Consequences of Completeness

The first application of the Axiom of Completeness is a result that may look like a more natural way to mathematically express the sentiment that the real line contains no gaps.

Theorem 1.4.1 (Nested Interval Property): For each $n \in N$, assume we are given a closed interval $I_m = [a_n, b_n] = \{x \in R : a_n \leq x \leq b\}$. Assume also that each I_n contains $I_{n+1.}$ Then, the resulting nested sequence of closed intervals

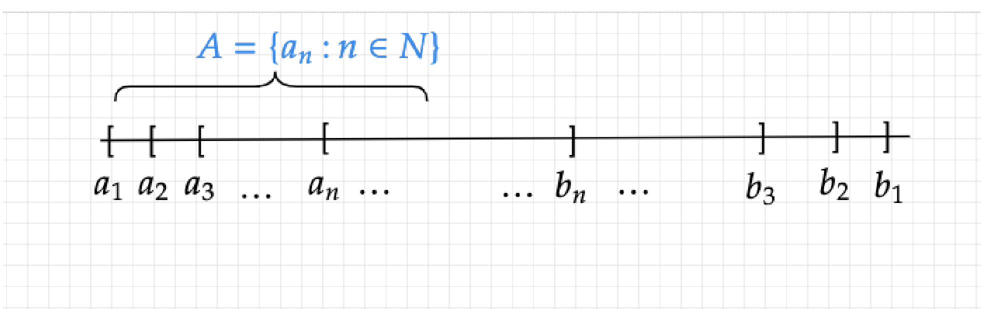
$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \ \dots$$

has a nonempty intersection; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof: In order to show that $\cap_{n=1}^{\infty} I_n$ is not empty, we are going to use the Axiom of Completeness (AoC) to produce a single real number x satisfying $x \in I_n$ for every $n \in N$. Now, AoC is a statement about bounded sets, and the one we want to consider is the set

$$A = \{a_n : n \in N\}$$

of left-hand endpoints of the intervals.



Because the intervals are nested, we see that every b_n serves as an upper bound for A. Thus, we are justified in setting

$$x = Sup\ A.$$

Now consider a particular $I_n = [a_n, b_n]$. Because x is an upper bound for A, we have $a_n \leq x$. The fact that each b_n is an upper bound for A and that x is the least upper bound implies $x \leq b_n$.

Although then, we have $a_n \leq x \leq b_n$, which mean $x \in I_n$ for every choice of $n \in N$. Hencem $x \in \cap_{n=1}^{\infty} I_n$, and the intersection is not empty. $\square$

The Density of $\mathbb{Q}$ in $\mathbb{R}$

The set Q is an extension of $N,$ and $\mathbb{R}$ in turn is an extension of $Q.$ The next few results indicate how N and Q sit inside of $R.$

Theorem 1.4.2 (Archimedean Property). (i) Given any number $x \in R,$ there exists an $n \in N$ satisfying $n > x.$
(ii) Given any real number $y > 0,$ there exists an $n \in N \, satisfying \, 1/n < y.$

Proof : Part(i) of the proposition states that N is not bounded above. There has never been any doubt about the truth of this, and it could be reasonably argued that we should not have to prove it at all, espicially in light of the fact that we have decided to take other familiar of $N, Z \, and \, Q$ as given.
The counterargument is that there is still a great deal of mystry about what the real numbers actually are. What we have said so far is that R is an extension of Q that maintains the algebic and order properties of the rationals but also posses the least upper bound property articulated in the Axiom of Completeness.
In the absence of any other information about $R,$ we have to consider the possibility that in extending Q we unwittingly acquired some new numbers that are upper bound for $N.$
In fact, as disorienting as it may sound, there are ordered field extensions of Q that include "numbers" bigger than every natural numer. Theorem 1.4.2 asserts that the real numbers do not contain such exotic creatures. The Axiom of Completeness, which we adopted to patch up holes in $Q,$ carries with it the implication that N is an unbounded subset of $R.$
And so the proof.
Assume, for contradiction, that N is bounded above. By the Axiom of Completeness(AoC),N should then have a least upper bound, and we can set $\alpha = sup \, N.$ If we consider $\alpha - 1,$ then we no longer have an upper bound(lemma 1.3.8),and therefore there exists $n \in N$ satisfying $\alpha - 1 < n.$ But this is equivalent to $\alpha < n + 1.$ Because $n + 1 \in N,$ we have a contradiction to the fact that α is supposed to be an upper bound for $N.$ (Notice that the contradiction here depends only on AoC and the fact that N is closed under addtition.
Part (ii) follows from (i) by letting $x = 1/y.$ □

This familiar property of N is the key to an extremely important fact about how Q fits inside of $R.$

Theorem 1.4.3 (Density of $Q \, in \, R$).