Notes :

Theorem 1.1.1 : There is no rational number where square is 2.

Proof : A rational number is any number that can be expressed in the form $\frac{p}{q}$, where p and q are integers.Thus, what the theorem asserts is that no matter how p and q are chosen, it is never the case that $\left(\frac{p}{q}\right)^2 = 2$. The line of attack is indirect using a type of argument referred using a type of argument referred to as a proof by CONTRADICTION. The idea is to assume that there is a rational

number whose square is 2 and then proceed along the logical lines until we reach a conclusion that is unacceptable. At this point, we will be forced to retrace our steps and reject the erronous assumptions that some rational number square is equal to 2. In short, we will prove that the theorem is true by demonstrating that is cannot be false. And so assume,for contradiction that there exists integers $p\ and\ q$ satisfying :

$$\left(\frac{p}{q}\right)^2 = 2. \qquad (1)$$

We may assume that $p\ and\ q$ have no common factor,because if they had, we would simple cancel it out and rewrite the fraction in lowest terms. Now, we can imply :

$$p^2 = 2q^2 \qquad (2)$$

from this,we can see that the integer p^2 is an even number (it is divisible by 2), & hence p must be even as well because the square of an odd number is odd. This allows us to write $p = 2r$, where r is also an integer. If we substitute $2r$ for p in the equation $p^2 = 2q^2,\ we\ get$

$$4r^2 = 2q^2 \Longrightarrow 2r^2 = q^2.$$

But now the absurdity is at hand. This last equation implies that $q^2\ is\ even,$ hence q must also be even. Thus we have shown that p & q are both even (i.e divisible by 2) when they were originally assumed to have no common factor. From this logical impasse we can only conclude that equation (1) cannot hold for any integers $p\ and\ q,$ and thus the theorem is proved.

Triangle Inequality : The absolute value function is so important that it merits the special notation

$|x|$ in place of the usual $f(x)\ or\ g(x)$. It is defined for every real number via the piecewise definition:

$$|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

With respect to multiplication and division, the absolute value function satisfies :

(i) $|ab| = |a||b|$ and

(ii) $|a + b| \leq |a| + |b|$

(ii) $|a+b| \le |a| + |b|$
for all choices a and b.

Q1. (a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational ?
(b) Where does the proof of theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ *is irrational ?*

Proof : (a) To prove : $\sqrt{3}$ is irrational. We will be using similar approach as given in Theorem 1.1.1. i.e, we prove by contradiction.

- $\exists$ integers $p\ and\ q\ such\ that : \left(\frac{p}{q}\right)^2 = 3$. We also assume that p and q have no common factor.
- Therefore, $p^2 = 3q^2,\ thus\ p^2\ is\ a\ multiple\ of\ 3 \implies p = 3r$ $(since\ p\ is\ an\ integers,\ and\ if\ p^2\ is\ a\ multiple\ of\ 3,\ so\ will\ p)$, where r is an integer. --------(2)
- Substituting $p = 3r\ in\ p^2 = 3q^2 \implies 9r^2 = 3q^2 \implies 3r^2 = q^2$.
- The equations implies that both $p\ and\ q$ are multiples of $3,$ which contradicts with our assumption that, $p\ and\ q$ have no common factor.
- From this logical impasse, we can conclude that $\left(\frac{p}{q}\right)^2 = 3$ cannot be true. Thus proving the theorem that $\sqrt{3}\ is\ irrational.$

Does a similar argument work to show $\sqrt{6}\ is\ irrational$?

- No, a similar approach do not work to prove $\sqrt{6}$ is irrational.
- To prove , our statement let $\left(\frac{p}{q}\right)^2 = 6 \implies p^2 = 6q^2 \implies p^2 = 2.3.q^2$
- Since, p^2 is a multiple of $2,$ so is $p.\ let\ p = 2r \implies (2r)^2 = 2.3.q^2 \implies 2r^2 = 3q^2$
- $\frac{2}{3}r^2 = q^2$. Thus there is no common factor between $p\ and\ q,$ hence our proof breaks down here,

  thus proving that $\sqrt{6}$ is rational (Check ?)

