

We can define mathematical operators on numbers as recursive functions.
For example addition is the following:

add 0 y = y add s(n) y = s(add n y)

We can write this as the following λ abstraction:

$$\text{add} = \lambda x, y : \text{nat} . \text{case}(x, z \mapsto y, s(v) \mapsto s(\text{add } v \ y))$$

This is a recursive function, so must be the fixpoint of another function A :

$$A = \lambda f : \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat} . \lambda x, y : \text{nat} . \text{case}(x, z \mapsto y, s(v) \mapsto s(f \ v \ y))$$

then add $x \ y = (\text{fix } A) \ x \ y$

So we write this in PCF as:

$$\text{fix } f : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat} . A : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$$

When we try to evaluate this term, we get the following, by the evaluation rule for fix:

$\text{fix } f : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat} . A : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat} = [f : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat} . A : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat} / f]A$. This expands to:

$$\begin{aligned} & \lambda x, y : \text{nat} . \text{case}(x, z \mapsto y, s(v) \mapsto \\ & s((\text{fix } f : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat} . A : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}) \ v \ y)) \end{aligned}$$

Therefore, expanding this infinitely gives all possible executions of the addition function.