$$f(w) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{\pi}{2} s\right) .$$

Substituting $s = \frac{1}{\pi} \ln w$, we get:

$$\begin{aligned} f(w) &= \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{\pi}{2}\left(\frac{1}{\pi} \ln w\right)\right) \\ &= \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{1}{2} \ln w\right) . \end{aligned}$$

Simplify the expression:

$$\begin{aligned} f(w) &= \frac{1}{2} + \frac{1}{2} \cdot \frac{w^{1/2} - w^{-1/2}}{w^{1/2} + w^{-1/2}} \\ &= \frac{1}{2} + \frac{1}{2} \cdot \frac{\sqrt{w} - \frac{1}{\sqrt{w}}}{\sqrt{w} + \frac{1}{\sqrt{w}}} \\ &= \frac{1}{2} + \frac{1}{2} \cdot \frac{w - 1}{w + 1}. \end{aligned}$$

Therefore, the mapping from w to the semi-infinite strip is:

$$f(w) = \frac{1}{2} + \frac{1}{2} \cdot \frac{w - 1}{w + 1}.$$

Composite Mapping

Combining the mappings from Step 1 and Step 3, we have:

$$\begin{aligned} \text{First, } w &= e^z, \\ \text{Then, } f(w) &= \frac{1}{2} + \frac{1}{2} \cdot \frac{e^z - 1}{e^z + 1}. \end{aligned}$$

Simplify the composite mapping:

$$\begin{aligned} f(z) &= \frac{1}{2} + \frac{1}{2} \cdot \frac{e^z - 1}{e^z + 1} \\ &= \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{z}{2}\right). \end{aligned}$$

Conclusion for Part (b)

The conformal map from $\{z = x + iy : 0 < y < \pi\}$ to the semi-infinite strip $\{\sigma + i\tau : 0 < \sigma < 1, \tau > 0\}$ is:

$$f(z) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{z}{2}\right).$$

■

■ **Example 29.5** Find a Schwarz-Christoffel transformation mapping the upper half-plane $\mathscr{H}$ onto the given domain D.
Solution

To find the Schwarz-Christoffel transformation for the modified domain D, we proceed as follows:
1. Analyze the Geometry of the Domain

The domain D now consists of:
- A vertical line segment from 0 to ia along the positive imaginary axis.
- A vertical line segment from 0 to $-i\infty$ along the negative imaginary axis.
- A horizontal line segment from 0 to $-\infty$ along the negative real axis.

2. Identify the Vertices and Angles

We identify the vertices of D and their corresponding interior angles:
- At $z = 0$: This is a vertex where three edges meet. The interior angle is $\frac{3\pi}{2}$.
- At $z = ia$: The angle here is π (since the path continues straight upward).
- At infinity along the negative real axis and negative imaginary axis.

3. Calculate the Exponents β_k

For each vertex, we calculate:

$$\beta_k = \frac{\text{Interior angle at vertex}}{\pi} - 1.$$

- At $z = 0$:

$$\beta_1 = \frac{\frac{3\pi}{2}}{\pi} - 1 = \frac{3}{2} - 1 = \frac{1}{2}.$$

- At $z = ia$:

$$\beta_2 = \frac{\pi}{\pi} - 1 = 0.$$

4. Write the Schwarz-Christoffel Transformation

The general form of the Schwarz-Christoffel transformation is:

$$f(z) = C \int \prod_k (z - z_k)^{\beta_k} \, dz + C_0,$$

where C and C_0 are constants, z_k are the pre-images of the vertices, and β_k are the exponents calculated above.
Substituting the values:

$$f(z) = C \int (z - 0)^{\beta_1} (z - ia)^{\beta_2} \, dz + C_0.$$

Since $\beta_2 = 0$, $(z - ia)^{\beta_2} = 1$. Therefore, the transformation simplifies to:

$$f(z) = C \int z^{\frac{1}{2}} \, dz + C_0.$$

5. Evaluate the Integral

Integrate $z^{\frac{1}{2}}$:

$$\int z^{\frac{1}{2}} \, dz = \frac{2}{3} z^{\frac{3}{2}} + \text{constant}.$$

Therefore, the Schwarz-Christoffel transformation becomes:

$$f(z) = C \left(\frac{2}{3} z^{\frac{3}{2}} \right) + C_0.$$

6. Determine the Constants

The constants C and C_0 can be determined based on boundary conditions or normalization requirements specific to the problem.

