Domain Proofs

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1 Single Element Domain

- $\{x\}$ is the set
- $\bot = x$
- $\sqsubseteq$ just contains (x, x)

1.1 $\forall x \in \{x\}.x \sqsubseteq x$

There is only one element, x and $x \sqsubseteq x$ is in the ordering.

1.2 $\sqsubseteq$ is a partial order

Reflexivity Only element is x and $x \sqsubseteq x$ is in the ordering

Antisymmetry x, y must both be x, as it is the only possible element, so $x = x$.

Transitivity x, y and z must all be x and $x \sqsubset x$ is in the ordering.

Therefore $\sqsubseteq$ is a partial order.

1.3 All chains must have a least upper bound

As x is the only possible element, all chains will be of the form

$$x \sqsubseteq x \sqsubseteq x \sqsubseteq \dots$$

Therefore we define $\sqcup\{x\} = x$.

Then we need an element z such that the following two statments are true. Let $z = \sqcup\{x\} = x$. Then:

- $\forall \mathbf{i}.\mathbf{x_i} \sqsubseteq \mathbf{z}$
  The only possible x_i is x, so we must have $x \sqsubseteq x$. This is in the ordering.
- $\forall \mathbf{y}.(\forall \mathbf{i}.\mathbf{x_i} \sqsubseteq \mathbf{y}) \Rightarrow \mathbf{z} \sqsubseteq \mathbf{y}$
  The only possible value of y is x. Therefore we must have $x \sqsubseteq x$, which is in the ordering.

Now we have proved all the conditions, so the single element domain is a domain.

2 Product of Domains

Given two domains $\mathbb{X} = X, \perp_X, \sqsubseteq_X$ and $\mathbb{Y} = Y, \perp_Y, \leq_Y$, we have the following:

- $X \times Y$ is the set
- $\perp = (\perp_X, \perp_Y)$
- $(x, y) \sqsubseteq (x', y')$ is defined when $x \sqsubseteq_X x'$ and $y \leq_Y y'$ are defined

2.1 $\forall (x, y) \in X \times Y. \perp \sqsubseteq (x, y)$

Because $\mathbb{X}$ is a domain, we know $\forall x \in X. \perp_X \sqsubseteq_X x$. Because $\mathbb{Y}$ is a domain, we know $\forall y \in Y. \perp_Y \leq_Y y$.

Therefore we have $\forall x, y. \perp_X \sqsubseteq_X x \wedge \perp_Y \leq_Y y$. This is the same as $\forall (x, y) \in X \times Y. (\perp_X, \perp_Y) \sqsubseteq (x, y)$

2.2 $\sqsubseteq$ is a partial order

Reflexivity For an element $(x, y) \in X \times Y$, we have $x \sqsubseteq_X x$ and $y \leq_Y y$ because $\mathbb{X}$ and $\mathbb{Y}$ are domains, so their orderings are reflexive. This means we have $(x, y) \sqsubseteq (x, y)$.

Antisymmetry For elements (x, y) and (x', y') we can assume $(x, y) \sqsubseteq (x', y')$ and $(x', y') \sqsubseteq (x, y)$. Expanding these definitions we have $x \sqsubseteq_X x' \wedge y \leq_Y y' \wedge x' \sqsubseteq_X x \wedge y' \leq_Y y$. If we reorder this we have:

$$x \sqsubseteq_X x' \wedge x' \sqsubseteq_X x \wedge y \leq_Y y' \wedge y' \leq_Y y$$

As the orderings on $\mathbb{X}$ and $\mathbb{Y}$ are antisymmetric, we can rewrite this as $x = x'$ and $y = y'$. Therefore we have $(x, y) = (x', y')$.

Transitivity For elements $(x, y), (x', y')$ and (x'', y'') we can assume $(x, y) \sqsubseteq (x', y')$ and $(x', y') \sqsubseteq (x'', y'')$. Expanding these definitions we have $x \sqsubseteq_X x' \wedge y \leq_Y y' \wedge x' \sqsubseteq_X x'' \wedge y' \leq_Y y''$. If we reorder this we have:

$$x \sqsubseteq_X x' \wedge x' \sqsubseteq_X x'' \wedge y \leq_Y y' \wedge y' \leq_Y y''$$

As the orderings on $\mathbb{X}$ and $\mathbb{Y}$ are transitive, we can rewrite this as $x \sqsubseteq_X x''$ and $y \leq_Y y''$. Therefore we can now define $(x, y) \sqsubseteq (x'', y'')$.

2.3 All chains have a least upper bound

Chains of $X \times Y$ will be of the form:

$$(x, y) \sqsubseteq (x', y') \sqsubseteq (x'', y'') \sqsubseteq \ldots$$

where $x \sqsubseteq_X x' \sqsubseteq_X x'' \ldots$ and $y \leq_Y y' \leq_Y y'' \ldots$.

Then we need an element z such that the following two statements are true. Let $z = \sqcup(x_i, y_i) = (\sqcup x_i, \sqcup y_i)$. Then:

- $\mathbf{\exists z \in X \times Y. \forall i. (x_i, y_i) \sqsubseteq z}$
  As $\mathbb{X}$ and $\mathbb{Y}$ are domains, we have $\forall i. x_i \sqsubseteq_X \sqcup x_i$ and $\forall i. y_i \leq_Y \sqcup y_i$. Therefore, for any (x, y) we have $\forall i. (x_i, y_i) \sqsubseteq (\sqcup x_i, \sqcup y_i)$.
- $\mathbf{\exists z \in X \times Y. \forall (x', y'). (\forall i. (x_i, y_i) \sqsubseteq (x', y')) \Rightarrow z \sqsubseteq (x', y')}$
  As $\mathbb{X}$ and $\mathbb{Y}$ are domains, we have $\forall x'. (\forall i. x_i \sqsubseteq_X x') \Rightarrow \sqcup x_i \sqsubseteq_X x'$ and $\forall y'. (\forall i. y_i \leq_Y y') \Rightarrow \sqcup y_i \leq_Y y'$.

Therefore if we assume $\forall (x', y'). (\forall i. (x_i, y_i) \sqsubseteq (x', y'))$, then we know $\sqcup x_i \sqsubseteq_X x'$ and $\sqcup y_i \leq_Y y'$. This is the definition of $(\sqcup x_i, \sqcup y_i) \sqsubseteq (x', y')$.

Now we have proved all the conditions, so the product of two domains is also a domain.