On Conformal Mappings and Vector Fields

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This thesis is submitted in partial fulfillment
of the requirements of the Research Honors Program
in the Department of Mathematics and Computer Science

Marietta College

Marietta, Ohio

April 28, 2008

This Research Honors thesis has been approved for

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Abstract

We seek to extend the applicability of the tools of complex analysis that have been developed to deal with problems in two-dimensional harmonic field theory. In order to ease the reader who has only a basic understanding of complex analysis into a working knowledge of its relevant applications to field theory, this material is introduced through the use of vector fields as common ground. Opportunities for using the mathematical tools being developed to solve physical problems are also highlighted by examples in order to aid comprehension and foster intuition. Established techniques used in solving problems involving point sources are then generalized to handle those involving interval sources.

Acknowledgements

Many people have contributed, both directly and indirectly, to the completion of this honors thesis and deserve to be recognized for their help.

To the Rickey and McCoy families: Thank you for your generous financial support. Without it I would not be in a position to complete this thesis.

To Craig Howald, Cavendish McKay, and Mark Miller: Thank you for serving on my thesis committee and working with me throughout the year.

To Dennis Kuhl, Holly Menzel, Matt Menzel, Roger Pitasky, and John Tynan: Thank you for helping me over the past four years to learn enough about mathematics and physics to enable me to pursue a career as a professional mathematician.

To Jacqueline Lane: Thank you for providing help throughout the years with all of the small things. It really adds up.

To Michael Brilleslyper: Thank you for inspiring this project and for providing your written work to help me learn this material.

To my fellow mathematics and physics majors: Thank you for taking classes with me. I wish you all the best.

To the Marietta College Track and Field athletes, coaches, and all of my other friends: Thank you for helping me keep my perspective on life over the past four years. Sanity is a very valuable thing.

And to my family members, especially my mother Diane, my father Rodney, and my brother Bradley: Thank you for supporting me in so many ways throughout my life. I really appreciate it. Seriously.

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Chapter 1

Vector Fields

Vector fields are ubiquitous in mathematical physics. Many physical problems involving fluid flow, heat flow, electromagnetism, gravitation, diffusion, and elasticity are essentially problems in field theory: given certain physical circumstances, what are the vector fields associated with the phenomenon in question? Developing mathematical tools to deal with certain classes of these physical problems and then applying these tools to specific representative problems to demonstrate their utility is an important function of the mathematical physicist. It is our purpose in this paper to explore applications of complex analysis and conformal mappings to two-dimensional harmonic field theory. Before we can do this, however, we must develop an understanding of vector fields in this context.

Vector fields are functions that assign to each point in a specified domain a vector. They come in many varieties, including force fields, electric fields, magnetic fields, and velocity fields, but they always indicate the strength and direction of a certain quantity at each point in a region of space. One of the most intuitive ways of thinking of vector fields is to think in terms of velocity fields governing fluid flows. In this case, a particle that is placed at any point in the vector field will be pushed in the direction of the vector field at that point. The vector field tells you the direction of the fluid flow at that point.

Random vector fields, with wildly varying vectors located at points that are arbitrarily close together, can be imagined, but they are difficult to deal with mathematically and do not govern typical physical phenomena. Physical vector fields are typically caused by some physical reality, one that is governed by underlying dynamics. As a result, there are many interesting physical phenomena that are governed by vector fields whose vectors vary in a natural, continuous way. It is the theory governing the behavior of a specific subset of these vector fields that we develop here, specifically the subset of two-dimensional irrotational incompressible vector fields in simply connected domains. The precise meaning of this assertion will become clear as the paper progresses.

1.1 Generalized Sources

As a starting point for our analysis of vector fields, we consider those vector fields that are generated by points at which field lines originate or terminate. A proton, for example, puts forth electric field lines radially in three dimensions, thereby generating an electric field. An electron likewise consumes electric field lines radially in three dimensions, producing another electric field. In both cases a physical quantity is being either produced or consumed, thereby generating a vector field. In the absence of other points of production, the net flow outwards from each such point does not vary with distance from that point because no field lines are being produced or consumed anywhere else. This consideration is implicit in any analysis done in this paper: field lines can only be produced or consumed at specific points in the plane.

With this consideration in mind, we can now use these examples as a basis for defining what is meant by a source or a sink, the names that we will give to such points of production or consumption. As we are only concerned here with two-dimensional vector fields, we are led to the following definitions.

Definition 1.1.1. A *source* is a point in the complex plane from which field lines emanate uniformly in every radial direction, provided there are no other flows to disturb it.

Definition 1.1.2. A *sink* is a point in the complex plane into which field lines vanish uniformly in every radial direction, provided there are no other flows to disturb it.

Definition 1.1.3. A *generalized source* is a point in the complex plane that is either a source or a sink.

Definition 1.1.4. The *strength* of a generalized source located at $z_0 = (x_0, y_0)$ in the plane is the flux of field lines outwards through any simple closed contour for which the generalized source at z_0 is the only one interior to the contour.

Every generalized source has a well-defined constant strength because physical considerations demand that field lines have some origin; field lines cannot appear or disappear without any apparent cause. As a result, the flux outwards through any "surface" gives a measure of the "volume" of field being produced within its interior. Thus the flux of field lines through any contour that serves as a boundary enclosing just one specific source will give a specific value, one that serves as a measure of the rate of production of field lines from that source. This value is what we defined as the strength of the source.

Theorem 1.1.5 (Divergence Theorem)**.** *Given a vector field $\vec{V}$ and a volume $\mathcal{V}$ whose boundary is a closed surface $\partial\mathcal{V}$,*

$$\int_{\mathcal{V}} (\vec{\nabla} \cdot \vec{V}) d\tau = \oint_{\partial\mathcal{V}} \vec{V} \cdot d\vec{A}. \tag{1.1.1}$$

This result is fundamental in the study of electromagnetism and is restated above for clarity. Further information regarding this critical theorem and its relation to physical applications through Gauss's Law can be found in [Gri99]. We use it here with 2-dimensional "volumes" and 1-dimensional boundary "surfaces" whenever we speak about the strength of a source: the strength of a generalized source at z_0 is the divergence of the vector field at z_0. This notion actually provides an alternative definition of a generalized source: any point in the plane at which the divergence of the vector field is nonzero is a generalized source for that vector field.

It is also worth noting that as field lines always vanish into a sink and emanate from a source, a sink is a generalized source with negative strength and a source is a generalized source with positive strength. A generalized source cannot be of strength zero because such a point is neither a source nor a sink.

Having established a foothold, we can now begin to work our way through the material from the ground up. We begin by determining the vector field generated by a source of a given strength directly from our definitions.

Theorem 1.1.6. *The vector field $\vec{V}(z)$ generated by a generalized source of strength S_0 located at the origin in the complex plane is*

$$\vec{V}(z) = \frac{S_0}{2\pi\bar{z}}. \tag{1.1.2}$$

Proof. The proof proceeds by taking the vector field $\vec{V}(z)$ and demonstrating that it possesses the defining characteristics of a vector field generated by a generalized source of strength S_0 located at the origin.

Let $z = re^{\imath\theta}$ (See A.1.2 and A.1.4). Then

$$\vec{V}(z) = \vec{V}(re^{\imath\theta}) = \frac{S_0}{2\pi r e^{-\imath\theta}} = \frac{S_0}{2\pi r}e^{\imath\theta}.$$

As the two-dimensional radial unit vector $\hat{r}$ can be expressed as $e^{\imath\theta}$ in the complex plane, we have that the given vector field emanates radially from the origin. Furthermore, its magnitude depends only upon distance, not direction, from the origin. Thus the vector field $\vec{V}(z)$ is indeed that of a generalized source. It remains only to demonstrate that the strength of this generalized source is S_0.

As the strength of this generalized source can be determined by calculating the flux through any simple closed contour surrounding the origin, consider a circular contour C_R of radius $R > 0$ centered at the origin. The flux outwards through this contour, which has boundary vector $d\vec{l} = Rd\theta\hat{r}$, is

$$\int_{C_R} \vec{V}(z) \cdot d\vec{l} = \int_{C_R} \frac{S_0}{2\pi R}\hat{r} \cdot Rd\theta\hat{r} = \int_0^{2\pi} \frac{S_0}{2\pi R} Rd\theta = S_0.$$

The strength of the generalized source given by the vector field $\vec{V}(z)$ is thus S_0, which completes the proof. □

This result is easily generalized using a coordinate transformation.

Corollary 1.1.7. *The vector field $\vec{V}(z)$ generated by a generalized source of strength S_0 located at the point z_0 in the complex plane is*

$$\vec{V}(z) = \frac{S_0}{2\pi(\bar{z} - \bar{z}_0)}. \tag{1.1.3}$$

Proof. Define $w := z - z_0$. The generalized source located at z_0 in the z-plane is located at the origin in the w-plane. Invoking Theorem 1.1.6, we conclude that

$$\vec{V}(w) = \frac{S_0}{2\pi\bar{w}} \Rightarrow \vec{V}(z) = \frac{S_0}{2\pi(\overline{z - z_0})} = \frac{S_0}{2\pi(\bar{z} - \bar{z}_0)}.$$

□

It may seem that by restricting ourselves to only two dimensions in order to later make use of the power of complex analysis in our calculations that physically realizable examples no longer exist, but this is not the case. Baseline examples certainly exist in fluid flow if one considers a hose pumping fluid into a shallow basin as a source and a drain drawing it out as a sink. Beyond this, however, any wire of uniform linear charge density is a generalized source in electrostatics.

Definition 1.1.8 (Coulomb's Law)**.** The force $\vec{F}$ exerted on a test charge Q by a source charge q located a distance r units away from Q and in the direction of $\hat{r}$ is

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{r} = Q\vec{E}, \tag{1.1.4}$$

where ϵ_0 is a constant known as the permittivity of free space and $\vec{E}$, known as the electric field, is defined by Equation 1.1.4.

Theorem 1.1.9. *An infinite wire of uniform linear charge density λ passing perpendicularly through the complex plane at some point acts as a source of electric field at that point of strength $\frac{\lambda}{\epsilon_0}$.*

Proof. We will demonstrate by direct calculation that the electric field generated by an infinite wire of uniform linear charge density λ passing through the origin is identical to that of a source at the origin of strength $\frac{\lambda}{\epsilon_0}$. The conclusion will then follow immediately because the designation of a specific point as the origin is arbitrary.

Let the wire pass through the origin along the z-axis, and consider a point in the complex plane a distance r from the origin. The field at this point due to the wire is the integral of the infinitesimal contributions at each point on the wire, each some distance ρ away from the point in the plane, and each one contributing to the overall field an amount

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{\rho^2}\hat{\rho} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dz}{\rho^2}\hat{\rho},$$

in accordance with Coulomb's Law, Equation 1.1.4.

As the z-components of these contributions cancel in pairs due to symmetry, and considerations of symmetry also lead to the conclusion that it is only the magnitude r of the displacement from the origin that affects the field, not the direction of this displacement, we have that the resulting electric field is

$$\vec{E} = \frac{\hat{r}}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\lambda}{r^2+z^2}\cos(\theta)dz,$$

where θ is the variable angle up from the xy-plane to the point z units along the wire from the origin. Using the fact that $z = r\tan(\theta)$, so that $dz = r\sec^2(\theta)d\theta$, the integral can be evaluated as

$$\begin{aligned}
\vec{E} &= \frac{\hat{r}}{4\pi\epsilon_0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\lambda}{r^2+(r\tan(\theta))^2}\cos(\theta)(r\sec^2(\theta)d\theta) \\
&= \frac{\lambda\hat{r}}{4\pi\epsilon_0 r}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec(\theta)}{1+\tan^2(\theta)}d\theta \\
&= \frac{\lambda\hat{r}}{4\pi\epsilon_0 r}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)d\theta = \frac{\lambda\hat{r}}{4\pi\epsilon_0 r}\sin(\theta)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\lambda\hat{r}}{2\pi\epsilon_0 r},
\end{aligned}$$

which, when written using complex variables, is simply

$$\vec{E} = \frac{\lambda z}{2\pi\epsilon_0|z|^2} = \frac{\lambda}{2\pi\epsilon_0\bar{z}}.$$

As this is precisely the vector field produced by a generalized source of strength $S_0 = \frac{\lambda}{\epsilon_0}$ located at the origin, the proof is complete. $\square$

These results are interesting, but the prospects of applying these results to more general problems may at first glance seem bleak. If the mathematics can only handle vector fields that are created by a lone generalized source, our tools are very limited in scope. A powerful physical principle, however, enables us to easily generalize our results to arbitrarily many generalized sources.

Definition 1.1.10. The *principle of superposition* states that the vector field determined by a complicated physical system is equal to the sum of the vector fields determined by its components.

