Logical Relations

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A logical relation in the Scott Model of PCF is defined by Streicher as the following:

Definition 1. *Let W be an arbitrary set. A **logical relation** of arity W (on the Scott Model) is a family of W-ary relations on each type:*

$$R = \{R_A \in \mathcal{P}({D_A}^W) \mid A \in Type\}$$

where for function types we have:

$$f \in R_{A \to B} = \forall d \in R_A.\ \lambda i \in W. f(i)(d(i)) \in R_B$$

A logical relation R of arity W is uniquely determined by R_{nat}, so for all subsets of ${D_{nat}}^W$ there is a unique R equal to the set.

Function Types For function types, for a function $f = (f_1, \ldots f_n)$ to be in the relation, if we apply it to **any** value that is in the relation of the type of its domain, for example, for arity 3:

$$(x, y, z) \in R_A$$

Then f applied to everything in these elements will be in the relation of the codomain, so we must have:

$$(f_1(x), f_2(y), f_3(z)) \in R_B$$

Then $f \in R_{A \to B}$

1 Main Lemma

For any denotation of a PCF term, we want to show that it is in the relations, so we want to show that $[\![\Gamma \vdash M : B]\!] \in R_{\Gamma \to B}$. An element of D_Γ is any tuple of substitutions $d^* = (d_1, \ldots d_n)$ for $x_1 : A_1, \ldots x_n : A_n$ in Γ. So $R_\Gamma \in \mathcal{P}(D_{A_1} \times \ldots \times D_{A_n})$.

We want all substitutions in a set of size W to be in the relation, so using the definition of $f \in R_{\Gamma \to B}$, we want to show that

$$\forall d \in R_\Gamma.\ \lambda i \in W.[\![\Gamma \vdash M : B]\!](d(i))$$

This says that for any position in the W-tuple, we have the denotation of M using the substitution d from the Wth position in the $d \in R_\Gamma$.

For W different substitutions we want to have

$$([\![\Gamma \vdash M : B]\!](d^*)_1, \ldots, [\![\Gamma \vdash M : B]\!](d^*)_W) \in R_B$$

Therefore the main lemma for λ terms is the following:

Lemma 1. *Let R be a logical relation of arity W on the Scott Model of PCF. Then for λ terms $\Gamma \vdash M : B$ and $d_j \in R_{A_j}$ for $j = 1, \ldots, n$*

$$\lambda i \in W.[\![\Gamma \vdash M : B]\!](d^*(i)) \in R_B$$

where $d^*(i) = d_1(i) \ldots d_n(i)$ and $\Gamma = x_1 : A_1, \ldots x_n : A_n$

Proof. By induction on λ terms:

Variables We have $x_1 : A_1, \ldots x_n : A_n \vdash x_j : A_j \vdash x_j : A_j$ and $d_j \in R_{A_j}$ for every j.

Need to show that $\lambda i \in W.[\![x_1 : A_1, \ldots x_n : A_n \vdash x_j : A_j]\!](d^*(i)) \in R_{A_j}$. $d^*(i)$ includes $d_j(i)$, so we rewrite this to $\lambda i \in W.d_j(i) \in R_{A_J}$. As we assumed that $d_j \in R_{A_j}$, then at each ith position, we already have $d_j(i) \in R_{A_j}$ as an assumption.

λ **Abstraction** Need to show that $\lambda i \in W.[\![\Gamma \vdash \lambda x : A.\ M : B]\!](d^*(i)) \in R_{A \to B}$. As this is a function type, we can rewrite this to:

$$\lambda d \in R_A.\lambda i \in W.\ [\![\Gamma \vdash \lambda x : A.M : B]\!](d^*(i))(d(i)) \in R_B$$

Let d be a substitution in R_A. By the definition of the denotational semantics for λ abstraction we can rewrite the goal to:

$$\lambda i \in W.\ [\![\Gamma, x : A \vdash M : B]\!](d^*(i), d(i)) \in R_B$$

We can use the induction hypothesis with $\Gamma, x : A \vdash M : B$ and $d_j \in R_{A_j}$ and $d \in R_A$ to get $\lambda i \in W.[\![\Gamma, x : A \vdash M : B]\!](d^*(i), d(i)) \in R_B$, which is the same as our goal.

