The background image is an aerial photograph of a dense urban skyline, likely Pittsburgh, featuring numerous skyscrapers of varying heights and architectural styles. In the foreground, a wide river flows through the city, with several boats visible on the water. A bridge spans the river across the middle ground. The overall scene is a mix of industrial and residential/commercial architecture.

MAT389H1 Fall 2024

Complex Analysis

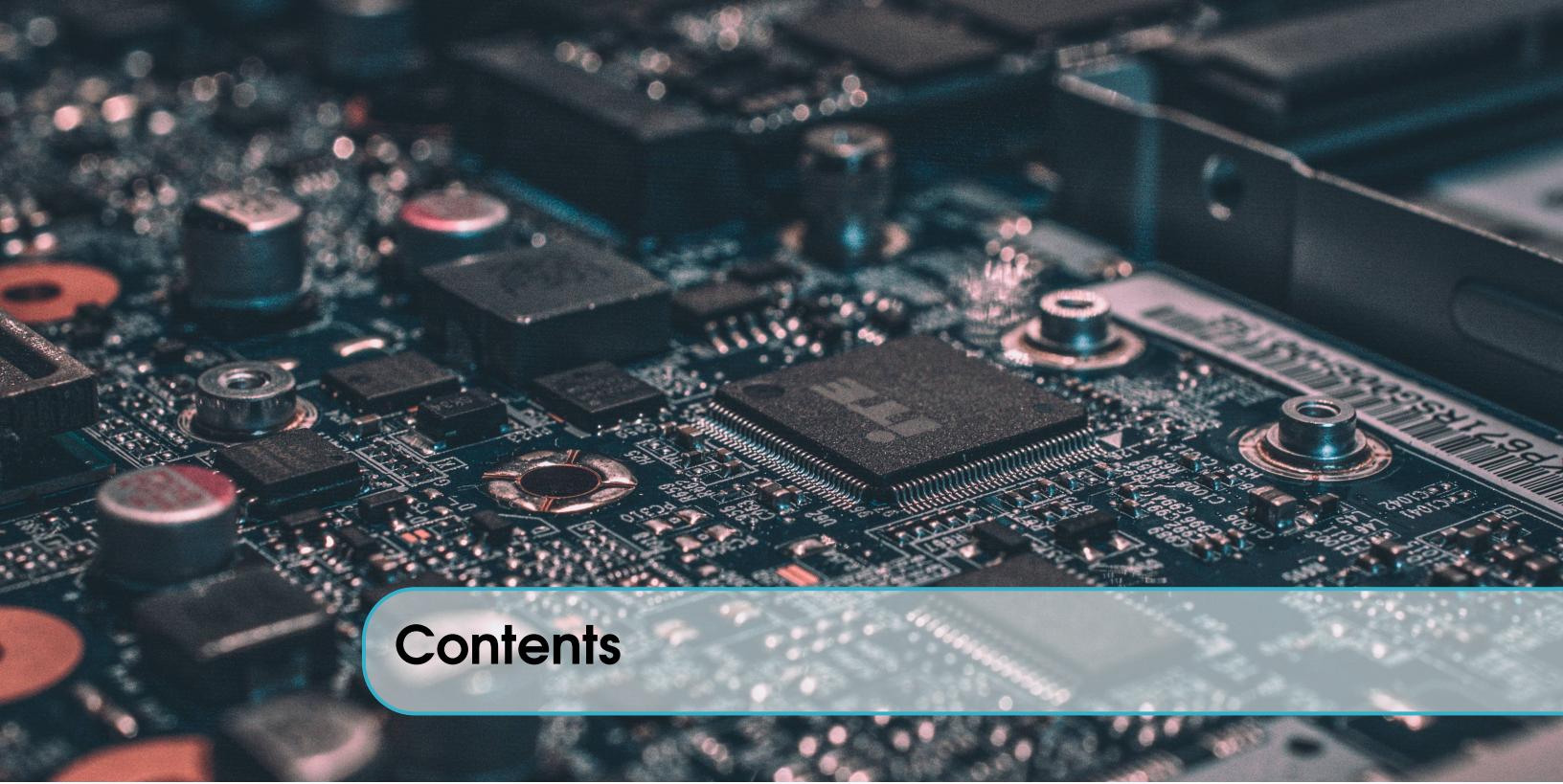
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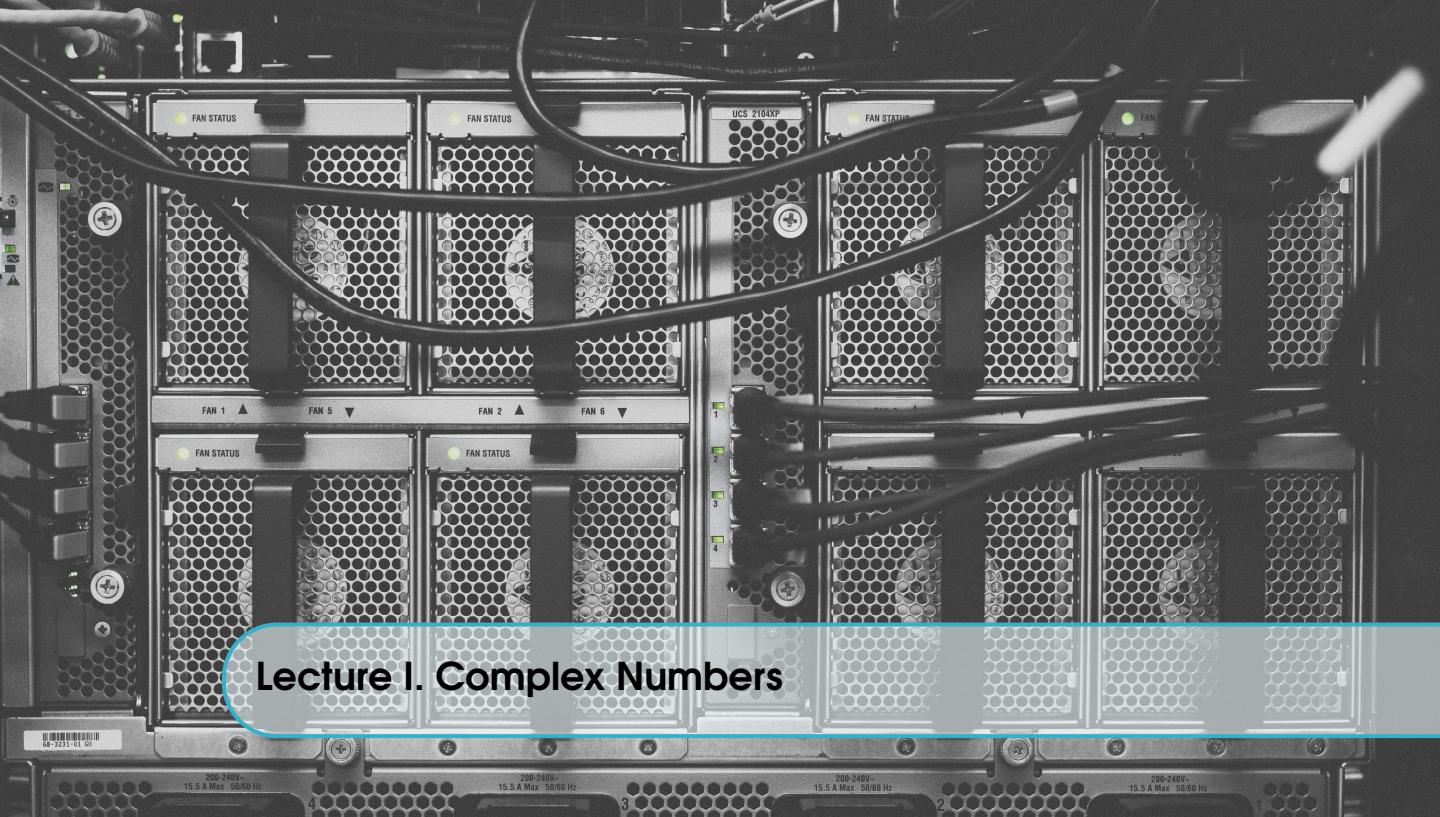
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I.I Introduction