Mathematically, the principle of superposition is nothing more than an assertion of the linearity of vector addition; however, physically, it is a powerful tool that can be used to break down a complicated problem in field theory into simpler component problems. If the fields determined by each of these lone components can be calculated, then the solution to the larger problem can be constructed by simply summing the vector fields produced by the components. In our case, this enables the vector field generated by any number of generalized sources to be determined by simply summing the vector fields generated by each generalized source.

Theorem 1.1.11. *The vector field $\vec{V}(z)$ generated by n generalized sources of strengths $S_1, S_2, \ldots, S_n$ located at $z_1, z_2, \ldots, z_n$, respectively, in the complex plane, is*

$$\vec{V}(z) = \sum_{k=1}^{n} \frac{S_k}{2\pi(\bar{z} - \bar{z_k})}. \tag{1.1.5}$$

Proof. This follows immediately from an application of the principle of superposition to Corollary 1.1.7. □

Theorem 1.1.11 can be used to apply Theorem 1.1.9 to specific physical problems. The following example of such an application is Problem 2.47 from [Gri99].

Problem 1.1.12. *What is the electric field produced by two parallel wires, one carrying uniform linear charge density λ and the other carrying uniform linear charge density $-\lambda$, that are a distance $2a$ apart?*

Solution. Choosing an appropriate coordinate system and invoking Theorem 1.1.9, we find that this problem corresponds to finding the field generated by a source of strength $\frac{\lambda}{\epsilon_0}$ located at $\imath a$ and a sink of strength $\frac{-\lambda}{\epsilon_0}$ located at $-\imath a$. By phrasing the problem in the context of generalized sources in two-dimensional field theory, the solution follows immediately from Theorem 1.1.11:

$$\vec{V}(z) = \frac{\lambda}{2\pi\epsilon_0(\bar{z} + \imath a)} - \frac{\lambda}{2\pi\epsilon_0(\bar{z} - \imath a)}.$$

An illustration of this vector field for the values $a = 1$, $\lambda = \epsilon_0$ is provided in Figure 1.1.

1.2 Complex Potential

In order to expand the class of problems that we are capable of tackling, we must now start to introduce tools and concepts from complex analysis. At first it may seem that complex variables are merely convenient bookkeeping devices; however, they actually significantly expand the class of problems that are amenable to solution by enabling the use conformal mappings, which make it possible for an unknown problem to be mapped onto a problem whose solution is known, thereby solving the original problem.

We start by defining several important concepts.

Definition 1.2.1. A *domain* is a connected open set.

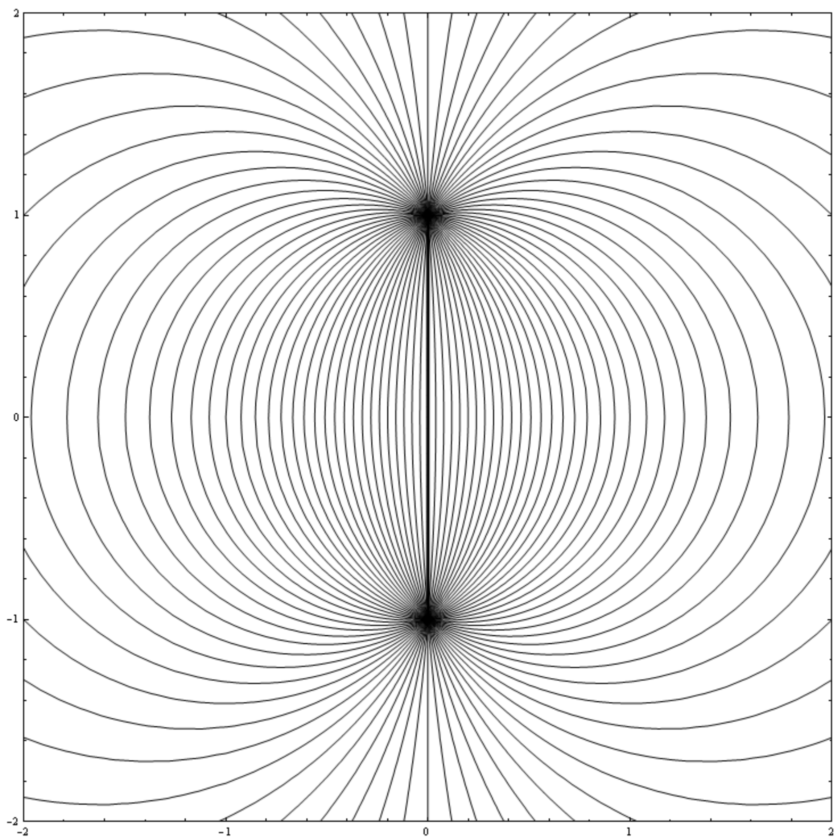

Figure 1.1: The flow of electric field lines produced by a source of strength 1 at $\imath$ and a sink of strength 1 at $-\imath$.

With respect to complex variables, a domain is, on a conceptual level, nothing more than a specific region in the complex plane on which a function is defined. It is important to note, however, that although this region can be any shape whatsoever, its boundary is not included in the domain by virtue of the requirement that a domain be an open set. It is also common to further qualify a domain as simply connected, which means that there are no holes in its shape.

Definition 1.2.2. A vector field $\vec{V}(z) = p(x, y)\hat{x} + q(x, y)\hat{y}$ is *incompressible* in a domain D if and only if $p_x(x_0, y_0) + q_y(x_0, y_0) = 0$ for all points $z_0 = (x_0, y_0)$ in D.

Definition 1.2.3. A vector field $\vec{V}(z) = p(x, y)\hat{x} + q(x, y)\hat{y}$ is *irrotational* in a domain D if and only if $q_x(x_0, y_0) - p_y(x_0, y_0) = 0$ for every point $z_0 = (x_0, y_0)$ in D.

The notion that a vector field is incompressible captures the fact that there is no net outflow through any closed surface whose interior is entirely contained within the domain of the vector field. Mathematically, incompressibility is most easily remembered as capturing the fact that the divergence of the vector field is zero. Likewise, the notion that a vector field is irrotational captures the fact that there is no tendency for the vector field to rotate. Mathematically, irrotationality is most easily remembered as capturing the fact that the curl of the vector field is zero.

It may be noted that if the divergence of a vector field were zero everywhere, including at points where generalized sources are located, then incompressible vector fields could have no generalized sources in their domain, for they would all have to be of strength zero in accordance with Theorem 1.1.5. It is, in fact, true that the points at which generalized sources are located cannot be included in the domains we will be considering without violating the incompressibility of those domains. Our solution to this problem is to consider the points at which generalized sources are located as being singularities in the vector field: as points outside of the legitimate domain of the vector field, but on the boundary of this domain. When describing domains, however, we shall take it for granted that such singularities exist within the domain as we describe it. This enables us to state plainly the shape and behavior of various domains without having to explicitly state these considerations each time.

These definitions enable us to state our next theorem, concerning the existence of a potential function for an incompressible vector field. This result is important as it sets the stage for later results concerning more general potential functions.

Potential functions are useful in vector field theory as they enable an unwieldy vector field to be expressed concisely in terms of a more manageable potential function, typically of lower dimensionality.

Theorem 1.2.4. *If a vector field $\vec{V}(z) = p(x,y)\hat{x} + q(x,y)\hat{y}$ is irrotational in a domain D, then there exists a scalar potential function $\phi(x,y)$ in D such that $\vec{V}(z) = \vec{\nabla}\phi(x,y)$ on D.*

Proof. Given the fact that $p_y(x,y) = q_x(x,y)$ on the domain D, we must find a potential function $\phi(x,y)$ defined on this domain that satisfies both $\phi_x(x,y) = p(x,y)$ and $\phi_y(x,y) = q(x,y)$. We do this by using any point (x_0, y_0) in D to define

$$\phi(x,y) := \int_{x_0}^{x} p(\xi,y)d\xi + \int_{y_0}^{y} q(x_0,\gamma)d\gamma + C,$$

a definition that guarantees that $\phi(x,y)$ is valid throughout the specified domain. By direct calculation, we then have that

$$\phi_x(x,y) = \frac{\partial}{\partial x}\left(\int_{x_0}^{x} p(\xi,y)d\xi + \int_{y_0}^{y} q(x_0,\gamma)d\gamma + C\right) = p(x,y)$$

and

$$\begin{aligned}
\phi_y(x,y) &= \frac{\partial}{\partial y}\left(\int_{x_0}^{x} p(\xi,y)d\xi + \int_{y_0}^{y} q(x_0,\gamma)d\gamma + C\right) \\
&= \int_{x_0}^{x} p_y(\xi,y)d\xi + q(x_0,y) = \int_{x_0}^{x} q_\xi(\xi,y)d\xi + q(x_0,y) \\
&= q(x,y) - q(x_0,y) + q(x_0,y) = q(x,y),
\end{aligned}$$

which is what we needed to demonstrate. □

Note that the scalar potential associated with any vector field can be adjusted by an arbitrary constant without changing the vector field that it produces. This freedom to define a convenient zero potential proves useful in solving physical problems and also holds true for other potential functions, as we will now observe as we generalize the scalar potential function.

Theorem 1.2.5. *If a vector field $\vec{V}(z) = p(x,y)\hat{x} + q(x,y)\hat{y}$ is continuous, irrotational, and incompressible in a domain D, and the partial derivatives p_x, p_y, q_x, and q_y are continuous in D, then there exists a function $\Omega(z) = \phi(x,y) + \imath\psi(x,y)$ that is analytic on D and that satisfies $\vec{V}(z) = \vec{\nabla}\phi(x,y) = \overline{\Omega'(z)}$.*

Proof. We proceed by finding $\Omega(z)$ from its derivative $\Omega'(z) = p - \imath q$. Given that $\vec{V}(z)$ is irrotational and incompressible, we have that $p_x = -q_y$ and $p_y = q_x$. It follows immediately from this that $\Omega'(z)$ satisfies the Cauchy-Riemann Equations (see A.2.1), namely

$$(p)_x = (-q)_y \text{ and } (p)_y = -(-q)_x.$$

As $\Omega'(z)$ is simply the reflection of $\vec{V}(z)$ across the x-axis, the continuity of $\vec{V}(z)$ implies that $\Omega'(z)$ is continuous in D. Altogether, we have that p_x, p_y, q_x, q_y, and $\Omega'(z)$ are continuous in D and that $\Omega'(z)$ satisfies the Cauchy-Riemann equations in D. These are precisely the Cauchy-Riemann conditions for differentiability for $\Omega'(z)$ (see A.2.2). It follows from this that $\Omega'(z)$ is differentiable on D. As a domain is by definition a connected open set, it follows that differentiability on D implies analyticity on D. Thus $\Omega'(z)$ is analytic on D.

It is known that any antiderivative of an analytic function is analytic, from which it follows that $\int \Omega'(z)dz = \Omega(z) + z_0$ is analytic for any complex constant z_0. As the value of this constant is not significant here, we absorb it into the function $\Omega(z)$. It is important, however, to remember that this function can be adjusted by an arbitrary complex constant without affecting the vector field that it produces.

We are left with the function $\Omega(z) = \phi(x, y) + \imath\psi(x, y)$, which is analytic on D with derivative $\Omega'(z) = \phi_x(x, y) - \imath\phi_y(x, y)$. We observe from this that the real part of $\Omega(z)$ is indeed the scalar potential $\phi(x, y)$ defined by the equation $\vec{V}(z) = \vec{\nabla}\phi(x, y)$. This verification completes the proof. $\square$

Definition 1.2.6. The analytic function $\Omega(z)$ defined by Theorem 1.2.5 in terms of the vector field $\vec{V}(z)$ is known as the *complex potential* of the vector field $\vec{V}(z)$.

Complex potentials are very useful ways of writing vector fields concisely. As noted in the proof of Theorem 1.2.5, it is important to keep in mind that a complex potential can be adjusted by any complex constant without changing the vector field that it produces. Adjusting this constant, however, does have some effect.

Definition 1.2.7. Each element of the set $\{\phi(x, y) = K; K \in \mathbb{R}\}$ is an *equipotential* of the complex potential $\Omega(z) = \phi(x, y) + \imath\psi(x, y)$.

Definition 1.2.8. Each element of the set $\{\psi(x, y) = K; K \in \mathbb{R}\}$ is a *streamline* of the complex potential $\Omega(z) = \phi(x, y) + \imath\psi(x, y)$.

If a real number is added to a complex potential, the value of the scalar potential ϕ along each equipotential curve increases by that amount. Likewise, if an imaginary number is added to a complex potential, the value of the stream function ψ along each streamline increases by the imaginary part of that number. This freedom makes it possible to specify a specific value along any one equipotential and one streamline in a given problem.

The overall flow pattern created by a vector field can be illustrated by creating a contour plot of either its equipotentials or its streamlines in a sense that is made clear by the next theorem.

