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Q.1) → $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Characteristic eqⁿ is $-\lambda^3 + S_1\lambda^2 - S_2\lambda + S_3 = 0$

S_1 = trace = $6+3+3 = 12$

S_2 = minors

$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$

$= (9-1) + (18-4) + (18-4)$

$= 8 + 14 + 14$

$= 36$

$S_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 6(9-1) + 2(-6+2) + 2(2-6)$

$= 6(8) + 2(-4) + 2(-4)$

$= 32$

Put into eqⁿ (1)

$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$

2	-1	12	-36	32
	↓	-2	20	-32
	-1	10	-16	0

10

$\lambda = 2$ $\quad -\lambda^2 + 10\lambda - 16 = 0$

$-\lambda^2 + 2\lambda + 8\lambda - 16 = 0$

$-\lambda(\lambda - 2) + 8(\lambda - 2) = 0$

$\lambda = 2 \quad \lambda = 8$

eigen values are $\lambda_1 = 2,\ \lambda_2 = 2,\ \lambda_3 = 8$

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Q.

Eigen vector for $\lambda = 2$

$|A - \lambda I| \bar{X}_1 = 0$

$$\left[\begin{matrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{matrix}\right] - \left[\begin{matrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{matrix}\right] \left[\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right] = \left[\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$$

$$\left[\begin{matrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{matrix}\right] \left[\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right] = \left[\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$$

$R_1 \to R_1/4$, $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 -$

$$\begin{matrix} -2 & 1 & -1 \\ 2 & -1 & 1 \\ \hline 0 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 2 & -1 & 1 \\ -2 & 1 & -1 \end{matrix}$$

$$\left[\begin{matrix} 1 & -1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}\right] \left[\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right]$$

$$x_1 - \frac{1}{2} x_2 + \frac{1}{2} x_3 = 0$$

Consider $x_2 = t$

$x_3 = s$

$$x_1 - \frac{t}{2} + \frac{s}{2} = 0$$

$$x_1 = \frac{t}{2} - \frac{s}{2}$$

$$\left[\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right] = \left[\begin{matrix} \frac{t}{2} - \frac{s}{2} \\ t \\ s \end{matrix}\right] = \left[\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}\right]$$

eigen vector for $\lambda = 2$

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Q.2) $A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$

Characterstic eq^n is

$|A - \lambda I|$

$= \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} -\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix}$

Characterstic of polynomial eq^n

$\begin{vmatrix} -\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (-\lambda)(1-\lambda)$

$= -\lambda + \lambda^2$

Characterstic eq^n of matrix.

$|A - \lambda I| = 0$

$-\lambda + \lambda^2 = 0$

$\lambda(-1 + \lambda) = 0$

$\lambda = 0 \qquad \lambda = 1$

eigen vector for $\lambda_1 = 1$

$|A - \lambda I| \, x = 0$

$\left[\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x_1 + 2x_2 = 0$

Consider $x_2 = t$

$x_1 = 2t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\bar{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ eigen value for $\lambda_1 = 1$

for $\lambda = 0$

$[A - \lambda I] X = 0$

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 0 & 1 & x_1 \\ 0 & 2 & x_2 \end{array}\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$$\left[\begin{array}{cc|c} 0 & 1 & x_1 \\ 0 & 0 & x_2 \end{array}\right]$$

$x_2 = 0$

$x_1 = 0$

$X_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ eigen vector '0'

Q.3) $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$

Characteristic eq^n is $-\lambda^3 + S_1\lambda^2 - S_2\lambda + S_3 = 0$

$S_1 = \text{trace} = 3 + 4 - 1 = 6$

$S_2 = \text{minors}$

$= \begin{vmatrix} 4 & 1 \\ -4 & -1 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$

$= (-4 + 4) + (-3 + 4) + (12 - 2)$

$= 1 + 10$

$= 11$

$S_3 = |A| = 3(-4 + 4) \mp 2(-1 + 2) + 2(-4 + 8)$

$= -2 + 8$

$= 6$

Put into eq^n

$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$$
\begin{array}{c|cccc}
1 & -1 & 6 & -11 & 6 \\
 & \downarrow & -1 & 5 & -6 \\
\hline
 & -1 & 5 & -6 & 0
\end{array}
$$

$\lambda = 1, \quad (-\lambda^2 + 5\lambda - 6) = 0$

$-\lambda^2 + 2\lambda + 3\lambda - 6 = 0$

$-\lambda(\lambda - 2) + 3(\lambda - 2) = 0$

$\lambda = 2 \quad \lambda = +3$

eigen values $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

eigen value of A^{-1} is

$1, \frac{1}{2}, \frac{1}{3}$

eigen values of A^T is $1, 2, 3$

Q.4)→

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Characteristic eq^n is $-\lambda^3 + S_1\lambda^2 - S_2\lambda + S_3 = 0$

$S_1 = \text{trace} = 3 + 2 + 4 = \cancel{10}\ 9$

$$S_2 = \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}$$

$= (12 - 6) + (8 - 3) + (6 - 2)$

$= 6 + 5 + 4$

$S_2 = 15$

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$$S_3 = \begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix} \begin{aligned} &= 2(12-6) - 1(8-6) + 1(6-9) \\ &= 12 - 2 - 3 \\ &= 7 \end{aligned}$$