Application Need to show that $\lambda i \in W.[\![\Gamma \vdash M(N) : B]\!](d^*(i)) \in R_B$.

Using the denotational semantics for application, we can rewrite the goal to:

$$\lambda i \in W.[\![\Gamma \vdash M : A \to B]\!](d^*(i))([\![\Gamma \vdash N : A]\!](d^*(i)) \in R_B$$

.

By induction on the denotations of M and N, we have $\lambda i \in W.[\![\Gamma \vdash M]\!](d^*(i)) \in R_{A \to B}$. and $\lambda i \in W.[\![\Gamma \vdash N]\!](d^*(i)) \in R_A$.

Therefore we have a $d \in R_A$, so by definition of $R_{A \to B}$ we have

$$\lambda i \in W.[\![\Gamma \vdash M]\!](d^*(i))([\![\Gamma \vdash N]\!](d^*(i)) \in R_B$$

which is our goal. □

Now we have proved the main lemma, we can show this holds specifically for closed terms:

Corollary 1. *If R is a logical relation of arity W and M is a closed λ term of type B then $\lambda i \in W.[\![M]\!] \in R_B$.*

Proof. We have $[\![\vdash M : B]\!]$ and there is nothing to substitute, so we use the main lemma to get $\lambda i \in W.[\![\vdash M : B]\!] <> \in R_B$, where $<>$ is the empty substitution. This is the same as $\lambda i \in W.[\![M]\!] \in R_B$, which is our goal. □

2 R-invariant

When a term M of type A is R-invariant, there is an element in R_A of the form $([\![\Gamma \vdash M]\!]d^*, \ldots, [\![\Gamma \vdash M]\!]d^*) \in R_A$

We can define this in general for an object $d \in D_A$:

Definition 2. *Let R is a logical relation of arity W. Then an object $d \in D_A$ is called R-invariant if*

$$\delta_W(d) = \lambda i \in W.\ d \in R_A$$

Therefore we can also say that if R is a logical relation of arity W and M is a closed term of type B then the denotation of M is R-invariant.

We can prove for terms that are not closed as long as the denotations of all the terms in the substitution are R-invariant, then the denotation of these whole terms are also R-invariant, also as a corollary of the main lemma:

Corollary 2. *Let R be a logical relation on the Scott Model of arity W and $\Gamma \vdash M : B$ a λ term. Then $[\![\Gamma \vdash M : B]\!](d^*(i))$ is $R-invariant$ whenever all $d \in d^*$ are.*

Proof. Assume all terms d in d^* are R invariant. Then $\delta(d_j) \in R_{A_j}$. Therefore we can use this as an assumption in the main lemma, which gives us:

$$\lambda i \in W.[\![\Gamma \vdash M]\!](\delta(d_1(i)), \dots \delta(d_n)(i)) = \lambda i \in W.[\![\Gamma \vdash M]\!](d^*(i)) \in R_B$$

□

Therefore an **element of the Scott model** is R-invariant as long as it is the denotation of a λ term that is R-invariant.

As we know that all the denotations of closed λ terms are R-invariant, any **closed** PCF term that can be written as a λ term will have an R-invariant denotation. If we can show that the constructs of PCF can be written as λ terms, then these can be composed to create λ terms for any closed PCF term, any any closed PCF term is R-invariant. (We prove this in Theorem 2)

Therefore, we want to see if the following terms are R-invariant:

- $zero$
- $\lambda x : Nat.succ(x)$
- $\lambda x : Nat.pred(x)$
- $\lambda x : Nat, y : Nat, z : Nat.if(x, y, z)$
- $\lambda f : A \to A.\ Y_A(f)$

Note that in Streicher's semantics, if is only defined on Natural numbers!

