

PROBABILITY THEORY

LECTURE 3

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OVERVIEW LECTURE 3

- ▶ Transforms
- ▶ Probability generating function
- ▶ Moment generating function
- ▶ Characteristic function
- ▶ Transforms and distributions with random parameters

TRANSFORMS

- ▶ Finding the distribution of sum of random variables is hard. Convolution is messy.
- ▶ Transforms are functions that *uniquely* describe probability distributions.
- ▶ If you know the transform, you know the distribution, and vice versa.
- ▶ $X \stackrel{d}{=} Y \iff g_X(t) = g_Y(t)$
- ▶ **Summation** of independent variables corresponds to **multiplication of transforms**. Nice!

PROBABILITY GENERATING FUNCTION

- Applies to **non-negative, integer-valued** random variables.

DEF The **probability generating function** of X is

$$g_X(t) = \mathbb{E}t^X = \sum_{n=0}^{\infty} t^n \cdot P(X = n)$$

- $g_X(t)$ is defined at least for $|t| \leq 1$.

TH If $g_X = g_Y$ then $p_X = p_Y$.

TH Let X_1, X_2, \dots, X_n be independent. Then

$$g_{X_1+X_2+\dots+X_n}(t) = \prod_{k=1}^n g_{X_k}(t)$$

PROBABILITY GENERATING FUNCTION, CONT.

COR Let X_1, X_2, \dots, X_n be independent and identically distributed. Then

$$g_{X_1+X_2+\dots+X_n}(t) = (g_X(t))^n$$

- The name probability generating function comes from:

$$P(X = n) = \frac{g_X^{(n)}(0)}{n!}$$

where $g_X^{(n)}(t)$ is the n th derivative of $g_X(t)$ wrt to t .

TH Factorial moments (if $E|X|^k < \infty$)

$$EX(X-1)\cdots(X-k+1) = g_X^{(k)}(1)$$

- Moments can be computed

$$EX = g_X'(1)$$

$$\text{Var}X = g_X''(1) + g_X'(1) - (g_X'(1))^2$$

PROBABILITY GENERATING FUNCTION - EXAMPLES

✓ binomial theorem: $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$.

📎 Bernoulli, $X \sim \text{Be}(p)$

$$g_X(t) = \sum_{n=0}^{\infty} t^n \cdot P(X = n) = t^0 q + t^1 p = q + pt$$

📎 Binomial, $X \sim \text{Bin}(n, p)$

$$g_X(t) = \sum_{k=0}^n t^k \binom{n}{k} p^k q^{n-k} = \sum_{k=0}^n \binom{n}{k} (pt)^k q^{n-k} = (q + pt)^n$$

📎 Let $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Be}(p)$ and $X = X_1 + \dots + X_n$, then

$$g_X(t) = \prod_{i=1}^n g_{X_i}(t) = \prod_{i=1}^n (q + pt) = (q + pt)^n$$

so $X \sim \text{Bin}(n, p)$.

PROBABILITY GENERATING FUNCTION - EXAMPLES

✓ Poisson prob func: $p(X = k) = e^{-m} m^k / k!$

✓ $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

⇒ If Poisson, $X \sim Po(m)$, then

$$g_X(t) = \sum_{k=0}^{\infty} t^k \frac{e^{-m} m^k}{k!} = e^{-m} \sum_{k=0}^{\infty} \frac{(mt)^k}{k!} = e^{m(t-1)}$$

⇒ If $X_1 \sim Po(m_1)$ independently of $X_2 \sim Po(m_2)$, then

$$g_{X_1+X_2}(t) = e^{m_1(t-1)} e^{m_2(t-1)} = e^{(m_1+m_2)(t-1)}$$

so $X_1 + X_2 \sim Po(m_1 + m_2)$.

MOMENT GENERATING FUNCTION

- ▶ $g_X(t)$ limited to non-negative integer-valued variables.

DEF **Moment generating function** of a variable X

$$\psi_X(t) = Ee^{tX}$$

if the expectation exist and is finite for $|t| < h$.

TH If $\psi_X(t)$ exists for $|t| < h$ for some $h > 0$, then

- ▶ All moments exist $E|X|^r < \infty$ for all $r > 0$
- ▶ $EX^n = \psi_X^{(n)}(0)$ for $n = 1, 2, \dots$
- ▶ Taylor expansion around $t = 0$ [note $\frac{\partial^k e^{tX}}{\partial t^k} = X^k e^{tX}$]

$$e^{tX} = 1 + \sum_{n=1}^{\infty} \frac{t^n X^n}{n!}$$

so

$$Ee^{tX} = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} EX^n$$

MOMENT GENERATING FUNCTION - EXAMPLES

➡ $X \sim \text{Be}(p)$

$$\psi_X(t) = Ee^{tX} = qe^{t \cdot 0} + pe^{t \cdot 1} = q + pe^t$$

- ▶ $\psi'_X(t) = pe^t$ so $E(X) = \psi'_X(0) = p$.
- ▶ $\psi''_X(t) = pe^t$ so $E(X^2) = \psi''_X(0) = p$.
- ▶ $\text{Var}(X) = E(X^2) - [E(X)]^2 = p - p^2 = pq$

➡ $X \sim \Gamma(p, a)$

$$\psi_X(t) = \frac{1}{(1 - at)^p}$$

- ▶ $\psi'_X(t) = \frac{ap}{(1-at)^{p+1}}$ so $E(X) = \psi'_X(0) = ap$.
- ▶ $\psi''_X(t) = \frac{a^2 p(p+1)}{(1-at)^{p+2}}$ so $E(X^2) = \psi''_X(0) = a^2 p(p+1)$.
- ▶ $\text{Var}(X) = E(X^2) - [E(X)]^2 = a^2 p(p+1) - a^2 p^2 = a^2 p$.

