Lemmas for Main Lemma

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Lemma 1. *If* $[\![e]\!] = \bot$ *and* $\vdash e : A$ *and* $e \mapsto^{\infty}$*, then* $e \in Good_A$

Proof. By induction on types. Base case will be for the type Nat. We need to show that $\vdash e : Nat$ (which we already have as an assumption) and $[\![e]\!] = n \Rightarrow e \mapsto^* \underline{n}$.

We can rewrite this as $[\![e]\!] \neq n \ \vee \ e \mapsto^* \underline{n}$

As we have $[\![e]\!] = \bot$ as an assumption and $\bot \notin \mathbb{N}$, we know that $[\![e]\!] \neq n$ for any $n \in \mathbb{N}$. Therefore $[\![e]\!] \neq n \ \vee \ e \mapsto^* \underline{n}$, so $[\![e]\!] = n \Rightarrow e \mapsto^* \underline{n}$ and $e \in Good_{Nat}$.

For the inductive case, we need to show that $e \in Good_{A \to B}$, so we need $\vdash e : A \to B$ (which we have as an assumption) and $\forall e' \in Good_A.\ e\ e' \in Good_B$.

Let $e' \in Good_A$. As $[\![e]\!] = \bot$ is the bottom element of $[A \to B]$, this is the same as $\lambda a \in A.\bot_B$. We can apply the inductive hypothesis to get $e\ e' \in Good_B$, as we know $[\![e\ e']\!] = \bot_B$ and $\vdash e\ e' \in B$ (from the typing rule for function application with $e' \in Good_A$), so to do this, we just need to prove $e\ e' \mapsto^{\infty}$, which we prove by contradiction:

Assume $e\ e' \mapsto \underline{n}$, for some $n \in \mathbb{N}$. Then by correctness, we have $[\![e\ e']\!] = n$. But $[\![e\ e']\!] = \bot_B$, so we have a contradiction. So $e\ e' \mapsto^{\infty}$.Now we apply the inductive hypothesis and get $e\ e' \in Good_B$.

So for any $e' \in Good_A$, we have $e\ e' \in Good_B$.

Now we have proved the lemma for any type. $\square$

Lemma 2. *If* $\gamma \in Good_{Ctx}(\Gamma)$ *and* $\Gamma \vdash e : A$ *then* $\vdash [\gamma](e) : A$

Proof. By induction on Γ. The base case is when Γ is the empty context. We have $\gamma = <>$, so we need to prove $\vdash <> e : A$. The empty substitution does nothing, so this is the same as $\vdash e : A$, which we already have as an assumption.

The inductive case is where we have $\gamma \in Good_{Ctx}(\Gamma, x : A)$. Let $\gamma = (\gamma', e'/x)$ where $\gamma' \in Good_{Ctx}(\Gamma)$ and $e' \in Good_A$.

From $e' \in Good_A$, we have $\vdash e' : A$. We can use weakening on each variable in the context individually to get $\Gamma \vdash e' : A$. We use type substitution with this and the assumption $\Gamma, x : A \vdash e : A$ to get $\Gamma \vdash [e'/x]e : A$.

Now we apply the inductive hypothesis of the theorem with $\gamma' \in Good_{Ctx}(\Gamma)$ to get $\vdash [\gamma'][e'/x']e : A = \vdash [\gamma]e : A$

Now we have proved the lemma for any type. $\square$