Natural Numbers

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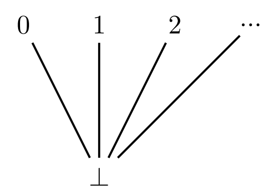
We have each number and a bottom element, $\perp$ that carries less information than a number. The only orderings we have are $\perp \sqsubseteq 0$, $\perp \sqsubseteq 1$, $\perp \sqsubseteq 2$, etc...



The set $\mathbb{N}_\perp = \{\perp\} \cup \mathbb{N}$ and the relation $\sqsubseteq$ is defined as

$$\sqsubseteq\, = \{(\perp, \perp)\} \cup \{(\perp, n), (n, n) \mid n \in \mathbb{N}\}$$

1 $\forall x \in \mathbb{N}_\perp.\ \perp \sqsubseteq x$

In the definition of our relation we have $\{(\perp, n) \mid n \in \mathbb{N}\}$ and $(\perp, \perp)$. so $\{(\perp, x)\ |x \in \mathbb{N}_\perp\} \subset \sqsubseteq$.

2 Prove $\sqsubseteq$ is a partial order

For $\sqsubseteq$ to be a partial order, it must be reflexive, antisymmetric and transitive.

Reflexivity $\sqsubseteq$ is *reflexive* by definition, as we have $\{(x, x) \mid x \in \mathbb{N}_\perp\}$ as a subset of $\sqsubseteq$.

Antisymmetry When $x = \perp$, the only possible y we can have such that $y \sqsubseteq x$, is $y = \perp$, as $n \sqsubseteq \perp$ is not defined in the relation for any n. Therefore $x = y = \perp$.

When $x = n$, the only possible value of y is n, so $x = y = n$.

Therefore $\sqsubseteq$ is *antisymmetric*

Transitivity If $x = \perp$, a possibility for y is $y = n$. Then we must have $z = n$ for (y, z) to be in $\sqsubseteq$. Then we need $\perp \sqsubseteq n$, which we have, as we have $(\perp, n)$, for any $n \in \mathbb{N}$, defined in the relation. y can also be $\perp$. Then we have both options for z. When $z = n$, we should have $\perp \sqsubseteq n$, which we have, as we have $(\perp, n)$ for any $n \in \mathbb{N}$ defined in the relation. When $z = \perp$, we just want $\perp \sqsubseteq \perp$, which is also in the definition of $\sqsubseteq$.

If $x = n$, then both y and z must also be equal to n for $x \sqsubseteq y$ and $y \sqsubseteq z$ to be defined. Therefore we should have $n \sqsubseteq n$. This is in the definition of $\sqsubseteq$.

Therefore $\sqsubseteq$ is *transitive*.

All three properties hold, so $\sqsubseteq$ is a partial order on $\mathbb{N}_\perp$.

3 All chains have a least upper bound (limit)

A chain formed from the set $\mathbb{N}_\perp$ can be of three types

- $\perp \sqsubseteq \ldots \sqsubseteq \perp$, for any number of $\perp$s
- $n \sqsubseteq \ldots \sqsubseteq n$, for any number of n , where n is the same number each time
- $\perp \sqsubseteq \ldots \sqsubseteq \perp \sqsubseteq n \sqsubseteq \ldots \sqsubseteq n$, for any number of $\perp$s followed by any number of identical ns

There are two things we must prove, for all chains ($\forall x.\ Chain(\mathbb{N}_\perp)$):

- $\exists z \in \mathbb{N}_\perp . \forall i . x_i \sqsubseteq z$
- $\exists z \in \mathbb{N}_\perp . \forall y . (\forall i . x_i \sqsubseteq y) \Rightarrow z \sqsubseteq y$

where z is the least upper bound of the chain. We can now prove this by case analysis on the different chains, proving the two properties for each case:

$\perp \sqsubseteq \ldots \sqsubseteq \perp$ For these chains, let $z = \perp$. The last element in the chain will always be $\perp$, so for every i we have $\perp \sqsubseteq \perp$. Therefore $\forall i . x_i \sqsubseteq \perp$.

For the second part, every element is $\perp$, so $x_i = \perp$ and $y = \perp$. Then we have $\perp \sqsubseteq \perp$ for $z \sqsubseteq y$. Therefore $\forall y . (\forall i . x_i \sqsubseteq y) \Rightarrow \perp \sqsubseteq y$ holds.

$n \sqsubseteq \ldots \sqsubseteq n$ For the chains just containing n, let $z = n$. The last element in the chain will always be n, so for every n we have $n \sqsubseteq n$. Therefore $\forall i.x_i \sqsubseteq n$.

For the second part, every element is n, so $x_i = n$ and $y = n$. Then we have $n \sqsubseteq n$ for $z \sqsubseteq y$. Therefore $\forall y.(\forall i.x_i \sqsubseteq y) \Rightarrow n \sqsubseteq y$ holds.

$\perp \sqsubseteq \ldots \sqsubseteq \perp \sqsubseteq n \sqsubseteq \ldots \sqsubseteq n$ For these chains, let $z = n$. The last element will be n. We have both $\perp \sqsubseteq n$ and $n \sqsubseteq n$ in the relation, so for any x, we have $x \sqsubseteq n$. Therefore $\forall i.x_i \sqsubseteq n$.

For the second part, $(\forall i.x_i \sqsubseteq y)$ is only true when $y = n$, so we only have to consider this case. Then we have $n \sqsubseteq n$ for $z \sqsubseteq y$. Therefore $\forall y.(\forall i.x_i \sqsubseteq y) \Rightarrow n \sqsubseteq y$ holds.

now we have proved that there exists a lower bound, z, for every possible chain, so our set $\mathbb{N}_\perp$ and partial order has a least upper bound for every chain in $Chain(\mathbb{N}_\perp)$, the set of possible chains we can form with $\mathbb{N}_\perp$.

We have proved that $\mathbb{N}_\perp$ with the ordering $\sqsubseteq$ is a pointed poset with a least upper bound for all of its chains, so it must be a domain.