The most difficult term to check is R-invariant is $\lambda f : A \to A.\ Y_A(f)$, and requires another property on logical relations.

3 Admissible Logical Relations

An admissible logical relation is the following:

Definition 3. *A logical relation R of base type and arity W is called **admissible** if $\delta_W(\bot) \in R_{Nat}$ and R_{Nat} is closed under suprema of directed sets*

This means that $(\bot, \dots, \bot) \in R_{Nat}$ and for any for subsets of elements in R_{Nat}, their least upper bounds are still in R.

We can prove the following theorem:

Theorem 1. *Let R be an admissible logical relation of arity W. Then for all types A we have:*

1. *$\delta_W(\bot) \in R_A$ and R_A is closed under suprema of directed sets*
2. *The interpretation of $\lambda f : A \to A.\ Y_A(f)$ is R-invariant*

Proof. We prove *1.* by induction on types. For base type, Nat, our goal is the same as the definition of admissible and we know R is admissible.

For function types $A \to B$, using the inductive hypothesis, we know that $\delta_W(\bot) \in R_A$ and $\delta_W(\bot) \in R_B$ and that both R_A and R_B are closed under suprema of directed sets.

We want to show that $\delta_W(\bot) \in R_{A\to B}$, which is the same as $\lambda i \in W.\bot \in R_{A\to B}$. As $A \to B$ is a function type, $\bot$ here is the function $\lambda x.\bot$. Therefore we must show that $\forall d \in R_A.\lambda i \in W.(\lambda x.\bot)(d(i)) \in R_B$, which is the same as $\forall d \in R_A.\lambda i \in W.\bot \in R_B$.

Let $d \in R_A$. Then we must show $\lambda i \in W.\bot \in R_B$. This is the same as $\delta_W(\bot) \in R_B$, which we already have.

To show that $R_{A\to B}$ is closed under suprema of directed sets, we can rewrite this as $\forall F \subseteq R_{A\to B}.\ \bigsqcup F \in R_{A\to B}$, where F is a directed subset of $R_{A\to B}$. Assume that F is a directed subset of $R_{A\to B}$. Then there must be some function $\bigsqcup F \in R_{A\to B}$, so we want to show that

$$\forall d \in R_A.\lambda i \in W.\bigsqcup F(i)(d(i)) \in R_B$$

Let d be an element of R_A. By definition of $R_{A\to B}$ we have $\lambda i \in W.f(i)(d(i)) \in R_B$ for any $f \in F$.

By induction we know that R_B is closed under suprema of all directed subsets, so there exists an upper bound, which is some function in F applied to d :

$$\bigsqcup_{f \in F} (\lambda i \in W. f(i)(d(i))) \in R_B$$

As all function types are represented in the Scott Model by domains of continuous functions, we know that $\bigsqcup F$ is continuous. Therefore $\bigsqcup_{f \in F}(\lambda i \in W. f(i)(d(i))) \in R_B = \lambda i \in W. \bigsqcup_{f \in F} f(i)(d(i)) = \lambda i \in W. \bigsqcup F(i)(d(i))$.

To prove *2.*, we must prove that $\delta_W([\![\lambda f : A \to A.\ Y_A(f)]\!]) \in R_{(A \to A) \to A}$. We can write this as:

$$\forall d \in R_{A \to A}. \lambda i \in W.\ \delta_W([\![\lambda f : A \to A.\ Y_A(f)]\!])(i)(d(i))$$

