Marginalization, Prediction, Decisions, Exponential family

PhD course in Statistical Inference

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- **Prediction**
- **Decision making**
- Bayesian inference for the **exponential family**

- **Posterior predictive density** for future $\tilde{y}$ given observed $\mathbf{y}$

$$p(\tilde{y}|\mathbf{y}) = \int_\theta p(\tilde{y}|\theta, \mathbf{y}) p(\theta|\mathbf{y}) d\theta$$

- If $p(\tilde{y}|\theta, \mathbf{y}) = p(\tilde{y}|\theta)$ [not true for time series], then

$$p(\tilde{y}|\mathbf{y}) = \int_\theta p(\tilde{y}|\theta) p(\theta|\mathbf{y}) d\theta$$

- **Parameter uncertainty** in $p(\tilde{y}|\mathbf{y})$ by **averaging over** $p(\theta|\mathbf{y})$.

- Predictive distribution is normal (next slide).
- Remember the posterior: $\theta|\mathbf{y} \sim N(\mu_n, \tau_n^2)$.
- Law of iterated expectation:

$$E(\tilde{y}|\mathbf{y}) = E_{\theta|\mathbf{y}}[E_{\tilde{y}|\theta}(\tilde{y})] = E_{\theta|\mathbf{y}}(\theta) = \mu_n$$

- The predictive variance of $\tilde{y}$ (total variance formula):

$$\begin{aligned} V(\tilde{y}|\mathbf{y}) &= E_{\theta|\mathbf{y}}[V_{\tilde{y}|\theta}(\tilde{y})] + V_{\theta|\mathbf{y}}[E_{\tilde{y}|\theta}(\tilde{y})] \\ &= E_{\theta|\mathbf{y}}(\sigma^2) + V_{\theta|\mathbf{y}}(\theta) \\ &= \sigma^2 + \tau_n^2 \end{aligned}$$

- In summary:

$$\tilde{y}|\mathbf{y} \sim N(\mu_n, \sigma^2 + \tau_n^2).$$

Simulation algorithm:

1. Generate a **posterior draw** of θ ($\theta^{(i)}$) from $N(\mu_n, \tau_n^2)$
2. Generate a **predictive draw** of $\tilde{y}$ ($\tilde{y}^{(i)}$) from $N(\theta^{(i)}, \sigma^2)$
3. Repeat Steps 1 and 2 N times to output:
   - Sequence of posterior draws: $\theta^{(1)}, ..., \theta^{(N)}$
   - Sequence of predictive draws: $\tilde{y}^{(1)}, ..., \tilde{y}^{(N)}$.

- Note: $\tilde{y}^{(i)} = \theta^{(i)} + \sigma Z_1 = (\mu_n + \tau_n Z_2) + \sigma Z_1$ where Z_1, Z_2 are $N(0, 1)$. So $\tilde{y}^{(i)}$ is normal.

- **Autoregressive process**

$$y_t = \mu + \phi_1(y_{t-1} - \mu) + ... + \phi_p(y_{t-p} - \mu) + \varepsilon_t,\ \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$$

Simulation algorithm. Repeat N times:

1. Generate a **posterior draw** of $\theta^{(1)} = (\phi_1^{(i)}, ..., \phi_p^{(i)}, \mu^{(i)}, \sigma^{(i)})$ from $p(\phi_1, ..., \phi_p, \mu, \sigma | \mathbf{y}_{1:T})$.

2. Generate a **predictive draw** of future time series by:
   - 2.1 $\tilde{y}_{T+1} \sim p(y_{T+1} | y_T, y_{T-1}, ..., y_{T-p}, \theta^{(i)})$
   - 2.2 $\tilde{y}_{T+2} \sim p(y_{T+2} | \tilde{y}_{T+1}, y_T, ..., y_{T-p}, \theta^{(i)})$
   - 2.3 $\tilde{y}_{T+3} \sim p(y_{T+3} | \tilde{y}_{T+2}, \tilde{y}_{T+1}, y_T, ..., y_{T-p}, \theta^{(i)})$
   - 2.4 ...

