

CIS2107: Computer Systems and Low-level programming

Information representation: Bits, Bytes, and Integers

Dr. Sian Jin

Computer and Information Sciences

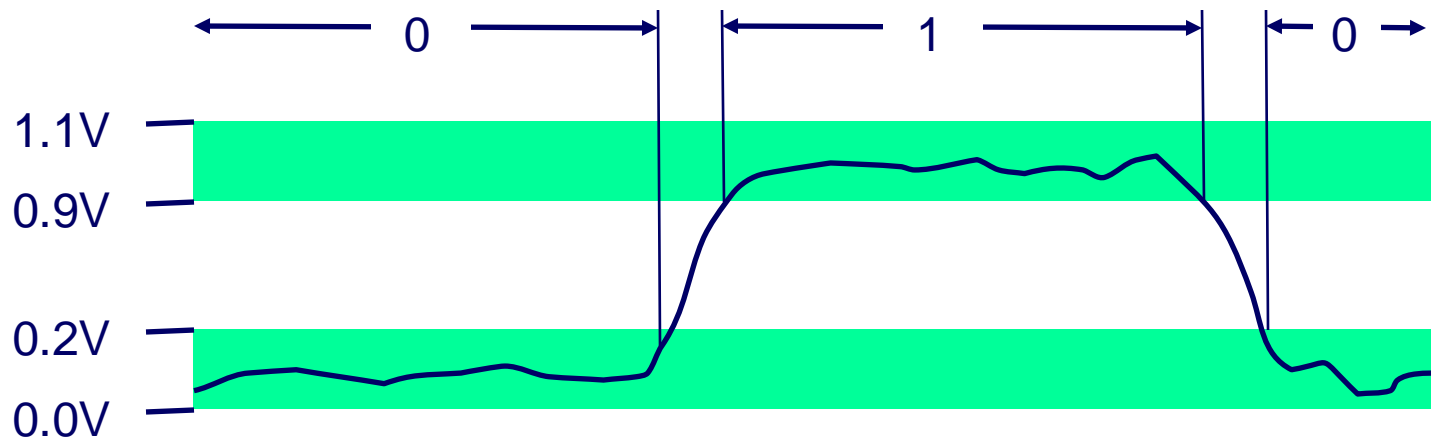
Temple University

Today: Bits, Bytes, and Integers

- **Representing information as bits**
- Bit-level manipulations
- **Integers**
 - Representation: unsigned and signed
 - Conversion, casting
 - Expanding, truncating
 - Addition, negation, multiplication, shifting
 - Summary

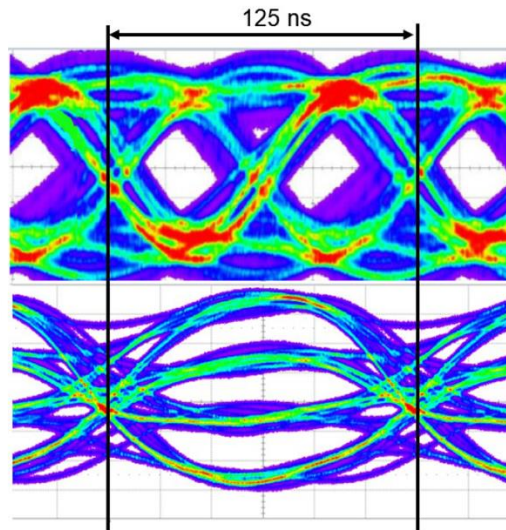
Everything is bits

- Each bit is 0 or 1
- By encoding/interpreting sets of bits in various ways
 - Computers determine what to do (instructions)
 - ... and represent and manipulate numbers, sets, strings, etc...
- **Why bits? Electronic Implementation**
 - Easy to store with bistable elements
 - Reliably transmitted on noisy and inaccurate wires



Everything is bits

- Each bit is 0 or 1
- By encoding/interpreting sets of bits in various ways
 - Computers determine what to do (instructions)
 - ... and represent and manipulate numbers, sets, strings, etc...
- **Why bits? Electronic Implementation**
 - Easy to store with bistable elements
 - Reliably transmitted on noisy and inaccurate wires



PAM4 in GDDR6X

For example, can count in binary

■ Base 2 Number Representation

- Represent 15213_{10} as 11101101101101_2
- Represent 1.20_{10} as $1.0011001100110011[0011]..._2$
- Represent 1.5213×10^4 as $1.1101101101101_2 \times 2^{13}$

Encoding Byte Values

■ Byte = 8 bits

- Binary 00000000_2 to 11111111_2
- Decimal: 0_{10} to 255_{10}
- Hexadecimal 00_{16} to FF_{16}
 - Base 16 number representation
 - Use characters '0' to '9' and 'A' to 'F'
 - Write $FA1D37B_{16}$ in C as
 - `0xFA1D37B`
 - `0xfa1d37b`

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Example Data Representations

C Data Type	Typical 32-bit	Typical 64-bit	x86-64
<code>char</code>	1	1	1
<code>short</code>	2	2	2
<code>int</code>	4	4	4
<code>long</code>	4	8	8
<code>float</code>	4	4	4
<code>double</code>	8	8	8
<code>long double</code>	–	–	10/16
<code>pointer</code>	4	8	8

Today: Bits, Bytes, and Integers

- Representing information as bits
- **Bit-level manipulations**
- Integers
 - Representation: unsigned and signed
 - Conversion, casting
 - Expanding, truncating
 - Addition, negation, multiplication, shifting
 - Summary

Boolean Algebra

■ Developed by George Boole in 19th Century

- Algebraic representation of logic
 - Encode “True” as 1 and “False” as 0

And

- $A \& B = 1$ when both $A=1$ and $B=1$

$\&$	0	1
0	0	0
1	0	1

Or

- $A | B = 1$ when either $A=1$ or $B=1$

$ $	0	1
0	0	1
1	1	1

Not

- $\sim A = 1$ when $A=0$

\sim	
0	1
1	0

Exclusive-Or (Xor)

