Fixpoint Theorem

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Theorem 1. *Every continuous function $f : X \to X$ has a least fixpoint, which is the limit of the chain $\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \ldots$*

Proof. Define the fixpoint function $fix(f) \equiv \sqcup f^i(\perp)$. This is the limit of the chain in the theorem. We know this limit exists because as f is continuous, $(X, \sqsubseteq, \perp)$ must form a domain, and by the definition of domain, all chains of X have a limit.

$\sqcup f^i(\perp)$ is a fixpoint For the limit to be a fixpoint we must have $f(\sqcup f^i(\perp)) = \sqcup f^i(\perp)$. As f is continuous, we have $f(\sqcup f^i(\perp)) = \sqcup f(f^i(\perp)) = \sqcup f^{i+1}(\perp)$. The chain formed by f^{i+1} is $f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \ldots$. This is the same as our original chain, but without $\perp$ at the start. Because X is a domain, we know that X has a least element, so $\forall x \in X.\ \perp \sqsubseteq x$. Therefore $\perp$ has no effect on the limit because every element is higher than it, so removing $\perp$ will not change the limit. This means that $\sqcup f^{i+1}(\perp) = \sqcup f^i(\perp)$.

$\sqcup f^i(\perp)$ is the least fixpoint Let x be an element of our chain such that $fix(x) = x$. Then for $\sqcup f^i(\perp)$ to be the least fixpoint, we must have $\sqcup f^i(\perp) \sqsubseteq x$, so x is an upper bound that is higher than $\sqcup f^i(\perp)$. To prove this, first we prove x is an upper bound, so $\forall n.\ f^n(\perp) \sqsubseteq x$. We prove by this by induction on n:

if $n = 0$, then $f^0(\perp) \sqsubseteq x$, This is the same as $\perp \sqsubseteq x$, which is true because $\perp$ is the least element of the chain.

Our inductive hypothesis is $f^n(\perp) \sqsubseteq x$. As f is continous, f is monotone, so $f(f^n(\perp)) \sqsubseteq f(x) = f^{n+1}(\perp) \sqsubseteq x$. Therefore we know that for any element $f^n(\perp)$ in the chain, $f^n(\perp) \sqsubseteq x$.

As $\sqcup f^i(\perp)$ is a least upper bound, we know that $\forall x \in X.\ \forall n.\ (f^n(\perp) \sqsubseteq x) \Rightarrow \sqcup f^i(\perp) \sqsubseteq x$. We have just proved the left hand side of this, so we now have $\sqcup f^i(\perp) \sqsubseteq x$.

Now we have proved that $\sqcup f^i(\perp)$ is the least fixpoint of f. $\square$