RESEARCH REPORT: DYNAMICS ON GRAPHS AND TREES

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1. Abstract

We take a look at the dynamics of trees and graphs. We started out with a basic tree and wrote code that would allow us to find the itinerary of any point on that tree. Using this we found interesting features relating to its itinerary and period. We then expanded to take a look at more general trees and developed theorems about the possible itineraries and its features such as the periodicity.

2. Introduction

Consider a continuous surjection $f : T \to T$, where T is any real interval. We can express T as the union of finitely many disjoint subintervals called *addresses*[1], for example, $T = A_0 \cup A_1 \cup A_2$, where $A_0, A_1, A_2 \subset T$ and $A_0 \cap A_1 \cap A_2 = \emptyset$. We denote the set of addresses by $W = \{A_0, A_1, A_2\}$.

Definition 2.1. The *itinerary* of a point $x \in T$ under f, denoted by $It_f(x)$, is a sequence of addresses $(A_n)_{n\in\mathbb{N}}$ such that $f^n(x) \in A_n \in W$.

Definition 2.2. The *period* of an itinerary $It_f(x)$ (if exists) is a subsequence of $It_f(x)$ that repeats infinitely.

Definition 2.3. We say $It_f(x)$ is *eventually periodic* if and only if there exists distinct $n, m \in \mathbb{N}$ such that $f^n(x) = f^m(x)$. Equivalently, $It_f(x)$ is *eventually periodic* if and only if there exists a subsequence of the itinerary that repeats infinitely.

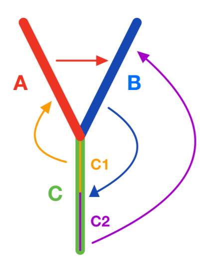
Now, we consider a concrete example. Let $T = [0, 30]$. Define $f : T \to T$ by

$$f(x) = \begin{cases} x + 10 & \text{if } x \in [0, 20) \\ 2(x - 20) & \text{if } x \in [20, 30] \end{cases}$$

We can express T as the union of disjoint subintervals by

$$T = A \cup B \cup C_1 \cup C_2$$

where $A = [0, 10), B = [10, 20), C_1 = [20, 25),$ and $C_2 = [25, 30]$.



[1]Used the definition from *On Iterated Maps of the Interval* by Milnor and Thurston

We ask *when is $It_f(x)$ periodic given the function and intervals defined above?*

We claim that $It_f(x)$ is eventually periodic if and only if x is rational.

Proof. Note that

$$\forall x \in A,\ f(x) \in B \qquad \forall x \in B,\ f(x) \in C_1 \cup C_2$$
$$\forall x \in C_1,\ f(x) \in A \qquad \forall x \in C_2,\ f(x) \in B$$

Therefore, we know that for any $n \in \mathbb{N}$

$$f^n(x) \in C_1 \Rightarrow\ f^{n+1}(x) \in A,\ f^{n+2}(x) \in B$$
$$f^n(x) \in C_2 \Rightarrow\ f^{n+1}(x) \in B$$

Define $\kappa : \mathbb{Z}_{>0} \to \mathbb{N}$ by

$$z \mapsto \text{the index of } z^{th} \text{ occurrence of } C_1 \text{ or } C_2$$

For an example sequence $(A, B, C_1, A, B, C_2, B, C_2)$, assuming 0-indexed, we have

$$\kappa(1) = 2 \quad \kappa(2) = 5 \quad \kappa(3) = 7$$

Since we know that the address(es) after $\kappa(n)$ and before $\kappa(n+1)$ is fixed, we only need to consider the pattern of the sequence containing just C_1 and C_2.
Let $(A_{\kappa(z)})_{z\in\mathbb{Z}_{>0}}$ be an ordered subset of $It_f(x)$ that contains only C_1 and C_2. We say $It_f(x)$ is eventually periodic if and only if $(A_{\kappa(z)})_{z\in\mathbb{Z}_{>0}}$ is eventually periodic. Note that

$$\forall x \in C_1,\ f^3(x) = 2(x-20) + 10 + 10 = 2x - 20 \in C_1 \cup C_2$$
$$\forall x \in C_2,\ f^2(x) = 2(x-20) + 10 = 2x - 30 \in C_1 \cup C_2$$