- Let θ be an **unknown quantity**. **State of nature**. Examples: Future inflation, Global temperature, Disease.
- Let $a \in \mathcal{A}$ be an **action**. Ex: Interest rate, Energy tax, Surgery.
- Choosing action a when state of nature is θ gives **utility**

$$U(a, \theta)$$

- Example:
  - θ is the number of items demanded of a product
  - a is the number of items in stock
  - Utility

$$U(a, \theta) = \begin{cases} p \cdot \theta - c_1(a - \theta) & \text{if } a > \theta \text{ [too much stock]} \\ p \cdot a - c_2(\theta - a)^2 & \text{if } a \leq \theta \text{ [too little stock]} \end{cases}$$

- Ad hoc decision rules: *Minimax*. *Minimax-regret* etc
- **Bayesian theory**: maximize the **posterior expected utility**:

$$a_{bayes} = \mathrm{argmax}_{a \in \mathcal{A}} \, E_{p(\theta|y)}[U(a,\theta)],$$

where $E_{p(\theta|y)}$ denotes the posterior expectation.

- Using simulated draws $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(N)}$ from $p(\theta|y)$:

$$E_{p(\theta|y)}[U(a,\theta)] \approx N^{-1} \sum_{i=1}^{N} U(a, \theta^{(i)})$$

- **Separation principle**:

1. First obtain $p(\theta|y)$
2. then form $U(a,\theta)$ and finally
3. choose a that maximes $E_{p(\theta|y)}[U(a,\theta)]$.

- **Model**

$$y_1, ..., y_n|\theta \overset{iid}{\sim} Pois(\theta)$$

- **Poisson distribution**

$$p(y) = \frac{\theta^y e^{-\theta}}{y!}$$

- **Likelihood** from iid Poisson sample $y = (y_1, ..., y_n)$

$$p(y|\theta) = \left[\prod_{i=1}^{n} p(y_i|\theta)\right] \propto \theta^{(\sum_{i=1}^{n} y_i)} \exp(-\theta n),$$

- ***Prior***

$$p(\theta) \propto \theta^{\alpha-1} \exp(-\theta\beta) \propto Gamma(\alpha, \beta)$$

which contains the info: $\alpha - 1$ counts in β observations.

- **Posterior**

$$
\begin{aligned}
p(\theta|y_1,\ldots,y_n) &\propto \left[\prod_{i=1}^{n} p(y_i|\theta)\right] p(\theta) \\
&\propto \theta^{\sum_{i=1}^{n} y_i} \exp(-\theta n)\theta^{\alpha-1}\exp(-\theta\beta) \\
&= \theta^{\alpha+\sum_{i=1}^{n} y_i - 1}\exp[-\theta(\beta+n)],
\end{aligned}
$$

  proportional to the $Gamma(\alpha+\sum_{i=1}^{n} y_i, \beta+n)$ distribution.

- **Prior-to-Posterior mapping**

$$
\begin{aligned}
\text{Model: } & y_1,\ldots,y_n|\theta \overset{iid}{\sim} Pois(\theta) \\
\text{Prior: } & \theta \sim Gamma(\alpha,\beta) \\
\text{Posterior: } & \theta|y_1,\ldots,y_n \sim Gamma\left(\alpha+\sum_{i=1}^{n} y_i, \beta+n\right).
\end{aligned}
$$

Poisson example - Bomb hits in London

$n = 576$, $\sum_{i=1}^{n} y_i = 229 \cdot 0 + 211 \cdot 1 + 93 * 2 + 35 * 3 + 7 * 4 + 1 \cdot 5 = 537$.

Average number of hits per region=$\bar{y} = 537/576 \approx 0.9323$.

$$p(\theta|y) \propto \theta^{\alpha+537-1} \exp[-\theta(\beta + 576)]$$

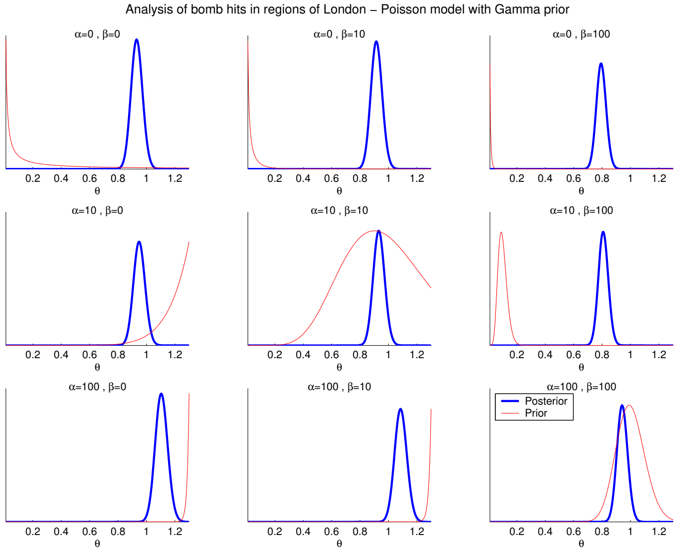
$$E(\theta|y) = \frac{\alpha + \sum_{i=1}^{n} y_i}{\beta + n} \approx \bar{y} \approx 0.9323,$$

