Notes: The Galaxy Clustering Effect on the Hellings-Downs Variance

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1 The case without galaxy clustering

Here, we summarize some basic concepts and notations that are commonly found in literature. We consider a universe without galaxy clustering and a stochastic GW background, causing a perturbation to the flat space-time metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 + h_{ij}(t,\mathbf{x})\mathrm{d}x^i\mathrm{d}x^j\,. \tag{1}$$

In the transverse-traceless synchronous gauge, the quantity h_{ij} can be written using the plane-wave expansion:

$$h_{ij}(t,\mathbf{x}) = \sum_{A=+,\times}\int \mathrm{d}f \int \mathrm{d}^2\boldsymbol{n}\, h_A(f,\boldsymbol{n})e^{2\pi i f(t-\boldsymbol{n}\cdot\mathbf{x})}e^A_{ij}(\mathbf{n})\,, \tag{2}$$

with $e^A_{ij}(\boldsymbol{n})$ being the polarization tensor and $h_A(f,\boldsymbol{n})$ being arbitrary complex functions satisfying $h_A(-f,\boldsymbol{n}) = h^*_A(f,\boldsymbol{n})$. We assume that $\langle h_A(f,\boldsymbol{n})\rangle = 0$, i.e. $h_A(f,\boldsymbol{n})$ is a random variable with zero mean value. Furthermore, the expectation value of the product of two $h_A(f,\boldsymbol{n})$ is given by:

$$\langle h^*_A(f,\mathbf{n})h_B(f',\mathbf{n}')\rangle = \frac{1}{2}\delta_{AB}\,\delta(f-f')\bar{S}_h(f)\frac{\delta^2(\mathbf{n},\mathbf{n}')}{4\pi}\,. \tag{3}$$

The appearing delta functions ensure that only gravitational waves with the same polarization state, at the same frequency and coming from the same direction are correlated. In other words, at each fixed value of f and $\boldsymbol{n}$, the quantity $h_A(f,\boldsymbol{n})$ is a Gaussian random variable with variance determined by the intensity $\bar{S}_h(f)$. There are no correlations with the random variable at any other value $(f',\boldsymbol{n}')$. We note that the intensity $\bar{S}_h(f)$ is related to the galaxy density via some astrophysical model.

2 Introducing galaxy clustering

In reality, the Universe is not perfectly homogeneous. Its galaxy density at each point $(\boldsymbol{n},z)$ is given by some realization of a Gaussian random field $\mathfrak{p}_g(\boldsymbol{n},z)$. This random field has the mean

$$\langle\mathfrak{p}_g(\boldsymbol{n},z)\rangle = \bar{\rho}_g(z)\,, \tag{4}$$

which is the background density of galaxies observed in the Universe at redshift z. The 2-point correlation function is given by

$$\langle(\mathfrak{p}_g(\boldsymbol{n},z)-\bar{\rho}_g(z))\,(\mathfrak{p}_g(\boldsymbol{n},z')-\bar{\rho}_g(z'))\rangle = \xi_g(\boldsymbol{n}\cdot\boldsymbol{n}',z,z')\,, \tag{5}$$

where $\xi_g(\boldsymbol{n}\cdot\boldsymbol{n}',z,z')$ is the galaxy correlation function, which depends on the angle between $\boldsymbol{n}$ and $\boldsymbol{n}'$ and the two redshifts z, z'. In the following, we will distinguish between the random field $\mathfrak{p}_g(\boldsymbol{n},z)$, and specific realizations of this random field denoted by $\rho_g(\boldsymbol{n},z)$.