Fix $O = 20$. Let $d(x) = |x - O|$ be the distance between x and O. We have that

$$\forall x \in C_1,\ d(f^3(x)) = 2d(x)$$
$$\forall x \in C_2,\ d(f^2(x)) = 2(d(x) - 5)$$

Therefore, we can define a function $g : [0,1] \to [0,1]$ in terms of the distance from O to describe the pattern of $(A_{\kappa(z)})_{z\in\mathbb{Z}_{>0}}$ by

$$g(x) = \begin{cases} 2x & \text{if } x \in [0, 0.5) \\ 2(x-0.5) & \text{if } x \in [0.5, 1] \end{cases}$$

Therefore, we have that for some $z \in \mathbb{Z}_{>0},\ x \in T$

$$A_{\kappa(z)} = C_1 \Leftrightarrow g^{\kappa(z)}(\frac{d(x)}{10}) < 0.5$$
$$A_{\kappa(z)} = C_2 \Leftrightarrow g^{\kappa(z)}(\frac{d(x)}{10}) \geq 0.5$$

Further, we observe that in binary format

$$\forall y \in [0_2, 0.1_2),\ g(y) \Leftrightarrow \text{remove the first fractional digit}$$
$$\forall y \in [0.1_2, 1_2],\ g(y) \Leftrightarrow \text{set the first fractional digit to 0 and remove it}$$
$$\Leftrightarrow \text{remove the first fractional digit}$$

Therefore, we have that for any $y \in [0,1]$ and some $z \in \mathbb{Z}_{>0}$, $g^{\kappa(z)}(y) < 0.5$ if and only if z^{th} fractional digit of y's binary expansion is 0; similarly, $g^{\kappa(z)}(y) \geq 0.5$ if and only if z^{th} fractional digit of y's binary expansion is 1. Hence, we have that $(A_{\kappa(z)})_{z\in\mathbb{Z}_{>0}}$ is eventually periodic if and only if y is rational. Then, it follows that for $x = d^{-1}(10y) \in [20, 30]$, $It_f(x)$ is eventually periodic if and only if x is rational. $\square$

Now, let's consider the length of the period under this map. Suppose x is a rational number with a finite number of fractional digits. We ask *how does the number of fractional digits of x affect the period of $It_f(x)$?* We claim that an additional fractional digit of x leads to an approximately 5 times longer period except for the following two cases:

- The additional fractional digit is 0 or 5.
- With the additional fractional digit, x has a finite binary representation.

Proof. Let x_k be a positive real number with k fractional digits (negative is similar). Then, we can represent x_k in terms of the whole part and fractional part as follows:

$$\begin{aligned} x_k &= \lfloor x_k \rfloor + \frac{m}{10^k} \\ &= \lfloor x_k \rfloor + \frac{1}{2^k} \cdot \frac{m}{5^k} \end{aligned}$$

where $\lfloor \cdot \rfloor$ is the floor function and m is an integer of k digits. Nontrivially, we will assume that m is always coprime to 5^k and $m < 5^k$. Let l_k be the length of the period of x_k in its binary representation. We claim that $l_{k+1} = 5l_k$.

Prior to proving the statement, we need to show a few lemmas. In the following proof, we use $(\cdot, \cdot)$ to denote the *greatest common divisor* of two positive integers.

Lemma 2.4. *If $(a, b) = 1$ and $(b, c) = 1$, then $(ab, c) = 1$.*

Lemma 2.5. *If $(a, b) = 1$ and $a > b$, then $(a - b, b) = 1$.*

The above two lemmas follow immediately from the Bézout's identity, and hence, their proofs will be omitted here.

Lemma 2.6. *l_k is the multiplicative order of 2 modulo 5^k.*

Proof. Consider the fractional part of x_k, denoted $\mathrm{frac}(x_k)$. The algorithm to convert a decimal fraction to a binary fraction is as follows:

(1) Multiply $\mathrm{frac}(x_k)$ by 2, denoted y.
(2) If $y > 1$, then append 1 in the result and go back to (1) with $y - 1$.
(3) If $y < 1$, then append 0 in the result and go back to (1) with y.
(4) If $y = 1$, then terminate.

