Math 51: Section 8 Exercises

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1 Exercise 8.1

This section was considered by the author to be too trivial to attempt. Move on to the next section.

2 Exercise 8.2

Consider the set $S = \{(x, y, z) \in \mathbf{R}^3 : x^3 + z^3 + 3y^2z^3 + 5xy = 0\}$.

(a) Give functions $f, h : \mathbf{R}^3 \to \mathbf{R}$ for which S is a level set of both $f(x, y, z)$ and $h(x, y, z)$.

The level set is when the function is set to a certain constant. To find f and h, we can simply evaluate them at different constants to satisfy the same set S. Thus, let

$$f(x, y, z) = x^3 + z^3 + 3y^2z^3 + 5xy$$

. Then, clearly $S = \{(x, y, z) \in \mathbf{R}^3 : f(x, y, z) = 0\}$. Similarly, we can evaluate and add different constants to satisfy the same set. For example, let

$$h(x, y, z) = x^3 + z^3 + 3y^2z^3 + 5xy + 1$$

. We see that because the constants cancel out, $S = \{(x, y, z) \in \mathbf{R}^3 : h(x, y, z) = 1\}$.

(b) By solving for z in terms of x and y, give a function $g : \mathbf{R}^2 \to \mathbf{R}$ for which S is the graph of g.

We can first solve for z. Doing this we get

$$z = \sqrt[3]{\frac{x^3 + 5xy}{1 + 3y^2}}$$

. We are asked to find g where S is the graph of z. Therefore, we can say that $z = g(x, y)$, as they are interchangable. Therefore, let

$$g(x, y) = \sqrt[3]{\frac{x^3 + 5xy}{1 + 3y^2}}$$

. It is clear to see that $\text{Graph}(g) = S = \{(x, y, z) \in \mathbf{R}^3 : x^3 + z^3 + 3y^2z^3 + 5xy = 0\}$.

3 Exercise 8.3

Consider the function $g : \mathbf{R} \to \mathbf{R}^2$ defined by $g(t) = \left(\frac{e^t+e^{-t}}{2}, \frac{e^t-e^{-t}}{2}\right)$

(a) Show that every point in the output of g lies on the hyperbola $x^2 - y^2 = 1$

We see that for the function g, the x-coordinate is $\left(\frac{e^t+e^{-t}}{2}\right)$ and the y-coordinate is $\left(\frac{e^t-e^{-t}}{2}\right)$. We can square both of these terms and plug it into the hyperbola and see if it equals 1 for all t. We see that $x^2 = \frac{e^{2t}+2+e^{-2t}}{4}$ and $y^2 = \frac{e^{2t}-2+e^{-2t}}{4}$. Therefore,

$$x^2 - y^2 = \frac{e^{2t}+2+e^{-2t}}{4} - \frac{e^{2t}-2+e^{-2t}}{4} = \frac{4}{4} = 1$$

. QED

(b) Are all the points in the hyperbola $\{(x, y) \in \mathbf{R}^2 : x^2 - y^2 = 1\}$ in the output of g? If "yes" then explain why, and if "no" explain why a specific point on the hyperbola is not in the output.

No, not all the points of the hyperbola are in the output of g. Consider the point (-1,0) which is on the hyperbola $x^2 - y^2 = 1$. This point cannot be attained in the output of $g : \mathbf{R} \to \mathbf{R^2}$, as for the x-coordinate to be negative, one such point $\{t \in \mathbf{R} : e^t + e^{-t} < 0\}$ must exist. However, exponents in $\mathbf{R}$ cannot be negative. So, such a t must not exist. Thus, the point (-1,0) on the hyperbola $x^2 - y^2 = 1$ is not on the output of g, meaning not all points of the hyperbola are in the output of g.

4 Exercise 8.4

Let $f : \mathbf{R^3} \to \mathbf{R}^3$ be the function $\mathbf{f}(x, y, z) = (x - y + 2z, 3x - 3y + 5z, 3x - 3y + 5z, 3x - 2y + 2z)$ and let $\mathbf{g} : \mathbf{R^3} \to \mathbf{R}^3$ be the function $\mathbf{f}(u, v, w) = (4u - 2v + w, 9u - 4v + w, 3u - v)$.