3 Conditional probabilities

In Section 1, the case without galaxy clustering is discussed, which corresponds to the specific realization $\mathfrak{p}_g(\boldsymbol{n},z) = \bar{\rho}_g(z)$, where the galaxy distribution is perfectly homogeneous and isotropic. Hence, eq. (3) should more precisely be written as

$$\langle h^*_A(f,\mathbf{n})h_B(f',\mathbf{n}')|\mathfrak{p}_g(\boldsymbol{n},z) = \bar{\rho}_g(z)\rangle = \frac{1}{2}\delta_{AB}\,\delta(f-f')\bar{S}_h(f)\frac{\delta^2(\mathbf{n},\mathbf{n}')}{4\pi}\,. \tag{6}$$

The expression on the left is a *conditional probability*: it represents the average over all realizations of h_A (i.e. the expectation value) of the product $h^*_A(f,\boldsymbol{n})h_B(f',\mathbf{n}')$, assuming a *fixed* realization of the density field $\mathfrak{p}_g(\boldsymbol{n},z)$.

On the right-hand side, $\bar{S}_h(f)$ is the variance (over all possible realization of h_A) of the field h_A, in a universe where the density distribution is perfectly homogeneous. Since the density distribution does not depend on direction, the variance is also independent on $\boldsymbol{n}$. Moreover, for a fixed realization of the density, h_A is a Gaussian random field.

Let us now consider another random realization $\mathfrak{p}_g(\boldsymbol{n},z) = \rho_g(\boldsymbol{n},z) = \bar{\rho}_g(z)(1+\delta_g(\boldsymbol{n},z))$, where $\delta_g(\boldsymbol{n},z)$ is the galaxy over-/underdensity. Now, eq. (3) changes to

$$\langle h_A^*(f,\mathbf{n})h_B(f',\mathbf{n}')|\mathfrak{p}_g(\boldsymbol{n},z)=\rho_g(\boldsymbol{n},z)\rangle = \frac{1}{2}\delta_{AB}\,\delta(f-f')S_h(f,\boldsymbol{n})\frac{\delta^2(\mathbf{n},\mathbf{n}')}{4\pi}\,. \tag{7}$$

For a given $\rho_g(\boldsymbol{n},z)$, this should be thought of as a relation defining the ensemble of the field $h_A(f,\mathbf{n})$, along with Gaussianity and the relation $\langle h_A(f,\mathbf{n})|\mathfrak{p}_g(\boldsymbol{n},z)=\rho_g(\boldsymbol{n},z)\rangle = 0$.

As before, this describes a Gaussian random field, with the variance at each point determined by the intensity $S_h(f,\boldsymbol{n})$ and no correlations between distinct angles, frequencies or polarization states. The only difference to before is the angle dependence of $S_h(f,\boldsymbol{n})$, given different galaxy densities in different directions.

For ease of notation, we define

$$\langle h_A^*(f,\mathbf{n})h_B(f',\mathbf{n}')\rangle_{h|\rho_g} \equiv \langle h_A^*(f,\mathbf{n})h_B(f',\mathbf{n}')|\mathfrak{p}_g(\boldsymbol{n},z)=\rho_g(\boldsymbol{n},z)\rangle\,. \tag{8}$$

We also write $S_h(f,\boldsymbol{n})$ as

$$S_h(f,\mathbf{n}) = \bar{S}_h(f) + \delta S_h(f,\mathbf{n}) = \bar{S}_h(f)\left[1+\int \mathrm{d}z\, b_{\mathrm{GW}}(f,z)\delta_g(\mathbf{n},z)\right]\,, \tag{9}$$

where $b_{\mathrm{GW}}(f,z)$ determines the relation between $\delta_g(\boldsymbol{n},z)$ and the GW intensity (generic quantity for now, will be determined when specifying an astrophysical model).