We have $f \in R_{A \to A}$, so we want to show:

$$\lambda i \in W.\ \delta_W([\![\lambda f : A \to A.\ Y_A(f)]\!])(i)(f(i))$$

Expanding out the δ_W gives us:

$$\lambda i \in W.\ [\![\lambda f : A \to A.\ Y_A(f)]\!] f(i)$$

And the denotational semantics for λ-abstraction gives us:

$$\lambda i \in W.\ (\lambda f \in D_{A \to A}.\ [\![Y_A(f)]\!])(f(i))$$

$f(i)$ is in $D_{A \to A}$ as it is in $R_{A \to A}$, so we now have:

$$\lambda i \in W.\ [\![Y_A(f(i))]\!]$$

Using the denotational semantics for fixpoint gives us:

$$\lambda i \in W.\ \mu(f(i))$$

Now we need to prove that $\lambda i \in W.\ \mu(f(i)) \in R_A$

As we have $\lambda i \in W.\ f(i)^n(\perp) \in R_A$ for any n, *(see below)* we know that $\lambda i \in W.\ \bigsqcup_{n \in \mathbb{N}} f(i)^n(\perp) = \lambda i \in W.\ \mu(f(i)) \in R_A$ $\square$

Lemma used in the proof:

Lemma 2. *For $f \in R_{A \to A}$ and all $n \in \mathbb{N}$, $\lambda i \in W.\ f(i)^n(\bot) \in R_A$*

Proof. By induction on n. The base case is $n = 0$. We must show $\lambda i \in W.\ \bot \in R_A = \delta_W(\bot) \in R_A$, which we already have.

For the inductive case, as $f \in R_{A \to A}$, we have $\forall d \in R_A.\lambda i \in W.\ f(i)(d(i)) \in R_A$. Let $d = \lambda i \in W.\ f(i)^n(\bot) \in R_A$, which we get from the inductive hypothesis. Now we have:

$$\lambda i \in W.\ f(i)((\lambda i \in W.f(i)^n(\bot))(i)) \in R_A$$

which is the same as $\lambda i \in W.\ f(i)(f(i)^n(\bot)) = \lambda i \in W.\ f(i)^{n+1}(\bot)$.

□

Now we know that $\lambda f : A \to A.\ Y_A(f)$ is R-invariant for any admissible R, we can prove that the interpretaion of closed PCF terms are R-invariant if all PCF constants are R-invariant, by the following theorem:

Theorem 2. *Let R be an admissible logical relation on the Scott model, such that the interpretations of the following terms:*

- $zero$
- $\lambda x : Nat.succ(x)$
- $\lambda x : Nat.pred(x)$
- $\lambda x : Nat, y : Nat, z : Nat.if(x, y, z)$

are all R-invariant. Then all interpretations of closed PCF-terms are R-invariant.

Proof. Let R be the admissible logical relation such that all the given PCF constants are R-invariant. By Theorem 2, we know that $\lambda f : A \to A.\ Y_A(f)$ is also R-invariant. Therefore all of the PCF constants are R-invariant, so closed PCF terms are also R invariant, as we can write them using as λ-terms, using the R-invariant constants. □

4 Logical Relation Examples

Given the following two logical relations:

$$(x, y, z) \in {R_{Nat}}^1 = x \uparrow y \ \wedge \ z = x \sqcap y$$

$$(x, y, z) \in {R_{Nat}}^2 = x = \bot \ \vee \ y = \bot \ \vee \ x = y = x$$

(where $x \sqcap y$ is the greatest lower bound of x and y and $x \uparrow y = \exists z.\ x \sqsubseteq z \wedge y \sqsubseteq z$)

we want to show that they are both admissible, so all the interpretations of all closed PCF terms are invariant in them.

$\delta_W(\perp)$ is in R^2, as everything is equal to $\perp$ and for R_1, $z = \perp$, so $x \uparrow y$ and $\perp \sqcap \perp = z$, so $\delta_W(\perp)$ is in R^1.

Streicher says for finite W there are no non-trivial directed subsets of D^W_{Nat}.

Therefore both of our relations will be admissible.

We first show that all PCF constants are R^2- invariant:

Zero We want to show that $\delta_W([\![zero]\!]) \in R^2_{Nat}$, so $\lambda i \in W.[\![zero]\!] \in R^2_{Nat}$.