- $A \wedge B = 1$ when either $A=1$ or $B=1$, but not both

\wedge	0	1
0	0	1
1	1	0

General Boolean Algebras

- **Operate on Bit Vectors**
 - Operations applied bitwise
- **All of the Properties of Boolean Algebra Apply**
- **Operations $\&$, $|$, \sim , \wedge Available in C**
 - Apply to any “integral” data type
 - long, int, short, char, unsigned
 - View arguments as bit vectors
 - Arguments applied bit-wise

Pop Quiz, try it yourself

■ Operations applied bitwise

$$\begin{array}{r} 01101001 \\ \& \underline{01010101} \end{array} \quad \begin{array}{r} 01101001 \\ | \underline{01010101} \end{array} \quad \begin{array}{r} 01101001 \\ \wedge \underline{01010101} \end{array} \quad \begin{array}{r} 01101001 \\ \sim \underline{01010101} \end{array}$$

■ Bit level operations (Char data type)

- $\sim 0x41$
- $\sim 0x00$
- $0x69 \& 0x55$
- $0x69 | 0x55$

And

- $A \& B = 1$ when both $A=1$ and $B=1$

$\&$	0	1
0	0	0
1	0	1

Not

- $\sim A = 1$ when $A=0$

\sim	0	1
0	1	0
1	0	1

Or

- $A | B = 1$ when either $A=1$ or $B=1$

$ $	0	1
0	0	1
1	1	1

Exclusive-Or (Xor)

- $A \wedge B = 1$ when either $A=1$ or $B=1$, but not both

\wedge	0	1
0	0	1
1	1	0

Contrast: Logic Operations in C

■ Contrast to Logical Operators

- `&&`, `||`, `!`
 - View 0 as “False”
 - Anything nonzero as “True”
 - Always return 0 or 1
 - Early termination

■ Examples (char data type)

- `!0x41` \leadsto `0x00`
- `!0x00` \leadsto `0x01`
- `!!0x41` \leadsto `0x01`

- `0x69 && 0x55` \leadsto `0x01`
- `0x69 || 0x55` \leadsto `0x01`
- `p && *p` (avoids null pointer access)

Contrast: Logic Operations in C

■ Contrast to Logical Operators

- `&&`, `||`, `!`
 - View 0 as “False”
 - Anything nonzero
 - Always returns 0 or 1
 - **Early exit**

■ Example

- `!0x41` \leadsto 0
- `!0x00` \leadsto 1
- `!!0x41` \leadsto 1

- `0x69 && 0x55` \leadsto 0x01
- `0x69 || 0x55` \leadsto 0x01
- `p && *p` (avoids null pointer access)

**Watch out for `&&` vs. `&` (and `||` vs. `|`)...
one of the more common oopsies in
C programming**

Shift Operations

■ Left Shift: $x \ll y$

- Shift bit-vector x left y positions
 - Throw away extra bits on left
 - Fill with 0's on right

■ Right Shift: $x \gg y$

- Shift bit-vector x right y positions
 - Throw away extra bits on right
- Logical shift
 - Fill with 0's on left
- Arithmetic shift
 - Replicate most significant bit on left

■ Undefined Behavior

- Shift amount < 0 or \geq word size

Argument x	01100010
$\ll 3$	00010000
Log. $\gg 2$	00011000
Arith. $\gg 2$	00011000

Argument x	10100010
$\ll 3$	00010000
Log. $\gg 2$	00101000
Arith. $\gg 2$	11101000

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
 - **Representation: unsigned and signed**
 - Conversion, casting
 - Expanding, truncating
 - Addition, negation, multiplication, shifting
 - Summary

Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

```
short int x = 15213;  
short int y = -15213;
```

Sign
Bit



■ C short 2 bytes long

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
y	-15213	C4 93	11000100 10010011

■ Sign Bit

- For 2's complement, most significant bit indicates sign
 - 0 for nonnegative
 - 1 for negative

Two-complement Encoding Example (Cont.)

x = 15213: 00111011 01101101
y = -15213: 11000100 10010011

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768

Sum

15213

-15213

Numeric Ranges

■ Unsigned Values

- $UMin = 0$
000...0
- $UMax = 2^w - 1$
111...1

■ Two's Complement Values

- $TMin = -2^{w-1}$
100...0
- $TMax = 2^{w-1} - 1$
011...1

■ Other Values

- Minus 1
111...1

Values for $W = 16$

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Values for Different Word Sizes

	W			
	8	16	32	64
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

■ Observations

- $|TMin| = TMax + 1$
 - Asymmetric range
- $UMax = 2 * TMax + 1$

■ C Programming

- `#include <limits.h>`
- Declares constants, e.g.,
 - `ULONG_MAX`
 - `LONG_MAX`
 - `LONG_MIN`
- Values platform specific

Unsigned & Signed Numeric Values

X	$B2U(X)$	$B2T(X)$
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	-8
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15	-1