Theorem 1.2.9. *Given any point $z_0 = x_0 + \imath y_0$ for which $\Omega'(z_0) \neq 0$, the equipotential $\phi(x,y) = \phi(x_0,y_0)$ and the streamline $\psi(x,y) = \psi(x_0,y_0)$ intersect orthogonally.*

Proof. A vector normal to the equipotential $\phi(x,y) = \phi(x_0,y_0)$ at $z_0 = x_0 + \imath y_0$ is $\vec{\nabla}\phi(x_0,y_0)$. Likewise, a vector normal to the streamline $\psi(x,y) = \psi(x_0,y_0)$ at $z_0 = x_0 + \imath y_0$ is $\vec{\nabla}\psi(x_0,y_0)$. If these two normal vectors are orthogonal, then the equipotential and the streamline from which they were obtained are also orthogonal at their point of intersection z_0. As two vectors are orthogonal precisely when they are both nonzero vectors and their dot product is zero, we must demonstrate that this holds for $\vec{\nabla}\phi(x_0,y_0) = \phi_x(x_0,y_0) + \imath\phi_y(x_0,y_0)$ and $\vec{\nabla}\psi(x_0,y_0) = \psi_x(x_0,y_0) + \imath\psi_y(x_0,y_0)$.

It follows immediately from the fact that

$$\Omega'(z_0) = \phi_x(x_0,y_0) - \imath\phi_y(x_0,y_0) = \psi_y(x_0,y_0) + \imath\psi_x(x_0,y_0)$$

is nonzero that neither normal vector is zero. Using the Cauchy-Riemann Equations for $\Omega(z)$, it is also clear that

$$\begin{aligned}
\vec{\nabla}\phi(x_0,y_0)\cdot\vec{\nabla}\psi(x_0,y_0) &= \phi_x(x_0,y_0)\psi_x(x_0,y_0) + \phi_y(x_0,y_0)\psi_y(x_0,y_0) \\
&= \psi_y(x_0,y_0)\psi_x(x_0,y_0) - \psi_x(x_0,y_0)\psi_y(x_0,y_0) \\
&= 0.
\end{aligned}$$

We conclude that the two normal vectors are orthogonal, which completes the proof. □

Recall that a vector field is at every point equal to the gradient of its scalar potential. As it has just been shown that this vector is at every point in the plane parallel to the streamline passing through that point, it follows that the family of streamlines of a vector field illustrates the overall flow pattern of the vector field. Using fluid flow as a basis for intuition, we can say that any object placed at rest in a fluid flow will flow along the streamline of that flow that passes through its initial position.

Chapter 2

Conformal Mappings

Conformal mappings are, for the purposes of two-dimensional field theory, nothing more than tools that are used to take problems that are stated in a wide variety of domains, such as disks, sectors, horizontal strips, or any other simply connected domain, except the full complex plane, onto the upper half plane. This is done because once the problem has been taken to the upper half plane it can be easily solved using standard techniques. The resulting solution can then be taken back to the original domain using the conformal mapping that was used to take it to the upper half plane.

This method works so long as the boundary of the original simply connected domain D is one of two things. If the boundary is a streamline, then the x-axis of the half plane will be a streamline after the mapping. In this case a solution can be found if the generalized sources are located anywhere in or on the boundary of D. On the other hand, if the boundary is an equipotential, then the x-axis of the half plane will be an equipotential after the mapping. In this case the generalized sources must be located in, but not on the boundary of, D.

The distinction between these two cases is slight. Fluid flow places emphasis on streamline analysis, whereas electrostatics places more emphasis on equipotential analysis. Both approaches are valuable, but we choose to focus on streamlines in order to be able to draw on the more natural intuition of fluid flow in describing vector fields as currents or flows. Focusing on a single case also allows us to maintain a clear focus and direction when articulating results formally.

Practically speaking, the only difference between the two cases is in the sign of the image generalized sources used in determining the complex potential of flows in the upper half plane. For many of the theorems later in this paper a corresponding result for the equipotential case can be obtained by simply switching the sign of the image generalized sources used in the proof of the theorem. Theorem 2.3.5, for example, can be generalized in this way. Restating each such theorem in this way is not done here, however, in order to keep the exposition of these techniques as brief and clear as possible.

Although this outline of where we are headed is a bit far-reaching, it is valuable in that it provides motivation for the theoretical developments that follow that might otherwise seem to be leading nowhere. Our goal is straightforward even if there is much work to be done before we can begin applying our ideas to problems: given any arrangement of generalized sources in a simply connected domain D that is not the full complex plane, there exists some conformal mapping that will take D to the upper half plane where the vector field of a corresponding problem can be found and translated back to the original domain D by means of the conformal mapping. This method of solution works so long as the vector field being considered is incompressible and irrotational.

We begin to explore this material in greater depth by first developing general theoretical results that underly later work. This theory is then supplemented by a study of specific conformal mappings that can be used to apply the theory. These tools are then used in tandem in order to attack various problems in two-dimensional harmonic field theory.

2.1 General Theory

Conformal mappings are essentially just complex functions that map one domain onto another in a particularly nice way. Many complex functions, however, have what are known as branch cuts, which are lines along which there is a discontinuity in the function value. Along such a branch cut a complex function is not differentiable, which interferes with the way in which it functions as a conformal mapping.

These lines actually separate one version of a given function from another version of the same essential function. Each such version of a function is called a branch of the underlying function. Choosing a different branch of a function can affect its behavior in important ways, particularly when the function is being used as a conformal mapping. The most important such functions are the argument function and the logarithm function.

There are several common branches associated with these functions. Underlying each such branch are the underlying argument and logarithm functions, which are actually function sets, or multi-valued functions.

Definition 2.1.1. Given any complex number z, define the set

$$arg(z) := \{\theta; z = re^{\imath\theta}\}. \tag{2.1.1}$$

If $\theta \in arg(z)$, then θ is *an argument* of z.

Definition 2.1.2. Given any complex number z, define the set

$$log(z) := \{ln|z| + \imath\theta; z = re^{\imath\theta}\}. \tag{2.1.2}$$

This set is the *general logarithm* and is written concisely as $log(z) = ln|z| + \imath arg(z)$.

These general functions need to be restricted to a specific branch in order to be used as actual functions, meaning as mathematical objects that send each input value to a single output value. The most common branch of these functions is that for which negative real numbers lie along the branch cut.

Definition 2.1.3. If $\theta \in arg(z)$ is restricted to $-\pi < \theta \leq \pi$, then the resulting branch of the argument function, denoted $Arg(z)$, gives the *principal argument* of z.

Definition 2.1.4. The *principal logarithm*, denoted $Log(z)$, is defined by

$$Log(z) := ln|z| + \imath Arg(z). \tag{2.1.3}$$

Any number of other branches of these functions can be defined, however, and in order to accommodate every possible case the following conventions are used.

Definition 2.1.5. If $\theta \in arg(z)$ is restricted to $\alpha < \theta \leq \alpha + 2\pi$, then the resulting branch of the argument function, denoted $Arg_\alpha(z)$, gives the α*-argument* of z.

Definition 2.1.6. The α*-branch of the logarithm*, denoted $Log_\alpha(z)$, is defined by

$$Log_\alpha(z) := ln|z| + \imath Arg_\alpha(z). \tag{2.1.4}$$

With these definitions, we can now tackle conformal mappings directly as we can use the general argument function as a way of representing the angle that a complex number makes with the x-axis.

Definition 2.1.7. A complex function $w = f(z)$ is said to be a *conformal mapping* from a domain D in the z-plane to a domain G in the w-plane if it preserves the magnitude and orientation of angles between oriented curves in the sense that given any two oriented curves C_1 and C_2 in D that intersect at z_0 in the z-plane, the oriented curves $f(C_1)$ and $f(C_2)$ in G intersect at the point $w_0 = f(z_0)$ at the same angle, both in terms of direction and orientation, in the w-plane.

Conformal mappings are intrinsically elegant in that they preserve much of the geometry of a domain. Their definition, however, says nothing about their existence or about how specific conformal mappings might be found that could be used in practice. Fortunately, the following theorem not only demonstrates that conformal mappings exist, but also provides a useful class of conformal mappings from which to draw when solving specific problems.

Theorem 2.1.8. *An analytic function $f(z)$ is a conformal mapping at every point z_0 in its domain such that $f'(z_0) \neq 0$.*

Proof. Let C_1 and C_2 be two smooth curves parameterized for $-1 \leq t \leq 1$ by $z_1(t)$ and $z_2(t)$, respectively, that pass through z_0 at $t = 0$. The images of these curves, $f(C_1)$ and $f(C_2)$, are then parameterized for $-1 \leq t \leq 1$ by $w_1(t) = f(z_1(t))$ and $w_2(t) = f(z_2(t))$, respectively, and intersect at $w_0 = f(z_0)$ at $t = 0$.

The velocity vectors $z_1'(t)$, $z_2'(t)$, $w_1'(t)$, and $w_2'(t)$ are tangent to their respective curves for all t. Thus, at $t = 0$, $z_1'(0)$ is the tangent to C_1 at z_0 and $z_2'(0)$ is the tangent to C_2 at z_0, so that the angle from C_1 to C_2 at z_0 is specified by the set

$$arg(z_2'(0)) - arg(z_1'(0)).$$

Likewise, $w_1'(0) = z_1'(0)f'(z_0)$ is the tangent to $f(C_1)$ at w_0 and $w_2'(0) = z_2'(0)f'(z_0)$ is the tangent to $f(C_2)$ at w_0, so that the angle from $f(C_1)$ to $f(C_2)$ at w_0 is specified by the set

$$arg(z_2'(0)f'(z_0)) - arg(z_1'(0)f'(z_0)).$$

As both C_1 and C_2 pass smoothly through the point z_0, neither $z_2'(0)$ nor $z_1'(0)$ are zero; however, if $f'(z_0) = 0$, then this set is undefined. We must, therefore, restrict ourselves to only those points z_0 for which $f'(z_0)$ is nonzero. Having made this restriction, we can now simplify this set to

$$\begin{aligned} & arg(z_2'(0)) + arg(f'(z_0)) - arg(z_1'(0)) - arg(f'(z_0)) \\ = \; & arg(z_2'(0)) - arg(z_1'(0)). \end{aligned}$$

As this is the very same set that specifies the angle from C_1 to C_2, we conclude that the sense and magnitude of the angle at z_0 is preserved by the mapping $f(z)$, and thus $f(z)$ is a conformal mapping. $\square$

The following theorem and its corollary make it much easier to find a conformal mapping that will map a given domain D onto a specific desired domain G by proving that a complicated mapping can be found in steps, building up the final result by composing any number of intermediate conformal mappings.

Theorem 2.1.9. *If $f(z)$ is an analytic function that maps the domain D to the domain G conformally and $g(z)$ is an analytic function that maps the domain G to the domain H conformally, then $h(z) := g(f(z))$ is an analytic function that maps D to H conformally.*

Proof. As each z in D is mapped to a value $f(z)$ in G, and any value $f(z)$ in G is mapped to a value $g(f(z))$ in H, we see that the composite function $h(z)$ does indeed map D to H. Thus, as there is no problem with invalid input, we can use the chain rule to calculate $h'(z) = f'(z)g'(f(z))$. As $f(z)$ is a conformal mapping from D to G, $f'(z) \neq 0$ for any z in D. Likewise, as $g(z)$ is a conformal mapping from G to H, $g'(f(z)) \neq 0$ for any $f(z)$ in G, where we recognize that $f(z)$ is simply an element in G. Thus $h'(z) \neq 0$ for any z in D. We conclude that $h(z)$ is a conformal mapping from D to H by Theorem 2.1.8, completing the proof. $\square$

Corollary 2.1.10. *For any positive integer n, given n analytic functions $f_1(z)$, $f_2(z), \ldots, f_n(z)$ that map the domains $D_1, D_2, \ldots, D_n$ to the domains $D_2, D_3, \ldots, D_{n+1}$ conformally, respectively, the composite function*

$$g(z) := f_n(f_{n-1}(\ldots(f_1(z))\ldots))$$

is analytic and maps D_1 to D_{n+1}.

Proof. This follows immediately from Theorem 2.1.9 by the principle of mathematical induction. $\square$

One more general result regarding conformal mappings must be stated before we can begin using these tools to solve specific problems. The following theorem is the foundation upon which applications of conformal mappings and complex analysis, particularly the use of complex potentials to represent vector fields, rests. It states, loosely speaking, that if a conformal mapping $w = f(z)$ takes one domain onto another where the complex potential is some $\Omega(w)$, then the original complex potential is nothing more than $\Omega(f(z))$. Although this may not seem essential, the invariance of complex potentials under conformal mappings is the very reason why it is worthwhile to rephrase problems in two-dimensional harmonic field theory in terms of complex variables.

