Some Notes from the Second Meeting

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1 Chains

$f : \mathbb{N} \rightarrow \mathbb{N}$ is a chain when $\forall i, j \in \mathbb{N}\ i \leq j \Rightarrow f(i) \leq f(j)$.

For a domain $(X, \perp, \sqsubseteq)$, $Chain(X)$ is the set of all chains that can be defined on it. We define it formally as:

$$Chain(X) = \{f \in \mathbb{N} \rightarrow X \mid \forall i.i \leq j \Rightarrow f(i) \leq f(j)\}$$

Upper bound of a chain If $f \in Chain(X)$ then y is an upper bound of f when $\forall i.f(i) \leq y$.

Least Upper bound of a chain If $f \in Chain(X)$ then $\sqcup f$ is the least upper bound of f when $\forall y.y = \sqcup f \Rightarrow \sqcup f \sqsubseteq y$.

2 Continuous Functions

Assume X and Y are domains. Then $Cont(X, Y) = \{f \in X \rightarrow Y\}$ is the set of continous functions from X to Y. This is defined as the functions where:

- $\forall x, x' \in X.x \sqsubseteq x' \Rightarrow f(x) \sqsubseteq f(x')$
- $x \in Chain(X) \Rightarrow f(\sqcup^{X \rightarrow Y} x_i) = \sqcup^{Y} f_i(x_i)$

So if any two elements of X are related, theit image in f will also be related in the same way.

For a chain of X, f applied to the lub of X is equal to the lub of the image of f.

Theory 1. *A Least Upper Bound of a chain is unique*

Proof. Assume there are two lubs a, a' of a chain a_i. Then we have $a \sqsubseteq a'$ and $a' \sqsubseteq a$. We know that $\sqsubseteq$ is a partial order and therefore antisymmetric. Therefore $a = a'$, so all lubs of a chain are equal to each other. $\square$

2.1 Functions example

The least upper bound for functions is:

$$\sqcup^{X \to Y} f_i = \lambda x. \sqcup^{Y} f_i(x)$$