Northeastern University, Boston, MA

College of Engineering

Department of Civil and Environmental Engineering

CIVE 7380: Performance Models and Simulation of Transportation Networks

Problem Set II

From: **Nathan David Obeng-Amoako**

To: Professor Haris N Koutsopoulos

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Department of Civil and
Environmental Engineering
400 Snell Engineering
360 Huntington Ave.
Boston, MA 02115

CIVE 7380
Problem Set #2
Due: Wednesday, February 5, 2025

1. Consider the discrete random variable X with PMF:

$$p_X(x) = \begin{cases} \dfrac{x^2}{a} & if\ x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & otherwise \end{cases}$$

   a. Find the value of *a*
   b. Find the E[X]
   c. Find the variance, Var(X), of X

2. A continuous random variable X has the following pdf:

$$f_X(x) = \begin{cases} \dfrac{x}{4} & if\ 1.0 < x \le 3.0 \\ 0 & otherwise \end{cases}$$

   a. Determine the numerical value of E[X]
   b. Find the P(A), where A is the event x ≥ 2

3. The random variable X, describing the time spent by cars paying for the toll at a tollbooth is exponentially distributed with parameter λ

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & otherwise \end{cases}$$

   a. What is the E[X]?
   b. Find the P(X ≥ E[X]).
   c. The toll authority has collected data on the actual times spent at the tollbooth by 10 cars as follows: 20 sec, 35 sec, 15 sec, 30 sec, 25 sec, 10 sec, 45 sec, 30 sec, 28 sec, 32 sec. Using this data approximate the value of λ in the above model.
   d. Plot the pdf and the cdf for the value of λ found in part d).

4. The inter-arrival times (time headways) for cars passing a checkpoint are independent random variables with pdf:

$$f_T(t) = \begin{cases} 2e^{-2t} & t \geq 0 \\ 0 & otherwise \end{cases}$$

Where, the inter-arrival times are measured in minutes.

a. Interpret the value of parameter 2 in the above pdf. What are the units?
b. Describe how the Poisson and the exponential distributions are related using the above as an example.
c. Determine the mean and the variance of the inter-arrival times (headways).
d. Given that no car has arrived in the last four minutes, determine the pdf of the time until the next arrival.

5. The inter-arrival times of cars at a tollbooth are exponentially distributed with an average time between arrivals of ½ minutes.
   a. What is the value of parameter λ for this problem?
   b. What is the variance of inter-arrival times?
   c. What is the probability that the time between two consecutive arrivals is between 0 and 1 minutes?
   d. What is the probability that 4 vehicles will arrive in 5 minutes? (HINT: remember the relationship between Poisson and exponential distributions).

6. An airport operating a single runway has a capacity of 24 landings/hr. The airport authority expects an increase in demand for landings to 25 landings per hour. Assuming that arrival times and service times are exponential:
   a. Can the current capacity accommodate the expected demand?
   b. The authority can increase the capacity of the airport, for example with additional air traffic controllers and minor improvements in the system. If the capacity is between 26 and 35 landings/hour the associated cost to provide that capacity (in $/hour) is given by:

      C = 22,000 + 2,300*Capacity

      where C is operating cost in $/hr and Capacity is the capacity in landings per hour (e.g. 35 landings/hr).

      The cost to the airlines landing in the airport (fuel and crew costs) is $47.00 per hour of delay. In addition, the delay cost to passengers in the planes is $5.00 per hour of delay per passenger. There are 200 passengers aboard each plane.
      Make a recommendation to the airport authority in terms of the capacity that minimizes the total cost (airport, airline, passengers). Consider all values of capacity between 26 and 35 landings per hour (to calculate total costs for comparison).

