Math 51: Section 8 Exercises

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1 Exercise 8.1

This section was considered by the author to be too trivial to attempt. Move on to the next section.

2 Exercise 8.2

Consider the set $S = \{(x, y, z) \in \mathbf{R}^3 : x^3 + z^3 + 3y^2z^3 + 5xy = 0\}$.

(a) Give functions $f, h : \mathbf{R}^3 \to \mathbf{R}$ for which S is a level set of both $f(x, y, z)$ and $h(x, y, z)$.

The level set is when the function is set to a certain constant. To find f and h, we can simply evaluate them at different constants to satisfy the same set S. Thus, let

$$f(x, y, z) = x^3 + z^3 + 3y^2z^3 + 5xy$$

. Then, clearly $S = \{(x, y, z) \in \mathbf{R}^3 : f(x, y, z) = 0\}$. Similarly, we can evaluate and add different constants to satisfy the same set. For example, let

$$h(x, y, z) = x^3 + z^3 + 3y^2z^3 + 5xy + 1$$

. We see that because the constants cancel out, $S = \{(x, y, z) \in \mathbf{R}^3 : h(x, y, z) = 1\}$.

(b) By solving for z in terms of x and y, give a function $g : \mathbf{R}^2 \to \mathbf{R}$ for which S is the graph of g.

We can first solve for z. Doing this we get

$$z = \sqrt[3]{\frac{x^3 + 5xy}{1 + 3y^2}}$$

. We are asked to find g where S is the graph of z. Therefore, we can say that $z = g(x, y)$, as they are interchangable. Therefore, let

$$g(x, y) = \sqrt[3]{\frac{x^3 + 5xy}{1 + 3y^2}}$$

. It is clear to see that Graph$(g) = S = \{(x, y, z) \in \mathbf{R}^3 : x^3 + z^3 + 3y^2z^3 + 5xy = 0\}$.

3 Exercise 8.3

Consider the function $g : \mathbf{R} \to \mathbf{R}^2$ defined by $g(t) = \left(\frac{e^t+e^{-t}}{2}, \frac{e^t-e^{-t}}{2}\right)$

(a) Show that every point in the output of g lies on the hyperbola $x^2 - y^2 = 1$

We see that for the function g, the x-coordinate is $\left(\frac{e^t+e^{-t}}{2}\right)$ and the y-coordinate is $\left(\frac{e^t-e^{-t}}{2}\right)$. We can square both of these terms and plug it into the hyperbola and see if it equals 1 for all t. We see that $x^2 = \frac{e^{2t}+2+e^{-2t}}{4}$ and $y^2 = \frac{e^{2t}-2+e^{-2t}}{4}$. Therefore,

$$x^2 - y^2 = \frac{e^{2t}+2+e^{-2t}}{4} - \frac{e^{2t}-2+e^{-2t}}{4} = \frac{4}{4} = 1$$

. QED

(b) Are all the points in the hyperbola $\{(x, y) \in \mathbf{R}^2 : x^2 - y^2 = 1\}$ in the output of g? If "yes" then explain why, and if "no" explain why a specific point on the hyperbola is not in the output.

No, not all the points of the hyperbola are in the output of g. Consider the point (-1,0) which is on the hyperbola $x^2 - y^2 = 1$. This point cannot be attained in the output of $g : \mathbf{R} \to \mathbf{R^2}$, as for the x-coordinate to be negative, one such point $\{t \in \mathbf{R} : e^t + e^{-t} < 0\}$ must exist. However, exponents in $\mathbf{R}$ cannot be negative. So, such a t must not exist. Thus, the point (-1,0) on the hyperbola $x^2 - y^2 = 1$ is not on the output of g, meaning not all points of the hyperbola are in the output of g.

4 Exercise 8.4

Let $f : \mathbf{R^3} \to \mathbf{R}^3$ be the function $\mathbf{f}(x, y, z) = (x - y + 2z, 3x - 3y + 5z, 3x - 3y + 5z, 3x - 2y + 2z)$ and let $\mathbf{g} : \mathbf{R^3} \to \mathbf{R}^3$ be the function $\mathbf{f}(u, v, w) = (4u - 2v + w, 9u - 4v + w, 3u - v)$.

(a) Show that $\mathbf{g}(\mathbf{f}(x, y, z)) = (x, y, z)$.

We can write these vectors top-down for simplicity.

$$g \circ f = \begin{bmatrix} 4u - 2v + w \\ 9u - 4v + w \\ 3u - v \end{bmatrix} \circ \begin{bmatrix} x - y + 2z \\ 3x - 3y + 5z \\ 3x - 2y + 2z \end{bmatrix}$$

which equals

$$\begin{bmatrix} 4(x - y + 2z) - 2(3x - 3y + 5z) + (3x - 2y + 2z) \\ 9(x - y + 2z) - 4(3x - 3y + 5z) + (3x - 2y + 2z) \\ 3(x - y + 2z) - (3x - 3y + 5z) \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(b) Show that $\mathbf{f}(\mathbf{g}(u, v, w)) = (u, v, w)$. Similarly,

$$f \circ g = \begin{bmatrix} x - y + 2z \\ 3x - 3y + 5z \\ 3x - 2y + 2z \end{bmatrix} \circ \begin{bmatrix} 4u - 2v + w \\ 9u - 4v + w \\ 3u - v \end{bmatrix}$$

equals

$$\begin{bmatrix} (4u-2v+w)-(9u-4v+w)+2(3u-v) \\ 3(4u-2v+w)-3(9u-4v+w)+5(3u-v) \\ 3(4u-2v+w)-2(9u-4v+w)+2(3u-v) \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

5 Exercise 8.5

Let S be a level set $\{(x,y,z) \in \mathbf{R}^3 : f(x,y,z) = c\}$ in $\mathbf{R}^3$.

(a) If S is also the graph $\{(x,y,z) \in \mathbf{R^3} : (x,y) \in D, z = g(x,y)\}$ of a function $g : D \to \mathbf{R}$ on some region D in R^2, explain why S meets each vertical line $\{(a,b,t) : t \in \mathbf{R}\}$ (for $(a,b) \in \mathbf{R}^2$) in at most one point.

(b) For the sphere $S = \{(x,y,z) \in \mathbf{R}^3 : x^2+y^2+z^2 = 4\}$ of radius 2 centered at the origin, explain both algebraically and geometrically why S violates the "vertical line test" in (a), so S is not the graph of a function.