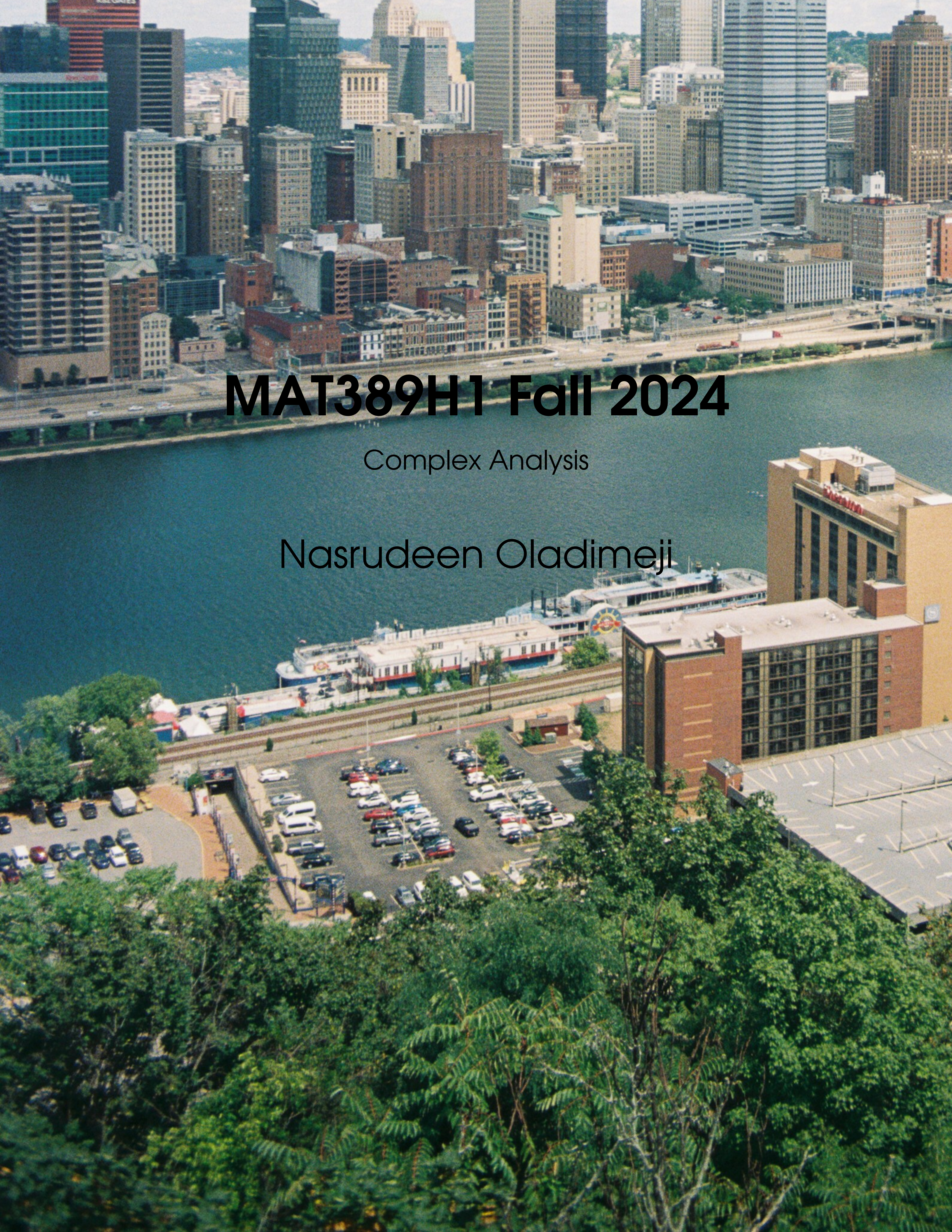


MAT389H1 Fall 2024

Complex Analysis

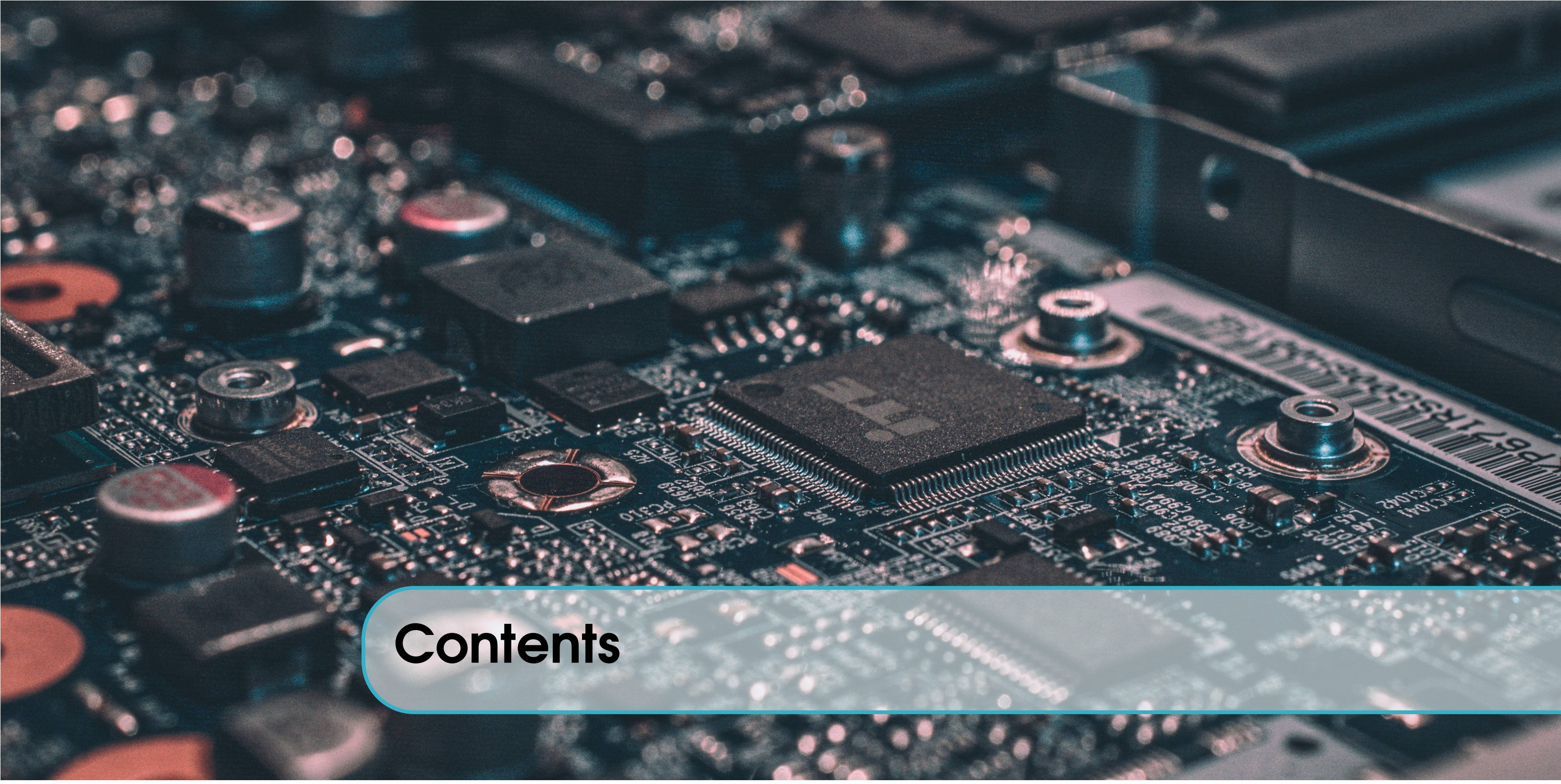
Nasrudeen Oladimeji

UNIVERSITY OF TORONTO

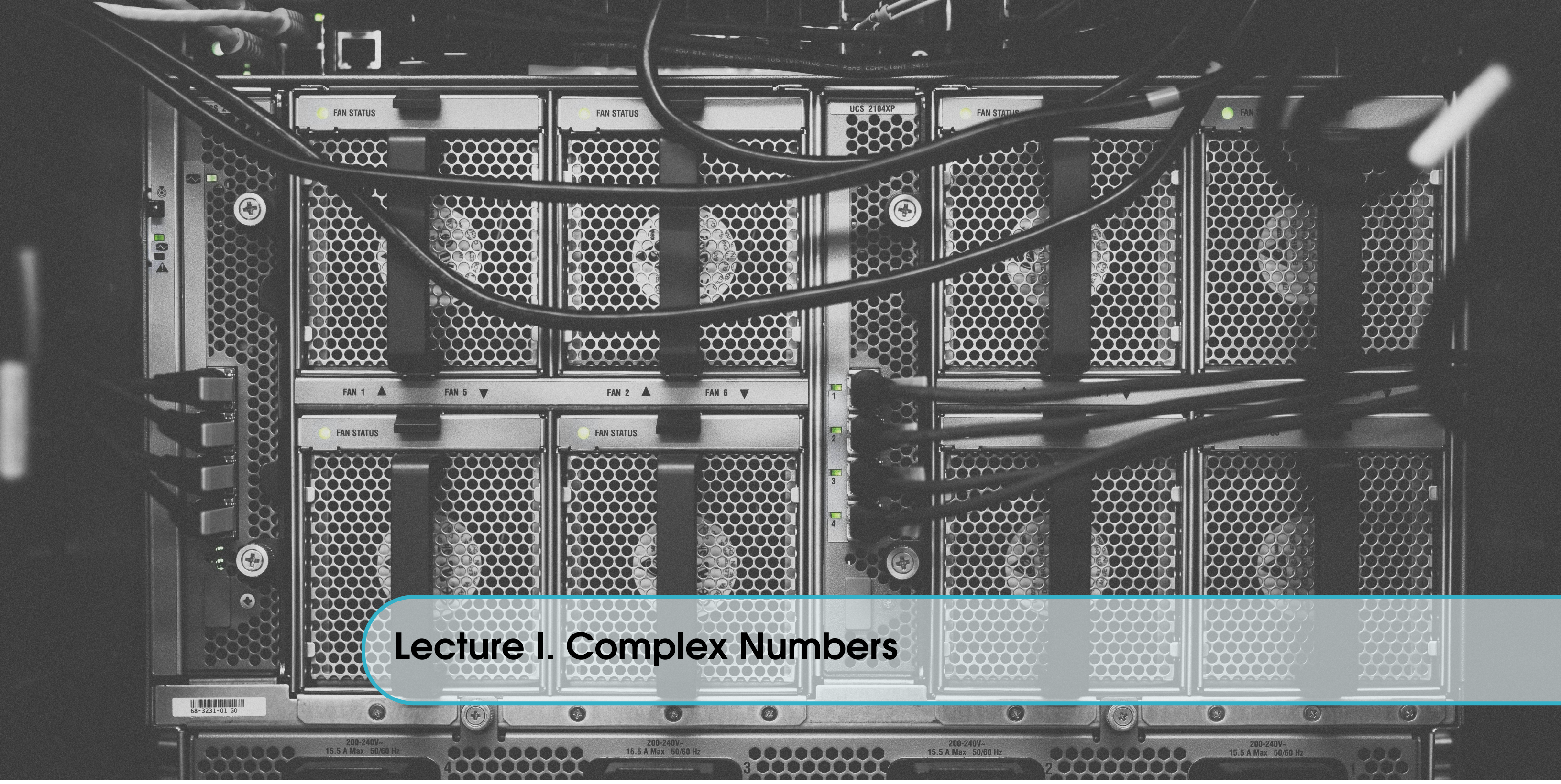
GITHUB.COM/NASR-905

Professor: Tristan Collins (tristan.collins@utoronto.ca)

First release, September 2024



Contents



Lecture I. Complex Numbers

I.I Introduction

Definition I.I.I — Complex Numbers. z is a complex number iff $z = a + bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.
The set of complex numbers is denoted by $\mathbb{C}$.

Definition I.I.II — Real and Imaginary Parts. If $z = a + bi$, then $\Re(z) = a \in \mathbb{R}$ and $\Im(z) = b \in \mathbb{R}$. where $\Re(z)$ is the real part of z and $\Im(z)$ is the imaginary part of z.

Definition I.I.III — Modulus. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$. $|z|$ is the modulus of z.

Definition I.I.IV — Conjugate. If $z = a + bi$, then $\bar{z} = a - bi$. $\bar{z}$ is the conjugate of z.

I.II Operations

Definition I.II.I — Addition and Subtraction. If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$.
Similarly $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$.

Definition I.II.II — Multiplication. If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then $z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$.
Note that $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

Definition I.II.III — Inversion. If $z = a + bi$, then $z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$.

Proof. Let's multiply by 1 in the form of the conjugate of z:

$$\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

∎

Definition I.II.IV — Division. For $z, w \in \mathbb{C}$, $\frac{w}{z} = w \cdot z^{-1} = \frac{w\overline{z}}{|z|^2}$.