7. The arrival rate at a toll booth is 75 veh/hr. The average time that each vehicle spends at the booth is 24 sec.
    a. What is the service rate at the toll booth?
    b. If the arrival and service patterns are deterministic what is the average queue length at the toll booth?
    c. Assume now that inter-arrival times are exponential with rate λ= 75 veh/hr. and that service times are exponential with rate μ equal to the one found in part a). Based on this information the operations at the toll booth can be modeled as an M/M/1 queue. Find the following quantities:
        i. Utilization ratio
        ii. Probability of an empty booth
        iii. Probability of exactly 1 vehicle in the system
        iv. Average waiting time in queue
        v. Average time in the system (waiting and service)
        vi. Expected number of vehicles in system
        vii. Expected number of vehicles in queue
        viii. Using the above values verify Little's formulae

Solution to Q1

Given:

$$p_X(x) = \begin{cases} \frac{x^2}{a}, & if\ x = -3, -2, -1, 0, 1, 2, 3 \\ 0, & otherwise \end{cases}$$

Find the value of a

The sum of all probabilities must equal 1. That is:

$$\sum_{x=-3}^{3} p_X(x) = 1$$

This implies that:

$$\frac{(-3)^2}{a} + \frac{(-2)^2}{a} + \frac{(-1)^2}{a} + \frac{(0)^2}{a} + \frac{(1)^2}{a} + \frac{(2)^2}{a} + \frac{(3)^2}{a} = 1$$

$$a = 28$$

Therefore:

$$p_X(x) = \begin{cases} \frac{x^2}{28}, & if\ x = -3, -2, -1, 0, 1, 2, 3 \\ 0, & otherwise \end{cases}$$

Find $E[X]$ and $Var(X)$

For a discrete random variable X:

$$E[X] = \sum_{x=-3}^{3} x \cdot p_X(x)$$

$$Var(X) = E[X^2] - (E[X])^2$$

x	$p_X(x)$	$x \cdot p_X(x)$	$x^2 \cdot p_X(x)$
-3	0.32143	-0.96429	2.89286
-2	0.14286	-0.28571	0.57143
-1	0.03571	-0.03571	0.03571
0	0.00000	0.00000	0.00000
1	0.03571	0.03571	0.03571
2	0.14286	0.28571	0.57143
3	0.32143	0.96429	2.89286
Sum	1.00000	0.00000	7.00000

Hence:

$E[X] = 0$

$Var(X) = 7 - 0 = 7$

Solution to Q2

Find $E[X]$

For a continuous random variable:

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

This implies that:

$$E[X] = \int_{1}^{3} x \cdot \frac{x}{4}\, dx$$

$$E[X] = \frac{1}{4}\int_{1}^{3} x^2\, dx$$

$$E[X] = 2.1\dot{6}$$

Find $P(X \geq 2)$

$$P(X \geq 2) = \int_{2}^{3} f_X(x)\, dx$$

$$P(X \geq 2) = \int_{2}^{3} \frac{x}{4}\, dx$$

$$P(X \geq 2) = \frac{1}{4}\int_{2}^{3} x\, dx$$

$$P(X \geq 2) = 0.625$$

Solution to Q3

Given:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

Find $E[X]$

For a continuous random variable:

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$E[X] = \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x}\, dx$$

I will apply integration by parts:

$$\int u\, dv = dv - \int v\, du$$

Let:

$u = x$ and $dv = \lambda e^{-\lambda x}\, dx$

This implies that:

$du = dx$ and $v = -e^{-\lambda x}$

Hence:

$$E[X] = \left[-xe^{-\lambda x}\right]_0^{\infty} + \int_{0}^{\infty} e^{-\lambda x}\, dx$$

Evaluating the boundary term:

$$\left[-xe^{-\lambda x}\right]_0^\infty = \lim_{x\to\infty}\left(-xe^{-\lambda x}\right) = 0$$

We now have:

$$E[X] = \int_0^\infty e^{-\lambda x}\,dx$$

$$E[X] = \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^\infty$$

As $x \to \infty$, $e^{-\lambda x} \to 0$. This implies that:

