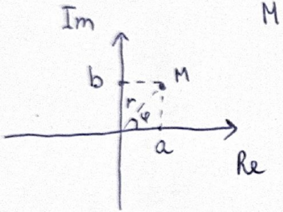
FFT

Комплексное число. Корень из комплексного числа.

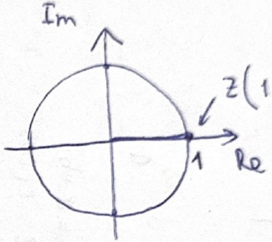
$$z = a + bi = r(\cos\varphi + i\sin\varphi),$$

$$r = \sqrt{a^2+b^2} \qquad \varphi = \operatorname{arctg}\frac{b}{a}$$

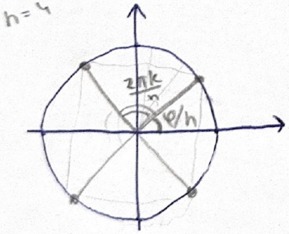
$$M(a,b) = a + bi$$



$$z = 1 + 0\cdot i = 1\,(\cos 2\pi k + i\sin 2\pi k)$$



$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\varphi+2\pi k}{n} + i\sin\frac{\varphi+2\pi k}{n}\right)$$

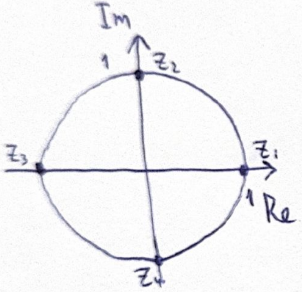


Корень n-степени из 1.

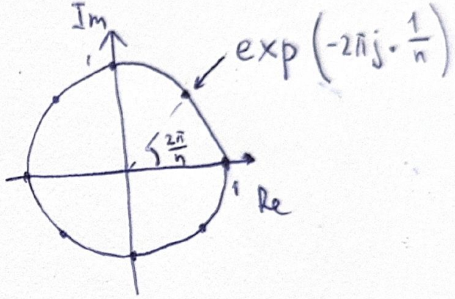
$\varphi = 0$

$$1 = 1\cdot\left(\cos 2\pi k + i \sin 2\pi k\right), \ k \in \mathbb{Z}$$

$n = 4$



$n = 8$



$$\sqrt[n]{1} = \sqrt[n]{|1|}\left(\cos\left(\frac{2\pi k + 0}{n}\right) + i \sin\left(\frac{2\pi k + 0}{n}\right)\right), \ k = 0, 1 \ldots n-1$$

Свойства корней единицы

Корни единицы

$$\exp\left(-2\pi j \frac{k}{N}\right) = W_N^k \quad , k = 0, 1 \ldots n-1$$

- $W_n^{n-k} = \exp\left(-2\pi j \frac{n-k}{n}\right) = \exp\left(+2\pi j \frac{k}{n}\right) = \overline{W_n^k}$
- $W_n^{2k} = \exp\left(-2\pi j \frac{2k/2}{n/2}\right) = \exp\left(-2\pi j \frac{k}{n/2}\right) = W_{\frac{n}{2}}^k$

Разложение X[u]

DFT:

$$X[u] = \sum_{n=0}^{N-1} x[n] W_N^{un} = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x[2n] W_N^{2un}}_{x[0]W_N^{0} + x[2]W_N^{2u}\dots} + \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x[2n+1] W_N^{u(2n+1)}}_{x[1]W_N^{u} + x[3]W_N^{3u}+\dots} =$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[2n] W_{\frac{N}{2}}^{un} + W_N^{u} \sum_{n=0}^{\frac{N}{2}-1} x[2n+1] W_{\frac{N}{2}}^{un} = E[u] + W_N^{u} O[u]$$

$$E[u] = \sum_{n=0}^{\frac{N}{4}-1} x[4n] W_N^{4un} + \sum_{n=0}^{\frac{N}{4}-1} x[4n+2] W_N^{u(4n+2)} = \sum_{n=0}^{\frac{N}{4}-1} x[4n] W_{\frac{N}{4}}^{un} + W_{\frac{N}{2}}^{u} \sum x[4n+2] W_{\frac{N}{4}}^{un}$$

$$= E'[u] + W_{\frac{N}{2}}^{u} O'[u]$$

$$O[u] = \sum_{n=0}^{\frac{N}{4}-1} x[4n+1] W_{\frac{N}{4}}^{un} + W_{\frac{N}{2}}^{un} \sum_{n=0}^{\frac{N}{4}-1} x[4n+3] W_{\frac{N}{4}}^{un} = E''[u] + W_{\frac{N}{2}}^{u} O''[u]$$

$$X\left[u+\frac{N}{2}\right] = \sum_{n=0}^{N/2-1} x[2n]\, W_N^{2n\left(u+\frac{N}{2}\right)} + \sum_{n=0}^{N/2-1} x[2n+1]\, W_N^{(2n+1)\left(u+\frac{N}{2}\right)}$$

$$W_N^{2nu+nN} = W_N^{2nu} = W_{\frac{N}{2}}^{nu} \qquad\qquad W_N^{(2n+1)u+nN+\frac{N}{2}} = W_N^{2nu}\cdot(-1)$$