(b)

- Let $\left(\frac{p}{q}\right)^2 = 4 \implies p^2 = 4q^2 \implies p^2 = 2.2.q^2 \implies (p = 2r)$
- $4r^2 = 4q^2 \implies r^2 = q^2$
- Thus, $p\ is\ a\ multiple\ of\ 2,\ and\ q\ is\ not,$ hence our assumption stands, that there is

no common multiple.[proof breaks down here], where in case of $\sqrt{2}$, we could contradict our assumption based on common multiple of p *and* q, but in this case, our assumption is proved true and there is no contradiction to prove otherwise.

Q2. Show that there is no rational number r satisfying $2^r = 3$.

Proof : Here again we will use, contradiction to prove the given statement.

- Let r *be a rational* . $r = \dfrac{p}{q}$ *with* $q \neq 0$.
- $2^{\frac{p}{q}} = 3 \implies$ (Multiplying with q power both side) $2^p = 3^q$.
- Thus, we reached an impasse , as the statement $2^p = 3^q$ cannot be equal as 2^p is even, and 3^q is odd, followed by the understanding that power of even is even, and odd is odd, and both cannot be equal.
- Hence, our assumption is wrong, thus proving the statement that there is no rational number r,
  satisfying $2^r = 3$.

Q3. Decide which of the following represent true statement about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \ldots$ are all sets containing an infinite number of elements, then the intersection $\cap_{n=1}^{\infty} A_n$ is infinite as well. True ✕ [Check this]
(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \ldots$ are all finite,nonempty sets of real numbers, then the intersection
$\cap_{n=1}^{\infty} A_n$ is finite and non-empty. **True.**

$m \in \cap_{n=1}^{\infty} A_n$. Means m belong to all sets preceding A_n.

$$A_1 = \{1, 2, 3, 4, \ldots\}$$
$$A_2 = \{2, 3, 4, \ldots\}$$
$$A_3 = \{3, 4, 5, \ldots\}$$
$$\vdots$$
$$A_n = \{n, n+1, n+2\}$$

(c) $A \cap (B \cap C) = (A \cap B) \cap C$ **True** (Associative property)
(d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ **True**
(e) $A \cap (B \cup C) = (A \cap B) \cup C$ **False**

- Let $x \in A \cap (B \cup C)$ *which implies* $x \in A$ *and* $x \in B \cup C \implies x \in B$ *or* $x \in C$
- Case 1 : $x \in A$ *and* $x \in B = x \in A \cap B$
- Case 2 : $x \in A$ *and* $x \in C = x \in A \cap C$
- Therefore, lhs : $x \in (A \cap B) \cup (A \cap C)$

Similarly, decoding rhs. : let $x \in (A \cap B) \cup C \implies x \in (A \cap B)\, or\, x \in C.$

- $[x \in A\, and\, x \in B\,]\, or\, x \in C \implies x \in A \cup C\, and\, x \in B \cup C \implies x \in (A \cup C) \cap (B \cup C)$
- [Follows from distributive law of set theory]

  Hence, $lhs \neq rhs.\ The\ equality\ is\ false.$

Q4. Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements , $A_i \cap A_j = \varnothing$ for all $i \neq j,\ and\ \cup_{i=1}^{\infty} A_i = N.$

Q6. (a) Verify the triangle inequality in the special case where a and b have the same sign.

Proof :

1. We prove $|-a-b| \leq |-a| + |-b|$.
   a) First we prove the equality .
      * LHS : $|-a-b| \implies -(-(a+b)) \implies (a+b) \sim$ absolute function.
      * rhs : $|-a| + |-b| \implies -(-a) + [-(-b)] \implies a+b$
      * Thus we prove the equality.

   (b) Now we prove $|-a-b| < |-a| + |-b|$.
      * $|a+b|^2 = |(-a)^2 + 2ab + (-b)^2| = |a^2| + 2|ab| + |b^2|$ ----------------(1)
      * $|-a-b| = (a+b)^2 = a^2 + 2ab + b^2$ -------------------(2)
      * Looking at equation (1) and (2), we see that two terms $a^2\ and\ b^2$ will be positive irrespective of the sign of variable. The only element we can compare from both equation is the third non-squared term.
      * $2|ab| \leq 2ab \implies |ab| \leq ab$ ~as absolute value of "ab" will always be positive ,---(3)
      * Therefore, from (1),(2) and (3) we can state that :
      * $|a+b|^2 \geq (a+b)^2 \implies |a+b| \geq (a+b)$ -------(4)
      * From triangle inequality we know that

        $|a| + |b| \geq |a+b|\ ,therefore,\ from\ (4),$

        it can be prove that $|-a-b| < |-a| + |-b|$.

