Adequacy of PCF

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Submitted in conformity with the requirements
for the degree of MSc Advanced Computer Science
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Abstract

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We give a proof of Adequacy, a theorem relating the denotational semantics of PCF, a typed language with recursion, to its operational semantics, using the logical relations technique. To do this, we first develop operational and denotational semantics of PCF, proving type safety and some necessary lemmas for this, and soundness of PCF. Finally we consider whether all possible elements in our denotational semantic model are denotations of syntactically correct PCF terms.

Keywords: *Computational Adequacy, Denotational Semantics, Domain Theory, Logical Relations, Operational Semantics, PCF*

Acknowledgements

Contents

Chapter 1

Introduction

Semantics is the study of the meaning of computer programs. We require this study because we can easily compile syntactically correct programs of a language into machine code, but this does not tell us how the programs actually behave. There are many different approaches to semantics, but the main three are operational [Plotkin, 1981], denotational [Scott, 1993] and axiomatic semantics [Hoare, 1969].

In this report, we are interested in the first two approaches. Operational semantics describes how a syntactic program is reduced to some other syntactic value either by using a transition relation, or "running" the program on an abstract machine. In denotational semantics, we construct a mathematical model of the language and analyse that instead of the syntax of the language.

In general, denotational semantics is considered to be more abstract, enabling us to remove detail that is unnecessary for analysing behaviour and just makes proofs less readable, whereas operational semantics can be more easily linked to an implementation of a language. Also, the operational semantics of a language are usually proved to be sound with relation to the Denotational Semantics, in a theorem called Correctness/Soundness (see Section 6.3). In the other direction the theorem is usually referred to as Completeness. We will see that a result this strong cannot be proved for all languages and instead we relate the denotational model of PCF to its operational semantics using a theorem called Adequacy (see Chapter 7). Therefore having both of these approaches to semantics is beneficial to analyse a the language in different ways. For example, later in the report we prove a property called Type Safety (see Chapter 5), for which we use just the operational semantics of the language, but to prove the operational semantics is sound we require both approaches.

1.1 History

Early examples of creating a mathematical model for a programming language include Landin's model for Algol 60 using the λ calculus [Landin, 1964, 1965]. and Gordon's denotational model for LISP [Gordon, 1973].

The language we model is PCF (Programming Computable Functions). It is a programming language that is analysed by the research community more often than it is actually compiled and used to write programs. However, it forms a subset of popular functional languages like ML [Milner, 1997] and Haskell [Marlow et al., 2010].

It was first discussed as a programming language and given semantics by Plotkin, [Plotkin, 1977], but is inspired by a "logic of computable functions" defined in [Scott, 1993], (and described by [Milner, 1973]), a very famous article that was widely known and written in 1969 (but not published until 1993 at Scott's request). In this paper, Scott defines a logical system and explains how it can modelled by a specific mathematical structure, partially ordered domains of continuous functions (which we define in section 2.1).

A theorem prover was created for LCF in the work of [Milner, 1972a], with an explanation of its working given by [Milner, 1972b], in which the syntax and semantics of a simple language are described in the logic. It was applied to describe and prove the correctness of a compiler in [Milner and Weyhrauch, 1972].

LCF was developed into the modern proof assistants HOL and Isabelle [Nipkow et al., 2002] and influenced many other systems. Also, it resulted in the creation of ML, which started as an aid for LCF, as discussed in [Gordon, 2000].

Therefore although PCF is a research language, it has many applications, so studying it is worthwhile. [Plotkin, 1977] defines the theorem of Adequacy, which is what we ultimately prove, by using a technique called Logical Relations, which is a different approach to Plotkin's proof, but also documented in modern textbooks such as [Streicher, 2006].

1.2 Structure

In Chapter 2, we explain *Domain Theory*, a area of mathematics studying the structures that we use to specify our denotational semantics and *Logical Relations*, a proof technique required for proving Adequacy

In chapter 3, we define the syntax of PCF and give the typing rules, which all syntactically correct PCF programs must satsify. In Chapter 4, we specify the operational semantics of PCF and in Chapter 5, we prove Type Safety, which states that all correctly typed PCF programs must have a valid evaluation in the operational semantics if they are expected to.

In Chapter 6, we define our denotational semantics of PCF and prove that the operational semantics are sound with respect to this. In Chapter 7, we prove the Adequacy theorem and in Chapter 8 contrast our proof with other approaches to the proof of Adequacy.

Finally our work is evaluated (in Chapter 10) and concluded (in Chapter 11).

Chapter 2

Background

There are some mathematical topics and proof techniques, which are very important in semantics, that are used in this report. To prove Adequacy, we need a model for our language in which we can work.

The most obvious choice is to model the language in sets, where we can use a set to model each type in PCF. However, for function types, if a function of some type never terminates, then its result will not be in the set of that type, and we cannot model it. Therefore we need some additional "undefined" element in the model to represent this.

We also want to compare different programs in the model, which we can do by using an information ordering, where more defined programs are higher in the ordering and the element representing non-termination is the lowest. The usefulness of the ordering will become apparent when we discuss recursion (see 2.4).

A good ordering to use is a partial order, which is defined as follows:

Definition 1. *A partial order on a set X is a binary relation $\sqsubseteq$ that it is reflexive, anti-symmetric and transitive, which are the following properties:*

- $\forall x \in X,\ x \sqsubseteq x$ *Reflexivity*
- $\forall x, y \in X.\ x \sqsubseteq y \wedge y \sqsubseteq x \rightarrow x = y$ *Antisymmetry*
- $\forall x, y, z \in X.\ x \sqsubseteq y \wedge y \sqsubseteq z \rightarrow x \sqsubseteq z$ *Transitivity*

Therefore, we need a structure which includes an underlying set, an partial order on the set and an undefined element - this is a **domain**.

2.1 Domain Theory

There is an entire area of mathematics devoted to exploring these structures called *Domain Theory*, as described in [Gunter, 1992] and [Hutton, 2014].

Generally domains can be defined in two different ways, either by using chains or by using directed sets.

Definition 2. *A **chain** is a sequence $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$ where every element is larger than the preceding one.*

Formally we define a chain as a function $\mathbb{N} \to X$, such that if $i \leq j$ then $x_i \sqsubseteq x_j$

As an example, given the lifted set $\mathbb{N}_\perp$ (a set with $\perp$ added) then a chain can be formed in three ways:

- $\perp \sqsubseteq \cdots \sqsubseteq \perp$, for any number of $\perp$s
- $n \sqsubseteq \cdots \sqsubseteq n$, for any number of n , where n is the same number each time
- $\perp \sqsubseteq \cdots \sqsubseteq \perp \sqsubseteq n \sqsubseteq \cdots \sqsubseteq n$, for any number of $\perp$s followed by any number of identical ns

This is because the ordering on the lifted set is only defined between numbers and themselves, $\perp \sqsubseteq \perp$ and $\perp \sqsubseteq n$ for some $n \in \mathbb{N}$.

Definition 3. *A **directed set** X, is a set where*

$$\forall x, y \in X.\ \exists z \in X.\ x \sqsubseteq z\ \wedge y\ \sqsubseteq z$$

Domains can be defined using either of these structures. If we take all the elements in a chain as a set, then the least upper bound will be the value of z for any two elements of the set. Therefore a chain is an example of a directed set.

For the rest of the report, we use the definition of domains that uses chains, as opposed to directed sets.

Definition 4. *A domain $(X, \perp, \sqsubseteq)$ consists of a set X, an element $\perp$ and a relation $\sqsubseteq \subseteq X_\perp \times X_\perp$ such that:*

- $\forall x \in X.\ \perp \sqsubseteq X$
- $\sqsubseteq$ *is a partial order*

- *All chains of elements of $X_\perp$ have a limit (i.e. a least upper bound). To prove this there are two properties we must prove, for some $z \in X_\perp$*
  - $\forall i.x_i \sqsubseteq z$ (z is the upper bound)
  - $\forall y.(\forall i.x_i \sqsubseteq y) \Rightarrow\ z \sqsubseteq y$ (z is the least upper bound)

The limit of a chain is usually written as $\bigsqcup x_n$, where x_n is a chain of length n and x_i is the element at the ith position in the chain.

2.2 Examples of Domains

2.2.1 Single Element Domain

In a single element domain the underlying set is $\{x\}$, so he only element $\perp$ can be is x and $\sqsubseteq$ just contains the pair (x, x)

Now we must prove the three conditions in the domain definition to show that $(\{x\}, x, \sqsubseteq)$ is a domain):

1. $\forall x \in \{x\}.\ x \sqsubseteq x$
   There is only one element, x, and $x \sqsubseteq x$ is in the ordering.
2. $\sqsubseteq$ **is a partial order**
   - Reflexivity
     The only element is x and $x \sqsubseteq x$ is in the ordering
   - Antisymmetry
     Any x and y must both be the element x, as it is the only possible element, so $x = x$.
   - Transitivity
     Any x, y and z must all be x and $x \sqsubseteq x$ is in the ordering.
3. **All chains must have a least upper bound**
   As x is the only possible element, all chains will be of the form

   $$x \sqsubseteq x \sqsubseteq x \sqsubseteq \ldots$$

   Let $\bigsqcup x_n = x$. Then we need an element z such that the following two statements are true, so let $z = \sqcup x_n = x$. Then:
   - $\forall i.x_i \sqsubseteq x$
     The only possible x_i is x, so we must have $x \sqsubseteq x$. This is in the ordering.

- $\forall y.(\forall i.x_i \sqsubseteq y) \Rightarrow x \sqsubseteq y$
  The only possible value of y is x. Therefore we must have $x \sqsubseteq x$, which is in the ordering.

Now we have proved all the conditions, so the single element domain is a domain.

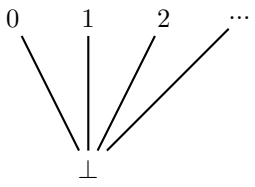
2.2.2 Flat Natural Numbers

A domain is a **flat** domain if the only ordering is between $\bot$ and the elements of the underlying set. In this example the underlying set is $\mathbb{N}$, the set of natural numbers, so the only orderings we have are $\bot \sqsubseteq 0$, $\bot \sqsubseteq 1$, $\bot \sqsubseteq 2$, etc...

Therefore the relation is defined as:

$$\sqsubseteq = \{(\bot, \bot)\} \cup \{(\bot, n), (n, n) \mid n \in \mathbb{N}\}$$

which gives us the following picture:



Now we must prove three conditions to show $(\mathbb{N}, \bot, \sqsubseteq)$ is a domain:

1. $\forall x \in \mathbb{N}_\bot.\ \bot \sqsubseteq x$
   In the definition of our relation we have $\{(\bot, n) \mid n \in \mathbb{N}\}$ and $(\bot, \bot)$. so $\{(\bot, x) \mid x \in \mathbb{N}_\bot\}$ is a subset of $\sqsubseteq$.

2. **Prove $\sqsubseteq$ is a partial order**
   For $\sqsubseteq$ to be a partial order, it must be reflexive, antisymmetric and transitive.

   - Reflexivity
     We know this from the definition of $\sqsubseteq$, as $\{(x, x) \mid x \in \mathbb{N}_\bot\}$ is a subset of $\sqsubseteq$.

   - Antisymmetry
     When $x = \bot$, the only possible y we can have such that $y \sqsubseteq x$, is $y = \bot$, as $n \sqsubseteq \bot$ is not defined in the relation for any n. Therefore $x = y = \bot$.

     When $x = n$, the only possible value of y is n, so $x = y = n$.

- Transitivity
  If $x = \bot$ and $y = n$, then we must have $z = n$ for (y, z) to be in $\sqsubseteq$. Then we need $\bot \sqsubseteq n$, which we have, as we have $(\bot, n)$, for any $n \in \mathbb{N}$, defined in the relation.

  If $x = \bot = y$, then we have two options for z. When $z = n$, we should have $\bot \sqsubseteq n$, which we have, as we have $(\bot, n)$ for any $n \in \mathbb{N}$ defined in the relation. When $z = \bot$, we just want $\bot \sqsubseteq \bot$, which is also in the definition of $\sqsubseteq$.

  If $x = n$, then both y and z must also be equal to n for $x \sqsubseteq y$ and $y \sqsubseteq z$ to be defined. Therefore we should have $n \sqsubseteq n$. This is in the definition of $\sqsubseteq$.

3. **All chains must have a least upper bound**
   For any chain of elements of $\mathbb{N}_\bot$, we must prove there is some $z \in \mathbb{N}_\bot$ satisfying the two statements. We prove this by case analysis on the different chains:

   (a) $\bot \sqsubseteq \cdots \sqsubseteq \bot$
   For these chains, let $z = \bot$. The last element in the chain will always be $\bot$, so for every i we have $\bot \sqsubseteq \bot$. Therefore $\forall i.x_i \sqsubseteq \bot$. For the second statement, as every element is $\bot$, $x_i = \bot$ and $y = \bot$, we have $\bot \sqsubseteq \bot$ for $z \sqsubseteq y$. Therefore $\forall y.(\forall i.x_i \sqsubseteq y) \Rightarrow \bot \sqsubseteq y$ holds.

   (b) $n \sqsubseteq \cdots \sqsubseteq n$
   For these chains, let $z = n$. The last element in the chain will always be n, so for every n we have $n \sqsubseteq n$. Therefore $\forall i.x_i \sqsubseteq n$.

   For the second part, every element is n, so $x_i = n$ and $y = n$. Then we have $n \sqsubseteq n$ for $z \sqsubseteq y$. Therefore $\forall y.(\forall i.x_i \sqsubseteq y) \Rightarrow n \sqsubseteq y$ holds.

   (c) $\bot \sqsubseteq \cdots \sqsubseteq \bot \sqsubseteq n \sqsubseteq \cdots \sqsubseteq n$
   For these chains, let $z = n$. The last element will be n. We have both $\bot \sqsubseteq n$ and $n \sqsubseteq n$ in the relation, so for any x, we have $x \sqsubseteq n$. Therefore $\forall i.x_i \sqsubseteq n$.

   For the second part, $(\forall i.x_i \sqsubseteq y)$ is only true when $y = n$, so we only have to consider this case. Then we have $n \sqsubseteq n$ for $z \sqsubseteq y$. Therefore $\forall y.(\forall i.x_i \sqsubseteq y) \Rightarrow n \sqsubseteq y$ holds.

2.2.3 Product of domains

Given two domains $\mathbb{X} = (X, \bot_X, \sqsubseteq_X)$ and $\mathbb{Y} = (Y, \bot_Y, \leq_Y)$, we have a new domain, where $X \times Y$ is the underlying set, the bottom element is $(\bot_X, \bot_Y)$ and $(x, y) \sqsubseteq (x', y')$ is defined when $x \sqsubseteq_X x'$ and $y \leq_Y y'$ are defined.

Now we must prove three conditions to show that $\mathbb{X} \times \mathbb{Y}$ is a domain:

1. $\forall (x, y) \in X \times Y.\ \bot \sqsubseteq (x, y)$
   Because $\mathbb{X}$ is a domain, we know $\forall x \in X.\ \bot_X \sqsubseteq_X x$ and because $\mathbb{Y}$ is a domain, we know $\forall y \in Y.\ \bot_Y \leq_Y y$.

   Therefore we have $\forall x, y.\ \bot_X \sqsubseteq_X x \wedge \bot_Y \leq_Y y$. This is the same as $\forall (x, y) \in X \times Y.(\bot_X, \bot_Y) \sqsubseteq (x, y)$

2. **$\sqsubseteq$ is a partial order**

   - Reflexivity
     For an element $(x, y) \in X \times Y$, we have $x \sqsubseteq_X x$ and $y \leq_Y y$ because $\mathbb{X}$ and $\mathbb{Y}$ are domains, so their orderings are reflexive. This means we have $(x, y) \sqsubseteq (x, y)$.

   - Antisymmetry
     For elements (x, y) and (x', y') we can assume $(x, y) \sqsubseteq (x', y')$ and $(x', y') \sqsubseteq (x, y)$. Expanding these definitions we have $x \sqsubseteq_X x' \wedge y \leq_Y y' \wedge x' \sqsubseteq_X x \wedge y' \leq_Y y$. If we reorder this we have:

     $$x \sqsubseteq_X x' \wedge x' \sqsubseteq_X x \wedge y \leq_Y y' \wedge y' \leq_Y y$$

     As the orderings on $\mathbb{X}$ and $\mathbb{Y}$ are antisymmetric, we can rewrite this as $x = x'$ and $y = y'$. Therefore we have $(x, y) = (x', y')$.

   - Transitivity
     For elements $(x, y), (x', y')$ and (x'', y'') we can assume $(x, y) \sqsubseteq (x', y')$ and $(x', y') \sqsubseteq (x'', y'')$. Expanding these definitions gives us $x \sqsubseteq_X x' \wedge y \leq_Y y' \wedge x' \sqsubseteq_X x'' \wedge y' \leq_Y y''$. If we reorder this we have:

     $$x \sqsubseteq_X x' \wedge x' \sqsubseteq_X x'' \wedge y \leq_Y y' \wedge y' \leq_Y y''$$

     As the orderings on $\mathbb{X}$ and $\mathbb{Y}$ are transitive, we can rewrite this as $x \sqsubseteq_X x''$ and $y \leq_Y y''$. Therefore we can now define $(x, y) \sqsubseteq (x'', y'')$.

3. **All chains have a least upper bound**
   Chains of $\mathbb{X} \times \mathbb{Y}$ will be of the form:

   $$(x, y) \sqsubseteq (x', y') \sqsubseteq (x'', y'') \sqsubseteq \ldots$$

   where $x \sqsubseteq_X x' \sqsubseteq_X x'' \ldots$ and $y \leq_Y y' \leq_Y y'' \ldots$.

   So we need an element z that makes the following two statements true. Let $z = \bigsqcup (x, y)_n = (\bigsqcup x_n, \bigsqcup y_n)$. Then:

- $\forall i.(x_i, y_i) \sqsubseteq z$
  As $\mathbb{X}$ and $\mathbb{Y}$ are domains, we have $\forall i.\ x_i \sqsubseteq_X \bigsqcup x_n$ and $\forall i.\ y_i \leq_Y \bigsqcup y_n$. Therefore, for any (x, y) we have $\forall i.(x_i, y_i) \sqsubseteq (\bigsqcup x_n, \bigsqcup y_n)$.

- $\forall (x', y').\ (\forall i.(x_i, y_i) \sqsubseteq (x', y')) \Rightarrow z \sqsubseteq (x', y')$
  As $\mathbb{X}$ and $\mathbb{Y}$ are domains, we have $\forall x'.\ (\forall i.\ x_i \sqsubseteq_X x') \Rightarrow \bigsqcup x_n \sqsubseteq_X x'$ and $\forall y'.\ (\forall i.\ y_i \leq_Y y') \Rightarrow \bigsqcup y_n \leq_Y y'$.

  Therefore if we assume $\forall (x', y').(\forall i.(x_i, y_i) \sqsubseteq (x', y'))$, then we know $\bigsqcup x_n \sqsubseteq_X x'$ and $\bigsqcup y_n \leq_Y y'$. This is the definition of $(\bigsqcup x_n, \bigsqcup y_n) \sqsubseteq (x', y')$.

Now we have proved all the conditions, so the product of two domains is also a domain.

2.3 Monotone and Continuous Functions

There are two different types of functions that we will use when modelling PCF functions:

Definition 5. *A **monotone** function, f, is a function that preserves the order of a partially ordered set.*

Given a chain $x_0 \sqsubseteq x_1 \sqsubseteq \ldots$, we can form the chain $f(x_0) \sqsubseteq f(x_1) \sqsubseteq \ldots$ using a monotone function.