As the arity of R^2 is 3, and $[\![zero]\!]$ is always 0, we need to show $(0,0,0) \in R^2_{Nat}$. $0 = 0 = 0$, so this is true.

Zero R^1_{Nat} Let $z = 0$. Then $0 \sqsubseteq 0$, so $x \uparrow y$ and $x \sqcap y = 0$. Therefore $(0,0,0) \in R^1_{nat}$.

Succ We want to show that $\delta_W([\![\lambda x.succ(x)]\!]) \in R^2_{Nat \to Nat} = \lambda i \in W.([\![\lambda x.succ(x)]\!]) \in R^2_{Nat \to Nat}$

This is equal to $\lambda i \in W.(\lambda n \in \mathbb{N}_\perp.\ [\![succ]\!]n) \in R^2_{Nat \to Nat}$, so we want to show that $(\lambda n \in \mathbb{N}_\perp.\ [\![succ]\!]n, \lambda n \in \mathbb{N}_\perp.\ [\![succ]\!]n, \lambda n \in \mathbb{N}_\perp.\ [\![succ]\!]n) \in R^2_{Nat \to Nat}$.

Expanding the definition of this reduces to:

$$\lambda n \in \mathbb{N}_\perp.\ [\![succ]\!]n = \perp \vee \lambda n \in \mathbb{N}_\perp.\ [\![succ]\!]n = \lambda n \in \mathbb{N}_\perp.\ [\![succ]\!]n = \lambda n \in \mathbb{N}_\perp.\ [\![succ]\!]n$$

The second half of this statement is obviously true, so the whole thing is and $\delta_w(([\![\lambda x.succ(x)]\!]) \in R^2_{Nat \to Nat}$.

Succ $R^1_{Nat \to Nat}$ Let $z = \lambda x \in \mathbb{N}_\perp.[\![succ]\!]n$. Then $x \sqsubseteq z\ \wedge y \sqsubseteq z$ and $x \sqcap y = z$.

Pred This case is exactly the same as succ but with $[\![pred]\!]$ instead of $[\![succ]\!]$.

If We want to show that $\delta_W([\![\lambda x : Nat, \lambda y : Nat, \lambda z : Nat.\ ifz(x,y,z)]\!]) \in R^2_{(Nat \to Nat \to Nat) \to Nat} = \lambda i \in W.\ [\![\lambda x : Nat, \lambda y : Nat, \lambda z : Nat.\ ifz(x,y,z)]\!] \in R^2_{(Nat \to Nat \to Nat) \to Nat}$

This is equal to $\lambda i \in W.(\lambda x, y, z \in \mathbb{N}_\perp.[\![ifz]\!](x,y,z)) \in R^2_{(Nat \to Nat \to Nat) \to Nat}$, which reduces to:

$$(\lambda x, y, z \in \mathbb{N}_\perp.[\![ifz]\!](x,y,z) = \perp) \vee$$

$$(\lambda x, y, z \in \mathbb{N}_\perp.[\![ifz]\!](x,y,z) = \lambda x, y, z \in \mathbb{N}_\perp.[\![ifz]\!](x,y,z) = \lambda x, y, z \in \mathbb{N}_\perp.[\![ifz]\!](x,y,z))$$

The second half is true as they are all the same function so map the same inputs to the same outputs.

If $R^1_{(Nat \to Nat \to Nat) \to Nat}$ Let $z' = \lambda x, y, z \in \mathbb{N}_\perp.[\![if]\!](x,y,z)$. Then $x' \sqsubseteq z' \wedge y' \sqsubseteq z'$ and $x' \sqcap y' = x' = y' = z'$, where x', y', z' are such that $(x', y', z') = (\lambda x, y, z \in \mathbb{N}_\perp.[\![if]\!](x,y,z), \lambda x, y, z \in \mathbb{N}_\perp.[\![if]\!](x,y,z), \lambda x, y, z \in \mathbb{N}_\perp.[\![if]\!](x,y,z))$.