The only case when this algorithm can terminate for $\mathrm{frac}(x_k)$ is when $m = 5^k$, but this does not happen by assumption.

We know that $(m, 5^k) = 1$ and $(2, 5^k) = 1$ for all $k \geq 1$. By Lemma 1, we have $(2m, 5^k) = 1$. More generally, we have $(2^n m, 5^k) = 1$ for all $n \geq 1$ by repeatedly using Lemma 1. By Lemma 2, we know that whenever $2^n m > 5^k$, we have $(2^n m - 5^k, 5^k) = 1$. Now, observe that if we were to apply the conversion algorithm to $\frac{m}{5^k}$, then (2) happens when $2^n m > 5^k$ and the resulting $y = \frac{2^n m - 5^k}{5^k}$; and (3) happens when $2^n m < 5^k$ and the resulting $y = \frac{2^n m}{5^k}$.

Consider the multiplicative group $(\mathbb{Z}/_{5^k}\mathbb{Z})^\times$, or $\mathbb{Z}_{5^k}^\times$. Let g be an arbitrary integer. We know that $g \in \mathbb{Z}_{5^k}^\times$ if and only if $(g, 5^k) = 1$. Therefore, we have $2^n m \in \mathbb{Z}_{5^k}^\times$ or $2^n m - 5^k \in \mathbb{Z}_{5^k}^\times$. Since $\mathbb{Z}_{5^k}^\times$ is a finite group, whose order is given by the number of integers coprime to 5^k, which is less than 5^k. By Lagrange's theorem, we have that

the order of every element of a finite group G divides the order of G. Therefore, we have $|2| = w$ for some positive integer $w < 5^k$.
Let $\overline{m}$ be the congruence class of m. If after some iterations of the algorithm, the numerator again lies in $\overline{m}$, then we know that the integers we pick from subsequent (2) and (3) will repeat, which essentially gives the period. Therefore, we conclude that the length of the period is the order of 2 in $\mathbb{Z}_{5^k}^{\times}$. □

Lemma 2.7. *Let p be a prime and $k \geq 1$. Let φ be Euler's totient function.*

$$\varphi(p^k) = p^k - p^{k-1}$$

Proof. Since p is a prime, we have $(a, p) = 1$ for all $0 < a < p$. By Lemma 1, we have if $(a, p^n) = 1$ for $n \geq 1$, then $(a, p^m) = 1$ for $m > n$. Conversely, if $(a, p^m) > 1$ for $m > 1$, then $(a, p^n) > 1$ for some $0 < n < m$. Therefore, we can deduce that such a can only be multiples of p. Since there are p^{k-1} distinct multiples of p that are less than p^k, we have the number of integers coprime to p^k be $p^k - p^{k-1}$. □

Lemma 2.8. *Let p be an odd prime, and let g be a primitive root modulo p. Then, g is a primitive root modulo every power of p unless $g^{p-1} \equiv 1$ (mod p^2)*[2].

Now, we have all the lemmas required to prove the claim that $l_{k+1} = 5l_k$. It is easy to check that 2 is a primitive modulo 5. By Lemma 5, we have that 2 is also a primitive modulo 5^k because $2^4 = 16 \not\equiv 1 \pmod{5^2}$. Therefore, by definition (of *primitive root*), we have that $|2| = \varphi(5^k) = 4 \cdot 5^{k-1}$. By Lemma 4, we know that $l_k = |2| = 4 \cdot 5^{k-1}$. Therefore, we have $l_{k+1} = 5l_k$.
Now, observe from the previous proof of the periodicity of $It_f(x)$ that every periodic itinerary can be encoded into a binary fraction, where 0 is C_1 and 1 is C_2, and the periodicity results directly from the binary expansion of x. Similarly, we can say that if the binary expansion of x_k has a period of l_k, then the repeating pattern consisting of C_1 and C_2 has length l_k. Since each occurrence of C_1 corresponds to one occurrence of A and B, and each occurrence of C_2 corresponds to one occurrence of B, we can compute the length of the period of $It_f(x_k)$ by $3k_1 + 2k_2$ where k_1, k_2 are the number of occurrences of 0's (or C_1's) and 1's (or C_2's) in the period of the binary expansion of x_k and $k_1 + k_2 = l_k = 4 \cdot 5^{k-1}$.
Suppose that 0's and 1's are evenly distributed in the period of the binary expansion. Let L_k be the length of the period of $It_f(x_k)$. We have

$$\begin{aligned} L_k &= 3k_1 + k_2 \\ &= 3 \cdot 2 \cdot 5^{k-1} + 2 \cdot 2 \cdot 5^{k-1} \\ &= 2 \cdot 5^k \end{aligned}$$