7. Conclusion

With the additional line segment from 0 to $-i\infty$ along the negative imaginary axis, the Schwarz-Christoffel transformation mapping the upper half-plane $\mathscr{H}$ onto the domain D is:

$$f(z) = C\left(\frac{2}{3}z^{\frac{3}{2}}\right) + C_0.$$

■

■ **Example 29.6** To find the Schwarz-Christoffel transformation mapping the upper half-plane $\mathscr{H}$ onto the domain D, we proceed as follows:

1. Analyze the Geometry of the Domain D

The domain D is composed of:

- Quadrant 1: $0 < \theta < \frac{\pi}{2}$
- Quadrant 2: $\frac{\pi}{2} < \theta < \pi$
- Quadrant 4: $-\frac{\pi}{2} < \theta < 0$
- An additional line segment from the origin in the direction $\theta = \frac{\pi}{4}$.

2. Identify the Vertices and Angles

We need to identify the vertices of the polygonal domain D and their corresponding interior angles. The vertices are:

- At $z = 0$: This point is a vertex where three edges meet—the rays at angles $-\frac{\pi}{2}$, 0, and $\frac{\pi}{2}$, with a slit along $\theta = \frac{\pi}{4}$. The interior angle at $z = 0$ is calculated as:

$$\text{Interior angle at } z = 0 = 2\pi - \left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \frac{5\pi}{4}.$$

- At infinity along $\theta = -\frac{\pi}{2}$ (negative imaginary axis): The interior angle is $\frac{\pi}{2}$.
- At infinity along $\theta = \pi$ (negative real axis): The interior angle is π.
- At infinity along $\theta = \frac{\pi}{4}$ (the slit): The interior angle is 0 (since the slit introduces a branch cut).

3. Calculate the Exponents β_k

For each vertex, we calculate:

$$\beta_k = \frac{\text{Interior angle at vertex}}{\pi} - 1.$$

- At $z = 0$:

$$\beta_1 = \frac{\frac{5\pi}{4}}{\pi} - 1 = \frac{5}{4} - 1 = \frac{1}{4}.$$

- At $\theta = -\frac{\pi}{2}$ (infinity along negative imaginary axis):

$$\beta_2 = \frac{\frac{\pi}{2}}{\pi} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}.$$

- At $\theta = \pi$ (infinity along negative real axis):

$$\beta_3 = \frac{\pi}{\pi} - 1 = 1 - 1 = 0.$$

- At $\theta = \frac{\pi}{4}$ (infinity along the slit):

$$\beta_4 = \frac{0}{\pi} - 1 = -1.$$

4. Write the Schwarz-Christoffel Transformation

The general Schwarz-Christoffel transformation is:

$$f(z) = C \int \prod_k (z - z_k)^{\beta_k} \, dz + C_0.$$

Since two of the vertices are at infinity, we adjust the mapping accordingly. Let's assign finite pre-images to the vertices:

- $z = -1$: Pre-image of $\theta = \pi$ (negative real axis).
- $z = 0$: Vertex at $z = 0$.
- $z = 1$: Pre-image of $\theta = \frac{\pi}{4}$ (the slit).
- $z = \infty$: Corresponds to $\theta = -\frac{\pi}{2}$ (negative imaginary axis).

The mapping becomes:

$$f(z) = C \int (z + 1)^{\beta_3} z^{\beta_1} (z - 1)^{\beta_4} \, dz + C_0.$$

Substituting the exponents:

$$\begin{aligned} f(z) &= C \int (z + 1)^0 \, z^{\frac{1}{4}} \, (z - 1)^{-1} \, dz + C_0 \\ &= C \int z^{\frac{1}{4}} (z - 1)^{-1} \, dz + C_0. \end{aligned}$$

5. Evaluate the Integral

The integral:

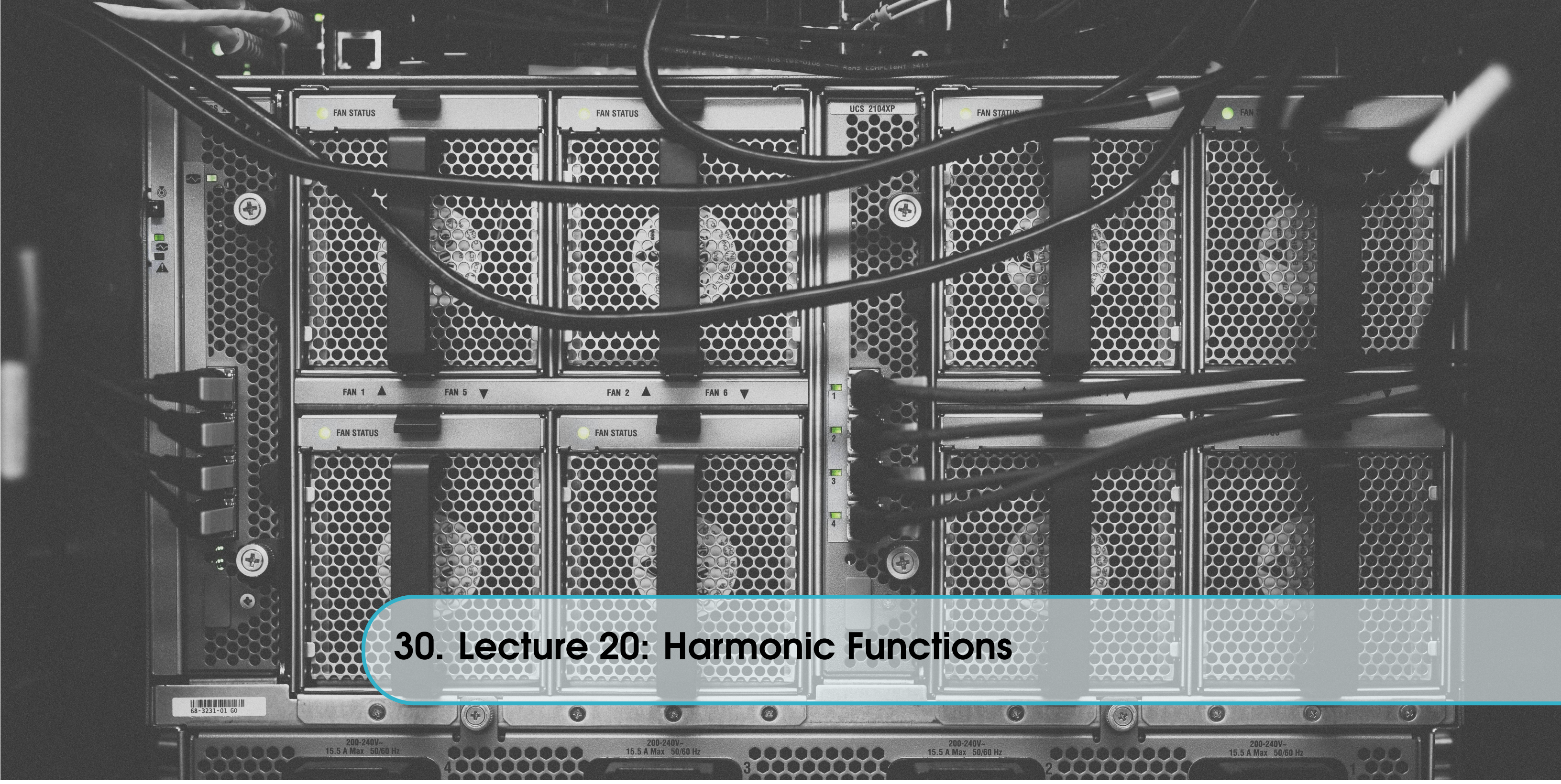
$$I = \int z^{\frac{1}{4}} (z - 1)^{-1} \, dz$$

can be evaluated using substitution or special functions, but it may involve complex analysis techniques. For the purpose of this problem, we can leave it in integral form or express it using known functions.

Therefore, the Schwarz-Christoffel transformation is:

$$f(z) = C \int z^{\frac{1}{4}} (z - 1)^{-1} \, dz + C_0,$$

where C and C_0 are constants determined by boundary conditions. ∎



30. Lecture 20: Harmonic Functions

Definition 30.0.1 — Harmonic Functions. if $D \subseteq \mathbb{C} = \mathbb{R}^2$ is open, then $u : D \to \mathbb{R}$ is **harmonic** if it satisfies Laplace's equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

R Harmonic functions are ubiquitous in physical applications.
for example, a harmonic function u may describe:

- The steady state temperature distribution $\mathbb{R}^2$.
  - Temperature at x is $\phi(x)$, $\quad x \in \partial D = \gamma$
- Electrostatic potential energy
- Planar stress
- Potential energy in an ideal fluid
- In general, harmonic functions can represent potential energy
- Particle motion where we look to minimize potential energy as fast as possible
  - $x(t)$ ais a trajectory of a particle in $\mathbb{R}^2$
  - $\dot{x}(t) = -\nabla u(x(t))$ where u is a harmonic function

Proposition 30.0.1 — Harmonic Functions Representing Particle motion. Say $x(t)$ is a trajectory of a particle in $\mathbb{R}^2$ and $\dot{x}(t) = -\nabla u(x(t))$ where u is a harmonic function.
We can try to find a function v such that:

$$\nabla v = (\nabla u)^{\perp} = \begin{pmatrix} -\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} \end{pmatrix}$$

Therefore v must satisfy:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Which are the Cauchy-Riemann equations!
v is called the **conjugate harmonic function** of u. The level sets of $\{v = \text{const}\}$ are trajectories of the particle u.

If $D = \{|z - z_0| < r\}$ is a disk, then

$$v(x, y) = -\int_{x_0}^{x} \frac{\partial u}{\partial y}(t, y_0)dt + \int_{y_0}^{y} \frac{\partial u}{\partial x}(x_0, t)dt$$

Then $f = u + iv$ is analytic in D.