Definition 6. *A **continuous** function f is a function which when applied to the limit of a chain gives the same result as the limit of the chain formed by applying f to every element of another chain. Formally:*

$$f(\bigsqcup x_n) = \bigsqcup (f(x_n))$$

Therefore continuous functions must also be monotone:

Theorem 1. *Comtinuous functions are monotone*

Proof. Given a chain $x_1 \sqsubseteq \cdots \sqsubseteq x_n$, its limit will be x_n. Then $f(x_n) = f(\bigsqcup (x_n)) = \bigsqcup (f x_n)$. Therefore every other element in $f(x_n)$'s chain must be related to it. If we remove x_n, we have the chain ending in x_{n-1}. The least upper bound of this chain is x_{n-1}, so we have $f(x_{n-1}) = f(\bigsqcup (x_{n-1})) = \bigsqcup (f x_{i-1})$. Everything below this element must be related to it. We repeat this for every lower element until we have $f(x_1)$ alone, which is related to itself. Therefore we get a chain $f(x_1) \sqsubseteq \cdots \sqsubseteq f(x_n)$, so f must be monotonic. □

2.3.1 Domain of Continuous Functions

We can form a domain of continuous functions between two other domains:

Given two domains, $\mathbb{X} = (X, \bot_X, \sqsubseteq_X)$ and $\mathbb{Y} = (Y, \bot_Y, \leq_y)$, we can form the set $\text{Cont}(X, Y) = \{f : X \to Y\}$, of continuous functions between the underlying sets, where:

- $\forall x, x' \in X.\ x \sqsubseteq_X x' \Rightarrow f(x) \leq_Y f(x')$ $\quad$ f preserves the ordering of chains in $\mathbb{X}$
- $x_n \in \text{Chain}(X) \Rightarrow f(\sqcup x_n) = \sqcup f(x_n)$ $\quad$ f is continuous

where $\text{Chain}(X)$ is the set of all possible chains we can form from $\mathbb{X}$.

$\bot_{X \to Y}$ is defined as the function $\bot = \lambda x.\bot(x)$, the function that loops on all inputs. The output of this function will always be $\bot$, because it does not terminate.

The relation $\sqsubseteq_C$ is defined as

$$\sqsubseteq_C = \{(f, g) \mid f, g \in \text{Cont}(X, Y)\ \wedge\ \forall x \in X.\ f(x) \leq_Y g(x)\}$$

Therefore our domain will be $(\text{Cont}(X, Y), \bot_{X \to Y}, \sqsubseteq_C)$. Now we must prove the three conditions:

1. $\forall f \in \text{Cont}(X, Y).\ \bot_{X \to Y} \sqsubseteq_C f$
   For all $x \in X$ we have $\bot \leq_Y f(x)$. As $\mathbb{Y}$ is a domain we know this holds for every element of Y and as the codomain of f is Y, every $f(x)$ is in Y. Therefore $\bot_{X \to Y} \sqsubseteq_C f$.

2. **Prove $\sqsubseteq_C$ is a partial order**
   As $\mathbb{Y}$ is a domain, we know that $\leq_Y$ is a partial order.

   - Reflexivity
     We need to prove that $\forall f \in \text{Cont}(X, Y).\ f \sqsubseteq_C f$. We can rewrite this using the definition of $\sqsubseteq_C$ to get

     $$\forall f \in \text{Cont}(X, Y).\ (\forall x \in X.\ f(x) \leq_Y f(x))$$

     Functions are single valued, so we know $\forall f.\ \forall x.\ f(x) = f(x)$ and as $\leq_Y$ is reflexive we know $\forall f. \forall x \in X.\ f(x) \leq_Y f(x)$. Therefore we have $f \sqsubseteq_C f$, for any $f \in \text{Cont}(X, Y)$.

   - Antisymmetry
     We need to prove that $\forall f, g \in \text{Cont}(X, Y).\ ((f \sqsubseteq_C g)\ \wedge\ (g \sqsubseteq_C f)) \Rightarrow f = g$. Rewriting this using the definition of $\sqsubseteq_C$ gives us

     $$\forall f, g \in \text{Cont}(X, Y).\ (\forall x \in X.\ ((f(x) \leq_Y g(x)) \wedge (g(x) \leq_Y f(x))) \Rightarrow f(x) = g(x))$$

$\leq_Y$ is antisymmetric, so we have $\forall x \in X.\ f(x) = g(x)$, for any values of f and g. Therefore $\sqsubseteq_C$ is also antisymmetric.

- Transitivity
  We need to prove that $\forall f, g, h \in \text{Cont}(X, Y).\ ((f \sqsubseteq_C g)\ \wedge\ (g \sqsubseteq_C h)) \Rightarrow f \sqsubseteq_C h$.
  Rewriting this using the definition of $\sqsubseteq_C$ gives us

$$\forall f, g, h \in \text{Cont}(X, Y).\ (\forall x \in X.\ ((f(x) \leq_Y g(x)) \wedge (g(x) \leq_Y h(x))) \Rightarrow f(x) \sqsubseteq_C h(x))$$

  As $\leq_Y$ is transitive, we have $\forall x \in X. f(x) \leq_Y h(x)$, for all f, g and h. Therefore $\sqsubseteq_C$ is also transitive.

3. **All chains have a least upper bound (limit)**

For any chain f_n, we must have some $z \in \text{Cont}(X, Y)$ such that:

- $\forall i. f_i \sqsubseteq_C z$
- $\forall g.(\forall i. f_i \sqsubseteq_C g) \Rightarrow z \sqsubseteq_C g$

Let $z = \lambda x. \bigsqcup^Y f_n(x)$, where $\sqcup^Y f_n(x)$ is the limit of the chain (in $\mathbb{Y}$) obtained by applying the functions in some chain of elements of $\text{Cont}(X, Y)$ to a certain element $x \in X$.

An example of a chain of such functions is:

$$f_1 \sqsubseteq_C f_2 \sqsubseteq_C \ldots \sqsubseteq_C \bigsqcup f_n$$

If we expand this using the definition of $\sqsubseteq_C$ we have

$$\forall x \in X.\ (f_1(x) \leq_y f_2(x) \leq_Y \cdots \leq_Y \bigsqcup f_n(x))$$

This is a set of chains in Chain($\mathbb{Y}$) where every chain contains the result of each function on a certain $x \in \mathbb{X}$. As $\mathbb{Y}$ is a domain, the least upper bound is defined for any chain using the elements of Y. Therefore we know that the least upper bound $\bigsqcup f_n(x)$ is defined. Now we can see that this is the same as our definition of z, which was $\lambda x. \bigsqcup^Y f_i(x)$.

For the second part of the proof, we can rewrite it using the definition of $\sqsubseteq_C$ as

$$\forall x \in X.\ (\forall g.(\forall i. f_i(x) \leq_Y g(x)) \Rightarrow z(x) \leq_Y g(x))$$

As $\mathbb{Y}$ is a domain, $(\forall i. f_i(x) \leq_Y g(x)) \Rightarrow z(x) \leq_Y g(x))$ holds for each of our individual chains for each $x \in X$. Therefore we have $\forall g.(\forall i. f_i \sqsubseteq_C g) \Rightarrow z \sqsubseteq_C g$

2.4 Fixpoint Theorem

Now we have a domain of continuous functions, we can use it to state and prove the fixpoint theorem. The following theorem is an important result in recursion theory, that we will use to model recursion in PCF. The chain given in the theorem is the chain obtained by repeatedly iterating a recursive function on its previous result. If an input of the function is computed using $f^n(\bot)$ and it needs more than n iterations then it will not terminate. As the chain can be infinitely long, it can model infinite (general) recursion.

Theorem 2. *Every continuous function $f : X \to X$ has a least fixpoint, which is the limit of the chain $\bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \ldots$*

Proof. Define the fixpoint function $fix(f) \equiv \bigsqcup f^n(\bot)$. This is the limit of the chain in the theorem. We know this limit exists because f is continuous, so $(X, \bot, \sqsubseteq)$ must form a domain (see Section 2.3.1), and by the definition of domain, all chains of $\mathbb{X}$ have a limit.

$\bigsqcup f^n(\bot)$ is a fixpoint For the limit to be a fixpoint we must have $f(\bigsqcup f^n(\bot)) = \bigsqcup f^n(\bot)$. As f is continuous, we have $f(\bigsqcup f^n(\bot)) = \bigsqcup f(f^n(\bot)) = \bigsqcup f^{n+1}(\bot)$. The chain formed by f^{n+1} is $f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \ldots$. This is the same as our original chain, but without $\bot$ at the start. Because $\mathbb{X}$ is a domain, we know that $\forall x \in X.\ \bot \sqsubseteq x$. Therefore $\bot$ has no effect on the limit because every element is higher than it, so removing $\bot$ will not change the limit. This means that $\bigsqcup f^{n+1}(\bot) = \bigsqcup f^n(\bot)$.

$\bigsqcup f^n(\bot)$ is the least fixpoint Let x be an element of our chain such that $fix(x) = x$. Then for $\bigsqcup f^n(\bot)$ to be the least fixpoint, we must have $\bigsqcup f^n(\bot) \sqsubseteq x$ (i.e. so x is an upper bound that is higher than $\bigsqcup f^n(\bot)$). First we prove x is an upper bound, so we must show $\forall n.\ f^n(\bot) \sqsubseteq x$. We prove by this by induction on n:

if $n = 0$, then $f^0(\bot) \sqsubseteq x$, This is the same as $\bot \sqsubseteq x$, which is true because $\bot$ is the least element of the chain.

Our inductive hypothesis is $f^n(\bot) \sqsubseteq x$. As f is continuous, f is monotone, so $f(f^n(\bot)) \sqsubseteq f(x) = f^{n+1}(\bot) \sqsubseteq x$. Therefore we know that for any element $f^n(\bot)$ in the chain, $f^n(\bot) \sqsubseteq x$.

As $\bigsqcup f^n(\bot)$ is a least upper bound, we know that $\forall x \in X.\ \forall n.\ (f^n(\bot) \sqsubseteq x) \Rightarrow \bigsqcup f^n(\bot) \sqsubseteq x$. We have just proved the left hand side of this, so we now have $\bigsqcup f^n(\bot) \sqsubseteq x$.

Now we have proved that $\bigsqcup f^n(\bot)$ is the least fixpoint of f. □

Now that we have proved the above theorem, we know enough Domain Theory to model recursive PCF programs between any types.

2.5 Logical Relations

Logical relations, developed in [Tait, 1967],[Plotkin, 1973],[Statman, 1985], are a proof technique that is used for proving properties that cannot be proved by structural induction alone, due to higher order constructions being present in the structure we are proving a property of.

They have been used to prove Strong Normalisation (i.e. that every expression terminates) of systems such as the Simply Typed λ Calculus, Type Safety and Program Equivalence.

2.5.1 Definition

We can define logical relations on program terms in PCF (or any other language) by defining relations on each type individually. We use Streicher's definition, as given in [Streicher, 2006]:

Definition 7. *Let W be an arbitrary set. A W-ary* ***logical relation*** *on the model of PCF is a family of relations*

$$R = (R_A \in \mathcal{P}([\![A]\!]^W) \mid A \in Type)$$

such that

$$f \in R_{A \to B} = \forall d \in R_A.\ \lambda i \in W.\ f(i)d(i) \in R_B$$

where $Type$ is the set of all possible types our programs can have, defined by induction.

Different logical relations are defined by defining the relation differently on the base type. For example, if the types are defined by the base type being Nat and other types $A \to B$ (formed any other types A and B), then the relation is defined by the definition of R_{Nat}.

Therefore we say that a logical relation R of arity W is uniquely determined by R_{Nat}, so for all subsets of $[\![\mathrm{Nat}]\!]^W$ there is a unique R equal to the set.

Logical Relations at Function Types For a function $f = (f_1, \dots f_n)$ to be in the relation, if we apply it to **any** value that is in the relation of the type of its domain, for example, for arity 3:

$$(x, y, z) \in R_A$$

Then f applied to everything in these elements will be in the relation of the codomain, so we must have:

$$(f_1(x), f_2(y), f_3(z)) \in R_B$$

Then $f \in R_{A \to B}$.

2.5.2 Examples and Non-Examples

Applying certain restrictions on R_{Nat} restricts the relations we can define on function types, for example:

- If $R_{\text{Nat}} = [\![\text{Nat}]\!]^n$, relations on function types can only have n arguments, if n is specified, otherwise there is no difference to the definition.
- If $R_{\text{Nat}} = \emptyset$, then all relations in R are the empty relation.

Union of two logical relations Given two logical relations R and S of the same arity, we could try to form a logical relation $R \cup S$. This is **not** a logical relation. For example, if we have $(f_1, \dots, f_n) \in R_{A \to B}$ (and not in $S_{A \to B}$) and $(d_1, \dots, d_n) \in S_A$, (and not in $R_{A \to B}$) then $(f_1(d_n), \dots, f_n(d_n))$ cannot be in either R or S.

Intersection of two logical relations Therefore if we restrict the logical relation to only contain tuples that are in both R and S then we do not have this problem, so this is a logical relation.

If the relations did not have the same arity, there will be no tuples that are in both relations that satisfy the definition, as if we have R of arity 3 and S of arity 5, then have, for example, $(f, g, h) \in R_{A \to B}$ and $(f, g, h, i, j) \in S_{A \to B}$, then we cannot say that (f, g, h) is in both, as this would not be in S anymore. Therefore, this would not be a logical relation.

Composition of two binary logical relations We define $R; S$ in the following way:

$$(R;S)(x,y) \Leftrightarrow \exists z.\ (R(x,z)\ \wedge\ S(z,y))$$

So for base type we have:

$$(R_{\mathrm{Nat}};S_{\mathrm{Nat}})(x,z) \Leftrightarrow \exists y.\ (R_{\mathrm{Nat}}(x,y)\ \wedge\ S_{\mathrm{Nat}}(y,z))$$

and for functions we have

$$(R_{A\to B};S_{A\to B})(f,h) \Leftrightarrow \exists g.\ (R_{A\to B}(f,g)\ \wedge\ S_{A\to B}(g,h))$$

where for any $(x,z) \in (R_A;S_A)$ we have

$$(R_B(f(x),g(y))\ \wedge\ S_B(g(y),h(z)))$$

So as functions in R are only applied to inputs that are in R and the same for S, and functions in both are only applied to inputs that are in both, then **this is a logical relation.**

2.5.3 Main Lemma

When using logical relations, we usually prove a general theorem about the relation, which we then give specific inputs to, to get the proof of our original property. This is called the Main Lemma (or sometimes the Fundamental Property/Theorem/Lemma). For program terms, we define the Main Lemma in the following way, as in [Streicher, 2006]. (Note that this definition uses the denotational semantics of PCF, so reading Chapter 6 first will make this definition much easier to understand):

For any denotation of a PCF term, we want to show that it is in the relation at its type, so we want to show that $[\![\Gamma \vdash e : A]\!] \in R_{\Gamma\to A}$. An element of $[\![\Gamma]\!]$ is any tuple of substitutions $d^* = (d_1, \ldots d_n)$ for $x_1 : A_1, \ldots x_n : A_n = \Gamma$. So $R_\Gamma \in \mathcal{P}([\![A_1]\!] \times \cdots \times [\![A_n]\!])$.

We want all substitutions in a set of size W to be in the relation, so using the definition of $f \in R_{\Gamma\to A}$, we want to show that

$$\forall d \in R_\Gamma.\ \lambda i \in W.[\![\Gamma \vdash e : A]\!](d(i))$$

This says that for any position in the W-tuple, we have the denotation of e using the substitution d from the ith position in the $d \in R_\Gamma$.

For W different substitutions we want to have

$$(\llbracket \Gamma \vdash e : A \rrbracket(d^*)_1, \ldots, \llbracket \Gamma \vdash e : A \rrbracket(d^*)_W) \in R_B$$

Therefore the main lemma (for $\lambda terms$??) is the following:

Lemma 1. *Let R be a logical relation of arity W on the Scott Model of PCF. Then for λ terms $\Gamma \vdash e : A$ and $d_j \in R_{A_j}$ for $j = 1, \ldots, n$*

$$\lambda i \in W. \llbracket \Gamma \vdash e : A \rrbracket(d^*(i)) \in R_A$$

where $d^*(i) = d_1(i) \ldots d_n(i)$ and $\Gamma = x_1 : A_1, \ldots x_n : A_n$

Chapter 3

Definition of PCF and Syntax

Programming Computable Functions (PCF) is a programming language that is based on the Simply Typed λ Calculus, with the addition of a "fix" operator, that allows us to write recursive functions using fixpoint recursion.

3.1 Definition of PCF

3.1.1 Types

There are some versions of PCF in different papers and books (such as [Plotkin, 1977], [Gunter, 1992]) where some use Booleans and Natural Numbers for the base types. Here we just use Natural numbers, where 0 represents false and any non zero number represents true. We define our types using the following grammar:

$$A ::= \text{Nat} \ | A \to B$$

3.1.2 Expressions

The allowable expressions include variables (represented by x), a constant z representing the number 0, and a successor function $s(e)$, which takes any expression as input and returns its successor.

case takes an expression e, which we assume is a numerical value. If e is zero, we return an expression e_0, otherwise we have the successor of some value x and we return the expression e_S.

Then we have function application, in which a function e is applied to an expression e', and λ-abstraction, which denotes a function e that takes an input x of type A.

Our last expression is the fixpoint expression, which takes a value x as input to a larger function e.

The grammar for expressions is:

$$e ::= \ x \mid z \mid s(e) \mid \text{ case } (e, z \mapsto e_0, s(x) \mapsto e_S) \mid e \ e' \mid \lambda x : A.e \mid \text{ fix } \ x : A.\ e$$

3.2 Type System for PCF

Γ is an example of a typing context, which is a function that maps variables to their types. For example, if we have an expression $x : A$ in the context, then $\Gamma(x) = A$. We write $\Gamma \vdash e : A$ for the context associated with an expression e of type A.

Therefore we need a typing rule for each expression in PCF, which if satisfied means that we are allowed to write that expression. The typing rules are given as inference rules, where our assumptions are above the line and the conclusion underneath:

$$\text{VARIABLES } \frac{\Gamma(x) = A}{\Gamma \vdash x : A} \qquad \text{ZERO } \frac{}{\Gamma \vdash z : \text{Nat}} \qquad \text{SUCC } \frac{\Gamma \vdash e : Nat}{\Gamma \vdash s(e) : \text{Nat}}$$

For *variables*, if x is in the domain of Γ, then we can conclude that we have a variable of that type.

z is a constant, so it needs no assumptions.

For *successor*, we must have an expression e of type Nat that we can apply the successor function to. We then know we have $s(e)$ of type Nat.

$$\text{CASE } \frac{\Gamma \vdash e : \text{Nat} \qquad \Gamma \vdash e_0 : A \qquad \Gamma, x : \text{Nat} \vdash e_S : A}{\Gamma \vdash \ \text{case } (e, z \mapsto e_0, s(x) \mapsto e_S) \ : A}$$

For case, we must have an expression e of type Nat to evaluate. Then we must have some other expressions e_0 and e_S to return, which can be of any type, as long as they are the same type. As the condition of e_S contains a specific x value, we must also know that this is well typed. Therefore we add $x : \text{Nat}$ to the context of e_S.

$$\textsc{application}\ \frac{\Gamma \vdash e : A \to B \qquad \Gamma \vdash e' : A}{\Gamma \vdash e\ e' : B} \qquad \textsc{abstraction}\ \frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x : A.e : A \to B}$$

For application, e must have a function type and e' must have the same type as the domain of e. Then we can apply e to e' to get the expression $e\ e'$ of the type of the codomain of e.