Theorem 2.1.11. *Let $w = f(z) = f(x + \imath y) = u(x, y) + \imath v(x, y)$ be an analytic function that maps the domain D to the domain G conformally, and let $\Omega_G(w) = \Phi(u, v)+\imath\Psi(u, v)$ be the complex potential in G. Then the complex potential in D, denoted $\Omega_D(z) = \phi(x, y)+\imath\psi(x, y)$, is given by $\Omega_G(f(z)) = \Phi(u(x, y), v(x, y))+\imath\Psi(u(x, y), v(x, y))$.*

Proof. We have by Definition 1.2.6 that, in G,

$$\begin{aligned}\vec{V}(w) &= \vec{\nabla}_w\Phi(u, v)\\ &= \Phi_u(u, v) + \imath\Phi_v(u, v)\\ &= \overline{\frac{d}{dw}\Big(\Omega_G(w)\Big)}\\ &= \overline{\Phi_u(u, v) + \imath\Psi_u(u, v)} = \overline{\Phi_u(u, v) - \imath\Phi_v(u, v)}\end{aligned}$$

because $\Omega_G(w)$ is the complex potential in G.

Similarly, we must show that, in D,

$$\begin{aligned}\vec{V}(z) &= \vec{\nabla}_z\Phi(u(x, y), v(x, y))\\ &= \Phi_x(u(x, y), v(x, y)) + \imath\Phi_y(u(x, y), v(x, y))\\ &= \overline{\frac{d}{dz}\Big(\Omega_G(f(z))\Big)},\end{aligned}$$

which is to say that we must verify that $\Omega_G(f(z))$ yields the correct vector field in D, namely that given by $\vec{\nabla}_z\Phi(u(x, y), v(x, y))$. By verifying this, we will have demonstrated that $\Omega_D(z) := \Omega_G(f(z))$ is the complex potential in D, which will complete the proof.

On the one hand, the vector field is

$$\begin{aligned}\vec{\nabla}_z\Phi(u(x, y), v(x, y)) &= \Phi_x(u(x, y), v(x, y)) + \imath\Phi_y(u(x, y), v(x, y))\\ &= \big(\Phi_u(u, v)u_x(x, y) + \Phi_v(u, v)v_x(x, y)\big)\\ &\quad +\imath\big(\Phi_u(u, v)u_y(x, y) + \Phi_v(u, v)v_y(x, y)\big)\end{aligned}$$

by an application of the chain rule.

On the other hand, the conjugate of the derivative with respect to z of the alleged complex potential in D, namely $\Omega_G(f(z))$, is

$$\begin{aligned}
\overline{\frac{d}{dz}\Big(\Omega_G(f(z))\Big)} &= \overline{\frac{d}{dw}\Big(\Omega_G(w)\Big)\frac{df(z)}{dz}} \\
&= \overline{\big(\Phi_u + \imath\Psi_u\big)\big(u_x + \imath v_x\big)} \\
&= \big(\Phi_u + \imath\Phi_v\big)\big(u_x - \imath v_x\big) \\
&= \big(\Phi_u u_x + \Phi_v v_x\big) + \imath\big(-\Phi_u v_x + \Phi_v u_x\big) \\
&= \big(\Phi_u u_x + \Phi_v v_x\big) + \imath\big(\Phi_u u_y + \Phi_v v_y\big)
\end{aligned}$$

by an application of our knowledge of $\Omega_G(w)$ and of the Cauchy-Riemann Equations. But this is exactly the same as the vector field previously calculated, just with the function arguments suppressed. We conclude that $\Omega_G(f(z))$ is the complex potential in D. $\square$

Corollary 2.1.10 and Theorem 2.1.11 are often used in combination to determine the complex potential associated with a specific physical system in a domain D. The unknown complex potential of interest in D is calculated by determining some sequence of conformal mappings whose composition maps D to G, in accordance with Corollary 2.1.10, where G is a domain in which the complex potential corresponding to that associated with the physical system of interest is easily calculated. This solution can then be taken back to the initial problem of interest by an application of Theorem 2.1.11. As the complex potential is a way of storing knowledge of both the scalar potential ϕ and the vector field $\vec{V}$ associated with a given physical problem, this method is useful in the study of many problems in two-dimensional vector analysis.

2.2 Specific Mappings

We are now in a position to investigate applications of conformal mappings to determining two-dimensional harmonic vector fields. In order to pursue these applications, however, we must first articulate several common conformal mappings that can be used as tools in this process. Each of these functions can be used in any number of different situations, often in combination, in order to determine the specific conformal mapping that is needed for the problem at hand. The myriad combinations and ways in which these mappings can interact make a clear, brief, and complete exposition impossible. As it is often most enlightening to experiment with the various aspects of each mapping independently, we opt for clarity and brevity here, leaving personal verification of the various theorems and assertions that follow, as well as the discovery of other possibilities, to the reader.

Although some exposition regarding the important qualities of each mapping is provided to give the reader a sense of the essential character of each mapping, this is by no means exhaustive. Additionally, only a few of the most common and useful mappings are provided below for reference and as an introduction to the beginner. If the reader desires a more thorough approach, there are many excellent sources that go over this material, such as [Bri04], [Bri08], [BW07], [MH06], [McG00], and [Nee97].

Theorem 2.2.1. *Given any domain D in the z-plane, the analytic function*

$$f(z) = R_0 e^{\imath\theta_0} z + z_0, \tag{2.2.1}$$

where $R_0 > 0$ and z_0 is any complex number, maps D conformally onto the domain G, where G is obtained from D by dilation about the origin by a factor of R_0, clockwise rotation about the origin by an angle θ_0, and translation in the complex plane by z_0. Such a mapping is a linear mapping.

Linear mappings are typically used to generalize other results to arbitrary locations and orientations in the plane, as was done in Corollary 1.1.7 using the base case provided in Theorem 1.1.6. A linear mapping simply takes a domain and moves it as a rigid shape to a new location, and often orientation, in the plane.

Theorem 2.2.2. *Given any positive integer n, the analytic function*

$$f(z) = z^n \tag{2.2.2}$$

maps $D := \{z = re^{\imath Arg(z)}; r > 0, -\frac{\pi}{n} < Arg(z) < \frac{\pi}{n}\}$ conformally onto $G := \{w = \rho e^{\imath Arg(w)}; \rho > 0, Arg(w) \neq \pi\}$.

The nth power mapping takes each point in a domain to a new point that has a modulus that is the original modulus taken to the nth power and that has an argument that is n times the original argument. Such mappings are useful in taking sectors of the plane to larger sectors of the plane, particularly in taking a sector of the plane located in Quadrant I to the upper half plane.

Theorem 2.2.3. *The analytic function*

$$f(z) = \frac{1}{z} \tag{2.2.3}$$

maps $D := \{z = re^{\imath arg(z)}; 0 < r < 1\}$ *conformally onto* $G := \{w = \rho e^{\imath arg(w)}; \rho > 1\}$.

The reciprocal mapping takes each point in a domain to a new point that has modulus that is the multiplicative inverse of the original modulus and that has an argument that is the additive inverse of the original argument. It can be thought of as an inversion with respect the unit circle, which is mapped onto itself. It can be used to take those points of modulus less than 1 to those points of modulus greater than 1.

Theorem 2.2.4. *Given any complex constants* a, b, c, *and* d *for which* $ad \neq bc$, *the analytic function*

$$f(z) = \frac{az+b}{cz+d}, \tag{2.2.4}$$

known as a Möbius transformation, maps any domain that is bounded by a circle or a line onto another domain that is bounded by a circle or a line.

One of the most common Möbius transformations is one that takes the unit disk onto the upper half plane. Other uses, however, abound, as any disk can be mapped onto any other disk using an appropriate Möbius transformation. Considering how portions of disks will map under such transformations opens up even more possibilities.

Möbius transformations are an incredibly important and widely studied class of transformations. In order to make use of them as conformal mappings, however, a technique to determine how to construct a specific Möbius transformation to fit a given situation must be developed. This is provided by the following theorem.

Theorem 2.2.5. *There exists a unique Möbius transformation that maps any three distinct points* z_1, z_2, z_3 *onto any three distinct points* w_1, w_2, w_3, *respectively. An implicit formula for this mapping is given by*

$$\frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1} = \frac{w-w_1}{w-w_3}\frac{w_2-w_3}{w_2-w_1}. \tag{2.2.5}$$

This implicit formula is incredibly useful in finding a Möbius transformation to fit a specific situation. As any three points in the complex plane will determine either a line, if they are collinear, or a circle, if they are not collinear, the points z_1, z_2, and z_3 determine the shape of the domain that is to be mapped. By considering the points in their natural order, traversing the curve that they determine from z_1 to z_2 to z_3, an orientation can be imposed on the circle or line. The same is true for w_1, w_2, and w_3. Thus, by choosing these six points such that the region to the left of the oriented curve $z_1 z_2 z_3$ is the desired domain D and the region to the left of the oriented curve $w_1 w_2 w_3$ is the desired domain G, the implicit formula given by equation 2.2.5 provides a conformal mapping from the domain D onto the domain G. This technique is illustrated in the following problem.

Problem 2.2.6. *What is a conformal mapping from the unit disk to the upper half plane?*

Solution. We begin by choosing three points on the unit circle in counterclockwise order and three points on the x-axis from left to right. These will be used in Equation 2.2.5 to ensure that the unit disk will be mapped onto the upper half plane by the resulting Möbius transformation. Many different choices are possible, but our particular choices will make the calculations as straightforward as possible.

So let $z_1 = -1$ go to $w_1 = -1$, $z_2 = -\imath$ go to $w_2 = 0$, and $z_3 = 1$ go to $w_3 = 1$. These choices lead to the equation

$$\begin{aligned} \frac{z+1}{z-1} \cdot \frac{-\imath-1}{-\imath+1} &= \frac{w+1}{w-1} \cdot \frac{0-1}{0+1} \\ \frac{1+\imath}{1-\imath} \cdot \frac{z+1}{z-1} &= \frac{w+1}{w-1} \end{aligned}$$

We proceed to isolate w in order to determine the explicit form of the desired conformal mapping.

$$\begin{aligned} \imath\frac{z+1}{z-1} &= \frac{w+1}{w-1} \\ (\imath z+\imath)(w-1) &= (w+1)(z-1) \\ \imath zw + \imath w - \imath z - \imath &= wz + z - w - 1 \\ \imath zw + \imath w - wz + w &= \imath z + \imath + z - 1 \\ w(-(1-\imath)z + (1+\imath)) &= (1+\imath)z - (1-\imath) \\ w &= \frac{(1+\imath)z - (1-\imath)}{-(1-\imath)z + (1+\imath)} \end{aligned}$$

We are left with our solution, namely that a conformal mapping that takes the unit disk onto the upper half plane is

$$w = f(z) = \frac{\imath z - 1}{-z + \imath}. \tag{2.2.6}$$

Theorem 2.2.7. *The analytic function*

$$f(z) = exp(z) \tag{2.2.7}$$

maps $D_\alpha := \{z = x + \imath y; \alpha < y < \alpha + 2\pi\}$ *conformally onto* $G_\alpha := \{w = \rho e^{\imath Arg_\alpha(w)}; \rho > 0, Arg_\alpha(w) \neq \alpha + 2\pi\}$ *for any real number* α.

The exponential function maps infinite horizontal strips in the plane onto sectors of the plane. Of particular use and importance is that it takes the infinite horizontal strip bordered by 0 and $\imath\pi$ onto the upper half plane.

Theorem 2.2.8. *The analytic function*

$$f(z) = Log_\alpha(z) \tag{2.2.8}$$

maps $D_\alpha := \{z = re^{\imath Arg_\alpha(z)}; r > 0, Arg_\alpha(z) \neq \alpha + 2\pi\}$ *conformally onto* $G_\alpha := \{w = u + \imath v; \alpha < v < \alpha + 2\pi\}$ *for any real number* α.

The α-branch of the logarithm function is an inverse of the exponential function for every real number α. It maps sectors of the plane onto infinite horizontal strips in the plane. Of particular use and importance is that the principal logarithm takes the upper half plane onto the infinite horizontal strip bordered by 0 and $\imath\pi$.