■ Equivalence

- Same encodings for nonnegative values

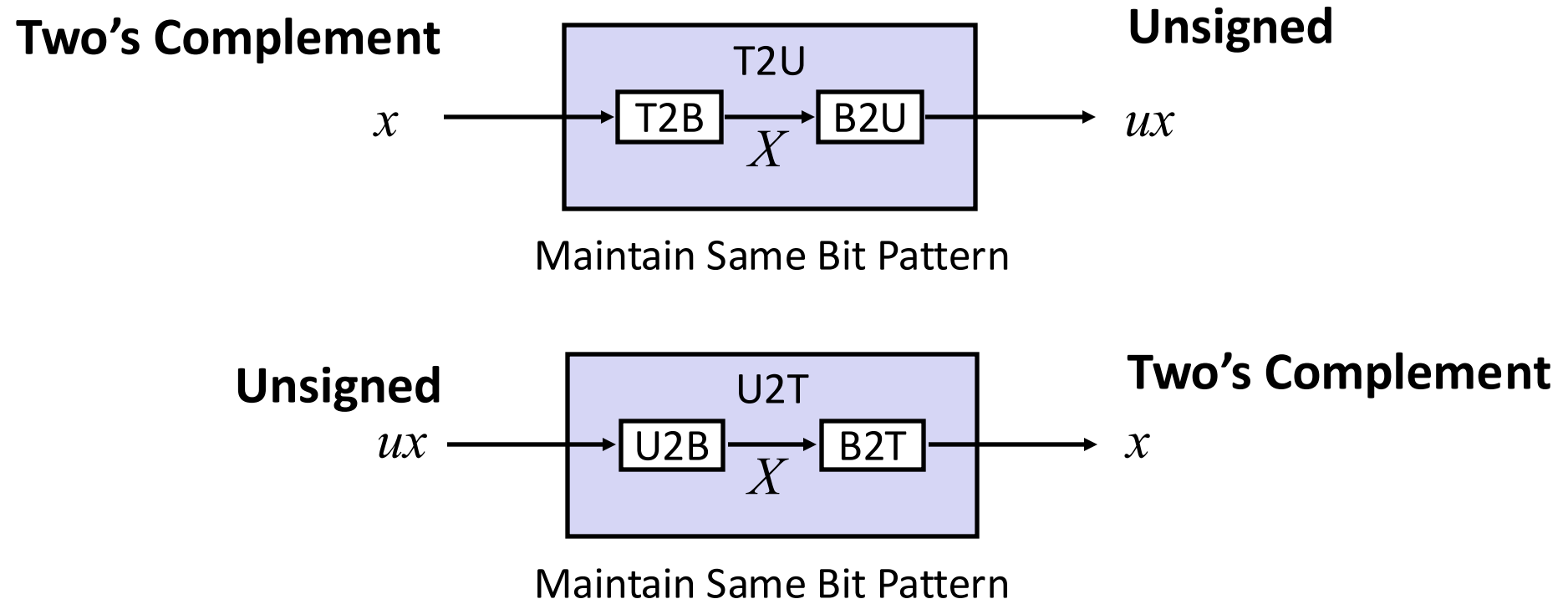
■ Uniqueness

- Every bit pattern represents unique integer value
- Each representable integer has unique bit encoding

■ \Rightarrow Can Invert Mappings

- $U2B(x) = B2U^{-1}(x)$
 - Bit pattern for unsigned integer
- $T2B(x) = B2T^{-1}(x)$
 - Bit pattern for two's comp integer

Mapping Between Signed & Unsigned



- Mappings between unsigned and two's complement numbers:
Keep bit representations and reinterpret

Mapping Signed \leftrightarrow Unsigned

Bits	Signed		Unsigned
0000	0		0
0001	1		1
0010	2		2
0011	3		3
0100	4		4
0101	5		5
0110	6		6
0111	7		7
1000	-8		8
1001	-7		9
1010	-6		10
1011	-5		11
1100	-4		12
1101	-3		13
1110	-2		14
1111	-1		15

T2U

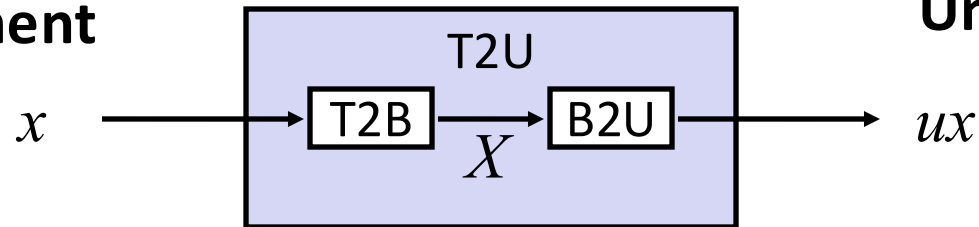
U2T

Mapping Signed \leftrightarrow Unsigned

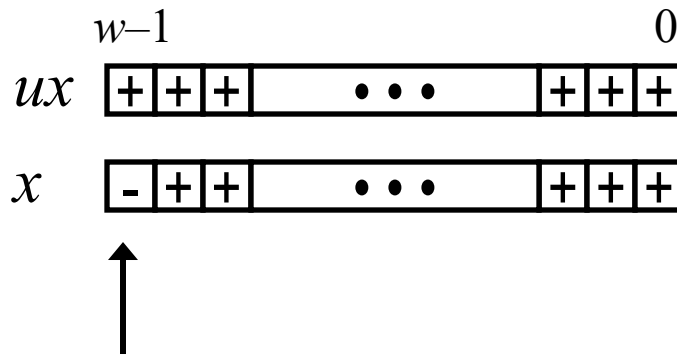
Bits	Signed		Unsigned
0000	0	$=$	0
0001	1		1
0010	2		2
0011	3		3
0100	4		4
0101	5		5
0110	6		6
0111	7		7
1000	-8	$+/- 16$	8
1001	-7		9
1010	-6		10
1011	-5		11
1100	-4		12
1101	-3		13
1110	-2		14
1111	-1		15