(a) Show that $\mathbf{g}(\mathbf{f}(x, y, z)) = (x, y, z)$.

We can write these vectors top-down for simplicity.

$$g \circ f = \begin{bmatrix} 4u - 2v + w \\ 9u - 4v + w \\ 3u - v \end{bmatrix} \circ \begin{bmatrix} x - y + 2z \\ 3x - 3y + 5z \\ 3x - 2y + 2z \end{bmatrix}$$

which equals

$$\begin{bmatrix} 4(x - y + 2z) - 2(3x - 3y + 5z) + (3x - 2y + 2z) \\ 9(x - y + 2z) - 4(3x - 3y + 5z) + (3x - 2y + 2z) \\ 3(x - y + 2z) - (3x - 3y + 5z) \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(b) Show that $\mathbf{f}(\mathbf{g}(u, v, w)) = (u, v, w)$. Similarly,

$$f \circ g = \begin{bmatrix} x - y + 2z \\ 3x - 3y + 5z \\ 3x - 2y + 2z \end{bmatrix} \circ \begin{bmatrix} 4u - 2v + w \\ 9u - 4v + w \\ 3u - v \end{bmatrix}$$

equals

$$\begin{bmatrix} (4u-2v+w)-(9u-4v+w)+2(3u-v) \\ 3(4u-2v+w)-3(9u-4v+w)+5(3u-v) \\ 3(4u-2v+w)-2(9u-4v+w)+2(3u-v) \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

5 Exercise 8.5

Let S be a level set $\{(x,y,z) \in \mathbf{R}^3 : f(x,y,z) = c\}$ in $\mathbf{R}^3$.

(a) If S is also the graph $\{(x,y,z) \in \mathbf{R^3} : (x,y) \in D, z = g(x,y)\}$ of a function $g : D \to \mathbf{R}$ on some region D in $\mathbf{R}^2$, explain why S meets each vertical line $\{(a,b,t) : t \in \mathbf{R}\}$ (for $(a,b) \in \mathbf{R}^2$) in at most one point.

We are already given that g is a function, which means for every (x,y) there is only one $g(x,y)$. So, if S is the graph of a function, it must also fulfill the requirements of a function. Thus, it must pass the vertical line test, meaning it meets the function at at most one point. We see that for $\{(a,b,t,) : t \in \mathbf{R}\}$, where $(a,b) \leftrightarrow (x,y) \in D$, this line is vertical. So, when $(a,b) \in D$, it intersects S in at most one point, and when $(a,b) \notin D$, it intersects S at exactly 0 points.

(b) For the sphere $S = \{(x,y,z) \in \mathbf{R}^3 : x^2+y^2+z^2 = 4\}$ of radius 2 centered at the origin, explain both algebraically and geometrically why S violates the "vertical line test" in (a), so S is not the graph of a function.

Algebraically, we cannot see that S is not a graph of a function if it is indeed a sphere, as by a definition of a function, one set of inputs can only have one output. If we define z as the dependent variable, it is clear to see that one value of x and y can provide two different outputs for z that satisfy the equation, namely the positive and negative counterparts of z. Geometrically, we see that a sphere can never pass the vertical line test, as there is a positive counterpart to every negative x, y, and z value on the other side, and vice versa.

6 Exercise 8.8

8.6 and 8.7 were involved with sketching countour maps, so I did not do it.

Let $f : \mathbf{R}^2 \to \mathbf{R}^3$ be given by

$$f(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

and let $g : \mathbf{R}^3 \to \mathbf{R}$ be given by

$$g(x,y,z) = x^2+y^2+z^2$$

.

(a) Calculate $g \circ f : \mathbf{R}^2 \to \mathbf{R}$.