MOMENT GENERATING FUNCTION, CONT.


TH If $\exists h > 0$ such that $\psi_X(t) = \psi_Y(t)$ for $|t| < h$, then $X \stackrel{d}{=} Y$.

TH If X_1, X_2, \dots, X_n are independent with moment generating functions that exist for $|t| < h$ for some $h > 0$, then

$$\psi_{X_1 + \dots + X_n}(t) = \prod_{i=1}^n \psi_{X_i}(t), \quad |t| < h$$

TH Moment generating function of a linear combination $a \cdot X + b$

$$\psi_{aX+b}(t) = e^{tb} \psi_X(at)$$

 If $X \sim \Gamma(d, p)$, what is the distribution of $Y = \sigma \cdot X$?

$$\psi_X(t) = \frac{1}{(1 - dt)^p}$$

$$\psi_Y(t) = \frac{1}{(1 - d\sigma t)^p},$$

which is the mgf of $\Gamma(d\sigma, p)$. Gamma family is closed under scaling.


THE CHARACTERISTIC FUNCTION

- ▶ Moment generating function is not defined for all random variable. No mgf for Cauchy or LogNormal.
- ▶ The **characteristic function** is more general and exists for any variable, **but complex valued**.

DEF The characteristic function of a random variable X is

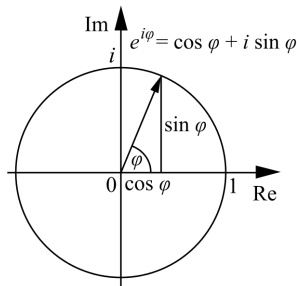
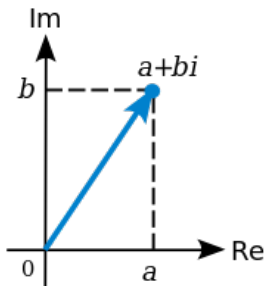
$$\varphi_X(t) = Ee^{itX} = E(\cos tX + i \sin tX)$$

where i is the imaginary number ($i^2 = -1$).

 $X \sim Be(p)$, then $\varphi_X(t) = (q + pe^{it})$.

COMPLEX NUMBERS

- ▶ Complex number $z = a + b \cdot i$
- ▶ $\text{Re}(z) = a$ is the real part of z
- ▶ $\text{Im}(z) = b$ is the imaginary part of z
- ▶ Complex conjugate $\bar{z} = a - b \cdot i$
- ▶ Addition: $z_1 + z_2 = a_1 + a_2 + (b_1 + b_2) \cdot i$
- ▶ Multiplication: $z_1 z_2 = a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i$
- ▶ Modulus: $|z| = \sqrt{a^2 + b^2}$. Length of vector.
- ▶ Complex exponentials: $e^{ix} = \cos x + i \cdot \sin x$



THE CHARACTERISTIC FUNCTION, CONT.

TH If $\varphi_X = \varphi_Y$ then $X \stackrel{d}{=} Y$.

TH Let F be the distribution function of X . If F is continuous at a and b , and $\int_{-\infty}^{\infty} |\varphi(t)| dt < \infty$ then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt$$

TH Characteristic function of a sums of independent variables

$$\varphi_{X_1+\dots+X_n}(t) = \prod_{i=1}^n \varphi_{X_i}(t)$$

TH Moments

$$\varphi_X^{(k)}(0) = i^k \cdot EX^k$$

TH Linear combinations

$$\varphi_{aX+b}(t) = e^{ibt} \varphi_X(at)$$

TRANSFORMS - DISTRIBUTIONS WITH RANDOM PARAMETERS

- ▶ Transforms are expected values (or t^X , e^{tX} or e^{itX}), so the law of iterated expectation is useful.
- 📎 Let $X|(N = n) \sim \text{Bin}(n, p)$ and $N \sim \text{Po}(\lambda)$. What is the marginal distribution of X ? X is non-negative and integer-valued, so $g_X(t)$ is defined.

$$g_X(t) = E\left(E(t^X|N)\right) = Eh(N)$$

where

$$h(n) = E(t^X|N = n) = (q + pt)^n.$$

We then have

$$g_X(t) = E\left((q + pt)^N\right) = g_N(q + pt) = e^{\lambda[(q+pt)-1]} = e^{\lambda p(t-1)}.$$

- 📎 $X|y \sim N(0, y)$ and $y \sim \text{Exp}(1)$, then $X \sim L(1/\sqrt{2})$. Prove using characteristic functions.