4 Total ensemble averages: 2-point

Until now, we have only considered averages over different realization of h_A but for fixed realization of the density $\mathfrak{p}_g(\boldsymbol{n},z)$. We are now interested to compute the full ensemble average, i.e. the average over all realizations of h and all realizations of the density:

$$\langle h_A^*(f,\mathbf{n})h_B(f',\mathbf{n}')\rangle = \left\langle \langle h_A^*(f,\mathbf{n})h_B(f',\mathbf{n}')\rangle_{h|\rho_g}\right\rangle_{\mathfrak{p}_g}\,. \tag{10}$$

Here we first compute the conditioned ensemble average, and then integrate over the probability distribution of $\mathfrak{p}_g(\boldsymbol{n},z)$. This way, we obtain

$$\langle h_A^*(f,\mathbf{n})h_B(f',\mathbf{n}')\rangle = \left\langle \frac{1}{2}\delta_{AB}\,\delta(f-f')S_h(f,\boldsymbol{n})\frac{\delta^2(\mathbf{n},\mathbf{n}')}{4\pi}\right\rangle_{\mathfrak{p}_g} = \frac{1}{2}\delta_{AB}\,\delta(f-f')\bar{S}_h(f)\frac{\delta^2(\mathbf{n},\mathbf{n}')}{4\pi}\,, \tag{11}$$

where we used eq. (9) and the fact that $\langle \delta_g(\boldsymbol{n},z)\rangle_{\mathfrak{p}_g} = 0$. Hence, on the 2-point level, we recover the same expression as in the original set-up summarized in section 1. As a consequence, this also means that galaxy clustering has no effect on the HD curve itself.

5 Variance of the HD curve from galaxy clustering

Here, we compute the variance of the HD curve arising from galaxy clustering. We emphasize that, in the following computation, we will never apply Isserlis' theorem on full ensemble averages. The only property that we use is that, for a fixed realization of the density field, the field h_A is Gaussian, as discussed in section 3. We will only apply Isserlis' theorem to the conditioned field, and we will compute full ensemble averages by splitting them as

$$\langle \dots \rangle = \langle \langle \dots \rangle_{h|\rho_g}\rangle_{\mathfrak{p}_g} \tag{12}$$

into ensemble averages over the conditioned field and ensemble averages over $\mathfrak{p}_g(\boldsymbol{n},z)$.

We follow the set-up presented in Appendix C of Ref. [1]. In particular, the quantity ρ_{ab} is given by

$$\begin{aligned}\rho_{ab} &= \overline{Z_a Z_b}\\ &= \sum_{A,A'}\int \mathrm{d}f\int \mathrm{d}f'\int \mathrm{d}\boldsymbol{n}\int \mathrm{d}\boldsymbol{n}' R_a^A(f,\boldsymbol{n})^* R_b^{A'}(f',\boldsymbol{n}')h_A^*(f,\boldsymbol{n})h_{A'}(f',\boldsymbol{n}')\mathrm{sinc}\left(\pi(f-f')T\right)\,,\end{aligned} \tag{13}$$

where the bar denotes an average over the total observation time T, $\mathrm{sinc}(x) \equiv \sin(x)/x$, and

$$R_a^A(f,\boldsymbol{n}) \equiv \left(1 - e^{-2\pi i f \tau_a (1+\boldsymbol{n}\cdot\boldsymbol{n}_a)}\right) F_a^A(\boldsymbol{n}) \,, \qquad F_a^A(\boldsymbol{n}) \equiv \frac{n_a^i n_a^j e_{ij}^A(\boldsymbol{n})}{2(1+\boldsymbol{n}\cdot\boldsymbol{n}_a)} \,. \tag{14}$$

Then, ρ_{ab}^2 is given by

$$\begin{aligned}\rho_{ab}^2 = \sum_{A,A',A'',A''} \iiiint \mathrm{d}f \mathrm{d}f' \mathrm{d}f'' \mathrm{d}f''' \iiiint \mathrm{d}\boldsymbol{n} \mathrm{d}\boldsymbol{n}' \mathrm{d}\boldsymbol{n}'' \mathrm{d}\boldsymbol{n}''' \mathrm{sinc}\left(\pi(f-f')T\right) \mathrm{sinc}\left(\pi(f''-f''')T\right) \\ \times R_a^A(f,\boldsymbol{n})^* R_b^{A'}(f',\boldsymbol{n}') R_a^{A''}(f'',\boldsymbol{n}'') R_b^{A'''}(f''',\boldsymbol{n}''')^* h_A^*(f,\boldsymbol{n}) h_{A'}(f',\boldsymbol{n}') h_{A''}(f'',\boldsymbol{n}'') h_{A'''}^*(f''',\boldsymbol{n}''') \,.\end{aligned} \tag{15}$$