Q Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not.

(a) Two real numbers satisfy $a < b$ **if and only if** $a < b + \epsilon$ for every $\epsilon > 0$.

Proof : Since we have the statement "if and only if" we shall prove the statement both way, as explained in the book example :

False

$(\implies)$ *If* $a < b$*, then for for every* $\epsilon > 0$*, it follows that* $a < b + \epsilon$

- For the proof of the first statement there is really not much to say. If $a < b$, and so certainly $a < b + \epsilon$ no matter what $\epsilon > 0$ is chosen.

$(\impliedby)$
If for every real number $\epsilon > 0$*, it follows that* $a < b + \epsilon$*, then we must have* $a < b$*.*

- We will prove the converse statement, by contradiction. The conclusion of the proposition in this direction states that $a < b$, so we assume $a \not< b$.
- As per our current assumption of
  $a \not< b$*, then two other possibilities can be true. Either*
  $a = b$ *or* $a > b$*.*
  1. If $a = b$, $a < b + \epsilon$ *is true.*
  - our assumption that $a = b$, can be true.

Hence, the statement is false.

- **(Not needed)2.** If $a > b$, then our statement $a < b + \epsilon$ can only be true for $\epsilon > a - b$.
  - Thus, for all values of $\epsilon < a - b$, the statement $a < b + \epsilon$ is untrue, in light of our current assumption of $a > b$. But the statement $a < b + \epsilon$ is true "for every $\epsilon > 0$".
  - This contradiction means that our assumption $a > b$ cannot be true.
  - Hence $a \not> b$.
- Since $a \neq b$ *and* $a \not> b$, therefore $a < b$, and the indirect proof is complete.

(b) Two real numbers satisfy $a < b$ **if** $a < b + \epsilon$ for every $\epsilon > 0$.

False, Same reasoning as the last statement. the statement $a = b$*, can also be true* .

(c) Two real numbers satisfy $a \leq b$ **if and only if** $a < b + \epsilon$ for every $\epsilon > 0$.

True.
The only inequality we have to prove by contradiction is $a > b$ not being true.
If $a > b$, then our statement $a < b + \epsilon$ can only be true for $\epsilon > a - b$.

- Thus, for all values of $\epsilon < a - b$, the statement $a < b + \epsilon$ is untrue, in light of our current assumption of $a > b$. But the statement $a < b + \epsilon$ is true "for every $\epsilon > 0$".
  - This contradiction means that our assumption $a > b$ cannot be true.
  - Hence $a \not> b$.

Q. Form the logical negation of each claim. One trivial way to do this is to simple add " It is not the case that ... " in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in N$ such that $a + 1/n < b$.

Negation : It is not the case that for all real numbers satisfying $a < b$, there exists an $n \in N$ such that $a + 1/n < b$.

The claim is true ...

- If $a < b$, *then the statement* $a + 1/n < b \implies 1/n < b - a \implies 1/(b-a) < n$.
- which implies for values of $n > 1/(b-a)$, the given inequality $a + 1/n <$ b *is true*.

Hence, if the claim is true, then the negation of the statement is untrue.

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in N$.

Negation : It is not the case that there exists a real number $x > 0$ such that $x < 1/n$ for all $n \in N$.

Claim is true.

For $n = 1$, $x > 0$ *and* $x < 1$ $x \in (0, 1)$. *Infinite number* of real numbers between natural numbers (0,1).

(c) Between any two real numbers there is a rational number .,

Q. Let $y_1 = 6$, and for each $n \in N$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in N$.

Proof : Using the rule we compute...

- $y_1 = 6$
- $y_2 = 2$
- $y_3 = -0.66$
- $y_4 = -2.44$
- $\vdots$
- $y_{12} = -5.8616$
- $\vdots$

The sequence just defined appears a the outset to be decreasing. For the terms computed, we have $y_4 < y_3 < y_2 < y_1$. To prove $y_n > -6$.

- To show $y_n > -6,$ since it is a decreasing sequence, if we can show $y_{n+1} > -6,$ then surely our requirement of y_n will be fulfilled.
- We now assume that $y_n > -6$. Let $y_n = -5.99$
- $y_{n+1} = (2y_n - 6)/3 \implies (2*(-5.99) - 6)/3 \implies -5.999$
- Thus $y_{n+1} > -6$. Therefore, $y_n > -6$. for all $n \in N$.