Theorem 30.0.2 A real-valued function is harmonic if and only if, on every disk, it is the real part of the analytic function.
Therefore, if I have a harmonic function, I can find an analytic function whose real part is the harmonic function and whose imaginary part is the conjugate harmonic function. Conversely, if I have an analytic function, I can find a harmonic function whose real part is the real part of the analytic function.

■ **Example 30.1** $u(x, y) = \frac{1}{2}\log(x^2 + y^2)$ is harmonic in $\{0 < x^2 + y^2 < 1\}$. Is it the real part of an analytic function?

No, it is not the real part of an analytic function defined on $\{0 < |z| < 1\}$, because $u = \Re(\log z)$, but $\log z = \log|z| + i\arg z$ is not continuous on $\{0 < |z| < 1\}$ because $\arg z$ is not continuous on $\{0 < |z| < 1\}$. This is due to the branch cut of the logarithm function, and it's an exception to the theorem.

■

R Due to the earlier theorem, properties of analytic functions are very closely (often equivalent) to properties of harmonic functions.

Lemma 30.0.3 — Maximum Modulus Principle for Harmonic Functions. if $u : D \to \mathbb{R}$ is harmonic and D is open and connected, then u cannot have a local max/min in D, unless u is constant.

$$\forall x \in D, \quad u(x) < \max_{\partial D} u \quad \text{if } u \text{ not constant}$$

$$\forall x \in D, \quad u(x) = \max_{\partial D} u \text{ iff } u \text{ is constant on } D$$

Proof. I'm not going to pretend I understand this proof, but here it is:
Suppose $x_* \in D$ is a local max of u. Then Hess $u(x_*)$ is negative semi-definite, so:

$$0 = \Delta u(x_*) = \text{Tr}\left(\text{Hess}\, u(x_*)\right) \le 0$$

■

Lemma 30.0.4 — Mean Value Property. if u is harmonic on $\{|z - z_0| < 2r\}$, then

$$\begin{aligned} u(z_0) &= \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta \\ &= \text{The average of } u \text{ on the circle of radius } r \text{ centered at } z_0 \end{aligned}$$

In fact: The mean value property is equivalent to being harmonic.

Lemma 30.0.5 If $u : D \to \mathbb{R}$ is harmonic on domain D and $\phi(\xi) : \tilde{D} \to D$ is analytic then $u \circ \phi$ is harmonic on $\tilde{D}$.

30.1 Harmonic Functions as Energy Minimizer

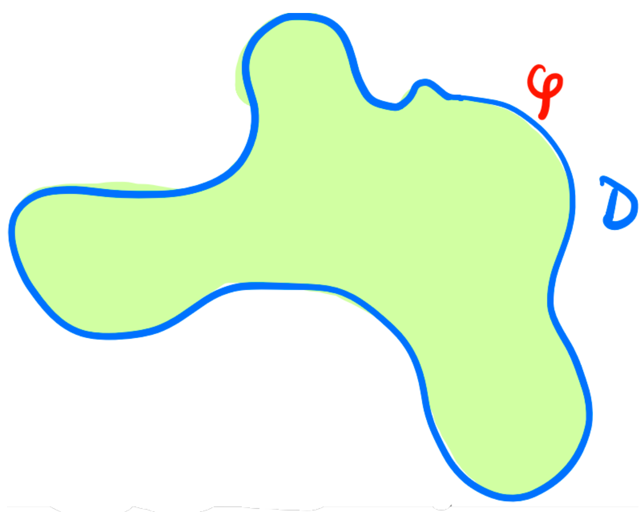


Figure 30.1: Heat distribution on a wire

Proposition 30.1.1 Let $D \subset \mathbb{R}^2$ be a domain, $\gamma = \partial D$, and $\phi : D \to \mathbb{R}$ be continuous. You can think of ∂D as a wire with a specified temperature, and ϕ is the heat distribution along D The natural energy (Dirichlet energy) associated with a function $u : D \to \mathbb{R} \mid u|_{\partial D} = \phi$ is:

$$\begin{aligned} \xi(u) &= \int_D |\nabla u|^2 dxdy \\ &= \int_D \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dxdy \end{aligned}$$

$$\begin{aligned} \therefore \text{ if } u &\sim Kg(\frac{m^2}{s^2}) && \text{has units of energy} \\ \nabla u &\sim (\frac{Kgm}{s^2}) && \text{force} \\ |\nabla u|^2 &\sim (\frac{Kg^2m^2}{s^4}) && \text{energy density} \\ \sqrt{\xi} &\sim (\frac{Kgm}{s^2}) && \text{energy} \end{aligned}$$

- $u(x, y)$ is our temperature distribution
- $\xi(u)$ Represents the total energy of u in D, it measure the smoothness of u. Essentially, the magnitude of the gradient of u at every point in D.
- We want to find u such that $u|_{\partial D} = \phi$ and $\xi(u)$ is minimized (because heat spreads out, thus we should reduce the magnitude of the gradient).
- This is a boundary value problem.