Relation between Signed & Unsigned

Two's Complement



Unsigned

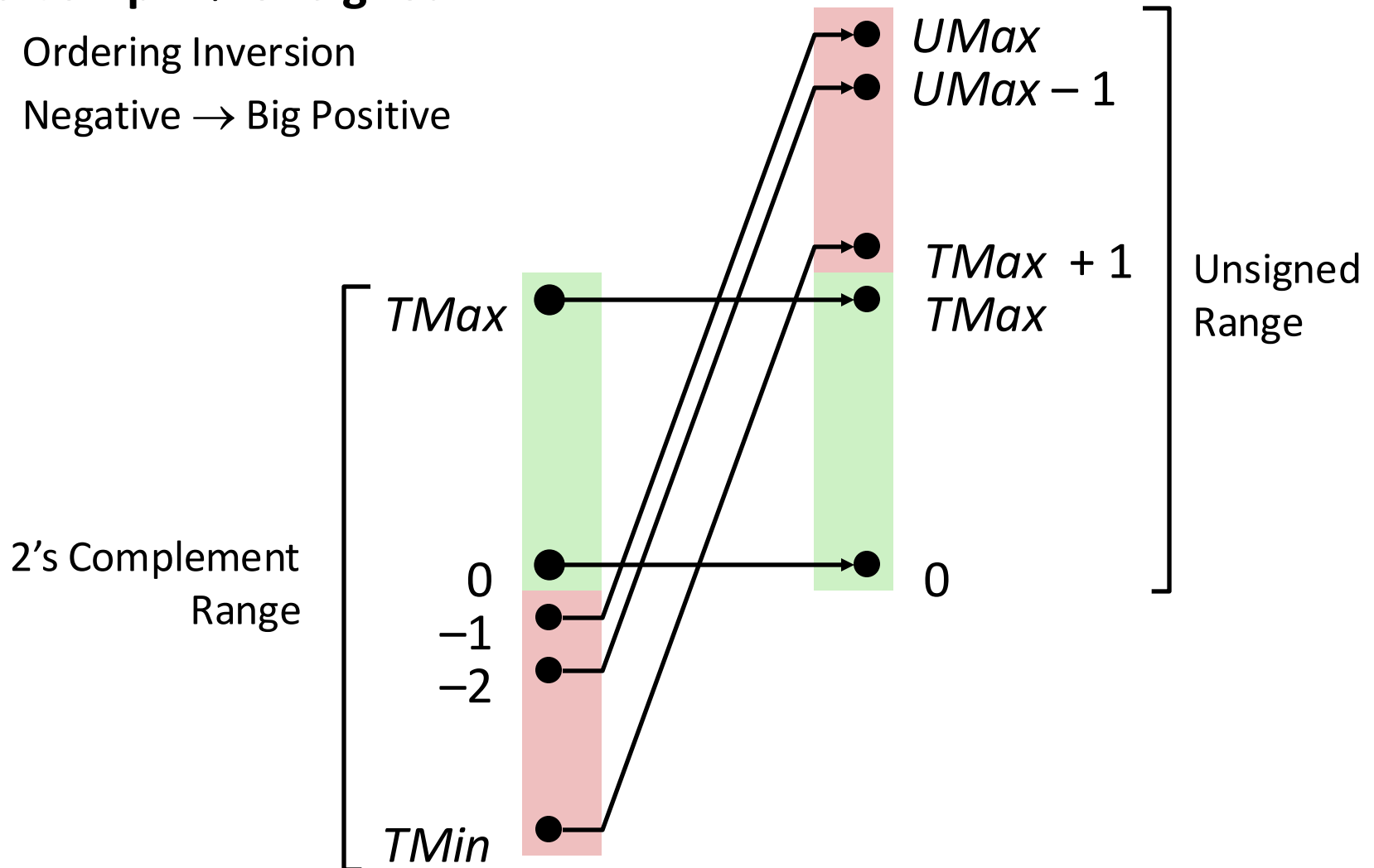


Large negative weight
becomes
Large positive weight

Conversion Visualized

■ 2's Comp. → Unsigned

- Ordering Inversion
- Negative → Big Positive



Signed vs. Unsigned in C

■ Constants

- By default are considered to be signed integers
- Unsigned if have “U” as suffix

`0U, 4294967259U`

■ Casting

- Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;  
unsigned ux, uy;  
tx = (int) ux;  
uy = (unsigned) ty;
```

- Implicit casting also occurs via assignments and procedure calls

```
tx = ux;  
uy = ty;
```

Casting Surprises

■ Expression Evaluation

- If there is a mix of unsigned and signed in single expression,
signed values implicitly cast to unsigned
- Including comparison operations $<$, $>$, $==$, $<=$, $>=$
- Examples for $W = 32$: **TMIN = -2,147,483,648** , **TMAX = 2,147,483,647**

■ Constant ₁	Constant ₂	Relation	Evaluation
0	0U	==	unsigned
-1	0	<	signed
-1	0U	>	unsigned
2147483647	-2147483647-1	>	signed
2147483647U	-2147483647-1	<	unsigned
-1	-2	>	signed
(unsigned)-1	-2	>	unsigned
2147483647	2147483648U	<	unsigned
2147483647	(int) 2147483648U	>	signed

Summary

Casting Signed \leftrightarrow Unsigned: Basic Rules

- Bit pattern is maintained
- But reinterpreted
- Can have unexpected effects: adding or subtracting 2^w
- Expression containing signed and unsigned int
 - `int` is cast to `unsigned`!!

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
 - Representation: unsigned and signed
 - Conversion, casting
 - **Expanding, truncating**
 - Addition, negation, multiplication, shifting
 - Summary

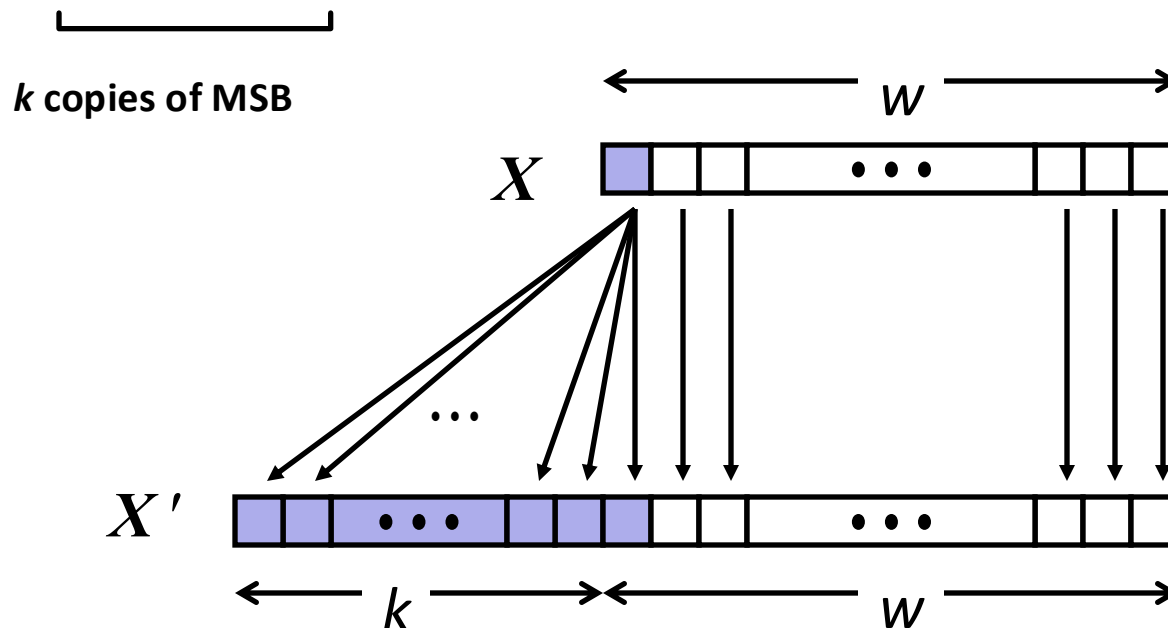
Sign Extension

■ Task:

- Given w -bit signed integer x
- Convert it to $w+k$ -bit integer with same value

■ Rule:

- Make k copies of sign bit:
- $X' = \underbrace{x_{w-1}, \dots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \dots, x_0$



Sign Extension Example