For λ abstraction, we need an expression e of some type B and we must know that the parameter x is in the context of this expression, to be able to use it in the λ abstraction. Then in the conclusion, as x is now bound to the expression, we remove it from the context of B.

$$\textsc{fix}\ \frac{\Gamma, x : A \vdash e : A}{\Gamma \vdash \text{ fix } x : A.\ e : A}$$

The fixpoint case is exactly the same as λ abstraction, where the x is bound to the fixpoint expression.

For any well typed PCF expression, we can obtain a derivation tree, where the root is the whole expression and the branches go up until we end up with the variables and constants of the expression at the leaves.

We also note that given a typing context Γ, an expression e and a type A such that we have the typing judgement $\Gamma \vdash e : A$, there is only one possible derivation of this judgement.

Chapter 4

Operational Semantics of PCF

Now we have defined the syntax and typing rules of PCF, we can use this to define its operational semantics.

We use the **Call By Name** evaluation strategy, which means that function arguments are placed into the body of the function and evaluated within the entire function's evaluation, instead of before.

The semantics we define are **small step** semantics, which means that in $e \mapsto e'$, the transition relation $\mapsto$ must take an expression e to another expression e' in only one step.

The first rules we have are **congruence rules**, which use the assumption $e \mapsto e'$ to replace e with e' in the whole expression. We can define these rules for any PCF expression that has an expression as a parameter, so this will be function application, successor and case:

$$\frac{e_0 \mapsto e_0'}{e_0\ e_1 \mapsto e_0'\ e_1} \qquad \frac{e \mapsto e'}{s(e) \mapsto s(e')}$$

$$\frac{e \mapsto e'}{\text{case } (e, z \to e_0, s(x) \to e_S) \mapsto \text{case } (e', z \to e_0, s(x) \to e_S)}$$

(Note that we could have given the typing contexts before each expression in the evaluation rules, but as they do not change, we can omit them. The same is also true for the rest of the rules we define below.)

Then we define rules on individual expressions. Note that the case rule above was only defined on expressions that reduce. The one defined below is only defined for **values**, which are expressions that have no applicable evaluation rule (including zero, successors of values,

and lambda abstractions). This ensures that there is only one possible rule to apply to any expression.

$$\frac{}{(\lambda x : A.\ e)\ e' \mapsto [e'/x]e}$$

The above rule states that given a function application, where the function is defined by a λ-abstracttion, we substitute e' for x in the expression e. This means that we replace every occurrence of x in e with the expression e'.

$$\frac{}{\text{case } (z, z \to e_0, s(x) \to e_S) \mapsto e_0}$$

$$\frac{}{\text{case } (s(v), z \to e_0, s(x) \to e_S) \mapsto [v/x]e_S}$$

The above rules give the evaluation of case, when we have a value as the condition expression. The first rule states that if $e = z$, then our result will be e_0. If $e = s(v)$, our result is e_S , but we must also substitute v for x in e_S (as e_S is defined for a bound variable x, which we now know is equal to v).

$$\frac{}{\text{fix } x : A.e \mapsto [\text{fix } x : A.e/x]e}$$

The above rule states that we replace the bound variable x in the expression e with the entire fixpoint expression. This replaces a parameter in e, which should be the recursive call, with the contents of the function, so that we can evaluate it all again. Then when we get to that point in the new evaluation, we will replace x with the whole expression. We can keep doing this infinitely, and keep expanding the evaluation in the following way:

$$\text{fix } x : A.e \mapsto [\text{fix } x : A.e/x]e \mapsto [[\text{fix } x : A.e/x]e/x]e \mapsto [[[\text{fix } x : A.e/x]e/x]e/x]e \ldots$$

We give an example of this behaviour below:

4.1 Example programs in PCF

4.1.1 Addition

In functional languages, we usually define mathematical operators on numbers as recursive functions. For example addition is the following function:

$$\text{add}\, 0\ y = y$$

$$\text{add}\, s(n)\ y = s(\text{add}\, n\ y)$$

We can write this as a λ abstraction:

$$\text{add} = \lambda x, y : \text{Nat}\,.\ case(x, z \mapsto y, s(v) \mapsto s(\text{add}\ \ v\ y))$$

This is a recursive function, so to define it in PCF, it must be the fixpoint of some other function A:

$$A = \lambda f : Nat \to Nat \to Nat.\ \lambda x, y : \text{Nat}\,.\ case(x, z \mapsto y, s(v) \mapsto s(f\ v\ y))$$

Therefore we can define this in PCF as:

$$\text{add}\, x\ y = (\text{fix}\, f : \text{Nat} \to \text{Nat} \to \text{Nat}\,.\ A : \text{Nat} \to \text{Nat} \to \text{Nat})\ x\ y$$

.

When we try to evaluate this term, we get the following, by the evaluation rule for fix:

$\text{fix}\, f : \text{Nat} \to \text{Nat} \to \text{Nat}\,.\ A : \text{Nat} \to \text{Nat} \to \text{Nat} = [\text{fix}\, f : \text{Nat} \to \text{Nat} \to \text{Nat}\,.\ A : \text{Nat} \to \text{Nat} \to \text{Nat}\,/f]A$. This expands to:

$$\lambda x, y : \text{Nat}\,.\ case(x, z \mapsto y, s(v) \mapsto$$

$$s((\text{fix}\, f : \text{Nat} \to \text{Nat} \to \text{Nat}\,.\ A : \text{Nat} \to \text{Nat} \to \text{Nat})\ v\ y))$$

Therefore expanding this infinitely gives all possible executions of the addition function.

Chapter 5

Type Safety

Type Safety is an important property of a programming language, as it proves that well typed programs do not go wrong. It is usually expressed as a property of the operational semantics, which we have defined in the previous chapter (see Chapter 4), and follows from the conclusion of two other lemmas we will prove; Type Preservation (5.2.1) and Type Progress (5.2.2)

5.1 Lemmas for Type Safety

There are two simple lemmas we must prove which will aid us in proving the lemmas of Type Safety:

5.1.1 Weakening

Weakening is the following theorem, which says that for an expression of type A in a context Γ, adding another variable x (of any type) to the context will not change the type of the expression:

Theorem 3. *If* $\Gamma \vdash e : A$ *then* $\Gamma, x : C \vdash e : A$

Proof. We prove this theorem by induction on derivation trees, so if we have a derivation tree for our assumption then there exists a derivation tree for the conclusion.

As there is only one derivation for a given judgement $\Gamma \vdash e : A$, we can use induction on the possible expressions:

Variables We rename x to y using α equivalence. We assume $\Gamma \vdash y : A$, giving us the following derivation tree:

$$\frac{\Gamma(y) = A}{\Gamma \vdash y : A}$$

of which $\Gamma(y) = A$ is a subtree. Γ is a function, so can also be represented by a set of (variable, type) pairs. Therefore $\Gamma, x : C$ is the set $\Gamma \ \cup \{(x, C)\}$. Therefore we define the function $(\Gamma, x : C)$, where $(\Gamma, x : C)(y) = A$ and for any other variable z, $(\Gamma, x : C)(z) = \Gamma(z)$. Then we just use the typing rule for variables to get the following derivation tree:

$$\frac{(\Gamma, x : C)(y) = A}{\Gamma, x : C \vdash y : A}$$

Now we have the required derivation tree, so weakening holds for variables.

Zero We assume $\Gamma \vdash z : \text{Nat}$, giving us the following derivation tree:

$$\frac{}{\Gamma \vdash z : \text{Nat}}$$

. The typing rule for zero says that no matter what Γ is, we always have zero, because there are no assumptions. Therefore we can have $\Gamma, x : C$ as the context and get the following derivation tree:

$$\frac{}{\Gamma, x : C \vdash z : \text{Nat}}$$

Now we have the required derivation tree, so weakening holds for zero.

Successor We assume $\Gamma \vdash s(e) : \text{Nat}$, giving us the following derivation tree from the typing rule:

$$\frac{\Gamma \vdash e : \text{Nat}}{\Gamma \vdash s(e) : \text{Nat}}$$

.

where $\Gamma \vdash e : \text{Nat}$ is a subtree. We can use the inductive hypothesis of weakening on this subtree to get $\Gamma, x : C \vdash e : \text{Nat}$. Then we use the typing rule for successor to get the following derivation tree:

$$\frac{\Gamma, x : C \vdash e : \text{Nat}}{\Gamma, x : C \vdash s(e) : \text{Nat}}$$

.

Now we have the required , so weakening holds for the successor function.

Case We assume $\Gamma \vdash$ case $(e, z \mapsto e_0, s(y) \mapsto e_S) : A$, (renaming x to y using alpha equivalence) giving us the following derivation tree from the typing rule:

$$\frac{\Gamma \vdash e : Nat \qquad \Gamma \vdash e_0 : A \qquad \Gamma, y : \text{Nat} \vdash e_S : A}{\Gamma \vdash \text{ case } (e, z \mapsto e_0, s(y) \mapsto e_S) \ : A}$$

giving us the subtrees $\Gamma \vdash e : \text{Nat}$, $\Gamma \vdash e_0 : A$ and $\Gamma, y : \text{Nat} \vdash e_S : A$ from the assumption of the typing rule. Using the inductive hypothesis on each of these, we get $\Gamma, x : C \vdash e : \text{Nat}$, $\Gamma, x : C \vdash e_0 : A$ and $\Gamma, y : \text{Nat}, x : C \vdash e_S : A$, so we can use the typing rule again with these assumptions:

$$\frac{\Gamma, x : C \vdash e : Nat \qquad \Gamma, x : C \vdash e_0 : A \qquad \Gamma, x : C, y : \text{Nat} \vdash e_S : A}{\Gamma, x : C \vdash \text{ case } (e, z \mapsto e_0, s(y) \mapsto e_S) \ : A}$$

Now we have the required derivation tree, so weakening holds for the case expression.

Application We assume $\Gamma \vdash e_0 \ e_1 : B$, giving us the following derivation tree:

$$\frac{\Gamma \vdash e_0 : A \to B \qquad \Gamma \vdash e_1 : A}{\Gamma \vdash e_0 \ e_1 : B}$$

which gives us the subtrees $\Gamma \vdash e_0 : A \to B$ and $\Gamma \vdash e_1 : A$. Using the inductive hypothesis on these trees gives us $\Gamma, x : C \vdash e_0 : A \to B$ and $\Gamma, x : C \vdash e_1 : A$, so we can just use the typing rule for application again to get the following derivation tree:

$$\frac{\Gamma, x : C \vdash e_0 : A \to B \qquad \Gamma, x : C \vdash e_1 : A}{\Gamma, x : C \vdash e_0 \ e_1 : B}$$

Therefore weakening holds for function application.

Abstraction We rename x to y using α equivalence. We assume $\Gamma \vdash \lambda y : A.\ e : B$, giving us the following derivation tree:

$$\frac{\Gamma, y : A \vdash e : B}{\Gamma \vdash \lambda y : A.\ e : A \to B}$$

which gives us the subtree $\Gamma, y : A \vdash e : B$. Using the inductive hypothesis, we get $\Gamma, y : A, x : C \vdash e : B$. Then we use the typing rule for λ abstraction to get the following tree:

$$\frac{\Gamma, y : A, x : C \vdash e : B}{\Gamma, x : C \vdash \lambda y : A.\ e : A \to B}$$

Now we have the required derivation tree, so weakening holds for λ abstraction.

Fixpoint We assume $\Gamma \vdash \text{fix}\ \ y : A.\ e : A$, renaming x to y using α equivalence. This gives us the following derivation tree:

$$\frac{\Gamma, y : A \vdash e : A}{\Gamma, y : A \vdash \text{fix}\ \ y : A.\ e : A}$$

As we have the subtree $\Gamma, y : A \vdash e : A$, we use the inductive hypothesis on this to get $\Gamma, x : C, y : A. \vdash e : A$. Then we use the typing rule for fix to get the following tree:

$$\frac{\Gamma, x : C, y : A \vdash e : A}{\Gamma, x : C, y : A \vdash \text{fix}\ \ y : A.\ e : A}$$

Therefore weakening holds for the fixpoint operator. Now we have proved weakening for derivation trees of any expression so weakening always holds.

□

5.1.2 Substitution Rules

Next we want to prove a lemma that shows that substitutions preserve the intended type of a given PCF expression.

But before we prove this, we must actually define rules for substitution on the level of each possible expression that can be formed in PCF, which are all instances of $[e/x]e'$. This notation says that an expression e replaces a variable x in another expression e'.

When defining the substitution rules, we must be careful that we avoid **variable capture** by bound variables. For example, given the following substitution:

$$[s(x)/y](\lambda x.x + y)$$

if we naively substitute $s(x)$ for y in $\lambda x.x + y$, we get $\lambda x.\ x + s(x)$ and the value of x is now the value assigned to the bound x. Therefore we can use **renaming** to avoid this.

Variables There are two cases for substitution in variables. The first is for when e' is the same as the variable being replaced:

$$[e/x]x = e$$

The second one is when e is a completely different variable, for which nothing happens:

$$[e/x]y = y$$

Zero For zero, any substitution will have no effect, as zero is a constant:

$$[e/x]z = z$$

Successor The successor function cannot be changed, so we substitute e in its argument:

$$[e/x]s(e') = s([e/x]e')$$

Case As we cannot change the case statement, we could substitute e in the expressions given as arguments to the case statement, in the following way:

$$[e/x]\ (\text{case}\ (e', z \mapsto e_0, s(x) \mapsto e_S)) = \text{case}\ ([e/x]e', z \to [e/x]e_0, s(x) \to [e/x]e_S)$$

But we must be careful with variable capture, as the derivation of e_S is $\Gamma, x : Nat \vdash e_S$. If we have $[s(x)/y]e_S$ and $e_S = x + y$, then we have $x + s(x)$ and the value of $s(x)$ is bound by $\Gamma, x : Nat$. Therefore we should rename $s(x)$ in the case statement to something that is not free in e_S.

Therefore our rule for case will be:

$(\text{case } (e', z \to e_0, s(x) \to e_S)) =$

$$\begin{cases} (\text{case } ([e/x]e', z \mapsto [e/x]e_0, s(x) \mapsto [e/x]e_S)) & \text{if } x \notin FV(e') \\ (\text{case } ([e/x]e', z \mapsto [e/x]e_0, s(y) \mapsto [e/x]e_S)) & \text{if } x \in FV(e') \end{cases}$$

where $y \notin FV(e')$.

For some expression e, $FV(e)$ is the set of free variables it contains.

Application We substitute e in the function and its argument, then apply the new function to the new argument:

$$[e/x](e_0\ e_1) = [e/x]e_0\ ([e/x]e_1)$$

λ **Abstraction** There are two cases, the first when the bound variable is x. This does nothing, as we are just rewriting the function using alpha equivalence in this case:

$$[e/x](\lambda x : A.\ e') = \lambda x : A.\ e'$$

The second case is when the bound variable is not equal to x, where we substitute e in the expression. If y is a free variable in e' then we must rename the bound variable to something else:

$[e/x](\lambda y : A.e') =$

$$\begin{cases} \lambda y : A.[e/x]e' & \text{if } y \notin FV(e') \\ \lambda z : A.[e/x]([z/y]e') & \text{if } y \in FV(e') \end{cases}$$

where $z \notin FV(e')$.

Fixpoint Fixpoint is similar to λ abstraction, so we have two cases. The first case is when the bound variable is x, for which we just rewrite the function using α equivalence. Therefore this rule does nothing:

$$[e/x]\,\text{fix}\ \ x : A.e' = \text{fix}\ \ e : A.e'$$

When the bound variable is not equal to x, we substitute e in the expression e'. If y is a free variable in e' then we must rename the bound variable to something else:

$[e/x](\text{fix } y : A.e') =$

$$\begin{cases} \text{fix } y : A.[e/x]e' & \text{if } y \notin FV(e') \\ \text{fix } z : A.[e/x]([z/y]e') & \text{if } y \in FV(e') \end{cases}$$

where $z \notin FV(e')$.

5.1.3 Substitution

Now we have all the rules, we can prove Substitution, which is the following theorem. It says that if we have an expression e that is well typed in the context Γ and an expression e' that is well typed in the context $\Gamma, x : A$, then the expression obtained by substituting e for x in e' will be derivable in the context Γ:

Theorem 4. *If* $\Gamma \vdash e : A$ *and* $\Gamma, x : A \vdash e' : C$ *then* $\Gamma \vdash [e/x]e' : C$

Proof. We can slso prove this by induction on derivation trees.

As there is only one derivation for a given judgement $\Gamma \vdash e : A$, again we can use induction on the possible values of e':

Variables We assume derivation trees exist for $\Gamma \vdash e : A$ and $\Gamma, x : A \vdash y : C$. As y has a different type to x, it cannot be equal to it, so there is only one case, as we can only use one of the substitution rules. The tree for y that is given is:

$$\frac{(\Gamma, x : A)(y) = C}{\Gamma, x : A \vdash y : C}$$

The function Γ is the set of pairs $(\Gamma, x : A) \backslash \{(x, A)\}$. As $x : A$ does not affect the value of y, we will still have $\Gamma(y) = C$. Using the typing rule for variables, we get the tree:

$$\frac{\Gamma(y) = C}{\Gamma \vdash y : C}$$

$\Gamma \vdash y : C$ will be the same as $\Gamma \vdash [e/x]y : C$ using the substitution rule for variables, so we have the derivation tree needed. Therefore variables satisfy substitution.

Zero We assume derivation trees exist for $\Gamma \vdash e : A$ and $\Gamma, x : A \vdash z : \text{Nat}$. As z is a constant, it exists in any context Γ, so we have a tree for $\Gamma \vdash z : \text{Nat}$. This is equal to $\Gamma \vdash [e/x]z : \text{Nat}$, as this is always zero no matter what e and x are. Therefore as we already have the tree for $\Gamma \vdash z : \text{Nat}$, we use it as the derivation tree for $\Gamma \vdash [e/x]z : \text{Nat}$.

Successor We assume derivation trees exist for $\Gamma \vdash e : A$ and $\Gamma, x : A \vdash s(e') : \text{Nat}$. The second tree is the following:

$$\frac{\Gamma, x : A \vdash e' : \text{Nat}}{\Gamma, x : A \vdash s(e') : \text{Nat}}$$

Therefore we have a subtree $\Gamma, x : A \vdash e' : \text{Nat}$. Using the induction hypothesis on this and $\Gamma \vdash e : A$, we have a derivation tree for $\Gamma \vdash [e/x]e' : \text{Nat}$. Using the typing rule for successor on this gives us the following tree:

$$\frac{\Gamma \vdash [e/x]e' : \text{Nat}}{\Gamma \vdash s([e/x]e') : \text{Nat}}$$

Using the substitution rule for successor, we know the bottom half is equal to $[e/x]s(e')$, so we get the following derivation tree:

$$\frac{\Gamma \vdash [e/x]e' : \text{Nat}}{\Gamma \vdash [e/x]s(e') : \text{Nat}}$$

which is a derivation tree for $\Gamma \vdash [e/x]s(e') : \text{Nat}$. Therefore substitution holds for successor function.