Table I.I: Properties of the Complex Conjugate

Property	Description
Conjugate of the Sum	$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
Conjugate Modulus	$z \cdot \overline{z} = \|z\|^2$
Conjugate of a Conjugate	$\overline{\overline{z}} = z$
Product of Conjugates	$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
Conjugate of a Quotient	$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
Real Part Conjugate	$Re(z) = \frac{z+\overline{z}}{2}$
Imaginary Part Conjugate	$Im(z) = \frac{z-\overline{z}}{2i}$
Real Number Check	$z = \overline{z} \iff z \in \mathbb{R}$
Imaginary Number Check	$z = -\overline{z} \iff z \in \mathbb{I}$
Function Linearity	If $\alpha = f(z)$ then $\overline{\alpha} = \overline{f(z)} = f(\overline{z})$

Table I.II: Properties of the Modulus in Complex Numbers

Property	Description
Positivity	$\|z\| \geq 0$, with equality if and only if $z = 0$
Triangle Inequality	$\|\|z_1\| - \|z_2\|\| \leq \|z_1 \pm z_2\| \leq \|z_1\| + \|z_2\|$
Multiplicative Property	$\|z_1 \cdot z_2\| = \|z_1\| \cdot \|z_2\|$
Division Property	$\left\|\frac{z_1}{z_2}\right\| = \frac{\|z_1\|}{\|z_2\|}$, for $z_2 \neq 0$
Conjugate	$\|z\| = \|\overline{z}\|$
Component Property	$-\|z\| \leq Re(z) \leq \|z\|$ $-\|z\| \leq Im(z) \leq \|z\|$
Cauchy-Schwarz Inequality	$\|z_1 w_1 + \cdots + z_n w_n\|^2 \leq \sum_{j=1}^{n} \|z_j\|^2 \sum_{j=1}^{n} \|w_j\|^2$

Proof. Proof of the Multiplicative Property of the Modulus:

$$\begin{aligned} |z_1 \cdot z_2|^2 &= (z_1 \cdot z_2) \cdot (\overline{z_1} \cdot \overline{z_2}) \\ &= z_1 \cdot \overline{z_1} \cdot z_2 \cdot \overline{z_2} \\ &= |z_1|^2 \cdot |z_2|^2 \end{aligned}$$

■

I.III Polar Representation

A complex number are vectors in $\mathbb{R}^2$, as such, they can be represented by a magnitude and a direction.

Definition I.III.I — Polar Form.

$$z = r(\cos(\theta) + i\sin(\theta)) \tag{I.I}$$

: $r = |z| \in \mathbb{R}^+$

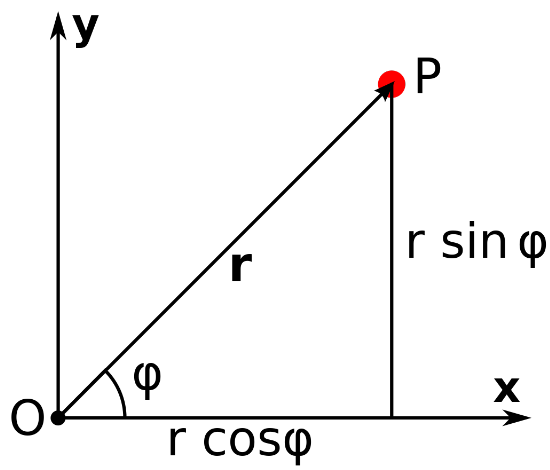


Figure Lecture I.1: Polar Coordinate Components

■ **Example I.I — Multiplying Complex Numbers in Polar Form.** Let $z_1 = r_1(\cos(\theta_1) + i\,\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\,\sin(\theta_2))$. Then:

$$z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1)\,\cos(\theta_2) - \sin(\theta_1)\,\sin(\theta_2) + i(\cos(\theta_1)\,\sin(\theta_2) + \sin(\theta_1)\,\cos(\theta_2))) \tag{I.II}$$

$$= r_1 r_2(\cos(\theta_1 + \theta_2) + i\,\sin(\theta_1 + \theta_2)) \tag{I.III}$$

Using the trig addition formula:

$\cos(\alpha+\beta) = \cos(\alpha)\,\cos(\beta) - \sin(\alpha)\,\sin(\beta)$ and $\sin(\alpha+\beta) = \cos(\alpha)\,\sin(\beta) + \sin(\alpha)\,\cos(\beta)$.

■

Theorem I.III.I — De Moivre's Theorem. if $z = r(\cos(\theta) + i\,\sin(\theta))$

$$z^n = r^n(\cos(\theta n) + i\,\sin(\theta n)) \tag{I.IV}$$

Proof. The following proof will illustrative the steps to inductive reasoning
Case of $n = 1$: $z^n = r^n(\cos(\theta n) + i\,\sin(\theta n)) = z = r(\cos(\theta) + i\,\sin(\theta))$
This is true by definition.
Assume that:
$z^{n-1} = r^{n-1}(\cos(\theta(n-1)) + i\,\sin(\theta(n-1)))$
Then from Equation (I.III) we can verify:

$$zz^{n-1} = rr^{n-1}(\cos(\theta(n-1) + \theta) + i\,\sin(\theta(n-1) + \theta))$$
$$z^n = r^n(\cos(\theta n) + i\,\sin(\theta n))$$

■

Definition I.III.II — Argument. The argument of a complex number $z = r(\cos(\theta) + i\sin(\theta))$ is any angle, $\arg(z) = \theta$, such that $z = r(\cos(\theta) + i\sin(\theta))$.