Thus, we have $L_{k+1} = 5L_k$. However, since the distribution of 0's and 1's in the binary expansion are yet to be determined, we shall only conclude that in general, $L_{k+1} \geq 5L_k$, or $L_{k+1} \approx 5L_k$.

□

3. Conjecture on a general case

For $\lambda \in (2, 4)$, define a function $f_\lambda : [-10, 20] \to [-10, 20]$ such that

[2]Excerpted from *A Course in Computational Algebraic Number Theory* by Cohen, p.26

$$f_\lambda(x) = \begin{cases} -x & \text{if } x \in [-10, 0] \\ x + 10 & \text{if } x \in [0, 10] \\ -2(x-10) & \text{if } x \in (10, 15] \\ \lambda(x-15) - 10 & \text{if } x \in (15, 20] \end{cases}$$

Let $A_0 = [-10, 0]$, $A_1 = [0, 10]$, $A_2 = [10, 15]$, $A_3 = [15, 20]$

Definition 3.1. The **iteration** of $x \in A_0$ under f_λ is the sequence $\{a_n^x\}_{n\in\omega} = w$ for $f^n(x) \in A_w$ where $w \in \{0, 1, 2, 3\}$, denoted as $\{It_n(x_\lambda)\}_{n\in\omega}$

Definition 3.2. For a f_λ, the **difference characteristic** of two iterations $\{It_n(x_\lambda)\}_{n\in\omega}$ and $\{It_n(y_\lambda)\}_{n\in\omega}$ is

$$K_y^x = min\{k \in \omega : It_k(x_\lambda) \neq It_k(y_\lambda)\}$$

Theorem 3.3. *K_y^x exists for $x \neq y$*

Proof. For $x \neq y$, assume $\{It_n(x_\lambda)\}_{n\in\omega} = \{It_n(y_\lambda)\}_{n\in\omega}$, then

$$|f_\lambda^3(x) - f_\lambda^3(y)| \neq 0$$

Note that if $f_\lambda^n(x) \in A_2$, then $f_\lambda^{n+3}(x) \in (10, 20)$ and

$$\begin{aligned} f_\lambda^{n+3}(x) &= -(-2(f_\lambda^n(x) - 10)) + 10 \\ &= 2f_\lambda^n(x) - 10 \\ |f_\lambda^{n+3}(x) - f_\lambda^{n+3}(y)| &= |2f_\lambda^n(x) - 10 - 2f_\lambda^n(y) + 10| \\ &= 2|f_\lambda^n(x) - f_\lambda^n(y)| \end{aligned}$$

If $f_\lambda^n(x) \in A_3$ and if $f_\lambda^{n+1}(x) \in A_0$, then $f_\lambda^{n+3}(x) \in (10, 20)$ and

$$\begin{aligned} f_\lambda^{n+3}(x) &= -(\lambda(f_\lambda^n(x) - 15) - 10) + 10 \\ &= -\lambda f_\lambda^n(x) + 15\lambda + 20 \\ |f_\lambda^{n+3}(x) - f_\lambda^{n+3}(y)| &= |-\lambda f_\lambda^n(x) + 15\lambda + 20 + \lambda f_\lambda^n(y) - 15\lambda - 20| \\ &= \lambda|f_\lambda^n(x) - f_\lambda^n(y)| \\ &\leq 2|f_\lambda^n(x) - f_\lambda^n(y)| \end{aligned}$$