```
short int x = 15213;  
int      ix = (int) x;  
short int y = -15213;  
int      iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
- C automatically performs sign extension

Summary:

Expanding, Truncating: Basic Rules

- **Expanding (e.g., short int to int)**
 - Unsigned: zeros added
 - Signed: sign extension
 - Both yield expected result
- **Truncating (e.g., unsigned to unsigned short)**
 - Unsigned/signed: bits are truncated
 - Result reinterpreted
 - Unsigned: mod operation
 - Signed: similar to mod
 - For small numbers yields expected behavior

Today: Bits, Bytes, and Integers

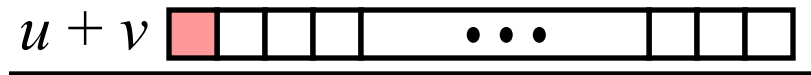
- Representing information as bits
- Bit-level manipulations
- **Integers**
 - Representation: unsigned and signed
 - Conversion, casting
 - Expanding, truncating
 - **Addition, negation, multiplication, shifting**
- Representations in memory, pointers, strings
- Summary

Unsigned Addition

Operands: w bits



True Sum: $w+1$ bits



Discard Carry: w bits



■ Standard Addition Function

- Ignores carry output

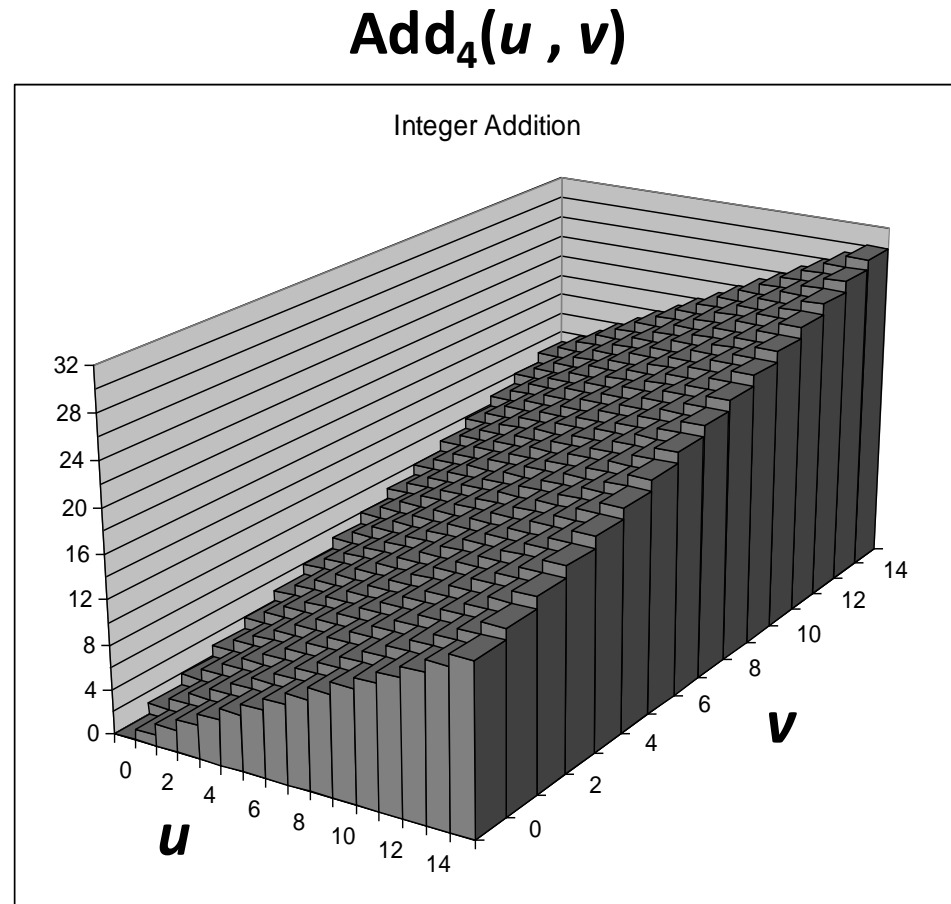
■ Implements Modular Arithmetic

$$s = \text{UAdd}_w(u, v) = u + v \bmod 2^w$$

Visualizing (Mathematical) Integer Addition

■ Integer Addition

- 4-bit integers u, v
- Compute true sum $\text{Add}_4(u, v)$
- Values increase linearly with u and v
- Forms planar surface