We now want to calculate

$$\left\langle \rho_{ab}^2 \right\rangle = \left\langle \left\langle \rho_{ab}^2 \right\rangle_{h|\rho_g} \right\rangle_{\mathfrak{p}_g} \,. \tag{16}$$

For the inner ensemble average, we can use Eq. (7), exactly following the calculations in Appendix C of Ref. [1]. The only difference is that now, we need to take the anisotropic term of $S_h(f,\boldsymbol{n})$ in Eq. (9) into account. More specifically, Eq. (C21) and (C22) of Ref. [1] change into

$$\begin{aligned}&\langle h_A^*(f,\boldsymbol{n}) h_{A'}(f',\boldsymbol{n}') h_{A''}(f'',\boldsymbol{n}'') h_{A'''}^*(f''',\boldsymbol{n}''')\rangle_{h|\rho_g} = \\ &\langle h_A^*(f,\boldsymbol{n}) h_{A'}(f',\boldsymbol{n}')\rangle_{h|\rho_g} \langle h_{A''}^*(-f'',\boldsymbol{n}'') h_{A'''}(-f''',\boldsymbol{n}''')\rangle_{h|\rho_g} \\ &\langle h_A^*(f,\boldsymbol{n}) h_{A''}(f'',\boldsymbol{n}'')\rangle_{h|\rho_g} \langle h_{A'}^*(-f',\boldsymbol{n}') h_{A'''}(-f''',\boldsymbol{n}''')\rangle_{h|\rho_g} \\ &\langle h_A^*(f,\boldsymbol{n}) h_{A'''}(-f''',\boldsymbol{n}''')\rangle_{h|\rho_g} \langle h_{A'}^*(-f',\boldsymbol{n}') h_{A''}(f'',\boldsymbol{n}'')\rangle_{h|\rho_g} \,,\end{aligned} \tag{17}$$

and

$$\begin{aligned}&\langle h_A^*(f,\boldsymbol{n}) h_{A'}(f',\boldsymbol{n}') h_{A''}(f'',\boldsymbol{n}'') h_{A'''}^*(f''',\boldsymbol{n}''')\rangle_{h|\rho_g} = \\ &\frac{1}{4}\delta_{AA'}\delta_{A''A'''}\frac{\delta^2(\boldsymbol{n},\boldsymbol{n}')}{4\pi}\frac{\delta^2(\boldsymbol{n}'',\boldsymbol{n}''')}{4\pi}\delta(f-f')\delta(f''-f''')S_h(f,\boldsymbol{n})S_h(f'',\boldsymbol{n}'') \\ &+\frac{1}{4}\delta_{AA''}\delta_{A'A'''}\frac{\delta^2(\boldsymbol{n},\boldsymbol{n}'')}{4\pi}\frac{\delta^2(\boldsymbol{n}',\boldsymbol{n}''')}{4\pi}\delta(f-f'')\delta(f'-f''')S_h(f,\boldsymbol{n})S_h(f',\boldsymbol{n}') \\ &+\frac{1}{4}\delta_{AA'''}\delta_{A'A''}\frac{\delta^2(\boldsymbol{n},\boldsymbol{n}''')}{4\pi}\frac{\delta^2(\boldsymbol{n}',\boldsymbol{n}'')}{4\pi}\delta(f+f''')\delta(f'+f'')S_h(f,\boldsymbol{n})S_h(f',\boldsymbol{n}') \,.\end{aligned} \tag{18}$$

