Consequences of Completeness

The first application of the Axiom of Completeness is a result that may look like a more natural way to mathematically express the sentiment that the real line contains no gaps.

Theorem 1.4.1 (Nested Interval Property): For each $n \in N$, assume we are given a closed interval $I_m = [a_n, b_n] = \{x \in R : a_n \leq x \leq b\}$. Assume also that each I_n contains $I_{n+1.}$Then, the resulting nested sequence of closed intervals

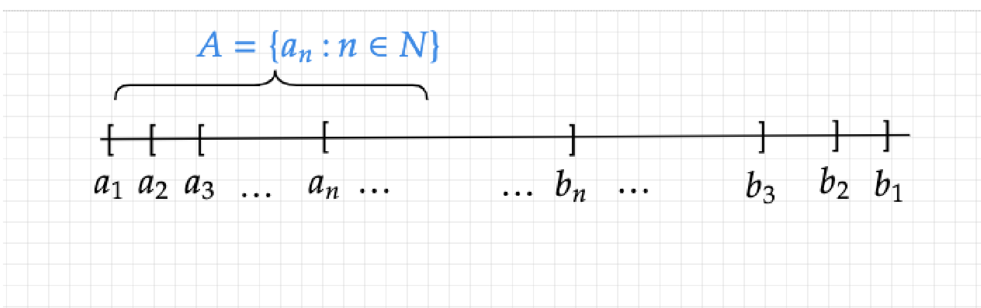
$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \ \ldots$$

has a nonempty intersection; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof: In order to show that $\cap_{n=1}^{\infty} I_n$ is not empty, we are going to use the Axiom of Completeness (AoC) to produce a single real number x satisfying $x \in I_n$ for every $n \in N$. Now, AoC is a statement about bounded sets, and the one we want to consider is the set

$$A = \{a_n : n \in N\}$$

of left-hand endpoints of the intervals.



Because the intervals are nested, we see that every b_n serves as an upper bound for A. Thus, we are justified in setting

$$x = Sup\ A.$$

Now consider a particular $I_n = [a_n, b_n]$. Because x is an upper bound for A, we have $a_n \leq x$. The fact that each b_n is an upper bound for A and that x is the least upper bound implies $x \leq b_n$.

Although then, we have $a_n \leq x \leq b_n$, which mean $x \in I_n$ for every choice of $n \in N$. Hencem $x \in \cap_{n=1}^{\infty} I_n$, and the intersection is not empty. $\square$

The Density of $\mathbb{Q}$ in $\mathbb{R}$

The set Q is an extension of N, and $\mathbb{R}$ in turn is an extension of Q. The next few results indicate how N and Q sit inside of R.

Theorem 1.4.2 (Archimedean Property). (i) Given any number $x \in R$, there exists an $n \in N$ satisfying $n > x$.
(ii) Given any real number $y > 0$, there exists an $n \in N satisfying\ 1/n < y$.

Proof : Part(i) of the proposition states that N is not bounded above. There has never been any doubt about the truth of this, and it could be reasonably argued that we should not have to prove it at all, especially in light of the fact that we have decided to take other familiar of $N, Z\ and\ Q$ as given.
The counterargument is that there is still a great deal of mystery about what the real numbers actually are. What we have said so far is that R is an extension of Q that maintains the algebric and order properties of the rationals but also posses the least upper bound property articulated in the Axiom of Completeness.
In the absence of any other information about R, we have to consider the possibility that in extending Q we unwittingly acquired some new numbers that are upper bound for N.
In fact, as disorienting as it may sound, there are ordered field extensions of Q that include "numbers" bigger than every natural number. Theorem 1.4.2 asserts that the real numbers do not contain such exotic creatures. The Axiom of Completeness, which we adopted to patch up holes in Q, carries with it the implication that N is an unbounded subset of R.
And so the proof.
Assume, for contradiction, that N is bounded above. By the Axiom of Completeness(AoC),N should then have a least upper bound, and we can set $\alpha = sup\ N$. If we consider $\alpha - 1$, then we no longer have an upper bound(lemma 1.3.8),and therefore there exists $n \in N$ satisfying $\alpha - 1 < n$. But this is equivalent to $\alpha < n + 1$. Because $n + 1 \in N$, we have a contradiction to the fact that α is supposed to be an upper bound for N. (Notice that the contradiction here depends only on AoC and the fact that N is closed under addition.
Part (ii) follows from (i) by letting $x = 1/y$. □

This familiar property of N is the key to an extremely important fact about how Q fits inside of R.