Put in eq^n (i)

$$-\lambda^3 + 9\lambda^2 - 15\lambda + 7 = 0$$

$$\begin{array}{c|cccc} 1 & -1 & 9 & -15 & 7 \\ & \downarrow & -1 & 8 & -7 \\ \hline & -1 & 8 & -7 & 0 \end{array}$$

$\lambda = 1$, $(-\lambda^2 + 8\lambda - 7) = 0$

$\lambda = 1$, $-\lambda^2 + \lambda + 7\lambda - 7 = 0$

$\cancel{\lambda(-1+\lambda}$ $-\lambda(\lambda - 1) + 7(\lambda - 1) = 0$

$\lambda = 1$, $\lambda = 7$

eigen values are $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 7$

Eigen vector For $\lambda = 1$

$$|A - \lambda I| \bar{X_1} = 0$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 + x_3 = 0$$

Consider

$$x_2 = t$$

$$x_3 = s$$

$$x_1 + t + s = 0$$

$$x_1 = -t-s$$

$$x_1 = \begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

basis for eigen space for $\lambda = 1$ is $\{(-2, 1, 1)\}$

Q.5) →

i) Low rank factorization for collaborative prediction :
This is what netflix does to predict what rating you'll have for a movies you have not yet watched. It uses the SVD, and throws away the smallest eigenvalues $A^T.A$

ii) The Google Page Rank algorithm :
The largest eigenvector of the graph of the internet is how the pages are ranked

→ Q.6) Inner product space :-
An inner product space is vector space along with an inner product.
Basic properties are :
i) for each fixed $u \in V$ the funⁿ that takes v to $\langle u, v \rangle$ is a linear map from V to F
ii) $\langle 0, u \rangle = 0$, $\forall \; u \in V$
iii) $\langle u, v+\omega \rangle = \langle u, v \rangle + \langle u, \omega \rangle$
iv) $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle$

→ Q.7)

eigen values 3, 2, 2

trace of matrix A = 7
determinant of A = 12
∴ By using properties.

→ Q.8) simplify $\langle 2u - 5v, 4u + 6v \rangle$
Let V be an inner product space.

i) Linearity
$\langle u, v \rangle \quad u \in V$
$2 \langle u, 4u + 6v \rangle - 5 \langle v, 4u + 6v \rangle$
ii) Conjugate symmetry 1st argument
$\langle u, v \rangle = \overline{\langle v, u \rangle}$
$8 \langle u, u \rangle + 12 \langle u, v \rangle - 20 \langle v, u \rangle - 30 \langle v$
iii) 2nd linearity
$2 \langle u, 4u \rangle + 2 \langle u, 6v \rangle - 5 \langle v, 4u \rangle$
$- 5 \langle v, 6v \rangle$

iv) conjugate symmetry
$8(u,u) + 12(u,v) - 20\langle u,v\rangle - 30\langle v,v\rangle$

$\therefore$ the inner product defined
$\langle 2u-5v, 4u+6v\rangle = 8\langle u,u\rangle - 8\langle u,v\rangle - 30\langle v,v\rangle$

This is simplified form of given inner product.

Q.9)
$\rightarrow$ i) Symmetry
$(u,v) = a_1.b_1 + a_2b_2 + \dots + a_nb_n$
$(v,u) = b_1.a_1 + b_2a_2 + \dots + b_na_n$
Two expressions are equal.
$\therefore$ inner product is symmetry

ii) Linearity with 1st argument.
$(au + bv, w) = (a_1u_1 + b_1v_1)w_1 + (a_2u_2 + b_2v_2)w_2 + \dots + (a_nu_n + b_nv_n)w_n$

Expanding, we get:
$a_1(u_1w_1) + b_1v_1w_1 + a_2u_2w_2 + b_2v_2w_2 + \dots + a_nu_nw_n + b_nv_nw_n$
$= a_1(u_1w_1) + (b_1(v_1w_1) + a_2(u_2w_2) + (b_2(v_2w_2) + \dots + a_n(u_nw_n) + b_n(v_nw_n)$
$= a_1(u,w) + b_1(v,w) + a_2(u,w) + b_2(v,w) + \dots + a_n(u,w) + b_n(v,w)$
$= a(u,w) + b\langle v,w\rangle$
Hence inner product is linear in first argument.

3) Conjugate symmetry :
sum of real no. ~~is~~ it is always real.
each term in sum is product of real numbers
so all terms are non-negative
∴ inner product is non-negative.

4) Non-degeneracy
if $(u,u) = a_1^2 + a_2^2 + \dots + a_n^2 = 0$
then each term sum must be zero.
∵ Since each term is satisfies all four properties.
of an inner prod
∴ R^n with defined inner product is an inner product space.

→ Q.10)

Now AI

To find a vector $u \in R^3$ such that $q(v) = < u, v >$ belongs to V, we need to find a vector u that satisfies the conditions imposed by V.

Let's denote $v = (x, y, z) \in R^3$. Then, $q(v) = x + 2y - 3z$. We want to find u such that $q(v) = < u, v >$.

Using the dot product formula, $< u, v > = u_1x + u_2y + u_3z$.

By comparing the coefficients of x, y, and z on both sides of the equation, we can determine the values of u_1, u_2, and u_3.

From $q(v) = x + 2y - 3z$, we have:

$u_1 = 1$
$u_2 = 2$
$u_3 = -3$

Thus, the vector $u = (1, 2, -3)$ belongs to R^3 and satisfies $q(v) = < u, v >$ for all $v \in R^3$.