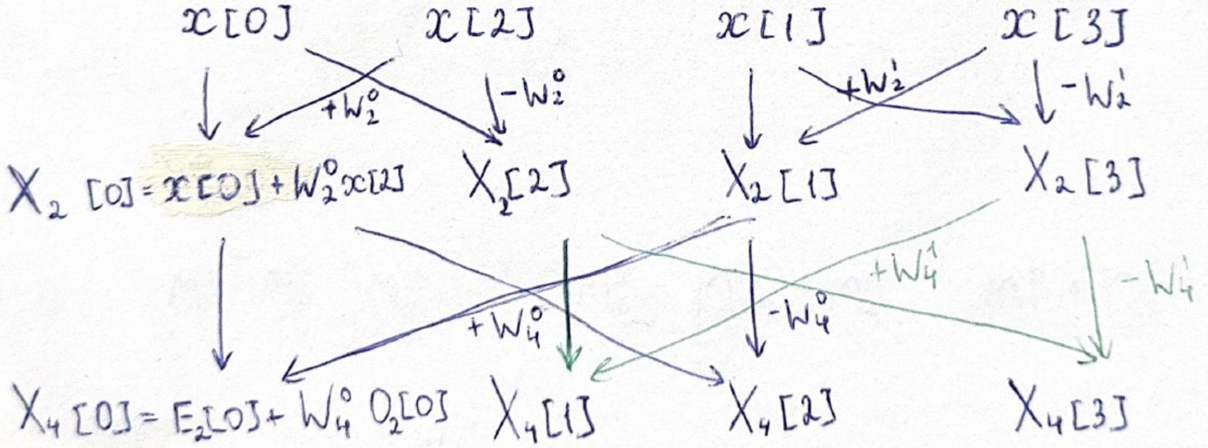
$$= E[u] - W_N^{u}\, O[u]$$

$$X_N[u] = E_{\frac{N}{2}}[u] + W_N^u\, O_{\frac{N}{2}}[u]$$

$$X_N[u+\tfrac{N}{2}] = E_{\frac{N}{2}}[u] - W_N^u\, O_{\frac{N}{2}}[u]$$

$$X_{\frac{N}{2}}[u] = E_{\frac{N}{2}}[u] + W_{\frac{N}{2}}^u[u]$$

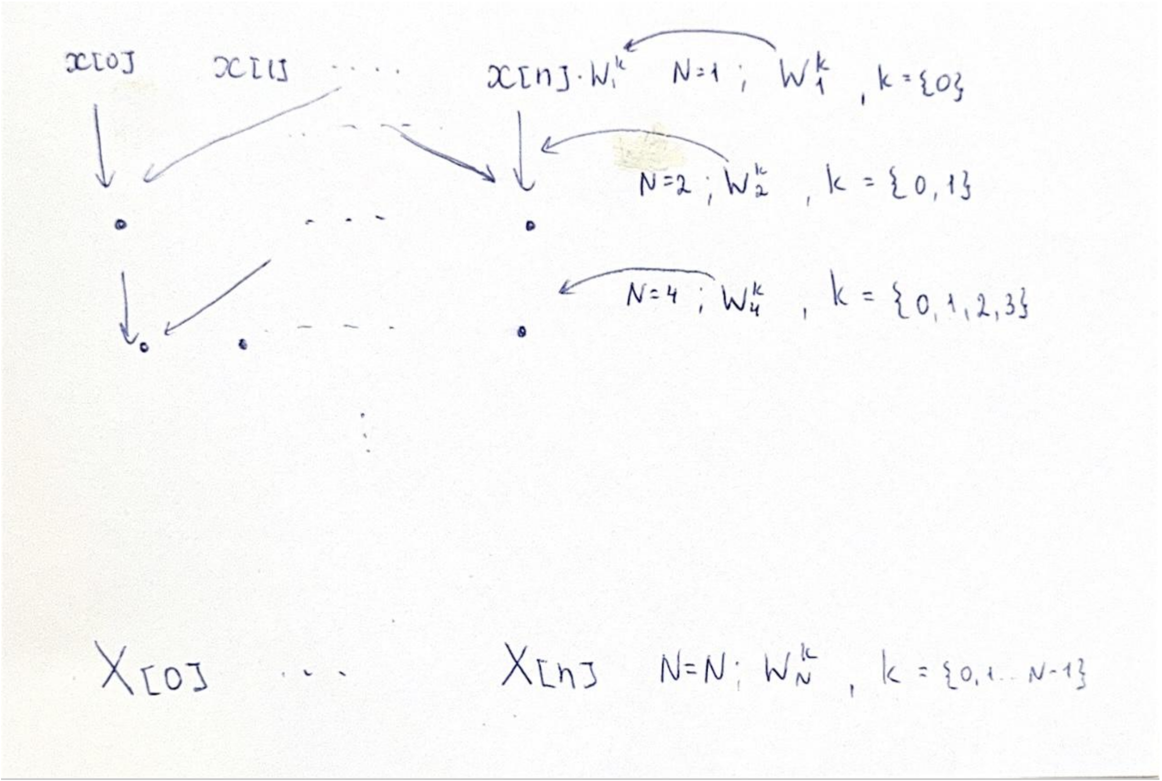
$N=4$:



$W_2^k \qquad k \subseteq [0,1]$

$W_4^k \qquad k \subseteq [0,1,2,3]$

На каждом уровне k < N



Вычисления W

Для пространства размерности N в fft достаточно вычислить N/4 экспонент:

$$W_N^0, W_N^1, \ldots, W_N^{N/4}$$

Остальные можно получить по свойствам:

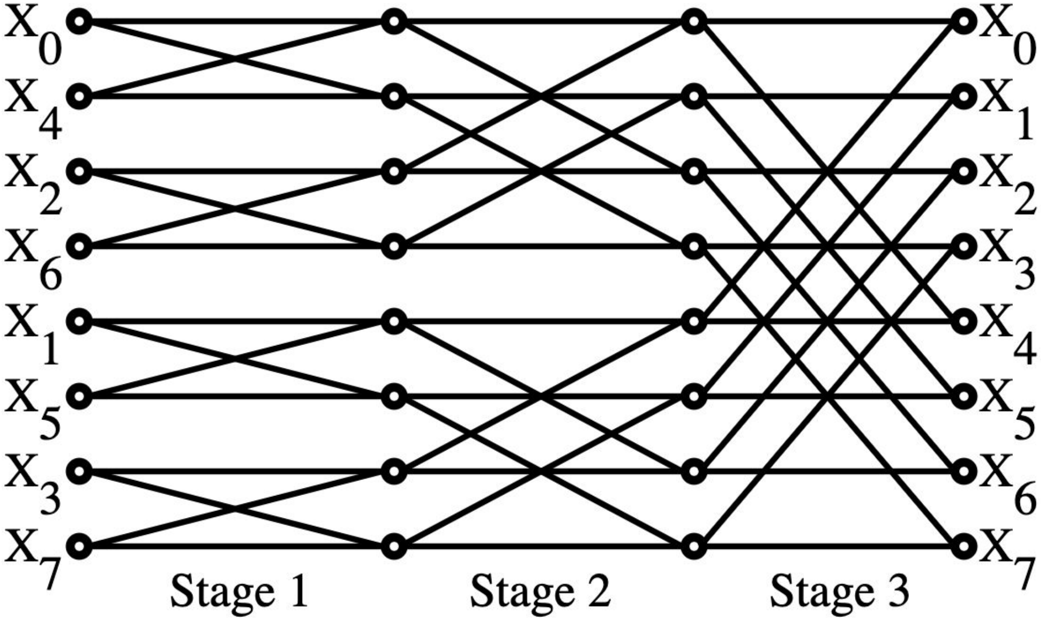
$$W_N^{N-k} = conj(W_N^k); W_N^{k+N/2} = -W_N^k$$

$$\underbrace{W_N^0, W_N^1 \cdots W_N^{N/4}}_{S_0}, \underbrace{W_N^{N/4+1} \cdots W_N^{N/2-1}}_{S_1}, \underbrace{W_N^{N/2}, W_N^{N/2+1}, \dots W_N^{N-1}}_{S_2}$$

Для S_1 : $W_N^{k+\frac{N}{4}} = W_N^k \cdot \exp\left(-\frac{\pi j}{2}\right) = -i W_N^k$

Для S_2 : $W_N^{k+\frac{N}{2}} = -W_N^k$

N=8:



W_2^k

W_4^k

W_8^k

$$W_2^k = W_8^{4k}; W_4^k = W_8^{2k}$$

$$W_8^k = e^{-\pi i k/4} = \begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}}(1-i) \\ -i \\ \frac{1}{\sqrt{2}}(-1-i) \\ -1 \\ \frac{1}{\sqrt{2}}(-1+i) \\ i \\ \frac{1}{\sqrt{2}}(1+i) \end{bmatrix}$$

2D fft

$$L[u,v] = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} l[n,m] \exp\left(-2\pi j\left(\frac{un}{N} + \frac{vm}{M}\right)\right) =$$

$$= \sum_{n=0}^{N-1} \exp\left(-2\pi j \frac{vm}{M}\right) \sum_{m=0}^{M-1} l[n,m] \cdot \exp\left(\frac{un}{N} \cdot (-2\pi j)\right) =$$

$$= fft_n\left(fft_m(l)\right)$$

Resources

FFT use in polynomial multiplication:

https://youtu.be/h7apO7q16V0?si=DMjA4b6AvThnaETI

DFT and FFT lecture:

https://www.robots.ox.ac.uk/~sjrob/Teaching/SP/l7.pdf

2D fft MatLab:

https://ch.mathworks.com/help/matlab/ref/fft2.html