SVC Final Project Derivations

Han Lu

April 25, 2025

consider the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$. to derive the Laplace transforms for derivatives, let us begin by attempting to derive the Laplace transform of $y''(t) = f(t)$. Recall that the formula for the Laplace transform is:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

for the values of s for which the integral converges. Thus, we can simply plug in our function into the formula, getting:

$$\begin{aligned}\mathcal{L}\{y''(t)\} &= \int_0^\infty e^{-st} y''(t) dt \\ &= e^{-st} y'(t)\Big|_0^\infty + s\int_0^\infty y'(t) e^{-st} dt \\ &= -y'(0) + sY'(s)\end{aligned} \tag{1}$$

We see that the derivative here involves $Y'(s)$, which is the Laplace transform of y'. So, let us calculate that.

$$\begin{aligned}\mathcal{L}\{y'(t)\} &= \int_0^\infty e^{-st} y'(t) dt \\ &= e^{-st} y(t)\Big|_0^\infty + s\int_0^\infty y(t) e^{-st} dt \\ &= -y(0) + sY(s)\end{aligned} \tag{2}$$

Plugging this back into what we have above, we get:

$$\begin{aligned}\mathcal{L}\{y''(t)\} &= -y'(0) + s(-y(0) + sY(s)) \\ &= s^2 Y(s) - sy(0) - y'(0)\end{aligned} \tag{3}$$

Assuming we have initial values and value this equation equals to, we can simply solve for $Y(s)$ and apply the inverse Laplace transform to find y.

Statement of a convolution. The convolution $f * g$ of two functions f and g is defined by:

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau \tag{4}$$

f is what we consider the kernel, and g the input function. Notice that the function is in terms of t, with τ being an arbitary variable of integration. The Convolution Theorem states that if $\mathcal{L}(f) = F$ and $\mathcal{L}(g) = G$, then

$$\mathcal{L}(f * g) = FG \tag{5}$$

The proof is omitted as it is beyond the scope of this presentation.

Statement of the Unit Step Function / Heaviside function

The unit step function is defined as the following piecewise function:

$$u(t-\tau) = \begin{cases} 0, t < \tau \\ 1, t \geq \tau \end{cases} \tag{6}$$

This is useful because piecewise functions can be rewritten in terms of the unit step function, then have the Laplace Transform taken of it. We have a theorem of that, which states that if g is defined on $[0,\infty)$, and $\mathcal{L}(g)$ exists greater than some s, then the Laplace Transform for the unit function times g exists for values greater than the same s, and is defined by:

$$\mathcal{L}\{u(t-\tau)g(t)\} = e^{-s\tau}\,\mathcal{L}(g(t+\tau)) \tag{7}$$

Statement of the Dirac-δ function.

The Dirac-δ function is defined using these three definitions:

$$\begin{gathered} \delta(x) = \begin{cases} +\infty, x = 0 \\ 0, x \neq 0 \end{cases} \\ \int_{-\infty}^{\infty} \delta(x)dx = 1 \\ \int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \end{gathered} \tag{8}$$

An interesting fact is that the dirac-δ function is the derivative of the Heaviside function, which makes sense intuitively.

Written Examples.

First example. We are asked to derive the solution of the following initial value problem:

$$f(t) = y'' + 2y' + 2y, y(0) = k_0, y'(0) = k_1$$

Applying the Laplace transforms to both sides and letting $\mathcal{L}(y) = Y$ and $\mathcal{L}(f) = F$ for simplicity, we get:

$$\mathcal{L}(y'') + 2\,\mathcal{L}(y') + 2\,\mathcal{L}(y) = \mathcal{L}\{f(t)\} \tag{9}$$

$$(s^2Y(s) - sy(0) - y'(0)) + 2(-y(0) + sY(s)) + 2Y(s) = F(s) \tag{10}$$

$$(s^2 + 2s + 2)Y(s) = F(s) + k_1 + k_0 s + 2k_0 \tag{11}$$

Isolating $Y(s)$,

$$\begin{aligned} Y(s) &= \frac{F(s)}{(s+1)^2+1} + \frac{k_1 + k_0 s + 2k_0}{(s+1)^2+1} \\ &= \frac{F(s)}{(s+1)^2+1} + \frac{k_0 s + (k_1 + 2k_0)}{(s+1)^2+1} \\ &= \frac{F(s)}{(s+1)^2+1} + \left(k_0 \cdot \frac{s+1-1}{(s+1)^2+1} + (k_1 + 2k_0) \cdot \frac{1}{(s+1)^2+1}\right) \\ &= \frac{F(s)}{(s+1)^2+1} + \left(k_0 \cdot \frac{s+1}{(s+1)^2+1} + (k_1 + k_0) \cdot \frac{1}{(s+1)^2+1}\right) \end{aligned} \tag{12}$$

Recall from the chart that:

$$\begin{aligned} \mathcal{L}\{e^{\lambda t} \cos \omega t\} &= \frac{s-\lambda}{(s-\lambda)^2 + \omega^2} \\ \mathcal{L}\{e^{\lambda t} \sin \omega t\} &= \frac{\omega}{(s-\lambda)^2 + \omega^2} \end{aligned} \tag{13}$$

We see that $\lambda = -1$ and $\omega = 1$ for both equations, leading us to the following equatoin:

$$\mathcal{L}^{-1}\left\{\frac{k_1 + k_0 s + 2k_0}{(s+1)^2+1}\right\} = k_0 e^{-t}[k_0 \cos(t) + (k_1 + k_0)\sin(t)] \tag{14}$$