$$E[X] = \frac{1}{\lambda}$$

Find the $P(X \geq E[X])$

$$P(X \geq E[X]) = \int_{1/\lambda}^\infty \lambda e^{-\lambda x}\,dx$$

$$P(X \geq E[X]) = \left[-e^{-\lambda x}\right]_{1/\lambda}^\infty$$

$$P(X \geq E[X]) = e^{-1}$$

$$P(X \geq E[X]) = 0.3679$$

Approximate the value of λ

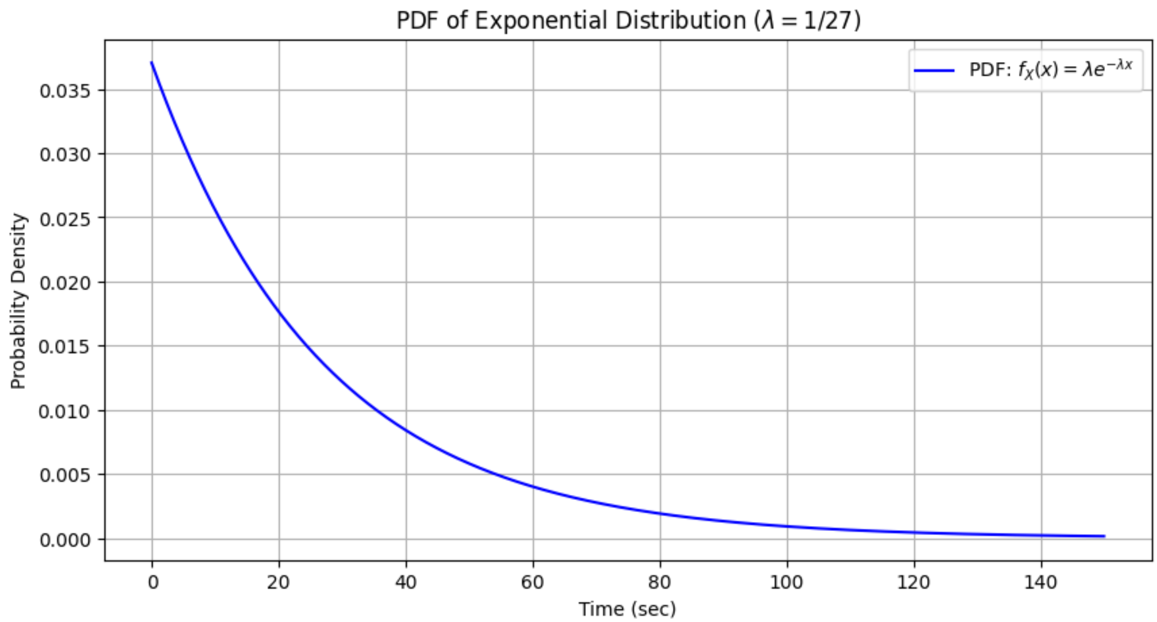
Mean service time:

$$\frac{1}{10}(20 + 35 + 15 + 30 + 25 + 10 + 45 + 30 + 28 + 32) = 27\ s$$

Service rate, $\lambda = 1/27$ or $0.037\ s^{-1}$

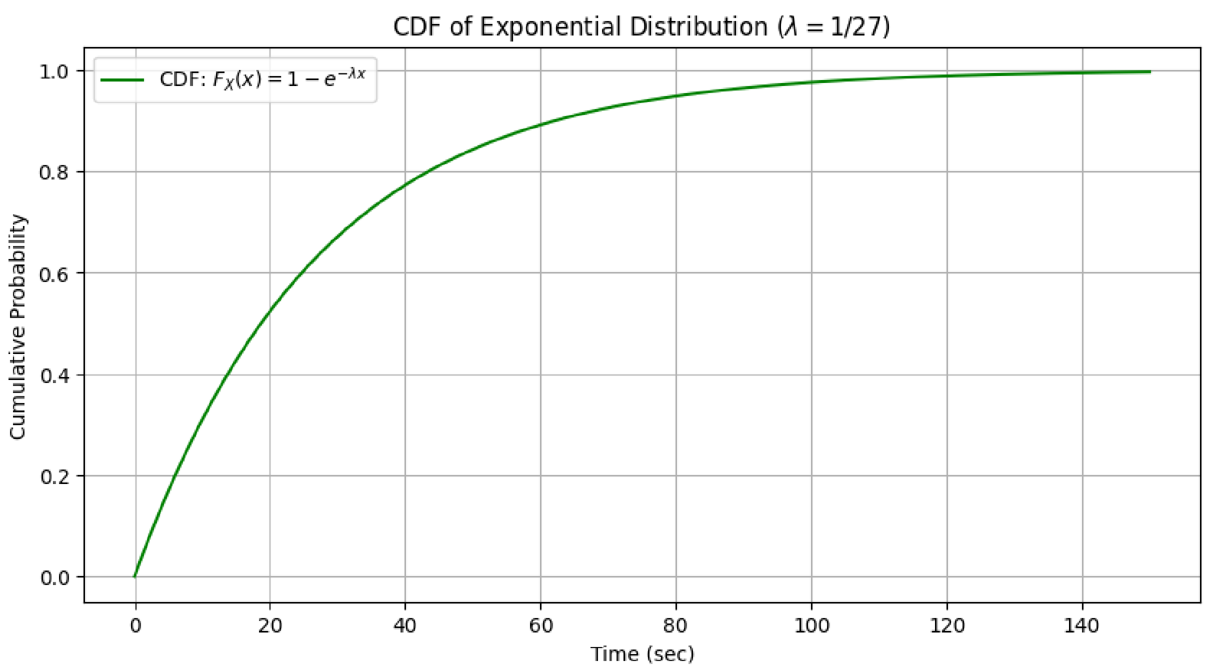
PDF Plot given $\lambda = 1/27$

$$f_X(x) = \frac{1}{27} e^{-\frac{x}{27}}$$



CDF Plot given $\lambda = 1/27$

$$F_X(x) = 1 - e^{-\frac{x}{27}}$$



Solution to Q4

Given:

$$f_T(t) = \begin{cases} 2e^{-2t}, & t \geq 0 \\ 0, & otherwise \end{cases}$$

Interpretation of the value of parameter 2

The 2 here stands for λ in the general equation, and it represents the average number of cars passing through the checkpoint every minute. Hence, the unit is $cars\ arrivals/minute$.

Relation between Poisson and exponential distributions

From the above example, the exponential distribution models the time between successive car arrivals at the checkpoint (0.5 minutes per arrival). On the other hand, the Poisson distribution models the number of car arrivals in a minute (2 arrivals per minute).

Mean and the variance of the inter-arrival times

For an exponential distribution:

$$Mean = \frac{1}{\lambda} = \frac{1}{2} = 0.5\ minutes$$

$$Variance = \frac{1}{\lambda^2} = \frac{1}{2^2} = 0.25\ minutes^2$$

Pdf of the time until the next arrival given that no car has arrived in the last four minutes

According to the memoryless property of the exponential distribution, the probability of an event occurring is independent of how much time has already passed. Mathematically:

$$P(T > t + s | T + t) = P(T > s)$$

This means that the pdf doesn't change. It remains:

$$f_T(t) = \begin{cases} 2e^{-2t}, & t \geq 0 \\ 0, & otherwise \end{cases}$$

Solution to Q5

Value of parameter λ

$$\lambda = \frac{1}{0.5} = 2$$

Variance of inter-arrival times

$$Variance = \frac{1}{\lambda^2} = \frac{1}{2^2} = 0.25\, minutes^2$$

The probability that the time between 2 consecutive arrivals is between 0 and 1 minute

$$P(0 \le T \le 1) = F(1) = F(0)$$

The CDF is: $F(x) = 1 - e^{-2t}$

Hence:

$$P(0 \le T \le 1) = (1 - e^{-2}) - (1 - e^{0}) = 0.135$$

Probability that 4 vehicles will arrive in 5 minutes

$$P(N = k) = \frac{(\lambda t)^k e^{-2t}}{k!}$$

$$P(N = 4) = \frac{(2 \times 5)^4 \times e^{-2\times 5}}{4!}$$

$$P(N = 4) = 0.0189$$

Solution to Q6

Airport operating cost in \$/hr, $C_{airport}$

$$C_{airport} = 22{,}000 + 2{,}300\mu$$

Delay cost in \$/hr, C_{delay}

$$C_{delay} = 47 + 5(200) = 1{,}047$$

Because arrival times and service times are exponential and there's only one server, then the associated queueing model to use is $\boldsymbol{M/M/1}$. This means that:

The average number of planes in the system, L, is given by:

$$L = \frac{\rho}{1-\rho}$$

The average delay per plane, W_q, is given by:

$$W_q = \frac{1}{\mu - \lambda}$$

Where:

ρ = utilization ratio

μ = service rate

λ = arrival rate

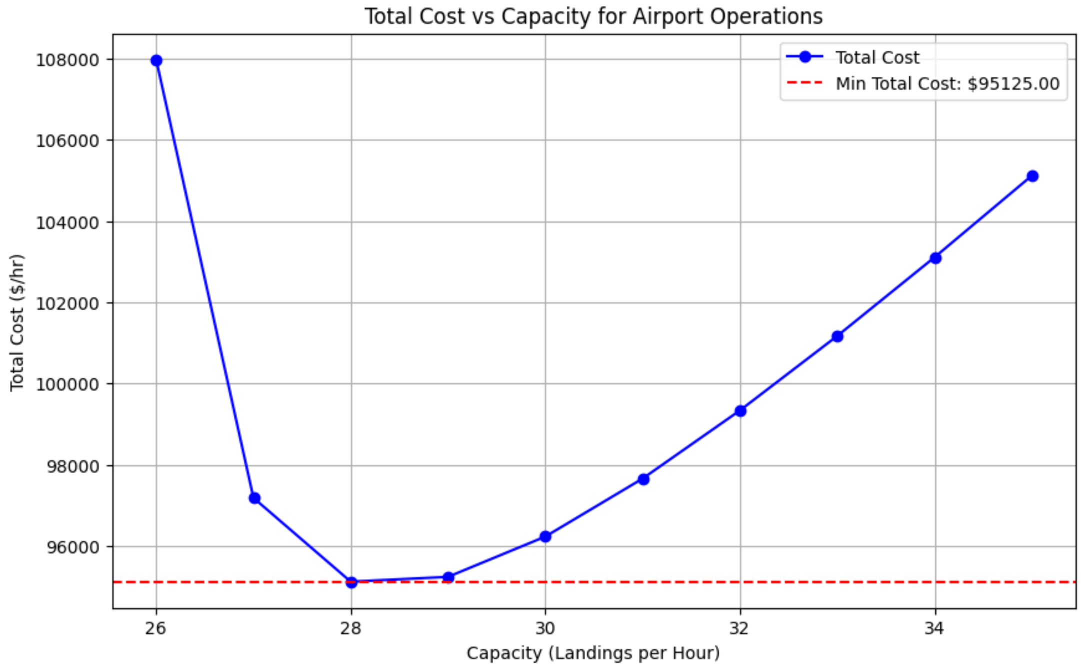
If planes arrive at a rate of λ planes/hour and are being delayed on average for W_q hours, then the average number of planes waiting in the queue (not including the plane being served) will be $L_q = \lambda W_q$. Hence, the total delay cost will be $\lambda W_q C_{delay}$.

Hence, the overall total cost (summation of airport operating cost and delay cost) will be:

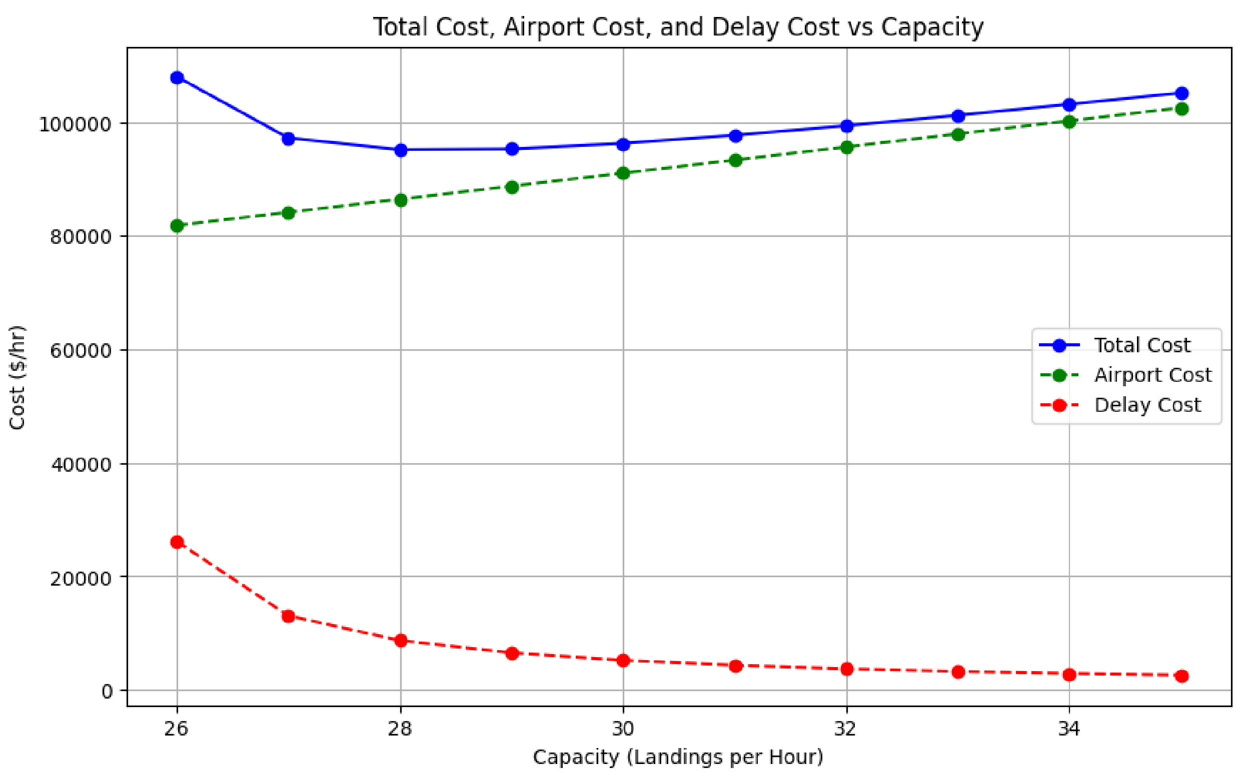
$$Total\ Cost = C_{airport} + \lambda \times W_q \times C_{delay}$$

$$Total\ Cost = 22{,}000 + 2{,}300\mu + 1{,}047\frac{25}{\mu - 25}$$

The minimum cost is found to be **$ 95,125.00** and it occurs at a capacity of 28 landings/hour. I recommend that the airport only increases capacity to 28 landings/hour to minimize cost.



The following plot shows how airport cost, delay cost and total cost vary with capacity.



Solution to Q7

Given:

$\lambda = 75\ veh/hr$

$\frac{1}{\mu} = 24\ s = \frac{24}{3,600} = \frac{1}{150}\ hr$

What is the service rate at the toll booth?

$\mu = 150\ veh/hr$

If the arrival and service patterns are deterministic what is the average queue length at the toll booth?

$Average\ queue\ length = \lambda \times \frac{1}{\mu} = 75 \times \frac{1}{150} = 0.5\ vehicles$

Part C (M/M/1 queueing policy)

Utilization ratio

$$\rho = \frac{\lambda}{\mu} = \frac{75}{150} = 0.5$$

Probability of an empty booth

$$P(0) = 1 - \rho = 0.5$$

Probability of exactly 1 vehicle in the system

$$P(1) = (1 - \rho)\rho = 0.25$$

Average waiting time in queue

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$W_q = \frac{75}{150(150 - 75)} = 24\ s$$

Average time in the system (waiting and service)

$$W = \frac{1}{\mu - \lambda}$$

$$W = \frac{1}{150 - 75} = 48\ s$$

Expected number of vehicles in the system

$$L = \frac{\lambda}{\mu - \lambda}$$

$$L = \frac{75}{150 - 75} = 1\ vehicle$$

Expected number of vehicles in queue

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$L_q = \frac{75^2}{150(150 - 75)} = 0.5\ vehicles$$

Verifying Little's formulae

$L = 1$ and $\lambda W = 75 \times \frac{1}{75} = 1$

Also, $L_q = 0.5$ and $\lambda W_q = 75 \times \frac{1}{150} = 0.5$

Verified!