Definition I.I.I — Complex Numbers. z is a complex number iff $z = a + bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.

The set of complex numbers is denoted by \mathbb{C} .

Definition I.I.II — Real and Imaginary Parts. If $z = a + bi$, then $\Re(z) = a \in \mathbb{R}$ and $\Im(z) = b \in \mathbb{R}$, where $\Re(z)$ is the real part of z and $\Im(z)$ is the imaginary part of z .

Definition I.I.III — Modulus. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$. $|z|$ is the modulus of z .

Definition I.I.IV — Conjugate. If $z = a + bi$, then $\bar{z} = a - bi$. \bar{z} is the conjugate of z .

I.II Operations

Definition I.II.I — Addition and Subtraction. If $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$, then $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$.

Similarly $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$.

Definition I.II.II — Multiplication. If $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$, then $z_1 \cdot z_2 = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$.

Note that $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

Definition I.II.III — Inversion. If $z = a + bi$, then $z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$.

Proof. Let's multiply by 1 in the form of the conjugate of z :

$$\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$



Definition I.II.IV — Division. For $z, w \in \mathbb{C}$, $\frac{w}{z} = w \cdot z^{-1} = \frac{w\bar{z}}{|z|^2}$.

Table I.I: Properties of the Complex Conjugate

Property	Description
Conjugate of the Sum	$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
Conjugate Modulus	$z \cdot \bar{z} = z ^2$
Conjugate of a Conjugate	$\bar{\bar{z}} = z$
Product of Conjugates	$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
Conjugate of a Quotient	$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
Real Part Conjugate	$Re(z) = \frac{z+\bar{z}}{2}$
Imaginary Part Conjugate	$Im(z) = \frac{z-\bar{z}}{2i}$
Real Number Check	$z = \bar{z} \iff z \in \mathbb{R}$
Imaginary Number Check	$z = -\bar{z} \iff z \in \mathbb{I}$
Function Linearity	If $\alpha = f(z)$ then $\bar{\alpha} = \overline{f(z)} = f(\bar{z})$

Table I.II: Properties of the Modulus in Complex Numbers

Property	Description
Positivity	$ z \geq 0$, with equality if and only if $z = 0$
Triangle Inequality	$ z_1 - z_2 \leq z_1 \pm z_2 \leq z_1 + z_2 $
Multiplicative Property	$ z_1 \cdot z_2 = z_1 \cdot z_2 $
Division Property	$\left \frac{z_1}{z_2}\right = \frac{ z_1 }{ z_2 }$, for $z_2 \neq 0$
Conjugate	$ z = \bar{z} $
Component Property	$- z \leq Re(z) \leq z $ $- z \leq Im(z) \leq z $
Cauchy-Schwarz Inequality	$ z_1 w_1 + \dots + z_n w_n ^2 \leq \sum_{j=1}^n z_j ^2 \sum_{j=1}^n w_j ^2$

Proof. Proof of the Multiplicative Property of the Modulus:

$$\begin{aligned} |z_1 \cdot z_2|^2 &= (z_1 \cdot z_2) \cdot (\bar{z}_1 \cdot \bar{z}_2) \\ &= z_1 \cdot \bar{z}_1 \cdot z_2 \cdot \bar{z}_2 \\ &= |z_1|^2 \cdot |z_2|^2 \end{aligned}$$

■

I.III Polar Representation

A complex number are vectors in \mathbb{R}^2 , as such, they can be represented by a magnitude and a direction.

Definition I.III.I — Polar Form.

$$z = r(\cos(\theta) + i \sin(\theta)) \quad (\text{I.I})$$

| : $r = |z| \in \mathbb{R}^+$

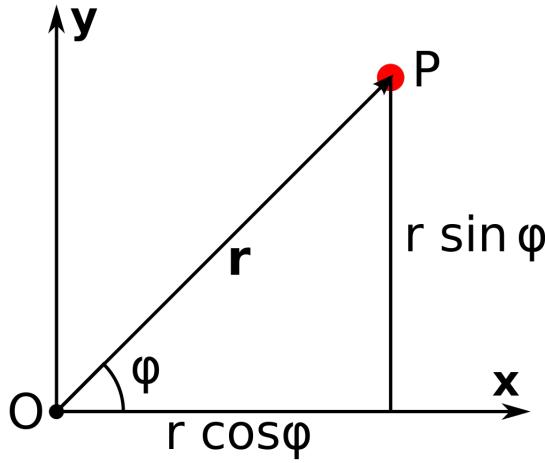


Figure Lecture I.1: Polar Coordinate Components

■ **Example I.I — Multiplying Complex Numbers in Polar Form.** Let $z_1 = r_1(\cos(\theta_1) + i \sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i \sin(\theta_2))$. Then:

$$z_1 \cdot z_2 = r_1 r_2 (\cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2) + i(\cos(\theta_1) \sin(\theta_2) + \sin(\theta_1) \cos(\theta_2))) \quad (\text{I.II})$$

$$= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \quad (\text{I.III})$$

Using the trig addition formula:

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) \text{ and } \sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta).$$

■

Theorem I.III.I — De Moivre's Theorem. if $z = r(\cos(\theta) + i \sin(\theta))$

$$z^n = r^n(\cos(\theta n) + i \sin(\theta n)) \quad (\text{I.IV})$$

Proof. The following proof will illustrate the steps to inductive reasoning

Case of $n = 1$: $z^n = r^n(\cos(\theta n) + i \sin(\theta n)) = z = r(\cos(\theta) + i \sin(\theta))$

This is true by definition.