The physical temperature distribution should minimize $\xi(u)$ over all functions on D such that $u|_{\partial D} = \phi$. We say u is minimized if, for any $\psi : D \to \mathbb{R}$ such that $\psi|_{\partial D} = 0$, we have:

$$\begin{aligned} \xi(u + t\psi) &\geq \xi(u) \quad \forall t \in \mathbb{R} \\ \int_D |\nabla u|^2 \ dxdy + 2t \int_D \nabla u \cdot \nabla \psi \ dxdy + t^2 \int_D |\nabla \psi|^2 \ dxdy &= \\ \xi(u) + 2t \int_D \nabla u \cdot \nabla \psi \ dxdy + t^2 \int_D |\nabla \psi|^2 \ dxdy &\geq \xi(u) \end{aligned}$$

We can determine that $\int_D \nabla u \cdot \nabla \psi \ \ dxdy = 0$ By considering that if $\xi(u)$ is minimized, then $\frac{d}{dt}\xi(u + t\psi) = 0$ at $t = 0$ because a minimum is a critical point.

$$\frac{d}{dt}\xi(u + t\psi) = \frac{d}{dt}\left[\int_D |\nabla u|^2 \ dxdy + 2t \int_D \nabla u \cdot \nabla \psi \ dxdy + t^2 \int_D |\nabla \psi|^2 \ dxdy\right]_{t=0} \tag{30.1}$$

$$0 = 2\int_D \nabla u \cdot \nabla \psi \ dxdy \tag{30.2}$$

Let's integrate $\int_D \nabla u \cdot \nabla \psi \ \ dxdy$ by parts:

$$\begin{aligned} 0 &= \int_D \nabla u \cdot \nabla \psi \ dxdy \\ &= \int_D \nabla \cdot (\psi \nabla u) \ dxdy \\ &= \int_{\partial D} \psi \nabla u \cdot \vec{\eta} dS - \int_D \psi \nabla \cdot (\nabla u) \ dxdy \end{aligned}$$

Where $\vec{\eta}$ is the outward normal to ∂D, but since $\psi|_{\partial D} = 0$, the first term is 0, and so:

$$\begin{aligned} 0 &= \int_D \psi \nabla \cdot (\nabla u) \ dxdy \\ &= \int_D \psi \Delta u \ dxdy \end{aligned}$$

But this should hold for **all** ψ where $\psi|_{\partial D} = 0$, so $\Delta u = 0$.