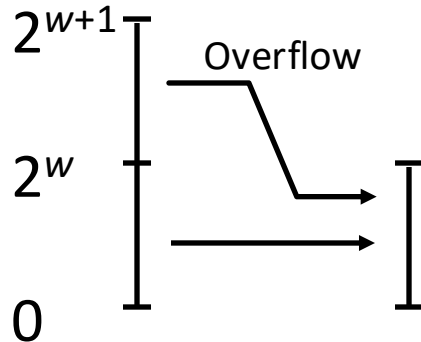


Visualizing Unsigned Addition

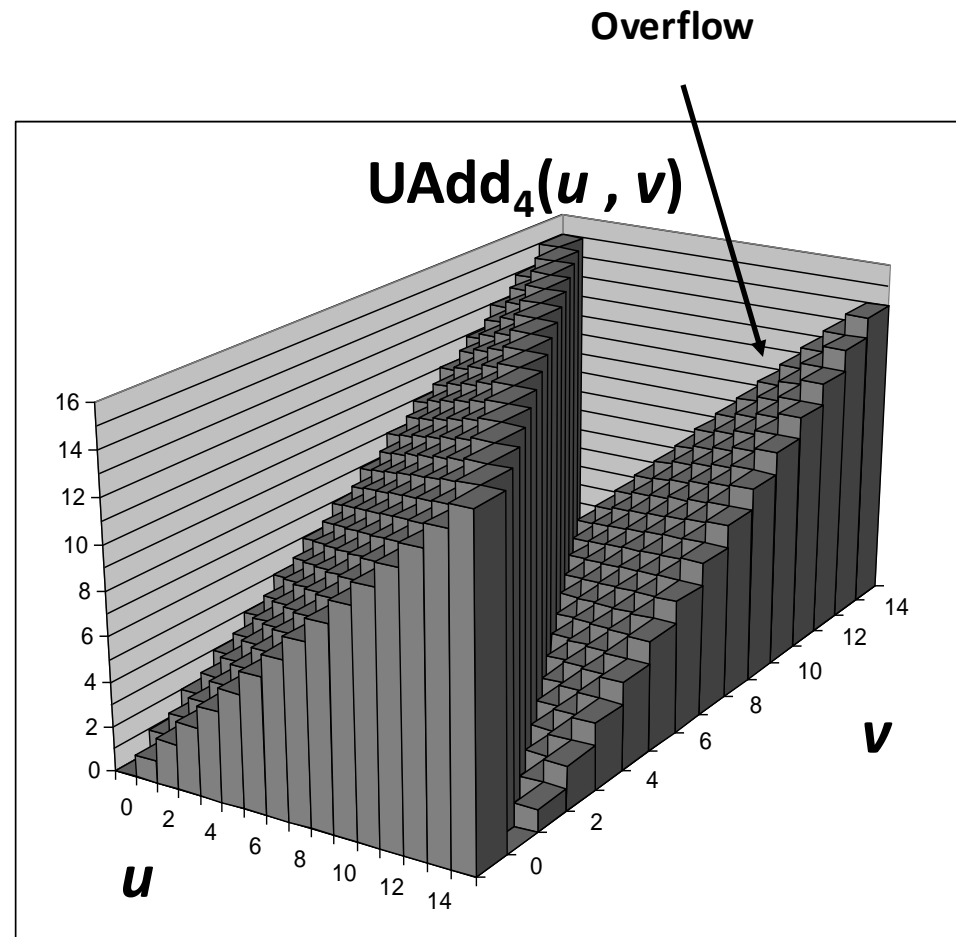
■ Wraps Around

- If true sum $\geq 2^w$
- At most once

True Sum



Modular Sum



Two's Complement Addition

Operands: w bits



True Sum: $w+1$ bits



Discard Carry: w bits



■ TAdd and UAdd have Identical Bit-Level Behavior

- Signed vs. unsigned addition in C:

```
int s, t, u, v;
```

```
s = (int) ((unsigned) u + (unsigned) v);
```

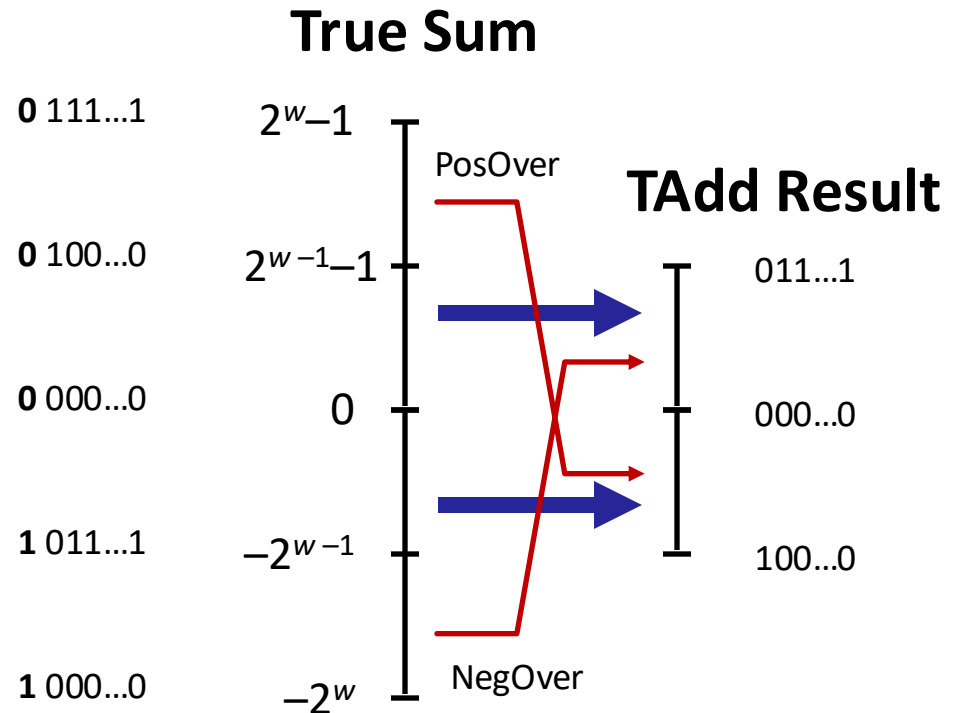
```
t = u + v
```

- Will give `s == t`

TAdd Overflow

■ Functionality

- True sum requires $w+1$ bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer



Visualizing 2's Complement Addition

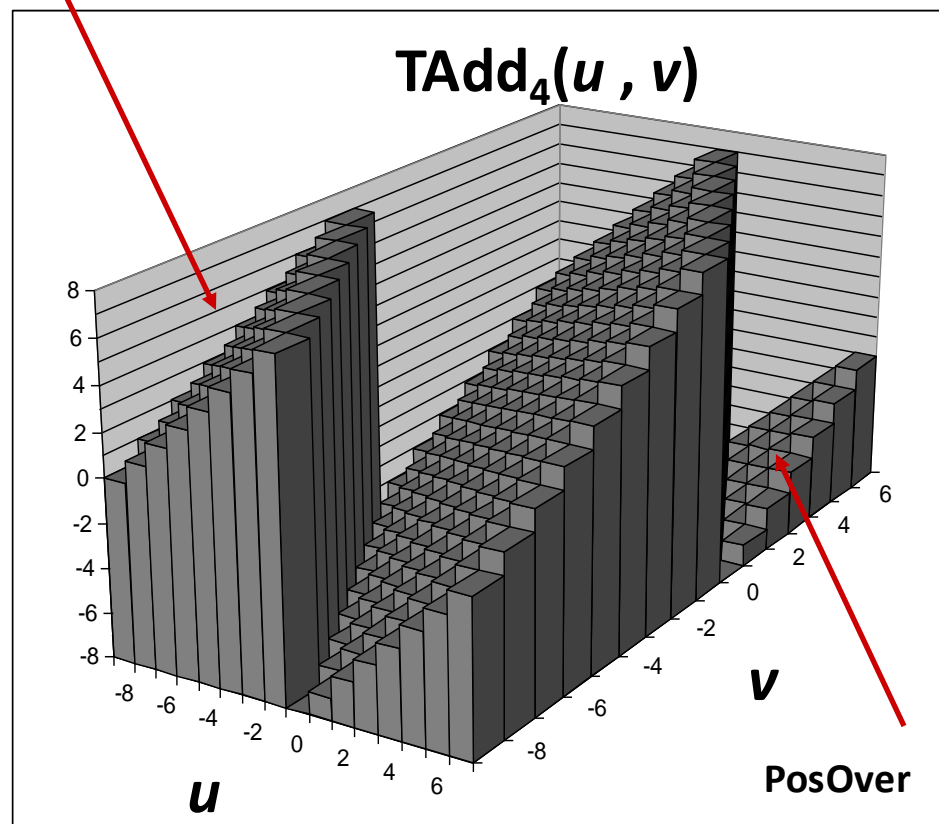
■ Values

- 4-bit two's comp.
- Range from -8 to +7

■ Wraps Around

- If $\text{sum} \geq 2^{w-1}$
 - Becomes negative
 - At most once
- If $\text{sum} < -2^{w-1}$
 - Becomes positive
 - At most once

NegOver



Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
 - Representation: unsigned and signed
 - Conversion, casting
 - Expanding, truncating
 - Addition, negation, multiplication, shifting
 - **Summary**

Arithmetic: Basic Rules

■ Addition:

- Unsigned/signed: Normal addition followed by truncate, same operation on bit level
- Unsigned: addition mod 2^w
 - Mathematical addition + possible subtraction of 2^w
- Signed: modified addition mod 2^w (result in proper range)
 - Mathematical addition + possible addition or subtraction of 2^w

■ Multiplication:

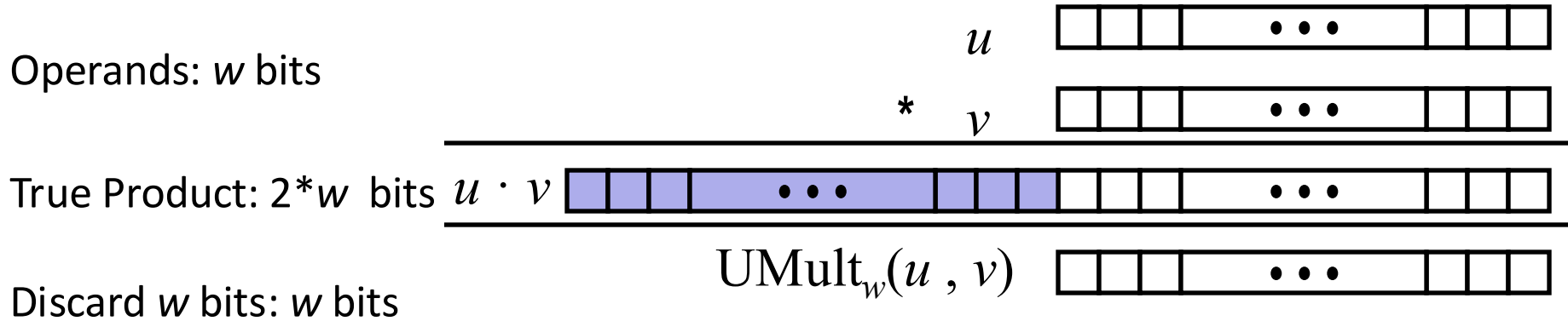
- Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
- Unsigned: multiplication mod 2^w
- Signed: modified multiplication mod 2^w (result in proper range)

Extras

Multiplication

- **Goal: Computing Product of w -bit numbers x, y**
 - Either signed or unsigned
- **But, exact results can be bigger than w bits**
 - Unsigned: up to $2w$ bits
 - Result range: $0 \leq x * y \leq (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
 - Two's complement min (negative): Up to $2w-1$ bits
 - Result range: $x * y \geq (-2^{w-1}) * (2^{w-1} - 1) = -2^{2w-2} + 2^{w-1}$
 - Two's complement max (positive): Up to $2w$ bits, but only for $(TMin_w)^2$
 - Result range: $x * y \leq (-2^{w-1})^2 = 2^{2w-2}$
- **So, maintaining exact results...**
 - would need to keep expanding word size with each product computed
 - is done in software, if needed
 - e.g., by “arbitrary precision” arithmetic packages