Squaring each term and writing it top down, we get:

$$\begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix} \circ \begin{bmatrix} x^2+y^2+z^2 \end{bmatrix} = \cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi + \cos^2\phi$$

simplifying, we get:

$$= \cos^2\theta\sin^2\phi + \sin^2\phi - \sin^2\phi\cos^2\theta + 1 - \sin^2\phi = 1$$

(b) Explain using (a) why each point $f(\theta, \phi)$ lies on the unit sphere in $\mathbf{R}^3$ centered at the origin. It turns out that every point in the unit sphere is in the output of f, but we are not asking you to show this.

From (a) we can see that a property of $f : \mathbf{R}^2 \to \mathbf{R}^3$ is that, when its outputs are taken as inputs for $g : \mathbf{R}^2 \to \mathbf{R}$, the output will equal 1, regardless of the values of θ and ϕ. The usage of θ and ϕ seems to be suggestive of using polar coordinates. If we consider polar coordinates in the form (r, θ, ϕ), we can see that

$$\text{cart} \leftrightarrow \text{sph} \begin{cases} x = r\cos\theta\sin\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\phi \end{cases}$$

f is supposedly lies on the unit sphere, so $r = 1$. So, f is a function that takes in angles and outputs the squares of the Cartesian coordinates from its polar form. So, it will always lie on the unit circle. Our need for (a) is to demonstrate that $r = 1$. We already know that f is always on a circle of some sort, as it is in the form of the cartesian coordinates, from polar coordinates, that take two angles as inputs. We can see that g involves several terms squared, strangely reminiscent of the distance formula for 3 dimensions, $d = \sqrt{x^2 + y^2 + z^2}$. So, we can say that g is the distance squared, which resolves the square root. We can, therefore, infer that $r = 1$, meaning it is a unit circle, with two reasons. (1) The distance squared is always 1, meaning the distance, r, always equals 1, as the square root of the distance squared, the distance, is always 1. (2) If r can vary, then $g \circ f$ will not remain constant through all θ and ϕ, as it is impacted by the square for different values of f, changing the value. Thus, every point of f lies on the unit sphere because $r = 1$ is always true.

7 Exercise 8.10

8.9 involved graphing, so I did not do it. Briefly justify whether each of the following statements is either true (i.e., always true) or false (i.e., sometimes not true):

(a) Let $f(x, y) = x^2 - y$ and let $g(x, y) = x^2 - y + 51$. If S_7 is the 7-level set of $f(x, y) = x^2 - y$ then it is also the c-level set of $g(x, y)$ for some c.

False. Although we can find some c for which equals the 7-level set of f, it is not true for all c, meaning is not always true. For example, we can see that $S_7 = \{(x, y) \in \mathbf{R}^2 : x^2 - y = 7\}$. Given a c-level set, S_c, the following identity is trivially untrue, as c can be any value, and there is at least one value, such as 1, that would make such an identity untrue: $S_c = S_7 = \{x(x, y) \in \mathbf{R}^2 : x^2 - y + 51 = c\} = \{x(x, y) \in \mathbf{R}^2 : x^2 - y = -50\} = \{x(x, y) \in \mathbf{R}^2 : x^2 - y = 7\}$.

(b) Let $f(x, y) = x^2 - y$ and let $g(x, y) = x^2 - y + 51$. If S_1 is the 1-level set of $f(x, y) = x^2 - y$ then it is also the 1 level set of $h(x, y)$.

False. This question is claiming that $S_1 = \{(x, y) \in \mathbf{R}^2 : x^2 - y = 1\} = \{(x, y) \in \mathbf{R}^2 : x^2 - y = -50\}$, which is untrue because $-50 \neq 1$.

(c) The graph of $f(x, y) = x^2 - y$ is the 2-level set of some function $F(x, y)$.

False. The level set means the function is set at some specific constant. If we set $F : \mathbf{R}^2 \to (R)$ and evaluate it at a certain constant, we will yield a graph in $\mathbf{R}^2$. However, the graph of f is clearly in $\mathbf{R}^3$, because it has two independent variables and one dependent variable, which are x, y, f respectively. So, this statement is not true because of the differences in dimension.