Applying the rightmost part of (9), we can simplify the first term too, recognizing that

$$\tfrac{1}{(s+1)^2+1} \leftrightarrow e^{-t} \sin t \text{ and } F(s) \leftrightarrow f(t)$$

Applying the convolution theorem,

$$\mathcal{L}\{f * g\} = FG \implies \mathcal{L}^{-1}\{FG\} = f * g$$

If we let $F = F(s)$, $G = \frac{1}{(s+1)^2+1}$, and f and g to be their transform pairs respectively, we can apply the convolution theorem to find the inverse Laplace transform. Doing this we get:

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{(s+1)^2+1}\right\} = f * e^{-t} \sin t \tag{15}$$

We can apply the flipped version of the convolution for convenience, as it can be shown that $f * g = g * f$. Applying the definition of a convolution, we see that:

$$(f * g)(t) = \int_0^t f(t-\tau)e^{-\tau} \sin \tau d\tau \tag{16}$$

Recall that we are trying to find the inverse Laplace transform of $Y(s)$, which is a transform pair with $y(t)$. So, it can be said that:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = k_0 e^{-t}[k_0 \cos(t) + (k_1 + k_0)\sin(t)] + \int_0^t f(t-\tau)e^{-\tau} \sin \tau d\tau \tag{17}$$

which is the derived solution for the initial value problem. To find a specific solution, we just need to be given k_0, k_1, and $f(t)$. For our case, let's assume they all equal 1. We can evaluate for $y(t)$.

$$\begin{aligned}
y(t) &= e^{-t}(\cos(t) + 2\sin(t)) + \int_0^t e^{-\tau}\sin\tau d\tau \\
&= e^{-t}(\cos(t) + 2\sin(t)) + \frac{1 - e^{-t}(\cos(t) + \sin(t))}{2} \\
&= \frac{1}{2} + e^{-t}\left(\frac{1}{2}\cos(t) + \frac{3}{2}\sin(t)\right)
\end{aligned} \tag{18}$$

Second Example. (Impulse Function) We are asked to find the solution $y(t)$ for the differential equation

$$y'' + y = 1 + 2\delta(t - \pi) - 3\delta(t - 2\pi), y(0) = -1, y'(0) = 2 \tag{19}$$

Recall the solution of a δ function is defined by

$$y(t) = \hat{y} + \alpha u(t - t_0)w(t - t_0) \tag{20}$$

for the differential equation

$$ay'' + by' + cy = f(t) + \alpha\delta(t - t_0), y(0) = k_0, y'(0) = k_1 \tag{21}$$

where $\hat{y}$ is the solution of

$$ay'' + by' + cy = f(t) \tag{22}$$

and w is defined as:

$$w = \mathcal{L}^{-1}\left\{\frac{1}{as^2 + bs + c}\right\} \tag{23}$$

Let us first solve $\hat{y}$. Taking the Laplace Transform of both sides and sovling for $Y(s)$, we get:

$$\begin{aligned}
\frac{1}{s}(s^2Y(s) - sy(0) - y'(0)) + Y(s) &= (s^2Y(s) - sy(0) - y'(0)) + Y(s) \\
Y(s) &= \frac{\frac{1}{s} - s + 2}{s^2 + 1} \\
&= \frac{1}{s(s^2 + 1)} - \frac{s}{s^2 + 1} + \frac{2}{s^2 + 1}
\end{aligned} \tag{24}$$

For the first section, $\frac{1}{s(s^2+1)}$, we have to use convolutions again. We can factor out $\frac{1}{s}$ from the equation. From the table of Laplace transforms, we know that

$$\tfrac{1}{s} \leftrightarrow 1 \text{ and } \tfrac{1}{s^2+1} \leftrightarrow \sin(t)$$

If we let $F = \frac{1}{s^2+1}$, $G = \frac{1}{s}$, and f and g to be their transform pairs respectively, we can apply the convolution theorem, which implies:

$$\mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s^2+1}\right\} = \sin(t) * 1 = \int_0^t \sin\tau d\tau = 1 - \cos t \tag{25}$$

From the table of Laplace transforms, going back to our differential equation, we know that:

$$\tfrac{s}{s^2+1} \leftrightarrow \cos t \text{ and } \tfrac{1}{s^2+1} \leftrightarrow \sin(t)$$

Plugging in these values and the value of the convolution, we get that:

$$\hat{y} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)} - \frac{s}{s^2+1} + \frac{2}{s^2+1}\right\} = 1 - 2\cos t + 2\sin t \tag{26}$$

To solve for w, we need a, b, and c, which are 1, 0, and 1 respectively. Therefore, we get that:

$$w = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t \tag{27}$$

When we recognize that $t_0 = \pi$ and $t_0 = 2\pi$, we can simply solve this equation for y by plugging into (20). Doing this we get:

$$y = 1 - 2\cos t + 2\sin t - 2u(t-\pi)\sin t - 3u(t-2\pi)\sin(t) \tag{28}$$

Additionally, we can rewrite this in piecewise form, which would be:

$$y = \begin{cases} 1 - 2\cos t + 2\sin t, & 0 \le t < \pi \\ 1 - 2\cos t, & \pi \le t < 2\pi \\ 1 - 2\cos t - 3\sin t, & t \ge 2\pi \end{cases} \tag{29}$$