If $f_\lambda^n(x) \in A_3$ and if $f_\lambda^{n+1}(x) \in A_1$, then $f_\lambda^{n+2}(x) \in (10, 20)$ and

$$\begin{aligned} f_\lambda n + 2(x) &= \lambda(f_\lambda^n(x) - 15) - 10 + 10 \\ &= \lambda(f_\lambda^n(x) - 15) - 10 \\ |f_\lambda^{n+2}(x) - f_\lambda^{n+2}(y)| &= |\lambda(f_\lambda^n(x) - 15) - 10 - \lambda(f_\lambda^n(y) - 15) + 10| \\ &= \lambda|f_\lambda^n(x) - f_\lambda^n(y)| \\ &\geq 2|f_\lambda^n(x) - f_\lambda^n(y)| \end{aligned}$$

Therefore,

$$\{It_n(x_\lambda)\}_{n\in\omega} = \{It_n(y_\lambda)\}_{n\in\omega} \Rightarrow 2^c|fx_\lambda^3(x) - fx_\lambda^3(y)| < \frac{1}{2} \text{ , } \forall c \in \omega$$

which is impossible.
Therefore,

$$x \neq y \Leftrightarrow \{It_n(x_\lambda)\}_{n\in\omega} \neq \{It_n(y_\lambda)\}_{n\in\omega} \Leftrightarrow \{k \in \omega : It_k(x_\lambda)\} \neq \emptyset$$

Hence, K^x_y always exists because $(\mathbb{N}, <)$ is a well-order. $\square$

Definition 3.4. For a f_λ, the **ordering characteristic** of two iterations $\{It_n(x_\lambda)\}_{n\in\omega}$ and $\{It_n(y_\lambda)\}_{n\in\omega}$ is the times of "3-0" appears before their iterations differ, i. e.

$$\varsigma^x_y = |\{n \in \omega : (n < K^x_y) \wedge (It_n(x_\lambda) = 3) \wedge (It_{n+1}(x_\lambda) = 0)\}|$$

Theorem 3.5. *$x < y$ iff $(-1)^{\varsigma^x_y}(It_{K^x_y}(x_\lambda) - It_{K^x_y}(y_\lambda)) < 0$*

Proof. Define $\|x\| = min\{|x|, x - 10\}$
For f_λ, and for $x < y$ consider $It_{K^x_y}(x_\lambda)$, $It_{K^x_y}(y_\lambda)$.
Note that if ς^x_y is even, then

$$\begin{aligned}
&\|f^m_\lambda(x)\| < \|f^m_\lambda(y)\|, \forall m < K^x_y \\
&\Rightarrow (It_{K^x_y}(x_\lambda) = 2 \wedge It_{K^x_y}(y_\lambda) = 3) \vee (It_{K^x_y}(x_\lambda) = 0 \wedge It_{K^x_y}(y_\lambda) = 1) \\
&\Rightarrow (x < y \Leftrightarrow It_{K^x_y}(x_\lambda) - It_{K^x_y}(y_\lambda) < 0)
\end{aligned}$$

Therefore, the only case that $\|f^m(x)\| > \|f^m(y)\|$ is that

$$f^{m-1}_\lambda(x),\ f^{m-1}_\lambda(y) \in A_3 \text{ and } f^m_\lambda(x),\ f^m_\lambda(y) \in A_0$$

So

$$x < y \Leftrightarrow (-1)^{\varsigma^x_y}(It_{K^x_y}(x_\lambda) - It_{K^x_y}(y_\lambda)) < 0$$

$\square$

Definition 3.6. For a set $T := \{\{a_n\}_{n\in\omega} : (a_n = 0 \Rightarrow a_{n+1} = 1) \wedge (a_n = 1 \Rightarrow (a_{n+1} = 2 \vee a_{n+1} = 3) \wedge (a_n = 2 \Rightarrow a_{n+1} = 0) \wedge (a_n = 3 \Rightarrow (a_{n+1} = 0 \vee a_{n+1} = 1)\}$
The **iterational order** on T is $(T, \prec)$ such that

$$\{It_n(x_\lambda)\}_{n\in\omega} \prec \{It_n(y_\lambda)\}_{n\in\omega} \Leftrightarrow (-1)^{\varsigma^x_y}(It_{K^x_y}(x_\lambda) - It_{K^x_y}(y_\lambda)) < 0$$

Proposition 3.7. *$(T, \prec)$ is a linear order.*

Theorem 3.8. *For*

$$\begin{aligned}
g : [2, 4] &\to T \\
\lambda &\mapsto \{It_n((-10)_\lambda)\}_{n\in\omega}
\end{aligned}$$

g is an isomorphism between sets.

