Continuous Functions

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Given two domains, $\mathbb{X} = (X, \bot_X, \sqsubseteq_X)$ and $\mathbb{Y} = (Y, \bot_Y, \leq_y)$ The set $Cont(X,Y) = \{f : X \to Y\}$ where:

- $\forall x, x' \in X.\ x \sqsubseteq_X x' \Rightarrow f(x) \leq_Y f(x')$
- $x \in Chain(X) \Rightarrow f(\sqcup x_i) = \sqcup f(x_i)$

The relation $\sqsubseteq_C$ is defined as

$$\sqsubseteq_C = \{(f,g)\ | f, g \in Cont(X,Y)\ \wedge\ \forall x \in X.\ f(x) \leq_Y g(x)\}$$

1 $\forall f \in Cont(X,Y).\ \bot \sqsubseteq_C f$

$\bot_{X \to Y}$ is defined as the function $\bot = \lambda x.\bot(x)$, the function that loops on all inputs. The output of this function will always be $\bot$, because it does not terminate. So for all $x \in X$ we have $\bot \leq_Y f(x)$. As $\mathbb{Y}$ is a domain we know this holds for every element of Y and as the codomain of f is Y, every $f(x)$ is in Y. Therefore $\bot \sqsubseteq_C f$.

2 Prove $\sqsubseteq_C$ is a partial order

For $\sqsubseteq_C$ to be a partial order, it must be reflexive, antisymmetric and transitive. As $\mathbb{Y}$ is a domain, we know that $\leq_Y$ is a partial order.

Reflexivity We need to prove that $\forall f \in Cont(X,Y).\ f \sqsubseteq_C f$. We can rewrite this using the definition of $\sqsubseteq_C$ to get

$$\forall f \in Cont(X,Y).\ (\forall x \in X.\ f(x) \leq_Y f(x))$$

Functions are single valued, so we know $\forall f.\forall x.\ f(x) = f(x)$ and as $\leq_Y$ is reflexive we know $\forall f.\forall x \in X.\ f(x) \leq_Y f(x)$. Therefore we have $f \sqsubseteq_C f$, for any $f \in Cont(X,Y)$.

Antisymmetry We need to prove that $\forall f, g \in Cont(X,Y).\ f \sqsubseteq_C g \wedge g \sqsubseteq_C f \Rightarrow f = g$. Rewriting this using the definition of $\sqsubseteq_C$ gives us

$$\forall f, g \in Cont(X,Y).\ (\forall x \in X.\ f(x) \leq_Y g(x) \wedge g(x) \leq_Y f(x) \Rightarrow f(x) = g(x))$$

$\leq_Y$ is antisymmetric, so we have $\forall x \in X.\ f(x) = g(x)$, for any values of f and g. Therefore $\sqsubseteq_C$ is also antisymmetric.

Transitivity We need to prove that $\forall f, g, h \in Cont(X,Y).\ f \sqsubseteq_C g \wedge g \sqsubseteq_C h \Rightarrow f \sqsubseteq_C h$. Rewriting this using the definition of $\sqsubseteq_C$ gives us

$$\forall f, g, h \in Cont(X,Y).\ (\forall x \in X.\ (f(x) \leq_Y g(x) \wedge g(x) \leq_Y h(x)) \Rightarrow f(x) \sqsubseteq_C h(x))$$

As $\leq_Y$ is transitive, we have $\forall x \in X. f(x) \leq_Y h(x)$, for all f, g and h. Therefore $\sqsubseteq_C$ is also transitive.

All three properties hold, so $\sqsubseteq_C$ is a partial order on $Cont(X,Y)$

3 All chains have a least upper bound (limit)

There are two things we must prove, for all chains f in $Chain(Cont(X,Y))$:

- $\exists z \in Cont(X,Y).\ \forall i. f_i \sqsubseteq_C z$
- $\exists z \in Cont(X,Y).\ \forall g. (\forall i. f_i \sqsubseteq_C g) \Rightarrow z \sqsubseteq_C g$

where z is the least upper bound of the chain. Let $z = \lambda x. \sqcup^Y f_i(x)$, where $\sqcup^Y f_i(x)$ is the limit of the chain obtained by applying the functions in $Cont(X,Y)$ to a certain element $x \in X$.

A chain of functions will be a chain of elements of the set $Cont(X,Y)$, for example

$$f_1 \sqsubseteq_C f_2 \sqsubseteq_C \ldots \sqsubseteq_C \sqcup f_i$$

If we expand this using the definition of $\sqsubseteq_C$ we have

$$\forall x \in X.\ (f_1(x) \leq_y f_2(x) \leq_Y \ldots \leq_Y \sqcup f_i(x))$$

This is a set of chains in $Chain(\mathbb{Y})$ where every chain contains the result of each function on a certain x value. As $\mathbb{Y}$ is a domain, for any chain using the elements of Y, rhe least upper bound is defined. Therefore we know that the least upper bound $\sqcup f_i(x)$ is defined. Now we can see that this is the same as our definition of z.

$$z = \lambda x. \sqcup^Y f_i(x)$$

For the second part of the proof, we can rewrite it using the definition of $\sqsubseteq_C$ as

$$\exists z \in Cont(X, Y).\ \forall x \in X.\ (\forall g.(\forall i.f_i(x) \leq_Y g(x)) \Rightarrow z(x) \leq_Y g(x))$$

As $\mathbb{Y}$ is a domain, $(\forall i.f_i(x) \leq_Y g(x)) \Rightarrow z(x) \leq_Y g(x))$ holds for each of our individual chains for each $x \in X$. Therefore we have $\exists z \in Cont(X, Y).\ \forall g.(\forall i.f_i \sqsubseteq_C g) \Rightarrow z \sqsubseteq_C g$