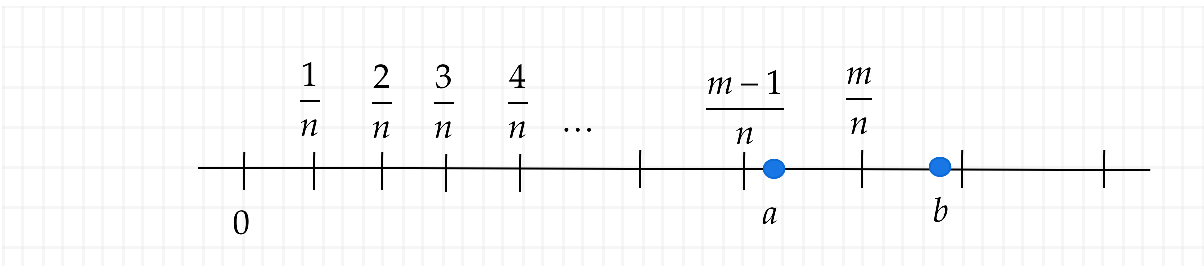
Theorem 1.4.3 (Density of $Q\ in\ R$). For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

Proof: A rational number is a quotient of integers, so we must produce $m \in Z$ and $n \in N$, so that

that

$$a < \frac{m}{n} < b \tag{1}$$

The first step is to choose the denominator n large enough so that consecutive increments of size $1/n$ are too close together to "step over" the interval (a, b).



Using the Archimedean Property (Theorem 1.4.2), we may pick $n \in N$ large enough so that

$$\frac{1}{n} < b - a \tag{2}$$

Inequality (1) (which we are trying to prove) is equivalent to $na < m < nb$. With n already chosen, the idea is to choose m to be the smallest integer greater than na. In other words, pick $m \in Z$, *so that*

$$m - 1 \leq^{(3)} na <^{(4)} m$$

Now, inequality (4) immediately yields $a < m/n$, which is half of the battle. Keeping in mind that inequality (2) is equivalent to $a < b - 1/n$, we can use (3) to write

$$\begin{aligned} m &\leq na + 1 \\ &< n\left(b - \frac{1}{n}\right) + 1 \\ &= nb \end{aligned}$$

Because $m < nb$ *implies* $m/n < b$, *we have* $a < m/n < b$, *as desired.*

The Existence of Square Roots

Theorem 1.4.5: *There exists a real number* $\alpha \in R$, *satisfying* $\alpha^2 = 2$.

Proof : Consider the set :

$$T = \left\{t \in R : t^2 < 2\right\}$$

and set $\alpha = supT$. We are going to prove $\alpha^2 = 2$ by ruling out the possibilities $\alpha^2 < 2$ and $\alpha^2 > 2$.
Keep in min that there are always two parts to the definition of $supT$, and they will both be important . The strategy is to demonstrate that $\alpha^2 < 2$ violates the fact that α is the upper bound for T, and $\alpha^2 > 2$ violates the fact that it is the least upper bound.

Let's first see what happens if we assume $\alpha^2 < 2$. In search of an element T that is larger than α.

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \ \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \ \alpha^2 + \frac{2\alpha + 1}{n}\end{aligned}$$

But now assume $\alpha^2 < 2$ gives us a little space in which to fit the $(2\alpha + 1)/n$ term and keep the total less than 2. Specifically, choose $n_0 \in N$ large enough so that

$$\frac{1}{n_0} < \frac{2 - \alpha^2}{2\alpha + 1}$$

This implies $(2\alpha + 1)/n_0 < 2 - \alpha^2$, and consequently that

$$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + \left(2 - \alpha^2\right) = 2.$$

Thus, $\alpha + 1/n_0 \in T$, contradicting the fact that α is an upper bound for T. We conclude that $\alpha^2 < 2$ cannot happen

Exercises

Q Recall that I stands for set of irrational numbers.
(a) Show that if $a, b \in Q,$ *then* ab *and* $a + b$ *are elements of* Q *as well*.
(b) Show that if $a \in Q$ and $t \in I,$ then $a + t \in I$ and $at \in I$ *as long as* $a \neq 0$.
(c) Part (a) can be summarized by saying that Q is closed under addition and multiplication. Is I closed under addition and multiplication? Given two irrational numbers s and $t,$ what can we say about $s + t$ and st.

Proof :
(a) Show that if $a, b \in Q,$ *then* ab *and* $a + b$ *are elements of* Q *as well*.