Lemma 3.9. *g is injective.*

Proof. For $\lambda_1 \neq \lambda_2$, suppose $\{It_n(1_{\lambda_1})\}_{n\in\omega} = \{It_n(1_{\lambda_2})\}_{n\in\omega}$, w.l.o.g. suppose $\lambda_1 < \lambda_2$,
then if $f^n_{\lambda_1}(-10), f^n_{\lambda_2}(-10) \in A_2$, then $f^{n+3}_{\lambda_1}(-10), f^{n+3}_{\lambda_2}(-10) \in (10, 20)$ and

$$\begin{aligned}
|f^{n+3}_{\lambda_1}(-10) - f^{n+3}_{\lambda_2}(-10)| &= |2f^n_{\lambda_1}(-10) - 10 - 2f^n_{\lambda_2}(-10) + 10| \\
&= 2|f^n_{\lambda_1}(-10) - f^n_{\lambda_2}(-10)|
\end{aligned}$$

If $f_{\lambda_1}^n(-10), f_{\lambda_2}^n(-10) \in A_3$ and if $f_{\lambda_1}^{n+1}(-10), f_{\lambda_2}^{n+1}(-10) \in A_1$, then $f_{\lambda_1}^{n+2}(-10), f_{\lambda_2}^{n+2}(-10) \in (10, 20)$ and

$$\begin{aligned}|f_{\lambda_1}^{n+2}(-10) - f_{\lambda_2}^{n+2}(-10)| &= |\lambda_1(f_{\lambda_1}^n(-10) - 15) - 10 - \lambda_2(f_{\lambda_2}^n(-10) - 15) + 10| \\ &\geq |\lambda_2(f_{\lambda_1}^n(-10) - 15) - 10 - \lambda_2(f_{\lambda_2}^n(-10) - 15) + 10| \\ &= \lambda_2|f_{\lambda_1}^n(-10) - f_{\lambda_2}^n(-10)| \\ &\geq 2|f_{\lambda_1}^n(-10) - f_{\lambda_2}^n(-10)|\end{aligned}$$

Therefore,

$$\{It_n((-10)_{\lambda_1})\}_{n\in\omega} = \{It_n((-10)_{\lambda_2})\}_{n\in\omega} \Rightarrow 2^c|f_{\lambda_1}^3(-10) - f_{\lambda_2}^3(-10)| < \frac{1}{2}\ , \forall c \in \omega$$

which is impossible. Hence,

$$\lambda_1 \neq \lambda_2 \Rightarrow \{It_n((-10)_{\lambda_1})\}_{n\in\omega} \neq \{It_n((-10)_{\lambda_2})\}_{n\in\omega}$$

which implies that f is injective. □

Definition 3.10. For a x, the **difference characteristic** of two iterations $\{It_n(x_{\lambda_1})\}_{n\in\omega}$ and $\{It_n(x_{\lambda_2})\}_{n\in\omega}$ is

$$K_{\lambda_2}^{\lambda_1} = min\{k \in \omega : It_k(x_{\lambda_1}) \neq It_k(x_{\lambda_2})\}$$

Definition 3.11. For a x, the **ordering characteristic** of two iterations $\{It_n(x_{\lambda_1})\}_{n\in\omega}$ and $\{It_n(x_{\lambda_1})\}_{n\in\omega}$ is the times of "3-0" appears before their iterations differ, i. e.

$$\varsigma_{\lambda_2}^{\lambda_1} = |\{n \in \omega : (n < K_{\lambda_2}^{\lambda_1}) \wedge (It_n(x_{\lambda_1}) = 3) \wedge (It_{n+1}(x_{\lambda_1}) = 0)\}|$$

Corollary 3.12. $K_{\lambda_2}^{\lambda_1}$ *exists.*

Lemma 3.13. $\lambda_1 < \lambda_2 \Rightarrow \{It_n((-10)_{\lambda_1})\}_{n\in\omega} \prec \{It_n((-10)_{\lambda_2})\}_{n\in\omega}$

Proof. For $\lambda_1 < \lambda_2$, consider $It_{K_{\lambda_2}^{\lambda_1}}((-10)_{\lambda_1})$, $It_{K_{\lambda_2}^{\lambda_1}}((-10)_{\lambda_2})$.