Case We rename x to y using alpha equivalence, then assume derivation trees exist for $\Gamma \vdash e : A$ and $\Gamma, x : A \vdash \text{case } (e', z \mapsto e_0, s(y) \mapsto e_S) \ : C$. The second derivation tree gives us the subtrees $\Gamma, x : A \vdash e' : \text{Nat}$, $\Gamma, x : A \vdash e_0 : C$ and $\Gamma, y : Nat, x : A \vdash e_S : C$. Using the induction hypothesis and the tree for $\Gamma \vdash e : A$, we get $\Gamma \vdash [e/x]e' : \text{Nat}$ and $\Gamma \vdash [e/x]e_0 : C$.

For $\Gamma, y : \text{Nat}, x : A \vdash e_S : C$, we need to change the context of e before we can apply the inductive hypothesis. We do this using our Weakening Lemmma we just proved (in Section 5.1.1), which gives us $\Gamma, y : \text{Nat} \vdash e : A$. Now we apply the inductive hypothesis with this to get $\Gamma, y : \text{Nat} \vdash [e/x]e_S : C$.

Now we can apply the typing rule to these trees to get the following derivation tree:

$$\frac{\Gamma \vdash [e/x]e' : \text{Nat} \qquad \Gamma \vdash [e/x]e_0 : C \qquad \Gamma, y : \text{Nat} \vdash [e/x]e_S : C}{\Gamma \vdash \ \text{case}\ ([e/x]e', z \mapsto [e/x]e_0, s(y) \mapsto [e/x]e_S)\ : C}$$

By the substitution rule for case, we can replace the bottom half of the tree with $\Gamma \vdash [e/x]\ \text{case}\ (e', z \mapsto e_0, s(y) \mapsto e_S)\ : C$. Therefore substitution holds for the case statement.

Application We assume derivation trees exist for $\Gamma \vdash e : A$ and $\Gamma, x : A \vdash e_0\ e_1 : C$. The second tree is the following:

$$\frac{\Gamma, x : A \vdash e_0 : B \to C \qquad \Gamma, x : A \vdash e_1 : B}{\Gamma, x : A \vdash e_0\ e_1 : C}$$

which contains the subtrees $\Gamma, x : A \vdash e_0 : B \to C$ and $\Gamma, x : A \vdash e_1 : B$. Combining each of these trees with $\Gamma \vdash e : A$ and the inductive hypothesis gives us $\Gamma \vdash [e/x]e_0 : B \to C$ and $\Gamma \vdash [e/x]e_1 : B$. Using the typing rule for function application with these trees gives us a derivation tree for $\Gamma \vdash [e/x]e_0([e/x]e_1) : C$. This is equal to $\Gamma \vdash [e/x](e_0\ e_1) : C$, so we have the derivation tree for this judgement.

Therefore substitution holds for function application.

Abstraction We rename x to y using α equivalence. Then derivation trees exist for $\Gamma \vdash e : A$ and $\Gamma, x : A \vdash \lambda y : B.\ e' : B \to C$. For the second tree we have the following:

$$\frac{\Gamma, x : A, y : B \vdash e' : C}{\Gamma, x : A \vdash \lambda y : B.\ e' : B \to C}$$

which gives us the subtree $\Gamma, x : A, y : B \vdash e' : C$. Then we use weakening on $\Gamma \vdash e : A$ to get $\Gamma, y : B \vdash e : A$, which when used with the inductive hypothesis gives us $\Gamma, y : B \vdash [e/x]e' : C$.

Applying the typing rule for λ abstraction to this gives us the following tree:

$$\frac{\Gamma, y : B \vdash [e/x]e' : C}{\Gamma \vdash \lambda y : B.\ [e/x]e' : B \to C}$$

.

Now there are two cases:

1. When $y \notin FV(e')$, the substitution rule gives us $\lambda y : B.\ [e/x]e' = [e/x]\lambda y : B.\ e'$, so we have a derivation tree for $\Gamma \vdash [e/x]\lambda y : B.\ e' : B \to C$

2. When $y \in FV(e')$, rewrite $\lambda y : B.\ [e/x]e' : B \to C$ as $\lambda z : B.[e/x]([z/y]e')$. Then this is the same as $[e/x]\lambda y : B.\ e'$, so we have a derivation tree for $\Gamma \vdash [e/x]\lambda y : B.\ e' : B \to C$

Fixpoint We rename x to y using alpha equivalence, then assume derivation trees exist for $\Gamma \vdash e : A$ and $\Gamma, x : A \vdash \text{fix } y : C.\ e' : C$. The second tree is the following:

$$\frac{\Gamma, y : C, x : A \vdash e' : C}{\Gamma, x : A \vdash \text{fix } y : C.e' : C}$$

giving us the subtree $\Gamma, y : C, x : A \vdash e' : C$. Then we use weakening on $\Gamma \vdash e : A$ to get $\Gamma, y : C \vdash e : A$, which when used with the inductive hypothesis gives us $\Gamma, y : C \vdash [e/x]e' : C$.

Applying the typing rule for fixpoint to this gives us the following derivation tree:

$$\frac{\Gamma, y : C \vdash [e/x]e' : C}{\Gamma \vdash \text{fix } y : C.[e/x]e' : C}$$

Now there are two cases:

1. When $y \notin FV(e')$, the substitution rule for fixpoint gives us $\Gamma \vdash \text{fix } y : C.[e/x]e' : C = \Gamma \vdash [e/x](\text{fix } y : C.e') : C$, so we have the required derivation tree and substitution holds for fixpoint.

2. When $y \in FV(e')$, rewrite $\text{fix } y : C.\ [e/x]e' : C$ as $\text{fix } z : C.[e/x]([z/y]e')$. Then this is the same as $[e/x]\,\text{fix } y : C.\ e'$, so we have a derivation tree for $\Gamma \vdash [e/x]\,\text{fix } y : C.\ e' : C$

Now we have proved substitution holds for derivation trees of any expression e'. □

5.2 Type Safety

Now we can prove the two lemmas that form the property of Type Safety.

5.2.1 Type Preservation

Type preservation says that if an expression e of type A is well typed in a context Γ, and it evaluates in one step to e', then e' will also have type A in Γ (i.e. we get the same type at the end of the evaluation as we had at the start):

Theorem 5. *If* $\Gamma \vdash e : A$ *and* $e \mapsto e'$, *then* $\Gamma \vdash e' : A$

Proof. We can prove this by induction on derivation trees.

There will be a derivation tree for each rule in the operational semantics, so we can check this statement for every evaluation rule on every possible expression:

Variables There are no rules in the operational semantics for when e is just a variable so we do nothing here.

Zero Same as for variables.

Successor We assume $\Gamma \vdash s(e) : \text{Nat}$, so we have the following tree (by the typing rule of successor):

$$\frac{\Gamma \vdash e : \text{Nat}}{\Gamma \vdash s(e) : \text{Nat}}$$

Then we assume $s(e) \mapsto s(e')$, so we have the following tree, using the congruence rule for successor:

$$\frac{e \mapsto e'}{s(e) \mapsto s(e')}$$

From these two trees, we get the subtrees $\Gamma \vdash e : \text{Nat}$ and $e \mapsto e'$. Using the inductive hypothesis of type preservation we get a tree for $\Gamma \vdash e' : \text{Nat}$. Then using the typing rule for successor with this we get a tree for $\Gamma \vdash s(e') : \text{Nat}$.

There are no other rules when the expression is $s(e)$, so type preservation holds for successor expressions.

Case There are three evaluation rules, which depend on the expression e being checked:

1. If e can be reduced, then we have the following tree from the typing rule for case:

$$\frac{\Gamma \vdash e : \text{Nat} \qquad \Gamma \vdash e_0 : A \qquad \Gamma, x : \text{Nat} \vdash e_S : A}{\Gamma \vdash \text{ case } (e, z \mapsto e_0, s(x) \mapsto e_S) \ : A}$$

and our second assumption uses the congruence evaluation rule for case as:

$$\frac{e \mapsto e'}{\text{case } (e, z \mapsto e_0, s(x) \mapsto e_S) \mapsto \text{case } (e', z \mapsto e_0, s(x) \mapsto e_S)}$$

Then we have subtrees for $\Gamma \vdash e : \text{Nat}$, $\Gamma \vdash e_0 : A$, $\Gamma, x : \text{Nat} \vdash e_S : A$ and $e \mapsto e'$.

Using the inductive hypothesis of type preservation, with the trees for $\Gamma \vdash e : \text{Nat}$ and $e \mapsto e'$ we get a tree for $\Gamma \vdash e' : \text{Nat}$. Then we apply the typing rule for case with this, $\Gamma \vdash e_0 : A$ and $\Gamma, x : \text{Nat} \vdash e_S : A$ to get a tree for $\Gamma \vdash \mathit{case}\ (e', z \to e_0, s(x) \to e_S) : A$

2. If $e = \text{case}\ (z, z \mapsto e_0, s(x) \mapsto e_S)$, then we get the following tree from the typing rule:

$$\frac{\Gamma \vdash z : \text{Nat} \qquad \Gamma \vdash e_0 : A \qquad \Gamma, x : \text{Nat} \vdash e_S : A}{\Gamma \vdash \ \text{case}\ (z, z \mapsto e_0, s(x) \mapsto e_S)\ \ : A}$$

and we also have the tree for the evaluation rule case $(z, z \mapsto e_0, s(x) \to e_S) \mapsto e_0$. Therefore we need a tree for $\Gamma \vdash e_0 : A$, which we already have, as a subtree of the first assumption.

3. If $e = \text{case}\ (s(v), z \mapsto e_0, s(x) \mapsto e_S)$, then the tree formed from its typing rule is:

$$\frac{\dfrac{\Gamma \vdash v : \text{Nat}}{\Gamma \vdash s(v) : \text{Nat}} \qquad \Gamma \vdash e_0 : A \qquad \Gamma, x : \text{Nat} \vdash e_S : A}{\Gamma \vdash \ \text{case}\ (s(v), z \mapsto e_0, s(x) \mapsto e_S)\ \ : A}$$

and we also have is the tree for the evaluation rule: case $(s(v), z \mapsto e_0, s(x) \mapsto e_S) \mapsto [v/x]e_S$.

We get the tree for $\Gamma \vdash [v/x]e_S : A$ by using the substitution lemma, with the subtrees for $\Gamma \vdash v : \text{Nat}$ and $\Gamma, x : \text{Nat} \vdash e_S : A$ as parameters.

Application There are two evaluation rules for function application:

1. When e is a function that can be reduced further, we have the following tree from its typing rule:

$$\frac{\Gamma \vdash e_0 : A \to B \qquad \Gamma \vdash e_1 : A}{\Gamma \vdash e_0\ e_1 : B}$$

and the following tree obtained from its evaluation rule:

$$\frac{e_0 \mapsto e_0'}{e_0\ e_1 \mapsto e_0'\ e_1}$$

We use the induction hypothesis with the subtrees for $\Gamma \vdash e_0 : A \to B$ and $e_0 \mapsto e_0'$ to get a subtree for $\Gamma \vdash e_0' : A \to B$. Then using this and the subtree for $\Gamma \vdash e_1 : B$, in the typing rule for function application, we get a subtree for $\Gamma \vdash e_0'\ e_1 : B$.

2. When $e = (\lambda x : A.\ e)\ e'$, we have the following tree from its typing rule:

$$\frac{\dfrac{\Gamma, x : A \vdash e : B}{\Gamma \vdash (\lambda x : A).e : A \to B} \qquad \Gamma \vdash e' : A}{\Gamma \vdash (\lambda x : A).e\ e' : B}$$

   And the tree for $(\lambda x : A).e\ e' \mapsto [e'/x]e$ from its evaluation rule. We have subtrees $\Gamma \vdash e' : A$ and $\Gamma, x : A \vdash e : B$, so we use these as parameters to the Substitution Lemma to get the tree for $\Gamma \vdash [e'/x]e : B$.

λ-abstraction has no evaluation rules when taken as a single expression (because it is a value).

Fixpoint When $e = \text{fix}\ x : A.e$, we have the following tree from its typing rule:

$$\frac{\Gamma, x : A \vdash e : A}{\Gamma \vdash \text{fix}\ x : A.\ e : A}$$

and the tree for fix $x : A.\ e \mapsto [\text{fix}\ x : A.e/x]e$ from its evaluation rule. We already have the tree for $\Gamma \vdash \text{fix}\ x : A.\ e : A$ as an assumption and $\Gamma, x : A \vdash e : A$ is a subtree of it, so we can use the substitution lemma with these parameters to get a tree for $\Gamma \vdash [\text{fix}\ x : A.e/x]e : A$

Now we have proved type preservation for all the rules in the operational semantics on all possible expressions. □

5.2.2 Type Progress

Type progress says that if an expression e of type A is well typed in a context Γ, then it must evaluate to another expression e' in one step, or be a value (so e cannot be evaluated further), where possible values are

$$v :: z \mid s(v) \mid \lambda x : A.\ e$$

which are numbers or non-recursive functions (but note this is not saying the values terminate, or that they have normal form):

Theorem 6. *If $\vdash e : A$ then $e \mapsto e'$ or e is a value.*

Proof. We can prove this by induction on derivation trees of e. When we have a derivation tree for a closed term e, we either get a derivation tree E, for evaluating it in one step to another expression, or e is a value.

Zero z is a value

Variables There are no closed terms that are variables, so this is vacuously true. This is because the context of a term is a set of $(variable, type)$ pairs, so when it is empty, there are no variables. Therefore we have no derivation tree for $\vdash x : A$, where x is a variable, so there is no case for variables.

Successor When we have an expression $s(e)$, we assume $\vdash s(e) : \text{Nat}$, giving us the following tree:

$$\frac{\vdash e : \text{Nat}}{\vdash s(e) : \text{Nat}}$$

so we have a subtree for $\vdash e : \text{Nat}$. We can use induction on this subtree to get $e \mapsto e'$ or e is a value. We then assume either side of this:

1. When $e \mapsto e'$ we use the congruence rule for successor to get a tree for $s(e) \mapsto s(e')$. Therefore $s(e) \mapsto s(e')$ or $s(e)$ is a value
2. When e is a value, v, we rewrite $s(e)$ to $s(v)$ This is a value, so $s(e) \mapsto s(e')$ or $s(e)$ is a value is true

Case When we have an expression case $(e, z \mapsto e_0, s(x) \mapsto e_S)$, we assume $\vdash$ case $(e, z \mapsto e_0, s(x) \mapsto e_S) : A$, giving us the following tree:

$$\frac{\vdash e : \text{Nat} \qquad \vdash e_0 : A \qquad x : \text{Nat} \vdash e_S : A}{\vdash \text{ case } (e, z \mapsto e_0, s(x) \mapsto e_S) \ : A}$$

so we have subtrees for $\vdash e : \text{Nat}$, $\vdash e_0 : A$ and $x : \text{Nat} \vdash e_S : A$.

By induction on $\vdash e : \text{Nat}$, we know that $e \mapsto e'$ or e is a value. We can assume either side of this:

1. When $e \mapsto e'$, we use the congruence rule for case to get case $(e', z \mapsto e_0, s(x) \mapsto e_S) \ : A$. Therefore we know our original expression maps to some other expression.
2. When e is a value there are two cases.:

(a) $e = z$. The evaluation rule for this has no assumption, so we already have a tree that maps case $(z, z \mapsto e_0, s(x) \mapsto e_S)$ to another expression, which is e_0.

(b) $e = s(v)$. The evaluation rule for this also has no assumption, so we already have a tree that maps case $(s(v), z \mapsto e_0, s(x) \mapsto e_S)$ to another expression, which is $[v/x]e_S$

Therefore, no matter what e is in the case expression, it always maps to another expression, e', so case $(e, z \mapsto e_0, s(x) \to e_S) \mapsto e'$ or case $(e, z \to e_0, s(x) \to e_S)$ is a value is true.

Application When we have an expression $e_0\ e_1$, we assume $\vdash e_0\ e_1 : B$ giving us the following tree:

$$\frac{\vdash e_0 : A \to B \qquad \vdash e_1 : A}{\vdash e_0\ e_1 : B}$$

so we have a subtree for $\vdash e_0 : A \to B$. We can use the inductive hypothesis on this to get $e_0 \mapsto e_0'$ or e_0 is a value. We can assume either side of this:

1. When $e_0 \mapsto e_0'$, we use the congruence rule for application to get a tree for $e_0\ e_1 \mapsto e_0'\ e_1$
2. When e_0 is a value it must be $\lambda x : A.\ e_0$, as the other values are not function types. The evaluation rule for $(\lambda x : A.\ e_0)\ e_1$ has no assumption, so we already have a tree that maps $(\lambda x : A.\ e_0)\ e_1$ to another expression, which is $[e_1/x]e_0$

Therefore, for every possible value of $e_0\ e_1$, we can evaluate it to another expression in one step, so $e_0\ e_1 \mapsto e_0'\ e_1$ or $e_0\ e_1$ is a value is true.

λ-abstraction $\lambda x : A.\ e$ is always a value, for any $x : A$ and $e : A$.

Fixpoint When we have an expression fix $x : A.\ e$, we assume $\vdash$ fix $x : A.\ e : A$, giving us the following tree:

$$\frac{x : A \vdash e : A}{\vdash \text{ fix } x : A.\ e : A}$$

There are no assumptions in the evaluation rule for fixpoint, so we already have the tree that maps fix $x : A.\ e$ to another expression, which is [fix $x : A.e/x]e$. Therefore we know that fix $x : A.\ e$ always maps to another expression.

Now we have proved type progress for all possible expressions. □

Now Type Safety can be assumed from the the two lemmas we have just proved. This approach was first used in [Wright and Felleisen, 1994].

The theorem for type safety says that an closed term either evaluates to a value in a finite number of steps or loops forever:

Theorem 7. *If* $\vdash e : A$ *then* $e \mapsto^* v$ *(where* $\vdash v : A$*) or* $e \mapsto^\infty$

Chapter 6

Denotational Semantics

Denotational semantics describe expressions in a programming language as functions in a mathematical model, as explained in Chapter 1. Now we use the domain theory we discussed in Section 2.1 to create that model for PCF:

6.1 Denotational Model of PCF

6.1.1 Denotation of Types

Our denotational semantics maps the types of PCF to a domain representing that type. We define a function:

$$[\![-]\!] : Type \to Domain$$

that maps a type to a domain. We defined our types by induction, so we have specify the different constructions of domains we use by induction on types:

1. The type of Natural numbers is the base type, so they are modelled by a single domain. We use the flat domain of Natural numbers, described in Section 2.2.2, where $\perp$ represents a term that does not terminate:

$$[\![\text{Nat}]\!] = \mathbb{N}_\perp$$

2. Function types are formed of other types. We model them using the domain of continuous functions, described in Section 2.3.1.

$$[\![A \to B]\!] = [\![A]\!] \to [\![B]\!]$$

(Where $[\![A]\!] \to [\![B]\!]$ is the same as $\mathrm{Cont}(A, B)$)

6.1.2 Denotation of Typing Contexts

We also need a domain for the typing contexts, which is given by the following function:

$$[\![-]\!]_{Ctx} : Context \to Domain$$

that maps a typing context to a domain. The domain will be a nested tuple, the size of which depends on the number of variables in Γ. Each variable's type is a domain, so the overall domain is a product of domains, described in Section 2.2.3. Therefore we define these domains by induction on the size of the given typing context:

The empty context is given by

$$[\![\cdot]\!]_{Ctx} = \mathbb{1}$$

the single element domain, which we defined in Section 2.2.1.

Adding a variable to a context Γ gives us the following domain:

$$[\![\Gamma, x : A]\!]_{Ctx} = [\![\Gamma]\!] \times [\![A]\!]$$

where $[\![\Gamma]\!]$ is a product of domains.