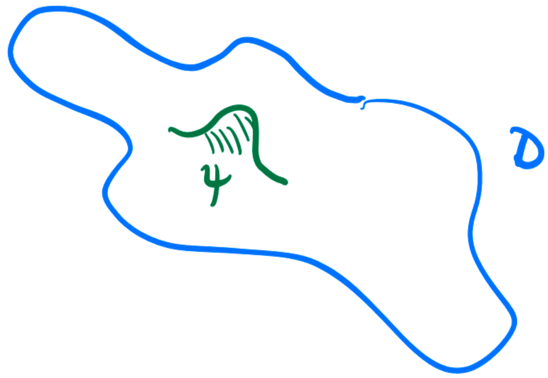


Figure 30.2: Boundary of domain D

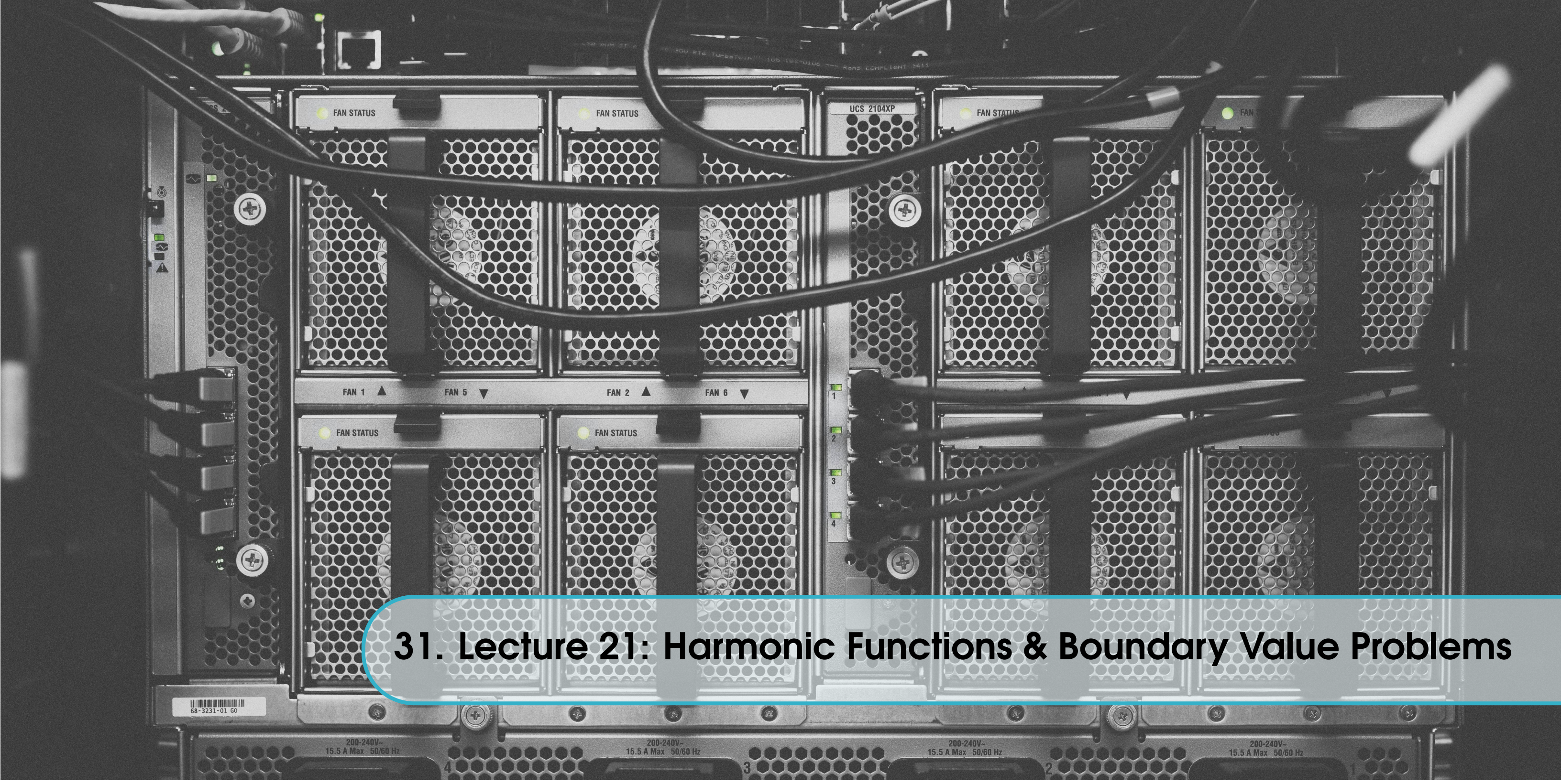
For most physical problems, we expect the energy should satisfy:

$$V(x) \sim |x|^2 \quad \text{near a minimizer}$$

Where V is the potential energy (or energy density) of the system.

$$\int_D V(\nabla u)dx = \int_D |\nabla u|^2 dx$$

Which is why harmonic functions appear so often in physics and engineering.



31. Lecture 21: Harmonic Functions & Boundary Value Problems

Proposition 31.0.1 — Boundary Value Problems. Suppose $D \in \mathbb{R}^2$ bounded domain and $\phi : \partial D \to R$ is given. How do we find (or does there exist) $u : D\mathbb{R}$:

$$\begin{cases} \Delta u = 0 \\ u|_{\partial D} = \phi \end{cases} \tag{31.1}$$

This becomes a boundary vlaue problem, which is a prototypical partial differential equation.

Theorem 31.0.2 — Solving Boundary Value Problems. **Strategy:** Integral Representation Formula Recall: if f is analytic on a simply connected domain D, with $\gamma = \partial D$ then:

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z} d\xi \quad \forall z \in D \tag{31.2}$$

$$\tag{31.3}$$

Therefore, if we know f on γ, we know f everywhere in D, and even have a formula for it!

Proposition 31.0.3 — Integral Representation Formula for Harmonic Functions. We try to apply this to harmonic functions. Let:

$$D = \{|z| \leq 1\}, \quad \gamma = \{e^{it} : 0 < t \leq 2\pi\} \tag{31.4}$$

if u is harmonic on D, then we can find f analytic such that $\Re(f) = u$. Then:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi \tag{31.5}$$

$$u = \Re(f) \tag{31.6}$$

$$= \Re\left(\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{it)}}{e^{it} - z} e^{it} dt\right) \tag{31.7}$$