Note that if $\varsigma_{\lambda_2}^{\lambda_1}$ is even, then

$$\begin{aligned}&\|f_{\lambda_1}^m(-10)\| < \|f_{\lambda_2}^m(-10)\|, \forall m < K_{\lambda_2}^{\lambda_1} \\ &\Rightarrow (It_{K_{\lambda_2}^{\lambda_1}}((-10)_{\lambda_1}) = 2 \wedge It_{K_{\lambda_2}^{\lambda_1}}((-10)_{\lambda_2}) = 3) \vee (It_{K_{\lambda_2}^{\lambda_1}}((-10)_{\lambda_1}) = 0 \wedge It_{K_{\lambda_2}^{\lambda_1}}((-10)_{\lambda_2}) = 1) \\ &\Rightarrow (\lambda_1 < \lambda_2 \Leftrightarrow It_{K_{\lambda_2}^{\lambda_1}}((-10)_{\lambda_1}) - It_{K_{\lambda_2}^{\lambda_1}}((-10)_{\lambda_2}) < 0)\end{aligned}$$

Therefore, the only case that $\|f_{\lambda_1}^m(-10)\| > \|f_{\lambda_2}^m(-10)\|$ is that

$$f_{\lambda_1}^{m-1}(-10),\ f_{\lambda_2}^{m-1}(-10) \in A_3 \text{ and } f_{\lambda_1}^m(-10),\ f_{\lambda_2}^m(-10) \in A_0$$

So

$$\begin{aligned}\lambda_1 < \lambda_2 &\Leftrightarrow (-1)^{\varsigma_{\lambda_2}^{\lambda_1}}(It_{K_{\lambda_2}^{\lambda_1}}((-10)_{\lambda_1}) - It_{K_{\lambda_2}^{\lambda_1}}((-10)_{\lambda_2}) < 0 \\ &\Leftrightarrow \{It_n((-10)_{\lambda_1})\}_{n\in\omega} \prec \{It_n((-10)_{\lambda_2})\}_{n\in\omega}\end{aligned}$$

□

Definition 3.14. the **iterational metric** of T is

$$d_{it}(\{a_n\}_{n\in\omega}, \{b_n\}_{n\in\omega}) = \begin{cases} \frac{1}{K_b^a} & \text{if } \{a_n\}_{n\in\omega} \neq \{b_n\}_{n\in\omega} \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.15. *d_{it} is a metric.*

Proof. It is trivial that

$$d_{it}(\{a_n\}_{n\in\omega}, \{a_n\}_{n\in\omega}) = 0$$
$$d_{it}(\{a_n\}_{n\in\omega}, \{b_n\}_{n\in\omega}) = d_{it}(\{b_n\}_{n\in\omega}, \{a_n\}_{n\in\omega}) = \frac{1}{K_b^a}$$
$$d_{it}(\{a_n\}_{n\in\omega}, \{b_n\}_{n\in\omega}) \leq 0$$

Now we want to show the triangle inequality.
For $\{a_n\}_{n\in\omega}$, $\{b_n\}_{n\in\omega}$, $\{c_n\}_{n\in\omega} \in T$, suppose:

$$d_{it}(\{a_n\}_{n\in\omega}, \{b_n\}_{n\in\omega}) = \frac{1}{k_1}$$
$$d_{it}(\{a_n\}_{n\in\omega}, \{c_n\}_{n\in\omega}) = \frac{1}{k_2}$$
$$d_{it}(\{b_n\}_{n\in\omega}, \{c_n\}_{n\in\omega}) = \frac{1}{k_3}$$

