Mathematical Explanation of the Trips Crossing Segment Formula

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Introduction

This document explains the formula used to determine whether a trip (k, m) crosses a given segment $i \to j$ in a circular loop of N stops. The explanation is presented in the context of public transportation, where trips are paths that traverse a set of stops in either a clockwise or counterclockwise direction.

The Formula

The trips crossing a segment $i \to j$ are given by:

$$\{(k, m) \mid k \neq m, \exists s \in \text{path}(k \to m), s = i \text{ and } (s + 1) \equiv j \pmod{N}\}.$$

Explanation of the Formula

The formula can be interpreted as follows:

1. (k, m): Represents a trip starting at stop k and ending at stop m, where $k \neq m$.
2. $\text{path}(k \to m)$: Refers to the sequence of stops traversed during the trip.
3. $\exists s \in \text{path}(k \to m)$: There exists a stop s within the path where $s = i$, indicating the segment start.
4. The relationship between i and j:
    - For clockwise direction: $(s + 1) \equiv j \pmod{N}$, ensuring that j is the next stop after i.
    - For counterclockwise direction: $(s - 1) \equiv j \pmod{N}$, ensuring that j is the stop before i.

Derivation of the Formula

The derivation is based on the following principles:

1. **Circular Loop Behavior:** In a circular loop, the stops are arranged sequentially such that stop N is followed by stop 1. This cyclic behavior is mathematically modeled using modular arithmetic.

2. **Segment Crossing:** A trip (k, m) crosses the segment $i \to j$ if:
   - Stop i is included in the trip's path.
   - The next stop in the path after i (clockwise) or before i (counterclockwise) is j.
3. **Path Representation:** The path $\text{path}(k \to m)$ is constructed by:
   - Starting at k.
   - Moving sequentially in the chosen direction (clockwise or counterclockwise).
   - Terminating at m.
4. **Conditions for Crossing:** The segment $i \to j$ is crossed if i appears in the path, and its subsequent or preceding stop (depending on direction) is j. Modular arithmetic ensures that the sequence wraps around for circular behavior.

Examples

Consider $N = 6$ stops arranged in a circular loop.

Clockwise Example

Given $i = 6$, $j = 1$, and the direction is clockwise:

1. j is calculated as $(i \mod N) + 1 = 1$.
2. Trip $(5, 2)$ with path $\{5, 6, 1, 2\}$ crosses $6 \to 1$.
3. Trip $(3, 5)$ with path $\{3, 4, 5\}$ does not cross $6 \to 1$.

Counterclockwise Example

Given $i = 6$, $j = 5$, and the direction is counterclockwise:

1. j is calculated as $((i - 2 + N) \mod N) + 1 = 5$.
2. Trip $(1, 5)$ with path $\{1, 6, 5\}$ crosses $6 \to 5$.
3. Trip $(4, 2)$ with path $\{4, 3, 2\}$ does not cross $6 \to 5$.

Conclusion

The formula leverages modular arithmetic to handle the circular nature of the stops and checks whether a segment is crossed by examining the path of trips.