- Let $a = \dfrac{m}{n}$, $b = \dfrac{r}{s}$ as mentioned $a, b \in Q$. , where $m, n, r, s \in Z \; and \; n, s \neq 0$
- $a*b = \dfrac{mr}{ns} \in Q,\; similarly,\; a + b = \dfrac{m}{n} + \dfrac{r}{s} = \dfrac{ms + nr}{ns} \in Q.$
- Hence, $ab \; \in Q \; and \; a + b \in Q$

(b) Show that if $a \in Q$ and $t \in I,$ then $a + t \in I$ and $at \in I \; as \; long \; as \; a \neq 0.$

- Let $a = \dfrac{m}{n} \; and \; t \in I.$ To prove $a + t \in I \; and \; at \in I.$ We prove the required result by contradiction.
- Let $a + t \in Q \implies \dfrac{m}{n} + t = \dfrac{p}{q} \implies t = \dfrac{p}{q} - \dfrac{m}{n} \implies \dfrac{np - mq}{nq} \in Q$
- Which is a contradiction with the given statement that $t \in I.$ Hence, our assumption $a + t \in I$ is incorrect, and $a + t \in I.$
- Similarly, $let \; at \in Q \implies \dfrac{m}{n} * t = \dfrac{r}{s} \implies t = \dfrac{nr}{ms} \in Q$
- Again, this is a contradiction with what we know about $t,\; i.e.,\; t \in I.$ Hence, our assumption $at \in Q$ is incorrect, and $at \in I.$

(c) Part (a) can be summarized by saying that Q is closed under addition and multiplication. Is I closed under addition and multiplication? Given two irrational numbers s and $t,$ what can we say about $s + t$ and $st.$

- Let $s = \sqrt{2} \; and \; t = -\sqrt{2} \; as \; given \; s, t \in I$
- $s*t = \sqrt{2}*(-\sqrt{2}) \implies -2 \in Z,\; similarly,\; s + t \; = \sqrt{2} + (-\sqrt{2}) = 0 \in Z$
- Hence, given two irrational numbers s and $t,$ "$s + t$" and "st" may or may not be irrational. Hence I is not closed under addition and multiplication.

Q. Let $A \subseteq R$ be nonempty and bounded above, and let $s \in R$ have the property that for all $n \in N,\; s + \dfrac{1}{n}$ is an upper bound for A and $s - \dfrac{1}{n}$ is not an upper bound for $A.$ Show that $s = Sup \; A.$

Proof : To prove the given statement, we will be using the **<u>Archimedean Property,</u>** which states:

$$Given \; any \; real \; number \; y > 0, there \; exists \; an \; n \in N \; satisfying \; 1/n < y.$$

a) Using Archimedean property, select $n \in N,\; such \; that \; \dfrac{1}{n} < s - a \quad \forall a \in A.$

- $\dfrac{1}{n} < s - a \implies s - \dfrac{1}{n} < a,$ which proves that $s - 1/n$ is not an upper bound of A. ---------(i)

b) Next using the equation of the given upper bound $a < s + \dfrac{1}{n}$-------(ii)

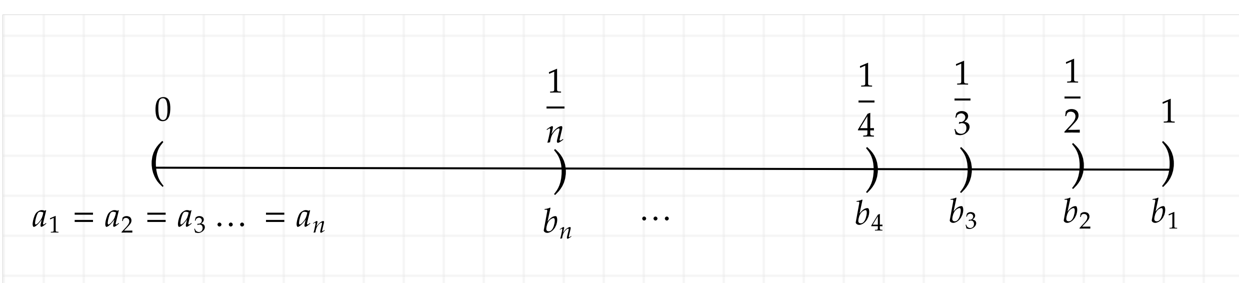
- Substituting value of $\dfrac{1}{n}$ $as\ s - a$, we get
- $a < s + \dfrac{1}{n} \implies a < s + s - a \implies 2a < 2s \implies a < s \ \ \forall a \in A$.
- Therefore, s is an upper bound for A.

c) Getting all statements , to prove $s = Sup\ A$.