With these changes, further following the calculations in Appendix C of Ref. [1] leads to (as a modification to Eq. (C25) therein):

$$\begin{aligned}\left\langle \rho_{ab}^2 \right\rangle_{h|\rho_g} =& \frac{1}{4}\iint \mathrm{d}f\,\mathrm{d}f' \int \mathrm{d}\boldsymbol{n}\, S_h(f,\boldsymbol{n})\rho_{ab}^s(\boldsymbol{n}) \int \mathrm{d}\boldsymbol{n}'\, S_h(f',\boldsymbol{n}')\rho_{ab}^s(\boldsymbol{n}') \\ &+ \iint \mathrm{d}f\,\mathrm{d}f' \mathrm{sinc}^2\left(\pi(f-f')T\right) \int \mathrm{d}\boldsymbol{n}\, S_h(f,\boldsymbol{n})\rho_{aa}^s(\boldsymbol{n}) \int \mathrm{d}\boldsymbol{n}'\, S_h(f',\boldsymbol{n}')\rho_{bb}^s(\boldsymbol{n}') \\ &+ \frac{1}{4}\iint \mathrm{d}f\,\mathrm{d}f' \mathrm{sinc}^2\left(\pi(f-f')T\right) \int \mathrm{d}\boldsymbol{n}\, S_h(f,\boldsymbol{n})\rho_{ab}^s(\boldsymbol{n}) \int \mathrm{d}\boldsymbol{n}'\, S_h(f',\boldsymbol{n}')\rho_{ab}^s(\boldsymbol{n}') \,,\end{aligned} \tag{19}$$

where

$$\rho_{ab}^s(\boldsymbol{n}) = \sum_{A=+,\times} F_a^A(\boldsymbol{n}) F_b^A(\boldsymbol{n}) \,. \tag{20}$$

We note again that each of the $S_h(f,\boldsymbol{n})$ appearing in the expression above can be expressed as the same of an isotropic and anisotropic contribution, see Eq. (9). We therefore split the expression for $\left\langle \rho_{ab}^2 \right\rangle_{h|\rho_g}$ into a "standard" and a "clustering" part,

$$\left\langle \rho_{ab}^2 \right\rangle_{h|\rho_g} = \left\langle \rho_{ab}^2 \right\rangle_{h|\rho_g}^{\mathrm{st.}} + \left\langle \rho_{ab}^2 \right\rangle_{h|\rho_g}^{\mathrm{clust.}} \,. \tag{21}$$

The standard terms,

$$\begin{aligned}\left\langle \rho_{ab}^2 \right\rangle_{h|\rho_g}^{\mathrm{st.}} =& \frac{1}{4}\iint \mathrm{d}f\,\mathrm{d}f' \int \mathrm{d}\boldsymbol{n}\, \bar{S}_h(f)\rho_{ab}^s(\boldsymbol{n}) \int \mathrm{d}\boldsymbol{n}'\, \bar{S}_h(f')\rho_{ab}^s(\boldsymbol{n}') \\ &+ \iint \mathrm{d}f\,\mathrm{d}f' \mathrm{sinc}^2\left(\pi(f-f')T\right) \int \mathrm{d}\boldsymbol{n}\, \bar{S}_h(f)\rho_{aa}^s(\boldsymbol{n}) \int \mathrm{d}\boldsymbol{n}'\, \bar{S}_h(f')\rho_{bb}^s(\boldsymbol{n}') \\ &+ \frac{1}{4}\iint \mathrm{d}f\,\mathrm{d}f' \mathrm{sinc}^2\left(\pi(f-f')T\right) \int \mathrm{d}\boldsymbol{n}\, \bar{S}_h(f)\rho_{ab}^s(\boldsymbol{n}) \int \mathrm{d}\boldsymbol{n}'\, \bar{S}_h(f')\rho_{ab}^s(\boldsymbol{n}') \,,\end{aligned} \tag{22}$$

are exactly those specified in Eq. (C25) of Ref. [1]. The clustering terms, on the other hand, arise from the anisotropic part of $S_h(f,\boldsymbol{n})$ and are given by