But this depends on $\Re(f)$ and $\Im(f)$ on γ. This is not helpful as we only know $\Re(f) = u$ on γ and having to find $\Im(f)$ adds another layer of complexity.
Trick: Consider:

$$G(\xi) = \frac{f(\xi)\bar{z}}{1 - \bar{z}\xi} \tag{31.8}$$

Is analytic as a function of ξ on D. Then by Cauchy's Theorem:

$$\int_\gamma \frac{f(\xi)\bar{z}}{1 - \bar{z}\xi} d\xi = 0 \tag{31.9}$$

Then:

$$f(z) = \frac{1}{2\pi i}\left[\int_\gamma \frac{f(\xi)}{\xi - z} + \frac{f(\xi)\bar{z}}{1 - \bar{z}\xi} d\xi\right] \tag{31.10}$$

$$= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[\frac{(1 - \xi\bar{z}) + (\xi - z)\bar{z}}{(\xi - z)(1 - \bar{z}\xi)}\right] d\xi \tag{31.11}$$

$$= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[\frac{(1 - |z|^2)}{(1 - \bar{z}\xi^{-1})(1 - z\xi)}\right]\frac{d\xi}{\xi} \tag{31.12}$$

On $\xi = e^{it}$ we have:

$$\frac{d\xi}{\xi} = idt \tag{31.13}$$

$$\xi^{-1} = e^{-it} = \overline{\xi} \tag{31.14}$$

So:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\underbrace{\frac{1 - |z|^2}{|1 - \bar{z}e^{it}|^2}}_{\text{Real}} dt \tag{31.15}$$

So:

$$u = \Re(f) = \frac{1}{2\pi}\int_0^{2\pi} \Re(f(e^{it}))\frac{1 - |z|^2}{|1 - \bar{z}e^{it}|^2}dt \tag{31.16}$$

Since we want $\Re(f) = u = \phi$ on ∂D, we have:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{it})\left[\frac{1 - |z|^2}{|1 - \bar{z}e^{it}|^2}\right] dt \tag{31.17}$$

Where $\frac{1-|z|^2}{|1-\bar{z}e^{it}|^2}$ is the Poisson Kernel for the disk.
If $z = re^{i\theta}$, then:

$$\frac{1 - |z|^2}{|1 - e^{it}\bar{z}|^2} = \frac{1 - r^2}{(1 - e^{it}\bar{z})\overline{(1 - e^{it}\bar{z})}} \tag{31.18}$$

$$= \frac{1 - r^2}{(1 - e^{it}\bar{z})(1 - e^{-it}z)} \tag{31.19}$$

$$= \frac{1 - r^2}{1 - e^{it} re^{-i\theta} - e^{-it} re^{i\theta} + e^{it} re^{-i\theta} e^{-it} re^{i\theta}} \tag{31.20}$$

$$= \frac{1 - r^2}{1 - e^{i(t-\theta)} - e^{i(\theta-t)} + r^2} \tag{31.21}$$

$$\rightarrow e^{it} = \cos(t) + i \sin(t) \tag{31.22}$$

$$= \frac{1 - r^2}{1 - \cos(t - \theta) - r\cos(t - \theta) + r^2} \tag{31.23}$$

$$= \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} \qquad = P_r(\theta - t) \tag{31.24}$$

Then:

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{it}) \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} dt \tag{31.25}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{it}) P_r(\theta - t) dt \tag{31.26}$$

Which solves the boundary value problem:

$$\begin{cases} \Delta u & = 0 \\ u|_{\partial D} & = \phi \\ \text{for } D & = \{|z| < 1\} \end{cases} \tag{31.27}$$

We only require that ϕ be bounded, piecewise continuous.

Theorem 31.0.4 — Integral Formula on the Upper Half-plane. If $\phi : \{\Im(z) = 0\} \rightarrow R$ is given, bounded, piecewise continuous. Then:

$$\boxed{W(\sigma + i\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(\tau) \left(\frac{\tau}{(\sigma - s)^2 + \tau^2} \right) ds \qquad (\tau? > 0)} \tag{31.28}$$

Solves the boundary value problem:

$$\begin{cases} \Delta w = 0 \quad \text{in } \mathbb{H} = \{\Im(z) > 0\} \\ w|_{\Im(z)=0} = \phi \end{cases} \tag{31.29}$$

Proof. Recall $\psi(z) = i\frac{(1+z)}{(1-z)}$ defines a conformal map:

$$\psi : \{|z| < 1\} \rightarrow \mathbb{H} = \{\Im(z) > 0\} \tag{31.30}$$

Then:

$$\tilde{\phi}(z) = \phi(\psi(z)) : \{|z| < 1\} \rightarrow \mathbb{R} \tag{31.31}$$

Is bounded, piecewise continuous on $|z| = 1$. Thus:

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\phi}(e^{it}) \frac{1 - |z|^2}{|1 - \bar{z}e^{it}|^2} dt$$

$$\text{solves} \begin{cases} \Delta u = 0 \text{ on } \{|z| < 1\} \\ u|_{\partial D} = \tilde{\phi} \end{cases}$$