This gives us all combinations of all possible values of each variable in Γ. If we want a specific valuation of the variables, we can refer to $d^* \in [\![\Gamma]\!]_{Ctx}$. $d^* = (d_1, \ldots, d_n)$ will be a tuple in the underlying set of the product domain of size n, where n is the number of domains in the product domain and also the number or variables in Γ.

6.1.3 Denotation of well typed terms

We map well-typed PCF expressions to the domain that models their type. Given a well typed term $\Gamma \vdash e : A$ we have the continuous function:

$$[\![\Gamma \vdash e : A]\!] \in [\![\Gamma]\!]_{Ctx} \to [\![A]\!]$$

So $[\![\Gamma \vdash e : A]\!]d^*$ gives us an element of $[\![A]\!]$, the domain that models e's type. We define this function by induction on each possible value of e:

Variables Given a context $\Gamma = x_0 : A_0, \ \ldots \ , x_n : A_n$, $[\![\Gamma]\!]_{Ctx}$ maps a tuple d^* in $[\![A_0]\!] \times \cdots \times [\![A_n]\!]$ to a value in $[\![A_i]\!]$:

$$[\![\Gamma \vdash x_i : A_i]\!] = \lambda d^* \in [\![\Gamma]\!].\pi_i(d^*)$$

We use the ith projection function to get the value of the ith variable in the context.

Zero z has the type Nat, the domain of which we have defined to be $\mathbb{N}_\perp$. As z is a constant, we always map it to the same value, which is 0, no matter what d^* is:

$$[\![\Gamma \vdash z : Nat]\!]d^* = 0$$

Successor When $\Gamma \vdash s(e) : \text{Nat}$ is a well typed term, then so is $\Gamma \vdash e : \text{Nat}$, so we can use $[\![\Gamma \vdash e : \text{Nat}]\!]$ in the definition of the denotational semantics for successor. As the domain of e is $\mathbb{N}_\perp$, we must consider the case where e maps to $\perp$, for which we would also have to map $s(e)$ to $\perp$:

$[\![\Gamma \vdash s(e) : \text{Nat}]\!]d^* = \text{Let } v = [\![\Gamma \vdash e : \text{Nat}]\!]d^* \text{ in}$

$$\begin{cases} v + 1 & \text{if } v \neq \perp \\ \perp & \text{if } v = \perp \end{cases}$$

Case When $\Gamma \vdash \text{case } (e, z \mapsto e_0, s(y) \mapsto e_S) : C$ is a well typed term, then so is $\Gamma \vdash e : \text{Nat}$, so we can use $[\![\Gamma \vdash e : \text{Nat}]\!]$ in the definition of the denotational semantics for case:

$[\![\Gamma \vdash \text{case } (e, z \mapsto e_0, s(y) \mapsto e_S) : C]\!]d^* = \text{Let } v = [\![\Gamma \vdash e : \text{Nat}]\!]d^*$
in

$$\begin{cases} [\![\Gamma \vdash e_0 : C]\!]d^* & \text{if } v = 0 \\ [\![\Gamma, y : \text{Nat} \vdash e_S : C]\!](d^*, n) & \text{if } v = n + 1 \\ \bot & \text{if } v = \bot \end{cases}$$

Application In this rule we already have a denotation for the function and for the element we are applying it to. The bottom element of our domain of functions is the function that loops on all inputs, $\lambda x \in X.\bot_Y$. Therefore the value of f will always be a function. Functions on domains can be applied to bottom elements, so we can still have $f(v)$ when $v = \bot$. Therefore there is only one case for function application:

$$[\![\Gamma \vdash e\ e' : B]\!]d^* = \text{Let } f = [\![\Gamma \vdash e : A \to B]\!]d^* \text{ in}$$
$$\text{Let } v = [\![\Gamma \vdash e' : A]\!]d^*$$
$$\text{in } f(v)$$

λ **abstraction** For λ abstraction, by its typing rule, we already have a denotation for $[\![\Gamma, x : A \vdash e : B]\!]d^*$. This is a function of type $[\![\Gamma]\!] \times [\![A]\!] \to [\![B]\!]$. The function we want to obtain is of type $[\![\Gamma]\!] \to ([\![A \to B]\!])$, so we must return a continuous function. We use currying, with our denotation of $\Gamma, x : A \vdash e : B$. As this is in a different context, we need our function to be in a context where the value of x is our $a \in [\![A]\!]$ that is the argument to our function, which is (d^*, a) :

$$[\![\Gamma \vdash \lambda x : A.e : A \to B]\!]d^* = \lambda a \in [\![A]\!].[\![\Gamma, x : A \vdash e : A]\!](d^*, a)$$

Fixpoint For fixpoint, by its typing rule we already have a denotation for $[\![\Gamma, x : A \vdash e : A]\!]d^*$ This is a function of type $[\![\Gamma]\!] \times [\![A]\!] \to [\![A]\!]$. The function we want to obtain is of type $[\![\Gamma]\!] \to [\![A]\!]$ To get an element of $[\![A]\!]$, we use the fixpoint function, $fix_{[\![A]\!]}$, which is a continuous function of type $([\![A]\!] \to [\![A]\!]) \to [\![A]\!]$. the function we give to the fixpoint is the one that maps any given $a \in [\![A]\!]$ to the denotation of $\Gamma, x : A \vdash e : A$ in a context where a is the value of x:

$$[\![\Gamma \vdash \text{fix } x : A.e : A]\!]d^* = fix_{[\![A]\!]}(\lambda a \in [\![A]\!].[\![\Gamma, x : A \vdash e : A]\!](d^*, a)$$

6.2 Substitution Theorem

The following theorem says that given a well typed expression $e : A$ and another expression $e' : C$, which is well typed in the context with $x : A$ added, then the denotation of e' with e substituted for x is the same as the denotation of the original expression in the context with $x : A$ added and valuation with the denotation of e as the value of x:

Theorem 8. *If* $\Gamma \vdash e : A$ *and* $\Gamma, x : A \vdash e' : C$ *and* $d^* \in [\![\Gamma]\!]$*, then* $[\![\Gamma \vdash [e/x]e' : C]\!]d^* = [\![\Gamma, x : A \vdash e' : C]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$

Proof. We prove this by induction on the value of e':

Variables There are two cases for variables:

1. For a variable $x : C$, C must be equal to A, so we get $[\![\Gamma \vdash [e/x]x : A]\!]d^* = [\![\Gamma \vdash e : A]\!]d^*$, from the substitution rule.

   On the right hand side, $[\![\Gamma, x : A \vdash x : A]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*) = \pi_i(d^*, [\![\Gamma \vdash e : A]\!]d^*)$. The value of this is the value of x, which is $[\![\Gamma \vdash e : A]\!]d^*$.

   Therefore $[\![\Gamma \vdash [e/x]x : A]\!]d^* = [\![\Gamma, x : A \vdash x : A]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*) = [\![\Gamma \vdash e : A]\!]d^*$

2. For a variable $y : C$, we have $[\![\Gamma \vdash [e/x]y : C]\!]d^* = [\![\Gamma \vdash y : C]\!]d^*$, by the substitution rule for variables. This is equal to $\pi_i(d^*)$, where $y : C$ is the ith element of Γ. If we extend the context Γ with $x : A$ and the valuation d^* with $[\![\Gamma \vdash e : A]\!]d^*$, then this does not affect $\pi_i(d^*)$, as each variable is independent. Therefore $[\![\Gamma \vdash [e/x]y : C]\!]d^* = [\![\Gamma, x : A \vdash y : C]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$.

Zero By the substitution rule for zero, $[\![\Gamma \vdash [e/x]z : \text{Nat}]\!]d^* = [\![\Gamma \vdash z : \text{Nat}]\!]d^*$. As z is a constant, its denotation will be the same for any Γ and d^*, so we always get 0. Therefore $[\![\Gamma \vdash [e/x]z : \text{Nat}]\!]d^* = [\![\Gamma, x : A \vdash z : \text{Nat}]\!](d^*, [\![\Gamma, \vdash e : A]\!]d^*) = 0$.

Successor Using the substitution rule, $[\![\Gamma \vdash [e/x]s(e') : \text{Nat}]\!]d^* = [\![\Gamma \vdash s([e/x]e') : \text{Nat}]\!]d^*$. The induction hypothesis is $[\![\Gamma \vdash [e/x]e' : C]\!]d^* = [\![\Gamma, x : A \vdash e' : C]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$, so we can use this to rewrite $[\![\Gamma \vdash s([e/x]e') : \text{Nat}]\!]d^*$ as the following function:

Let $v = [\![\Gamma, x : A \vdash e' : C]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$ in

$$\begin{cases} v + 1 & \text{if } v \neq \bot \\ \bot & \text{if } v = \bot \end{cases}$$

This function is also the definition of $[\![\Gamma, x : A \vdash s(e') : C]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$.

Therefore $[\![\Gamma \vdash [e/x]s(e') : \text{Nat}]\!]d^* = [\![\Gamma, x : A \vdash s(e') : C]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$

Case Using the substitution rule for case, $[\![\Gamma \vdash [e/x](\text{case } (e', z \mapsto e_0, s(y) \mapsto e_S) : C]\!]d^* = [\![\Gamma \vdash (\text{case } ([e/x]e', z \mapsto [e/x]e_0, s(y) \mapsto [e/x]e_S) : C]\!]d^*$. We can use induction on all the expressions with substitutions to get the following definition of $[\![\Gamma \vdash (\text{case } ([e/x]e', z \mapsto [e/x]e_0, s(y) \mapsto [e/x]e_S) : C]\!]d^*$:

Let $v = [\![\Gamma, x : A \vdash e' : \text{Nat}]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$ in

$$\begin{cases} [\![\Gamma, x : A \vdash e_0 : C]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*) & \text{if } v = 0 \\ [\![\Gamma, y : \text{Nat}, x : A \vdash e_S : C]\!](d^*, n, [\![\Gamma \vdash e : A]\!]d^*) & \text{if } v = n + 1 \\ \bot & \text{if } v = \bot \end{cases}$$

This function is also the definition of $[\![\Gamma, x : A \vdash [e/x](\text{case } (e', z \mapsto e_0, s(v) \mapsto e_S)) : C]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$

Therefore $[\![\Gamma \vdash [e/x](\text{case } (e', z \mapsto e_0, s(v) \mapsto e_S) : C]\!]d^* = [\![\Gamma, x : A \vdash \text{case } (e', z \mapsto e_0, s(v) \mapsto e_S) : C]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$

Application Using the substitution rule for application, $[\![\Gamma \vdash [e/x](e_0\ e_1) : B]\!]d^* = [\![\Gamma \vdash [e/x]e_0([e/x]e_1) : B]\!]d^*$. We can use induction on $[\![\Gamma \vdash [e/x]e_0 : A \to B]\!]$ and $[\![\Gamma \vdash [e/x]e_1 : A]\!]$ to rewrite the denotation as the following:

Let $f = [\![\Gamma, x : A \vdash e_0 : A \to B]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$ in

Let $v = [\![\Gamma, x : A \vdash e_1 : A]\!](d^*[\![\Gamma \vdash e : A]\!]d^*)$

in $f(v)$

This function is also the definition of $[\![\Gamma, x : A \vdash e_0\ e_1 : B]\!](d^*[\![\Gamma \vdash e : A]\!]d^*)$.

Therefore, $[\![\Gamma \vdash [e/x](e_0\ e_1) : B]\!]d^* = [\![\Gamma, x : A \vdash e_0\ e_1 : B]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$

λ-Abstraction Using the substitution rule we have $[\![\Gamma \vdash [e/x](\lambda y : A.e') : A \to B]\!]d^* = [\![\Gamma \vdash \lambda y : A.([e/x]e') : A \to B]\!]d^*$. We can use induction with $[\![\Gamma \vdash \lambda[e/x]e' : B$, to rewrite the denotation as the following:

$$\lambda a \in [\![A]\!].[\![\Gamma, y : A, x : A \vdash e : B]\!](d^*, a, [\![\Gamma \vdash e : A]\!]d^*)$$

This is also the definition of $[\![\Gamma, x : A \vdash \lambda y : A.\ e' : A \to B]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$

Therefore $[\![\Gamma \vdash [e/x](\lambda y : A.e') : A \to B]\!]d^* = [\![\Gamma, x : A \vdash \lambda y : A.\ e' : A \to B]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*)$

Fixpoint Using the substitution rule for fixpoint, $[\![\Gamma \vdash [e/x](\text{fix}\ \ y : C.e' : C)]\!]d^* = [\![\Gamma \vdash \text{fix}\ \ y : C.[e/x]e' : C]\!]d^*$. The denotation of this is the following:

$$fix_{[\![C]\!]}(\lambda c \in [\![C]\!].[\![\Gamma, y : C \vdash [e/x]e' : C]\!](d^*, c)$$

We can use induction on $[\![\Gamma, y : C \vdash [e/x]e' : C]\!]$ to rewrite the denotation as the following:

$$fix_{[\![C]\!]}(\lambda c \in [\![C]\!].[\![\Gamma, y : C, x : A \vdash e' : C]\!](d^*, c, [\![\Gamma \vdash e : A]\!]d^*)$$

This is also the definition of $[\![\Gamma, x : A \vdash (fix\ y : C.e' : C)]\!](d^*, [\![\Gamma \vdash e : C]\!]d^*)$.

Therefore $[\![\Gamma \vdash [e/x](\text{fix}\ \ y : C.\ e' : C)]\!]d^* = [\![\Gamma, x : A \vdash (fix\ y : C.e' : C)]\!](d^*, [\![\Gamma \vdash e : A]\!]d^*/x)$.

Now we have proved the theorem for every case of e'. □

6.3 Soundness

In general, soundness says that if we have a mapping from one expression to another in the operational semantics, then these expressions must also have equal denotations in the denotational semantics (so the Operational Semantics are sound with relation to the Denotational Semantics).

Our Soundness theorem is the following theorem, which says that for a well typed expression e, if it maps to another expression e', then its denotation will be equal to that of the new expression in the same context:

Theorem 9. *If* $\Gamma \vdash e : A$ *and* $e \mapsto e'$ *and* $d^* \in [\![\Gamma]\!]$*, then* $[\![\Gamma \vdash e : A]\!]d^* = [\![\Gamma \vdash e' : A]\!]d^*$

Proof. By induction on $e \mapsto e'$, so there is a case for each evaluation rule:

Variables have no rules in the operational semantics, so there are no cases here.

Zero is a value, so it has no evaluation rules. Therefore there are no cases for zero.

Successor We use a congruence rule for successor, so when $s(e) \mapsto s(e')$ we also know that $e \mapsto e'$. From $\Gamma \vdash s(e) : \text{Nat}$, we know that $\Gamma \vdash e : \text{Nat}$. Therefore we can use induction on this to get $[\![\Gamma \vdash e : A]\!]d^* = [\![\Gamma \vdash e' : A]\!]d^*$.

We can use this to rewrite $[\![\Gamma \vdash s(e) : \text{Nat}]\!]d^*$ as:

Let $v = [\![\Gamma \vdash e' : \text{Nat}]\!]d^*$ in

$$\begin{cases} v + 1 & \text{if } v \neq \bot \\ \bot & \text{if } v = \bot \end{cases}$$

Which is the same as $[\![\Gamma \vdash s(e') : \text{Nat}]\!]d^*$

Case There are three cases for case:

1. When e is an expression that can be reduced, we use a congruence rule for case, so when case $(e, z \mapsto e_0, s(y) \mapsto e_S) \mapsto$ case $(e', z \mapsto e_0, s(y) \mapsto e_S)$ we also know that $e \mapsto e'$. From $\Gamma \vdash$ case $(e, z \mapsto e_0, s(y) \mapsto e_S) : C$, we know that $\Gamma \vdash e : \text{Nat}$. Therefore we can use induction to get $[\![\Gamma \vdash e : \text{Nat}]\!]d^* = [\![\Gamma \vdash e' : \text{Nat}]\!]d^*$.

   We can use this to rewrite $[\![\Gamma \vdash$ case $(e, z \mapsto e_0, s(y) \mapsto e_S) : C]\!]d^*$ as:

Let $v = [\![\Gamma \vdash e' : \text{Nat}]\!]d^*$ in

$$\begin{cases} [\![\Gamma \vdash e_0 : C]\!]d^* & \text{if } v = 0 \\ [\![\Gamma, y : \text{Nat} \vdash e_S : C]\!](d^*, n) & \text{if } v = n + 1 \\ \bot & \text{if } v = \bot \end{cases}$$

Which is the same as $[\![\Gamma \vdash \text{case } (e', z \mapsto e_0, s(y) \mapsto e_S) : C]\!]d^*$.

2. When $e = z$, we have $[\![\Gamma \vdash \text{case}(z, z \mapsto e_0, s(y) \mapsto e_S) : C]\!]d^*$ which is:

Let $v = [\![\Gamma \vdash z : \text{Nat}]\!]d^*$ in

$$\begin{cases} [\![\Gamma \vdash e_0 : C]\!]d^* & \text{if } v = 0 \\ [\![\Gamma, y : \text{Nat} \vdash e_S : C]\!](d^*, n) & \text{if } v = n + 1 \\ \bot & \text{if } v = \bot \end{cases}$$

As $[\![\Gamma' \vdash z : \text{Nat}]\!]d^*$ is always 0, this can be simplified to $[\![\Gamma \vdash e_0 : C]\!]d^*$, which is the result of the evaluation rule.

3. When $e = s(v)$, we have $[\![\Gamma \vdash \text{case}(s(v), z \mapsto e_0, s(y) \mapsto e_S) : C]\!]d^*$ which is:

Let $v' = [\![\Gamma \vdash s(v) : \text{Nat}]\!]d^*$ in

$$\begin{cases} [\![\Gamma \vdash e_0 : C]\!]d^* & \text{if } v' = 0 \\ [\![\Gamma, y : \text{Nat} \vdash e_S : C]\!](d^*, n) & \text{if } v' = v + 1 \\ \bot & \text{if } v' = \bot \end{cases}$$

where $n = [\![\Gamma \vdash v : Nat]\!]d^*$.

There are two possibilities for the value of v'.

(a) If $v' = \bot$ then the function will return $\bot$

(b) Otherwise $v' = n + 1$, where $n = [\![\Gamma \vdash v : \text{Nat}]\!]d^*$. With this we can simplify the definition of the expression to $[\![\Gamma, y : \text{Nat} \vdash e_S : C]\!](d^*, n)$

This is the same as:

$$[\![\Gamma, y : Nat \vdash e_S : C]\!](d^*, [\![\Gamma \vdash v : Nat]\!]d^*)$$

Using the substitution lemma, we get $[\![\Gamma \vdash [v/y]e_S]\!]d^*$.

Application There are two cases for application:

1. We use a congruence rule for function application, so when $e_0\ e_1 \mapsto e_0'\ e_1$ we also know that $e \mapsto e'$. From $\Gamma \vdash e_0\ e_1 : B$, we know that $\Gamma \vdash e_0 : A \to B$. Therefore we can use induction on this to get $[\![\Gamma \vdash e_0\ e_1 : A]\!]d^* = [\![\Gamma' \vdash e_0'\ e_1 : A]\!]d^*$.

   We can use this to rewrite $[\![\Gamma \vdash e_0\ e_1 : B]\!]d^*$ as:

   Let $f = [\![\Gamma \vdash e_0' : A \to B]\!]d^*$ in

   $$\text{Let } v = [\![\Gamma \vdash e_1 : A]\!]d^*$$
   $$\text{in } f(v)$$

   which is the same as $[\![\Gamma \vdash e_0'\ e_1 : B]\!]d^*$

2. When $e = \lambda x : A.e$, it is a value, so cannot be reduced further by the congruence rule. We use the semantic rule:

   $$\frac{}{(\lambda x : A.\ e)\ e' \mapsto [e'/x]e}$$

   so we need a denotation $[\![\Gamma \vdash [e'/x]e]\!]d^*$.