Theorem 2.2.9 (Schwarz-Christoffel)**.** *Let G be a domain in the w-plane bounded by straight lines and line segments and specified by the n points $w_1, w_2, \ldots, w_n$, all finite except possibly w_n. If the region thus defined has exterior angle $-\pi < \alpha_k < \pi$ at the vertex w_k for each positive integer $k \leq n-1$, then there exist points $x_1, x_2, \ldots, x_{n-1}$ and complex constants A and B such that the function*

$$f(z) = B + A\int (z - x_1)^{\frac{-\alpha_1}{\pi}}(z - x_2)^{\frac{-\alpha_2}{\pi}} \cdots (z - x_{n-1})^{\frac{-\alpha_{n-1}}{\pi}} dz \tag{2.2.9}$$

is analytic with derivative

$$f'(z) = A(z - x_1)^{\frac{-\alpha_1}{\pi}}(z - x_2)^{\frac{-\alpha_2}{\pi}} \cdots (z - x_{n-1})^{\frac{-\alpha_{n-1}}{\pi}}, \tag{2.2.10}$$

satisfies $w_n = f(\infty)$, and maps the upper half plane $D := \{z = x + \imath y; y > 0\}$ conformally onto G.

Schwarz-Christoffel transformations are a particularly general class of mappings. There is a Schwarz-Christoffel transformation, for example, that maps the upper half plane onto any polygon whatsoever. Although the computations involved in calculating such a Schwarz-Christoffel transformation are usually rather laborious, and we will thus not use any such mappings in this paper, it would be derelict to omit such a powerful theorem. Recent advances in generalizations of Schwarz-Christoffel mappings have made it possible to numerically determine conformal mappings involving multiply connected domains, which significantly broadens the applicability of conformal mappings in solving problems in two-dimensional harmonic field theory. Further information relating to these advances can be found in [Cas08] and a development of introductory Schwarz-Christoffel mappings can be found in [MH06].

2.3 Determining Vector Fields

Using these conformal mappings, we can now proceed to use Theorem 2.1.11 to solve various problems in two-dimensional field theory. By progressing from simple problems that follow easily from our earlier work with vector fields, we will find that conformal mappings enable us to extend these basic solutions to solve more complicated problems.

Theorem 2.3.1. *The complex potential $\Omega(z)$ of the vector field produced by n generalized sources of strengths $S_1, S_2, \ldots, S_n$ located at $z_1, z_2, \ldots, z_n$, respectively, in the complex plane, is*

$$\Omega(z) = \sum_{k=1}^{n} \frac{S_k}{2\pi} Log(z - z_k). \tag{2.3.1}$$

Proof. Theorem 1.1.11 states that the vector field produced by such an arrangement is given by Equation 1.1.5:

$$\vec{V}(z) = \sum_{k=1}^{n} \frac{S_k}{2\pi(\bar{z} - \bar{z}_k)}.$$

Taking the complex conjugate of this equation and then integrating the result, we arrive at the desired complex potential:

$$\Omega(z) = \sum_{k=1}^{n} \frac{S_k}{2\pi} Log(z - z_k).$$

□

Problem 2.3.2. *What is the complex potential of generalized sources of strengths* $2\pi, 4\pi, -3\pi$, *and* $-\pi$ *located at* $1, -1-2\imath, -2+3\imath$, *and* 0, *respectively, in the complex plane?*

Solution. The solution is given by Equation 2.3.1 with an appropriate substitution of parameters:

$$\Omega(z) = Log(z-1) + 2Log(z-(-1-2\imath)) - \frac{3}{2}Log(z-(-2+3\imath)) - \frac{1}{2}Log(z).$$

A contour plot of this vector field is provided in Figure 2.1.

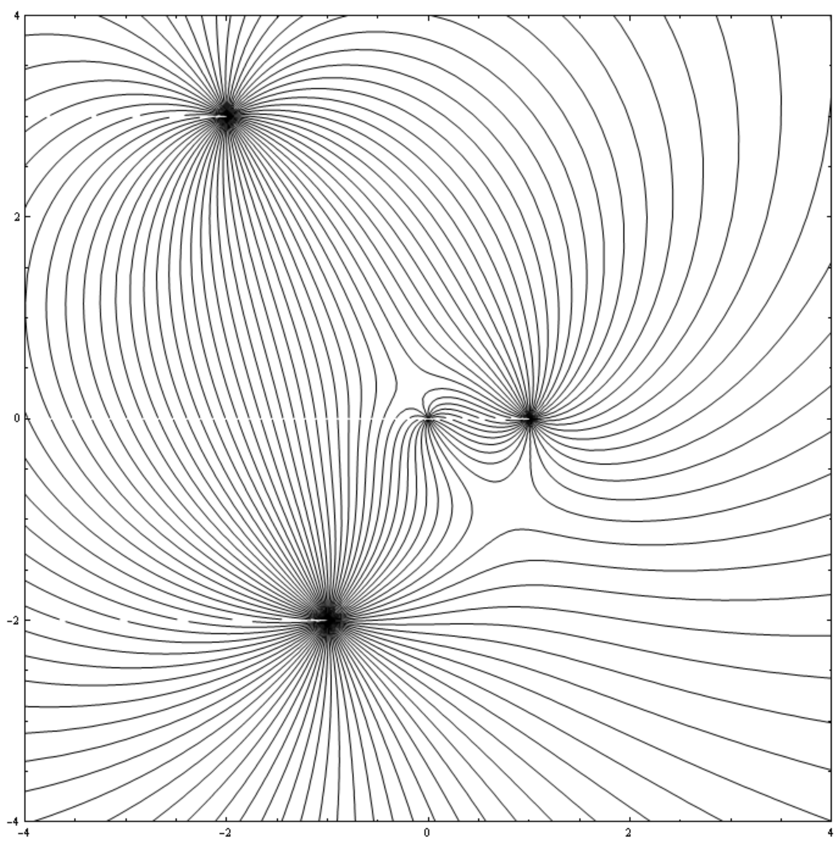

Figure 2.1: The flow of the vector field produced by the generalized sources of Problem 2.3.2.

Determining the complex potential of sources located in the full complex plane is straightforward, but not immediately applicable to more general problems. The difficulty is that finding a conformal mapping from a given domain to the full complex plane is not usually possible. It is, however, possible to find a conformal mapping from any simply connected domain to the upper half plane. This follows from Equation 2.2.6 and a deep result known as the Riemann Mapping Theorem.

Theorem 2.3.3 (Riemann Mapping)**.** *If D is a simply connected domain, then there exists a conformal mapping from D to the unit disk.*

The proof of Theorem 2.3.3 is beyond the scope of this paper, but its implications are relevant to our present purpose. By using the upper half plane as a standard domain for which the complex potential of any collection of sources is known, we can determine other complex potentials by mapping them onto the upper half plane, something that we know must be possible if the problem is stated in terms of a simply connected domain.

The only difficulty lies in determining the complex potential of any collection of sources in the upper half plane, a case in which the x-axis serves as a wall through which no field lines may flow. This statement of the behavior of the x-axis, however, is completely consistent with the behavior of a streamline: no field lines flow through a streamline because streamlines of a flow never intersect. If two streamlines were to intersect, the vector field would not be well-defined at the point of intersection as it would have two distinct values, one for each streamline passing through the point. This would contradict the fact that the vector field is by its very nature a function, assigning only a single vector to each point in the plane.

This notion that the x-axis is nothing more than a streamline of the flow enables us to invoke the principle of superposition to determine the complex potential of any arrangement of generalized sources in the upper half plane.

Theorem 2.3.4. *The complex potential $\Omega(z)$ of a generalized source of strength S_0 located at $z_0 = x_0$, where x_0 is real, on the boundary of the upper half plane, is*

$$\Omega(z) = \frac{S_0}{2\pi} Log(z - x_0). \tag{2.3.2}$$

Proof. As the generalized source lies on the x-axis, the x-axis is a natural streamline of the flow it produces. We can thus simply invoke Theorem 2.3.1. □

Theorem 2.3.5. *The complex potential $\Omega(z)$ of a generalized source of strength S_0 located at $z_0 = x_0 + \imath y_0$, where $y_0 \neq 0$, in the upper half plane is*

$$\Omega(z) = \frac{S_0}{2\pi} Log(z^2 - 2Re[z_0]z + |z_0|^2). \tag{2.3.3}$$

Proof. In order to make the x-axis a streamline of the flow, we need to perfectly balance out the vertical contributions of the generalized source at z_0 along the x-axis. We can do this by introducing an image source at $\overline{z_0}$ of equal strength and applying the principle of superposition and Theorem 2.3.1 to these two sources. This leads to

$$\begin{aligned}\Omega(z) &= \frac{S_0}{2\pi}(Log(z-z_0)+Log(z-\overline{z_0}))\\ &= \frac{S_0}{2\pi}Log((z-z_0)(z-\overline{z_0}))\\ &= \frac{S_0}{2\pi}Log(z^2-z(z_0+\overline{z_0})+|z_0|^2)\\ &= \frac{S_0}{2\pi}Log(z^2-2Re[z_0]z+|z_0|^2),\end{aligned}$$

which completes the proof. □

Problem 2.3.6. *What is the complex potential of generalized sources of strengths* -2π *and* 2π *located at* 0 *and* $\imath$*, respectively, in the upper half plane?*

Solution. The principle of superposition makes this problem a simple matter of applying Theorem 2.3.4 in combination with Theorem 2.3.5 for the specific case at hand. We thus arrive at the complex potential

$$\Omega(z) = Log(z^2+1) - Log(z).$$

A contour plot of this vector field is provided in Figure 2.2.

It is important to note that although the source strength of the on-axis sink and off-axis source in Problem 2.3.6 were apparently of equal strength, the flow diagram shown in Figure 2.2 clearly indicates that the outflow from the off-axis source dominated the inflow of the on-axis sink. The difficulty is that the strength of a source, as it is presently defined, is an intrinsic strength. The flux of field lines outwards from an on-axis generalized source is into both the upper half plane and the lower half plane. Thus the effective strength in the upper half plane of an on-axis generalized source is only half of its intrinsic strength. This leads us to the following definition.

Definition 2.3.7. The *effective strength* of a source in a domain D is the flux of field lines into D.

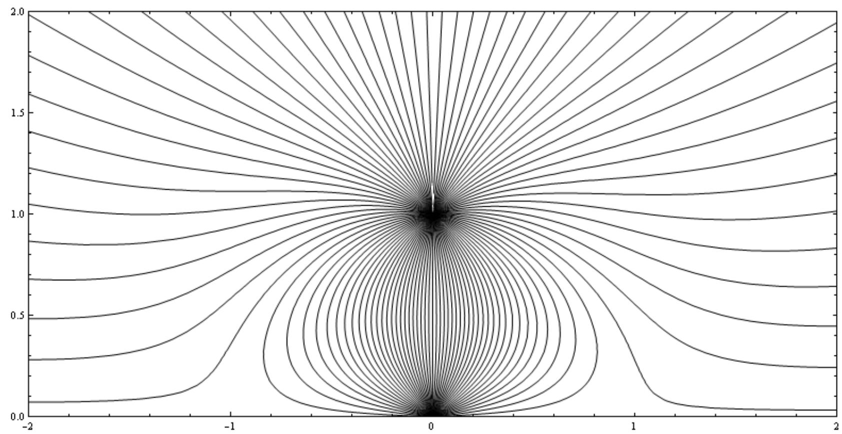

Figure 2.2: The flow of the vector field produced by the generalized sources of Problem 2.3.6.

This concept of effective strength is clear and easy to apply once it has been taken into account, but without it errors would arise when specific problems were considered.

Problem 2.3.8. *What flow pattern is produced when two hoses of equal strength, spaced two feet apart, pump water into a large shallow rectangular basin from a wall along the edge of the basin, and all of the water goes down a drain located one perpendicular foot away from the drain on the right?*

Solution. This problem can be rephrased in terms of finding the complex potential of 3 generalized sources of strengths 2π, 2π, and -2π located at -1,1, and $1+\imath$, respectively. Although the relative effective strengths given in the problem statement indicate that the sink should be twice as strong as either source, a consideration of effective strength enables the correct mathematical statement of the problem just stated to be obtained.

Once this restatement of the problem has been made, the principle of superposition can be used in combination with Theorem 2.3.4 and Theorem 2.3.5 to obtain a quick solution. The complex potential is

$$\Omega(z) = Log(z+1) + Log(z-1) - Log(z^2 - 2z + 2).$$

A contour plot of the vector field is provided in Figure 2.3.