Then $w(\xi) = u(\psi^{-1}(\xi))$ is the desired function, such that:

$$\begin{cases} \Delta w = 0 \text{ in } \mathbb{H} \\ w|_{\Im(\xi)=0} = \tilde{\phi}(\psi^{-1}(\xi)) = \phi(\xi) = \phi(\psi(\psi^{-1}(\xi))) = \phi(\xi) \end{cases} \tag{31.32}$$

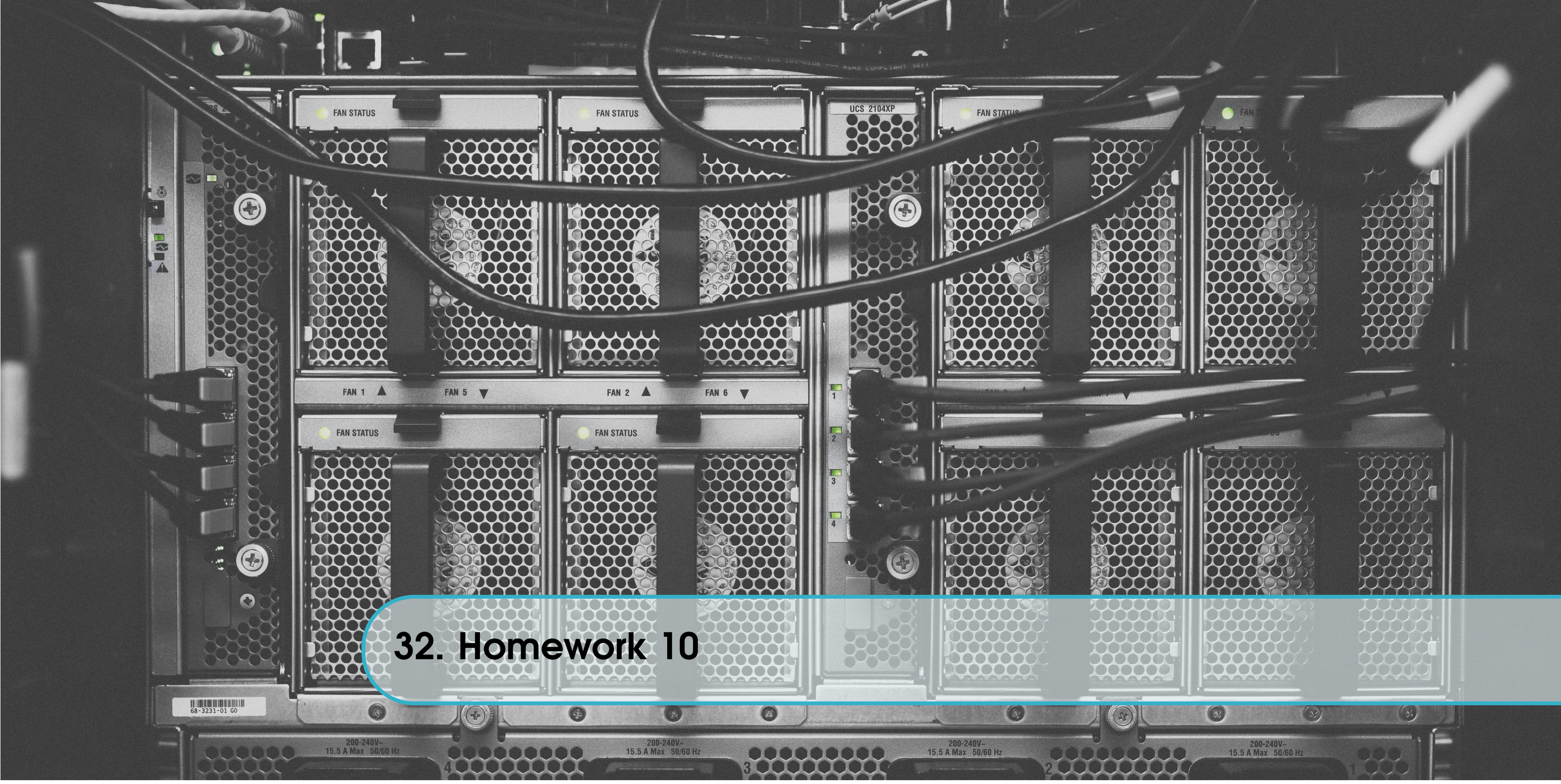
And:

$$\psi^{-1}(\xi) = \frac{\xi - i}{\xi + i} \tag{31.33}$$

So:

$$w(\sigma + i\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(t) \left(\frac{\tau}{(\sigma - t)^2 + \tau^2} \right) dt \tag{31.34}$$

Then plug into the Rep. Formula for u and simplify.
The key idea is to use conformal maps to transport the representation formula from the disk to other domains ■



32. Homework 10