   As we have the denotation of $[\![\Gamma \vdash (\lambda x : A.e)\ e' : B]\!]d^*$, we have $f = [\![\Gamma \vdash \lambda x : A.e : A \to B]\!]d^*$ and $v = [\![\Gamma \vdash e' : A]\!]d^*$.

   $f = \lambda a \in [\![A]\!].[\![\Gamma, x : A \vdash e : A]\!](d^*, a)$, so $f\ v$ is $[\![\Gamma, x : A \vdash e : A]\!](d^*, [\![\Gamma \vdash e' : A]\!]d^*)$

   By the substitution lemma, this is the same as $[\![\Gamma \vdash [e'/x]e]\!]d^*$

λ-Abstraction is a value, so has no evaluation rules. Therefore there are no cases.

Fixpoint As $[\![\Gamma \vdash \text{fix } x : A.e : A]\!]d^*$ is a fixpoint operator, we know that $f(fix(f)) = fix(f)$, so we can rewrite it as:

$$(\lambda a \in [\![A]\!].[\![\Gamma, x : A \vdash e : A]\!](d^*, a))[fix_{[\![A]\!]}(\lambda a \in [\![A]\!].[\![\Gamma, x : A \vdash e : A]\!](d^*, a))]$$

which is equal to:

$$[\![\Gamma, x : A \vdash e : A]\!](d^*, fix_{[\![A]\!]}(\lambda a \in [\![A]\!].[\![\Gamma, x : A \vdash e : A]\!](d^*, a)))$$

which is equal to:

$$[\![\Gamma, x : A \vdash e : A]\!](d^*, [\![\Gamma \vdash \text{fix } x : A.e : A]\!]d^*)$$

Using the substitution lemma, this is the same as

$$[\![\Gamma \vdash [fix\ x : A.e/x]e : A]\!]d^*$$

The evaluation rule is

$$\frac{}{fix\ x : A.e \mapsto [fix\ x : A.e/x]e}$$

so this is the denotation we need. □

Chapter 7

Adequacy

Now that we have defined our operational semantics (in Chapter 4) and our denotational semantics (in Chapter 6), we can relate them using the Adequacy Theorem:

Theorem 10. *If* $\vdash e : \text{Nat}$ *(i.e. e is a closed term of type* Nat*), then* $\forall n \in \mathbb{N}.\ [\![e]\!] = n \Leftrightarrow e \mapsto^* \underline{n}$

where $[\![e]\!] = [\![\vdash e : \text{Nat}]\!]$ and $\underline{n}$ represents the numeral n, (as opposed to the actual natural number n).

The theorem has two directions. We have mostly proved the backwards direction because of the Soundness proof we proved in the previous chapter (see Section 6.3). Therefore we have proved half of Adequacy.

Therefore, the right to left direction of adequacy is a corollary of the soundness proof:

Corollary 1. *If* $\vdash e : \text{Nat}$ *and* $[\![e]\!] = n$ *then* $e \mapsto^* \underline{n} \Rightarrow [\![e]\!] = n$

Proof. We rewrite $e \mapsto^* \underline{n}$ as $e \mapsto e_0 \mapsto \cdots \mapsto e_m \mapsto \underline{n}$, for any $m \geq 0$. Applying soundness to these evaluations gives us $[\![e]\!] = \cdots = [\![e_m]\!] = [\![\underline{n}]\!]$. Therefore we have $[\![e]\!] = [\![\underline{n}]\!]$.

Now we need to prove $\forall n \in \mathbb{N}.[\![\underline{n}]\!] = n$, which we prove by induction on n:

If $n = 0$, then $[\![\underline{0}]\!] = [\![z]\!] = 0$, by the definition of the denotational semantics for zero.

If $n = n + 1$, then $[\![\underline{n+1}]\!] = [\![s(\underline{n})]\!]$. By the inductive hypothesis we have $[\![\underline{n}]\!] = n$, so $[\![s(\underline{n})]\!] = n + 1$. The definition of $[\![s(\underline{n})]\!]$ is $[\![\underline{n}]\!] + 1$. Therefore $[\![\underline{n+1}]\!] = n + 1$.

Therefore $\forall n \in \mathbb{N}.[\![\underline{n}]\!] = n$, so $[\![e]\!] = [\![\underline{n}]\!] = n$.

□

To prove the forwards direction, we could naively try to prove it by induction on the denotations of each expression. However, although we only consider closed expressions of base type, if it has sub-expressions then they may not be closed or of type Nat, so we have no inductive hypothesis to provide in these cases.

Therefore, instead we use the Logical Relations approach that we described in Section 2.5.

7.1 Logical Relation

The logical relation we define is a binary logical relation between denotations of PCF expressions and closed expressions of PCF. It is defined as the following:

Definition 8. *We define a family of binary relations $R_A \subseteq [\![A]\!] \times \{e \mid \vdash e : A\}$, defined by induction on types for each type as follows:*

$$dR_{\text{Nat}}e \Leftrightarrow \forall n \in \mathbb{N}.\ d = n \Rightarrow e \mapsto^* n$$

$$fR_{A\to B}e \Leftrightarrow \forall d \in [\![A]\!].\ \forall e' \in \{e \mid \vdash e : A\}.\ dR_A e' \Rightarrow f(d)R_B e\ e'$$

The forwards direction of Adequacy is given by the definition of the relation on expressions of type Nat.

As we previously described in section 2.5, to prove our actual property, we prove a more general Main Lemma on the logical relation first. Our Main Lemma for this relation will be the following:

Lemma 2. *If $\Gamma = x_1 : A_1, \ldots, x_n : A_n$, $\Gamma \vdash e : A$ and $d_1 R_{A_1} e_1,\ d_2 R_{A_2} e_2, \ldots,\ d_n R_{A_n} e_n$, then*

$$[\![\Gamma \vdash e : A]\!](d_1, \ldots, d_n)\ R_A\ [e_1/x_1, \ldots, e_n/x_n]e$$

In the lemma we are given an expression e that is well typed in a typing context Γ and some denotations d_i that are related to some expressions e_i.

The denotation of that expression applied to substitutions $(d_1, \ldots d_n)$ for all the free variables in Γ will be related to the expression e with the expressions that correspond to each

denotation (according to our assumption) substituted for the free variables in e.

We will prove this by induction on each possible expression e.

7.1.1 Substitution Function

In the Main Lemma, we make multiple substitutions in the expression e, but we only defined our substitution function (in Chapter 5) on single substitutions $[e'/x]e$. Therefore we define substituting multiple variables in the following way, where $\gamma = [e_1/x_1, \ldots e_n/x_n]$:

$$[\gamma](zero) = zero$$

$$[\gamma](x) = \begin{cases} [\gamma](x) & \text{if } x \in dom(\gamma) \\ x & otherwise \end{cases}$$

This says that if x is present in γ, (there is a substitution for it), replace x with the result of the substitution in γ. Otherwise there is nothing to replace x with, so we return the variable unaltered.

$$[\gamma](s(e)) = s([\gamma](e))$$

$$[\gamma](\text{case}(e, z \mapsto e_0, s(v) \mapsto e_S)) = \text{case}([\gamma](e), z \mapsto [\gamma](e_0), s(v) \mapsto [\gamma](e_S))$$

$$[\gamma](e\ e') = ([\gamma]e)([\gamma]e')$$

$$[\gamma](\lambda x : A.e) = \lambda x : A.[\gamma]e$$

$$[\gamma](\text{fix}\ \ x : A.e) = \text{fix}\ \ x : A.[\gamma]e$$

For a non empty $\gamma = e_1/x_1, \ldots e_n/x_n$, we have:

$$[e_1/x_1, \ldots, e_n/x_n]e = [e_1/x_1]([e_2/x_2](\ldots [e_n/x_n]e)$$

If we add a variable to the substitution we can define this in the following way:

$$[t/x][\gamma]e = [\gamma, t/x]e$$

7.2 Lemmas for Main Lemma

We are almost ready to prove the Main Lemma, but first we prove some lemmas that we will use in the proof:

7.2.1 Bottom Element Lemma

Lemma 3. *For any type A and $\Gamma \vdash e : A$, $\bot\ R_A\ e$*

Proof. By induction on types. For Nat, we want to show $\forall n \in \mathbb{N}.\ \bot = n \Rightarrow e \mapsto^* n$. Because $\bot \notin \mathbb{N}$, this statement is vacuously true.

For $R_{A\to B}$, we have that $\bot$ is the function $\lambda x : A.\bot$, so we want to show $\forall d \in [\![A]\!].\ \forall e' \in \{e \mid \vdash e : A\}.\ dR_A e' \Rightarrow (\lambda x : A.\bot)(d) R_B e\ e'$. Assume d is an element of the domain representing type A and e' is an expression of type A such that $d\ R_A e'$. We want to show that $\bot\ R_B\ e\ e'$ and by the inductive hypothesis, we know for any $e : B$ that $\bot\ R_B\ e$, so we can just use this with $e\ e'$ to get our conclusion. □

7.2.2 Expansion Lemma

Lemma 4. *If $\Gamma \vdash e : A$ and $e \mapsto e'$ and $dR_A e'$ then $dR_A e$*

Proof. By induction on types. For base case we assume $dR_{\text{Nat}} e'$, so we have $\forall n \in \mathbb{N}.\ d = n \Rightarrow e' \mapsto^* n$. Let $n = d$. Then, as $e \mapsto e'$ and $e' \mapsto^* n$, we know that $e \mapsto^* n$. Therefore, $dR_{\text{Nat}} e$.

For the inductive case, assume $e_0 \mapsto e_0'$ and $f\ R_{A\to B}\ e_0$, so we have $\forall a \in [\![A]\!].\ \forall e' \in \{e \mid \vdash e : A\}.\ a\ R_A\ e' \Rightarrow f(a)\ R_B\ e_0\ e'$. Let a be an element in the domain representing A and e_1 be an expression of type A such that $aR_A e_1$. Then $f(a)\ R_B\ e_0\ e_1$.

By the inductive hypothesis we know that for any $e \mapsto e'$ for expressions of type B, that they are both related to the same denotation. Therefore we have $f(a)\ R_B\ e_0'\ e_1$, so we know $\forall a \in [\![A]\!].\ \forall e' \in \{e \mid \vdash e : A\}.\ a\ R_A\ e' \Rightarrow f(a)\ R_B\ e_0'\ e'$, so $f\ R_{A\to B}\ e_0'$. □

7.2.3 Chains Lemma

Lemma 5. *For an expression $e : A$ and a chain $x_0 \sqsubseteq x_1 \sqsubseteq \ldots$, if $x_n R_A e$, then $\bigsqcup x_n\ R_A\ e$*

Proof. By induction on types. For base type, Nat, we assume we have a chain of elements in $\mathbb{N}_\bot$ such that $x_n\ R_{\text{Nat}} e$. Therefore for any element in the chain we know $\forall n \in \mathbb{N}.\ x_n = n \Rightarrow e \mapsto^* n$. We have two cases depending on the values of $\bigsqcup x_n$:

1. $\bigsqcup x_n = n$. Then we know that $e \mapsto^* \sqcup x_n$, as any $x_n R_{\text{Nat}} e$
2. $\bigsqcup x_n = \bot$. By Lemma 3 we know that $\bot R_{\text{Nat}} e$ for any e

The inductive case is for $R_{A\to B}$. Assume we have a chain $f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ of elements in $[\![A]\!] \to [\![B]\!]$ and $d\ R_A\ e'$ for some $d \in [\![A]\!]$ and $e : A$. Then we need to show $\bigsqcup f_n(d)\ R_B\ e\ e'$. By induction we know for any expression of type B and a chain $f_1(d) \sqsubseteq f_2(d) \sqsubseteq \ldots$ of elements of $[\![B]\!]$, $\bigsqcup f_n(d)$ is related to that expression. Therefore we have $\bigsqcup f_n(d)\ R_B\ e\ e'$. □

7.3 Main Lemma

Finally, we can prove the Main Lemma (Lemma 2 above):

Proof. By induction on the possible values of e.

Variables For a variable $x_1 : A_1, \ldots, x_n : A_n \vdash x : A$, assume for any $i = 1, \ldots, n$ that $d_i R_{A_i} e_i$. Then we want to show:

$$(\lambda(d_1, \ldots, d_n) \in [\![\Gamma]\!].\ \pi_i(d_1, \ldots, d_n))(d_1, \ldots, d_n)\ R_A\ [e_1/x_1, \ldots, e_n/x_n]x$$

As we have $\Gamma \vdash x : A$, by the typing rule for variables we have $\Gamma(x) = A$, so $x \in dom(\Gamma)$ and $\exists i.\ d_i R_{A_i} e_i$. Then on the right hand side we have $[e_i/x]x$. Therefore we want to show

$$d_i R_{A_i} e_i$$

which we have as an assumption.

Zero For $z : \mathrm{Nat}$, we want to show:

$$[\![\Gamma \vdash z : \mathrm{Nat}]\!](d_1, \ldots, d_n)\ R_{\mathrm{Nat}}\ [e_1/x_1, \ldots, e_n/x_n]z$$

Expanding the definitions gives us $0\ R_{\mathrm{Nat}}\ z$, so we must show $\forall n \in \mathbb{N}.0 = n \Rightarrow z \mapsto^* n$, which is the case as z reduces to n in zero steps. Therefore $0\ R_{\mathrm{Nat}}\ z$.

Successor For $x_1 : A_1, \ldots, x_n : A_n \vdash s(e) : \mathrm{Nat}$, assume for any $i = 1, \ldots, n$ that $d_i R_{A_i} e_i$. Then we want to show:

$$[\![\Gamma \vdash s(e) : \mathrm{Nat}]\!](d_1, \ldots, d_n)\ R_{\mathrm{Nat}}\ [e_1/x_1, \ldots, e_n/x_n](s(e))$$

Which expands to $\forall n \in \mathbb{N}.\ [\![\Gamma \vdash s(e) : \mathrm{Nat}]\!](d_1, \ldots, d_n) = n \Rightarrow [e_1/x_1, \ldots, e_n/x_n]s(e) \mapsto^* n$. By the inductive hypothesis, we know:

$$[\![\Gamma \vdash e : \text{Nat}]\!](d_1, \ldots, d_n)\ R_{\text{Nat}}\ [e_1/x_1, \ldots, e_n/x_n]e$$

This expands to $\forall n \in \mathbb{N}.\ [\![\Gamma \vdash e : \text{Nat}]\!](d_1, \ldots, d_n) = n \Rightarrow [e_1/x_1, \ldots, e_n/x_n]e \mapsto^* n$. (If the denotation of e is $\bot$ then this is vacuously true.)

Therefore there will be two cases:

1. If $[\![\Gamma \vdash e : \text{Nat}]\!](d_1, \ldots, d_n) = \bot$, then we must show $\bot R_{\text{Nat}}[e_1/x_1, \ldots, e_n/x_n]s(e)$, which we get from Lemma 3

2. If $[\![\Gamma \vdash e : \text{Nat}]\!](d_1, \ldots, d_n) = v$, then we must show $v+1\ R_{\text{Nat}}\ [e_1/x_1, \ldots, e_n/x_n]s(e)$. Let $n = v+1$. From the inductive hypothesis we know that $[e_1/x_1, \ldots, e_n/x_n]e \mapsto^* v$. Using the congruence evaluation rule for successor, we get $s([e_1/x_1, \ldots, e_n/x_n]e) \mapsto s(v)$ and $s(v)$ is the same as $v+1$. Therefore we have $v+1\ R_{Nat}s(v)$, so if we use this with the congruence rule in Lemma 4, we have $v+1\ R_{Nat}s([e_1/x_1, \ldots, e_n/x_n]e)$

Case For $x_1 : A_1, \ldots, x_n : A_n \vdash \text{case}(e, z \mapsto e_0, s(x) \mapsto e_S)$, assume for any $i = 1, \ldots, n$ that $d_i R_{A_i} e_i$. Then we want to show:

$$[\![\Gamma \vdash \text{case}(e, z \mapsto e_0, s(x) \mapsto e_S) : A]\!](d_1, \ldots, d_n)$$

$$R_A$$

$$[e_1/x_1, \ldots, e_n/x_n]\,\text{case}(z \mapsto e_0, s(x) \mapsto e_S)$$

The result of the denotation depends on the value of e, so we have three cases:

1. $[\![\Gamma \vdash e : \text{Nat}]\!](d_1, \ldots, d_n) = \bot$, then we must show $\bot\ R_A\ [e_1/x_1, \ldots, e_n/x_n]$ $\text{case}(z \mapsto e_0, s(x) \mapsto e_S)$, which we get by applying Lemma 3.

2. $[\![\Gamma \vdash e : \text{Nat}]\!](d_1, \ldots, d_n) = 0$, then we want to show $[\![\Gamma \vdash e_0 : A]\!](d_1, \ldots, d_n)$ $R_A[e_1/x_1, \ldots, e_n/x_n]\,\text{case}(z \mapsto e_0, s(x) \mapsto e_S)$. As we have $\Gamma \vdash e_0 : A$ from the typing rule of case, we can get $[\![\Gamma \vdash e_0 : A]\!](d_1, \ldots, d_n)\ R_A[e_1/x_1, \ldots, e_n/x_n]e_0$ by the induction hypothesis. We can now use this in Lemma 4, with the operational semantics rule for case when the expression is zero, to get

$$[\![\Gamma \vdash e_0 : A]\!](d_1, \ldots, d_n)\ R_A[e_1/x_1, \ldots, e_n/x_n]\,\text{case}(z \mapsto e_0, s(x) \mapsto e_S)$$

3. $[\![\Gamma \vdash e : \text{Nat}]\!](d_1, \ldots, d_n) = n+1$ we want to show

$$[\![\Gamma, x : \text{Nat} \vdash e_S : \text{Nat}]\!](d_1, \ldots, d_n, d)$$

$$R_A$$

$$[e_1/x_1, \ldots, e_n/x_n] \operatorname{case}(e, z \mapsto e_0, s(x) \mapsto e_S)$$

From the induction hypothesis we have

$$[\![\Gamma, x : \text{Nat} \vdash e_S : \text{Nat}]\!](d_1, \ldots, d_n, d) \;\; R_A \;\; [e_1/x_1, \ldots, e_n/x_n, e/x]e_S$$

We can now use this in Lemma 4, with the operational semantics rule for case when the expression is not zero to get

$$[\![\Gamma, x : \text{Nat} \vdash e_S : \text{Nat}]\!](d_1, \ldots, d_n, d)$$

$$R_A$$

$$[e_1/x_1, \ldots, e_n/x_n] \operatorname{case}(e, z \mapsto e_0, s(x) \mapsto e_S)$$

.