If $\frac{1}{k_3} < \frac{1}{k_2}$, then $k_2 < k_3$. In this case, $c_m = b_m$ for all $m < k_3$. Therefore, $k_1 = k_2$. So

$$\frac{1}{k_1} + \frac{1}{k_3} \geq \frac{1}{k_2}$$

And therefore the triangle inequality holds. □

Theorem 3.16. *g is continuous.*

Proof. For all $\varepsilon > 0$, for all $\lambda \in [2, 4]$, there is an $\sigma < \frac{1}{5\times 2^{2k+1}}$ for some $k > \frac{1}{\epsilon}$ such that for all λ' such that $|\lambda' - \lambda| < \sigma$,

$$\begin{aligned}
|f_{\lambda'}^k(-10) - f_\lambda^k(-10)| &\leq |f_{\lambda'}^3(-10) - f_\lambda^3(-10)| \times 4^k \\
&= 5 \times |\lambda' - \lambda| \times 4^k \\
&< 5 \times \frac{1}{5 \times 2^{2k+1}} \times 4^k \\
&= \frac{1}{2}
\end{aligned}$$

So $d_{it}(\{It_n((-10)_{\lambda_1})\}_{n\in\omega}, \{It_n((-10)_{\lambda_2})\}_{n\in\omega}) < \frac{1}{k} < \varepsilon$

□

Corollary 3.17. *g is surjective.*

Proof. Note that $g(2) = minT$ and $g(4) = maxT$. Suppose there is an $\{a_n\}_{n\in\omega} \in T$ such that $\{a_n\}_{n\in\omega} \notin im(g)$, then neither $T_1 = \{\lambda \in [2, 4] : g(\lambda) \prec \{a_n\}_{n\in\omega}\}$ nor $T_2 = \{\lambda \in [2, 4] : g(\lambda) \succ \{a_n\}_{n\in\omega}\}$ is empty. Also, $T_1 \cup T_2 = [2, 4]$. By Lemma 2. 12, either T_1 has a maximum or T_2 has a minimum is true. W.l.o.g, assume T_1 has a maximum.
Note that for all $\{b_n\}_{n\in\omega} \in T$ such that

$$d(\{b_n\}_{n\in\omega}, g(maxT_1)) < d(g(maxT_1, \{a_n\}_{n\in\omega})$$

we have

$$\{b_n\}_{n\in\omega} \prec \{a_n\}_{n\in\omega}$$

Therefore, for $\lambda = maxT_1$, for all $\sigma > 0$, there is an λ' such that $|\lambda' - \lambda| < \sigma$ and $d(g(\lambda), g(\lambda')) > d(g(maxT_1, \{a_n\}_{n\in\omega})$, which is contradiction with the fact that g is continuous. Therefore, g is surjective.

□

Proof for $x < y$ iff x is closer than y to 0 in I_a

Proof. Consider any points x, y on I_a
Saying x is closer than y to 0 is the same as:

$$|x - 0| < |y - 0|$$

$$|x| < |y|$$

Since x, y are in the range of $[0, 1]$ this is:

$$x < y$$

This is known to be a linear order for numbers hence is a valid linear order. □

Proof for points closer to 0 map to points closer to 0 except I_{c2} to I_a

Proof. Consider any point x, y that are not on I_{c2} such that $x < y$
Let $x_{new} = f(x)$ and $y_{new} = f(y)$
We Know that 0 must map to 0 and that the map is continuous
This means that all points from 0 to y map to points in between 0 and y_{new}
since x is in the range of $(0, y)$

$$x_{new} < y_{new}$$

Consider any point x, y that are in I_{c2} and map to I_a such that $x < y$
Since the functions is continuous and I_{c1} maps to all of I_a,
The start of I_{c2} must map to the end of I_a which is 1
Since the function is continuous and a bijection,
All points between the start of I_c2 and y should map to points between 1 and y_{new}
since x is in the range of $(y, 1)$

$$x_{new} > y_{new}$$

Consider any point x, y that are in I_{c2} and map to I_b such that $x < y$
Let z be the point in I_{c2} that maps to 0 in I_a and I_b
Since the function is continuous and a bijection all points from the start of I_{c2} to z map to all of I_a
Since x, y map to I_b both must be $> z$
All points between z and y should map to points between 0 and y_{new}
since x is in the range of (z, y)

$$x_{new} < y_{new}$$

□

4. Conclusion

We have thus developed various theorem about trees of the structure mentioned above and plan to continue the research beyond this semester. We would like to look at various other types of trees and also expand to different functions such as quadratic and polynomial functions.

5. References

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