There are other subtle difficulties that arise in applying these techniques to determine specific vector fields. For example, if the domain in which the problem is

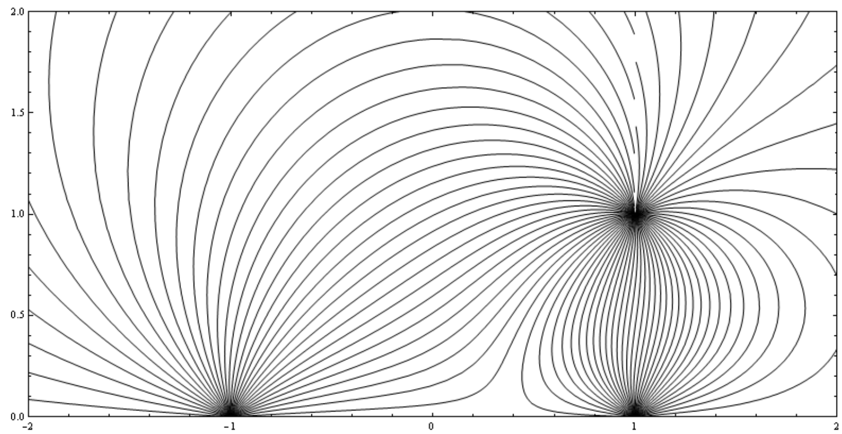

Figure 2.3: The flow of the vector field produced by the generalized sources of Problem 2.3.8.

stated is bounded, then the net overall flow out of the domain must be zero, otherwise it would be impossible for the boundary of this domain to be a streamline of the vector field. This makes our distinction between the intrinsic strength and the effective strength of a source significant, for without taking this into consideration the mathematical problem could not be properly stated.

Problem 2.3.9. *What flow is produced by generalized sources of strengths* 2π *and* -4π *located at* 0 *and* 1*, respectively, if the unit disk serves as a flow boundary?*

Solution. This problem is well-posed as the net effective strength is zero, a fact that we will see more clearly by using the mapping from the unit disk to the upper half plane given in Equation 2.2.6,

$$w = f(z) = \frac{\imath z - 1}{-z + \imath},$$

to find the image of this flow. As it takes the source at 0 to a source at $f(0) = \imath$ and the sink at 1 to a sink at $f(1) = 1$, we can use Theorem 2.3.4 and Theorem 2.3.5 to state the complex potential of the image flow:

$$\Omega(w) = Log(w^2 + 1) - 2Log(w - 1).$$

We can then use this result to find the complex potential of the original problem by invoking Theorem 2.1.11:

$$\Omega(z) = Log\bigg(\Big(\frac{\imath z - 1}{-z + \imath}\Big)^2 + 1\bigg) - 2Log\bigg(\frac{\imath z - 1}{-z + \imath} - 1\bigg).$$

Contour plots of these fields are provided in Figure 2.4 and Figure 2.5.

Another difficulty can arise if the domain is unbounded and the net overall flow is nonzero. In this case, such overflow needs to be explicitly taken into account by considering one or more generalized sources at one or more points infinitely distant from the origin. This is necessary because a point that is infinitely distant from the origin may map under the conformal mapping being used to a finite point. In order to correctly determine the complex potential of the image field such additional generalized sources must be included.

Problem 2.3.10. *What flow is produced by a generalized source of strength* 2π *located at* 0 *if the* x*-axis and the line* $y = \pi$ *serve as boundaries of the flow?*

Solution. This problem statement is deceptive in that the positive net overall flow leads to the existence of two additional generalized sources that must be taken into consideration. Specifically, there are generalized sources of strengths $-\pi$ and $-\pi$ located at $-\infty + \imath\frac{\pi}{2}$ and $\infty + \imath\frac{\pi}{2}$, respectively.

With this modification, the problem can be solved without difficulty by using the mapping $w = f(z) = exp(z)$ in accordance with Theorem 2.2.7. The source at 0 is mapped to a source at 1, the sink at $-\infty + \imath\frac{\pi}{2}$ is mapped to a sink at 0, and the sink at $\infty + \imath\frac{\pi}{2}$ is mapped to a sink at $\imath\infty$. This last sink can be neglected when determining the complex potential of the image flow, but it is essential that the sink at 0 be included if the correct vector field is to be obtained.

We find, using Theorem 2.3.4 and Theorem 2.3.5, that the complex potential of the image flow is

$$\Omega(w) = Log(w - 1) - \frac{1}{2}Log(w).$$

Theorem 2.1.11 then allows us to conclude that the complex potential of the original flow in the channel is

$$\Omega(z) = Log(e^z - 1) - \frac{z}{2}.$$

Contour plots of these fields are provided in Figure 2.6 and Figure 2.7.

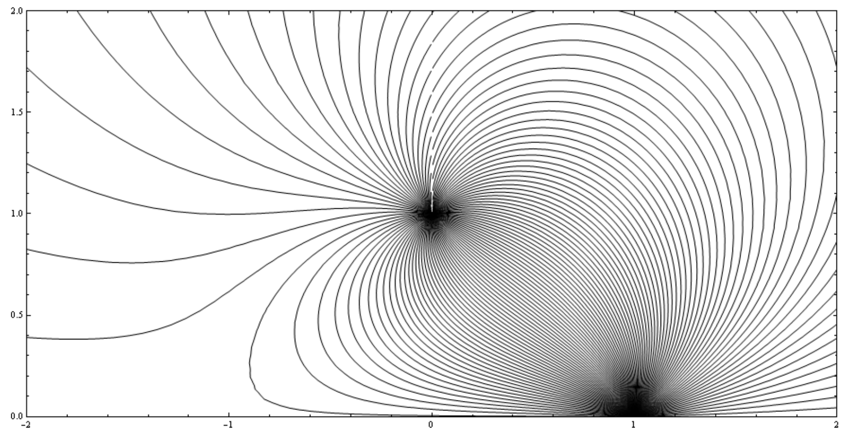

Figure 2.4: The image of the flow of the vector field produced by the generalized sources of Problem 2.3.9.

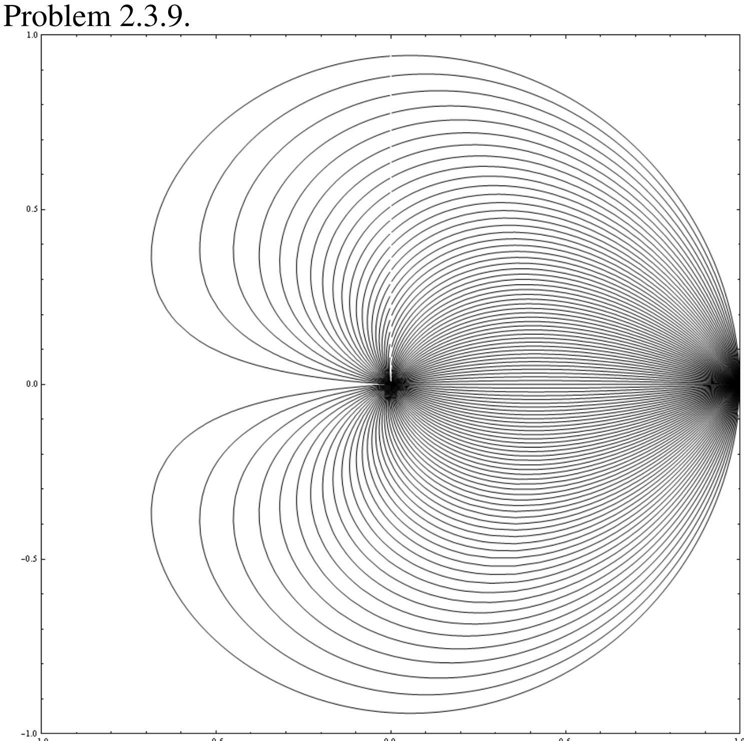

Figure 2.5: The flow of the vector field produced by the generalized sources of Problem 2.3.9.

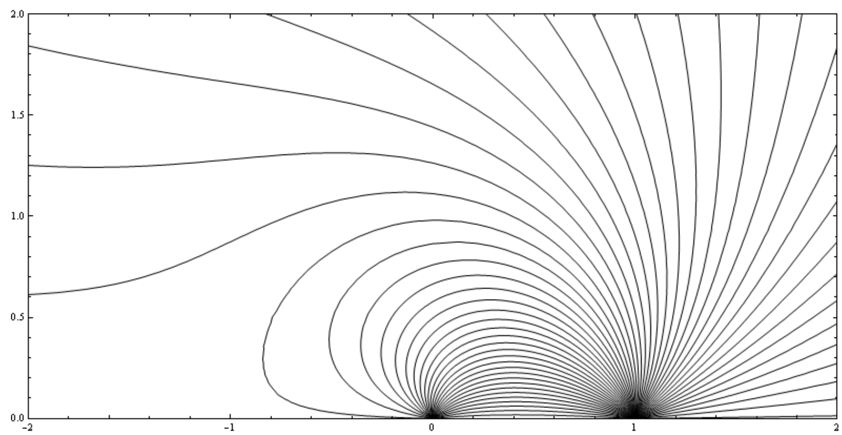

Figure 2.6: The image of the flow of the vector field produced by the generalized sources of Problem 2.3.10.

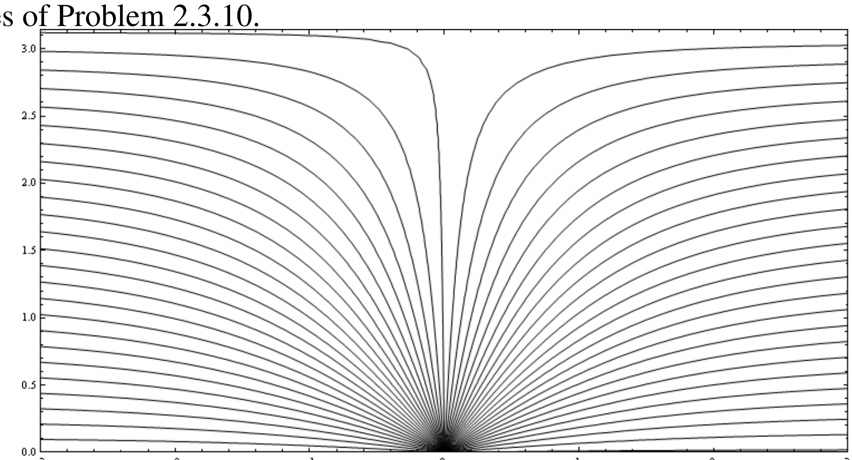

Figure 2.7: The flow of the vector field produced by the generalized sources of Problem 2.3.10.

Chapter 3

Interval Sources

Developing techniques to solve problems that involve generalized sources is useful and fruitful; however, these are far from the only problems to which the techniques we have developed are amenable. Rather than considering points at which field lines flow into a given domain, it is natural to consider the case where field lines emanate from or terminate into a continuous interval. Such interval sources, as we shall call them, have the capacity to represent a far broader class of physical problems than point sources. For example, rather than determining the field produced by wires of uniform linear charge density, as in Theorem 1.1.9, we would be in a position to attack problems involving sheets of uniform surface charge density.

Before such applications are possible, however, the mathematics governing these interval sources and how they behave under conformal mappings must be determined. We strive here to apply the techniques developed thus far to this new problem as much as is possible.

3.1 Along the X-Axis

As a first step, we consider the case where the interval source runs along the x-axis from $x = a$ to $x = b$. By constructing the interval source as the limiting case of a great many point sources, evenly spaced throughout the interval, we determine the complex potential of the interval source.

Theorem 3.1.1. *The complex potential of an interval source of strength S_0 and* *uniform density on* $[a,b]$ *in the complex plane is given by*

$$\Omega(z) = \frac{S_0}{2\pi(b-a)}\Big((z-a)Log(z-a) - (z-b)Log(z-b)\Big). \tag{3.1.1}$$

Proof. Our first task is to partition the interval $[a,b]$. With this purpose in mind, let n be a positive integer, so that

$$x_k := a + \frac{b-a}{n}k,$$

where $0 \le k \le n$. These $n+1$ points are evenly spaced from $x_0 = a$ to $x_n = b$. We want to select one point from each of the n intervals thus determined so that a generalized source can be placed at each of these points. As we then let n increase without bound, our n generalized sources will then become indistinguishable from the interval source whose complex potential we wish to determine.

We thus define these interior points to be

$$\widetilde{x}_k := \frac{x_k + x_{k+1}}{2} = a + \frac{b-a}{n}k + \frac{b-a}{2n},$$

where now $0 \le k \le n-1$.

As we want the final interval source to be of uniform density, we must give each such component source the same strength. For n sources of total strength S_0, this equal strength is $\frac{S_0}{n}$. We thus have by Theorem 1.1.11 that

$$\vec{V}(z) = \overline{\Omega'(z)} = \frac{S_0}{2\pi n}\sum_{k=0}^{n-1}\frac{1}{\overline{z} - \widetilde{x}_k}.$$

Our goal is to turn this sum into a Riemann integral. We do so by further defining

$$\Delta\widetilde{x}_k := \widetilde{x}_{k+1} - \widetilde{x}_k = \frac{b-a}{n},$$

where $0 \le k \le n-2$. We can now use this to replace n in our sum, so that we have

$$\Omega'(z) = \frac{S_0}{2\pi(b-a)}\sum_{k=0}^{n-2}\frac{\Delta\widetilde{x}_k}{z - \widetilde{x}_k} + \frac{S_0}{2\pi n}\cdot\frac{1}{z - \widetilde{x}_{n-1}}.$$

If we now let n increase without bound, the extra term approaches 0 and we are left with the definite integral

$$\Omega'(z) = \frac{S_0}{2\pi(b-a)}\int_a^b \frac{dx}{z-x},$$

from which it follows that the complex potential we are looking for is simply

$$\Omega(z) = \frac{S_0}{2\pi(b-a)} \int \left(\int_a^b \frac{dx}{z-x} \right) dz.$$

Evaluating this double integral in stages, we have that

$$\int_a^b \frac{dx}{z-x} = -Log(z-x)\Big|_a^b = Log(z-a) - Log(z-b),$$

which can then be integrated with respect to z to obtain our desired result:

$$\Omega(z) = \frac{S_0}{2\pi(b-a)} \Big((z-a)Log(z-a) - (z-b)Log(z-b) \Big)$$

□

This result can be combined with previous results regarding point sources to solve a wider variety of problems.