Unsigned Multiplication in C



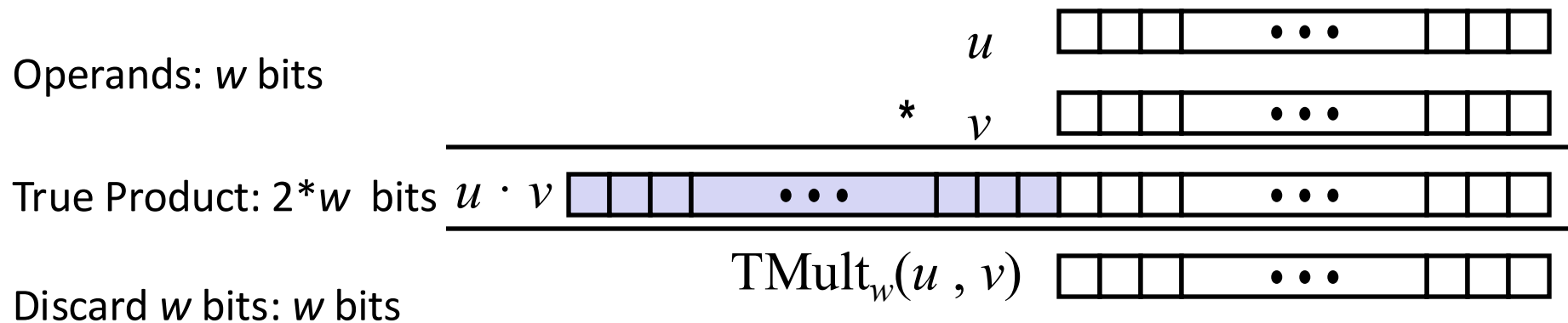
■ Standard Multiplication Function

- Ignores high order w bits

■ Implements Modular Arithmetic

$$\text{UMult}_w(u, v) = u \cdot v \bmod 2^w$$

Signed Multiplication in C



■ Standard Multiplication Function

- Ignores high order w bits
- Some of which are different for signed vs. unsigned multiplication
- Lower bits are the same

Power-of-2 Multiply with Shift

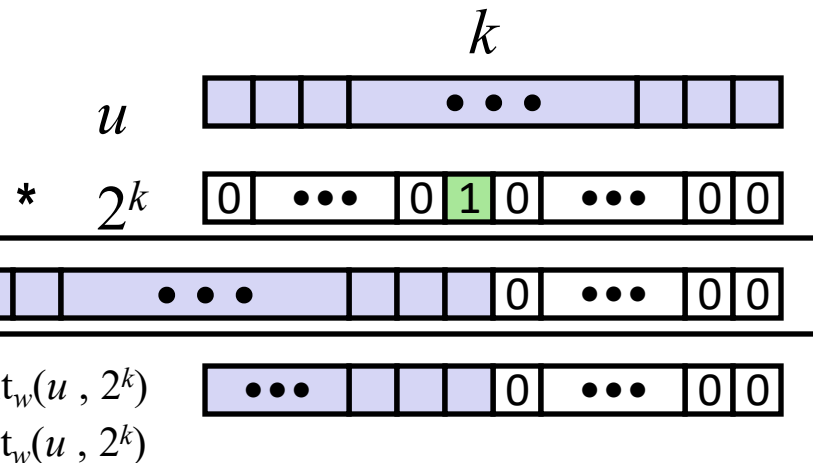
■ Operation

- $u \ll k$ gives $u * 2^k$
- Both signed and unsigned

Operands: w bits

True Product: $w+k$ bits

Discard k bits: w bits



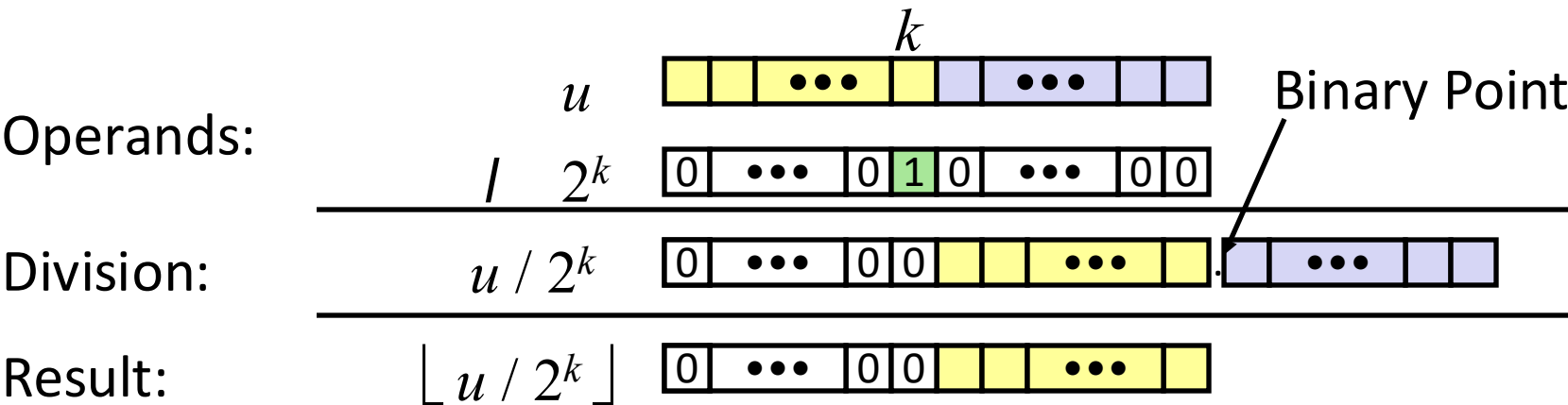
■ Examples

- $u \ll 3 \quad == \quad u * 8$
- $(u \ll 5) - (u \ll 3) == \quad u * 24$
- Most machines shift and add faster than multiply
 - Compiler generates this code automatically

Unsigned Power-of-2 Divide with Shift

■ Quotient of Unsigned by Power of 2

- $u \gg k$ gives $\lfloor u / 2^k \rfloor$
- Uses logical shift



	Division	Computed	Hex	Binary
x	15213	15213	3B 6D	00111011 01101101
x >> 1	7606.5	7606	1D B6	00011101 10110110
x >> 4	950.8125	950	03 B6	00000011 10110110
x >> 8	59.4257813	59	00 3B	00000000 00111011