$$\begin{aligned}\left\langle\rho_{ab}^2\right\rangle_{h|\rho_g}^{\text{clust.}} =& \iint \mathrm{d}z\,\mathrm{d}z'\left(\tilde{h}^2(z)\tilde{h}^2(z')+\tilde{g}^4(z,z')\right)\iint \mathrm{d}\boldsymbol{n}\,\mathrm{d}\boldsymbol{n'}\delta_g(\boldsymbol{n},z)\delta_g(\boldsymbol{n'},z')\rho_{ab}^s(\boldsymbol{n})\rho_{ab}^s(\boldsymbol{n'})\\ &+4\iint \mathrm{d}z\,\mathrm{d}z'\tilde{g}^4(z,z')\iint \mathrm{d}\boldsymbol{n}\,\mathrm{d}\boldsymbol{n'}\delta_g(\boldsymbol{n},z)\delta_g(\boldsymbol{n'},z')\rho_{aa}^s(\boldsymbol{n})\rho_{bb}^s(\boldsymbol{n'})\\ &+\text{terms linear in }\delta_g(\boldsymbol{n},z)\,,\end{aligned} \tag{23}$$

where

$$\begin{aligned}\tilde{h}^2(z)&=\int_0^\infty \mathrm{d}f\,\bar{S}_h(f)b_{\mathrm{GW}}(f,z)\,,\\ \tilde{g}^4(z,z')&=\frac{1}{4}\iint \mathrm{d}f\,\mathrm{d}f'\mathrm{sinc}^2\left(\pi(f-f')T\right)\bar{S}_h(f)\bar{S}_h(f')b_{\mathrm{GW}}(f,z)b_{\mathrm{GW}}(f,z')\,.\end{aligned} \tag{24}$$

Note that we do not write down the full expression for the terms linear in δ_g in Eq. (27), they are irrelevant since they vanish in the next step due to the fact that $\langle\delta_g(\boldsymbol{n},z)\rangle_{\mathfrak{p}_g}=0$. More precisely, taking the ensemble average $\langle\ldots\rangle_{\mathfrak{p}_g}$ leads to

$$\left\langle\rho_{ab}^2\right\rangle=\left\langle\left\langle\rho_{ab}^2\right\rangle_{h|\rho_g}\right\rangle_{\mathfrak{p}_g}=\left\langle\rho_{ab}^2\right\rangle^{\text{st.}}+\left\langle\rho_{ab}^2\right\rangle^{\text{clust.}}\,, \tag{25}$$

where (since the standard terms are not affected by the density field)

$$\left\langle\rho_{ab}^2\right\rangle^{\text{st.}}=\left\langle\rho_{ab}^2\right\rangle_{h|\rho_g}^{\text{st.}} \tag{26}$$

and

$$\begin{aligned}\left\langle\rho_{ab}^2\right\rangle^{\text{clust.}} =& \iint \mathrm{d}z\,\mathrm{d}z'\left(\tilde{h}^2(z)\tilde{h}^2(z')+\tilde{g}^4(z,z')\right)\iint \mathrm{d}\boldsymbol{n}\,\mathrm{d}\boldsymbol{n'}\xi_g(|\boldsymbol{n}-\boldsymbol{n'}|,z,z')\rho_{ab}^s(\boldsymbol{n})\rho_{ab}^s(\boldsymbol{n'})\\ &+4\iint \mathrm{d}z\,\mathrm{d}z'\tilde{g}^4(z,z')\iint \mathrm{d}\boldsymbol{n}\,\mathrm{d}\boldsymbol{n'}\,\xi_g(|\boldsymbol{n}-\boldsymbol{n'}|,z,z')\rho_{aa}^s(\boldsymbol{n})\rho_{bb}^s(\boldsymbol{n'})\,,\end{aligned} \tag{27}$$

where we have used Eq. (5) for the 2-point correlation function of δ_g and performed a change of variable, $\boldsymbol{n'}\mapsto\boldsymbol{\vartheta}=\boldsymbol{n'}-\boldsymbol{n}$.