Problem 3.1.2. *What flow is produced by a generalized source of strength -2π located at $\imath$ and an interval source of strength 6π of uniform density on the interval $[-1, 2]$ if the x-axis serves as a boundary of the flow?*

Solution. We have from Theorem 2.3.5 and Theorem 3.1.1 that the complex potential of this configuration is given by

$$\Omega(z) = (z+1)Log(z+1) - (z-2)Log(z-2) - Log(z^2+1).$$

A contour plot of this flow is provided in Figure 3.1.

Theorem 3.1.1, stated in slightly different form by Brilleslyper in [Bri08], is useful as a starting point for interval source analysis, but it does not enable us to deal with the full spectrum of problems that can arise from interval sources. Questions regarding nonuniform source densities, for example, naturally arise when we start to think about how interval sources might map under conformal mappings. If we map an interval source in a domain D to another domain G, how does the mapping affect the density of the image interval source? This question cannot be answered with mathematical rigor and detail until we first develop an understanding of nonuniform interval sources in simpler settings. We begin this work by determining the complex potential of an interval source on $[a, b]$ with a normalized relative density function $\lambda(x)$.

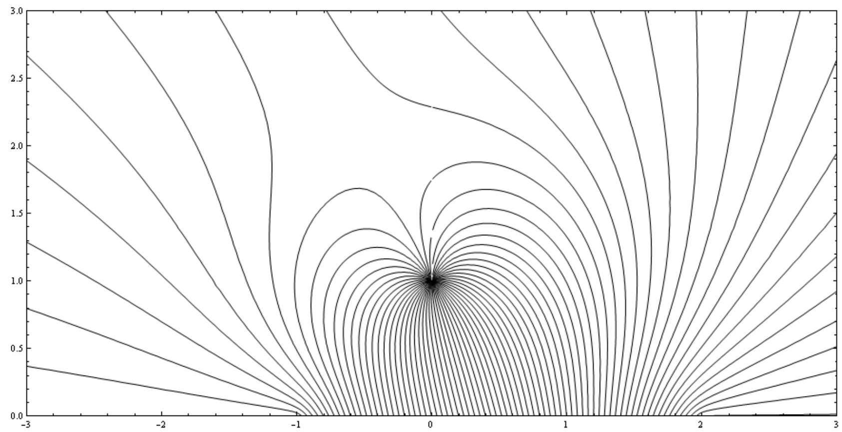

Figure 3.1: The flow of the vector field produced by the configuration of Problem 3.1.2.

Theorem 3.1.3. *The complex potential of an interval source on $[a, b]$ in the complex plane is given by*

$$\Omega(z) = \frac{S_0}{2\pi(b-a)} \int \left(\int_a^b \frac{\lambda(x)}{z-x} dx \right) dz, \tag{3.1.2}$$

where $\lambda(x)$ is a normalized relative density satisfying

$$\int_a^b \lambda(x) dx = b - a. \tag{3.1.3}$$

Proof. As in the proof of Theorem 3.1.1, we begin by partitioning the interval (a, b). With this purpose in mind, let n be a positive integer, so that

$$x_k := a + \frac{b-a}{n} k,$$

where $0 \leq k \leq n$. These $n+1$ points are evenly spaced from $x_0 = a$ to $x_n = b$. We want to select one point from each of the n intervals thus determined so that a generalized source can be placed at each of these points. As we then let n increase without bound, our n generalized sources will then become indistinguishable from the interval source whose complex potential we wish to determine.

We thus define these interior points to be

$$\widetilde{x}_k := \frac{x_k + x_{k+1}}{2} = a + \frac{b-a}{n} k + \frac{b-a}{2n},$$

where now $0 \le k \le n-1$.

As we want the final interval source to have density $\lambda(x)$, we must give each such component source a weighted strength based on the value of λ at that specific point. For n sources of total strength S_0, the mean strength of each generalized source is $\frac{S_0}{n}$. We weight each such point in accordance with the value of λ at that point by giving the generalized source at point $\widetilde{x}_k$ strength $\frac{S_0}{n}\lambda(\widetilde{x}_k)$. This weighting scheme makes the density λ a density relative to the mean point strength $\frac{S_0}{n}$. Regardless of the specific relative density λ, however, we must still have the specified total interval strength S_0. This imposes the normalization condition that

$$\sum_{k=0}^{n-1} \frac{S_0}{n}\lambda(\widetilde{x}_k) = S_0.$$

Returning for a moment to the vector field, we have by Theorem 1.1.11 that

$$\vec{V}(z) = \overline{\Omega'(z)} = \frac{S_0}{2\pi n}\sum_{k=0}^{n-1} \frac{\lambda(\widetilde{x}_k)}{\overline{z} - \widetilde{x}_k}.$$

Our goal is to turn this sum into a Riemann integral. We do so by further defining

$$\Delta\widetilde{x}_k := \widetilde{x}_{k+1} - \widetilde{x}_k = \frac{b-a}{n},$$

where $0 \le k \le n-2$. We can now use this to replace n in our sum, so that we have

$$\Omega'(z) = \frac{S_0}{2\pi(b-a)}\sum_{k=0}^{n-2} \frac{\lambda(\widetilde{x}_k)\Delta\widetilde{x}_k}{z - \widetilde{x}_k} + \frac{S_0}{2\pi n}\cdot\frac{\lambda(\widetilde{x}_{n-1})}{z - \widetilde{x}_{n-1}}.$$

If we now let n increase without bound, the extra term approaches 0 and we are left with the definite integral

$$\Omega'(z) = \frac{S_0}{2\pi(b-a)}\int_a^b \frac{\lambda(x)dx}{z-x},$$

from which it follows that the complex potential we are looking for is simply

$$\Omega(z) = \frac{S_0}{2\pi(b-a)}\int \left(\int_a^b \frac{\lambda(x)dx}{z-x}\right)dz.$$

This is our formula for the complex potential of an interval source on $[a, b]$ with relative density $\lambda(x)$, but we cannot neglect the normalization condition articulated earlier. Applying the same substitution for n as with the complex potential and taking the limit as n approaches infinity, we arrive at the final normalization condition that

$$\int_a^b \lambda(x)dx = b - a.$$

□

Such a formula for a general density is interesting from a theoretical standpoint, and it is clear that Equation 3.1.2 can be used to compute the complex potential of a specific density, just as was done for $\lambda(x) = 1$ in Theorem 3.1.1 and Problem 3.1.2, but a nonuniform specific case would be useful in answering some questions regarding the specific way in which such general density interval sources can behave. For example, there is nothing to prevent $\lambda(x)$ from taking on negative values at some points on a given interval, so long as the normalization condition given in Equation 3.1.3 is still met. Is this possibility indicative of an error in the proof of Theorem 3.1.3? If not, what would such a situation look like?

We address these issues by computing the complex potential for an interval source on $[a, b]$ of density $\lambda(x) = Ax$, where A is a normalization constant, and illsutrating the result with a sample problem.

Theorem 3.1.4. *The complex potential of an interval source of strength S_0 and linear density $\lambda(x) = Ax$ on (a, b) in the complex plane is given by*

$$\Omega(z) = \frac{S_0}{2\pi(b^2 - a^2)}\Big((z^2 - a^2)Log(z - a) - (z^2 - b^2)Log(z - b) - (b - a)z\Big). \tag{3.1.4}$$

Proof. We must first determine the value of the constant A using the normalization condition, Equation 3.1.3. This leads to

$$\int_a^b Axdx = \frac{A}{2}(b^2 - a^2) = b - a,$$

from which it follows that

$$A = \frac{2}{b + a}.$$

This now enables us to set $\lambda(x) = \frac{2}{b+a}x$ in Equation 3.1.2 to state the complex potential in terms of a double integral as

$$\Omega(z) = \frac{S_0}{\pi(b^2 - a^2)} \int \left(\int_a^b \frac{x}{z - x} dx \right) dz.$$

Now we only have to calculate this integral. The definite integral with respect to x is

$$\begin{aligned}
\int_a^b \frac{x}{z - x} dx &= \int_a^b \left(-1 + \frac{z}{z - x} \right) dx \\
&= -x - zLog(z - x) \Big|_a^b \\
&= -(b - a) + zLog(z - a) - zLog(z - b).
\end{aligned}$$

This must now be integrated with respect to z. As the first term is just a constant and the integral of the third term follows immediately from that of the second term, the only real calculation to be done is in determining the integral of the second term. This can be done as follows.

$$\begin{aligned}
\int zLog(z - a)dz &= \int \Big((z - a)Log(z - a) + aLog(z - a) \Big) dz \\
&= \frac{(z - a)^2}{2} Log(z - a) - \frac{(z - a)^2}{4} \\
&\quad + a(z - a)Log(z - a) - az.
\end{aligned}$$

When this is used to determine the desired integral, we find that the result simplifies to

$$\frac{1}{2}\Big((z^2 - a^2)Log(z - a) - (z^2 - b^2)Log(z - b) - (b - a)z \Big),$$

where an additional constant summand of $\frac{1}{4}(b^2 - a^2)$ has been discarded for convenience because adjusting a complex potential by any complex constant does not affect the vector field that it produces.

If we now multiply this result through by the constant coefficient that we set aside during our intermediate calculations, we arrive at the desired result:

$$\Omega(z) = \frac{S_0}{2\pi(b^2 - a^2)} \Big((z^2 - a^2)Log(z - a) - (z^2 - b^2)Log(z - b) - (b - a)z \Big).$$

□

Problem 3.1.5. *What flow is produced by an interval source of strength 6π of linear density $\lambda(x) = Ax$ on the interval $[-1, 2]$ if the x-axis serves as a boundary of the flow?*

Solution. We have from Theorem 3.1.4 that the complex potential for this flow is

$$\Omega(z) = (z^2 - 1)Log(z + 1) - (z^2 - 4)Log(z - 2) - 3z. \tag{3.1.5}$$

A contour plot of this flow is provided in Figure 3.2.

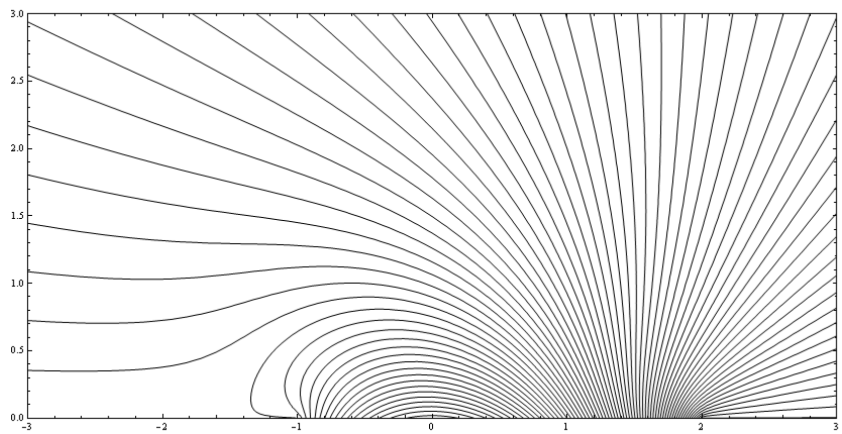

Figure 3.2: The flow of the vector field produced by the configuration of Problem 3.1.5.

Figure 3.2 is illustrative of how a nonuniform interval source can behave. The density is negative on $[-1, 0)$, zero at 0, and positive on $(0, 2]$. This results in a swirling effect, but the overall flow is still outward, in accordance with the positive total strength. By allowing the density to take on negative values on the interval, we can calculate a single complex potential that gives us the desired behavior, rather than breaking the problem into several interval sources, each of solely positive or negative density.

3.2 Along Simple Contours

In order to be able to apply these techniques more generally using conformal mappings, the method of images, and the principle of superposition, we must determine the complex potential of an interval source along any simple contour in the complex plane. We can then proceed to find how conformal mappings affect the complex potential of such an interval source, thereby completing our generalization from point sources to interval sources, at least theoretically. Computing specific cases for these more complicated examples is more difficult and will not be done here, but the success in calculating the complex potential of an interval source on $[a, b]$ with linear density, particularly as illustrated in Figure 3.2, offers some credence to these new theoretical developments in the absence of verification through the solution of specific problems.

