Northeastern

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**CIVE 7380**
**Problem Set #2**
**Due: Wednesday, February 5, 2025**

1. Consider the discrete random variable X with PMF:

$$p_X(x) = \begin{cases} \dfrac{x^2}{a} & if\ \ x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & otherwise \end{cases}$$

   a. Find the value of $a$
   b. Find the E[X]
   c. Find the variance, Var(X), of X

2. A continuous random variable X has the following pdf:

$$f_X(x) = \begin{cases} \dfrac{x}{4} & if\ 1.0 < x \le 3.0 \\ 0 & otherwise \end{cases}$$

   a. Determine the numerical value of E[X]
   b. Find the P(A), where A is the event x ≥ 2

3. The random variable X, describing the time spent by cars paying for the toll at a tollbooth is exponentially distributed with parameter $\lambda$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & otherwise \end{cases}$$

   a. What is the E[X]?
   b. Find the P(X ≥ E[X]).
   c. The toll authority has collected data on the actual times spent at the tollbooth by 10 cars as follows: 20 sec, 35 sec, 15 sec, 30 sec, 25 sec, 10 sec, 45 sec, 30 sec, 28 sec, 32 sec. Using this data approximate the value of $\lambda$ in the above model.
   d. Plot the pdf and the cdf for the value of $\lambda$ found in part d).

4. The inter-arrival times (time headways) for cars passing a checkpoint are independent random variables with pdf:

$$f_T(t) = \begin{cases} 2e^{-2t} & t \geq 0 \\ 0 & otherwise \end{cases}$$

Where, the inter-arrival times are measured in minutes.

   a. Interpret the value of parameter 2 in the above pdf. What are the units?
   b. Describe how the Poisson and the exponential distributions are related using the above as an example.
   c. Determine the mean and the variance of the inter-arrival times (headways).
   d. Given that no car has arrived in the last four minutes, determine the pdf of the time until the next arrival.

5. The inter-arrival times of cars at a tollbooth are exponentially distributed with an average time between arrivals of ½ minutes.
   a. What is the value of parameter $\lambda$ for this problem?
   b. What is the variance of inter-arrival times?
   c. What is the probability that the time between two consecutive arrivals is between 0 and 1 minutes?
   d. What is the probability that 4 vehicles will arrive in 5 minutes? (HINT: remember the relationship between Poisson and exponential distributions).

6. An airport operating a single runway has a capacity of 24 landings/hr. The airport authority expects an increase in demand for landings to 25 landings per hour. Assuming that arrival times and service times are exponential:
   a. Can the current capacity accommodate the expected demand?
   b. The authority can increase the capacity of the airport, for example with additional air traffic controllers and minor improvements in the system. If the capacity is between 26 and 35 landings/hour the associated cost to provide that capacity (in $/hour) is given by:

      C = 22,000 + 2,300*Capacity

      where C is operating cost in $/hr and Capacity is the capacity in landings per hour (e.g. 35 landings/hr).

      The cost to the airlines landing in the airport (fuel and crew costs) is $47.00 per hour of delay. In addition, the delay cost to passengers in the planes is $5.00 per hour of delay per passenger. There are 200 passengers aboard each plane.
      Make a recommendation to the airport authority in terms of the capacity that minimizes the total cost (airport, airline, passengers). Consider all values of capacity between 26 and 35 landings per hour (to calculate total costs for comparison).

7. The arrival rate at a toll booth is 75 veh/hr. The average time that each vehicle spends at the booth is 24 sec.
   a. What is the service rate at the toll booth?
   b. If the arrival and service patterns are deterministic what is the average queue length at the toll booth?
   c. Assume now that inter-arrival times are exponential with rate $\lambda$= 75 veh/hr. and that service times are exponential with rate $\mu$ equal to the one found in part a). Based on this information the operations at the toll booth can be modeled as an M/M/1 queue. Find the following quantities:
      i. Utilization ratio
      ii. Probability of an empty booth
      iii. Probability of exactly 1 vehicle in the system
      iv. Average waiting time in queue
      v. Average time in the system (waiting and service)
      vi. Expected number of vehicles in system
      vii. Expected number of vehicles in queue
      viii. Using the above values verify Little's formulae