From Equation (I.I), we observe that r is unique (because we constrained it to just positive values). θ, however, is not unique.

Definition I.III.III — Principle Orientation. We say θ is the principle orientation of z if $\theta \in [-\pi, \pi)$
In this range, θ is unique.

Definition I.III.IV — Vector Dot Product. The dot product of two vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ is given by:

$$a \cdot b = \Re(a\bar{b}) \tag{I.V}$$

$$\cos\theta = \frac{a \cdot b}{|a||b|} \tag{I.VI}$$

Corollary I.III.II — Perpendicular Vectors. Complex variables z and w are perpendicular if $\Re(z\overline{w}) = 0$.

R [Complex Numbers to Solve Polynomial Equations] Over $\mathbb{C}$, every equation of the form $z^n = a$ has n solutions.

■ **Example I.II — Solving $z^n = -1$.** Let $z = r(\cos(\theta) + i\sin(\theta))$. Then:

$$\begin{aligned}
& z^n = r^n(\cos(\theta n) + i\sin(\theta n)) = -1 \\
\implies & r^n = 1 \text{ and } \cos(\theta n) + i\sin(\theta n) = -1 \\
\implies & r = 1 \text{ and } \cos(\theta n) = -1 \text{ and } \sin(\theta n) = 0 \\
\implies & \theta n = \pi + 2\pi k \text{ for } k \in \mathbb{Z} \\
\implies & \theta = \frac{\pi + 2\pi k}{n} \text{ for } k \in \mathbb{Z}
\end{aligned}$$

We can now find the principle solutions for Z

$$\therefore\ \theta_0 = \frac{\pi}{n},\ \theta_1 = \frac{3\pi}{n}, \ldots, \theta_{n-1} = \frac{(2n-1)\pi}{n}$$

■

R Roots of Unity The solutions to $z^n = 1$ are called the nth roots of unity. Plotting these solutions splits the complex plane into n equal parts.

I.IV Subsets of the Plane

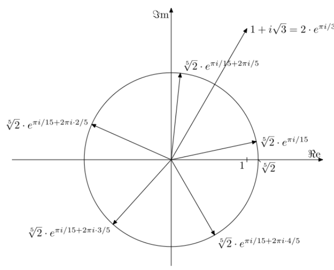


Figure Lecture I.2: Complex Fifth Roots of Unity

Definition I.IV.I — Open Disc. An open disc of radius R centered at z_0 is the set of all z such that $D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R \subset \mathbb{C}\}$.

Definition I.IV.II — Interior Point. A point z_0 is an interior point of a set $A \subset \mathbb{C}$ if there exists an open disc centered at z_0 that is contained in A.

$$z_0 \text{is an interior point of } A \text{ if} \exists D_{>0}(z_0) \in A$$

Definition I.IV.III — Open Set. A set $A \subset \mathbb{C}$ is open if every point in A is an interior point.
I.e. there are no 'hard lines' in the set.

■ **Example I.III — Open Disc.** Show that the disc $D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R\}$ is an open set.

Proof. Let $z_1 \in D$. Then $|z_1 - z_0| < R$. Let $r = R - |z_1 - z_0|$. Then $r > 0$.
Let $z_2 \in D$ be any point in D, such that $|z_2 - z_1| < r$. Then:

$$\begin{aligned} |z_2 - z_0| &\leq |z_2 - z_1| + |z_1 - z_0| \\ &< r + R - r = R \end{aligned}$$

Therefore $z_2 \in D$ and D is open. ■

■

Definition I.IV.IV — Boundary (∂D). The boundary of a set A is the set of all points z such that every open disc centered at z contains points in A and points not in A.
The boundary of A is denoted by ∂A and a boundary point z is denoted by $z \in \partial A$.

$$z_0 \text{is an boundary point of } A \text{ if} \exists z \in D_R(z_0) : z \notin A \forall R > 0$$

Definition I.IV.V — Closed Set. A set D is closed if it contains all its boundary points.

R A set can be both open and closed ($\mathbb{C}$, $\emptyset$), open and not closed, closed and not open, or neither open nor closed.

Theorem I.IV.I — Properties of Open and Closed Sets.

1. D is open iff $\mathbb{C} \setminus D$ is closed.
2. D is closed iff $\mathbb{C} \setminus D$ is open.
3. D is open if and only if it contains none of its boundary points.