The above equation can be further simplified using the Limber approximation, see e.g. Sec. 2.4.2 in [2]. This approximation (commonly used in cosmology for galaxy clustering and weak lensing, where it is very precise up to very large scales) is based on the fact that the correlation $\xi_g(\mathbf{k},z,z')$ falls off fast for $z\neq z'$, i.e. it is zero outside of a correlation length L. Assuming that $\tilde{h}^2(z)$ and $\tilde{g}^4(z,z')$ do not vary appreciably within this correlation length, we can thus simplify the expression as (cf. Eq. (2.83) in [2]):

$$\begin{aligned}\left\langle\rho_{ab}^2\right\rangle^{\text{clust.}} =& \int \mathrm{d}z\left(\tilde{h}^2(z)^2+\tilde{g}^4(z,z)\right)\int\frac{k\,\mathrm{d}k}{2\pi}P_g(k,z)\iint \mathrm{d}\boldsymbol{n}\,\mathrm{d}\boldsymbol{n'}\rho_{ab}^s(\boldsymbol{n})\rho_{ab}^s(\boldsymbol{n'})J_0\big(r_z|\boldsymbol{n}-\boldsymbol{n'}|k\big)\\ &+4\int \mathrm{d}z\,\tilde{g}^4(z,z)\int\frac{k\,\mathrm{d}k}{2\pi}P_g(k,z)\iint \mathrm{d}\boldsymbol{n}\,\mathrm{d}\boldsymbol{n'}\rho_{aa}^s(\boldsymbol{n})\rho_{bb}^s(\boldsymbol{n'})J_0\big(r_z|\boldsymbol{n}-\boldsymbol{n'}|k\big)\,,\end{aligned} \tag{28}$$

where $P_g(k,z)$ is the galaxy power spectrum (related to $\xi_g(d,z=z')$ via Fourier transformation).

Alternative form of Eq. (27)**:** An alternative to the Limber approximation would be to directly use the angular galaxy power spectrum $C_l(z,z')$: Using

$$\xi_g(\vartheta,z,z')=\frac{1}{4\pi}\sum_l(2l+1)C_l(z,z')\mathcal{P}_l(\cos\vartheta)\,, \tag{29}$$

with $\mathcal{P}_l$ being the Legendre polynomials, we can write the above as

$$\begin{aligned}&\left\langle\rho_{ab}^2\right\rangle^{\text{clust.}} =\\ &\iint \mathrm{d}z\,\mathrm{d}z'\left(\tilde{h}^2(z)\tilde{h}^2(z')+\tilde{g}^4(z,z')\right)\sum_l\frac{2l+1}{4\pi}C_l(z,z')\iint \mathrm{d}\boldsymbol{n}\,\mathrm{d}\boldsymbol{n'}\,\mathcal{P}_l(\boldsymbol{n}\cdot\boldsymbol{n'})\rho_{ab}^s(\boldsymbol{n})\rho_{ab}^s(\boldsymbol{n'})\\ &+4\iint \mathrm{d}z\,\mathrm{d}z'\tilde{g}^4(z,z')\sum_l\frac{2l+1}{4\pi}C_l(z,z')\iint \mathrm{d}\boldsymbol{n}\,\mathrm{d}\boldsymbol{n'}\mathcal{P}_l(\boldsymbol{n}\cdot\boldsymbol{n'})\rho_{aa}^s(\boldsymbol{n})\rho_{bb}^s(\boldsymbol{n'})\,,\end{aligned} \tag{30}$$

References

[1] B. Allen, "Variance of the Hellings-Downs correlation," *Phys. Rev. D* **107** no. 4, (2023) 043018, `arXiv:2205.05637 [gr-qc]`.

[2] M. Bartelmann and P. Schneider, "Weak gravitational lensing," *Phys. Rept.* **340** (2001) 291–472, `arXiv:astro-ph/9912508 [astro-ph]`.