Application For $x_1 : A_1, \ldots, x_n : A_n \vdash e\ e' : B$, assume for any $i = 1, \ldots, n$ that $d_i R_{A_i} e_i$. Then we want to show:

$$[\![\Gamma \vdash e\ e' : B]\!](d_1, \ldots, d_n)$$

$$R_B$$

$$[e_1/x_1, \ldots, e_n/x_n]e\ e'$$

Using the denotational semantics and substitution function we can rewrite this as:

$$[\![\Gamma \vdash e : A \to B]\!](d_1, \ldots, d_n)([\![\Gamma \vdash e' : B]\!](d_1, \ldots, d_n))$$

$$R_B$$

$$([e_1/x_1, \ldots, e_n/x_n]e)([e_1/x_1, \ldots, e_n/x_n]e')$$

By the inductive hypothesis we have

$$[\![\Gamma \vdash e : A \to B]\!](d_1, \ldots, d_n) \; R_{A \to B} \; [e_1/x_1, \ldots, e_n/x_n]e$$

Expanding this gives us $\forall d \in [\![A]\!].\ \forall e' \in \{e \mid \vdash e : A\}.\ dR_A e' \Rightarrow [\![\Gamma \vdash e : A \to B]\!](d_1, \ldots, d_n)(d)\ R_B\ [e_1/x_1, \ldots, e_n/x_n]e\ e'$. Let $d = [\![\Gamma \vdash e' : B]\!](d_1, \ldots, d_n)$ and $e' =$

$[e_1/x_1, \ldots, e_n/x_n]e'$. Then we have

$$[\![\Gamma \vdash e : A \to B]\!](d_1, \ldots, d_n)([\![\Gamma \vdash e' : B]\!](d_1, \ldots, d_n)\ R_B\ ([e_1/x_1, \ldots, e_n/x_n]e)([e_1/x_1, \ldots, e_n/x_n]e')$$

λ-Abstraction For $x_1 : A_1, \ldots, x_n : A_n \vdash \lambda x : A.\ e : A \to B$, assume for any $i = 1, \ldots, n$ that $d_i R_{A_i} e_i$. Then we want to show:

$$[\![\Gamma \vdash \lambda x : A.\ e : A \to B]\!](d_1, \ldots, d_n)\ R_{A \to B}\ [e_1/x_1, \ldots, e_n/x_n](\lambda x : A.\ e)$$

Expanding the definition of the logical relation gives us

$$\forall d \in [\![A]\!].\ \forall e' \in \{e \mid\ \vdash e : A\}.\ dR_A e' \Rightarrow$$
$$[\![\Gamma, x : A \vdash e : B]\!](d_1, \ldots, d_n)(d)\ R_B\ [x_1/t_1, \ldots, x_n/t_n](\lambda x : A.\ e)\ e'$$

Let d be an element of the domain of type A and e' be an expression of type A such that $dR_A e'$.

Then by the using the denotational semantics for λ abstraction, we want to show:

$$(\lambda d \in [\![A]\!].\ [\![\Gamma, x : A \vdash e : B]\!](d_1, \ldots, d_n, d))(d)\ R_B\ [e_1/x_1, \ldots, e_n/x_n](\lambda x : A.\ e)\ e'$$

which is the same as:

$$[\![\Gamma, x : A \vdash e : B]\!](d_1, \ldots, d_n, d)\ R_B\ [e_1/x_1, \ldots, e_n/x_n, e'/x]\ e$$

which we get by the inductive hypothesis, because from the evaluation rule we have

$$[e_1/x_1, \ldots, e_n/x_n](\lambda x : A.\ e)\ e' \mapsto [e'/x][e_1/x_1, \ldots, e_n/x_n]e = [e_1/x_1, \ldots, e_n/x_n, e'/x]e$$

so we can use Lemma 4.

Fixpoint For $x_1 : A_1, \ldots, x_n : A_n \vdash \mathrm{fix}\, x : A.\ e : A$, assume for any $i = 1, \ldots, n$ that $d_i R_{A_i} e_i$. Then we want to show:

$$[\![\Gamma \vdash \mathrm{fix}\, x : A.\ e : A]\!](d_1, \ldots, d_n)\ R_A\ [e_1/x_1, \ldots, e_n/x_n](\mathrm{fix}\, x : A.\ e : A)$$

The denotation of the left hand side is $fix(\lambda a \in [\![A]\!].\ [\![\Gamma, x : A \vdash e : A]\!](d_1, \ldots, d_n, a))$.

By the fixpoint theorem, we know this is the same as

$$\bigsqcup_n (\lambda a \in [\![A]\!].\ [\![\Gamma, x : A \vdash e : A]\!](d_1, \ldots, d_n, a))^n(\bot)$$

If we can prove that

$$\forall n \in \mathbb{N}.\ (\lambda a \in [\![A]\!].\ [\![\Gamma, x : A \vdash\ e : A]\!](d_1 \ldots d_n, a))^n(\bot)\ R_A\ [e_1/x_1, \ldots e_n/x_n]\,\mathrm{fix}\, x : A.\ e : A$$

then we can use Lemma 5, to get the above result.

We prove this by induction on n. When $n = 0$, we need to show $\bot\ R_A\ [e_1/x_1, \ldots e_n/x_n]\,\mathrm{fix}\, x : A.\ e : A$, for which we just use Lemma 3.

For the inductive case, we must show

$$(\lambda a \in [\![A]\!].\ [\![\Gamma, x : A \vdash e : A]\!](d_1 \ldots d_n, a))^{n+1}(\bot)\ R_A\ [e_1/x_1, \ldots e_n/x_n]\,\mathrm{fix}\, x : A.\ e : A$$

We can rewrite the left hand side as $(\lambda a \in [\![A]\!].\ [\![\Gamma, x : A \vdash e : A]\!](d_1 \ldots d_n, a))(\lambda a \in [\![A]\!].\ [\![\Gamma, x : A \vdash e : A]\!](d_1 \ldots d_n, a))^n(\bot))$, which is the same as:

$$[\![\Gamma, x : A \vdash e : A]\!](d_1 \ldots d_n, (\lambda a \in [\![A]\!].\ [\![\Gamma, x : A \vdash e : A]\!](d_1 \ldots d_n, a))^n(\bot)))$$

We know that

$$(\lambda a \in [\![A]\!].\ [\![\Gamma, x : A \vdash e : A]\!](d_1 \ldots d_n, a))^n(\bot)))\ R_A\ [e_1/x_1, \ldots e_n/x_n]\,\mathrm{fix}\, x : A.\ e : A$$

by the inductive hypothesis, so we can use this as an assumption in the inductive hypothesis of the Main Lemma to get:

$$[\![\Gamma, x : A \vdash e : A]\!](d_1 \ldots d_n, (\lambda a \in [\![A]\!].\ [\![\Gamma, x : A \vdash e : A]\!](d_1 \ldots d_n, a))^n(\bot)))$$

$$R_A$$

$$[e_1/x_1, \ldots e_n/x_n, [e_1/x_1, \ldots e_n/x_n]\,\mathrm{fix}\, x : A.\ e : A/x]e$$

because

$$[e_1/x_1, \ldots e_n/x_n] \operatorname{fix} x : A.\ e : A$$

$$= \operatorname{fix} x : A.[e_1/x_1, \ldots e_n/x_n]e : A$$

$$\mapsto [\operatorname{fix} x : A.[e_1/x_1, \ldots e_n/x_n]e/x][e_1/x_1, \ldots e_n/x_n]e : A$$

$$= [[e_1/x_1, \ldots e_n/x_n] \operatorname{fix} x : A.e/x][e_1/x_1, \ldots e_n/x_n]e : A$$

$$= [e_1/x_1, \ldots e_n/x_n, [e_1/x_1, \ldots e_n/x_n] \operatorname{fix} x : A.e/x]e : A$$

We can use Lemma 4, to get

$$(\lambda a \in [\![A]\!].\ [\![\Gamma, x : A \vdash e : A]\!](d_1 \ldots d_n, a))^{n+1}(\bot)\ R_A\ [e_1/x_1, \ldots e_n/x_n] \operatorname{fix} x : A.\ e : A$$

.

□

Chapter 8

Other Proofs of Adequacy

There have been other proofs for Adequacy for PCF completed. Here we discuss some of the other approaches:

8.1 Proof using "Computable Terms"

The following definition of a computable term in PCF is given by [Gunter, 1992] and is adpated from [Plotkin, 1977]:

Definition 9. *A computable term is defined by the following statements:*

1. *If e is a closed PCF term of type* Nat*, then e is computable if $[\![e]\!] = n \Rightarrow e \mapsto^* n$*
2. *If e is a closed PCF term of type $A \to B$, it is computable if whenever e' is a closed computable term of type A, then $e\ e'$ is a closed computable term of type B*
3. *If e is an open term with free variables $x_1, \ldots, x_n$ of types $A_1, \ldots A_n$ then it is computable if $[e_1/x_1, \ldots e_n/x_n]e$ is computable when $\forall i.\ e_i$ are closed computable terms*

By combining each part of the definition of computable terms, we get that $e : A \to B$ is computable only if $\sigma e(e'_1, \ldots e'_n)$ is (where $\forall i.\ e'_i : A_i$ is closed computable and σ is a substitution formed of closed computable terms). This is because starting from the open term e, using part 3 of the definition we know that everything in σ is closed computable, so if σe is closed computable, then so is e. Then we use part 2 to prove σe is closed computable. We get $\sigma e(e'_1)$ is closed computable, then $(\sigma e(e'_1))(e'_2)$ is closed computable and so on until we have applied every argument and the whole result is closed computable, so σe is too.

Therefore if open function terms substituted with a closed computable substitution are closed computable and the application of them to any closed computable terms will give a closed computable term as a result, then the open function term is also closed computable. This will come in useful for proving terms of types other than base type are computable:

Theorem 11. *Every PCF term is computable*

Proof. By induction on e. The cases for variables, zero, succ are basically the same as in our proof.

For case, we prove $\sigma\,\mathrm{case}(e, z \to e_0, s(v) \to e_S)(e'_1, \ldots e'_n)$ is computable (where $(e'_1, \ldots e'_n)$ are all closed computable and σ contains only closed computable terms) to prove all possible case terms are computable. (This also works in closed case terms σ is just the empty substitution).

Function application just follows from the inductive hypothesis (on σe and $\sigma e'$ for open terms) and part 2 of the definition.

For λ abstraction, we prove $(\lambda x : A.\ \sigma e)(e'_1 \ldots e'_n)$ is computable if $e'_1 \ldots e'_n$ are closed computable and all the free variables of e have been substituted by closed computable terms in σ, by using parts 2 and 3 of the definition, so any $\lambda x : A.\ e$ is closed computable.

Fixpoint is the hardest case. We want to prove $(\mathrm{fix}\, x : A.\ \sigma e)(e'_1 \ldots e'_n)$ is computable if $e'_1 \ldots e'_n$ are closed computable and all the free variables of e have been substituted, λ abstraction. To do this we use the syntactic approximation described in **??**, so we now want to prove $\sigma M^n(e'_1 \ldots e'_n)$ is closed computable.

We prove this by induction on n, as we did in our Adequacy proof. $[\![M^0]\!] = \bot$, so the proof follows in a similar way to our non-termination lemma **??**.

For the inductive case, we assume σM^{k-1} is closed computable, and then evaluate σM^k in the operational semantics to get $[\sigma, \sigma M^{k-1}/x]e$.

By the inductive hypothesis, we know σM^{k-1} and e are closed computable, so $[\sigma, \sigma M^{k-1}/x]e$ is too.

Now we know σM^k is computable, we must show $\sigma M^k(e'_1, \ldots, e'_n)$ is too. The denotation of M^k is $\bigsqcup nM^n$, so we must prove $[\![M^n]\!] = n$ □

8.2 PCF with fixpoint constant

We could have also defined PCF in a different way, by using constants to replace certain constructors in the language.

For example, in our definition of PCF, we have terms $\succ\ e$, for some expression e. Here $\succ$ is a constructor, so $\succ$ itself is not a term, but once we attach an expression to it, it is a term, $succ\ e$.

This is the same for zero, case and fix.

Alternatively, we could have defined these operations as constants, for example, $\succ$: Nat $\to$ Nat is a function in the language that has a pre defined definition that never changes. We then also define $zero$: Nat, case : Nat $\to$ Nat $\to$ Nat (just on Natural numbers) and fix : $(A \to A) \to A$.

This is the way PCF is defined in [Streicher, 2006]. Also Plotkin uses this definition, with the addition of a Boolean base type in [Plotkin, 1977].

8.3 Proof using Binary Logical Relation

The proof of Adequacy given in [Streicher, 2006], as well as using a different definition of PCF, also uses a different logical relation.

Chapter 9

Non-Definability of Parallel-Or

In semantics, there is a theorem called Full Abstraction, in which if two expressions are operationally equal, then they are denotationally equal too.

9.1 The Domain Model for PCF is not fully abstract

This theorem does not hold in PCF. We can show this by giving an example of a function that we can define in the denotational model, but cannot define in PCF.

The example we use is the function Parallel Or:

9.1.1 Parallel Or

Parallel or (por) is the function defined in the following way (where $0 = true$):

$$por\ x\ y = \begin{cases} 0 & \text{if } x = 0 \vee y = 0 \\ 1 & \text{if } x = y = 1 \\ \bot & \text{otherwise} \end{cases}$$

which gives us the following truth table:

	0	1	$\bot$
0	0	0	0
1	0	1	$\bot$
$\bot$	0	$\bot$	$\bot$

As we are using call by name evaluation, (see Chapter 4) *por* evaluates x and y while it is evaluating the *por* function, and not before. However, as the function depends on x and y, it must evaluate them before *por* is fully evaluated. If the first argument we evaluate diverges, then the second may be true, but we will never know.

We can formally prove that *por* is non-definable by using logical relations, but first we need another definition on logical relations:

9.2 R-invariant

When an expression e of type A is R-**invariant**, there is an element in R_A of the form $(\llbracket \Gamma \vdash e \rrbracket d^*, \ldots, \llbracket \Gamma \vdash e \rrbracket d^*) \in R_A$

We can define this in general for an denotation $d \in \llbracket A \rrbracket$:

Definition 10. *Let R be a logical relation of arity W. Then an denotation $d \in \llbracket A \rrbracket$ is called R-invariant if*

$$\delta_W(d) = \lambda i \in W.\ d \in R_A$$

Therefore we can also say that if R is a logical relation of arity W and e is a closed term of type A then the denotation of e is R-invariant.

For non-closed expressions, we can prove that as long as the denotations of all the expressions in the substitution are R-invariant, then the denotation of the entire non closed expression is also R-invariant, as a corollary of the main lemma (see Lemma 1):

Corollary 2. *[Streicher, 2006] Let R be a logical relation on the Scott Model of arity W and $\Gamma \vdash e : B$ be a λ term. Then $\llbracket \Gamma \vdash e : B \rrbracket(d^*(i))$ is R-invariant whenever all $d \in d^*$ are.*

Therefore an **element of the denotational model** is R-invariant as long as it is the denotation of a λ term that is R-invariant.

As we know that all the denotations of closed λ terms are R-invariant, any **closed** PCF term that can be written as a λ term will have an R-invariant denotation. If we can show that the syntax of PCF can be written as λ terms, then these λ terms can be composed to create λ terms for any closed PCF term, so any closed PCF term will be R-invariant.

Therefore, we want to ensure that the following expressions are R-invariant:

- z
- $\lambda x : \mathrm{Nat}\, .s(x)$

- $\lambda x : \mathrm{Nat}, e_0 : A, e_S : A.\, \mathrm{case}(e, z \to e_0, s(x) \to e_S)$
- $\lambda e : A.\ \mathrm{fix}\, x : A\ e$

The most difficult expression to check is R-invariant is $\lambda e : A.\ \mathrm{fix}\, x : A\ e$, so we require another property on logical relations:

9.3 Admissible Logical Relations

An admissible logical relation is the following:

Definition 11. *A logical relation R of arity W is called **admissible** if $\delta_W(\bot) \in R_{Nat}$ and for all chains $d_1 \sqsubseteq d_2 \sqsubseteq \ldots$, if $d_i \in R_{\mathrm{Nat}}$, then $\bigsqcup d_n \in R_{\mathrm{Nat}}$.*

In [Streicher, 2006], the above definition is defined for directed sets. If R_{Nat} is closed under directed suprema, this is the same as saying that if we form chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ of W-tuples of elements in $\mathbb{N}_\bot$'s underlying set, if $d_i \in R_{\mathrm{Nat}}$, then $\bigsqcup d_n \in R_{\mathrm{Nat}}$. This is how obtained our definition of admissible.

We can prove the following theorem:

Theorem 12. *Let R be an admissible logical relation of arity W. Then for all types A we have:*

1. *$\delta_W(\bot) \in R_A$ and R_A for all chains $d_1 \sqsubseteq d_2 \sqsubseteq \ldots$, if $d_i \in R_A$, then $\bigsqcup d_n \in R_A$*
2. *The interpretation of $\lambda x : A.\lambda e : A.\, \mathrm{fix}\, x : A.\ e : A$ is R-invariant*

Proof. We prove *1.* by induction on types. For base type, Nat, our goal is the same as the definition of admissible and we know R is admissible.

For function types $A \to B$, using the inductive hypothesis, we know that $\delta_W(\bot) \in R_A$ and $\delta_W(\bot) \in R_B$ and that for R_A, for all chains $d_1 \sqsubseteq d_2 \sqsubseteq \ldots$, if $d_i \in R_A$, then $\bigsqcup d_n \in R_A$ and that this holds for R_B too, as we assume they are admissible.

We want to show that $\delta_W(\bot) \in R_{A\to B}$, which is the same as $\lambda i \in W.\bot \in R_{A\to B}$. As $A \to B$ is a function type, $\bot$ here is the function $\lambda x.\bot$. Therefore we must show that $\forall d \in R_A.\lambda i \in W.(\lambda x.\bot)(d(i)) \in R_B$, which is the same as $\forall d \in R_A.\lambda i \in W.\bot \in R_B$.

Let $d \in R_A$. Then we must show $\lambda i \in W.\bot \in R_B$. This is the same as $\delta_W(\bot) \in R_B$, which we already have.

To show that the limit of chains of elements in $R_{A\to B}$ is also in $R_{A\to B}$, we first assume we have a chain of functions $f_1 \sqsubseteq f_2 \sqsubseteq \ldots$, where $f_m \in R_{A\to B}$ for every element in the chain.

Expanding any one of these definitions gives us:

$$\forall d \in R_A.\ \lambda i \in W. f_m(i)d(i) \in R_B$$

Let d be a tuple of elements of $[\![A]\!]$'s underlying set that is in R_A. Then we have $\lambda i \in W.\ f_m(i)d(i) \in R_B$, so we can have a chain of elements of R_B (by monotonicity). By the inductive hypothesis, we know that limit of a chain formed of any elements of R_B is also in R_B. Therefore the element $\lambda i \in W.\ \bigsqcup f_n(i)d(i) \in R_B$ (by continuity).

Therefore $\forall d \in R_A.\ \lambda i \in W.\ \bigsqcup f_n(i)d(i) \in R_B$, so $\bigsqcup f_n \in R_{A \to B}$.

Note that this is a more general form of the function types case of the Chains Lemma (see Lemma 5)

Fix case as in written notes

We prove *2* by □

Now we have proved this, we can prove the following:

Theorem 13. *[Streicher, 2006] Let R be an admissible logical relation such that the interpretations of the terms:*

- z
- $\lambda x : \mathrm{Nat}\,.s(x)$
- $\lambda x : \mathrm{Nat}, e_0 : A, e_S : A.\,\mathrm{case}(e, z \to e_0, s(x) \to e_S)$

are all R-invariant. Then all interpretaions of closed PCF terms are R-invariant.

9.4 Logical Relation Examples

Given the following two logical relations, defined at base types:

$$(x, y, z) \in {R_{Nat}}^1 = x \uparrow y \ \wedge\ z = x \sqcap y$$

$$(x, y, z) \in {R_{Nat}}^2 = x = \bot \ \vee\ y = \bot \ \vee\ x = y = x$$

(where $x \sqcap y$ is the greatest lower bound of x and y and $x \uparrow y = \exists z.\ x \sqsubseteq z \ \wedge\ y \sqsubseteq z$)

we want to show that they are both admissible, so all the interpretations of all closed PCF terms are invariant in them.