and

$$SD(\theta|y) = \left(\frac{\alpha + \sum_{i=1}^{n} y_i}{(\beta + n)^2}\right)^{1/2} = \frac{(\alpha + \sum_{i=1}^{n} y_i)^{1/2}}{(\beta + n)} \approx \frac{(537)^{1/2}}{576} \approx 0.0402.$$

if α and β are small compared to $\sum_{i=1}^{n} y_i$ and n.

Poisson bomb hits in London

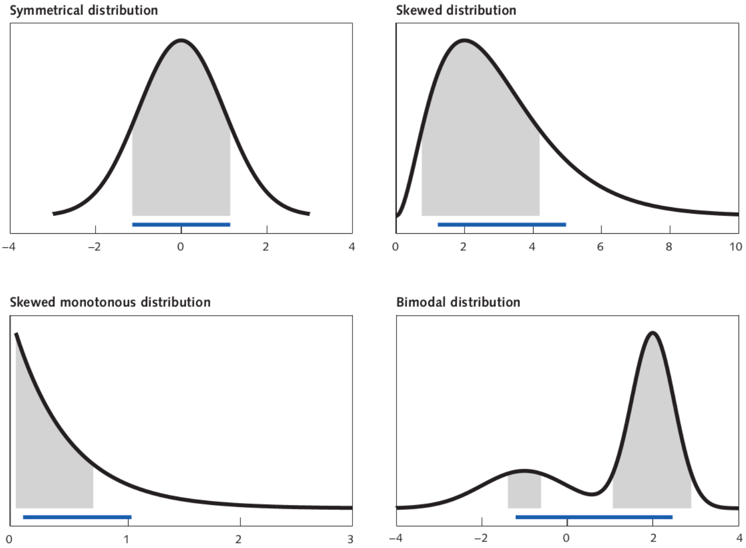


Poisson example - posterior intervals

- **Bayesian 95% credible interval**: the probability that the unknown parameter θ lies in the interval is 0.95.
- Approximate 95% **credible interval** for θ (for small α and β):

$$E(\theta|y) \pm 1.96 \cdot SD(\theta|y) = [0.8535; 1.0111]$$

- An exact 95% **equal-tail interval** is $[0.8550; 1.0125]$ (assuming $\alpha = \beta = 0$)
- **Highest Posterior Density** (**HPD**) interval contains the θ values with highest pdf.
- An exact Highest Posterior Density (HPD) interval is $[0.8525; 1.0144]$. Obtained numerically, assuming $\alpha = \beta = 0$.


Symmetrical distribution
–4
–2
0
2
4
Skewed distribution
0
2
4
6
8
10
Skewed monotonous distribution
0
1
2
3
Bimodal distribution
–4
–2
0
2
4

- Normal likelihood: Normal prior$\rightarrow$Normal posterior.
- Bernoulli likelihood: Beta prior$\rightarrow$Beta posterior.
- Poisson likelihood: Gamma prior$\rightarrow$Gamma posterior.

- **Conjugate priors**: A prior is conjugate to a model if the prior and posterior belong to the same distributional family.

- Formal definition: Let $\mathcal{F} = \{p(y|\theta), \theta \in \Theta\}$ be a class of sampling distributions. A family of distributions $\mathcal{P}$ is **conjugate** for $\mathcal{F}$ if

$$p(\theta) \in \mathcal{P} \Rightarrow p(\theta|\mathbf{x}) \in \mathcal{P}$$

holds for all $p(y|\theta) \in \mathcal{F}$.

- **Exponential family** in the canonical parametrization

$$p(x|\theta) = h(x)\exp\left(\theta^T \mathbf{t}(x) - A(\theta)\right)$$

where $A(\theta) = -\ln a(\theta)$ in Rolf's notation.

- **Likelihood**

$$p(x_1, \dots, x_n|\theta) = \left[\prod_{i=1}^{n} h(x_i)\right] \exp\left(\theta^T \sum_{i=1}^{n} \mathbf{t}(x_i) - nA(\theta)\right)$$

- **Conjugate prior**

$$p(\theta) = H(\tau_0, n_0)\exp\left(\theta^T \tau_0 - n_0 A(\theta)\right),$$

where τ_0 and n_0 are prior hyperparameters and $H(\tau_0, n_0)$ is the normalizing constant which is known to exist if $n_0 > 0$.