Assume that:

$$z^{n-1} = r^{n-1}(\cos(\theta(n-1)) + i \sin(\theta(n-1)))$$

Then from Equation (I.III) we can verify:

$$zz^{n-1} = rr^{n-1}(\cos(\theta(n-1) + \theta) + i \sin(\theta(n-1) + \theta))$$

$$z^n = r^n(\cos(\theta n) + i \sin(\theta n))$$

■

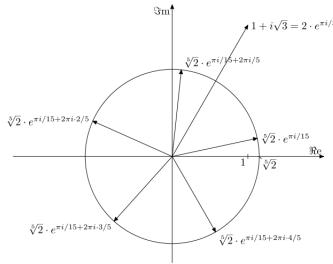


Figure Lecture I.2: Complex Fifth Roots of Unity

From Equation (I.I), we observe that r is unique (because we constrained it to just positive values). θ , however, is not unique.

Definition I.III.II — Principle Orientation. We say θ is the principle orientation of z if $\theta \in [-\pi, \pi)$

In this range, θ is unique.

Definition I.III.III — Vector Dot Product. The dot product of two vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ is given by:

$$a \cdot b = \Re(a\bar{b}) \quad (\text{I.V})$$

R [Complex Numbers to Solve Polynomial Equations] Over \mathbb{C} , every equation of the form $z^n = a$ has n solutions.

■ **Example I.II — Solving $z^n = -1$.** Let $z = r(\cos(\theta) + i \sin(\theta))$. Then:

$$\begin{aligned} z^n &= r^n(\cos(\theta n) + i \sin(\theta n)) = -1 \\ \implies r^n &= 1 \text{ and } \cos(\theta n) + i \sin(\theta n) = -1 \\ \implies r &= 1 \text{ and } \cos(\theta n) = -1 \text{ and } \sin(\theta n) = 0 \\ \implies \theta n &= \pi + 2\pi k \text{ for } k \in \mathbb{Z} \\ \implies \theta &= \frac{\pi + 2\pi k}{n} \text{ for } k \in \mathbb{Z} \end{aligned}$$

We can now find the principle solutions for Z

$$\therefore \theta_0 = \frac{\pi}{n}, \theta_1 = \frac{3\pi}{n}, \dots, \theta_{n-1} = \frac{(2n-1)\pi}{n}$$

■

R Roots of Unity The solutions to $z^n = 1$ are called the n th roots of unity. Plotting these solutions splits the complex plane into n equal parts.

Definition I.IV.I — Open Disc. An open disc of radius r centered at z_0 is the set of all z such that $D = \{ |z - z_0| < R \}$.

Definition I.IV.II — Interior Point. A point z_0 is an interior point of a set $D \subset \mathbb{C}$ if there exists an open disc centered at z_0 that is contained in D .

Definition I.IV.III — Open Set. A set $D \subset \mathbb{C}$ is open if every point in D is an interior point.

■ **Example I.III — Open Disc.** Show that the disc $D = \{ z \in \mathbb{C} : |z - z_0| < R \}$ is an open set.

Proof. Let $z_1 \in D$. Then $|z_1 - z_0| < R$. Let $r = R - |z_1 - z_0|$. Then $r > 0$.

Let $z_2 \in D$ such that $|z_2 - z_1| < r$. Then:

$$\begin{aligned}|z_2 - z_0| &\leq |z_2 - z_1| + |z_1 - z_0| \\ &< r + R - r = R\end{aligned}$$

Therefore $z_2 \in D$ and D is open. ■

Definition I.IV.IV — Boundary (∂D). The boundary of a set D is the set of all points z such that every open disc centered at z contains points in D and points not in D .

The boundary of D is denoted by ∂D and a boundary point z is denoted by $z \in \partial D$.

Definition I.IV.V — Closed Set. A set D is closed if it contains all its boundary points.



A set can be both open and closed (\mathbb{C}, \emptyset), open and not closed, closed and not open, or neither open nor closed.

Theorem I.IV.I — Properties of Open and Closed Sets.

- 1. D is open iff $\mathbb{C} \setminus D$ is closed.
- 2. D is closed iff $\mathbb{C} \setminus D$ is open.
- 3. D is open if and only if it contains none of its boundary points.