- And $s + \dfrac{1}{n}$ is an upper bound of A such that $s < s + \dfrac{1}{n}$
    - $[s - \dfrac{1}{n} < s < s + \dfrac{1}{n}\ ]$
    - n (b) we deduced that s is an upper bound for set A, which satisfies statement(i) of the axiom of completeness. stating that s must be an upper bound for A.
    - In (a) we proved that $s - \dfrac{1}{n}$ $is\ not\ an\ upper\ bound\ for\ A$, which satisfies the requirement of statement (ii) of the axiom of completeness. [That, any real number less than s cannot be an upper bound for the set A]
    - Since, both the statements for axiom of completeness is satisfied, hence we proved that
      $s = Sup\ A.$ $\square$

Q. Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \varnothing$. Notice that this demonstrate that the interval in the nested interval property must be closed for the Theorem to hold.

Proof : Let us write down sets for new initial n:



$$
\begin{gathered}
I_1 = n = 1 \implies (0,1) \\
I_2 = n = 2 \implies (0,1/2) \\
I_3 = n = 3 \implies (0,1/3) \\
\vdots \\
I_n = n = n-1 \implies (0,1/(n-1))
\end{gathered}
$$

$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \ldots$

Here, the intervals for the nested interval are open intervals

$I_n = (a_n, b_n) = \{x \in R : 0 < x < b_n\}$.

- As we are performing intersection of sets, a real number $x \in I_n$, may not be included in the set following I_n. i.e, $n \notin I_{n+1}$.
- As $n \to \infty$, for n very large, $\frac{1}{n} \approx 0$. But, since $0 \notin I_1. \ldots I_n \ldots I_{n+m}$. We get an empty set for the intersection of the nested sets.
- Hence, $\bigcap_{n=1}^{\infty}(0,1/n) = \varnothing$. □

Q. Let $a < b$ be real numbers and consider the set $T = Q \cap [a,b]$. Show that $sup\ T = b$.

Proof:

- Theorem of Density of $Q\ in\ R\ states\ that:$ *For every two real numbers* $a\ and\ b$ with $a < b$, there exists a rational number r satisfying $a < r < b$.
- For the given question the set T being defined as

$$
\begin{gathered}
T = Q \cap [a,b] \\
= Q \cap [a < r < b] = r.
\end{gathered}
$$

To prove $sup\ T = b$, we need to prove two part of the definition of Supremum.

(a)To prove $sup\ T = b$, b is an upper bound of the set T.

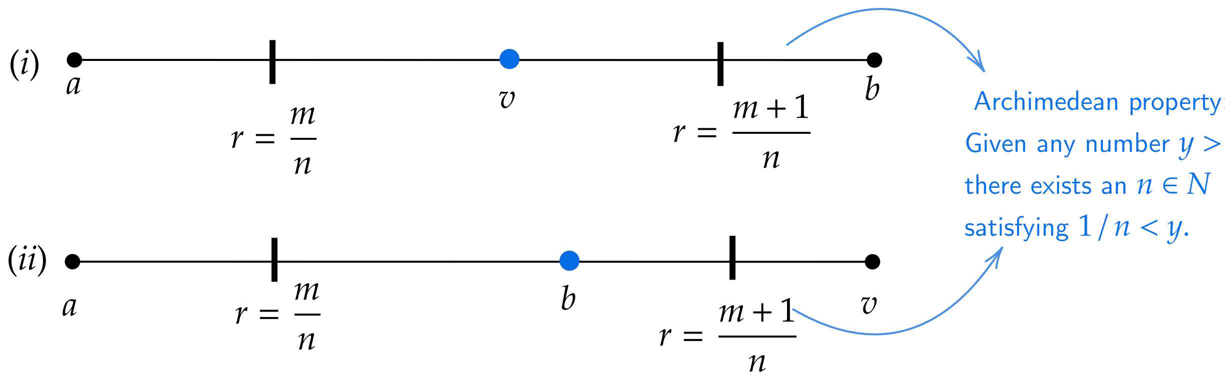
  * From the theorem of Density of Q in R, we know $a < r < b$, hence from this ,we can state that b is an upper bound of the set T.

(b) if v is any other upper bound of set T, then $b \leq v$.