- **Conjugate prior**

$$p(\theta) = H(\tau_0, n_0) \exp\left(\theta^T \tau_0 - n_0 A(\theta)\right)$$

- **Posterior**

$$p(\theta | x_1, \ldots, x_n) \propto \exp\left[\theta^T \left(\tau_0 + \sum_{i=1}^{n} \mathbf{t}(x_i)\right) - (n_0 + n) A(\theta)\right]$$

- **Prior-to-posterior updating**

$$\tau_0 \Longrightarrow \tau_n = \tau_0 + \sum_{i=1}^{n} \mathbf{t}(x_i)$$

$$n_0 \Longrightarrow n_0 + n$$

- **Exponential family** in the non-canonical parametrization

$$p(x|\theta) = h(x) \exp\left(\phi(\theta)^T \mathbf{t}(x) - A(\theta)\right)$$

- **Conjugate prior**

$$p(\theta) = H(\tau_0, n_0) \exp\left(\phi(\theta)^T \tau_0 - n_0 A(\theta)\right)$$

- **Bernoulli likelihood**

$$\begin{aligned} p(x_1, \dots, x_n|\theta) &= \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} \\ &= \exp\left(\log\left(\frac{\theta}{1-\theta}\right)\sum_{i=1}^{n} x_i - n\log\left(\frac{1}{1-\theta}\right)\right) \\ &= \exp\left(\phi(\theta)\sum_{i=1}^{n} x_i - nA(\theta)\right) \end{aligned}$$

where $\phi = \log\left(\frac{\theta}{1-\theta}\right)$ and $A(\theta) = \log\left(\frac{1}{1-\theta}\right)$.

- **Conjugate prior** $p(\phi)$

$$\exp\left(\phi(\theta)\tau_0 - n_0 A(\theta)\right) = \exp\left(\log\left(\frac{\theta}{1-\theta}\right)\tau_0 - n_0 \log\left(\frac{1}{1-\theta}\right)\right) = \theta^{\tau_0}(1-\theta)^{n_0-\tau_0}$$

- **Exponential family** in the canonical parametrization

$$p(x|\theta) = h(x) \exp\left(\theta^T \mathbf{t}(x) - A(\theta)\right)$$

- **Conjugate prior**

$$p(\theta) = H(\tau_0, n_0) \exp\left(\theta^T \tau_0 - n_0 A(\theta)\right),$$

- **Prior mean** of $\mu \equiv E(x|\theta) = \nabla A(\theta)$

$$\mathrm{E}(\mu) = \tau_0 / n_0$$

Proof: $\nabla p(\theta) = p(\theta)\left(\tau_0 - n_0 \nabla A(\theta)\right)$, so
$\int \nabla p(\theta) d\theta = \left(\tau_0 - n_0 E\left[\nabla A(\theta)\right]\right) = 0$ (Green's theorem).

- **Posterior mean of** of $\mu \equiv E(x|\theta)$

$$\mathrm{E}(\mu|x_1, ..., x_n) = \frac{\tau_0 + \sum_{i=1}^n \mathbf{t}(x_i)}{n_0 + n} = w\tau_0 + (1 - w)\hat{\mu}_{ML},$$

where $\hat{\mu}_{ML} = \frac{1}{n}\sum_{i=1}^n \mathbf{t}(x_i)$ and $w = n_0/(n_0 + n)$.

- **Predictive distribution** of x_{n+1}

$$
\begin{aligned}
p(x_{n+1}|x_{1:n}) &= \int p(x_{n+1}|\theta)p(\theta|x_{1:n})d\theta \\
&= \int h(x_{n+1}) \exp\left(\theta^T \mathbf{t}(x_{n+1}) - A(\theta)\right) H(\tau_n, n_0 + n) \exp\left(\theta^T \tau_n - (n_0 + n)A(\theta)\right) d \\
&= h(x_{n+1})H(\tau_n, n_0 + n) \int \exp\left(\theta^T (\tau_n + \mathbf{t}(x_{n+1})) - (n_0 + n + 1)A(\theta)\right) d\theta \\
&= h(x_{n+1})H(\tau_n, n_0 + n)/H(\mathbf{t}(x_{n+1}) + \tau_n, n_0 + n + 1)
\end{aligned}
$$