We begin by determining the complex potential of an interval source along a simple contour.

Theorem 3.2.1. *The complex potential of an interval source in the complex plane along the simple parameterized contour $C(t)$, where t runs from t_0 to T, is*

$$\Omega(z) = \frac{S_0}{2\pi(T - t_0)} \int \left(\int_{t_0}^{T} \frac{\lambda(t)}{z - C(t)} dt \right) dz, \tag{3.2.1}$$

where $\lambda(t)$ is a normalized relative density function satisfying

$$\int_{t_0}^{T} \lambda(t) dt = T - t_0. \tag{3.2.2}$$

Proof. We begin by parameterizing the simple contour along which the interval source is to lie as $C(t) = x(t) + \imath y(t)$ for $t \in [t_0, T]$.

The proof now proceeds similarly to that of Theorem 3.2.1. We must partition the interval $[t_0, T]$. With this purpose in mind, let n be a positive integer, so that

$$t_k := t_0 + \frac{T - t_0}{n} k,$$

where $0 \le k \le n$. These $n + 1$ points are evenly spaced from t_0 to $t_n = T$ and correspond to $n + 1$ points that run along the contour $C(t)$ from $C(t_0)$ to $C(T)$. We want to select one point from each of the n segments of contour thus determined so that a generalized source can be placed at each of these points. As we then let n increase without bound, our n generalized sources will then become indistinguishable from the interval source whose complex potential we wish to determine.

We thus define the interior parameter values

$$\widetilde{t}_k := \frac{t_k + t_{k+1}}{2} = t_0 + \frac{T - t_0}{n} k + \frac{T - t_0}{2n},$$

where now $0 \leq k \leq n-1$, each of which corresponds to a point source located at $C(\widetilde{t}_k)$.

As we want the final interval source to have density $\lambda(t)$, we must give each such component source a weighted strength based on the value of λ at that specific point. For n sources of total strength S_0, the mean strength of each generalized source is $\frac{S_0}{n}$. We weight each such point in accordance with the value of λ corresponding to that point by giving the generalized source at point $C(\widetilde{t}_k)$ strength $\frac{S_0}{n}\lambda(\widetilde{t}_k)$. This weighting scheme makes the density λ a density relative to the mean point strength $\frac{S_0}{n}$. Regardless of the specific relative density λ, however, we must still have the specified total interval strength S_0. This imposes the normalization condition that

$$\sum_{k=0}^{n-1} \frac{S_0}{n}\lambda(\widetilde{t}_k) = S_0.$$

Returning for a moment to the vector field, we have by Theorem 1.1.11 that

$$\vec{V}(z) = \overline{\Omega'(z)} = \frac{S_0}{2\pi n}\sum_{k=0}^{n-1} \frac{\lambda(\widetilde{t}_k)}{\overline{z} - C(\widetilde{t}_k)}.$$

Our goal is to turn this sum into a Riemann integral. We do so by further defining

$$\Delta\widetilde{t}_k := \widetilde{t}_{k+1} - \widetilde{t}_k = \frac{T - t_0}{n},$$

where $0 \leq k \leq n-2$. We can now use this to replace n in our sum, so that we have

$$\Omega'(z) = \frac{S_0}{2\pi(T - t_0)}\sum_{k=0}^{n-2} \frac{\lambda(\widetilde{t}_k)\Delta\widetilde{t}_k}{z - C(\widetilde{t}_k)} + \frac{S_0}{2\pi n}\cdot\frac{\lambda(\widetilde{t}_{n-1})}{z - C(\widetilde{t}_{n-1})}.$$

If we now let n increase without bound, the extra term approaches 0 and we are left with the definite integral

$$\Omega'(z) = \frac{S_0}{2\pi(T - t_0)}\int_{t_0}^{T} \frac{\lambda(t)dt}{z - C(t)},$$

from which it follows that the complex potential we are looking for is simply

$$\Omega(z) = \frac{S_0}{2\pi(T - t_0)}\int\left(\int_{t_0}^{T} \frac{\lambda(t)dt}{z - C(t)}\right)dz.$$

This is our formula for the complex potential of an interval source on $C(t)$ for $t \in [t_0, T]$ with relative density $\lambda(t)$, but we cannot neglect the normalization condition articulated earlier. Applying the same substitution for n as with the complex potential and taking the limit as n approaches infinity, we arrive at the final normalization condition that

$$\int_{t_0}^{T} \lambda(t)dt = T - t_0.$$

□

As with point sources, we are primarily concerned with determining the complex potential of interval sources located in the upper half plane because we can find a conformal mapping from any simply connected domain that is not the entire complex plane to the upper half plane. Theorem 3.1.3 handles the case where the contour lies completely on the x-axis, and now Theorem 3.2.1 allows us to find the complex potential generated in the upper half plane by any contour that never touches the x-axis, as stated in the following corollary.

Corollary 3.2.2. *The complex potential of an interval source in the upper half plane along the simple parameterized contour $C(t)$, where t runs from t_0 to T, is*

$$\Omega(z) = \frac{S_0}{2\pi(T - t_0)} \int \left(\int_{t_0}^{T} \left(\frac{\lambda(t)}{z - C(t)} + \frac{\lambda(t)}{z - \overline{C(t)}} \right) dt \right) dz, \tag{3.2.3}$$

where $\lambda(t)$ is a normalized relative density function satisfying

$$\int_{t_0}^{T} \lambda(t)dt = T - t_0. \tag{3.2.4}$$

Proof. As the reflection across the x-axis of the contour $C(t)$ is the contour $\overline{C(t)}$, which must have the same density $\lambda(t)$ as $C(t)$ in order to serve as an image interval source, the result follows immediately from an application of the method of images and Theorem 3.2.1 to the problem being considered. □

Our generalization from point sources to interval sources is almost complete. All that remains is to determine the complex potential of an interval source in a simply connected domain D that must be taken to the upper half plane by a conformal mapping. This is done by first considering the case where the image of the contour lies along the x-axis. We can then use Theorem 3.1.3 to solve this case in the w-plane and bring it back to the z-plane using the conformal mapping, just as with point sources. In the case where the image of the contour never touches the x-axis we can likewise use Corollary 3.2.4 to solve the problem in the w-plane and bring it back to the z-plane via the conformal mapping.

Theorem 3.2.3. *The complex potential of an interval source along a simple parameterized contour $C(t)$ that traces out the boundary of a domain D, where t runs from t_0 to T, is*

$$\Omega(z) = \frac{S_0}{2\pi(T - t_0)} \int \left(\int_{t_0}^{T} \frac{\lambda(t)}{f(z) - f(C(t))} dt \right) f'(z) dz, \tag{3.2.5}$$

where $\lambda(t)$ is a normalized relative density function satisfying

$$\int_{t_0}^{T} \lambda(t) dt = T - t_0 \tag{3.2.6}$$

and $f(z)$ is a conformal mapping from D onto the upper half plane.

Proof. This proof follows from that of Theorem 3.2.1 nearly exactly. The only difference is that we first map the contour $C(t)$ in the domain D in the z-plane to the contour $f(C(t))$ along the u-axis on the boundary of the upper half of the w-plane. The complex potential in the w-plane is thus

$$\Omega(w) = \frac{S_0}{2\pi(T - t_0)} \int \left(\int_{t_0}^{T} \frac{\lambda(t)}{w - f(C(t))} dt \right) dw$$

by Theorem 3.2.1.

This complex potential is then taken back to the z-plane by using the conformal mapping $w = f(z)$ to replace w with $f(z)$ and dw with $f'(z)dz$. The final result is that the complex potential in D is

$$\Omega(z) = \frac{S_0}{2\pi(T - t_0)} \int \left(\int_{t_0}^{T} \frac{\lambda(t)}{f(z) - f(C(t))} dt \right) f'(z) dz,$$

where $\lambda(t)$ is the same normalized relative density function as before. □

Corollary 3.2.4. *The complex potential of an interval source along a simple parameterized contour $C(t)$ that lies completely within a domain D, where t runs from t_0 to T, is*

$$\Omega(z) = \frac{S_0}{2\pi(T - t_0)} \int \left(\int_{t_0}^{T} \left(\frac{\lambda(t)}{f(z) - f(C(t))} + \frac{\lambda(t)}{f(z) - \overline{f(C(t))}} \right) dt \right) f'(z) dz, \tag{3.2.7}$$

where $\lambda(t)$ is a normalized relative density function satisfying

$$\int_{t_0}^{T} \lambda(t) dt = T - t_0 \tag{3.2.8}$$

and $f(z)$ is a conformal mapping from D onto the upper half plane.

Proof. This proof is nothing more than a combination of ideas that have been developed previously. Theorem 3.2.3 states how the conformal mapping affects the complex potential, and applying the method of images to extend this result is identical to what was done with the method of images to extend Theorem 3.2.1 to Corollary 3.2.2. The complex potential follows immediately from these ideas. Also, as the relative density function $\lambda(t)$ is being used just as it was before, the normalization condition has not changed. □

Theorem 3.2.3 and Corollary 3.2.4 state concisely all that has been determined regarding the behavior of interval sources in this paper. They are the greatest extensions of previous results regarding point sources to interval sources that can presently be provided.

Chapter 4

Conclusion

Mathematical techniques from complex analysis reduce the problem of determining a two-dimensional irrotational incompressible vector field generated in a simply connected domain by point and interval sources a matter of determining a conformal mapping from the domain being considered to the upper half plane. Once this has been accomplished, the complex potential can be found by direct calculation using the theorems presented in this paper. This complex potential yields the desired vector field. Although these calculations can be difficult, if a problem in vector field theory is amenable to these techniques, it is essentially a solved problem.

This does not, however, mean that there is nothing interesting yet to discover with respect to this material. More explicit results might be obtained by considering special cases of the general formulas provided for interval sources, for example. Exploring applications to equipotential theory could also yield new and interesting results. Additionally, it would be useful to have illuminating examples of interval sources that did not lie along the x-axis to illustrate the validity of the theorems presented here.

Appendix A

Introductory Complex Analysis

Certain fundamental background material is provided here to remind the reader of certain useful results and notational conventions common in the study of complex analysis. This list is far from exhaustive. Thorough explanations and motivations for all of this material and more can be found in any introductory textbook on complex analysis, including [MH06], [McG00], and [Nee97].

A.1 Definitions and Notations

Definition A.1.1. A *complex number* z is anything of the form

$$z = x + \imath y, \tag{A.1.1}$$

where x and y are real numbers and $\imath$ is the positive square root of negative one.

Theorem A.1.2 (Euler Identity)**.** *Given any real number θ, the exponential function satisfies*

$$e^{\imath\theta} = \cos(\theta) + \imath\sin(\theta). \tag{A.1.2}$$

Definition A.1.3. The *modulus* of a complex number $z = x + \imath y$, denoted $|z|$, is defined as

$$|z| := \sqrt{x^2 + y^2}. \tag{A.1.3}$$

Theorem A.1.4. *Any complex number can be written in polar form as*

$$z = |z|e^{\imath\theta}. \tag{A.1.4}$$

Definition A.1.5. A *complex function* mapping each point z in a domain D to a point $w = f(z)$ in a domain G is anything of the form $f(z) = u(x, y) + \imath v(x, y)$, where $u(x, y)$ and $v(x, y)$ are real functions of the two real variables x and y.

A.2 Differentiability and Analyticity

Theorem A.2.1 (Cauchy-Riemann Equations)**.** *If a complex function $f(z) = u(x, y) + \imath v(x, y)$ is analytic in a domain D, then it satisfies the Cauchy-Riemann equations*

$$\begin{aligned} u_x(x, y) &= v_y(x, y) \text{ and} \\ u_y(x, y) &= -v_x(x, y) \end{aligned}$$

at every point in D, where subscripts denote partial derivatives.

Theorem A.2.2 (Cauchy-Riemann Conditions for Differentiability)**.** *If a complex function $f(z) = u(x, y) + \imath v(x, y)$ is continuous, satisfies the Cauchy-Riemann equations, and has continuous partial derivatives $u_x(x, y)$, $u_y(x, y)$, $v_x(x, y)$, and $v_y(x, y)$ throughout a domain D, then it is differentiable at every point in D.*

Definition A.2.3. A two-dimensional real function $u(x, y)$ is *harmonic* if it satisfies

$$u_{xx}(x, y) + u_{yy}(x, y) = 0. \tag{A.2.1}$$

Theorem A.2.4. *If a complex function $f(z) = u(x, y) + \imath v(x, y)$ is analytic in a domain D, then $u(x, y)$ is harmonic on D and $v(x, y)$ is harmonic on D.*

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