  * Let v be any other upper bound for the set T. The possibilities of placing v within the set T,
  * From the drawing below, case (ii), is clearly states that $b \leq v$.

    hence, $set\ T$, will only have a single element $T = \left[\frac{m}{n}\right] < b$. Hence, v is some other upper bound of set T, $satisfying\ b \leq v$.
  * for case (i), we see that if $v \leq b$, then the set T, $also\ get\ extended$.
  * such that $T = \left[\frac{m}{n}, \frac{m+1}{n}\right]$, hence we see that $v < \frac{m+1}{n}$, hence, v cannot be an upper bound for set T, and both elements of set $T < b$, hence b is the

least upper bound for the set T.



Q. Using exercise 1.4.1, supply a proof for corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ *and* $b - \sqrt{2}$.

Proof : *(My approach was wrong, so the solution here is the right solution from Abbott)*

Corollary 1.4.4: *Given any two real numbers*
$a < b$*, there exists an irrational number t satisfying*
$a < t < b$.

Consider $a - \sqrt{2}$ *and* $b - \sqrt{2}$, then by rational density theorem,there exists a rational number r such that

$$a - \sqrt{2} < r < b - \sqrt{2}$$

adding $\sqrt{2}$ we get

$$a < r + \sqrt{2} < b$$

Since r is rational and $\sqrt{2}$ is irrational, from 1.4.1(b) we get $r + \sqrt{2}$ is irrational.

Q.Recall that a set B is *dense* in R if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in R? Take $p \in Z$, and $q \in N$ in every case.

(a)The set of all rational numbers p/q *with* $q \leq 10$.

Q. Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = supT$.

Proof:
Solving for the case when $\alpha^2 > 2$. This time we write.

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2}$$
$$> \alpha^2 - \frac{2\alpha}{n}$$

Select n_0 *large such that*

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$$

This implies $(2\alpha)/n_0 < \alpha^2 - 2$, and consequently that

$$\left(\alpha - \frac{1}{n_0}\right)^2 > \alpha^2 - \left(\alpha^2 - 2\right) = 2.$$

Thus, $\alpha - 1/n_0 \ < \alpha$ *and also* $\alpha - 1/n_0 \ > 2$, contradicting the fact that α is an least upper bound for T. We conclude that $\alpha^2 > 2$ cannot happen.

Q.Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

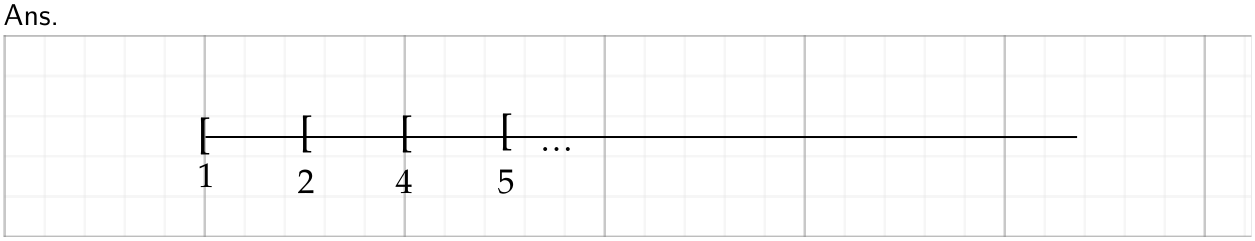
(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \ \dots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

Ans: This request is impossible, as nested intervals need to be closed, for $\bigcap_{n=1}^{\infty} J_n \neq \emptyset$. With open intervals this request is not possible.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \ \dots$ *with* $\bigcap_{n=1}^{\infty} L_n = \emptyset$

(An unbounded close interval has the form $[a, \infty) = \{x \in R : x \geq a\}$.)

Ans.



$$\begin{aligned} L_1 &= [1, \infty) \\ L_2 &= [2, \infty) \\ L_3 &= [3, \infty) \\ &\vdots \\ L_n &= [n, \infty) \\ &\vdots \\ L_{n+k} &= [n+k, \infty) \end{aligned}$$

As we can see that $1 \in L_1,\ but\ 1 \notin L_2,\ similarly,\ n \in L_n, but\ n \notin L_{n+1.}$
Therefore, as

$$n \to \infty, the\ \bigcap_{n=1}^{\infty} L_n = \varnothing,\ as\ subsequent\ sets\ will\ continue\ to\ omit\ elements\ of\ precedng\ sets.$$

Hence, the quest is possible, and is demonstrated by the above given example.