At function type, the definition will be as for a general logical relation:

$$f \in R^1_{A\to B} = \forall d \in R^1_A.\ \lambda i \in W. f(i)(d(i)) \in R^1_B$$

$$f \in R^2_{A\to B} = \forall d \in R^2_A.\ \lambda i \in W. f(i)(d(i)) \in R^2_B$$

R_1 is admissible For R_1, $z = \bot$, so $x \uparrow y$ and $\bot \sqcap \bot = z$, so $\delta_W(\bot)$ is in R^1.

Given a chain of numbers/$\bot$s, in R_1, their limits must also be in R_1. (trivial?)

R_2 is admissible $\delta_W(\bot)$ is in R^2, as everything is equal to $\bot$.

Have written on paper all possible elements of each, should I put it here too?

Now we must show that the interpretations of all the λ definitions of the constructors are in each relation:

9.4.1 Constructors are in R_1

Lemma 6. *$[\![z]\!]$ is R^1 invariant*

Proof. We must show that $(0,0,0) \in R^1_{\text{Nat}}$. Let $z = 0$. Then $0 \sqsubseteq 0$, so $x \uparrow y$. As x and y are both 0, their glb will be too, so $x \bigsqcap y = 0 = z$. Therefore $(0,0,0) \in R^1_{\text{Nat}}$. □

Lemma 7. *$[\![\lambda x : \text{Nat}\,.\ s(x)]\!]$ is R^1 invariant.*

Proof. We want to show that $(\lambda n \in \mathbb{N}_\bot.\ [\![x : \text{Nat} \vdash s(x) : \text{Nat}]\!](n/x), \lambda n \in \mathbb{N}_\bot.\ [\![x : \text{Nat} \vdash s(x) : \text{Nat}]\!](n/x), \lambda n \in \mathbb{N}_\bot.\ [\![x : \text{Nat} \vdash s(x) : \text{Nat}]\!](n/x)) \in R^1_{\text{Nat}\to\text{Nat}}$.

As these are functions of type Nat $\to$ Nat, we must show that for any $(x,y,z) \in R^1_{\text{Nat}}$, that the denotations we obtain when these values replace n in each of the functions are still related by the relation. If any of these values are $\bot$, the denotation will also be $\bot$. If any of (x,y,z) are n, then the result of their denotation function will be $n+1$. For $x \uparrow y$ to be true, if either x or y are n then the other one must also be n. If $z = x \sqcap y$, then if x or y is n, then z must be n. The conditions in R^1_{Nat} still be true if we add 1 to the value that is n, as numbers can only be related to themselves or $\bot$, so it does not matter what number n is.

Therefore, for any $(x, y, z) \in R^1_{\text{Nat}}$,

$$((\lambda n \in \mathbb{N}_\perp.\ [\![x : \text{Nat} \vdash s(x) : \text{Nat}]\!](n/x))(x),$$
$$(\lambda n \in \mathbb{N}_\perp.\ [\![x : \text{Nat} \vdash s(x) : \text{Nat}]\!](n/x))(y),$$
$$(\lambda n \in \mathbb{N}_\perp.\ [\![x : \text{Nat} \vdash s(x) : \text{Nat}]\!](n/x))(z))$$
$$\in R^1_{\text{Nat}}$$

as the *bot*s stay the same and we add 1 to the *n*s, so all the properties are preserved. This means that the denotation of the λ term for successor is R^1 invariant. □

Lemma 8. $[\![\lambda e : \text{Nat}, x : \text{Nat}, e_0 : A, e_S : A.\, \text{case}(e, z \mapsto e_0, s(x) \mapsto e_S)]\!]$ *is* R^1 *invariant.*

Proof. We want to show that

$$\lambda i \in W.($$
$$\lambda n \in \mathbb{N}_\perp, x' \in \mathbb{N}_\perp, e'_0 \in [\![A]\!], e'_S \in [\![A]\!].$$
$$[\![e : \text{Nat}, x : \text{Nat}, e_0 : A, e_S : A \vdash \text{case}(e, z \mapsto e_0, s(x) \mapsto e_S)]\!]$$
$$(n/e, x'/x, e'_0/e_0, e'_S/e_S))$$

.

$$\in R^1_{\text{Nat}\to\text{Nat}\to A\to A\to A}$$

As this is a function type, we assume that we have $(x, y, z) \in R^1_{\text{Nat}}$. Then we can replace n with x, y and z. If any of these values were $\perp$, the resulting denotation is $\lambda x' \in \mathbb{N}_\perp, e'_0 \in [\![A]\!], e'_S \in [\![A]\!].\perp$.

If it is 0, then we have

$$\lambda x' \in \mathbb{N}_\perp, e'_0 \in [\![A]\!], e'_S \in [\![A]\!].$$
$$[\![e : \text{Nat}, x : \text{Nat}, e_0 : A, e_S : A \vdash e_0]\!]$$
$$(n/e, x'/x, e'_0/e_0, e'_S/e_S)$$

and for e_S we have the same but we swap out the denotation of e_0 for the denotation of e_S.

The denotation of the function with $\perp$ will have less information than the other two functions, which are equally informative, so the properties in $(x, y, z) \in R^1_{\text{Nat}}$ will be preserved. Therefore the λ term for case is R^1 invariant. □

9.4.2 Constructors are in R_2

Lemma 9. *$[\![z]\!]$ is R^2 invariant*

Proof. We want to show that $(0,0,0) \in R^2_{\text{Nat}}$, which we know as $0 = 0 = 0$. □

Lemma 10. *$[\![\lambda x.\ s(x)]\!]$ is R^2 invariant*

Proof. Given some $(x,y,z) \in R^2_{\text{Nat}}$, then if any of these values are $\bot$, then

$$(\lambda n \in \mathbb{N}_\bot.\ [\![x : \text{Nat} \vdash s(x) : \text{Nat}]\!](n/x))(\bot)$$

will also be $\bot$, so the successor term is in R^2_{Nat}.

If they are equal, then the denotation of the successor terms will either all be the same $n+1$, or will all be $\bot$, so will also be in R^2_{Nat}.

Therefore for all $(x,y,z) \in R^2_{\text{Nat}}$ that are given as parameters to $\lambda i \in W.[\![\lambda x.\ s(x)]\!]$, the resulting denotations will also be in R^2_{Nat}, so $\lambda i \in W.[\![\lambda x.\ s(x)]\!] \in R^2_{\text{Nat}\to\text{Nat}}$, and $[\![\lambda x.\ s(x)]\!]$ is R^2 invariant. □

Lemma 11. *$[\![\lambda e : \text{Nat}, x : \text{Nat}, e_0 : A, e_S : A.\, \text{case}(e, z \mapsto e_0, s(x) \mapsto e_S)]\!]$ is R^2 invariant.*

Proof. For some $(x,y,z) \in R^2_{\text{Nat}}$, if one of them is equal to $\bot$, then the denotation of the case expression with it will be $\lambda x \in \text{Nat},\ e_0 \in [\![A]\!], e_S \in [\![A]\!].\ \bot$, which is the function that does not terminate, so will be $\bot$ in $R^2_{\text{Nat}\to\text{Nat}\to A\to A\to A}$.

Otherwise, $x = y = z - n$, so the denotation of all the case expression is either

$$\lambda x' \in \mathbb{N}_\bot, e'_0 \in [\![A]\!], e'_S \in [\![A]\!].$$
$$[\![e : \text{Nat}, x : \text{Nat}, e_0 : A, e_S : A \vdash e_0]\!]$$
$$(n/e, x'/x, e'_0/e_0, e'_S/e_S)$$

or

$$\lambda x' \in \mathbb{N}_\bot, e'_0 \in [\![A]\!], e'_S \in [\![A]\!].$$
$$[\![e : \text{Nat}, x : \text{Nat}, e_0 : A, e_S : A \vdash e_S]\!]$$
$$(n/e, x'/x, e'_0/e_0, e'_S/e_S)$$

so they will all be the same function. Therefore the λ term is R^2 invariant.

□

Now we know that all PCF constructor λ terms are all R^1 invariant, so by the theorem, all denotations of closed PCF terms are R^1 invariant.

9.4.3 Stable PCF functions

Now we can go prove a lemma which says that all first order PCF terms are *stable*. Stable means that binary infima of consistent pairs are preserved (i.e. given x and y such that $x \uparrow y$, then if $z = x \sqcap y$, then $f(z) = f(x) \sqcap f(y)$):

Lemma 12. *[Streicher, 2006] For every expression e of first order type, (i.e. of type* $\text{Nat} \to \text{Nat} \to \cdots \to \text{Nat}$*, (for k* Nat*s)), we have:*

$$[\![e]\!](x_1 \sqcap y_1) \dots (x_k \sqcap y_k) = [\![e]\!](x_1) \dots (x_n) \sqcap [\![e]\!](y_1) \dots (y_n)$$

for all x^* *and* $y^* \in [\![\text{Nat} \times \cdots \times \text{Nat}]\!]$*, with* $x^i \uparrow y^i$ *for all* $i = 1, \dots, k$

Proof. As we know that all the λ terms of the constructors are R^1 invariant and R^1 is admissible, then the denotation of any closed PCF term is R^1 invariant. This means for any function $[\![e]\!]$ that describes the denotation of a closed PCF term, and any $(x, y, z) \in R^1_{\text{Nat}}$, that

$$[\![e]\!](x) \uparrow [\![e]\!](y) \ \wedge \ [\![e]\!](z) = [\![e]\!](x) \sqcap [\![e]\!](y)$$

We know every $x^i \uparrow y^i$ so we have $(x^i, y^i, x^i \sqcap y^i) \in R^1_{\text{Nat}}$.

Applying the definition of $([\![e]\!], [\![e]\!], [\![e]\!]) \in R^1_{\text{Nat}\to\dots\text{Nat}}$ we get:

$$([\![e]\!](x^*), [\![e]\!](y^*), [\![e]\!](x^* \sqcap y^*)) \in R^1_{\text{Nat}}$$

Therefore we have $([\![e]\!](x_1, \dots x_n), [\![e]\!](y_1, \dots y_n), [\![e]\!](x_1 \sqcap y_1), \dots (x_n \sqcap y_n)) \in R^1_{\text{Nat}}$.

Assuming $x^* \sqcap y^* = (x_1 \sqcap y_1), \dots (x_n \sqcap y_n)$

By the second part of R^1_{Nat}'s definition, we have:

$$[\![e]\!](x_1 \sqcap y_1), \dots (x_n \sqcap y_n) = [\![e]\!](x_1, \dots x_n) \sqcap [\![e]\!](y_1, \dots y_n)$$

□

Now we can use this to show the non definability of parallel or:

Corollary 3. *There are no PCF definable functions f of type* $[\![\text{Nat} \to \text{Nat} \to \text{Nat}$ *where:*

- $f0\bot = 0$
- $f\bot 0 = 0$
- $f11 = 1$

Proof. By contradiction. Assume there is a function f that satisfies the above conditions and it is definable in PCF. Then $f0\bot = 0 = f\bot 0$, so using Lemma 12 we have

$$f\bot\bot = f(0 \sqcap \bot)(\bot \sqcap 0) = f0\bot \sqcap f\bot 0 = 0 \sqcap 0 = 0$$

However we also have $f11 = 1$ and by monotonicty of f we should have $f\bot\bot \sqsubseteq f11$ which is $0 \sqsubseteq 1$ which is not in the relation for $\mathbb{N}_\bot$. Therefore we have a contradiction and there is no PCF definable f satisfying the constraints. □

The constraints in the above corollary characterize parallel or, so we cannot define parallel or in our PCF.

9.4.4 Showing por cannot be defined with R^2

As all the λ-terms for PCF constructors are in R^2 and it is admissible, it is also the case that the denotation of any closed PCF term of type Nat $\to$ Nat $\to$ Nat is R^2 invariant. If this term is f, then we cannot satisfy all the por conditions.

$(\bot, 0, 1)$ and $(0, \bot, 1)$ are in R^2_{Nat}, as both tuples contain $\bot$, so we should have $f((\bot, 0, 1)(0, \bot, 1) \in R^2_{\text{Nat}}$.

However, $(f\bot 0, f0\bot, f11) = (0, 0, 1)$, so it is not in R^2_{Nat}.

Chapter 10

Evaluation

10.1 What did I learn/achieve?

Before I started this project I knew very little about domain theory, but I knew it featured prominently in research in programming language semantics. Now I know what a domain is (see 2.1), how to prove given structures are domains and how certain domains can be used to give models of programming languages (see Chapter 6).

I also knew nothing about Logical Relations (see 2.5), but now know how to construct them and prove properties using them.

I now also know that Type Safety is expressed by two lemmas (see 5.2.1 and 5.2.2) and how to prove them. Type Safety is an important theorem in programming language semantics, so now I can apply this knowledge to prove type safety for other type systems.

I previously proved Soundness (see 6.3) before (for an imperative language in my mini-project), but now I have proved it for a new language and semantics, so I am applying the skills from my mini project to a new situation. Type Safety and Soundness are common theorems in semantics, so I am also finding it easier to read new papers in the area now.

Finishing the Adequacy proof (see Theorem 10) was rewarding, as in the more simple language I previously studied, we proved a theorem called Completeness, that was less challenging to prove and required less mathematical background.

10.2 Evaluation of the Product

My product is the proofs and analysis of these proofs. The proofs on domains were mostly proofs that the definitions were satisfied. Proving partial orders was quite easy as the properties they must satisfy were studied in the Maths Techniques module in 2nd year. Proving that all chains have limits was the most difficult bit. The proof of the fixpoint theorem (see 2.4) was the most difficult domain theory proof, but it had clear stages that made it easier to prove.

Many of the proofs on the operational semantics and typing rules were by induction. Having now completed more difficult proofs, they seem easier than they were at the time. The number of cases involved make the proofs very long, but each case uses the inductive hypothesis in a similar way.

The Adequacy proof was the hardest proof, as I had to come up with a logical relation that described our property in a way that it would be correct and actually used to prove Adequacy. The first attempt I made at this did not give enough information for the fixpoint case, as it was only defined on the syntax and we couldn't use our fixpoint theorem (see 2.4) in the proof. I spent a lot of time trying to make this proof work, but ultimately I had to try another approach. In all the other proofs in my dissertation, I did not hit a wall like this, so this proof took significantly longer.

Eventually I realised that we needed to include the semantics in the definition of the relation and a binary relation , like that defined by [Streicher, 2006] would be required, which made the proof much easier.

10.3 Evaluation of the Process

My problem statement was to prove Adequacy for PCF. I completed this by defining the operational and denotational semantics for PCF, then proving both directions of the theorem. Therefore I believe I have fully solved the problem stated in the introduction.

Every thing we proved in the preceeding sections is necessary to show that our domains are appropriate for modelling the language, its typing system is correct and that the semantics can be related and ordered in a way that no proof needs content given later in the report to understand.

10.4 Review of Project Plan

I followed my project plan up until the later stages, when we got to Adequacy. This proof took much longer than expected, the reason being that I underestimated its difficulty. As I was unfamiliar with the logical relations technique, it was hard to know exactly how long it would take to prove a theorem in this way.

Luckily I had no further work planned past this point, as Adequacy was the main goal of the report. This ensured I had enough time to fall back on this, as I had not overplanned my time with no room to manoeuvre.

10.5 Limitations and ideas for future work

Having now proved Adequacy for PCF in the Domain Theoretic model, I could apply my knowledge to other models and/or other languages.

For example FPC is a language similar to PCF in which we have recursive types, so I could try to prove the same theorem with this addition to the possible types we can form.

I could also try different ways of constructing the logical relation to describe Adequacy.

Chapter 11

Conclusion

In the report we defined the syntax of PCF and gave the typing rules, which all syntactically correct PCF programs must satsify. We specified the operational semantics of PCF and we proved Type Safety, which states that all correctly typed PCF programs must have a valid evaluation in the operational semantics if they are expected to. We defined our denotational semantics of PCF and proved that the operational semantics are sound with respect to this. We proved the Adequacy theorem on our semantics and we discussed other approaches to the proof of Adequacy.

Therefore we have fully solved the problem stated in the introduction (in Chapter 1).

Bibliography

Michael JC Gordon. Evaluation and denotation of pure lisp programs: a worked example in semantics. 1973.

Mike Gordon. Proof, language, and interaction. chapter From LCF to HOL: A Short History, pages 169–185. MIT Press, Cambridge, MA, USA, 2000. ISBN 0-262-16188-5. URL `http://dl.acm.org/citation.cfm?id=345868.345890`.

Carl A Gunter. *Semantics of programming languages: Structures and Techniques*. MIT press, 1992.

Charles Antony Richard Hoare. An axiomatic basis for computer programming. *Communications of the ACM*, 12(10):576–580, 1969.

G. Hutton. Introduction to Domain Theory (Course Notes). 2014.

P. J. Landin. Correspondence between algol 60 and church's lambda-notation: Part i. *Commun. ACM*, 8(2):89–101, February 1965. ISSN 0001-0782. doi: 10.1145/363744.363749. URL `http://doi.acm.org/10.1145/363744.363749`.

Peter J Landin. The mechanical evaluation of expressions. *The Computer Journal*, 6(4): 308–320, 1964.

Simon Marlow et al. Haskell 2010 language report. 2010.

R. Milner. *The Definition of Standard ML: Revised*. MIT Press, 1997. ISBN 9780262631815.

Robin Milner. Logic for Computable Functions: Description of a Machine Implementation. Technical report, Stanford, CA, USA, 1972a.

Robin Milner. Implementation and Applications of Scott's Logic for Computable Functions. *SIGACT News*, (14):1–6, January 1972b. ISSN 0163-5700.

Robin Milner. Models of LCF. Technical report, Stanford, CA, USA, 1973.

Robin Milner and Richard Weyhrauch. Proving compiler correctness in a mechanized logic. *Machine Intelligence*, 7:51–70, 1972.

Tobias Nipkow, Lawrence C Paulson, and Markus Wenzel. *Isabelle/HOL: a proof assistant for higher-order logic*, volume 2283. Springer Science & Business Media, 2002.

Gordon D. Plotkin. Lambda-definability and logical relations. Memorandum SAI–RM–4, University of Edinburgh, Edinburgh, Scotland, October 1973.

Gordon D. Plotkin. LCF considered as a programming language. *Theoretical computer science*, 5(3):223–255, 1977.

Gordon D Plotkin. A structural approach to operational semantics, 1981.

Dana S. Scott. A Type-theoretical Alternative to ISWIM, CUCH, OWHY. *Theor. Comput. Sci.*, 121(1-2):411–440, December 1993. ISSN 0304-3975. doi: 10.1016/0304-3975(93)90095-B. URL `http://dx.doi.org/10.1016/0304-3975(93)90095-B`.

R. Statman. Logical relations and the typed -calculus. *Information and Control*, 65(2):85 – 97, 1985. ISSN 0019-9958.

Thomas Streicher. *Domain-theoretic foundations of functional programming*. World Scientific, 2006.

W. W. Tait. Intensional interpretations of functionals of finite type i. *The Journal of Symbolic Logic*, 32:198–212, 8 1967. ISSN 1943-5886.

A.K. Wright and M. Felleisen. A syntactic approach to type soundness. *Information and Computation*, 115(1):38 – 94, 1994. ISSN 0890-5401.